Lab -1

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In this lab we numerically and analytically analyzed two radioactive decay reactions happening in a compartment setting, where the product of the first radioactive decay becomes the reactant for the second reaction. We made observations for different decay rates of the two radioactive substances involved in the reaction and also made some approximations based on certain assumptions .

I. INTRODUCTION

There are many systems in the physical world where the rate of change of a substance is directly proportional to the amount of substance . One of such scenarios occur in the radioactive decay reaction of a substance . Here we consider a system where we have some initial amount of substance A, A decays to another substance B and then B further decays to another substance C, thus forming a chain of reactions. The dynamics of the system like the the time of radioactivity and the amount of substances at different time steps depend on decay constants of A and B and also on the initial amount of A i.e. A_0 taken.

II. MODEL

We can analyze the given problem using difference equations:-

$$A_{t+\Delta t} = A_t - (\alpha_A A_t).\Delta t \tag{1}$$

$$B_{t+\Delta t} = (\alpha_A A_t).\Delta t - (\alpha_B B_t).\Delta t \tag{2}$$

$$C_{t+\Delta t} = (\alpha_B B_t).\Delta t \tag{3}$$

We have used these difference equations for the simulation and plotting of graphs. Eq. (1) depicts the amount of A after a time interval of Δt, as A is decaying its amount will decrease with time.Eq. (2) depicts the amount of B after a time interval of Δt, decay of A is growth for B, as B is formed from A and then B also decays proportional to the amount of B formed. Eq. (3) depicts the amount of C after a time interval of Δt, decay of B is growth for C, as C is formed from B.

Assumptions made in this model are that we are taking sufficient number of identical atoms of A in the beginning so that we can avoid the randomness factor involved in the decay of single radioactive atom and all throughout the radioactive chain reactions , we are avoiding randomness in the radioactive decay.

III. RESULTS

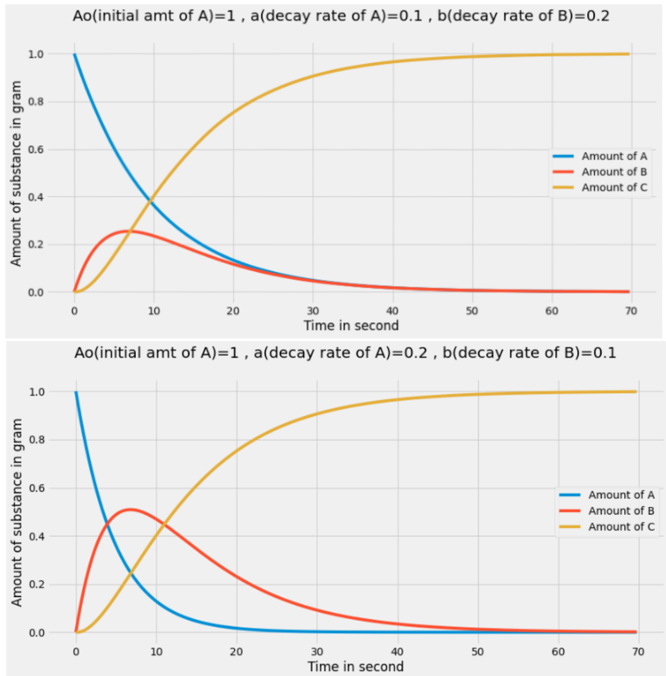

FIG. 1: Amount of A,B and C with respect to time in Case 1: $a < b$ and Case2: $a > b$ for both starting with same A_0

Fig. 1 shows the graph formed by the computer program by simulating the difference equations[(1),(2),(3)], we have discretized the problem by plotting the amount of substances after every time step of 0.25 second.

We see in Fig 1 that amount of A exponentially decreases as A is purely decaying and we know that by solving the differential equation corresponding to the above difference equation[(1)], we will get exponential term. Amount of B increases for some time because decay of A will result in forming of B, decay of B will be significant after a certain amount of B is formed and then decay of B will dominate over formation of B and amount of B will start decreasing.Amount of C will exponentially increase after the time decay of B starts.

Also we observe in several cases when $a < b$, the ratio of $\frac{B}{A} \approx \frac{a}{b-a}$. Intuitively when $a < b$, after some time both A and B should decay at the same rate because

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initially small amount of A decays to form B, so there is small amount of B and larger decay constant of B and decay of B is the product of decay rate constant of B and amount of B . Also we can see that amount of B will be always less than that of A in cases when $a < b$. Decay of A will be approximately equal to that of B because it has smaller decay rate and relatively greater amount than B at any point in time.
. Analytically solving for A and B gives us that:-

$$B = \frac{a.A_0}{b-a}.(e^{-at} - e^{-bt}) \tag{4}$$

$$A = A_0 e^{-at} \tag{5}$$

As $b > a$ and $t \to \infty$ so $B \approx \frac{aA_0e^{-at}}{b-a}$ and so $\frac{B}{A} \approx \frac{a}{b-a}$. Also for finding the maximum amount of B we can take the derivative of (4) and equate it to 0.The maximum amount of B occurs at the time:-

$$t_{max} = \frac{\ln(a) - \ln(b)}{a-b} \tag{6}$$

Max amount of B can be found by putting t_{max} in (4).
Also we can notice in Case2 of Fig 1 that when $a > b$,such an approximation cannot be formed because in Eq. (4) if we approximate the first $e^{-at} = 0$ then the amount of B will become -ve which is not possible.The below figure demonstrates the case for radioactive chain reaction of $Bi->Po->Pb$ where a¡b:- .
The approximate maximum amount of B using sim-

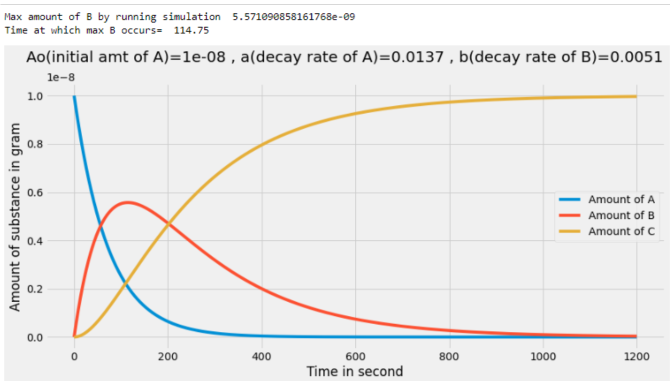

FIG. 2: Radioactive chain reaction:- $Bi->Po->Pb$ where $A = Bi, B = Po$ and $C = Pb$

ulation =5.571 $* 10^{-9}$ and the approximate time at which max B occurs =114.75 sec using simulation . By putting values of a and b in the analytical formula of max time (6),we get time of max amount of $B = 114.90$ seconds. Hence the analytical and experimental results are matching.

We observe from (1) and (2) that at any point in time, A would be disintegrating to form B and B would be disintegrating to form C . So total no of disintegrations per second will be:-

$$\frac{dD}{dt} = aA + bB \tag{7}$$

For finding maximum no of disintegrations, we will equate (6) to 0 and will find that:-

$$t_{max} = \frac{ln(a-b+1)}{(a-b)} \tag{8}$$

Keeping b=0.2 fix in the above equation , we find that the time of maximum total radioactivity(i.e. t_{max}) monotonically decreases as a(decay constant of A) goes from 0.1 to 1. Intuitively we can understand that when decay rate of A increases, the time of maximum total radioactivity decreases as reaction will complete in short duration of time because there will be more number of disintegrations in shorter period of time. In 1 case the reaction is getting completed in about 70 seconds whereas in 2,it completes in about 1200 seconds.

For the case when $a << b$(a is much much smaller than b), then $e^{-at} \approx 1$ and $e^{-bt} \approx 0$, So in this case we can approximate $A \approx A_0$ and $B \approx \frac{aA_0}{(b-a)}$.This is demonstrated in the below figure:-
In the above simulation , the maximum amount of

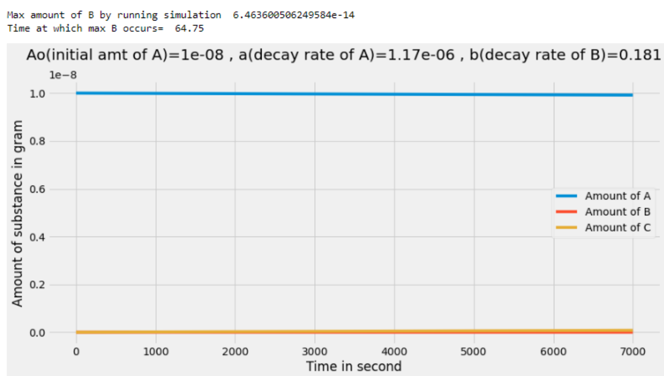

FIG. 3: Radioactive chain reaction:- $Ra->Rn->Po$ where $A = Ra, B = Rn$ and $C = Po$

$B = 6.465 * 10^{-14}$ and this maximum B, first occured at t=64.75 seconds. In fact we can see that amount of B will remain this only for the rest of the time of the reaction.By putting values of a and b in the analytical formula of max time (6),we get time of max amount of B = 66 seconds.Hence the analytical and experimental results are matching.

IV. CONCLUSIONS

We modelled the radioactive chain reactions using differential equations and for for simulating them on computer we have used corresponding difference equations and have discretized time as small time steps so that we can plot amount of substance with respect to time for the chain of reaction . We have studied the radioactive chain reaction in a compartment model setting, where the product due to radioactive decay of 1st substance creates another radioactive substance which decays to create another substance. We analyzed the effects of varying the rate constants of the two radioactive substances. We saw for the cases when (decay constant of A)$a < b$(decay constant of B) , $a > b$ and $a << b$. We simulated programs for each of the above case and experimentally found the maximum amount of substance B that will be formed and also found at what time maximum amount of B will be formed. We then compared these experimental values with the analytically found values and concluded that both were matching. We had also seen that how the time at which the maximum disintegrations happen varies with the rate constants(a and b).We found that the time at which the maximum total radioactivity occurs decreases with the increase in value of rate constants of A and B.