Lab - 7: Random Walk

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MC312, Modeling and Simulation

In this lab we numerically, analytically and visually analyze the random-walk model for 1-D and 2-D by performing Monte-Carlo simulation. We considered two different type of random walk namely biased random walk and unbiased random walk and produced visual simulation of the same.

I. INTRODUCTION

Random walk refers to the apparently random movement of an entity[1]. Random walk is an event that is function of time in which we have an entity that moves on a grid of cell. For 1-D we have assumed that walker can go only 1-step forward or 1-step backward and in for 2-D it can go only 4 or 8 neighbouring cells. Also we have considered a case where step length is variable and it is coming from a normal distribution. Then we perform Monte Carlo simulation of random walk to observe final outcome. Random walk is stochastic process, so we study the outcomes in terms of statistical quantities such as average, variance, probability, standard deviation etc.

II. MODEL

A. Random Walk in 1 Dimension

We start with the simplest problem of 1D random walk. We assume that the entity starts at origin of the number line. It can only move forward and backward by 1 unit of distance. For unbiased random walk we are making an assumption that probability of moving forward and backward is equal. We want to study the results after entity took N steps. Suppose, d_i represents the step taken by entity after i^{th} step. d_i will either be +1 or -1. +1 represents that entity moves one step forward and -1 represents 1 step backward. Then we can write total displacement(or final position) d_{tot} of the entity after N step as

$$d_{tot} = d_1 + d_2 + d_3 + d_4 + ... + d_N \tag{1}$$

Obviously, d_{tot} will vary every time we repeat the simulation. So a typical N-step Random Walk can have many possible paths. So for calculating any specific quantity like mean or variance of final position in N-step Random Walk, we have to take many simulations of the same N-step Random Walk and then take the average. If we repeat the experiment many times or in other words, if we perform Monte Carlo simulations of random walk, then average total displacement will be zero which is explained below. Taking average both the side in Eq. (1)

$$< d_{tot} >=< d_1 > + < d_2 > + < d_3 > + < d_4 > +...+ < d_N > \tag{2}$$

As d_i takes values either 1 or -1 randomly, for Monte Carlo simulations, we can consider $< d_i >$ to be zero. Hence, $d_{tot} = 0$. This result does not provide much information about final displacement. So, we look for d_{tot}^2. We can write,

$$< d_{tot}^2 >= \sum_{i=1}^{N} < d_i^2 > +2 \sum_{i=1}^{N} \sum_{j=1}^{i-1} < d_i d_j > \tag{3}$$

Now, d_i^2 can only be +1. Hence $< d_i^2 >= 1$. On the other hand, product of d_i adn d_j can have two possible value -1 and 1 with equal probability. Therefore, $< d_i d_j >= 0$. Now, we can write

$$< d_{tot}^2 >= \sum_{i=1}^{N} 1 + 0 \tag{4}$$

$$< d_{tot}^2 >= N \tag{5}$$

In biased random walk, entity is biased to move toward particular direction, either in forward or backward direction. Let us assume that entity is biased towards moving forwards with probability p. Now, we define $P(x, N)$ as finding entity at position x after N steps. Probability that entity takes k steps forward and $N - k$ steps backward, then using binomial distribution, we write

$$P(k, N) = \binom{N}{k} \cdot p^k \cdot (1 - p)^{N-k} \tag{6}$$

Here, we have take k step forward and N - k steps backward to have a final position at x so,

$$1 \cdot k - 1 \cdot (N - k) = x \tag{7}$$

$$x = 2 \cdot (k) - N \implies k = \frac{N + x}{2} \tag{8}$$

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As we keep on increasing N, Monte Carlo simulation tends to produce Gaussian distribution. We calculate mean and variance,

$$mean = N \cdot (2p - 1) \tag{9}$$

$$variance = 4Np(1 - p) \tag{10}$$

B. Random Walk in 2 Dimensions

In 2D random walk the entity moves on 2D grid. We can thought 2D random walk as independent 1D random walk in two direction. The entity starts at origin and can move along North, East, West and South direction at every step. Going East and West can be considered as 1D random walk on x axis and going North and South can be considered as 1D random walk on y axis. We can show same calculation as shown in 1D random walk to prove that net average displacement is zero. Also, we calculate Pythagorean distance $\sqrt{x^2 + y^2}$ from the origin. 2D random walk has also two variants, Unbiased and biased random walk. In next section, we have shown simulations results for both model.

III. RESULTS

A. 1D Random Walk

1. **Unbiased Random Walk:** For the first case we consider an unbiased random walk i.e. the events of the walker going in +ve x direction or -ve x direction are equally probable.
   Mean Distance travelled: On implementing the case of unbiased random walk for 1D we found that as the number of steps increases the mean distance travelled also increases.

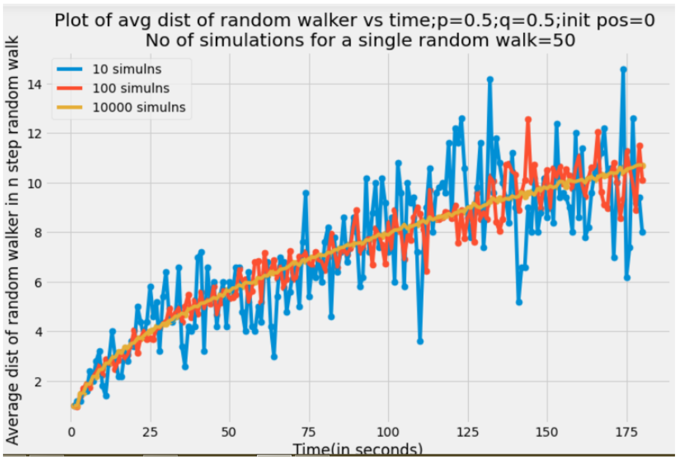

FIG. 1: Average Distance travelled

Average position: From Fig. 3 we can see that as the number of steps increases the average position of the walker tends towards 0. This should be the case because for very large number of steps the number of moves in the +ve x direction would be almost equal to the number of moves in the -ve x direction.

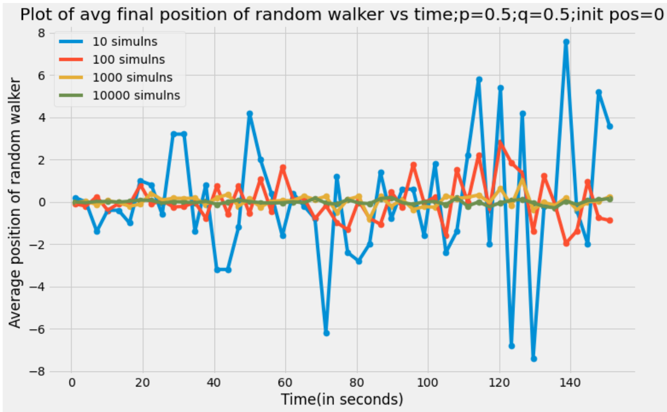

FIG. 2: Average Final Position of the walker in unbiased 1-D random walk

Variance in distance: From the Fig 4 we can see that for more number of simulations the plot of variance tends towards the line y=x, because with increase in number of steps the walker has more possible options to travel and hence the variance increases.

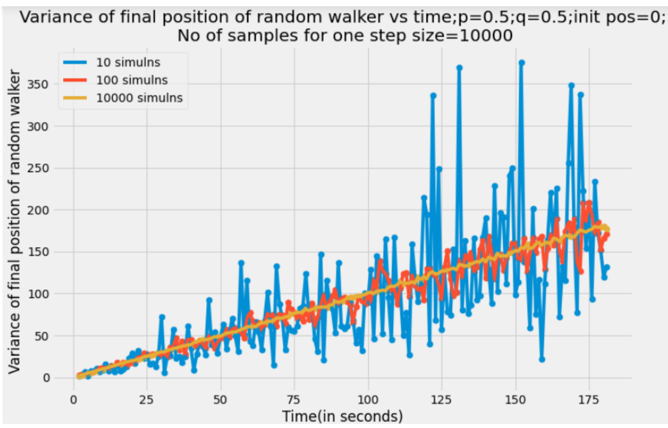

FIG. 3: Variance in the final position in the 1-D Random Walk

Histogram:

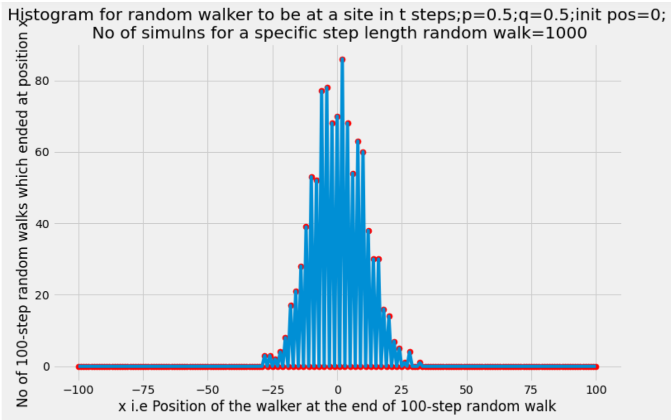

FIG. 4: Histogram showing the probability distribution for unbiased 100 step 1D Random Walk.

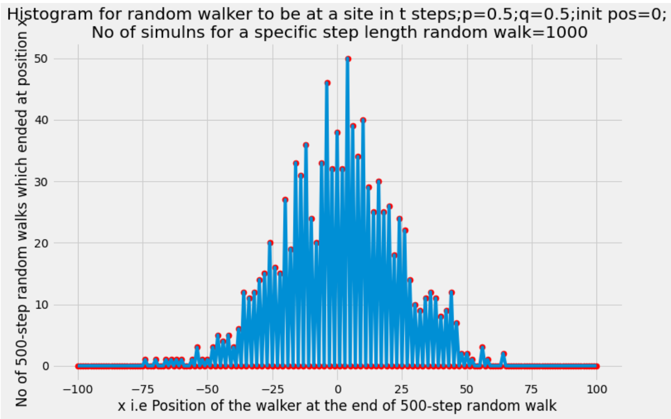

FIG. 5: Histogram showing the probability distribution for unbiased 500 step 1D Random Walk.

From the above figures, we see that histograms obtained are approximately of binomial distribution and as we increase N , it will converge to the bell-shaped normal distribution. Also we know that as N increases , variance also increases linearly , and hence the bell-shape will be wider as we increase N(because variance increases correspondingly which means final position random variable is spread across many different values .)

2. **Biased Random Walk:** Now we look at the case of biased random walk, where the walker is not equally likely to go in any direction. As the biasing probability increases, walker tends to move towards positive +ve x direction.

   Mean Distance travelled: On implementing the case of biased random walk for 1D with probability p(probability of moving forward) = 0.6 and q(probability of moving backward) = 0.4.

   Average position: From Fig. 6 we can see that this time as the number of steps increases the average position of the walker is not close to 0 but is in

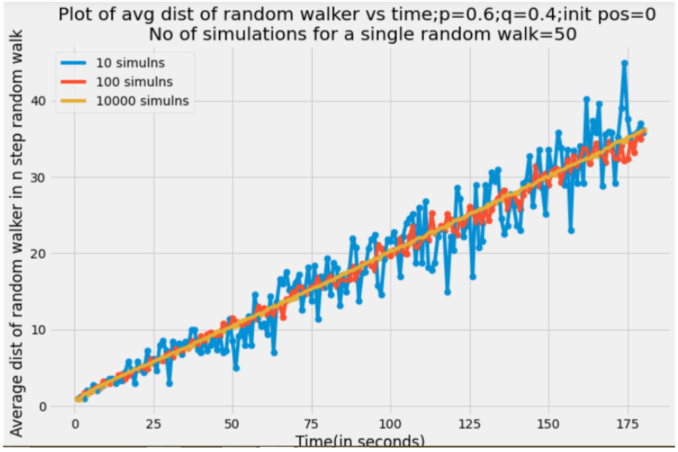

FIG. 6: Average Distance travelled for biased 1-D Random Walk p = 0.6 and q = 0.4

the +ve x direction. This is because the probability of the walker going in the +ve x direction is more as compared to that going in the -ve x direction.

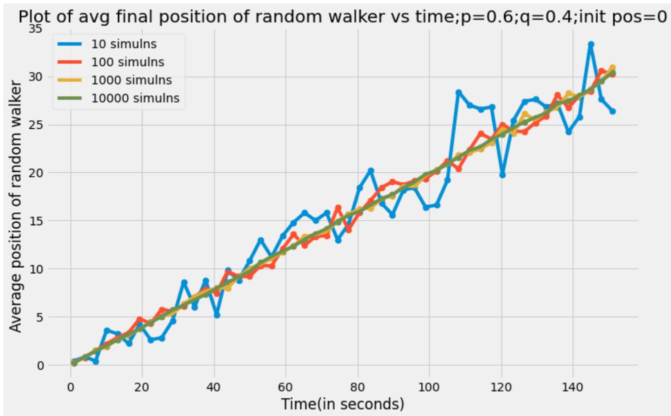

FIG. 7: Average Final Position of the1-D walker with p=0.6 and q=0.4

From the above figure, we can see that average final position of the random walker takes more positive value when we increase no of steps.This is because the distribution from the which values of steps are coming is favouring +ve step 60 % of time , so as we increase no of steps , the walker's final position is likely to take a higher positive value on an average.

Variance in position: From the Fig 4 we can see that for more number of simulations the plot of variance tends towards the line y=x, because with increase in number of steps the walker has more possible options to travel and hence the variance increases.

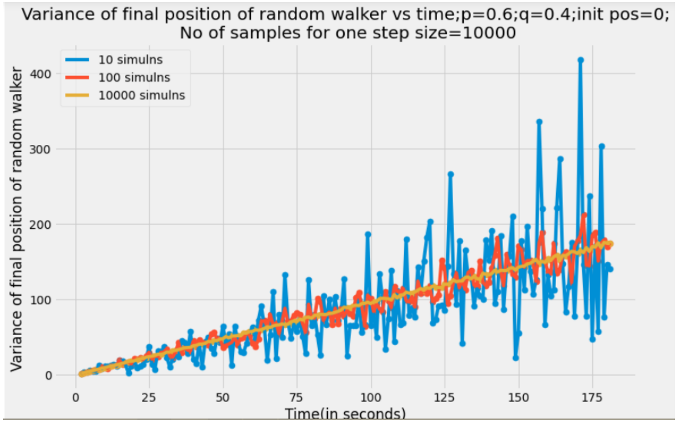

FIG. 8: Variance in the final position for different number of simulations

Histogram:

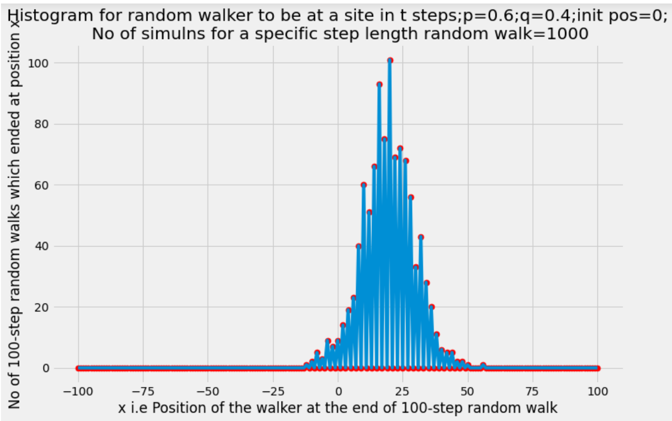

FIG. 9: Histogram showing the probability distribution for biased 1D walk when p=0.6 and q=0.4.

The above figure shows histogram or frequency distribution(or probability distribution) for a biased random walker to be at a position at x in a 100-step random walk. We have done 1000 simulations for a 100-step random walk and have plotted the frequency for the random walker to be at different positions. As like previous cases it turns out to be a binomial distribution(which in limit when no of steps tends to infinity will be a gaussian distribution). By comparing with 4 of the unbiased walk case, we see that the histogram has only shifted towards right and the bell shape in case of biased walk is centered at the mean$(N.(p-q) = 100(0.6-0.4) = 20)$ of the biased distribution.

3. **1-D unbiased Random Walk with step length coming from Normal Distribution:** Here instead of assuming step length to be 1 we are taking the step length values coming from a normal distribution whose mean is 0 and variance is 1.

Mean Distance travelled: Here also we observe that mean distance increases with no of steps.

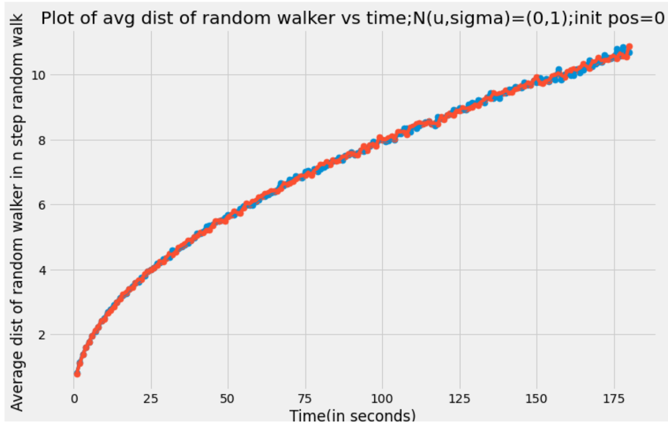

FIG. 10: 1000 simulations performed for a specific step Random Walk

Variance in position: Like the previous cases, here also variance linearly increases with no of steps, which is consistent with the formula derived in class.Which shows that it behaves independently of the distribution from which step length is coming, and this behaviour will remain same whenever the individual step length are independent random variables.

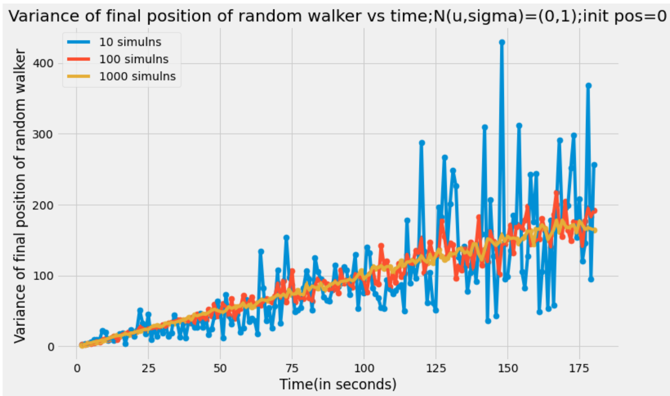

FIG. 11: variance of final position where step length is normal

B. 2D Random Walk

1. **Unbiased Random Walk:** For the first case we consider unbiased random walk i.e. the walker has an equal probability of going in any of the 4 directions.
Mean Distance travelled: The following figure shows the average distance travelled by the hiker with increasing number of steps. It can be seen that as the number of steps increases the mean distance travelled also increases.

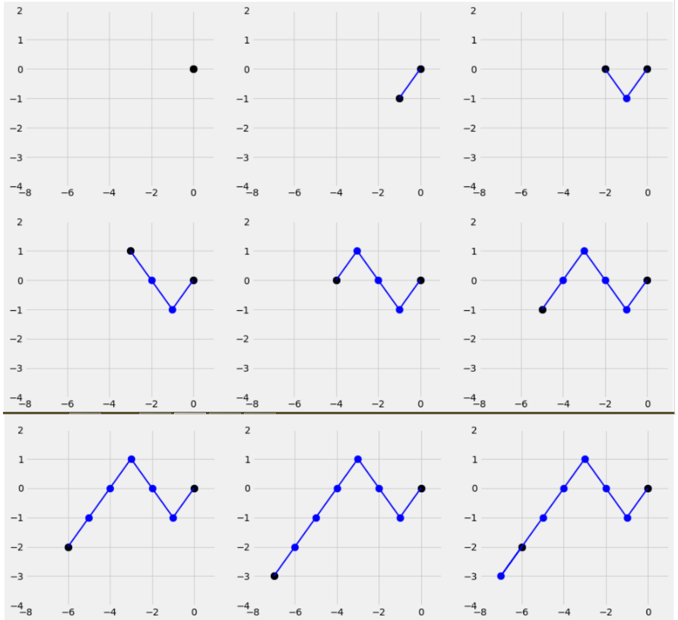

FIG. 12: Simulation of unbiased 2-D Random Walk, at each step 4 possible directions

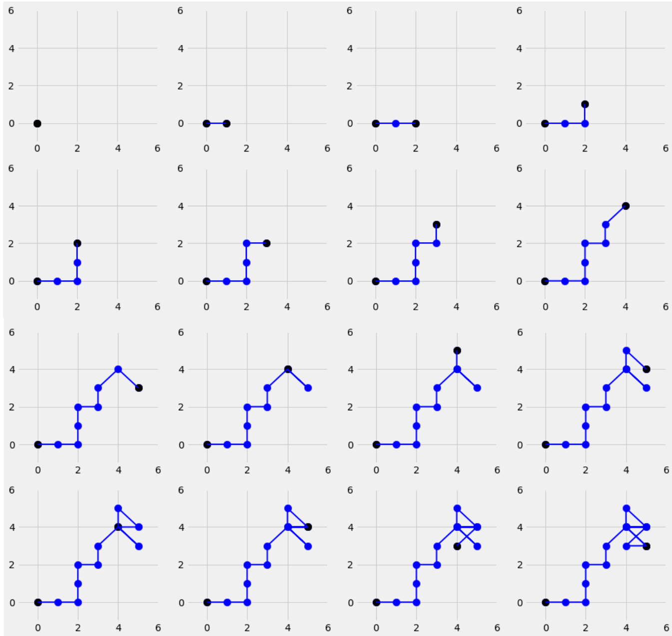

FIG. 14: Simulation of unbiased 2-D Random Walk, at each step 8 possible directions

Average Distance Travelled: In 2-D random walk also, the average distance of the random walker follows similar behaviour as with previous cases, it is like $\sqrt{N}$, i.e. it increases with no of steps.

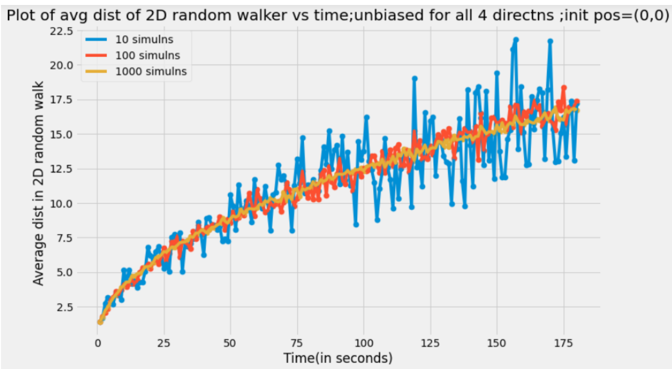

FIG. 13: Average distance travelled in 2-D Random Walk

2. **Biased Random Walk:** For the case of biased random walk we consider the case that a hiker is standing in the dark and we are given the probabilities of directions in which he tends to move: N: 0.19; NE: 0.24; E: 0.17; SE: 0.10; S: 0.02; SW: 0.03; W: 0.10; NW: 0.15. The figure shown below shows the random walk generated by giving the probabilities as input.

   Observation: As we see in 14, the random walker tends to be in right(or in east direction) because, the probability of going towards east + probability of going towards northeast + probability of going towards southeast is 0.51, where as the probability of going towards west + probability of going towards northwest + probability of going towards southwest is 0.28. So in the simulations we see that the random walker generally goes towards east direction, this is an example of biased 2-D Random Walk.

IV. CONCLUSIONS

We have observed that unbiased random has zero average displacement. Variance is linear function of number of steps. In other words, Average distance after N the step is approximately $\pm\sqrt{N}$. When N is significantly increased, distribution tends to be Gaussian distribution following Central Limit Theorem. Hence, we see stationary graph. On the other hand, biased random walk has non zero average displacement. The slop of variance decreases in case of biased random walk. The slope of variance is maximum in case of unbiased random walk. We observe similar thing in 2D random walk.

[1] A. Shiflet and G. Shiflet, *Introduction to Computational Science: Modeling an Simulation for the Sciences*, Princeton University Press.3, 276 (2006).