2nd Report, Computer Simulations

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June 30, 2023

1 Multivariate stable distributions

We will analyze the Multivariate stable vectors. At first we will show a method used to simulate such vectors with illustrations to show their behavior. Next we will look closer at the 2-dimensional densities, empirical and theoretical. At the end we will shortly discuss a method of estimation of spectral measure with few examples.

1.1 Simulation of stable vectors

Let's begin with a definition of multivariate stable distribution. A d-dimensional stable vector is defined using a spectral measure Γ and a shift vector $\mu^0 \in \mathbb{R}^d$. We will denote it as

$$X \sim S_{\alpha,d}(\Gamma, \mu^0). \tag{1}$$

The characteristic function of X is then

$$\phi_X(t) = \mathbf{E}(\exp\{i\langle X, t\rangle\}). \tag{2}$$

We will examine the case with a discrete spectral measure

$$\Gamma(\cdot) = \sum_{j=0}^{n} \gamma_j \delta_{s_j}(\cdot). \tag{3}$$

Then the characteristic function takes the form

$$\phi(t) = \exp\left(-\sum_{j=1}^{n} \psi_\alpha(\langle t, s_j\rangle)\gamma_j\right), \tag{4}$$

where

$$\psi_\alpha(u) = \begin{cases} |u|^\alpha(1 - \imath sign(u))\tan\frac{\pi\alpha}{2}, & \alpha \neq 1, \\ |u|(1 + \imath\frac{2}{\pi} sign(u))\log|u|, & \alpha = 1. \end{cases} \tag{5}$$

Then X can be expressed as

$$X = \begin{cases} \sum_{j=1}^{n} \gamma_j^{1/\alpha} Z_j s_j + \mu^0, & \alpha \neq 1, \\ \sum_{j=1}^{n} \gamma_j^{1/\alpha} (Z_j + \frac{2}{\pi} \log \gamma_j) s_j + \mu^0, & \alpha \neq 1, \end{cases} \tag{6}$$

where $Z_1, \ldots, Z_n$ are one dimensional α-stable random variables. We will use $Z_i \sim S_\alpha(\beta = 1, \gamma = 1, \delta = 0)$. So the only thing needed to simulate multivariate stable distribution is the ability to generate a vector of univariate variables. Now let's take a look on how the vectors look depending on α and the spectral measure. For visualization simplicity we will only look at 2-dimensional case and present it with a scatter plot, 3D histogram and a density heatmap. All three were generated using `python` with `plt.scatter`, `np.histogram2d` with `bar3d` functionality and `plotly.express.density_contour` respectively. Here we analyze the case where

$$\begin{aligned} \gamma &= [0.25, 0.125, 0.25, 0.25, 0.125, 0.25] \\ \mathbf{s} &= [(1,0), (\frac{1}{2}, \frac{\sqrt(3)}{2}), (-\frac{1}{2}, \frac{\sqrt(3)}{2}), (-1,0), (-\frac{1}{2}, -\frac{\sqrt(3)}{2}), (\frac{1}{2}, -\frac{\sqrt(3)}{2})] \end{aligned} \tag{7}$$

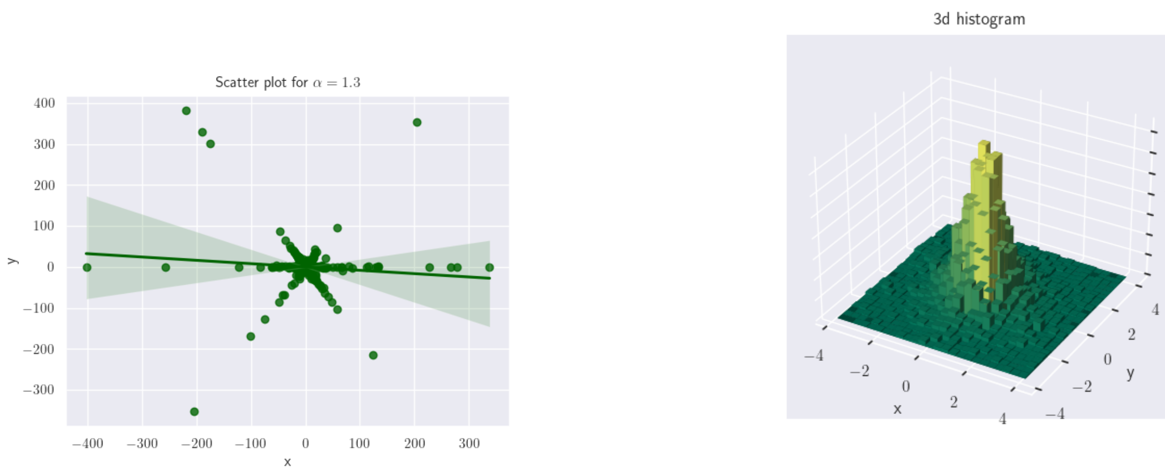


Figure 1: Plots for $\alpha = 1.3$

Now we will try to change the parameters to observe vectors with different spectral measure. We take the same values, but in different orders. From now on we take vector s as

$$s_j = \left(\cos(2\pi \frac{j-1}{m}), \sin(2\pi \frac{j-1}{m}) \right), \tag{8}$$

where m is the number of masses in the spectral measure. So for $m = 6$ and $\Gamma = [0.25, 0.25, 0.125, 0.125, 0.25, 0.25]$ we have

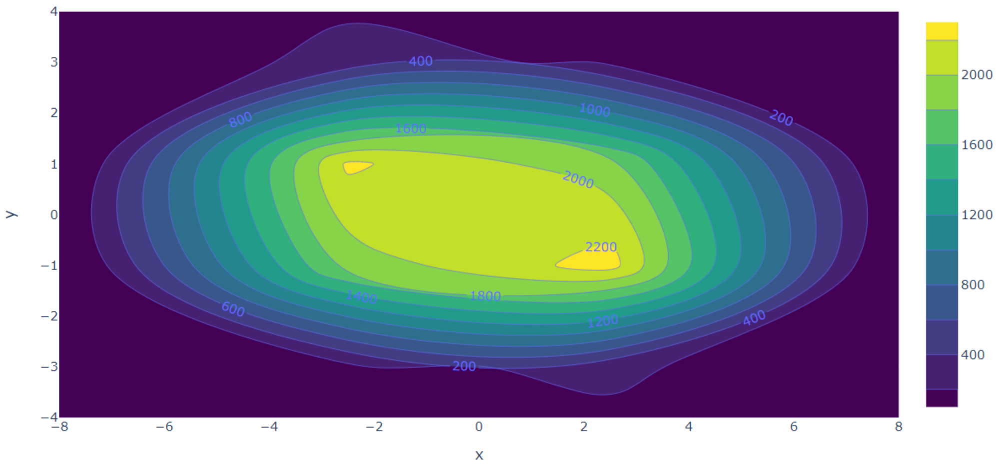


Figure 2: Heatmap for $\alpha = 1.3$

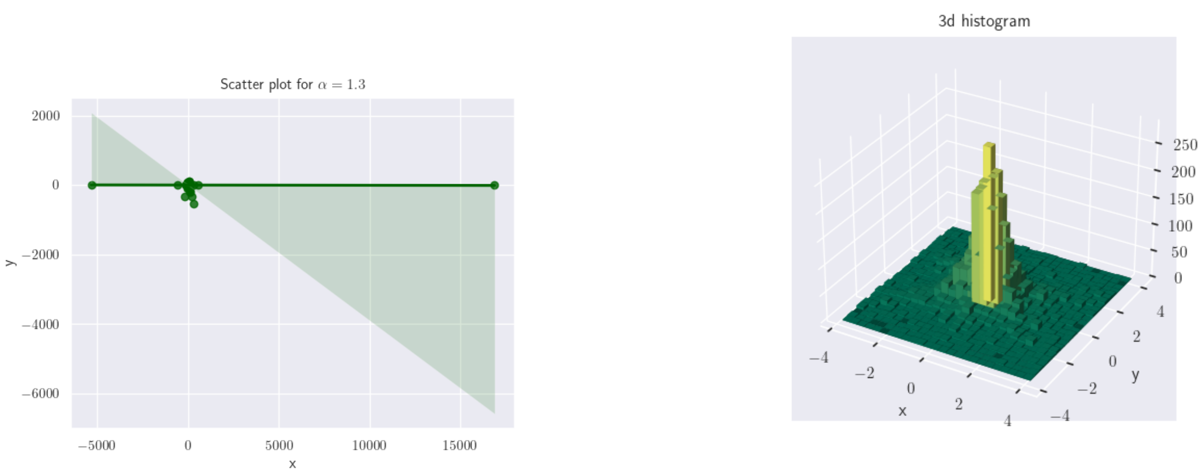


Figure 3: Plots for $\alpha = 1.3$

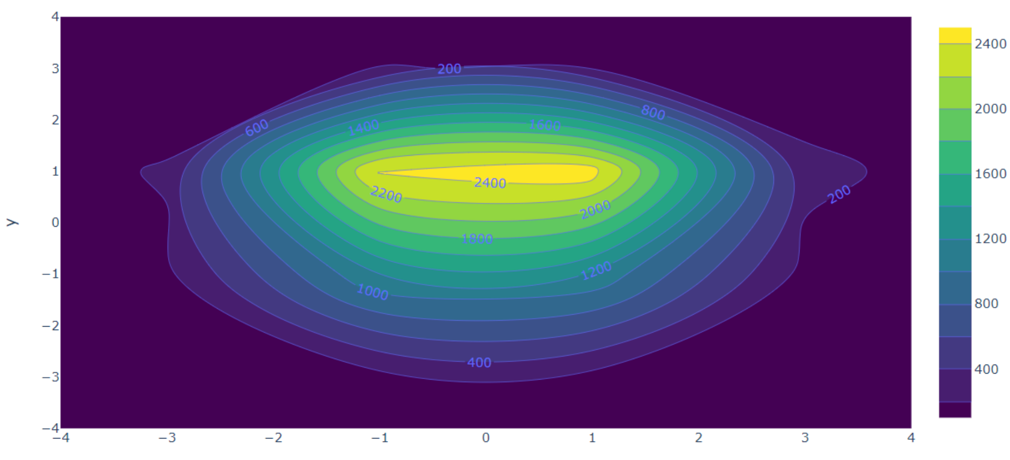


Figure 4: Heatmap for $\alpha = 1.3$

We will present some more results for different spectral measures and values of $alpha$ this time using only density map. Here we will also present the unsymmetrical case with $m = 5$ -point masses.

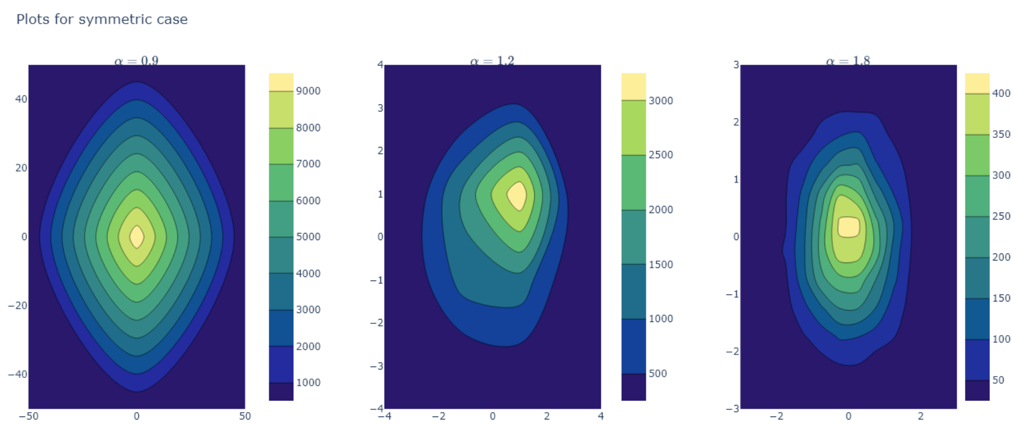


Figure 5

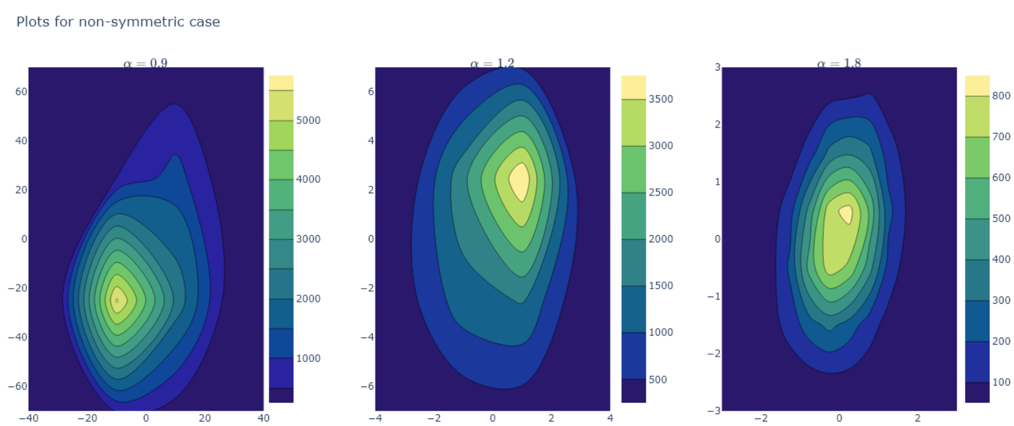
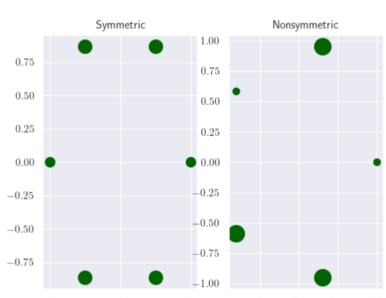


Figure 6

We can easily observe that all the parameters have a big influence on the resulting 2-dimensional vector.



Also it's visible that we get far more regular histogram shapes in the symmetric cases. Additionally if we drew mass points with specific spectral measure we could observe that the plots reflect that parameter. When comparing earlier presented histogram contours with scatter of mass points, we can see that they strictly correspond to each other, showing us how big role plays the spectral measure. We can also observe that the dependency seems higher for greater values of α.