3rd Report, Computer Simulations

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In this report, we will shortly describe, and then compare, two methods of estimation of the Hurst index for Fractional Brownian Motion (further called FBM). Before we proceed with analysis results, let us first start with introducing FBM and the Hurst index. It is also important to note that all the computations were performed in Python.

1 Fractional Brownian Motion

Fractional Brownian motion (FBM) is a generalization of Brownian Motion, where increments are not independent. It can be described by the following stochastic representation

$$B_H(t) = \frac{1}{\Gamma(H+\frac{1}{2}}\left(\int_{-\infty}^{0}\left[(t-s)^{H-1/2} - (-s)^{H-1/2}\right]dB(s) + \int_0^t (t-s)^{H-1/2} dB(s)\right), \tag{1}$$

where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\exp(-x)dx$ and $0 < H < 1$ is the Hurst parameter, which we will describe in net section

Often the Fractional Brownian Motion is also described by it's autocovariance function γ

$$\gamma(k) = \frac{1}{2}\left(|k-1|^{2H} - 2|k|^{2H} + |k+1|^{2H}\right). \tag{2}$$

To get a better understanding of how this process behaves, we will show a few trajectories and basic properties of FBM. To simulate the sample, due to performance and accuracy of methods, we decided to use Davies and Harte method.

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2 Hurst index

Hurst parameter H describes the memory of a process. Different values of this attribute correspond to particular processes

1. $0 < H < \frac{1}{2}$ – short memory process,
2. $H = \frac{1}{2}$ – geometric random walk,
3. $\frac{1}{2} < H < 1$ – long-range dependent process.

In particular the fractional Brownian Motion with $H = \frac{1}{2}$ is the classical Brownian Motion.

This parameter is also closely related to a specific property of a process – self-similarity. That can give a better understanding to what H really is. A process $X = (X(t))_{t>0}$ is called self-similar (ss) if it satisfies

$$X(at) \stackrel{d}{=} a^H X(t), \tag{3}$$

where H is the Hurst parameter.

3 Lag variance method

Let's denote an arbitrary lag of the time series X_t by τ. Then, the variance for lag τ can be expressed as

$$\mathrm{Var}(\tau) = \mathrm{Var}(X_{t+\tau} - X_t). \tag{4}$$

We can write down the relationship between the variance and the lag

$$\mathrm{Var}(\tau) \approx \tau^{2H}, \tag{5}$$

where H — Hurst index. Having this relationship we can easily estimate the Hurst index by simply computing the slope of $\log(\mathrm{Var}(\tau))$ against $\log(\tau)$. Obtained slope, divided by 2, is our H estimator.

4 Rescaled range method

This method is based on the self-similarity property of considered process $X = (X(t))_{t>0}$. First we have to find the mean ($M = \frac{1}{n}\sum_1^n X_i$) and standard deviation $S(n) = \frac{1}{n}\sum_1^n (X_i - M)^2$). Next we construct a mean adjusted series $Y_i = X_i - M$ and a cumulative deviation $Z_t = \sum_1^n Y_i$. We use that to calculate the range

$$R(n) = \max(Z_1, Z_2, \ldots, Z_n) - \min(Z_1, Z_2, \ldots, Z_n) \tag{6}$$

Finally from that we can calculate the rescaled range $P(n) = \frac{R(n)}{S(n)}$ and from that obtain the estimation of Hurst parameter

$$\widehat{H} = \frac{\log P(n)}{\log n} \tag{7}$$

References

[1] Amjat H. Hamza and Munaf Y. Hmood (2021) *Comparison of Hurst exponent estimation methods*.

[2] http://epchan.blogspot.com/2016/04/mean-reversion-momentum-and-volatility.html.