Universal constructions: limits and colimits

Consider and arbitrary but fixed category $\mathbf{K}$ for a while.

Initial and terminal objects

An object $I \in |\mathbf{K}|$ is *initial* in $\mathbf{K}$ if for each object $A \in |\mathbf{K}|$ there is exactly one morphism from I to A.

Examples:

- $\emptyset$ is initial in $\mathbf{Set}$.
- For any signature $\Sigma \in |\mathbf{AlgSig}|$, T_Σ is initial in $\mathbf{Alg}(\Sigma)$.
- For any signature $\Sigma \in |\mathbf{AlgSig}|$ and set of Σ-equations Φ, the initial model of $\langle\Sigma, \Phi\rangle$ is initial in $\mathbf{Mod}(\Sigma, \Phi)$, the full subcategory of $\mathbf{Alg}(\Sigma)$ determined by the class $Mod(\Sigma, \Phi)$ of all models of Φ.

Look for initial objects in other categories.

Fact: *Initial objects, if exist, are unique up to isomorphism:*

- *Any two initial objects in* $\mathbf{K}$ *are isomorphic.*
- *If* I *is initial in* $\mathbf{K}$ *and* I' *is isomorphic to* I *in* $\mathbf{K}$ *then* I' *is initial in* $\mathbf{K}$ *as well.*

Terminal objects

An object $I \in |\mathbf{K}|$ is *terminal* in $\mathbf{K}$ if for each object $A \in |\mathbf{K}|$ there is exactly one morphism from A to I.

$$\boxed{\text{terminal} = co\text{-initial}}$$

Dualise those for initial objects.

Exercises:

- Look for terminal objects in standard categories.
- Show that terminal objects are unique to within an isomorphism.
- Look for categories where there is an object which is both initial and terminal.

Products

A *product* of two objects $A, B \in |\mathbf{K}|$, is any object $A \times B \in |\mathbf{K}|$ with two morphisms (*product projections*) $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ such that for any object $C \in |\mathbf{K}|$ with morphisms $f_1 : C \to A$ and $f_2 : C \to B$ there exists a unique morphism $h : C \to A \times B$ such that $h;\pi_1 = f_1$ and $h;\pi_2 = f_2$.

$$
\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
& C \arrow[ul, "f_1"] \arrow[u, "\exists!\ h"] \arrow[ur, "f_2"'] &
\end{tikzcd}
$$

In $\mathbf{Set}$*, Cartesian product is a product*

We write $\langle f_1, f_2 \rangle$ for h defined as above. Then: $\langle f_1, f_2 \rangle;\pi_1 = f_1$ and $\langle f_1, f_2 \rangle;\pi_2 = f_2$. Moreover, for any h into the product $A \times B$: $h = \langle h;\pi_1, h;\pi_2 \rangle$.

Essentially, this equationally defines a product!

Fact: *Products are defined to within an isomorphism (which commutes with projections).*

Exercises

- Product commutes (up to isomorphism): $A \times B \cong B \times A$
- Product is associative (up to isomorphism): $(A \times B) \times C \cong A \times (B \times C)$
- What is a product of two objects in a preorder category?
- Define the product of any family of objects. What is the product of the empty family?
- For any algebraic signature $\Sigma \in |\mathbf{AlgSig}|$, try to define products in $\mathbf{Alg}(\Sigma)$, $\mathbf{PAlg_s}(\Sigma)$, $\mathbf{PAlg}(\Sigma)$. Expect troubles in the two latter cases...
- Define products in the *category of partial functions*, $\mathbf{Pfn}$, with sets (as objects) and partial functions as morphisms between them.
- Define products in the *category of relations*, $\mathbf{Rel}$, with sets (as objects) and binary relations as morphisms between them.
  - BTW: What about products in $\mathbf{Rel}^{op}$?

Coproducts

coproduct = *co*-product

A *coproduct* of two objects $A, B \in |\mathbf{K}|$, is any object $A + B \in |\mathbf{K}|$ with two morphisms (*coproduct injections*) $\iota_1 : A \to A + B$ and $\iota_2 : B \to A + B$ such that for any object $C \in |\mathbf{K}|$ with morphisms $f_1 : A \to C$ and $f_2 : B \to C$ there exists a unique morphism $h : A + B \to C$ such that $\iota_1;h = f_1$ and $\iota_2;h = f_2$.

In $\mathbf{Set}$, *disjoint union is a coproduct*

$$\begin{tikzcd}
A \arrow[r, "\iota_1"] \arrow[dr, "f_1"'] & A+B \arrow[d, "\exists!\ h"] & B \arrow[l, "\iota_2"'] \arrow[dl, "f_2"] \\
& C &
\end{tikzcd}$$

We write $[f_1, f_2]$ for h defined as above. Then: $\iota_1;[f_1, f_2] = f_1$ and $\iota_2;[f_1, f_2] = f_2$. Moreover, for any h from the coproduct $A + B$: $h = [\iota_1;h, \iota_2;h]$.

Essentially, this equationally defines a product!

Fact: *Coproducts are defined to within an isomorphism (which commutes with injections).*

Exercises: Dualise!

Equalisers

An *equaliser* of two "parallel" morphisms $f, g : A \to B$ is a morphism $e : E \to A$ such that $e;f = e;g$, and such that for all $h : H \to A$, if $h;f = h;g$ then for a unique morphism $k : H \to E$, $k;e = h$.

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
H \arrow[u, "\exists!"', "k"] \arrow[ur, "h"'] & &
\end{tikzcd}$$

- Equalisers are unique up to isomorphism.
- Every equaliser is mono.
- Every epi equaliser is iso.

In $\mathbf{Set}$, given functions $f, g : A \to B$, define $\boxed{E = \{a \in A \mid f(a) = g(a)\}}$

The inclusion $e : E \hookrightarrow A$ is an equaliser of f and g.

Define equalisers in $\mathbf{Alg}(\Sigma)$.

Try also in: $\mathbf{PAlg_s}(\Sigma)$, $\mathbf{PAlg}(\Sigma)$, $\mathbf{Pfn}$, $\mathbf{Rel}$, ...

Coequalisers

A *coequaliser* of two "parallel" morphisms $f, g : A \to B$ is a morphism $c : B \to C$ such that $f;c = g;c$, and such that for all $h : B \to H$, if $f;h = g;h$ then for a unique morphism $k : C \to H$, $c;k = h$.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "c"] \arrow[dr, "h"'] & C \arrow[d, "\exists! \;k"] \\ & & H \end{tikzcd}$$

- Coequalisers are unique up to isomorphism.
- Every coequaliser is epi.
- Every mono coequaliser is iso.

In $\mathbf{Set}$, given functions $f, g : A \to B$,
let $\equiv \subseteq B \times B$ be the least equivalence such that $\boxed{f(a) \equiv g(a) \text{ for all } a \in A}$
The quotient function $[\_]_\equiv : B \to B/\!\equiv$ is a coequaliser of f and g.

Define coequalisers in $\mathbf{Alg}(\Sigma)$.
Try also in: $\mathbf{PAlg_s}(\Sigma)$, $\mathbf{PAlg}(\Sigma)$, $\mathbf{Pfn}$, $\mathbf{Rel}$, ...

Most general unifiers are coequalisers in $\mathbf{Subst}_\Sigma$

Pullbacks

A *pullback* of two morphisms with common target $f : A \to C$ and $g : B \to C$ is an object $P \in |\mathbf{K}|$ with morphisms $j : P \to A$ and $k : P \to B$ such that $j;f = k;g$, and such that for all $P' \in |\mathbf{K}|$ with morphisms $j' : P' \to A$ and $k' : P' \to B$, if $j';f = k';g$ then for a unique morphism $h : P' \to P$, $h;j = j'$ and $h;k = k'$.

In $\mathbf{Set}$, given functions $f : A \to C$ and $f : B \to C$, define $\boxed{P = \{\langle a, b\rangle \in A \times B \mid f(a) = g(b)\}}$

Then P with obvious projections on A and B, respectively, is a pullback of f and g.

Define pullbacks in $\mathbf{Alg}(\Sigma)$.

Try also in: $\mathbf{PAlg_s}(\Sigma)$, $\mathbf{PAlg}(\Sigma)$, $\mathbf{Pfn}$, $\mathbf{Rel}$, ...

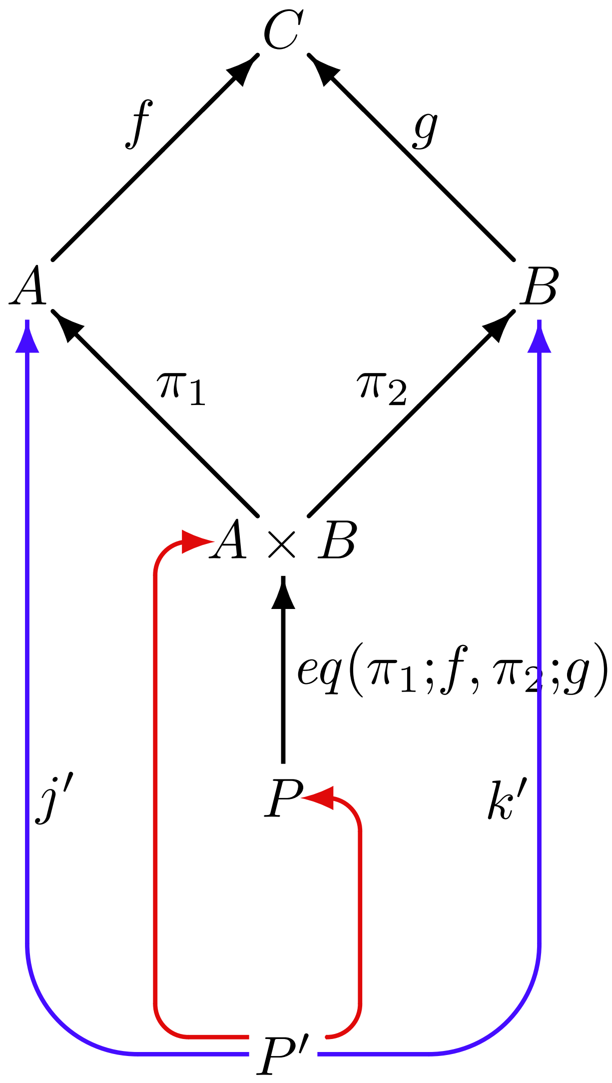
Wait for a hint to come...

$$\begin{tikzcd}
& C & \\
A \arrow[ur, "f"] & & B \arrow[ul, "g"'] \\
& P \arrow[ul, "j"] \arrow[ur, "k"'] & \\
& P' \arrow[u, "\exists!\, h"] \arrow[uul, "j'"] \arrow[uur, "k'"'] &
\end{tikzcd}$$

Few facts

- Pullbacks are unique up to isomorphism.
- If $\mathbf{K}$ has all products (of pairs of objects) and all equalisers (of pairs of parallel morphisms) then it has all pullbacks (of pairs of morphisms with common target).
- If $\mathbf{K}$ has all pullbacks and a terminal object then it has all binary products and equalisers. HINT: to build an equaliser of $f, g : A \to B$, consider a pullback of $\langle id_A, f\rangle, \langle id_A, g\rangle : A \to A \times B$.
- Pullbacks translate monos to monos: if the following is a pullback square and f is mono then f' is mono as well.

$$\begin{tikzcd}
\bullet & \bullet \arrow[l] \\
\bullet \arrow[u, "f"] & \bullet \arrow[l] \arrow[u, "f'"']
\end{tikzcd}$$



Pushouts

pushout $=$ *co*-pullback

A *pushout* of two morphisms with common source $f : C \to A$ and $g : C \to A$ is an object $P \in |\mathbf{K}|$ with morphisms $j : A \to P$ and $k : B \to P$ such that $f;j = g;k$, and such that for all $P' \in |\mathbf{K}|$ with morphisms $j' : A \to P'$ and $k' : B \to P'$, if $f;j' = g;k'$ then for a unique morphism $h : P \to P'$, $j;h = j'$ and $k;h = k'$.

In $\mathbf{Set}$, given two functions $f : A \to C$ and $g : B \to C$, define the least equivalence $\equiv$ on $A \uplus B$ such that $\boxed{f(c) \equiv g(c) \text{ for all } c \in C}$ The quotient $(A \uplus B)/\equiv$ with compositions of injections and the quotient function is a pushout of f and g.

$$
\begin{tikzcd}
 & P' & \\
 & P \arrow[u, "\exists ! \, h"] & \\
A \arrow[ur, "j"] \arrow[uur, "j'"] & & B \arrow[ul, "k"'] \arrow[uul, "k'"'] \\
 & C \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}
$$

Dualise facts for pullbacks!

Example

$$\begin{tikzcd}[column sep=small]
& \{0,[1=1],[2=3],2,3\} & \\
\{0,1,2,3\} \arrow[ur, "{\{0\mapsto 0,1\mapsto[1=1],\ 2\mapsto[2=3],3\mapsto 3\}}"] & \mathsf{PO} & \{1,2,3\} \arrow[ul, "{\{1\mapsto[1=1],\ 2\mapsto 2,3\mapsto[2,3]\}}"'] \\
& \{a,b\} \arrow[ul, "{\{a\mapsto 1,b\mapsto 2\}}"] \arrow[ur, "{\{a\mapsto 1,b\mapsto 3\}}"'] &
\end{tikzcd}$$

Pushouts put objects together taking account of the indicated sharing

Example in AlgSig

$$\begin{tikzcd}
\boxed{\begin{array}{l}\textbf{sort}\ \ String\\ \textbf{ops}\ \ a,\ldots,z : String;\\ _{}^\frown_ : String \times String\\ \hfill \to String\end{array}} \arrow[r] \arrow[dr, phantom, "\text{PO}"] & \boxed{\begin{array}{l}\textbf{sorts}\ \ String, Nat, Array[String]\\ \textbf{ops}\ \ a,\ldots,z : String;\\ _{}^\frown_ : String \times String \to String;\\ empty : Array[String];\\ put : Nat \times String \times Array[String]\\ \hfill \to Array[String];\\ get : Nat \times Array[String] \to String\end{array}} \\
\boxed{\textbf{sort}\ \ Elem} \arrow[u] \arrow[r] & \boxed{\begin{array}{l}\textbf{sorts}\ \ Elem, Nat, Array[Elem]\\ \textbf{ops}\ \ empty : Array[Elem];\\ put : Nat \times Elem \times Array[Elem]\\ \hfill \to Array[Elem];\\ get : Nat \times Array[Elem] \to Elem\end{array}} \arrow[u]
\end{tikzcd}$$

Graphs

A graph consists of sets of nodes and edges, and indicate source and target nodes for each edge

$$\Sigma_{Graph} = \textbf{sorts}\ nodes, edges$$
$$\textbf{opns}\ source : edges \to nodes$$
$$target : edges \to nodes$$

Graph is any Σ_{Graph}-algebra.

The category of graphs:

$$\mathbf{Graph} = \mathbf{Alg}(\Sigma_{Graph})$$

For any small category $\mathbf{K}$, define its *graph*, $G(\mathbf{K})$

For any graph $G \in |\mathbf{Graph}|$, define *the category of paths in* G, $\mathbf{Path}(G)$:

- objects: $|G|_{nodes}$
- morphisms: *paths* in G, i.e., sequences $n_0 e_1 n_1 \ldots n_{k-1} e_k n_k$ of nodes $n_0, \ldots, n_k \in |G|_{nodes}$ and edges $e_1, \ldots, e_k \in |G|_{edges}$ such that $source(e_i) = n_{i-1}$ and $target(e_i) = n_i$ for $i = 1, \ldots, k$.

Diagrams

A diagram in $\mathbf{K}$ is a graph with nodes labelled with $\mathbf{K}$-objects and edges labelled with $\mathbf{K}$-morphisms with appropriate sources and targets.

A *diagram* D consists of:

- a graph $G(D)$,
- an object $D_n \in |\mathbf{K}|$ for each node $n \in |G(D)|_{nodes}$,
- a morphism $D_e : D_{source(e)} \to D_{target(e)}$ for each edge $e \in |G(D)|_{edges}$.

For any small category $\mathbf{K}$, define its *diagram*, $D(\mathbf{K})$, with graph $G(D(\mathbf{K})) = G(\mathbf{K})$

BTW: A diagram D *commutes* (or is *commutative*) if for any two paths in $G(D)$ with common source and target, the compositions of morphisms that label the edges of each of them coincide.

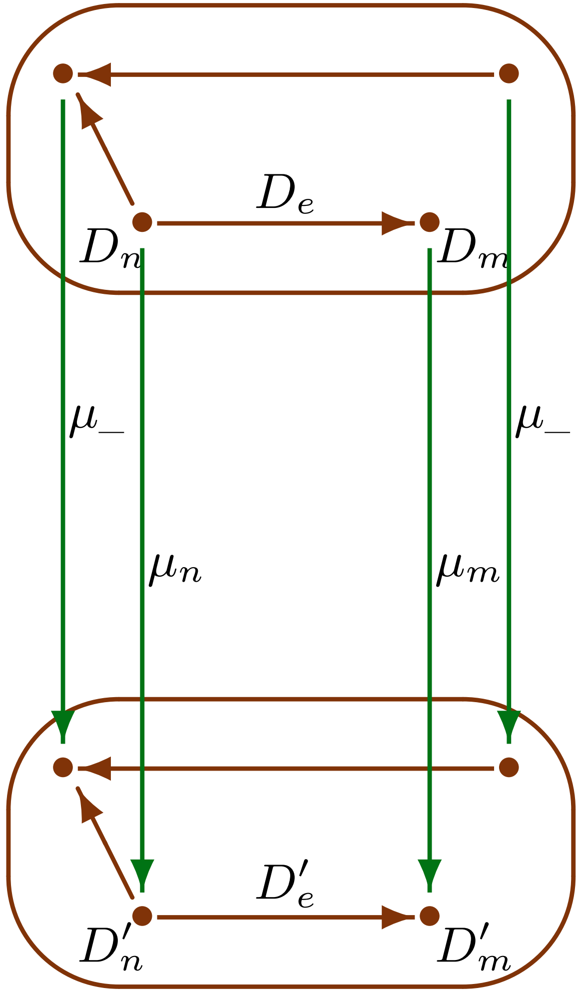
Diagram categories

Given a graph G with nodes $N = |G|_{nodes}$ and edges $E = |G|_{edges}$, the *category of diagrams of shape G in* $\mathbf{K}$, $\mathbf{Diag}^G_{\mathbf{K}}$, is defined as follows:

- objects: all diagrams D in $\mathbf{K}$ with $G(D) = G$
- morphisms: for any two diagrams D and D' in $\mathbf{K}$ of shape G, a morphism $\mu : D \to D'$ is any family $\mu = \langle \mu_n : D_n \to D'_n \rangle_{n \in N}$ of morphisms in $\mathbf{K}$ such that for each edge $e \in E$ with $source_{G(D)}(e) = n$ and $target_{G(D)}(e) = m$,

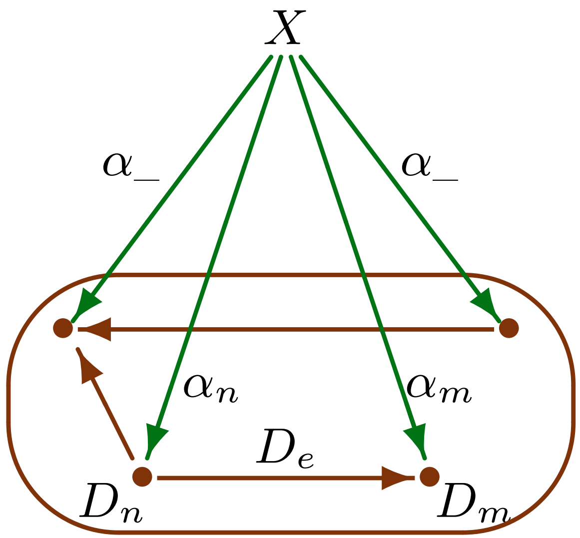
$$\boxed{\mu_n;D'_e = D_e;\mu_m}$$

.

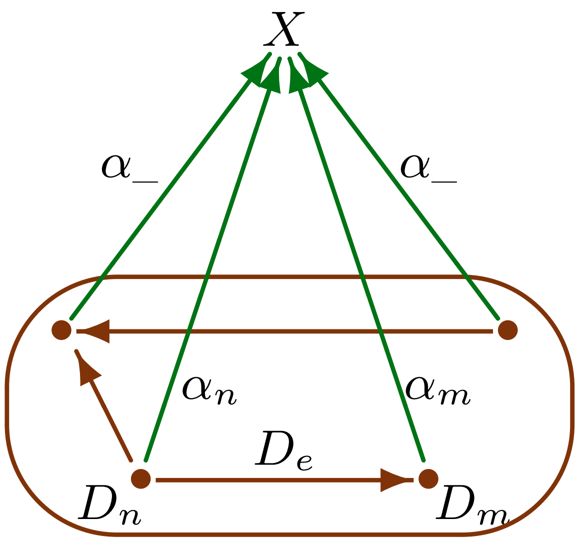


Let D be a diagram over $G(D)$ with nodes $N = |G(D)|_{nodes}$ and edges $E = |G(D)|_{edges}$.

Cones and cocones

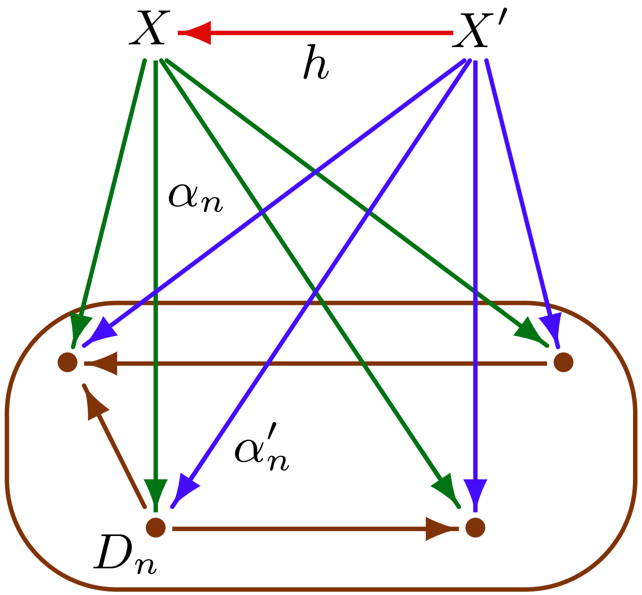


A *cone* on D (in $\mathbf{K}$) is an object $X \in |\mathbf{K}|$ together with a family of morphisms $\langle \alpha_n : X \to D_n \rangle_{n \in N}$ such that for each edge $e \in E$ with $source_{G(D)}(e) = n$ and $target_{G(D)}(e) = m$, $\alpha_n;D_e = \alpha_m$.

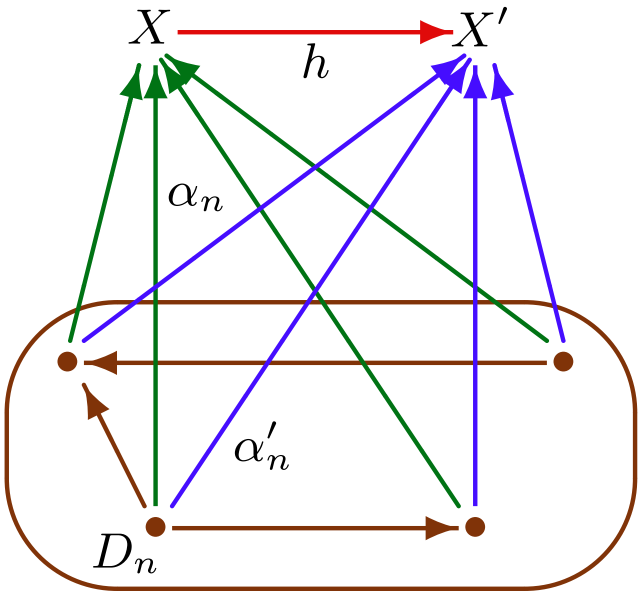


A *cocone* on D (in $\mathbf{K}$) is an object $X \in |\mathbf{K}|$ together with a family of morphisms $\langle \alpha_n : D_n \to X \rangle_{n \in N}$ such that for each edge $e \in E$ with $source_{G(D)}(e) = n$ and $target_{G(D)}(e) = m$, $\alpha_n = D_e;\alpha_m$.

Limits and colimits



A *limit* of D (in $\mathbf{K}$) is a cone $\langle \alpha_n : X \to D_n \rangle_{n \in N}$ on D such that for all cones $\langle \alpha'_n : X' \to D_n \rangle_{n \in N}$ on D, for a unique morphism $h : X' \to X$, $h;\alpha_n = \alpha'_n$ for all $n \in N$.



A *colimit* of D (in $\mathbf{K}$) is a cocone $\langle \alpha_n : D_n \to X \rangle_{n \in N}$ on D such that for all cocones $\langle \alpha'_n : D_n \to X' \rangle_{n \in N}$ on D, for a unique morphism $h : X \to X'$, $\alpha_n;h = \alpha'_n$ for all $n \in N$.

Some limits

diagram	limit	in $\mathbf{Set}$
(empty)	*terminal object*	$\{*\}$
$A \qquad B$	*product*	$A \times B$
$A \underset{g}{\overset{f}{\rightrightarrows}} B$	*equaliser*	$\{a \in A \mid f(a) = g(a)\} \hookrightarrow A$
$A \xrightarrow{f} C \xleftarrow{g} B$	*pullback*	$\{(a, b) \in A \times B \mid f(a) = g(b)\}$

. . . & colimits

diagram	colimit	in $\mathbf{Set}$
(empty)	*initial object*	$\emptyset$
$A \qquad B$	*coproduct*	$A \uplus B$
$A \underset{g}{\overset{f}{\rightrightarrows}} B$	*coequaliser*	$B \longrightarrow B/{\equiv}$ where $f(a) \equiv g(a)$ for all $a \in A$
$A \xleftarrow{f} C \xrightarrow{g} B$	*pushout*	$(A \uplus B)/{\equiv}$ where $f(c) \equiv g(c)$ for all $c \in C$

Exercises

- For any diagram D, define the *category of cones over* D, $\mathbf{Cone}(D)$:
  - objects: all cones over D
  - morphisms: a morphism from $\langle \alpha_n : X \to D_n \rangle_{n \in N}$ to $\langle \alpha'_n : X' \to D_n \rangle_{n \in N}$ is any $\mathbf{K}$-morphism $h : X \to X'$ such that $h;\alpha'_n = \alpha_n$ for all $n \in N$.
- Show that limits of D are terminal objects in $\mathbf{Cone}(D)$. Conclude that limits are defined uniquely up to isomorphism (which commutes with limit projections).
- Construct a limit in $\mathbf{Set}$ of the following diagram:

$$A_0 \xleftarrow{f_0} A_1 \xleftarrow{f_1} A_2 \xleftarrow{f_2} \cdots$$

- Show that limiting cones are *jointly mono*, i.e., if $\langle \alpha_n : X \to D_n \rangle_{n \in N}$ is a limit of D then for all $f, g : A \to X$, $f = g$ whenever $f;\alpha_n = g;\alpha_n$ for all $n \in N$.

Dualise all the exercises above!

Completeness and cocompleteness

A category $\mathbf{K}$ *is (finitely) complete if any (finite) diagram in* $\mathbf{K}$ *has a limit.*

A category $\mathbf{K}$ *is (finitely) cocomplete if any (finite) diagram in* $\mathbf{K}$ *has a colimit.*

- If $\mathbf{K}$ has a terminal object, binary products (of all pairs of objects) and equalisers (of all pairs of parallel morphisms) then it is finitely complete.
- If $\mathbf{K}$ has products of all families of objects and equalisers (of all pairs of parallel morphisms) then it is complete.

Prove completeness of $\mathbf{Set}$, $\mathbf{Alg}(\Sigma)$, $\mathbf{AlgSig}$, $\mathbf{Pfn}$, ...

When a preorder category is complete?

BTW: If a small category is complete then it is a preorder.

Dualise the above!