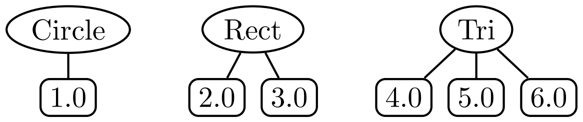
CS251 Programming Languages **Handout # 21**
Randy Shull
Wellesley College

Sum-of-Product Data Types

1 Introduction

Every general-purpose programming language must allow the processing of values with different structure that are nevertheless considered to have the same "type". For example, in the processing of simple geometric figures, we want a notion of a "figure type" that includes circles with a radius, rectangles with a width and height, and triangles with three sides. Abstractly, figure values might be depicted as shown below:



The name in the oval is a *tag* that indicates which kind of figure the value is, and the branches leading down from the oval indicate the *components* of the value. Such types are known as **sum-of-product data types** because they consist of a sum of tagged types, each of which holds on to a product of components. They are also known as **algebraic data types**.

In OCAML we can declare a new `figure` type that represents these sorts of geometric figures as follows:

```
type figure =
    Circ of float (* radius *)
  | Rect of float * float (* width, height *)
  | Tri of float * float * float (* side1, side2, side3 *)
```

Such a declaration is known as a **data type** declaration. It consists of a series of `|`-separated clauses of the form

`constructor-name` `of` *`component-types`*,

where *constructor-name* must be capitalized. The names `Circ`, `Rect`, and `Tri` are called the **constructors** of the `figure` type. Each serves as a function-like entity that turns components of the appopriate type into a value of type `figure`. For example, we can make a list of the three figures depicted above:

```
# let figs = [Circ 1.; Rect (2.,3.); Tri(4.,5.,6.)];; (* List of sample figures *)
val figs : figure list = [Circ 1.; Rect (2., 3.); Tri (4., 5., 6.)]
```

It turns out that constructors are *not* functions and cannot be manipulated in a first-class way. For example, we cannot write

```
List.map Circ [7.;8.;9.] (* Does not work, since Circ is not a function *)
```

However, we can always embed a constructor in a function when we need to. For example, the following does work:

```
List.map (fun r -> Circ r) [7.;8.;9.] (* This works *)
```

We manipulate a value of the `figure` type by using the OCAML `match` construct to perform a case analysis on the value and name its components. For example, Fig. 1 shows how to calculate figure perimeters and scale figures.

Using data type declarations, we can create user-defined versions of some of the built-in OCAML types that we have been using. For instance, here is a definition of the `option` type:

```
# let pi = 3.14159;;
val pi : float = 3.14159

(* Use pattern matching to define functions on sum-of-products datatype values *)
# let perim fig = (* Calculate perimeter of figure *)
    match fig with
      Circ r -> 2.*.pi*.r
    | Rect (w,h) -> 2.*.(w+.h)
    | Tri (s1,s2,s3) -> s1+.s2+.s3;;
val perim : figure -> float = <fun>

# List.map perim figs;;
- : float list = [6.28318; 10.; 15.]

# let scale n fig = (* Scale figure by factor n *)
  match fig with
    Circ r -> Circ (n*.r)
  | Rect (w,h) -> Rect (n*.w, n*.h)
  | Tri (s1,s2,s3) -> Tri (n*.s1, n*.s2, n*.s3);;
val scale : float -> figure -> figure = <fun>

# List.map (scale 3.) figs;;
- : figure list = [Circ 3.; Rect (6., 9.); Tri (12., 15., 18.)]

# List.map (FunUtils.o perim (scale 3.)) figs;;
- : float list = [18.84954; 30.; 45.]
```

Figure 1: Manipulations of `figure` values.

```
# type 'a option = None | Some of 'a;;
type 'a option = None | Some of 'a
```

This is an example of a parameterized data type declaration in which the `'a` can be instantiated to any type. `option` is an example of a `type constructor`.

```
# None;;
- : 'a option = None
# Some 3;;
- : int option = Some 3
# Some true;;
- : bool option = Some true
```

We can even construct our own version of a list type (though it won't support the infix `::` operator or the square-bracket list syntax):

```
# type 'a myList = Nil | Cons of 'a * 'a myList;;
type 'a myList = Nil | Cons of 'a * 'a myList
```

Note that lists are a recursive data type. Data types are implicitly recursive without the need for any keyword like `rec`. Here are our lists in action:

```
# let ns = Cons(1, Cons(2, Cons(3, Nil)));;
val ns : int myList = Cons (1, Cons (2, Cons (3, Nil)))
# let bs = Cons('a', Cons('b', Cons('c', Nil)));;
val bs : char myList = Cons ('a', Cons ('b', Cons ('c', Nil)))
```

```
# let ss = Cons("d", Cons("n", Cons("t", Nil)));;
val ss : string myList = Cons ("d", Cons ("n", Cons ("t", Nil)))

# let rec map f xs =
    match xs with
      Nil -> Nil
    | Cons(x,xs') -> Cons(f x, map f xs');;
      val map : ('a -> 'b) -> 'a myList -> 'b myList = <fun>

# map ((+) 1) ns;;
- : int myList = Cons (2, Cons (3, Cons (4, Nil)))

# map ((^) "a") ss;;
- : string myList = Cons ("ad", Cons ("an", Cons ("at", Nil)))
```

Data type declarations may be mutually recursive if they are glued together with **and**. For example, here is the definition of data types and constructors for lists of even length and odd length:

```
# type 'a evenList = ENil | ECons of 'a * ('a oddList)
  and 'a oddList = OCons of 'a * ('a evenList);;
type 'a evenList = ENil | ECons of 'a * 'a oddList

type 'a oddList = OCons of 'a * 'a evenList

# OCons(2, ECons(3, OCons(5, ENil)));;
- : int oddList = OCons (2, ECons (3, OCons (5, ENil)))

# ECons(2, ENil);;
Characters 9-13:
  ECons(2, ENil);;
           ^^^^

This expression has type 'a evenList but is here used with type int oddList
```

The last example shows that the type system cannot be fooled by declaring an odd-length list to have even length.

Sometimes it is helpful for data type declarations to have multiple type parameters. These must be enclosed in parenthesis and separated by commas. For example:

```
# type ('a,'b) swaplist = SNil | SCons of 'a * (('b,'a) swaplist);;
type ('a, 'b) swaplist = SNil | SCons of 'a * ('b, 'a) swaplist

# let alts = SCons(1, SCons(true, SCons(2, SCons(false, SNil))));;
val alts : (int, bool) swaplist =
  SCons (1, SCons (true, SCons (2, SCons (false, SNil))))

# let stail xs =
    match xs with
      SNil -> raise (Failure "attempt to take tail of empty swaplist")
    | SCons(x,xs') -> xs';;
val stail : ('a, 'b) swaplist -> ('b, 'a) swaplist = <fun>

# stail alts;;
- : (bool, int) swaplist = SCons (true, SCons (2, SCons (false, SNil)))

# let srev xs =
    let rec loop olds news =
      match olds with
        SNil -> news
      | SCons(x,xs') -> loop xs' (SCons(x,news))
    in loop xs SNil;;
val srev : ('a, 'a) swaplist -> ('a, 'a) swaplist = <fun>
```

Note that the OCAML type reconstruction process forces both `swaplist` type parameters to be the same. Intuitively, this is because the type of the first element of the reversed list depends on whether the list length is even or odd. When these two types are unified, `srev` works on any length of list.

```
# srev alts;;
Characters 5-9:
  srev alts;;
       ^^^^
This expression has type (int, bool) swaplist but is here used with type
  (int, int) swaplist
```

Invoking `srev` on `alts` fails because it has type `(int, bool) swaplist`, which is not match the form `('a, 'a) swaplist`. However, we can use `srev` on lists that do match this form:

```
# srev (SCons(true, SCons(false, SNil)));;
- : (bool, bool) swaplist = SCons (false, SCons (true, SNil))

# srev (SCons(1, SCons(2, SCons(3, SNil))));;
- : (int, int) swaplist = SCons (3, SCons (2, SCons (1, SNil)))
```

Data type declarations are different from type abbreviations, even though both are are introduced via the `type` keyword. Consider the following:

```
# type 'a doubloon1 = 'a * 'a;; (* Type abbreviation *)
type 'a doubloon1 = 'a * 'a

# type 'a doubloon2 = Doubloon of 'a * 'a;; (* Data type declaration *)
type 'a doubloon2 = Doubloon of 'a * 'a
```

The presence of the capitalized constructor name `Doubloon` (as well as the keyword `of`[1]) is the syntactic marker that indicates that `doubloon2` is a sum-of-products data type rather than a type abbreviation. Recall that constructor names *must* be capitalized in OCAML.

Note that `doubloon2` is an example of a data type with just a single constructor. Such data types can be useful as a simple data abstraction mechanism in cases where it is desirable to distinguish representations that happen to have the same concrete type.[2] For example:

```
# let swap1 ((x,y) : 'a doubloon1) = (y, x);;
val swap1 : 'a doubloon1 -> 'a * 'a = <fun>

# let swap2 d = match d with Doubloon (x,y) -> Doubloon (y,x);;
val swap2 : 'a doubloon2 -> 'a doubloon2 = <fun>

# swap1 (1,2);;
- : int * int = (2, 1)

# swap2 (Doubloon(1,2));;
- : int doubloon2 = Doubloon (2, 1)

# swap1 (Doubloon(1,2));;
Characters 7-20:
  swap1 (Doubloon(1,2));;
         ^^^^^^^^^^^^^
This expression has type 'a doubloon2 but is here used with type
  'b doubloon1 = 'b * 'b
```

[1]The `of` keyword is not required for nullary constructors, such as `None` in the `option` data type.

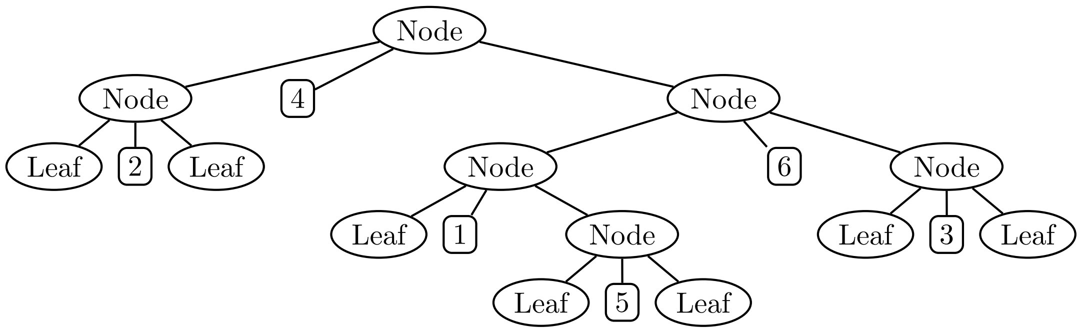
[2]However, this is a very crude form of abstraction, since the concrete representation can be manipulated via pattern matching. For true data abstraction, the module mechanism described in Handout #23 should be used.

```
# swap2 (1,2);;
Characters 7-10:
  swap2 (1,2);;
         ^^^

This expression has type int * int but is here used with type 'a doubloon2
```

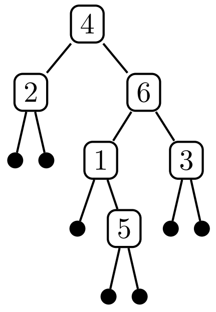
2 Binary Trees

2.1 Background

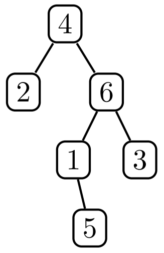
A classic example of a parameterized recursive data type is the **binary tree**. A binary tree is either (1) a leaf or (2) a node with a left subtree, value, and right subtree. Here is a depiction of a sample binary tree:



The depiction is more compact if we put each value in the node position and use ● for each leaf:



The notation is even more compact if we do not draw the leaves:



2.2 Binary Trees in OCAML

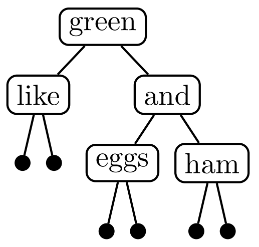
We can express the type of a binary tree in OCAML as follows:

```
(* Binary tree datatype abstracted over type of node value *)
type 'a bintree =
    Leaf
  | Node of 'a bintree * 'a * 'a bintree (* left subtree, value, right subtree *)
```

Here is how we create our sample binary tree of integers in OCAML:

```
# let int_tree =
      Node(Node(Leaf, 2, Leaf),
           4,
           Node(Node(Leaf, 1, Node(Leaf, 5, Leaf)),
                6,
                Node(Leaf, 3, Leaf)));;
val int_tree : int bintree =
  Node (Node (Leaf, 2, Leaf), 4,
   Node (Node (Leaf, 1, Node (Leaf, 5, Leaf)), 6, Node (Leaf, 3, Leaf)))
```

Of course, we can create a binary tree of strings instead, such as:



```
# let string_tree =
      Node(Node(Leaf, "like", Leaf),
           "green",
           Node(Node(Leaf, "eggs", Leaf),
                "and",
                Node(Leaf, "ham", Leaf)));;
val string_tree : string bintree =
  Node (Node (Leaf, "like", Leaf), "green",
   Node (Node (Leaf, "eggs", Leaf), "and", Node (Leaf, "ham", Leaf)))
```

Now we'll define some simple functions that manipulate trees in Figs. 2–5. In each case we will use the `match` construct to test whether a given tree is a leaf or a node and to extract the components of the node.

```
(* Returns number of nodes in tree *)
# let rec nodes tr =
    match tr with
      Leaf -> 0
    | Node(l,v,r) -> 1 + (nodes l) + (nodes r);;
val nodes : 'a bintree -> int = <fun>

# nodes int_tree;;
- : int = 6

# nodes string_tree;;
- : int = 5

(* Returns height of tree *)
# let rec height tr =
    match tr with
      Leaf -> 0
    | Node(l,v,r) -> 1 + max (height l) (height r);;
val height : 'a bintree -> int = <fun>

# height int_tree;;
- : int = 4

# height string_tree;;
- : int = 3

(* Returns sum of nodes in tree of integers *)
# let rec sum tr =
    match tr with
      Leaf -> 0
    | Node(l,v,r) -> v + (sum l) + (sum r);;
val sum : int bintree -> int = <fun>


# sum int_tree;;
- : int = 21
```

Figure 2: Binary tree functions, part 1.

```
(* Returns pre-order list of leaves *)
# let rec prelist tr =
    match tr with
      Leaf -> []
    | Node(l,v,r) -> v :: (prelist l) @ (prelist r);;
val prelist : 'a bintree -> 'a list = <fun>

# prelist int_tree;;
- : int list = [4; 2; 6; 1; 5; 3]

# prelist string_tree;;
- : string list = ["green"; "like"; "and"; "eggs"; "ham"]

(* Returns in-order list of leaves *)
# let rec inlist tr =
    match tr with
      Leaf -> []
    | Node(l,v,r) -> (inlist l) @ [v] @ (inlist r);;
val inlist : 'a bintree -> 'a list = <fun>

# inlist int_tree;;
- : int list = [2; 4; 1; 5; 6; 3]

# inlist string_tree;;
- : string list = ["like"; "green"; "eggs"; "and"; "ham"]

(* Returns post-order list of leaves *)
# let rec postlist tr =
    match tr with
      Leaf -> []
    | Node(l,v,r) -> (postlist l) @ (postlist r) @ [v];;
val postlist : 'a bintree -> 'a list = <fun>

# postlist int_tree;;
- : int list = [2; 5; 1; 3; 6; 4]

# postlist string_tree;;
- : string list = ["like"; "eggs"; "ham"; "and"; "green"]
```

Figure 3: Binary tree functions, part 2.

```
(* Map a function over every value in a tree *)
# let rec map f tr =
    match tr with
      Leaf -> Leaf
    | Node(l,v,r) -> Node(map f l, f v, map f r);;
val map : ('a -> 'b) -> 'a bintree -> 'b bintree = <fun>

# map (( * ) 10) int_tree;;
- : int bintree =
Node (Node (Leaf, 20, Leaf), 40,
 Node (Node (Leaf, 10, Node (Leaf, 50, Leaf)), 60, Node (Leaf, 30, Leaf)))

# map String.uppercase string_tree;;
- : string bintree =
Node (Node (Leaf, "LIKE", Leaf), "GREEN",
 Node (Node (Leaf, "EGGS", Leaf), "AND", Node (Leaf, "HAM", Leaf)))

# map String.length string_tree;;
- : int bintree =
Node (Node (Leaf, 4, Leaf), 5,
 Node (Node (Leaf, 4, Leaf), 3, Node (Leaf, 3, Leaf)))

# map ((flip String.get) 0) string_tree;;
- : char bintree =
Node (Node (Leaf, 'l', Leaf), 'g',
 Node (Node (Leaf, 'e', Leaf), 'a', Node (Leaf, 'h', Leaf)))
```

Figure 4: Binary tree functions, part 3.

```
(* Divide/conquer/glue on trees *)
# let rec fold glue lfval tr =
    match tr with
      Leaf -> lfval
    | Node(l,v,r) -> glue (fold glue lfval l)
                          v
                          (fold glue lfval r);;
val fold : ('a -> 'b -> 'a -> 'a)
           -> 'a -> 'b bintree -> 'a = <fun>

(* Alternative definition *)
# let sum = fold (fun l v r -> l + v + r) 0;;
val sum : int bintree -> int = <fun>
(* can define nodes, height similarly *)

(* Alternative definition *)
# let prelist tr =
    fold (fun l v r -> v :: l @ r) [] tr;;
val prelist : 'a bintree -> 'a list = <fun>
(* can define inlist, postlist similarly *)

# let toString valToString tr =
    fold (fun l v r -> "(" ^ l ^ " "
                           ^ (valToString v)
                           ^ " " ^ r ^ ")")
         "*"
         tr;;
val toString : ('a -> string) -> 'a bintree -> string = <fun>

# toString string_of_int int_tree;;
- : string = "((* 2 *) 4 ((* 1 (* 5 *)) 6 (* 3 *)))"

# toString FunUtils.id string_tree;;
- : string = "((* like *) green ((* eggs *) and (* ham *)))"
```

Figure 5: Binary tree functions, part 4.

2.3 Binary Search Trees

A common use of binary trees in practice is to implement **binary search trees** (BSTs). The **BST condition** holds at a non-leaf binary tree node if (1) all elements in the left subtree of the node are $\leq$ the node value and (2) all elements in the right subtree of the node are $\geq$ the node value. A tree is a BST if the BST condition holds at all non-leaf nodes in the tree.

For example, below are some of the many possible BSTs containing the numbers 1 through 7. Note that a BST is not required to be balanced in any way.

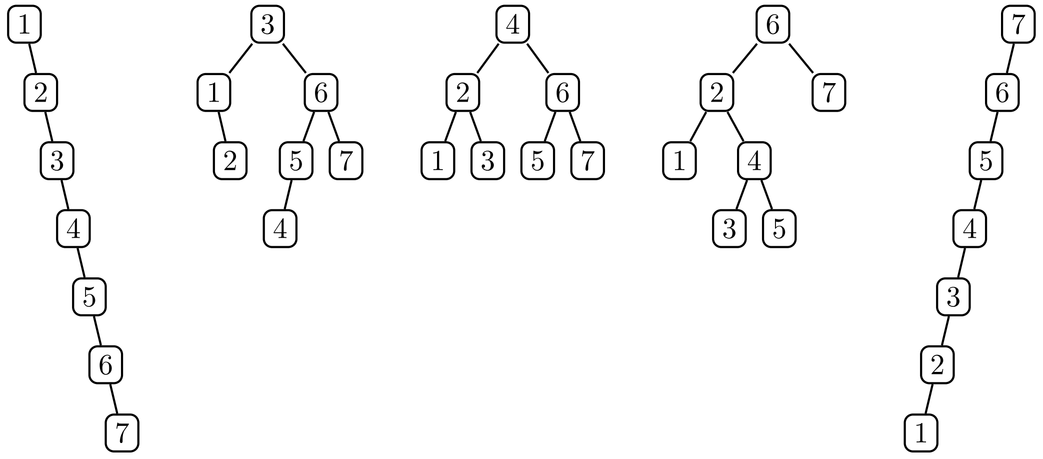


Fig. 6 presents a `BSTSet` module in which sets are represented as binary search trees. (This is the `BSTSet` module mentioned in the discussion of sets in Handout #23.) The combination of sum-of-product data types, pattern matching on these types, and higher-order functions makes this implementation remarkably concise. It is worthwhile for you to compare this implementation to a corresponding implementation in other languages, such as Java or C.

```
module BSTSet : SET = struct

  open Bintree (* exports bintree datatype and functions from previous section *)

  type 'a set = 'a bintree

  let empty = Leaf

  let singleton x = Node(Leaf, x, Leaf)

  let rec insert x t =
    match t with
      Leaf -> singleton x
    | Node(l,v,r) -> if x = v then t
                     else if x < v then Node(insert x l, v, r)
                     else Node(l, v, insert x r)

  let rec member x s =
    match s with
      Leaf -> false
    | Node(l,v,r) -> if x = v then true
                     else if x < v then member x l
                     else member x r

  (* Assume called on non-empty tree *)
  let rec deleteMax t =
    match t with
      Leaf -> raise (Failure "shouldn't happen")
    | Node(l,v,Leaf) -> (v, l)
    | Node(l,v,r) -> let (max, r') = deleteMax r in
                       (max, Node(l,v,r'))

  let rec delete x s =
    match s with
      Leaf -> Leaf
    | Node(l,v,Leaf) when v = x -> l
    | Node(Leaf,v,r) when v = x -> r
    | Node(l,v,r) -> if x = v then
                       let (pred, l') = deleteMax l in Node(l', pred, r)
                     else if x < v then Node(delete x l, v, r)
                     else Node(l, v, delete x r)

  let rec toList s = inlist s

  let rec union s1 s2 = ListUtils.foldr insert s2 (postlist s1)
  (* In union and difference, postlist helps to preserve balance more than
     inlist or prelist in a foldr.  For a foldl, prelist would be best. *)

  let rec intersection s1 s2 =
    ListUtils.foldr (fun x s -> if not (member x s2) then delete x s
                                else s)
                    s1
                    (inlist s1)

  let rec difference s1 s2 = ListUtils.foldr delete s1 (postlist s2)

  let rec fromList xs = ListUtils.foldr insert empty xs

  let rec toString eltToString s = StringUtils.listToString eltToString (toList s)

end
```

Figure 6: An implementation of the SET signature using binary search trees.

3 Expression Trees

The most common kind of trees that we will manipulate in this course are trees that represent the structure of programming language expressions (and other kinds of program phrases). Here we begin to explore some of the concepts and techniques used for describing and representing expressions.

3.1 Abstract Syntax Trees

Fig. 7 describes (in English) the abstract structure of expressions for a simple expression language that we'll call EL. There are two kinds of expressions in EL: **integer expressions** that denote integers, and **boolean expressions** that denote boolean truth values (i.e., true or false).

Integer Expressions

An EL integer expression is one of:

- an *intlit* — an integer literal (numeral) *num*;
- a *variable reference* — a reference to an integer variable named *name*
- an *arithmetic operation* — an application of a *rator*, in this case a binary *arithmetic operator*, to two integer *rand* expressions, where an arithmetic operator is one of:
  - addition,
  - subtraction,
  - multiplication,
  - division,
  - remainder;
- a *conditional* — a choice between integer *then* and *else* expressions determined by a boolean *test* expression.

An EL boolean expression is one of:

- a *boollit* — a boolean literal *bool* (i.e., a true or false constant);
- a *negation* — the negation of a boolean expression *negand*;
- a *relational operation* — an application of *rator*, in this case a binary *relational operator*, to two integer *rand* expressions, where a relational operator is one of:
  - less-than,
  - equal-to,
  - greater-than;
- a *logical operation* — an application of a *rator*, in this case a binary *logical operator*, to two boolean *rand* expressions, where a logical operator is one of:
  - and,
  - or.

Figure 7: An abstract grammar for EL programs.

An integer expression in EL can be constructed out of various kinds of components. Some of the components, like integer literals, variable references, and arithmetic operators, are **primitive** — they cannot be broken down into subparts.[3] Other components, such as arithmetic operations and

[3]Numerals can be broken down into digits, but we will ignore this detail.

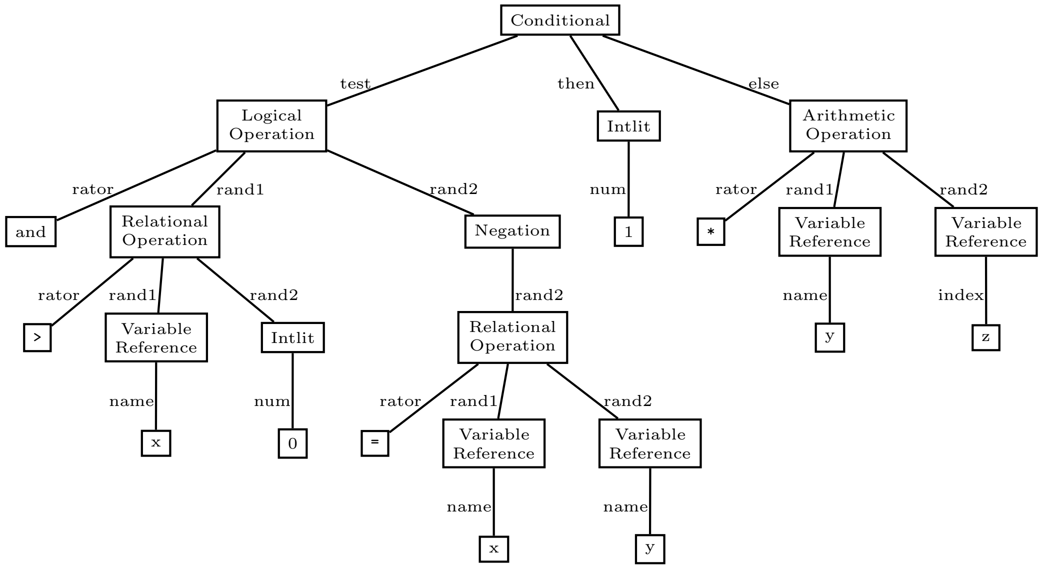


Figure 8: An abstract syntax tree for an EL integer expression.

conditional expressions, are **compound** — they are constructed out of constituent components. The components have names; e.g., the subparts of an arithmetic operation are the **rator** (short for "operator") and two **rands** (short for "operands"), while the subexpressions of the conditional expression are the **test** expression, the **then** expression, and the **else** expression.

Boolean expressions in EL are also either primitive (i.e., logical operators and boolean literals) or compound (negations, relational operations, and logical operations).

The structural description in Fig. 7 constrains the ways in which integer and boolean expressions may be "wired together." Boolean expressions can appear only as the test expression of a conditional, the negand of a negation, or the operands of a logical operation. Integer expressions can appear only as the operands of arithmetic or relation operations, or as the then or else expressions of a conditional.

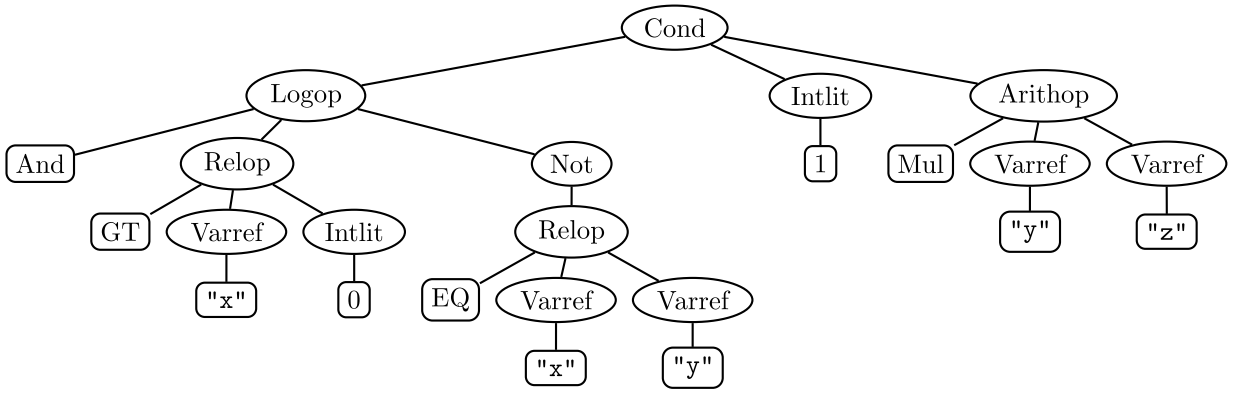
A specification of the allowed wiring patterns for the syntactic entities of a language is called a **grammar**. Fig. 7 is said to be an **abstract grammar** because it specifies the logical structure of the syntax but does not give any indication how individual expressions in the language are actually written down in a concrete form.

Parsing an expression with an abstract grammar results in a value called an **abstract syntax tree (AST)**. As we will see later, abstract syntax trees are easy to inspect and disassemble, making them ideal substrates for defining the meaning of program phrases in terms of their parts.

Consider an EL expression that denotes 1 if the integer variable `x` has a value that is positive and not equal to the value of the integer variable `y`, and otherwise denotes the product of the integer variables `y` and `z`. The abstract syntax tree for this program appears in Fig. 8. Each node of the tree corresponds to a numerical or boolean expression. The leaves of the tree stand for primitive expressions, while the intermediate nodes represent compound expressions. The labeled edges from a parent node to its children show the relationship between a compound phrase and its components. The AST is defined purely in terms of these relationships; the particular way that the nodes and edges of a tree are arranged on the page is immaterial.

3.2 Data Types for Expressions

We can easily recast any AST as a sum-of-product tree by dropping the edge labels and fixing the left-to-right order of components for compound nodes. For example, the AST from Fig. 8 can be expressed as the following sum-of-product tree:



We have shortened many of the node labels to make the tree take less space.

Based on this observation, we can describe any EL expressions using the following OCAML data type declarations:

```
type intExp = (* integer expressions *)
    Intlit of int (* value *)
  | Varref of string (* name *)
  | Arithop of arithRator * intExp * intExp (* rator, rand1, rand2 *)
  | Cond of boolExp * intExp * intExp (* test, then, else *)

and boolExp = (* boolean expressions *)
    Boollit of bool (* value *)
  | Not of boolExp (* negand *)
  | Relop of relRator * intExp * intExp (* rator, rand1, rand2 *)
  | Logop of logRator * boolExp * boolExp (* rator, rand1, rand2 *)

and arithRator = Add | Sub | Mul | Div | Rem (* arithmetic operators *)

and relRator = LT | EQ | GT (* relational operators *)

and logRator = And | Or (* logical operators *)
```

For instance, we can use these data types to express our example EL integer expression:

```
Cond(Logop(And,
           Relop(GT, Varref "x", Intlit 0),
           Not(Relop(EQ, Varref "x", Varref "y"))),
     Intlit 1,
     Arithop(Mul, Varref "y", Varref "z"))
```

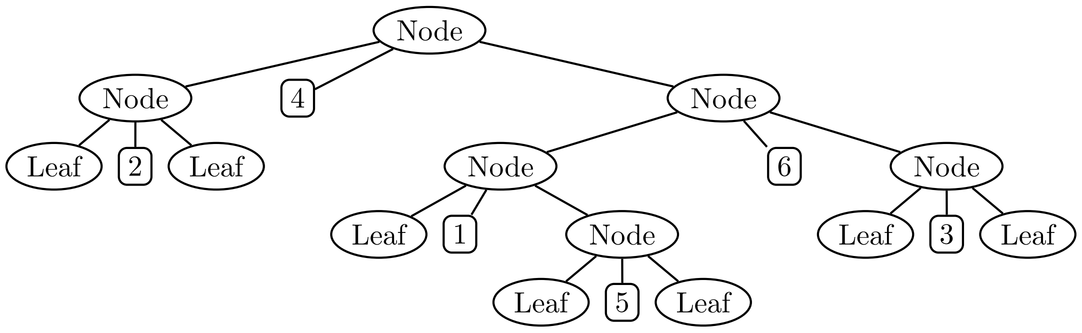
We will spend much of the rest of the semester studying how to manipulate expression trees. We'll see that manipulating expressions trees is similar to the sorts of tree manipulation you're familiar with from *CS230*. The main difference is that expression trees have many more kinds of nodes than the trees you've studied before. This means that you'll need more cases (one per node label) when writing programs to manipulate expression trees.

4 S-Expressions

4.1 Motivation: The Parsing Problem

Constructing trees using OCAML data type constructors is rather verbose. In many cases, we'd prefer to have a more compact notation for our trees.

For example, consider the binary tree:



We can create this in OCAML using constructors:

```
Node(Node(Leaf, 2, Leaf),
     4,
     Node(Node(Leaf, 1, Node(Leaf, 5, Leaf)),
          6,
          Node(Leaf, 3, Leaf))
```

But we'd prefer to use more concise tree notations, like the following:

```
((* 2 *) 4 ((* 1 (* 5 *)) 6 (* 3 *))) ; The ``compact'' notation

((2) 4 ((1 (5)) 6 (3)))  ; The ``dense'' notation
```

As another example, consider the sample EL integer expression tree from the previous page. Rather than express it via OCAML constructors, we'd like to use a more concise expression notation. Here are some examples:

```
if x>0 && !(x=y) then 1 else y*z ; Standard infix notation

(if ((x > 0) && (! (x = y))) then 0 else (y * z)) ; Fully parenthesized infix notation

x 0 > x y = ! && (1) (y z *) if; Postfix notation

if && > x 0 ! = x y 1 * y z ; Prefix notation

(if (&& (> x  0) (! (= x y))) 1 (* y z)) ; Fully parenthesized prefix notation
```

In all of these notations, we use the same abbreviated notiations for constructors: `if` for `Cond`, `!` for `Not`, `&&` for logical operations with `And`, `*` for arithmetic operations with `Mul`, and `>` and `=` for relational operations with `GT` and `EQ`, respectively. The main difference between the above notations is (1) whether operations are written in prefix, infix, or postfix form; (2) whether operator precedence is determined by precedence rules or by requiring parenthesis; and (3) whether parentheses are optional or required for all compound expressions.

To use *any* character-based notation for binary trees and EL expressions (either the verbose OCAML notation or the more concise notations considered above), it is necessary to decompose a character string using one of these notations into fundamental **tokens** and then **parse** these tokens into the desired OCAML constructor tree. The problem of transforming a linear character string into a constructor tree is called the **parsing problem**.

In *CS235*, we studied a standard two-step solution to the parsing problem.

1. Define a **lexical analyzer** (a.k.a. lexer, scanner, or tokenizer) that decomposes a character string into tokens. Using a scanner generator like Lex, it is possible to describe the structure of the tokens at a high level and automatically generate the lexical analyzer from this description.

2. Define a **parser** that arranges tokens into trees according to a specification of the grammar of the language. Using a parser generator like Yacc, it is possible to automatically generate

the parser from a description of the grammar along with rules for resolving any ambiguities in the grammar.

We could adopt the same solution in this course, but then we would spend a lot of our time defining tokens and grammars and worrying about precedence rules and other distractions.

So instead, we will adopt the following strategy: we will develop a standard parenthesized notation (known as s-expressions) for representing trees , and then use this standard notation to represent all trees (including all expression trees) throughout the rest of the course. This way we only need to define one scanner and one parser for the whole course. It will still be necessary to translate back and forth between general s-expression trees and particular OCAML constructor trees, but it is *much* easier to translate between different kinds of trees than to parse linear character strings to trees.

4.2 Overview of S-Expressions

A **symbolic expression** (**s-expression** for short) is a simple notation for representing tree structures using linear text strings containing matched pairs of parentheses. Each leaf of a tree is an **atom**, which (to first approximation) is any sequence of characters that does not contain a left parenthesis (‘`(`’), a right parenthesis (‘`)`’), or a whitespace character (space, tab, newline, etc.).[4] Examples of atoms include `x`, `this-is-an-atom`, `anotherKindOfAtom`, `17`, `3.14159`, `4/3*pi*r^2`, `a.b[2]%3`, `'Q'`, and `"a (string) atom"`. A node in an s-expression tree is represented by a pair of parentheses surrounding zero or s-expressions that represent the node's subtrees. For example, the s-expression

```
((this is) an ((example) (s-expression tree)))
```

designates the structure depicted in Fig. 9. Whitespace is necessary for separating atoms that

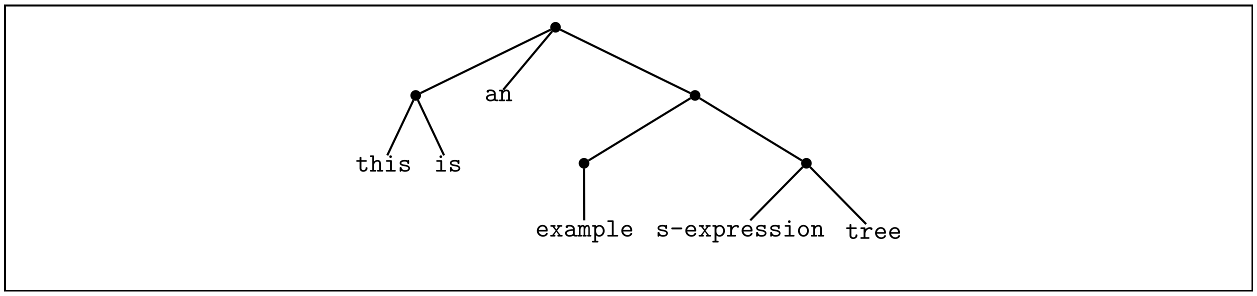


Figure 9: Viewing `((this is) an ((example) (s-expression tree)))` as a tree.

appear next to each other, but can be used liberally to enhance (or obscure!) the readability of the structure. Thus, the above s-expression could also be written as

```
((this is)
 an
 ((example)
  (s-expression
   tree)))
```

or (less readably) as

[4]As we shall see, string and character literals *can* contain parentheses and whitespace characters.

```
               (
     (          this
is) an  (   (          example
          ) (
s-expression tree       )
  )
          )
```
without changing the structure of the tree.

S-expressions were pioneered in LISP as a notation for data as well as programs (which we have seen are just particular kinds of tree-shaped data!). We shall see that s-expressions are an exceptionally simple and elegant way of solving the parsing problem — translating string-based representations of data structures and programs into the tree structures they denote.[5] For this reason, all the mini-languages we study later in this course have a concrete syntax based on s-expressions.

The fact that LISP dialects (including SCHEME) have a built-in primitive for parsing s-expressions (`read`) and treating them as literals (`quote`) makes them particularly good for manipulating programs (in any language) written with s-expressions. It is not quite as convenient to manipulate s-expression program syntax in other languages, such as OCAML, but we shall see that it is still far easier than solving the parsing problem for more general notations.

4.3 Representing S-Expressions in OCAML

As with any other kind of tree-shaped data, s-expressions can be represented in OCAML as values of an appropriate data type. The OCAML data type representing s-expression trees is presented in Fig. 10.

```
type sexp =
        Int of int
      | Flt of float
      | Str of string
      | Chr of char
      | Sym of string
      | Seq of sexp list
```

Figure 10: OCAML s-expression data type.

There are five kinds of atoms, distinguished by type; these are the leaves of s-expression trees:

1. integer literals, constructed via `Int`;
2. floating point literals, constructed via `Flt`;
3. string literals, constructed via `Str`;
4. character literals, constructed via `Chr`; and
5. symbols, which are name tokens (as distinguished from quoted strings), constructed via `Sym`.

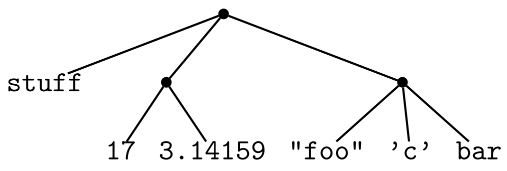
The nodes of s-expression trees are represented via the `Seq` constructor, whose `sexp list` argument denotes any number of s-expression subtrees.

[5]There are detractors who hate s-expressions and claim that LISP stands for **L**ots of **I**rritating **S**illy **P**arenthesis. Apparently such people lack a critical aesthetic gene that prevents them from appreciating beautiful designs. Strangely, many such people seem to prefer the far more verbose encoding of trees in XML notation discussed later in the course. Go figure!

For example, consider the s-expression given by the concrete notation

```
(stuff (17 3.14159) ("foo" 'c' bar))
```

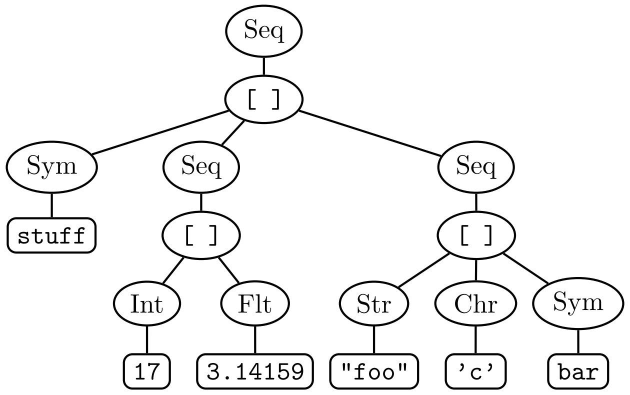
which would be expressed in general tree notation as



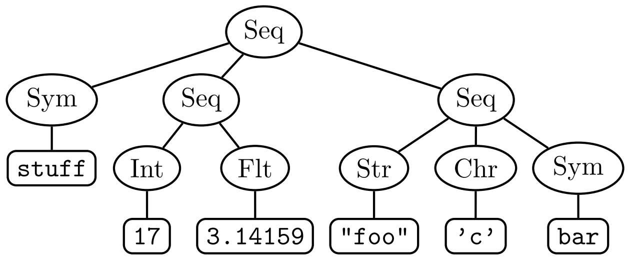
This s-expression can be written in in OCAML as

```
Seq [Sym("stuff");
     Seq [Int(17); Flt(3.14159)];
     Seq [Str("foo"); Chr('c'); Sym("bar")]]
```

which corresponds to the following constructor tree:



In the constructor tree, nodes labeled `[ ]` represent lists whose elements are shown as the children of the node. Since it's cumbersome to write such list nodes explicitly, we will often omit the explicit `[ ]` nodes and instead show `Seq` nodes as having any number of children:



The Sexp module in ~/cs251/utils/Sexp.ml contains several handy utilities for manipulating sexp trees. In particular, it contains functions for **parsing** s-expression strings into sexp trees and for **unparsing** sexp trees into s-expression trees. The signature SEXP for the Sexp module is presented in Fig. 11. We will not study the implementation of the Sexp functions, but will use them as black boxes.

Here are some sample invocations of functions from the Sexp module:

```
# let s = Sexp.stringToSexp "(stuff (17 3.14159) (\"foo\" 'c' bar))";;
val s : Sexp.sexp =
  Sexp.Seq
   [Sexp.Sym "stuff"; Sexp.Seq [Sexp.Int 17; Sexp.Flt 3.14159];
    Sexp.Seq [Sexp.Str "foo"; Sexp.Chr 'c'; Sexp.Sym "bar"]]

# Sexp.sexpToString s;;
- : string = "(stuff (17 3.14159) (\"foo\" 'c' bar))"

# Sexp.sexpToString' 20 s;;
- : string =
"(stuff (17 3.14159)\n       (f̈oo¨ 'c'\n              bar\n              )\n       )"

# let ss = Sexp.stringToSexps "stuff (17 3.14159) (\"foo\" 'c' bar)";;
val ss : Sexp.sexp list =
  [Sexp.Sym "stuff"; Sexp.Seq [Sexp.Int 17; Sexp.Flt 3.14159];
   Sexp.Seq [Sexp.Str "foo"; Sexp.Chr 'c'; Sexp.Sym "bar"]]

# Sexp.sexpsToString ss;;
- : string = "stuff\n\n(17 3.14159)\n\n(\"foo\" 'c' bar)"

# Sexp.readSexp();;
(a b
   (c d e)
   (f (g h))
   i)
- : Sexp.sexp =
Sexp.Seq
 [Sexp.Sym "a"; Sexp.Sym "b";
  Sexp.Seq [Sexp.Sym "c"; Sexp.Sym "d"; Sexp.Sym "e"];
  Sexp.Seq [Sexp.Sym "f"; Sexp.Seq [Sexp.Sym "g"; Sexp.Sym "h"]];
  Sexp.Sym "i"]
```

```
module type SEXP = sig

  (* The sexp type is exposed for the world to see *)
  type sexp =
      Int of int
    | Flt of float
    | Str of string
    | Chr of char
    | Sym of string
    | Seq of sexp list

  exception IllFormedSexp of string
  (* This exception is used for all errors in s-expression manipulation *)

  val stringToSexp : string -> sexp
  (* (stringToSexp <str>) returns the sexp tree represented by the s-expression
     <str>. Raise an IllFormedSexp exception if <str> is not a valid
     s-expression string. *)

  val stringToSexps : string -> sexp list
  (* (stringToSexps <str>) returns the list of sexp trees represented by <str>, which
     is a string containing a sequence of s-expressions. Raise an IllFormedSexp
     exception if <str> not a valid representation of a sequence of s-expressions. *)

  val fileToSexp : string -> sexp
  (* (fileToSexp <filename>) returns the sexp tree represented by the s-expression
     contents of the file named by <filename>. Raises an IllFormedSexp exception
     if the file contents is not a valid s-expression. *)

  val fileToSexps : string -> sexp list
  (* (fileToSexps <filename>) returns the list of sexp trees represented by the
     contents of the file named by <filename>. Raises an IllFormedSexp exception if
     the file contents is not a valid representation of a sequence of s-expressions. *)

  val sexpToString : sexp -> string
  (* (sexpToString <sexp>) returns an s-expression string representing <sexp> *)

  val sexpToString' : int -> sexp -> string
  (* (sexpToString' <width> <sexp>) returns an s-expression string representing
     <sexp> in which an attempt is made for each line of the result to be
     <= <width> characters wide. *)

  val sexpsToString : sexp list -> string
  (* (sexpsToString <sexps>) returns string representations of the sexp trees
     in <sexps> separated by two newlines. *)

  val sexpToFile : sexp -> string -> unit
  (* (sexpsToFile <sexp> <filename>) writes a string representation of <sexp>
     to the file name <filename>. *)

  val readSexp : unit -> sexp
  (* Reads lines from standard input until a complete s-expression has been
     found, and returns the sexp tree for this s-expresion. *)

end
```

Figure 11: The SEXP signature.

4.4 S-Expression Representations of Sum-of-Product Trees

We will mainly use s-expressions for representing the trees implied by sum-of-product data type constructor invocations in a standard format. We will represent a tree node with tag *tag* and subtrees $t_1 \ldots t_n$ by an s-expression of the form:

(*tag* *<s-expression for* t_1*>* ... *<s-expression for* t_n*>*)

For instance, consider the representations of `fig` values presented in Fig. 12.

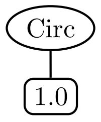
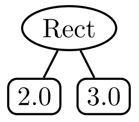
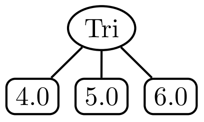
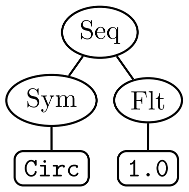
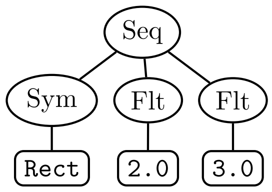
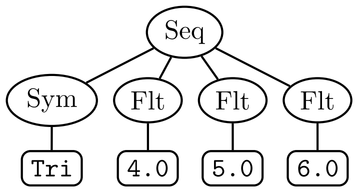
Constructor Invocation	`Circ 1.0`	`Rect (2.0, 3.0)`	`Tri(4.0, 5.0, 6.0)`
Constructor Tree			
S-expression Tree			
S-expression	`(Circ 1.0)`	`(Rect 2.0 3.0)`	`(Tri 4.0 5.0 6.0)`

Figure 12: Representations of `fig` values.

In order to use the s-expression representation of figures, we will need a way to convert between `fig` constructor trees and s-expression constructor trees:

```
let toSexp fig =
  match fig with
    Circ r -> Seq [Sym "Circ"; Flt r]
  | Rect (w,h) -> Seq [Sym "Rect"; Flt w; Flt h]
  | Tri (s1,s2,s3) -> Seq [Sym "Tri"; Flt s1; Flt s2; Flt s3]

let fromSexp sexp =
  match sexp with
    Seq [Sym "Circ"; Flt r] -> Circ r
  | Seq [Sym "Rect"; Flt w; Flt h] -> Rect (w,h)
  | Seq [Sym "Tri"; Flt s1; Flt s2; Flt s3] -> Tri (s1,s2,s3)
  | _ -> raise (Failure ("Fig.fromSexp -- can’t handle sexp:\n"
                          ^ (sexpToString sexp)))
```

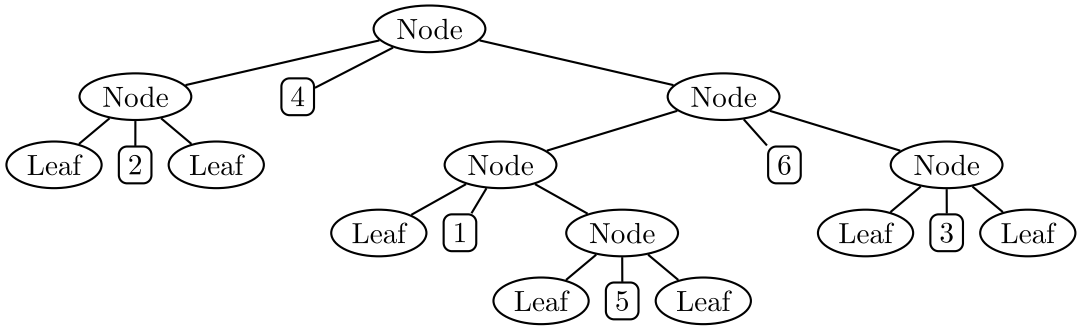
In what context would we use `toSexp` and `fromSexp`? In any context where we wish to interactively manipulated `fig` values specified in files or keyboard input from users. For example, suppose we want to interactively scale figures as shown below:

```
# Fig.interactiveScale();;
Enter a scaling factor> 2.3

Enter a sequence of figures (in s-expression format) on one line>
(Circ 1.0) (Rect 2.0 3.0) (Tri 4.0 5.0 6.0)

Here are the scaled results:
(Circ 2.3)
(Rect 4.6 6.9)
(Tri 9.2 11.5 13.8)
- : unit = ()
```

Here's how we would defined `interactiveScale` in OCAML:

```
let interactiveScale () =
  let _ = StringUtils.print "Enter a scaling factor> " in
  let s = read_float() (* standard function for reading a float from input *) in
  let _ = StringUtils.println
            "\nEnter a sequence of figures (in s-expression format) on one line> " in
  let line = read_line() (* standard function for reading line (as string) from input *) in
  let figs = map fromSexp (stringToSexps line) in
  let scaled_figs = map (scale s) figs in
  let _ = StringUtils.println "\nHere are the scaled results:" in
     for_each (fun fig -> StringUtils.println (sexpToString (toSexp fig))) scaled_figs
```

What about binary trees? Here is a binary tree example:



Using the above conventions, this would be represented via the s-expression:

```
(Node (Node (Leaf) 2 (Leaf))
      4
     (Node (Node (Leaf) 1 (Node (Leaf) 5 (Leaf)))
           6
           (Node (Leaf) 3 (Leaf))))  ; Let's call this the ``verbose'' notation
```

Fig. 13 shows how to convert between this "verbose" s-expression notation and `bintree` values.

But this is not a very compact representation! As mentioned above, we can often develop more compact representations for particular data types. For instance, here are other s-expressions representing the binary tree above:

```
((* 2 *) 4 ((* 1 (* 5 *)) 6 (* 3 *))) ; The ``compact'' notation

((2) 4 ((1 (5)) 6 (3)))  ; The ``dense'' notation
```

Figs. 14–15 show functions that convert between binary trees and these more concise s-expression notations.

```
let rec toVerboseSexp eltToSexp tr =
  match tr with
   Leaf -> Seq [Sym "Leaf"]
 | Node(l,v,r) -> Seq [Sym "Node";
                       toVerboseSexp eltToSexp l;
                       eltToSexp v;
                       toVerboseSexp eltToSexp r]

let rec fromVerboseSexp eltFromSexp sexp =
  match sexp with
    Seq [Sym "Leaf"] -> Leaf
  | Seq [Sym "Node"; leftx; valx; rightx] ->
       Node(fromVerboseSexp eltFromSexp leftx,
            eltFromSexp valx,
            fromVerboseSexp eltFromSexp rightx)
  | _ -> raise (Failure ("Bintree.fromVerboseSexp"
                         ^ "-- can't handle sexp:\n"
                         ^ (sexpToString sexp)))
```

Figure 13: Functions for converting between binary trees and verbose s-expressions.

```
let rec toCompactSexp eltToSexp tr =
  match tr with
    Leaf -> Sym "*"
  | Node(l,v,r) -> Seq [toCompactSexp eltToSexp l;
                        eltToSexp v;
                        toCompactSexp eltToSexp r]

let rec fromCompactSexp eltFromSexp sexp =
  match sexp with
    Sym "*" -> Leaf
  | Seq [leftx; valx; rightx] ->
      -> Node(fromCompactSexp eltFromSexp leftx,
              eltFromSexp valx,
              fromCompactSexp eltFromSexp rightx)
  | _ -> raise (Failure ("Bintree.fromCompactSexp"
                         " -- can't handle sexp:\n"
                         ^ (sexpToString sexp)))
```

Figure 14: Functions for converting between binary trees and compact s-expressions.

```
let rec toDenseSexp eltToSexp s =
  match s with
    Leaf -> Seq [] (* Special case for tree that's a leaf *)
  | Node(Leaf,v,Leaf) -> Seq [eltToSexp v]
  | Node(l,v,Leaf) -> Seq [toDenseSexp eltToSexp l;
                           eltToSexp v]
  | Node(Leaf,v,r) -> Seq [eltToSexp v;
                           toDenseSexp eltToSexp r]
  | Node(l,v,r) -> Seq [toDenseSexp eltToSexp l;
                        eltToSexp v;
                        toDenseSexp eltToSexp r]

let rec fromDenseSexp eltFromSexp sexp =
  match sexp with
    Seq [] -> Leaf
  | Seq [valx] -> Node(Leaf, eltFromSexp valx, Leaf)
  | Seq [(Seq _) as leftx; valx] ->
      Node(fromDenseSexp eltFromSexp leftx,
           eltFromSexp valx,
           Leaf)
  | Seq [valx; (Seq _) as rightx] ->
      Node(Leaf,
           eltFromSexp valx,
           fromDenseSexp eltFromSexp rightx)
  | Seq [leftx; valx; rightx] ->
      Node(fromDenseSexp eltFromSexp leftx,
           eltFromSexp valx,
           fromDenseSexp eltFromSexp rightx)
  | _ -> raise (Failure ("Bintree.fromDenseSexp"
                         " -- can't handle sexp:\n"
                         ^ (sexpToString sexp)))
```

Figure 15: Functions for converting between binary trees and dense s-expressions.