Lab -4

Kashyap Halavadia (202003040)* and Jay Patel (202003019)†
Dhirubhai Ambani Institute of Information & Communication Technology,
Gandhinagar, Gujarat 382007, India
MC312, Modeling and Simulation

In this lab, we modelled the population growth in a system using the logistic equation and a certain modification of it. In first part we analysed the effect of the rate of harvesting on the population growth , also we found the threshold value. If harvesting is performed at a rate which is above threshold value then for any initial value , the population will become extinct. Also we made some modifications in the basic logistic equation model in the second part.

I. INTRODUCTION

In first part we used logistic equation for modelling population growth, in the problem we had to find the threshold value of the harvesting rate, in the two different cases. First, when the harvesting rate is constant and the second in which the harvesting rate is proportional to the population at time t.

II. MODEL

We used logistic equation in the first part.All the assumptions of the logistic model are also made here, that the inflow and outflow from the population is only through birth and death respectively. In the death rate variable we capture the interaction factor among the population, it becomes dominant when population size is large enough. In the first part there were two cases :-

A. Using Logistic equation

1. *Harvesting rate is constant*

$$\frac{dx}{dt} = rx(1 - \frac{x}{k}) - h \tag{1}$$

Where r is growth rate ,k is carrying capacity and h(constant) is rate of harvesting. We make the above equation dimensionless to get rid of the parameters, and we get:-

$$\frac{dX}{dT} = X(1 - X) - H \tag{2}$$

Where $X = x/k$, $T = tr$ and $H = \frac{h}{rk}$

2. *Harvesting rate is proportional to x*

$$\frac{dx}{dt} = rx(1 - \frac{x}{k}) - \epsilon x \tag{3}$$

Where r is growth rate ,k is carrying capacity and ϵ(constant) is rate of harvesting. We make the above equation dimensionless to get rid of the parameters, and we get:-

$$\frac{dX}{dT} = X((1 - \alpha) - X) \tag{4}$$

Where $X = x/k$, $T = tr$ and $\alpha = \frac{\epsilon}{r}$

B. Introducing cubic order non-linearity

Since the Logistic equation predicts that any initial population over zero leads to a non zero final population. A more realistic situation would be that a nonzero final population results only if the initial population is above a threshold. To model this we write the beow equation :-

$$\frac{dX}{dT} = -X(X - k_1)(X - k_2) \tag{5}$$

Where k_1 and k_2 are fixed points , also we can say that k_1 is threshold value of the initial population and k_2 is carrying capacity of the system. For any initial value of the population below k_1, it will become extinct and for any other initial value, it will converge to the carrying capacity.

III. RESULTS

By simulating the equation (2) in the program , we get the graph of x vs t as below:-

The below figure(1) shows that with H=2, if we start with any $X_0 < 0.276$ (with initial population $x_0 < X_0k$), then it will become extinct and if we start with $X_0 >$

---

*Electronic address: 202003040@daiict.ac.in
†Electronic address: 202003019@daiict.ac.in

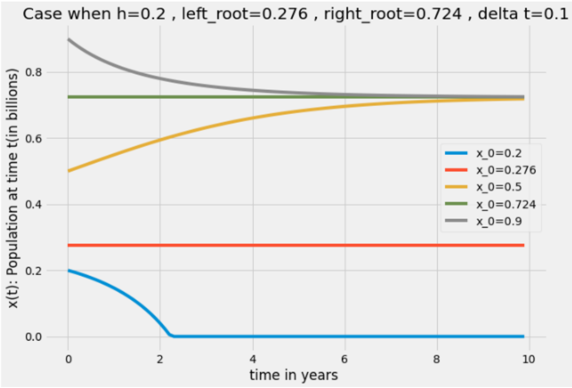

FIG. 1: x(t) vs t graph when H=0.2, note that H=h/rk

0.276 i.e. (with initial population $x_0 > X_0k$) , then it will converge to 0.724 billion(carrying capacity). Hence we can say that 0.276 is unstable fixed point and 0.724 is stable fixed point.

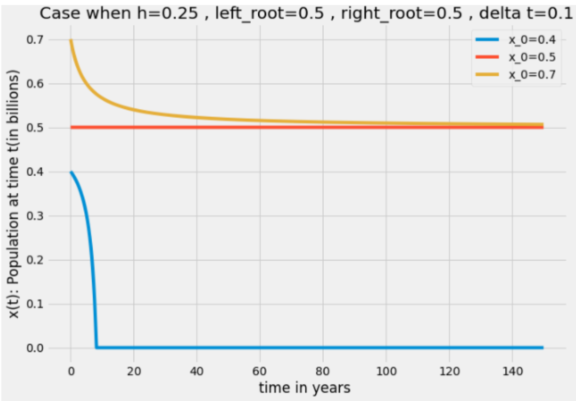

FIG. 2: x(t) vs t graph when H=0.25(threshold value of H), note that H=h/rk

From above fig 2,we can see that for any $X_0 < 0.5$ or $(x_0 < 0.5k)$, will extinct to 0, and any $X_0 > 0.5$ $(x_0 > 0.5k$) will converge to 0.5.

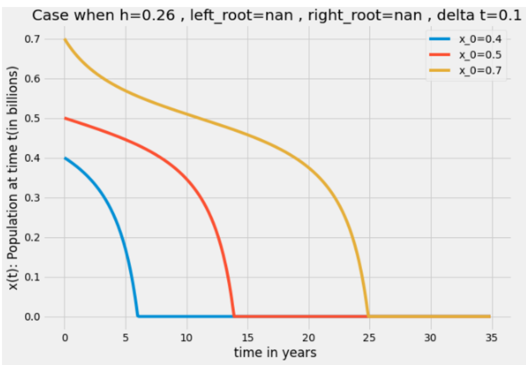

FIG. 3: x(t) vs t graph when H=0.26(greater than threshold value of H), note that H=h/rk

From above fig 3,we can see that for any $X_0 < 0.5$or $(x_0 = Xk)$, will extinct to 0.That is there is no fixed point in the system. As the value of H has exceeded 0.25, there will be no roots for dX/dT.So we can conclude that H=0.25 or h=0.25rk is the maximum harvesting rate that can be done, because if harvesting is done at a rate above it then the population will become extinct.

Harvesting rate is proportional to x(t)

By simulating the equation (4) in the program , we get the graph of x vs t as below:-

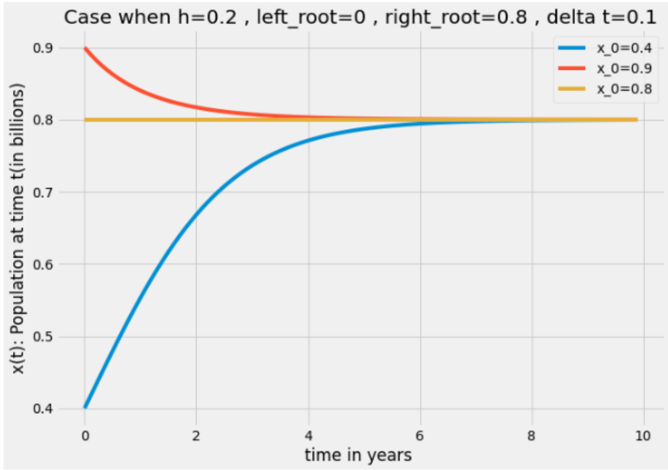

FIG. 4: x(t) vs t graph when $H = 0.2$ or $\alpha = 0.2$, note that $\alpha = \epsilon/r$.Graph will be of same nature when $0 < \alpha < 1$ or $0 < \epsilon < r$

In the above figure 4 , we can see that for any positive value of X_0, X converges to 0.8 (initial population converges to 0.8 k billion)

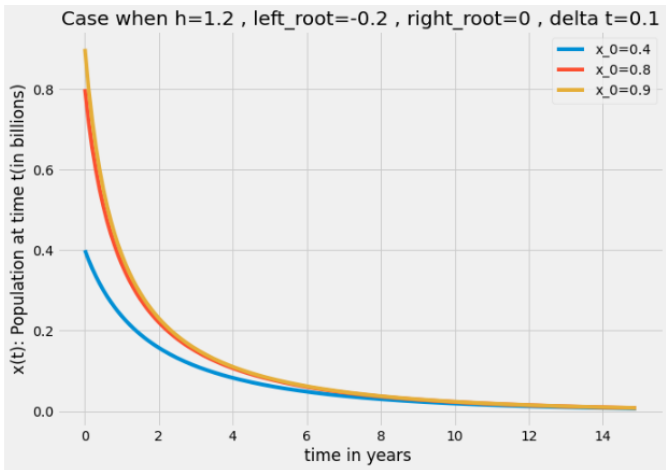

FIG. 5: x(t) vs t graph when $H = 1.2$ or $\alpha = 1.2$, note that $\alpha = \epsilon/r$. Graph will be of same nature when $\alpha > 1$ or $\epsilon > r$

From 5, we can see if we start with any positive X_0, X will converge to 0, i.e. the population will become extinct.So $\alpha < 1$ needs to be satisfied while harvesting in this case i.e. $\epsilon/r < 1$ i.e. $\epsilon < r$. The rate of harvesting must be less than growth rate of the population to prevent extinction.

Introducing cubic order non linearity

To have the feature that the population will converge to carrying capacity only if it is above certain threshold can be captured by the (5). By simulating the (5) in program for $k_1 = 1$ and $k_2 = 2$, we get the following graph:-

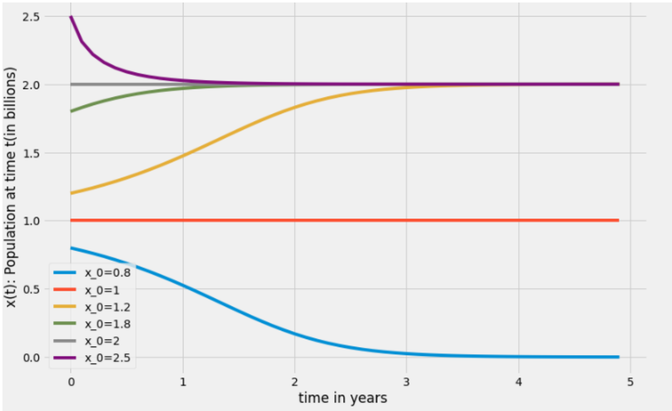

FIG. 6: x(t) vs t graph when there are 3 fixed points at x=0,1,2.Stable fixed points at x=0 and x=2 and unstable at x=1.Note that plot for $X_0 = 0.8$ is converging to 0 and is not converging to 2, unlike in the case of logistic equation.

From 6, we can see that any initial $X_0 < 1$, will converge to 0, and any $X_0 > 1$ will converge to the carrying capacity i.e. 2. So, here we can say that the threshold value of the initial population is 1 billion and the carrying capacity=2 billion. Whereas in case of logistic equation model, any non-zero population would have converged to carrying capacity whereas here it converges to carrying capacity only when it is above certain threshold value.

IV. CONCLUSIONS

We studied and analysed population growth using logistic equation and some of its variants. In the first part we analysed what can be the maximum harvesting rate that is feasible, in the sense that even after harvesting at that max rate the population will not become extinct. And in the second part we modified the logistic equation to make it more realistic by adding the feature of threshold population.