Functors

and

natural transformations

functors	$\rightsquigarrow$	*category morphisms*
natural transformations	$\rightsquigarrow$	*functor morphisms*

Functors

A *functor* $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ from a category $\mathbf{K}$ to a category $\mathbf{K}'$ consists of:

- a function $\mathbf{F} : |\mathbf{K}| \to |\mathbf{K}'|$, and
- for all $A, B \in |\mathbf{K}|$, a function $\mathbf{F} : \mathbf{K}(A, B) \to \mathbf{K}'(\mathbf{F}(A), \mathbf{F}(B))$

such that:

Make explicit categories in which we work at various places here

- $\mathbf{F}$ preserves identities, i.e.,

$$\boxed{\mathbf{F}(id_A) = id_{\mathbf{F}(A)}}$$

  for all $A \in |\mathbf{K}|$, and

- $\mathbf{F}$ preserves composition, i.e.,

$$\boxed{\mathbf{F}(f;g) = \mathbf{F}(f);\mathbf{F}(g)}$$

  for all $f : A \to B$ and $g : B \to C$ in $\mathbf{K}$.

We really should differentiate between various components of F

Examples

- *identity functors*: $\mathbf{Id}_{\mathbf{K}} : \mathbf{K} \to \mathbf{K}$, for any category $\mathbf{K}$
- *inclusions*: $\mathbf{I}_{\mathbf{K} \hookrightarrow \mathbf{K}'} : \mathbf{K} \to \mathbf{K}'$, for any subcategory $\mathbf{K}$ of $\mathbf{K}'$
- *constant functors*: $\mathbf{C}_A : \mathbf{K} \to \mathbf{K}'$, for any categories $\mathbf{K}, \mathbf{K}'$ and $A \in |\mathbf{K}'|$, with $\mathbf{C}_A(f) = id_A$ for all morphisms f in $\mathbf{K}$
- *powerset functor*: $\mathbf{P} : \mathbf{Set} \to \mathbf{Set}$ given by
  - $\mathbf{P}(X) = \{Y \mid Y \subseteq X\}$, for all $X \in |\mathbf{Set}|$
  - $\mathbf{P}(f) : \mathbf{P}(X) \to \mathbf{P}(X')$ for all $f : X \to X'$ in $\mathbf{Set}$,
    $\mathbf{P}(f)(Y) = \{f(y) \mid y \in Y\}$ for all $Y \subseteq X$
- *contravariant powerset functor*: $\mathbf{P}_{-1} : \mathbf{Set}^{op} \to \mathbf{Set}$ given by
  - $\mathbf{P}_{-1}(X) = \{Y \mid Y \subseteq X\}$, for all $X \in |\mathbf{Set}|$
  - $\mathbf{P}_{-1}(f) : \mathbf{P}(X') \to \mathbf{P}(X)$ for all $f : X \to X'$ in $\mathbf{Set}$,
    $\mathbf{P}_{-1}(f)(Y') = \{x \in X \mid f(x) \in Y'\}$ for all $Y' \subseteq X'$

Examples, cont'd.

- *projection functors*: $\pi_1 : \mathbf{K} \times \mathbf{K}' \to \mathbf{K}$, $\pi_2 : \mathbf{K} \times \mathbf{K}' \to \mathbf{K}'$
- *list functor*: $\mathbf{List} : \mathbf{Set} \to \mathbf{Monoid}$, where $\mathbf{Monoid}$ is the category of monoids (as objects) with monoid homomorphisms as morphisms:
  - $\mathbf{List}(X) = \langle X^*, {}^\frown, \epsilon \rangle$, for all $X \in |\mathbf{Set}|$, where X^* is the set of all finite lists of elements from X, ${}^\frown$ is the list concatenation, and ϵ is the empty list.
  - $\mathbf{List}(f) : \mathbf{List}(X) \to \mathbf{List}(X')$ for $f : X \to X'$ in $\mathbf{Set}$,
    $\mathbf{List}(f)(\langle x_1, \ldots, x_n \rangle) = \langle f(x_1), \ldots, f(x_n) \rangle$ for all $x_1, \ldots, x_n \in X$
- *totalisation functor*: $\mathbf{Tot} : \mathbf{Pfn} \to \mathbf{Set}_*$, where $\mathbf{Set}_*$ is the subcategory of $\mathbf{Set}$ of sets with a distinguished element $*$ and $*$-preserving functions
  - $\mathbf{Tot}(X) = X \uplus \{*\}$
  - $\mathbf{Tot}(f)(x) = \begin{cases} f(x) & \text{if it is defined} \\ * & \text{otherwise} \end{cases}$

Define $\mathbf{Set}_*$ as the category of algebras

Examples, cont'd.

- *carrier set functors*: $|\_| : \mathbf{Alg}(\Sigma) \to \mathbf{Set}^S$, for any algebraic signature $\Sigma = \langle S, \Omega \rangle$, yielding the algebra carriers and homomorphisms as functions between them
- *reduct functors*: $\_|_\sigma : \mathbf{Alg}(\Sigma') \to \mathbf{Alg}(\Sigma)$, for any signature morphism $\sigma : \Sigma \to \Sigma'$, as defined earlier
- *term algebra functors*: $\mathbf{T}_\Sigma : \mathbf{Set} \to \mathbf{Alg}(\Sigma)$ for all (single-sorted) algebraic signatures $\Sigma \in |\mathbf{AlgSig}|$

  Generalise to many-sorted signatures

  - $\mathbf{T}_\Sigma(X) = T_\Sigma(X)$ for all $X \in |\mathbf{Set}|$
  - $\mathbf{T}_\Sigma(f) = f^\# : T_\Sigma(X) \to T_\Sigma(X')$ for all functions $f : X \to X'$
- *diagonal functors*: $\Delta^G_\mathbf{K} : \mathbf{K} \to \mathbf{Diag}^G_\mathbf{K}$ for any graph G with nodes $N = |G|_{nodes}$ and edges $E = |G|_{edges}$, and category $\mathbf{K}$
  - $\Delta^G_\mathbf{K}(A) = D^A$, where D^A is the "constant" diagram, with $D^A_n = A$ for all $n \in N$ and $D^A_e = id_A$ for all $e \in E$
  - $\Delta^G_\mathbf{K}(f) = \mu^f : D^A \to D^B$, for all $f : A \to B$, where $\mu^f_n = f$ for all $n \in N$

Hom-functors

Given a *locally small* category $\mathbf{K}$, define

$$\boxed{\mathbf{Hom}_{\mathbf{K}} : \mathbf{K}^{op} \times \mathbf{K} \to \mathbf{Set}}$$

a binary *hom-functor*, contravariant on the first argument and covariant on the second argument, as follows:

- $\mathbf{Hom}_{\mathbf{K}}(\langle A, B\rangle) = \mathbf{K}(A, B)$, for all $\langle A, B\rangle \in |\mathbf{K}^{op} \times \mathbf{K}|$, i.e., $A, B \in |\mathbf{K}|$
- $\mathbf{Hom}_{\mathbf{K}}(\langle f, g\rangle) : \mathbf{K}(A, B) \to \mathbf{K}(A', B')$, for $\langle f, g\rangle : \langle A, B\rangle \to \langle A', B'\rangle$ in $\mathbf{K}^{op} \times \mathbf{K}$, i.e., $f : A' \to A$ and $g : B \to B'$ in $\mathbf{K}$, as a function given by $\mathbf{Hom}_{\mathbf{K}}(\langle f, g\rangle)(h) = f;g;h$.

$$\begin{tikzcd}
& A \arrow[dd, "h"] & & A' \arrow[ll, "f"'] \arrow[dd] & \\
\mathbf{K}(A,B) & & & & \mathbf{K}(A',B') \\
& B \arrow[rr, "g"] \arrow[rr, bend right, "\mathbf{Hom}_{\mathbf{K}}(f,g)"'] & & B' &
\end{tikzcd}$$

Also: $\mathbf{Hom}_{\mathbf{K}}(A, \_) : \mathbf{K} \to \mathbf{Set}$

$\mathbf{Hom}_{\mathbf{K}}(\_, B) : \mathbf{K}^{op} \to \mathbf{Set}$

Functors preserve. . .

- Check whether functors preserve:
  - monomorphisms
  - epimorphisms
  - (co)retractions
  - isomorphisms
  - (co)cones
  - (co)limits
  - . . .
- A functor is (finitely) continuous if it preserves all existing (finite) limits.

  Which of the above functors are (finitely) continuous?

Dualise!

Functors compose. . .

Given two functors $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ and $\mathbf{G} : \mathbf{K}' \to \mathbf{K}''$, their *composition* $\mathbf{F};\mathbf{G} : \mathbf{K} \to \mathbf{K}''$ is defined as expected:

- $(\mathbf{F};\mathbf{G})(A) = \mathbf{G}(\mathbf{F}(A))$ for all $A \in |\mathbf{K}|$
- $(\mathbf{F};\mathbf{G})(f) = \mathbf{G}(\mathbf{F}(f))$ for all $f : A \to B$ in $\mathbf{K}$

Cat, *the category of (sm)all categories*

– objects: (sm)all categories
– morphisms: functors between them
– composition: as above

Characterise isomorphisms in **Cat**

Define products, terminal objects, equalisers and pullback in **Cat**

Try to define their duals

Comma categories

Given two functors with a common target, $\mathbf{F}:\mathbf{K1}\to\mathbf{K}$ and $\mathbf{G}:\mathbf{K2}\to\mathbf{K}$, define their *comma category*

$$\boxed{(\mathbf{F},\mathbf{G})}$$

– objects: triples $\langle A_1, f:\mathbf{F}(A_1)\to\mathbf{G}(A_2), A_2\rangle$, where $A_1\in|\mathbf{K1}|$, $A_2\in|\mathbf{K2}|$, and $f:\mathbf{F}(A_1)\to\mathbf{G}(A_2)$ in $\mathbf{K}$

– morphisms: a morphism in $(\mathbf{F},\mathbf{G})$ is any pair $\langle h_1,h_2\rangle:\langle A_1, f:\mathbf{F}(A_1)\to\mathbf{G}(A_2), A_2\rangle\to\langle B_1, g:\mathbf{F}(B_1)\to\mathbf{G}(B_2), B_2\rangle$, where $h_1:A_1\to B_1$ in $\mathbf{K1}$, $h_2:A_2\to B_2$ in $\mathbf{K2}$, and $\mathbf{F}(h_1);g=f;\mathbf{G}(h_2)$ in $\mathbf{K}$.

– composition: component-wise

$\mathbf{K1}$: $\quad$ $\mathbf{K}$: $\quad$ $\mathbf{K2}$:

$$\begin{tikzcd}
A_1 \arrow[d, "h_1"'] & \mathbf{F}(A_1) \arrow[r, "f"] \arrow[d, "\mathbf{F}(h_1)"'] & \mathbf{G}(A_2) \arrow[d, "\mathbf{G}(h_2)"] & A_2 \arrow[d, "h_2"] \\
B_1 & \mathbf{F}(B_1) \arrow[r, "g"] & \mathbf{G}(B_2) & B_2
\end{tikzcd}$$

Examples

- The category of graphs as a comma category:

$$\boxed{\mathbf{Graph} = (\mathbf{Id}_{\mathbf{Set}}, \mathbf{CP})}$$

where $\mathbf{CP} : \mathbf{Set} \to \mathbf{Set}$ is the (Cartesian) product functor ($\mathbf{CP}(X) = X \times X$ and $\mathbf{CP}(f)(\langle x, x' \rangle) = \langle f(x), f(x') \rangle$). Hint: write objects of this category as $\langle E, \langle source, target \rangle : E \to N \times N, N \rangle$

- The category of algebraic signatures as a comma category:

$$\boxed{\mathbf{AlgSig} = (\mathbf{Id}_{\mathbf{Set}}, (\_)^+)}$$

where $(\_)^+ : \mathbf{Set} \to \mathbf{Set}$ is the non-empty list functor ($(X)^+$ is the set of all non-empty lists of elements from X, $(f)^+(\langle x_1, \ldots, x_n \rangle) = \langle f(x_1), \ldots, f(x_n) \rangle$). Hint: write objects of this category as $\langle \Omega, \langle arity, sort \rangle : \Omega \to S^+, S \rangle$

Define $\mathbf{K}^{\to}$, $\mathbf{K}{\downarrow}A$ as comma categories. The same for $\mathbf{Alg}(\Sigma)$.

Cocompleteness of comma categories

Fact: *If* $\mathbf{K1}$ *and* $\mathbf{K2}$ *are (finitely) cocomplete categories,* $\mathbf{F} : \mathbf{K1} \to \mathbf{K}$ *is a (finitely) cocontinuous functor, and* $\mathbf{G} : \mathbf{K2} \to \mathbf{K}$ *is a functor then the comma category* $(\mathbf{F}, \mathbf{G})$ *is (finitely) cocomplete.*

Proof (idea):

Construct coproducts and coequalisers in $(\mathbf{F}, \mathbf{G})$, using the corresponding constructions in $\mathbf{K1}$ and $\mathbf{K2}$, and cocontinuity of $\mathbf{F}$.

> *State and prove the dual fact,*
> *concerning completeness of comma categories*

Coproducts:

$$
\begin{tikzcd}
 & A_1 \arrow[dl, "\iota_{A_1}"'] & \mathbf{F}(A_1) \arrow[rrr, "f"] \arrow[dr, "\mathbf{F}(\iota_{A_1})"'] & & & \mathbf{G}(A_2) \arrow[dl, "\mathbf{G}(\iota_{A_2})"] & A_2 \arrow[dr, "\iota_{A_2}"] & \\
A_1 + B_1 & & & \mathbf{F}(A_1 + B_1) \arrow[r] & \mathbf{G}(A_2 + B_2) & & & A_2 + B_2 \\
 & B_1 \arrow[ul, "\iota_{B_1}"] & \mathbf{F}(B_1) \arrow[rrr, "g"] \arrow[ur, "\mathbf{F}(\iota_{B_1})"] & & & \mathbf{G}(B_2) \arrow[ul, "\mathbf{G}(\iota_{B_2})"'] & B_2 \arrow[ur, "\iota_{B_2}"'] &
\end{tikzcd}
$$

Coequalisers:

$$
\begin{tikzcd}
A_1 \arrow[d, shift right, "h_1"'] \arrow[d, shift left, "h_1'"] & \mathbf{F}(A_1) \arrow[rr, "f"] \arrow[d, shift right, "\mathbf{F}(h_1)"'] \arrow[d, shift left, "\mathbf{F}(h_1')"] & & \mathbf{G}(A_2) \arrow[d, shift right, "\mathbf{G}(h_2)"'] \arrow[d, shift left, "\mathbf{G}(h_2')"] & A_2 \arrow[d, shift right, "h_2"'] \arrow[d, shift left, "h_2'"] \\
B_1 \arrow[d, "c_1"'] & \mathbf{F}(B_1) \arrow[rr, "g"] \arrow[d, "\mathbf{F}(c_1)"'] & & \mathbf{G}(B_2) \arrow[d, "\mathbf{G}(c_2)"] & B_2 \arrow[d, "c_2"] \\
C_1 & \mathbf{F}(C_1) \arrow[rr] & & \mathbf{G}(C_2) & C_2
\end{tikzcd}
$$

Indexed categories

An *indexed category* is a functor $\boxed{\mathcal{C} : \mathbf{Ind}^{op} \to \mathbf{Cat}}$.

Standard example: $\mathbf{Alg} : \mathbf{AlgSig}^{op} \to \mathbf{Cat}$

The Grothendieck construction: Given $\mathcal{C} : \mathbf{Ind}^{op} \to \mathbf{Cat}$, define a category $\mathbf{Flat}(\mathcal{C})$:

- objects: $\langle i, A \rangle$ for all $i \in |\mathbf{Ind}|$, $A \in |\mathcal{C}(i)|$
- morphisms: a morphism from $\langle i, A \rangle$ to $\langle j, B \rangle$, $\langle \sigma, f \rangle : \langle i, A \rangle \to \langle j, B \rangle$, consists of a morphism $\sigma : i \to j$ in $\mathbf{Ind}$ and a morphism $f : A \to \mathcal{C}(\sigma)(B)$ in $\mathcal{C}(i)$
- composition: given $\langle \sigma, f \rangle : \langle i, A \rangle \to \langle i', A' \rangle$ and $\langle \sigma', f' \rangle : \langle i', A' \rangle \to \langle i'', A'' \rangle$, their composition in $\mathbf{Flat}(\mathcal{C})$, $\langle \sigma, f \rangle;\langle \sigma', f' \rangle : \langle i, A \rangle \to \langle i'', A'' \rangle$, is given by

$$\boxed{\langle \sigma, f \rangle;\langle \sigma', f' \rangle = \langle \sigma;\sigma', f;\mathcal{C}(\sigma)(f') \rangle}$$

Fact: *If $\mathbf{Ind}$ is complete, $\mathcal{C}(i)$ are complete for all $i \in |\mathbf{Ind}|$, and $\mathcal{C}(\sigma)$ are continuous for all $\sigma : i \to j$ in $\mathbf{Ind}$, then $\mathbf{Flat}(\mathcal{C})$ is complete.*

Try to formulate and prove a theorem concerning cocompleteness of $\mathbf{Flat}(\mathcal{C})$

Natural transformations

Given two parallel functors $\mathbf{F}, \mathbf{G} : \mathbf{K} \to \mathbf{K}'$, a *natural transformation* from $\mathbf{F}$ to $\mathbf{G}$

$$\boxed{\tau : \mathbf{F} \to \mathbf{G}}$$

is a family $\tau = \langle \tau_A : \mathbf{F}(A) \to \mathbf{G}(A) \rangle_{A \in |\mathbf{K}|}$ of $\mathbf{K}'$-morphisms such that for all $f : A \to B$ in $\mathbf{K}$ (with $A, B \in |\mathbf{K}|$), $\boxed{\tau_A;\mathbf{G}(f) = \mathbf{F}(f);\tau_B}$

Then, τ is a *natural isomorphism* if for all $A \in |\mathbf{K}|$, τ_A is an isomorphism.

$\mathbf{K}$:

$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd}$$

$\mathbf{K}'$:

$$\begin{tikzcd} \mathbf{F}(A) \arrow[r, "\tau_A"] \arrow[d, "\mathbf{F}(f)"'] & \mathbf{G}(A) \arrow[d, "\mathbf{G}(f)"] \\ \mathbf{F}(B) \arrow[r, "\tau_B"] & \mathbf{G}(B) \end{tikzcd}$$

Examples

- *identity transformations*: $id_{\mathbf{F}} : \mathbf{F} \to \mathbf{F}$, where $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$, for all objects $A \in |\mathbf{K}|$, $(id_{\mathbf{F}})_A = id_A : \mathbf{F}(A) \to \mathbf{F}(A)$
- *singleton functions*: $sing : \mathbf{Id}_{\mathbf{Set}} \to \mathbf{P}$ $(: \mathbf{Set} \to \mathbf{Set})$, where for all $X \in |\mathbf{Set}|$, $sing_X : X \to \mathbf{P}(X)$ is a function defined by $sing_X(x) = \{x\}$ for $x \in X$
- *singleton-list functions*: $sing^{\mathbf{List}} : \mathbf{Id}_{\mathbf{Set}} \to |\mathbf{List}|$ $(: \mathbf{Set} \to \mathbf{Set})$, where $|\mathbf{List}| = \mathbf{List};|\_| : \mathbf{Set}(\to \mathbf{Monoid}) \to \mathbf{Set}$, and for all $X \in |\mathbf{Set}|$, $sing^{\mathbf{List}}_X : X \to X^*$ is a function defined by $sing^{\mathbf{List}}_X(x) = \langle x \rangle$ for $x \in X$
- *append functions*: $append : |\mathbf{List}|;\mathbf{CP} \to |\mathbf{List}|$ $(: \mathbf{Set} \to \mathbf{Set})$, where for all $X \in |\mathbf{Set}|$, $append_X : (X^* \times X^*) \to X^*$ is the usual append function (list concatenation) polymorphic functions between algebraic types

Polymorphic functions

Work out the following generalisation of the last two examples:

- for each algebraic type scheme $\forall\alpha_1 \ldots \alpha_n \cdot T$, built in `Standard ML` using at least products and algebraic data types (no function types though), define the corresponding functor $[\![T]\!] : \mathbf{Set}^n \to \mathbf{Set}$
- argue that in a representative subset of `Standard ML`, for each polymorphic expression $E : \forall\alpha_1 \ldots \alpha_n \cdot T \to T'$ its semantics is a natural transformation $[\![E]\!] : [\![T]\!] \to [\![T']\!]$

Theorems for free!
(see Wadler 89)

Yoneda lemma

Given a locally small category $\mathbf{K}$, functor $\mathbf{F} : \mathbf{K} \to \mathbf{Set}$ and object $A \in |\mathbf{K}|$:

$$\boxed{Nat(\mathbf{Hom}_{\mathbf{K}}(A, \_), \mathbf{F}) \cong \mathbf{F}(A)}$$

natural transformations from $\mathbf{Hom}_{\mathbf{K}}(A, \_)$ *to* $\mathbf{F}$*, between functors from* $\mathbf{K}$ *to* $\mathbf{Set}$*, are given exactly by the elements of the set* $\mathbf{F}(A)$

EXERCISES:

- Dualise: for $\mathbf{G} : \mathbf{K}^{op} \to \mathbf{Set}$,

$$\boxed{Nat(\mathbf{Hom}_{\mathbf{K}}(\_, A), \mathbf{G}) \cong \mathbf{G}(A)}$$

.

- Characterise all natural transformations from $\mathbf{Hom}_{\mathbf{K}}(A, \_)$ to $\mathbf{Hom}_{\mathbf{K}}(B, \_)$, for all objects $A, B \in |\mathbf{K}|$.

Proof

- For $a \in \mathbf{F}(A)$, define $\tau^a : \mathbf{Hom_K}(A, \_) \to \mathbf{F}$, as the family of functions $\tau^a_B : \mathbf{K}(A, B) \to \mathbf{F}(B)$ given by $\tau^a_B(f) = \mathbf{F}(f)(a)$ for $f : A \to B$ in $\mathbf{K}$.

  This is a natural transformation, since for $g : B \to C$ and then $f : A \to B$,

  $$\mathbf{F}(g)(\tau^a_B(f)) = \mathbf{F}(g)(\mathbf{F}(f)(a))$$
  $$= \mathbf{F}(f;g)(a) = \tau^a_C(f;g)$$
  $$= \tau^a_C(\mathbf{Hom_K}(A, g)(f))$$

  $\mathbf{K}$: $\quad$ **Set**:

  $$\begin{tikzcd} B \arrow[d, "g"'] & \mathbf{K}(A,B) \arrow[r, "\tau^a_B"] \arrow[d, "(\_);g = \mathbf{Hom_K}(A,g)"'] & \mathbf{F}(B) \arrow[d, "\mathbf{F}(g)"] \\ C & \mathbf{K}(A,C) \arrow[r, "\tau^a_C"] & \mathbf{F}(C) \end{tikzcd}$$

  Then $\tau^a_A(id_A) = a$, and so for distinct $a, a' \in \mathbf{F}(A)$, τ^a and $\tau^{a'}$ differ.

- If $\tau : \mathbf{Hom_K}(A, \_) \to \mathbf{F}$ is a natural transformation then $\tau = \tau^a$, where we put $a = \tau_A(id_A)$, since for $B \in |\mathbf{K}|$ and $f : A \to B$, $\tau_B(f) = \mathbf{F}(f)(\tau_A(id_A))$ by naturality of τ:

  $$\begin{tikzcd} A \arrow[d, "f"'] & \mathbf{K}(A,A) \arrow[r, "\tau_A"] \arrow[d, "(\_);f = \mathbf{Hom_K}(A,f)"'] & \mathbf{F}(A) \arrow[d, "\mathbf{F}(f)"] \\ B & \mathbf{K}(A,B) \arrow[r, "\tau_B"] & \mathbf{F}(B) \end{tikzcd}$$

Compositions

vertical composition:

From:

$$\begin{tikzcd}[column sep=huge] \mathbf{K} \arrow[r, bend left=70, "\mathbf{F}"{name=U}] \arrow[r, "\mathbf{F}'"{name=M}] \arrow[r, bend right=70, "\mathbf{F}''"'{name=D}] & \mathbf{K}' \arrow[from=U, to=M, phantom, "\tau\rightsquigarrow"] \arrow[from=M, to=D, phantom, "\sigma\rightsquigarrow"] \end{tikzcd}$$

to:

$$\begin{tikzcd}[column sep=huge] \mathbf{K} \arrow[r, bend left=70, "\mathbf{F}"{name=U}] \arrow[r, bend right=70, "\mathbf{F}''"'{name=D}] & \mathbf{K}' \arrow[from=U, to=D, phantom, "\tau;\sigma\rightsquigarrow"] \end{tikzcd}$$

horizontal composition:

From:

$$\begin{tikzcd}[column sep=huge] \mathbf{K} \arrow[r, bend left=70, "\mathbf{F}"{name=U1}] \arrow[r, bend right=70, "\mathbf{F}'"'{name=D1}] & \mathbf{K}' \arrow[r, bend left=70, "\mathbf{G}"{name=U2}] \arrow[r, bend right=70, "\mathbf{G}'"'{name=D2}] & \mathbf{K}'' \arrow[from=U1, to=D1, phantom, "\tau\rightsquigarrow"] \arrow[from=U2, to=D2, phantom, "\sigma\rightsquigarrow"] \end{tikzcd}$$

to:

$$\begin{tikzcd}[column sep=huge] \mathbf{K} \arrow[r, bend left=70, "\mathbf{F};\mathbf{G}"{name=U}] \arrow[r, bend right=70, "\mathbf{F}';\mathbf{G}'"'{name=D}] & \mathbf{K}'' \arrow[from=U, to=D, phantom, "\tau\cdot\sigma\rightsquigarrow"] \end{tikzcd}$$

Vertical composition

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbf{K} \arrow[r, bend left=60, "\mathbf{F}"] \arrow[r, "\mathbf{F}'" near end] \arrow[r, bend right=60, "\mathbf{F}''"'] \arrow[r, phantom, shift left=4, "\Downarrow\tau"] \arrow[r, phantom, shift right=4, "\Downarrow\sigma"] & \mathbf{K}'
\end{tikzcd}
$$

The *vertical composition* of natural transformations $\tau : \mathbf{F} \to \mathbf{F}'$ and $\sigma : \mathbf{F}' \to \mathbf{F}''$ between parallel functors $\mathbf{F}, \mathbf{F}', \mathbf{F}'' : \mathbf{K} \to \mathbf{K}'$

$$\boxed{\tau;\sigma : \mathbf{F} \to \mathbf{F}''}$$

is a natural transformation given by $\boxed{(\tau;\sigma)_A = \tau_A;\sigma_A}$ for all $A \in |\mathbf{K}|$.

$$
\begin{array}{ll}
\mathbf{K}: & \mathbf{K}': \\
\begin{tikzcd}[row sep=huge]
A \arrow[d, "f"'] \\
B
\end{tikzcd}
&
\begin{tikzcd}[row sep=huge, column sep=huge]
\mathbf{F}(A) \arrow[r, "\tau_A"] \arrow[d, "\mathbf{F}(f)"'] & \mathbf{F}'(A) \arrow[r, "\sigma_A"] \arrow[d, "\mathbf{F}'(f)"] & \mathbf{F}''(A) \arrow[d, "\mathbf{F}''(f)"] \\
\mathbf{F}(B) \arrow[r, "\tau_B"] & \mathbf{F}'(B) \arrow[r, "\sigma_B"] & \mathbf{F}''(B)
\end{tikzcd}
\end{array}
$$

Horizontal composition

$$\begin{tikzcd} \mathbf{K} \arrow[rr, bend left, "\mathbf{F}"] \arrow[rr, bend right, "\mathbf{F}'"'] \arrow[rr, phantom, "\tau\Downarrow"] && \mathbf{K}' \arrow[rr, bend left, "\mathbf{G}"] \arrow[rr, bend right, "\mathbf{G}'"'] \arrow[rr, phantom, "\sigma\Downarrow"] && \mathbf{K}'' \end{tikzcd}$$

The *horizontal composition* of natural transformations $\tau : \mathbf{F} \to \mathbf{F}'$ and $\sigma : \mathbf{G} \to \mathbf{G}'$ between composable pairs of parallel functors $\mathbf{F}, \mathbf{F}' : \mathbf{K} \to \mathbf{K}'$, $\mathbf{G}, \mathbf{G}' : \mathbf{K}' \to \mathbf{K}''$

$$\boxed{\tau{\cdot}\sigma : \mathbf{F};\mathbf{G} \to \mathbf{F}';\mathbf{G}'}$$

is a natural transformation given by $\boxed{(\tau{\cdot}\sigma)_A = \mathbf{G}(\tau_A);\sigma_{\mathbf{F}'(A)} = \sigma_{\mathbf{F}(A)};\mathbf{G}'(\tau_A)}$ for all $A \in |\mathbf{K}|$.

Multiplication by functor:

– $\tau{\cdot}\mathbf{G} = \tau{\cdot}id_{\mathbf{G}} : \mathbf{F};\mathbf{G} \to \mathbf{F}';\mathbf{G}$,
  i.e., $(\tau{\cdot}\mathbf{G})_A = \mathbf{G}(\tau_A)$

– $\mathbf{F}{\cdot}\sigma = id_{\mathbf{F}}{\cdot}\sigma : \mathbf{F};\mathbf{G} \to \mathbf{F};\mathbf{G}'$,
  i.e., $(\mathbf{F}{\cdot}\sigma)_A = \sigma_{\mathbf{F}(A)}$

$\mathbf{K}'$:

$$\begin{tikzcd} \mathbf{F}(A) \arrow[d, "\tau_A"'] \\ \mathbf{F}'(A) \end{tikzcd}$$

$\mathbf{K}''$:

$$\begin{tikzcd} \mathbf{G}(\mathbf{F}(A)) \arrow[r, "\sigma_{\mathbf{F}(A)}"] \arrow[d, "\mathbf{G}(\tau_A)"'] \arrow[dr, "(\tau{\cdot}\sigma)_A"] & \mathbf{G}'(\mathbf{F}(A)) \arrow[d, "\mathbf{G}'(\tau_A)"] \\ \mathbf{G}(\mathbf{F}'(A)) \arrow[r, "\sigma_{\mathbf{F}'(A)}"] & \mathbf{G}'(\mathbf{F}'(A)) \end{tikzcd}$$

Show that indeed, $\tau{\cdot}\sigma$ is a natural transformation

Functor categories

Given two categories $\mathbf{K}, \mathbf{K}'$, define the *category of functors from $\mathbf{K}'$ to $\mathbf{K}$*, $\mathbf{K}^{\mathbf{K}'}$, as follows:

- objects: functors from $\mathbf{K}'$ to $\mathbf{K}$
- morphisms: natural transformations between them
- composition: vertical composition of the natural transformations

Exercises:

- View the category of S-sorted sets, $\mathbf{Set}^S$, as a functor category
- Show how any functor $\mathbf{F} : \mathbf{K}'' \to \mathbf{K}'$ induces a functor $(\mathbf{F};\_) : \mathbf{K}^{\mathbf{K}'} \to \mathbf{K}^{\mathbf{K}''}$
- Check whether $\mathbf{K}^{\mathbf{K}'}$ is (finitely) (co)complete whenever $\mathbf{K}$ is so.
- Check when $(\mathbf{F};\_) : \mathbf{K}^{\mathbf{K}'} \to \mathbf{K}^{\mathbf{K}''}$ is (finitely) (co)continuous, for a given functor $\mathbf{F} : \mathbf{K}'' \to \mathbf{K}'$

Diagrams as functors

Each diagram D over graph G in category $\mathbf{K}$ yields a functor $\mathbf{F}_D : \mathbf{Path}(G) \to \mathbf{K}$ given by:

- $\mathbf{F}_D(n) = D_n$, for all nodes $n \in |G|_{nodes}$
- $\mathbf{F}_D(n_0 e_1 n_1 \ldots n_{k-1} e_k n_k) = D_{e_1};\ldots;D_{e_k}$, for paths $n_0 e_1 n_1 \ldots n_{k-1} e_k n_k$ in G

Moreover:

- for distinct diagrams D and D' of shape G, $\mathbf{F}_D$ and $\mathbf{F}_{D'}$ are different
- all functors from $\mathbf{Path}(G)$ to $\mathbf{K}$ are given by diagrams over G

Diagram morphisms $\mu : D \to D'$ between diagrams of the same shape G are exactly natural transformations $\mu : \mathbf{F}_D \to \mathbf{F}_{D'}$.

$$\boxed{\mathbf{Diag}^G_{\mathbf{K}} \cong \mathbf{K}^{\mathbf{Path}(G)}}$$

Diagrams are functors from small (shape) categories

Double law

Given:

$$\begin{tikzcd}[column sep=huge]
\mathbf{K} \arrow[r, bend left=60, "\mathbf{F}"{name=F}] \arrow[r, "\mathbf{F}'"{name=Fp}] \arrow[r, bend right=60, "\mathbf{F}''"'{name=Fpp}] & \mathbf{K}' \arrow[r, bend left=60, "\mathbf{G}"{name=G}] \arrow[r, "\mathbf{G}'"{name=Gp}] \arrow[r, bend right=60, "\mathbf{G}''"'{name=Gpp}] & \mathbf{K}''
\arrow[from=F, to=Fp, Rightarrow, squiggly, "\tau"'] \arrow[from=Fp, to=Fpp, Rightarrow, squiggly, "\tau'"'] \arrow[from=G, to=Gp, Rightarrow, squiggly, "\sigma"'] \arrow[from=Gp, to=Gpp, Rightarrow, squiggly, "\sigma'"']
\end{tikzcd}$$

then:

$$\boxed{(\tau\cdot\sigma);(\tau'\cdot\sigma') = (\tau;\tau')\cdot(\sigma;\sigma')}$$

$$\begin{tikzcd}[column sep=huge]
\mathbf{K} \arrow[r, bend left=60, "\mathbf{F};\mathbf{G}"{name=U}] \arrow[r, bend right=60, "\mathbf{F}'';\mathbf{G}''"'{name=D}] & \mathbf{K}''
\arrow[from=U, to=D, Rightarrow, squiggly, "(\tau\cdot\sigma);(\tau'\cdot\sigma')"', "(\tau;\tau')\cdot(\sigma;\sigma')"]
\end{tikzcd}$$

This holds in $\mathbf{Cat}$, which is a paradigmatic example of a two-category.

A category $\mathbf{K}$ is a *two-category* when for all objects $A, B \in |\mathbf{K}|$, $\mathbf{K}(A, B)$ is again a category, with *1-morphisms* (the usual $\mathbf{K}$-morphisms) as objects and *2-morphisms* between them. Those 2-morphisms compose vertically (in the categories $\mathbf{K}(A, B)$) and horizontally, subject to the double law as stated here.

In two-category $\mathbf{Cat}$, we have $\mathbf{Cat}(\mathbf{K}', \mathbf{K}) = \mathbf{K}^{\mathbf{K}'}$.

Equivalence of categories

- Two categories $\mathbf{K}$ and $\mathbf{K}'$ are *isomorphic* if there are functors $\mathbf{F}: \mathbf{K} \to \mathbf{K}'$ and $\mathbf{G}: \mathbf{K}' \to \mathbf{K}$ such that $\mathbf{F};\mathbf{G} = \mathbf{Id}_{\mathbf{K}}$ and $\mathbf{G};\mathbf{F} = \mathbf{Id}_{\mathbf{K}'}$.
- Two categories $\mathbf{K}$ and $\mathbf{K}'$ are *equivalent* if there are functors $\mathbf{F}: \mathbf{K} \to \mathbf{K}'$ and $\mathbf{G}: \mathbf{K}' \to \mathbf{K}$ and natural isomorphisms $\eta: \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$ and $\epsilon: \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$.
- A category is *skeletal* if any two isomorphic objects are identical.
- A *skeleton* of a category is any of its maximal skeletal subcategory.

Fact: *Two categories are equivalent iff they have isomorphic skeletons.*

All "categorical" properties are preserved under equivalence of categories