## What is Categorical Type Theory?

André Joyal

IAS and UQÀM

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#### Introduction

Motivation: To understand Martin-Löf type theory.

Conceptual mathematics  $\rightarrow$  category theory.

Two questions:

- Is type theory soluble in categorical logic?
- Is category theory soluble in type theory?

I will not discuss the second question here.

#### Overview

#### Aspects of categorical logic:

- Cartesian closed categories
- Essentially algebraic theories
- Locally cartesian closed categories
- ▶ Tribes

#### Homotopical logic:

- Weak factorization systems
- Homotopical algebra
- Pre-typoi
- Typoi
- Univalent typoi

### Aspects of categorical logic

The basic principles of categorical logic was expressed in Lawvere's paper *Adjointness in Foundation* (1969). I will use these principles implicitly.

- Algebraic theories
- Cartesian closed categories
- Essentially algebraic theories
- Locally cartesian closed categories
- Tribes
- Π-tribes

## Monoids and groups in categories(1)

Eckmann and Hilton: groups and cogroups in the category of pointed topological spaces ( $\sim$  1962).

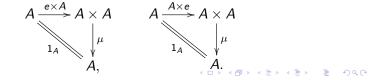
Diagrammatic language:

A monoid object in a category  $\mathcal C$  is an object  $A \in \mathcal C$  equipped with two operations  $\mu: A \times A \to A$  and  $e: \top \to A$  such that the following diagrams commute,

Associativity:

$$\begin{array}{ccc}
A \times A \times A & \xrightarrow{\mu \times A} A \times A \\
 & & \downarrow^{\mu} \\
 & & A \times A \xrightarrow{\mu} & A.
\end{array}$$

Left and right units,



# Monoids and groups in categories(2)

A group object in  $\mathcal C$  is a monoid  $(G,\mu,e)$  equipped with an operation of inversion  $\theta:G\to G$  such that the following diagrams commute,

$$\begin{array}{c|c}
G \times G & \xrightarrow{\theta \times G} & G \times G \\
\downarrow^{(1_G,1_G)} & & \downarrow^{\mu} \\
G & \longrightarrow \top & \xrightarrow{e} & G
\end{array}$$

$$\begin{array}{c}
G \times G & \xrightarrow{G \times \theta} & G \times G \\
\downarrow^{(1_G,1_G)} & & \downarrow^{\mu} \\
G & \longrightarrow \top & \xrightarrow{e} & G
\end{array}$$

#### Cartesian product

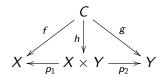
Recall that the cartesian product  $X \times Y$  of two objects X and Y in a category  $\mathcal C$  is an object  $X \times Y$  equipped with a pair of maps

$$X \stackrel{p_1}{\longleftarrow} X \times Y \stackrel{p_2}{\longrightarrow} Y$$

having the following universal property: for any object  $C \in \mathcal{C}$  and any a pair of maps

$$X \stackrel{f}{\longleftarrow} C \stackrel{g}{\longrightarrow} Y$$
,

there is a unique map  $h = \langle u, v \rangle : C \to X \times Y$  such that  $p_1 h = u$  and  $p_2 h = v$ .



#### Cartesian category

A category  $\ensuremath{\mathcal{C}}$  is said to be  $\ensuremath{\textit{cartesian}}$  if it has finite cartesian products

Recall that an object  $\top$  is said to be *terminal* if for every object  $A \in \mathcal{C}$ , there is a unique map

$$!:A\to \top$$
.

We may say that a map  $a : T \to A$  is a *point* of A, or a *term* of type A and write a : A.

A functor between cartesian categories  $F: \mathcal{C} \to \mathcal{D}$  is said to be *cartesian* if it preserves (finite) cartesian products.



## Algebraic theories (1)

#### Lawvere (1963):

- ▶ An algebraic theory  $\mathbb{T}$  generates cartesian category  $\mathcal{C}(\mathbb{T})$ ;
- ▶ the objects of  $C(\mathbb{T})$  are finite products of sorts  $U \times V \times W$ , ...;
- the maps in  $\mathcal{C}(\mathbb{T})$  are the operations of  $\mathbb{T}$ ;
- ▶ two maps  $f, g: X \to Y$  in  $\mathcal{C}(\mathbb{T})$  are equal if they are *provably* equal operations in  $\mathbb{T}$ ;
- ▶ a model of  $\mathbb{T}$  is a cartesian functor  $M : \mathcal{C}(\mathbb{T}) \to \mathbf{Set}$ ;
- ▶ A morphism of models  $F \rightarrow G$  is a natural transformation.

Completeness theorem: two maps  $f,g:X\to Y$  in  $\mathcal{C}(\mathbb{T})$  are equal if and only if M(f)=M(g) for every model  $M:\mathcal{C}(\mathbb{T})\to \mathbf{Set}$ .

Follows from Yoneda lemma!

# Algebraic theories (2)

**Example**: The cartesian theory of monoids Mon is the cartesian category generated by one object  $A \in Mon$  and two operations  $\mu: A \times A \to A$  and  $e: \top \to A$  (the multiplication and unit) satisfying:

Associativity:

$$\begin{array}{ccc}
A \times A \times A \xrightarrow{\mu \times A} A \times A \\
 & \downarrow^{\mu} \\
 & A \times A \xrightarrow{\mu} A
\end{array}$$

Left and right units,

$$A \xrightarrow{e \times A} A \times A \qquad A \xrightarrow{A \times e} A \times A$$

$$\downarrow^{\mu}$$

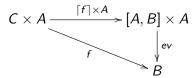
$$A \xrightarrow{A}$$

$$\downarrow^{\mu}$$

$$A \xrightarrow{A}$$

## Cartesian closed categories (1)

If A and B are two objects of a cartesian category C, we shall say that an object [A,B] equipped with a map  $\epsilon:[A,B]\times A\to B$  is the *exponential* of B by A if for every object  $C\in \mathcal{C}$  and every map  $f:C\times A\to B$  there exists a unique map  $\lceil f\rceil:C\to [A,B]$  such that  $\epsilon(\lceil f\rceil\times A)=f$ .



We shall write  $\lambda^A f = \lceil f \rceil$ . This defines a natural bijection between

the maps 
$$C \times A \rightarrow B$$
 and the maps  $C \rightarrow [A, B]$ .

# Cartesian closed categories (2)

A cartesian category C is said to be *closed* if the object [A, B] exists for every pair of objects  $A, B \in C$ .

#### Examples of cartesian closed categories

- the category of sets Set
- the category of (small) catégories Cat (Lawvere)
- the category of groupoids Grpd
- Every category  $[\mathbb{C}, \mathbf{Set}]$
- ▶ The category of simplicial sets  $[\Delta^{op}, \mathbf{Set}]$

### Essentially algebraic theories (1)

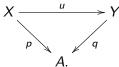
Ehresmann: Theory of sketches ( $\sim$ 1968),

Gabriel and Ulmer: Locally presentable categories ( $\sim$  1971)

#### Slice categories

Recall that the *slice category*  $\mathcal{C}/A$  has for objects the pairs (X, p), where p is a map  $X \to A$  in  $\mathcal{C}$ . The map  $p: X \to A$  is called the *structure map* of (X, p).

A morphism  $(X, p) \rightarrow (Y, q)$  in  $\mathcal{C}/A$  is a map  $u : X \rightarrow Y$  in  $\mathcal{C}$  such that qu = p,



#### Push-forward

To every map  $f:A\to B$  in a category  $\mathcal C$  we can associate a *push-forward* functor

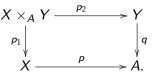
$$f_!: \mathcal{C}/A \to \mathcal{C}/B$$

by putting  $f_1(X, p) = (X, fp)$  for every map  $p: X \to A$ ,

$$\begin{array}{ccc}
X & \longrightarrow & X \\
\downarrow p & & \downarrow fp \\
A & \longrightarrow & B.
\end{array}$$

#### Fiber products

Recall that the *fiber product* of two maps  $X \to A$  and  $Y \to A$  in a category  $\mathcal C$  is their cartesian product  $X \times_A Y$  as objects of the category  $\mathcal C/A$ .



The square is also called a *pullback square*.

#### Base changes

In a category with finite limits  $\mathcal C$  the push-forward functor  $f_!:\mathcal C/A\to\mathcal C/B$  has a right adjoint

$$f^{\star}: \mathcal{C}/B \to \mathcal{C}/A$$

for any map  $f:A\to B$ . The functor  $f^*$  takes a map  $p:X\to B$  to the map  $p_1:A\times_BX\to A$  in a pullback square

$$\begin{array}{cccc}
A \times_B X & \xrightarrow{p_2} & X \\
\downarrow^{p_1} & & \downarrow^{p} \\
A & \xrightarrow{f} & B.
\end{array}$$

The map  $p_1$  is said to be the *base change* of the map  $p: X \to B$  along the map  $f: A \to B$ .

## Essentially algebraic theories (1)

An essentially algebraic theory is defined to be a category with finite limits.

A *morphism* of essentially algebraic theories  $F: \mathcal{C} \to \mathcal{D}$  is defined to be a functor which preserves finite limits.

A *model* of an essentially algebraic theory  $\mathcal C$  is defined to be a morphism  $M:\mathcal C\to \mathbf{Set}.$ 

For example, the notion of *category* is essentially algebraic, but not algebraic.

### Duality

Let C be a category with finite limits.

For every  $A \in \mathcal{C}$  the functor  $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$  is a model of  $\mathcal{C}$ , since it preserves finite limits.

In fact, the functor  $A \mapsto \mathcal{C}(A,-)$  is a contravariant equivalence between the category  $\mathcal{C}$  and the category of finitely presentable models of  $\mathcal{C}$ .

For example, the finite limit theory of categories is the opposite of the category of finitely presentable categories.

## Locally cartesian closed categories(1)

A category with finite limits C is said to be *locally cartesian closed* if the category C/A is cartesian closed for every object  $A \in C$ .

A category with finite limits  $\mathcal C$  is locally cartesian closed if and only if the base change functor  $f^*:\mathcal C/B\to\mathcal C/A$  has a right adjoint

$$f_{\star}: \mathcal{C}/A \to \mathcal{C}/B$$

for every map  $f: A \to B$  in C. The functor  $f_*$  is a *internal product* along  $f: A \to B$ ,

$$f_{\star} = \Pi_f : \mathcal{C}/A \to \mathcal{C}/B,$$

## Locally cartesian closed categories(2)

#### Examples of locally cartesian closed categories

- the category of sets Set
- Every category  $[\mathbb{C}, \mathbf{Set}]$
- ▶ The category of simplicial sets  $[\Delta^{op}, \mathbf{Set}]$
- A Grothendieck topos

The category **Cat** is *not* locally cartesian closed.

The category **Grpd** is *not* locally cartesian closed.

## Tribes(1)

A class of maps  $\mathcal F$  in a category  $\mathcal C$  is said to be *closed under base changes* if the base change  $A\times_B X\to A$  of every map  $X\to B$  in  $\mathcal F$  along any map  $f:A\to B$  in  $\mathcal C$  exists and belongs to  $\mathcal F$ ,

#### **Definition**

A *tribe structure* on a category with terminal object  $\mathcal C$  is a class of maps  $\mathcal F\subseteq\mathcal C$  which satisfies the following conditions:

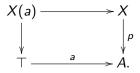
- every isomorphism belongs to  $\mathcal{F}$ ;
- $ightharpoonup \mathcal{F}$  is closed under composition and base changes;
- ▶ the map  $X \to \top$  belongs to  $\mathcal{F}$  for every object  $X \in \mathcal{C}$ .

A map in  $\mathcal{F}$  is a *family* or a *fibration* of the tribe.

A *tribe* is a category C equipped tribe structure F.

## Tribes(2)

The *fiber* of a fibration  $p: X \to A$  at a point  $a: T \to A$  is the object X(a) defined by the pullback square



The fibration  $p: X \to A$  can be regarded as an *internal family of objects*  $(X(a): a \in A)$  parametrized by the codomain of p.

We denote by  $\mathcal{C}(A)$  the full subcategory of  $\mathcal{C}/A$  whose objects are the fibrations  $X \to A$ . An object of  $\mathcal{C}(A)$  is an *internal family* of objects of A.

# Tribes (3)

If  $u: A \to B$  is a map in C, then the base-change functor

$$u^*: \mathcal{C}(B) \to \mathcal{C}(A)$$

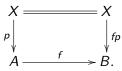
is an operation of *substitution of parameters* for internal families, since we have

$$u^{\star}(Y)(a) = Y(u(a))$$

for every every fibration  $Y \to B$  and every  $a \in A$ .

### Tribes (4)

To every fibration  $f: A \to B$  in tribe  $\mathcal{C}$  we can associate a push-forward functor  $f_!: \mathcal{C}(A) \to \mathcal{C}(B)$  by putting  $f_!(X,p) = (X,fp)$ ,



The functor  $f_{!}$  is a summation along f,

$$f_! = \Sigma_f : \mathcal{C}(A) \to \mathcal{C}(B).$$

Formally,

$$f_{!}(X)(b) = \sum_{f(a)=b} X(a)$$

for every fibration  $X \to A$  and every  $b \in B$ .

# Tribes (5)

If C is a tribe, then the category C(A) has the structure of a tribe for every object  $A \in C$ .

By definition, a morphism  $f:(X,p)\to (Y,q)$  in  $\mathcal{C}(A)$  is a fibration if the map  $f:X\to Y$  is a fibration in  $\mathcal{C}$ .

# Tribes (6)

#### Definition

A morphism of tribes  $F:\mathcal{C}\to\mathcal{D}$  is a functor which

- takes fibrations to fibrations;
- preserves base changes of fibrations;
- preserves terminal objects.

For example, the base change functor  $u^* : \mathcal{C}(B) \to \mathcal{C}(A)$  is a morphism of tribes for any map  $u : A \to B$  in a tribe  $\mathcal{C}$ .

#### Tribes are essentially algebraic theories

A set model of a tribe  $\mathcal C$  is a functor  $F:\mathcal C\to Set$  preserving the pullback squares defined by base-changes of fibrations.

For example, the theory of categories takes the form of a tribe  $\mathbb{T}(\mathit{Cat})$  if the map

$$(s,t):A\rightarrow O\times O$$

is declared to be a fibration.

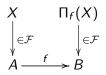
The objects and morphisms in  $\mathbb{T}(Cat)$  can be described explicitly.

The equality relation between two maps in  $\mathbb{T}(Cat)$  is decidable.

# $\Pi$ -tribes (1)

#### Definition

We say that a tribe  $\mathcal C$  is a  $\Pi$ -tribe if the product of a fibration  $X \to A$  along any fibration  $f: A \to B$  exists and the structure map  $\Pi_f(X) \to B$  is a fibration.



If C is a  $\Pi$ -tribe, then so is the tribe C(A) for every object  $A \in C$ .

# Π-tribes (2)

If  $f:A\to B$  is a fibration in a  $\Pi$ -tribe  $\mathcal C$ , then the base change functor  $f^\star:\mathcal C(B)\to\mathcal C(A)$  has a right adjoint  $f_\star$ , The functor  $f_\star$  is a product along f,

$$f_{\star} = \Pi_f : \mathcal{C}(A) \to \mathcal{C}(B).$$

Formally,

$$f_*(X)(b) = \prod_{f(a)=b} X(a)$$

for every fibration  $X \to A$  and every  $b \in B$ .

A Π-tribe is cartesian closed.

# Π-tribes (3)

Example of  $\Pi$ -tribes.

A locally cartesian closed category is a  $\Pi$ -tribe in which every map is a fibration.

#### Definition

A functor between groupoids  $F: \mathbb{A} \to \mathbb{B}$  is an *iso-fibration* if for every object  $A \in \mathbb{A}$  and every morphism  $g \in \mathbb{B}$  with domain F(A) there exists a morphism  $f \in \mathbb{A}$  with domain A such that F(f) = g.

The category of small groupoids **Grpd** becomes a  $\Pi$ -tribe if the fibrations are taken to be the iso-fibrations.

# Π-tribes (4)

#### Definition

A morphism of  $\Pi$ -tribes  $F: \mathcal{C} \to \mathcal{D}$  is a functor which preserves

- terminal objects, fibrations and base changes of fibrations;
- the internal product  $\Pi_f(X)$ .

For example, the base change functor  $u^* : \mathcal{C}(B) \to \mathcal{C}(A)$  is a morphism of  $\Pi$ -tribes for any map  $u : A \to B$  in a  $\Pi$ -tribe  $\mathcal{C}$ .

### Homotopical logic

- Weak factorization systems
- Quillen model categories
- Pre-typoi
- ▶ Typoi
- Univalent typoi

## Weak factorisation systems(1)

#### **Definition**

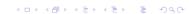
A map  $u: A \to B$  in a category  $\mathcal C$  is said to have the *left lifting* property with respect to a map  $f: X \to Y$ , and f is to have the right lifting property with respect to u, if every commutative square



has a diagonal filler  $d: B \rightarrow X$ , du = a and fd = b.



We shall denote this relation by  $u \cap f$ .



# Weak factorisation systems(2)

For any class of maps  $\mathcal{S} \subseteq \mathcal{C}$ , let us put

$$S^{\pitchfork} = \{ f \in \mathcal{C} : \forall u \in \mathcal{S} \ u \pitchfork f \}$$
$$^{\pitchfork} S = \{ u \in \mathcal{C} : \forall f \in \mathcal{S} \ u \pitchfork f \}$$

#### Definition

A pair  $(\mathcal{L}, \mathcal{R})$  of classes of maps in a category  $\mathcal{C}$  is said to be a weak factorization system if the following two conditions are satisfied

- $ightharpoonup \mathcal{R} = \mathcal{L}^{\pitchfork}$  and  $\mathcal{L} = {}^{\pitchfork}\mathcal{R}$
- ▶ Every map  $f: A \rightarrow B$  in C admits a factorization  $f = pu: A \rightarrow E \rightarrow B$  with  $u \in L$  and  $p \in R$ .

## Homotopical algebra(1)

Recall that a class  $\mathcal W$  of maps in a category  $\mathcal E$  is said to have the 3-for-2 property if whenever two sides of a commutative triangle



belongs to  $\mathcal{W}$ , then so is the third (3 apples for the price of two!).

# Homotopical algebra(2)

Quillen (1967)

### Definition

Recall that a *Quillen model structure* on a category  $\mathcal{E}$  consists on three class of maps  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  respectively called the *cofibrations*, the *weak equivalences* and the *fibrations*, such that the following conditions are satisfied:

- W has the 3-for-2 property;
- ▶ the pair  $(W \cap C, F)$  is a weak factorisation system;
- ▶ the pair  $(C, W \cap F)$  is a weak factorisation system.

A model category is a category  $\mathcal{E}$  equipped with a Quillen model structure  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ .

# Homotopical algebra and type theory (1)

Theorem (Awodey-Warren):

Martin-Löf type theory can be interpreted in a model category:

- types are interpreted as fibrant objects;
- display maps are interpreted as fibrations;
- ▶ the identity type  $Id_A \rightarrow A \times A$  is a path object for A;
- ▶ the reflexivity term  $r: A \rightarrow Id_A$  is an acyclic cofibration.

# Homotopical algebra and type theory(2)

Let  $\mathcal{C}(\mathbb{T})$  be the syntactic category of Martin-Löf type theory.

Let  $\mathcal{F}$  be the class of display maps in  $\mathcal{C}(\mathbb{T})$ .

Theorem (Gambino-Garner):

Every map  $f:A\to B$  in  $\mathcal{C}(\mathbb{T})$  admits a factorization

 $f = pu : A \to E \to B$  with  $u \in {}^{\pitchfork}\mathcal{F}$  and  $p \in \mathcal{F}$ .

# Pre-typoi(1)\*

We say that a map in a tribe  $C = (C, \mathcal{F})$  is anodyne if it belongs to the class  ${}^{\pitchfork}\mathcal{F}$ .

### **Definition**

We say that a tribe  $\mathcal C$  is a pre-typos\* if the following two conditions are satisfied

- the base change of an anodyne map along a fibration is anodyne;
- every map f : A → B admits a factorization f = pu : A → E → B with u an anodyne map and p a fibration.
- $(\star)$  Any idea for a better name?

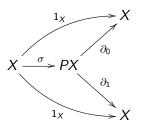


### Pre-typoi(2)

A path object for an object X in a pre-typos  $\mathcal C$  is a factorisation

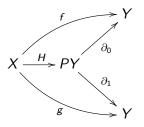
$$\langle \partial_0, \partial_1 \rangle \sigma : X \to PX \to X \times X$$

of the diagonal  $X \to X \times X$  as an anodyne map  $\sigma: X \to PX$  followed by a fibration  $\langle \partial_0, \partial_1 \rangle: PX \to X \times X$ .



### Pre-typoi(3)

A homotopy between two maps  $f, g: X \to Y$  is a map  $H: X \to PY$  such that  $\partial_0 H = f$  and  $\partial_1 H = g$ ,



We shall write  $H: f \sim g$  or  $f \sim g$ .

### Pre-typoi(4)

The homotopy relation  $f \sim g$  is a congruence on the arrows of the category  $\mathcal{C}$ .

This defines a category  $Ho(\mathcal{C}) = \mathcal{C}/\sim$ .

A map  $f: X \to Y$  in  $\mathcal C$  is a homotopy equivalence if it is invertible in  $Ho(\mathcal C)$ .

An object  $X \in \mathcal{C}$  is *contractible* if the map  $X \to \top$  is a homotopy equivalence.

# Pre-typoi(5)

#### Definition

A morphism of pre-typoi  $F:\mathcal{C}\to\mathcal{D}$  is a functor which preserves

- terminal objects, fibrations and base changes of fibrations;
- the homotopy relation.

For example, the base change functor  $u^* : \mathcal{C}(B) \to \mathcal{C}(A)$  is a morphism of pre-typoi for any map  $u : A \to B$  of a pre-typos  $\mathcal{C}$ .

# Typoi(1)\*

#### Definition

A pre-typos  $\mathcal C$  is a  $typos^*$  if it is a  $\Pi$ -tribe and the product functor  $\Pi_f:\mathcal C(A)\to\mathcal C(B)$  preserves the homotopy relation for every fibration  $f:A\to B$ .

If C is a typos, then so is the tribe C(A) for any object  $A \in C$ .

 $(\star)$  Named after a joke by Steve Awodey. Do you know a better name?

### Typoi(2)

Theorem (Hoffman and Streicher)

The category of groupoids **Grpd** has the structure of a typos in which the fibrations are the isofibrations.

# Typoi(3)

#### Definition

A morphism of typoi  $F: \mathcal{C} \to \mathcal{D}$  is a functor which preserves

- terminal objects, fibrations and base changes of fibrations;
- the homotopy relation;
- the internal products  $\Pi_f(X)$ .

For example, the base change functor  $u^* : \mathcal{C}(B) \to \mathcal{C}(A)$  is a morphism of typoi for any map  $u : A \to B$  in a typos  $\mathcal{C}$ .

# Typoi(4)

If  $u: A \to B$  is a map in a typos C, then the functor

$$Ho(u^*): Ho(\mathcal{C}(B)) \to Ho(\mathcal{C}(A))$$

has a both a left adjoint and a right adjoint.

The functor

$$A \mapsto Ho(\mathcal{C}(A))$$

is a hyper-doctrine in the sense of Lawvere!

### Typoi(5)

A typos C may contain a sub-typos of *small fibrations*.

A small fibration  $p: U' \to U$  is *universal* if for every small fibration  $f: X \to B$  there exists a (homotopy) cartesian square:

$$\begin{array}{c|c}
X \xrightarrow{\chi'} & U' \\
\downarrow^f & \downarrow^p \\
B \xrightarrow{\chi} & U.
\end{array}$$

Martin-Löf axiom: There exists a universal fibration U.

### Univalent typoi

The pair  $(\chi, \chi')$  is *classifying* the fibration  $f: X \to B$ .

A fibration  $p: U' \to U$  is univalent if the pair  $(\chi, \chi')$  classifying a fibration is homotopy unique.

Voevodsky axiom: The universal fibration  $U' \rightarrow U$  is univalent.

Theorem (Voevodsky)

The category of Kan complexes **Kan** has the structure of a univalent typos in which the fibrations are the Kan fibrations.

### Conclusions

Homotopy type theory depends on the notion of weak factorization system, at least implicitly.

The notion of weak factorization system first appeared in homotopical algebra; it arose from a *praxis*, not from adjointness principles.

Homotopy type theory is soluble in category theory, but not in a logical framework exclusively based on adjointness principles.

Thanks!