Institutions

Tuning up the logical system

- various sets of formulae (Horn-clauses, first-order, higher-order, modal formulae, ...)
- various notions of algebra (partial algebras, relational structures, error algebras, Kripke structures, ...)
- various notions of signature (order-sorted, error, higher-order signatures, sets of propositional variables, ...)
- (various notions of signature morphisms)

No best logic for everything

Solution:

Work with an arbitrary logical system

Institutions

Abstract model theory

for specification and programming

Goguen & Burstall: $1980 \to 1992$

- a standard formalization of the concept of the underlying logical system for specification formalisms and most work on foundations of software specification and development from algebraic perspective;
- a formalization of the concept of a logical system for foundational studies:
  - truly abstract model theory
  - proof-theoretic considerations
  - building complex logical systems

Some institutional topics

- **Institutions: intuitions and motivations**

  Goguen & Burstall $\sim 1980 \to 1992$

- **Very abstract model theory**

  Tarlecki ~ 1986, Diaconescu *et al* $\sim 2003 \to \ldots$

- **Structured specifications**

  Clear ~ 1980, Sannella & Tarlecki $\sim 1984 \to \ldots$, Casl ~ 2004

  for Casl see: LNCS 2900 & 2960

- **Moving between institutions**

  Goguen & Burstall $\sim 1983 \to 1992$, Tarlecki $\sim 1986, 1996$, Goguen & Rosu ~ 2002

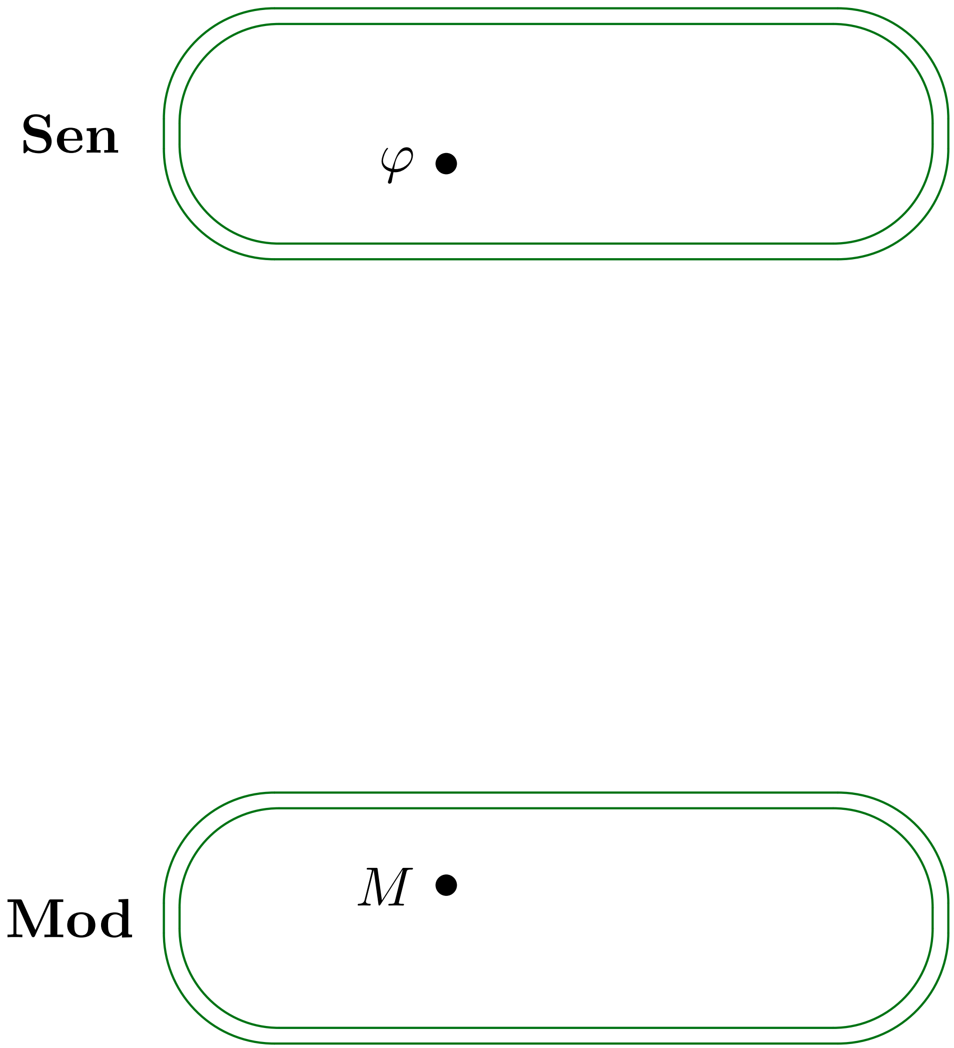
- **Heterogeneous specifications**

  Sannella & Tarlecki ~ 1988, Tarlecki $\sim 2000 \to \ldots$, Mossakowski $\sim 2002 \to \ldots$

  ...to be continued by **Till Mossakowski** (Hets)

...apologies for missing some names and for inaccurate years...

Institution: abstraction



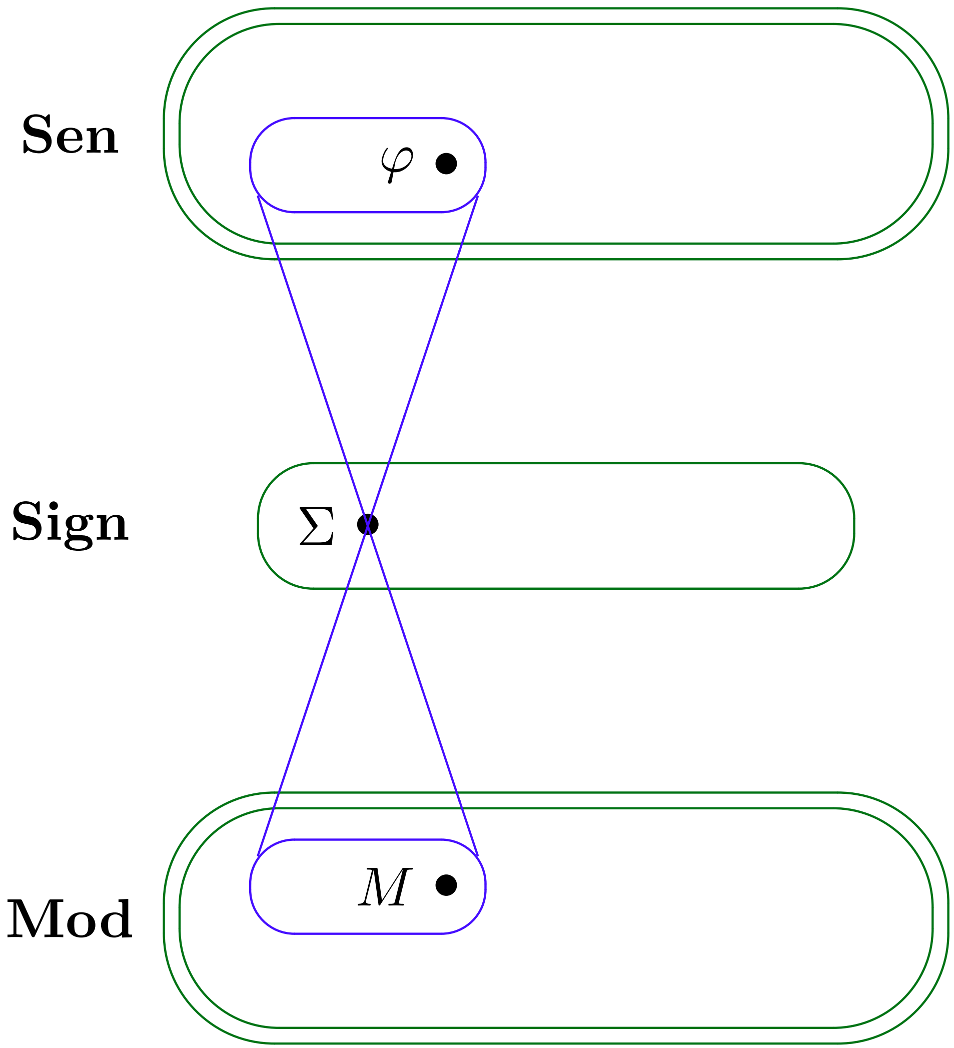
plus *satisfaction relation:*

$$\boxed{M \models \varphi}$$

and so the usual Galois connection between classes of models and sets of sentences, with the standard notions induced ($Mod(\Phi)$, $Th(\mathcal{M})$, $Th(\Phi)$, $\Phi \models \varphi$, etc).

- Also, possibly adding (sound) consequence: $\Phi \vdash \varphi$ (implying $\Phi \models \varphi$) to deal with proof-theoretic aspects.

Institution: first insight



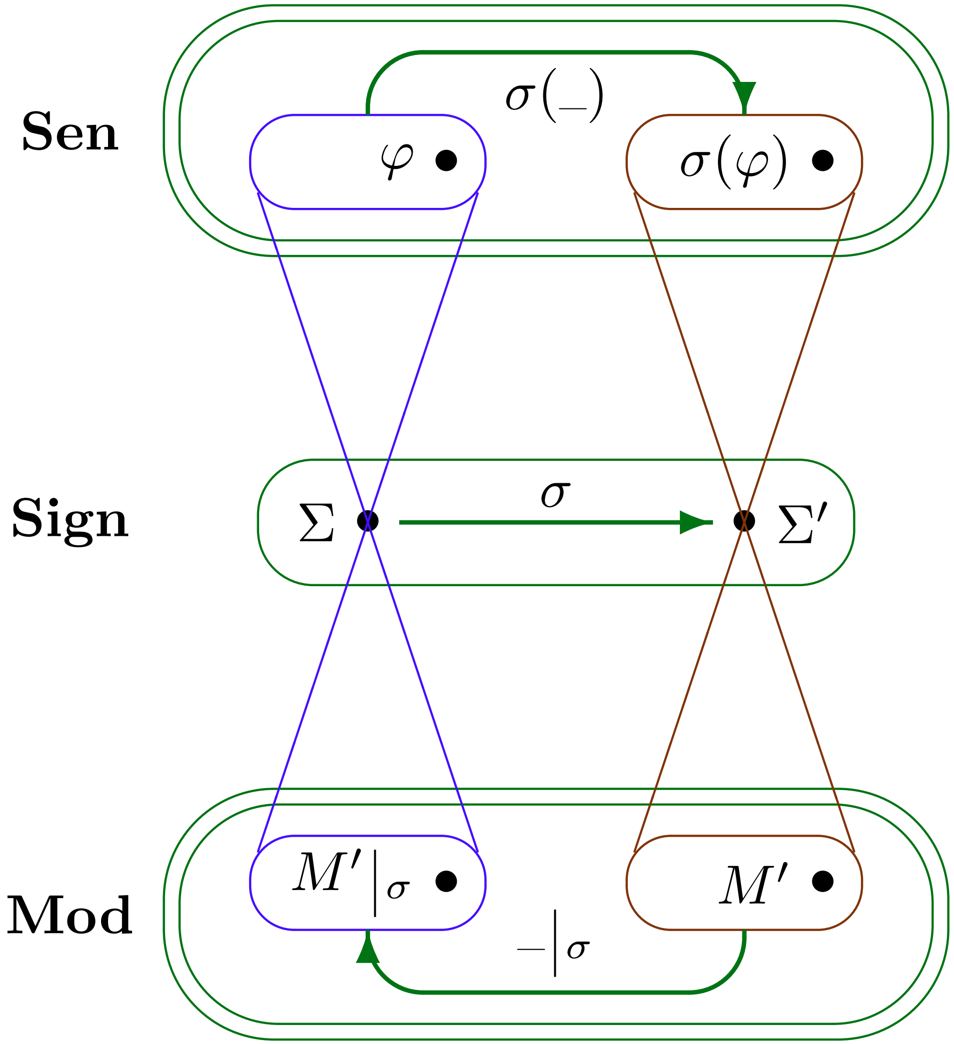
plus *satisfaction relation:*

$$\boxed{M \models_\Sigma \varphi}$$

and so, for each signature, the usual Galois connection between classes of models and sets of sentences, with the standard notions induced ($Mod_\Sigma(\Phi)$, $Th_\Sigma(\mathcal{M})$, $Th_\Sigma(\Phi)$, $\Phi \models_\Sigma \varphi$, etc).

- Also, possibly adding (sound) consequence: $\Phi \vdash_\Sigma \varphi$ (implying $\Phi \models_\Sigma \varphi$) to deal with proof-theoretic aspects.

Institution: key insight



imposing the *satisfaction condition:*

$$\boxed{M' \models_{\Sigma'} \sigma(\varphi) \text{ iff } M'|_\sigma \models_\Sigma \varphi}$$

Truth is invariant
under change of notation

and independent of
any additional symbols around

Institution

- a category $\mathbf{Sign}$ of *signatures*
- a functor $\mathbf{Sen}\colon \mathbf{Sign} \to \mathbf{Set}$
  - $\mathbf{Sen}(\Sigma)$ is the set of Σ-*sentences*, for $\Sigma \in |\mathbf{Sign}|$
- a functor $\mathbf{Mod}\colon \mathbf{Sign}^{op} \to \mathbf{Cat}$
  - $\mathbf{Mod}(\Sigma)$ is the category of Σ-*models*, for $\Sigma \in |\mathbf{Sign}|$
- for each $\Sigma \in |\mathbf{Sign}|$, Σ-*satisfaction relation* $\models_\Sigma \subseteq |\mathbf{Mod}(\Sigma)| \times \mathbf{Sen}(\Sigma)$

subject to the *satisfaction condition*:

$$\boxed{M'|_\sigma \models_\Sigma \varphi \iff M' \models_{\Sigma'} \sigma(\varphi)}$$

where $\sigma\colon \Sigma \to \Sigma'$ in $\mathbf{Sign}$, $M' \in |\mathbf{Mod}(\Sigma')|$, $\varphi \in \mathbf{Sen}(\Sigma)$,
$M'|_\sigma$ stands for $\mathbf{Mod}(\sigma)(M')$, and $\sigma(\varphi)$ for $\mathbf{Sen}(\sigma)(\varphi)$.

Typical institutions

- **EQ** — equational logic
- **FOEQ** — first-order logic (with predicates and equality)
- **PEQ**, **PFOEQ** — as above, but with partial operations
- **HOL** — higher-order logic
- logics of constraints (fitted via signature morphisms)
- **CASL** — the logic of CASL: partial first-order logic with equality, predicates, generation constraints, and subsorting

  CASL subsorting: the sets of sorts in signatures are *pre-ordered*; in every model M, $s \leq s'$ yields an injective *subsort embedding* (*coercion*) $em_M^{s\leq s'} : |M|_s \to |M|_{s'}$ such that $em_M^{s\leq s} = id_{|M|_s}$ for each sort s, and $em_M^{s\leq s'};em_M^{s'\leq s''} = em_M^{s\leq s''}$, for $s \leq s' \leq s''$; plus partial projections and subsort membership predicates derived from the embeddings.

Somewhat less typical institutions:

- modal logics
- three-valued logics
- programming language semantics:
  - **IMP**: imperative programming language with sets of computations as models and procedure declararions as sentences
  - **FPL**: functional programming language with partial algebras as models and the usual axioms with extended term syntax allowing for local recursive function definitions

Temporal logic

Institution $\mathbf{TL}$:

extremely simplified version
and oversimplified presentation

- signatures $\mathcal{A}$: (finite) sets of *actions*;
- models $\mathcal{R}$: sets of *runs*, finite or infinite sequences of (sets of) actions;
- sentences φ: built from atomic statements a (action $a \in \mathcal{A}$ happens) using the usual propositional and temporal connectives, including $\mathbf{X}\varphi$ (an action happens and then φ holds) and $\varphi\mathbf{U}\psi$ (φ holds until ψ holds)
- satisfaction $\mathcal{R} \models \varphi$: φ holds at the beginning of every run in $\mathcal{R}$

WATCH OUT!

Under some formalisations, satisfaction condition may fail!

Care is needed in the exact choice of sentences considered, morphisms (between sets of actions) allowed, and reduct definitions.

Perhaps unexpected examples:

- no sentences
- no models
- no signatures
- trivial satisfaction relations

- sets of sentences as sentences
- sets of sentences as signatures
- classes of models as sentences
- sets of sentences as models
- . . .

Let's fix an institution $\mathbf{I} = (\mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}, \langle \models_\Sigma \rangle_{\Sigma \in |\mathbf{Sign}|})$ for a while.

Semantic entailment

$$\boxed{\Phi \models_\Sigma \varphi}$$

> *Σ-sentence φ is a semantic consequence of a set of Σ-sentences Φ if φ holds in every Σ-models that satisfies Φ.*

BTW:

- *Models* of a set of sentences: $Mod(\Phi) = \{M \in |\mathbf{Mod}(\Sigma)| \mid M \models \Phi\}$
- *Theory* of a class of models: $Th(\mathcal{C}) = \{\varphi \mid \mathcal{C} \models \varphi\}$
- $\Phi \models \varphi \iff \varphi \in Th(Mod(\Phi))$
- Mod and Th form a *Galois connection*

Semantic equivalences

Equivalence of sentences: for $\Sigma \in |\mathbf{Sign}|$, $\varphi, \psi \in \mathbf{Sen}(\Sigma)$ and $\mathcal{M} \subseteq |\mathbf{Mod}(\Sigma)|$,

$$\boxed{\varphi \equiv_{\mathcal{M}} \psi}$$

if for all Σ-models $M \in \mathcal{M}$, $M \models \varphi$ iff $M \models \psi$. For $\varphi \equiv_{|\mathbf{Mod}(\Sigma)|} \psi$ we write:

$$\boxed{\varphi \equiv \psi}$$

Semantic equivalence

Equivalence of models: for $\Sigma \in |\mathbf{Sign}|$, $M, N \in |\mathbf{Mod}(\Sigma)|$, and $\Phi \subseteq \mathbf{Sen}(\Sigma)$,

$$\boxed{M \equiv_{\Phi} N}$$

if for all $\varphi \in \Phi$, $M \models \varphi$ iff $N \models \varphi$. For $M \equiv_{\mathbf{Sen}(\Sigma)} N$ we write:

$$\boxed{M \equiv N}$$

Elementary equivalence

Compactness, consistency, completeness...

- Institution $\mathbf{I}$ is *compact* if for each signature $\Sigma \in |\mathbf{Sign}|$, set of Σ-sentences $\Phi \subseteq \mathbf{Sen}(\Sigma)$, and Σ-sentences $\varphi \in \mathbf{Sen}(\Sigma)$,

  if $\Phi \models \varphi$ then $\Phi_{\mathit{fin}} \models \varphi$ for some finite $\Phi_{\mathit{fin}} \subseteq \Phi$

- A set of Σ-sentences $\Phi \subseteq \mathbf{Sen}(\Sigma)$ is *consistent* if it has a model, i.e.,

  $$\boxed{Mod(\Phi) \neq \emptyset}$$

- A set of Σ-sentences $\Phi \subseteq \mathbf{Sen}(\Sigma)$ is *complete* if it is a maximal consistent set of Σ-sentences, i.e., Φ is consistent and

  for $\Phi \subseteq \Phi' \subseteq \mathbf{Sen}(\Sigma)$, if Φ' is consistent then $\Phi = \Phi'$

Fact: *Any complete set of Σ-sentences $\Phi \subseteq \mathbf{Sen}(\Sigma)$ is a theory: $\Phi = Th(Mod(\Phi))$.*

Preservation of entailment

Fact:

$$\boxed{\Phi \models_\Sigma \varphi \implies \sigma(\Phi) \models_{\Sigma'} \sigma(\varphi)}$$

for $\sigma\colon \Sigma \to \Sigma'$, $\Phi \subseteq \mathbf{Sen}(\Sigma)$, $\varphi \in \mathbf{Sen}(\Sigma)$.

If the reduct $\_|_\sigma\colon |\mathbf{Mod}(\Sigma')| \to |\mathbf{Mod}(\Sigma)|$ *is surjective, then*

$$\boxed{\Phi \models_\Sigma \varphi \iff \sigma(\Phi) \models_{\Sigma'} \sigma(\varphi)}$$

Adding provability

Add to institution:

- *proof-theoretic entailment*:

$$\vdash_\Sigma \subseteq \mathcal{P}(\mathbf{Sen}(\Sigma)) \times \mathbf{Sen}(\Sigma)$$

  for each signature $\Sigma \in |\mathbf{Sign}|$, closed under
  - weakening, reflexivity, transitivity (cut)
  - translation along signature morphisms

Require:

- *soundness*: $\Phi \vdash_\Sigma \varphi \implies \Phi \models_\Sigma \varphi$

(?) *completeness*: $\Phi \models_\Sigma \varphi \implies \Phi \vdash_\Sigma \varphi$

Presentations (basic specifications)

$$\boxed{\langle \Sigma, \Phi \rangle}$$

- signature Σ, to determine the static module interface
- axioms (Σ-sentences) $\Phi \subseteq \mathbf{Sen}(\Sigma)$, to determine required module properties

Use strong enough logic to capture the "right" class of models, excluding undesirable "modules"

Presentation morphisms

Presentation morphism:

$$\boxed{\sigma : \langle \Sigma, \Phi \rangle \to \langle \Sigma', \Phi' \rangle}$$

is a signature morphism $\sigma : \Sigma \to \Sigma'$ such that for all $M' \in \mathbf{Mod}(\Sigma')$:

$$\boxed{M' \in Mod(\Phi') \implies M'|_\sigma \in Mod(\Phi)}$$

Then $\_|_\sigma : Mod(\Phi') \to Mod(\Phi)$

Fact: *A signature morphism* $\sigma : \Sigma \to \Sigma'$ *is a presentation morphism* $\sigma : \langle \Sigma, \Phi \rangle \to \langle \Sigma', \Phi' \rangle$ *if and only if* $\Phi' \models \sigma(\Phi)$ *.*

BTW: for all presentation morphisms $\Phi \models_\Sigma \varphi \implies \Phi' \models_{\Sigma'} \sigma(\varphi)$

Conservativity

A presentation morphism:

$$\boxed{\sigma : \langle \Sigma, \Phi \rangle \to \langle \Sigma', \Phi' \rangle}$$

is *conservative* if for all Σ-sentences φ: $\boxed{\Phi' \models_{\Sigma'} \sigma(\varphi) \implies \Phi \models_{\Sigma} \varphi}$

A presentation morphism $\sigma : \langle \Sigma, \Phi \rangle \to \langle \Sigma', \Phi' \rangle$ *admits model expansion* if for each $M \in Mod(\Phi)$ there exists $M' \in Mod(\Phi')$ such that $M'|_\sigma = M$

(i.e., $\_|_\sigma : Mod(\Phi') \to Mod(\Phi)$ is surjective).

Fact: *If $\sigma : \langle \Sigma, \Phi \rangle \to \langle \Sigma', \Phi' \rangle$ admits model expansion then it is conservative.*

In general, the equivalence does not hold!

Fact: *If $\langle \Sigma, \Phi \rangle$ is complete and $\langle \Sigma', \Phi' \rangle$ is consistent then any presentation morphism $\sigma : \langle \Sigma, \Phi \rangle \to \langle \Sigma', \Phi' \rangle$ is conservative.*

Categories of presentations & of theories

- $\mathbf{Pres}$: the *category of presentations* in $\mathbf{I}$ has presentations as objects and presentation morphisms as morphisms, with identities and composition inherited from $\mathbf{Sign}$, the category of signatures.
- $\mathbf{TH}$: the *category of theories* in $\mathbf{I}$ is the full subcateogry of $\mathbf{Pres}$ with theories (presentations with sets of sentences closed under consequence) as objects.

$\mathbf{Pres}$ and $\mathbf{TH}$ are equivalent:

$$id_{\Sigma} : \langle \Sigma, \Phi \rangle \to \langle \Sigma, Th(Mod(\Phi)) \rangle$$
is an isomorphism in $\mathbf{Pres}$

Fact: *The forgetful functors from* $\mathbf{Pres}$ *and* $\mathbf{TH}$*, respectively, to* $\mathbf{Sign}$ *preserve and create colimits.*

Fact: *If the category* $\mathbf{Sign}$ *of signatures is cocomplete, so are the categories* $\mathbf{Pres}$ *of presentations and* $\mathbf{TH}$ *of theories.*

Proof hint

in $\mathbf{Sign}$:

$$
\begin{tikzcd}
 & \Sigma' & \\
\Sigma_1 \arrow[ur, "\sigma_2'"] & \text{PO} & \Sigma_2 \arrow[ul, "\sigma_1'"'] \\
 & \Sigma \arrow[ul, "\sigma_1"] \arrow[ur, "\sigma_2"'] &
\end{tikzcd}
$$

in $\mathbf{Pres}$:

$$
\begin{tikzcd}
 & \langle \Sigma', \sigma_2'(\Phi_1) \cup \sigma_1'(\Phi_2) \rangle & \\
\langle \Sigma_1, \Phi_1 \rangle \arrow[ur, "\sigma_2'"] & \text{PO} & \langle \Sigma_2, \Phi_2 \rangle \arrow[ul, "\sigma_1'"'] \\
 & \langle \Sigma, \Phi \rangle \arrow[ul, "\sigma_1"] \arrow[ur, "\sigma_2"'] &
\end{tikzcd}
$$

Logical connectives

- $\mathbf{I}$ *has negation* if for every signature $\Sigma \in |\mathbf{Sign}|$ and Σ-sentence $\varphi \in \mathbf{Sen}(\Sigma)$, there is a Σ-sentence "$\neg\varphi$" $\in \mathbf{Sen}(\Sigma)$ such that for all Σ-models $M \in |\mathbf{Mod}(\Sigma)|$, $M \models$ "$\neg\varphi$" iff $M \not\models \varphi$.
- $\mathbf{I}$ *has conjunction* if for every signature $\Sigma \in |\mathbf{Sign}|$ and Σ-sentences $\varphi, \psi \in \mathbf{Sen}(\Sigma)$, there is a Σ-sentence "$\varphi \wedge \psi$" $\in \mathbf{Sen}(\Sigma)$ such that for all Σ-models $M \in |\mathbf{Mod}(\Sigma)|$, $M \models$ "$\varphi \wedge \psi$" iff $M \models \varphi$ and $M \models \psi$.
- . . . *implication*, *disjunction*, *falsity*, *truth* . . .

Fact: *For any signature morphism $\sigma : \Sigma \to \Sigma'$ and Σ-sentence $\varphi \in \mathbf{Sen}(\Sigma)$ $\sigma($"$\neg\varphi$"$)$ and "$\neg\sigma(\varphi)$" are equivalent.*

Similarly, for Σ-sentences $\varphi, \psi \in \mathbf{Sen}(\Sigma)$), $\sigma($"$\varphi \wedge \psi$"$)$ and "$\sigma(\varphi) \wedge \sigma(\psi)$" are equivalent.

Similarly for other connectives. . .

For any institution $\mathbf{I}$, define its closures: under negation $\mathbf{I}^{\neg}$, under conjunction $\mathbf{I}^{\wedge}$, etc.

Free variables and quantification

Standard algebra	Institution $\mathbf{I}$
algebraic signature $\Sigma = \langle S, \Omega \rangle$	signature $\Sigma \in \|\mathbf{Sign}\|$
S-sorted set of variables X	signature extension $\iota : \Sigma \to \Sigma(X)$
open Σ-formula φ with variables X	$\Sigma(X)$-sentence φ
Σ-algebra M	Σ-model $M \in \|\mathbf{Mod}(\Sigma)\|$
valuation of variables $v : X \to \|M\|$ in M	ι-expansion M^v of M, i.e., $M^v \in \|\mathbf{Mod}(\Sigma(X))\|$, $M^v\|_\iota = M$ *($M^v_x = v(x)$ for variable/constant $x \in X$)*
satisfaction of formula φ in M under v: $M \models^v_\Sigma \varphi$	satisfaction of "open formula" φ $M^v \models_{\Sigma(X)} \varphi$

A characterisation of such signature extensions:

$\sigma : \Sigma \to \Sigma'$ is *representable* iff $\mathbf{Mod}(\Sigma')$ has an initial model and
$\_|_\sigma : (\mathbf{Mod}(\Sigma')\uparrow M') \to (\mathbf{Mod}(\Sigma)\uparrow(M'|_\sigma))$ is iso for $M' \in |\mathbf{Mod}(\Sigma')|$

Quantification

Let $\mathcal{I}$ be a class of signature morphisms. For decency, assume that it forms a subcategory of $\mathbf{Sign}$ and is closed under pushouts with arbitrary signature morphisms.

- $\mathbf{I}$ *has universal quantification along* $\mathcal{I}$ if for every signature morphism $\theta : \Sigma \to \Sigma'$ in $\mathcal{I}$ and Σ'-sentence $\psi \in \mathbf{Sen}(\Sigma')$, there is a Σ-sentence "$\forall\theta{\cdot}\psi$" $\in \mathbf{Sen}(\Sigma)$ such that for all Σ-models $M \in |\mathbf{Mod}(\Sigma)|$, $M \models$ "$\forall\theta{\cdot}\psi$" iff for all Σ'-models with $M'|_\theta = M$, $M' \in |\mathbf{Mod}(\Sigma')|$, $M' \models \psi$.
- $\mathbf{I}$ *has existential quantification along* $\mathcal{I}$ if for $\theta : \Sigma \to \Sigma'$ in $\mathcal{I}$ and Σ'-sentence $\psi \in \mathbf{Sen}(\Sigma')$, there is a Σ-sentence "$\exists\theta{\cdot}\psi$" $\in \mathbf{Sen}(\Sigma)$ such that for all Σ-models $M \in |\mathbf{Mod}(\Sigma)|$, $M \models$ "$\exists\theta{\cdot}\psi$" iff for some Σ'-model $M' \in |\mathbf{Mod}(\Sigma')|$ with $M'|_\theta = M$, $M' \models \psi$.

Fact: *For any* $\sigma : \Sigma \to \Sigma_1$, $\sigma($"$\forall\theta{\cdot}\psi$"$)$ *and* "$\forall\theta'{\cdot}\sigma'(\psi)$" *are equivalent, where the following is a pushout in* $\mathbf{Sign}$ *with* $\theta' \in \mathcal{I}$:

$$\begin{tikzcd} \Sigma' \arrow[r, "\sigma'"] & \Sigma'_1 \\ \Sigma \arrow[u, "\theta"] \arrow[r, "\sigma"'] \arrow[ur, phantom, "PO"] & \Sigma_1 \arrow[u, "\theta'"'] \end{tikzcd}$$

Similarly for existential quantification.

$\boxed{\textit{AMALGAMATION NEEDED!}}$

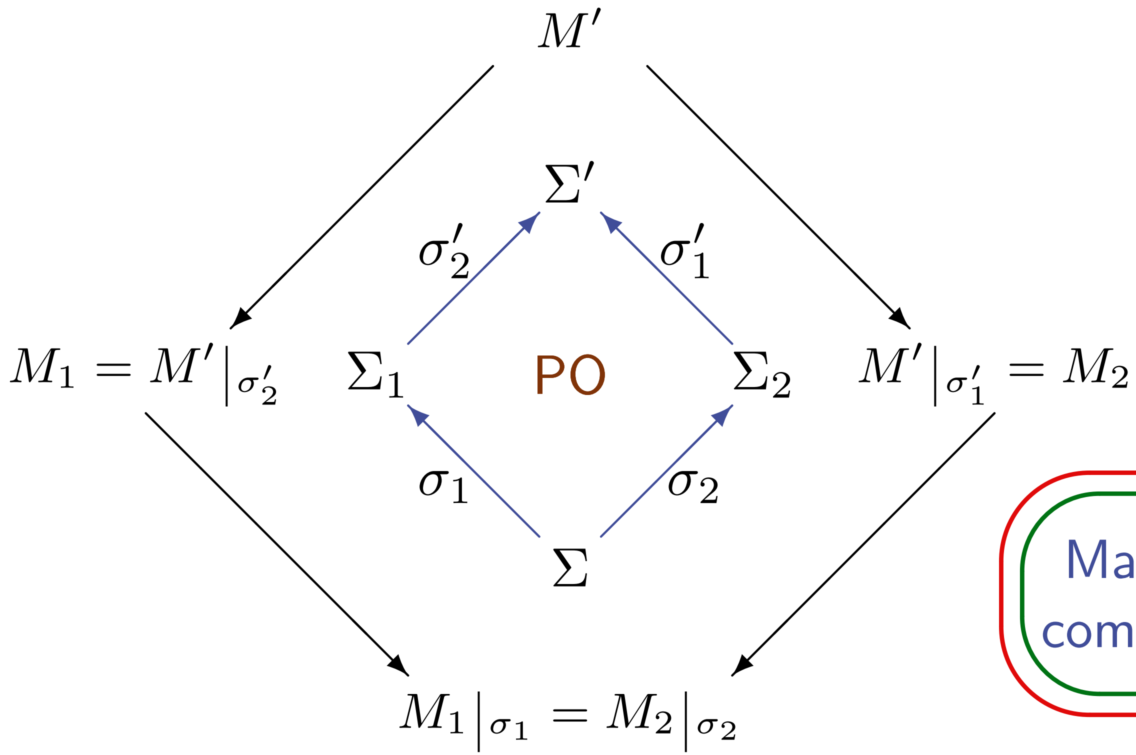
Define $\mathbf{I}^{FO}$, *"first-order closure" of* $\mathbf{I}$

Amalgamation for algebras

$$
\begin{tikzcd}
 & & A' \arrow[dll] \arrow[drr] & & \\
 & & \Sigma_1 \cup \Sigma_2 & & \\
A_1 = A'|_{\Sigma_1} \arrow[drr] & \Sigma_1 \arrow[ur] & \text{PO} & \Sigma_2 \arrow[ul] & A'|_{\Sigma_2} = A_2 \arrow[dll] \\
 & & \Sigma_1 \cap \Sigma_2 \arrow[ul] \arrow[ur] & & \\
 & & A_1|_{\Sigma_1 \cap \Sigma_2} = A_2|_{\Sigma_1 \cap \Sigma_2} & &
\end{tikzcd}
$$

Fact: *For any algebras* $A_1 \in |\mathbf{Alg}(\Sigma_1)|$ *and* $A_2 \in |\mathbf{Alg}(\Sigma_2)|$ *with common interpretation of common symbols* $A_1|_{\Sigma_1 \cap \Sigma_2} = A_2|_{\Sigma_1 \cap \Sigma_2}$*, there is a unique "union" of* A_1 *and* A_2*,* $A' \in |\mathbf{Alg}(\Sigma_1 \cup \Sigma_2)|$ *with* $A'|_{\Sigma_1} = A_1$ *and* $A'|_{\Sigma_2} = A_2$*.*

Amalgamation



May be sensibly stated for any commuting square of morphisms

In $\mathbf{I}$, *amalgamation property* holds for the pushout above if for all $M_1 \in |\mathbf{Mod}(\Sigma_1)|$ and $M_2 \in |\mathbf{Mod}(\Sigma_2)|$ with $M_1|_{\sigma_1} = M_2|_{\sigma_2}$, there is a unique $M' \in |\mathbf{Mod}(\Sigma')|$ with $M'|_{\sigma_1'} = M_2$ and $M'|_{\sigma_2'} = M_1$.

Adding amalgamation

Assume:

- the model functor $\mathbf{Mod}\colon \mathbf{Sign}^{op} \to \mathbf{Cat}$ is *continuous* (maps colimits of signatures to limits of model categories)

Fact: $\mathbf{Alg}\colon \mathbf{AlgSig}^{op} \to \mathbf{Cat}$ *is continuous.*

Amalgamation property: *Amalgamation property follows for a pushout in* $\mathbf{Sign}$ *if* $\mathbf{Mod}$ *maps it to a pullback in* $\mathbf{Cat}$*:*

$$\begin{tikzcd}
\Sigma_1 \arrow[r, "\sigma_2'"] & \Sigma' & & \mathbf{Mod}(\Sigma_1) \arrow[d, "-|_{\sigma_1}"'] \arrow[dr, phantom, "\text{PB}"] & \mathbf{Mod}(\Sigma') \arrow[l, "-|_{\sigma_2'}"'] \arrow[d, "-|_{\sigma_1'}"] \\
\Sigma \arrow[u, "\sigma_1"] \arrow[r, "\sigma_2"'] \arrow[ur, phantom, "\text{PO}"] & \Sigma_2 \arrow[u, "\sigma_1'"'] \arrow[r, mapsto, "\mathbf{Mod}"] & {} & \mathbf{Mod}(\Sigma) & \mathbf{Mod}(\Sigma_2) \arrow[l, "-|_{\sigma_2}"]
\end{tikzcd}$$

Adding interpolation

$\mathbf{I}$ has the *interpolation property* for a pushout in $\mathbf{Sign}$

$$\begin{tikzcd}
 & \Sigma' & \\
\Sigma_1 \arrow[ur, "\sigma_2'"] & \text{PO} & \Sigma_2 \arrow[ul, "\sigma_1'"'] \\
 & \Sigma \arrow[ul, "\sigma_1"] \arrow[ur, "\sigma_2"'] &
\end{tikzcd}$$

if for all $\varphi_1 \in \mathbf{Sen}(\Sigma_1)$ and $\varphi_2 \in \mathbf{Sen}(\Sigma_2)$ such that $\sigma_2'(\varphi_1) \models_{\Sigma'} \sigma_1'(\varphi_2)$ there is $\theta \in \mathbf{Sen}(\Sigma)$ such that $\varphi_1 \models_{\Sigma_1} \sigma_1(\theta)$ and $\sigma_2(\theta) \models_{\Sigma_2} \varphi_2$.

Fact: $\mathbf{FOEQ}$ *has the interpolation property for all pushouts of pairs of morphisms, where at least one of the morphisms is injective on sorts.*

Spell out a version with a set of interpolants

Craig interpolation theorem

Consistency theorem

$\mathbf{I}$ has the *consistency property* for a pushout in $\mathbf{Sign}$

$$\begin{tikzcd}
 & \Sigma' & \\
\Sigma_1 \arrow[ur, "\sigma_2'"] & \text{PO} & \Sigma_2 \arrow[ul, "\sigma_1'"'] \\
 & \Sigma \arrow[ul, "\sigma_1"] \arrow[ur, "\sigma_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \langle \Sigma', \sigma_2'(\Phi_1) \cup \sigma_1'(\Phi_2) \rangle & \\
\langle \Sigma_1, \Phi_1 \rangle \arrow[ur, "\sigma_2'"] & \text{PO} & \langle \Sigma_2, \Phi_2 \rangle \arrow[ul, "\sigma_1'"'] \\
 & \langle \Sigma, \Phi \rangle \arrow[ul, "\sigma_1"] \arrow[ur, "\sigma_2"'] &
\end{tikzcd}$$

if for all $\Phi \subseteq \mathbf{Sen}(\Sigma)$ and consistent $\Phi_1 \subseteq \mathbf{Sen}(\Sigma_1)$ and $\Phi_2 \subseteq \mathbf{Sen}(\Sigma_2)$ such that $\sigma_1 : \langle \Sigma, \Phi \rangle \to \langle \Sigma_1, \Phi_1 \rangle$ is a conservative presentation morphism and $\sigma_2 : \langle \Sigma, \Phi \rangle \to \langle \Sigma_2, \Phi_2 \rangle$ is a presentation morphism, $\langle \Sigma', \sigma_2'(\Phi_1) \cup \sigma_1'(\Phi_2) \rangle$ is consistent.

Robinson consistency theorem (for first-order logic)

Fact: *In any compact institution with falsity, negation and conjunction, Craig interpolation and Robinson consistency properties are equivalent.*

The method of diagrams

Institution $\mathbf{I}$	Standard algebra
Given a signature Σ and Σ-model M, build signature extension $\iota : \Sigma \to \Sigma(M)$	*(adding elements of $\lvert M \rvert$ as constants)*
and a $\Sigma(M)$-presentation E_M	*(all ground atoms true in M^M, the natural ι-expansion of M)*
so that the reduct by ι yields isomorphism $\mathbf{Mod}(\Sigma(M), E_M) \to (\mathbf{Mod}(\Sigma){\uparrow}M)$	*(then the reduct by ι yields isomorphism $\mathbf{Alg}(\Sigma(M), E_M) \to (\mathbf{Alg}(\Sigma){\uparrow}M)$)*
... and everything is natural ...	*(everything is natural)*
Now: M has a "canonical" ι-expansion which is initial in $\mathbf{Mod}(\Sigma(M), E_M)$	*(M^M, reachable ι-expansion of M, is initial in $\mathbf{Alg}(\Sigma(M), E_M)$)*

Equipped with the method of diagrams, one can do a lot!

Abstract abstract model theory

Providing new insights and abstract formulations for classical model-theoretic concepts and results

- amalgamation over pushouts
- the method of elementary diagrams
- existence of free extensions
- interpolation results
- Birkhoff variety theorem(s)
- Beth definability theorem
- logical connectives, free variables, quantification
- completeness for *any* first-order logic
- . . .

in any institution
with various bits of extra structure,
under some technical assumptions. . .

WORK IN AN ARBITRARY INSTITUTION

. . . adding extra structure and assumptions only if really needed . . .

Revised rough analogy

module interface	$\rightsquigarrow$	signature
module	$\rightsquigarrow$	model
module specification	$\rightsquigarrow$	class of models