Freyd is Kleisli, for Arrows

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July 2, 2006

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2-categorical characterization of Kleisli categories

(Eilenberg-Moore) algebras for Arrows

- Interfaces for *structured computations* (as opposed to *pure functions*):
 - ☐ **monads** [Moggi'91] for structured output
 - comonads for structured input
 - (monad + comonad + distr. law)
 for structured input/output

Arrows [Hughes'00] generalize them

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All come with notions of

- ☐ Kleisli categories
- ☐ Eilenberg-Moore algebras
- Arrows [Hughes'00] generalize them

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- ☐ Kleisli categories
- ☐ Eilenberg-Moore algebras
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Question

What are { Kleisli categories (Eilenberg-Moore) algebras

for Arrows?

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■ *Kleisli* for Arrows:

Freyd categories

[Power, Robinson, Thielecke]

 \square {Arrows on \mathbb{C} }

≅ [Heunen&Jacobs, MFPS'06]

{Freyd categories on \mathbb{C} }

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 \square {Arrows on \mathbb{C} }

Kleisli! [This paper] ≅ [Heunen&Jacobs, MFPS'06]

 $\{$ Freyd categories on $\mathbb{C}\}$

□ 2-categorical characterization (Cf. [Street'72])

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 $\{$ Freyd categories on $\mathbb{C}\}$

- □ 2-categorical characterization (Cf. [Street'72])
- (Eilenberg-Moore) algebras for Arrows:

our notion is comparable to monad-algebras



□ Not carried by specific object(s)

Introducing Arrows

Use of monads [Moggi'91]

Kleisli category for monads

Strong monad

Use of comonads, (monad + comonad + distr. law)

Arrows [Hughes'00]

Arrows generalize (co)monads

Comparing strong monad vs. Arrow Arrows, like Monads, are monoids [Heunen&Jacobs,MFPS'06]

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 \mathbb{C} : category of types and pure functions. $\mathbb{C} = \mathbf{Sets}, \mathbf{Cpo}$, etc.

- **Monad** $T:\mathbb{C}\to\mathbb{C}$ is a functor equipped with:

 - \square extension $(X \xrightarrow{f} TY) \xrightarrow{(-)^*} (TX \xrightarrow{f^*} TY)$

Use of monads [Moggi'91]

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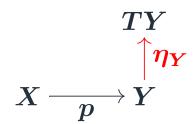
Kleisli categories for Arrows

2-categorical characterization of Kleisli categories

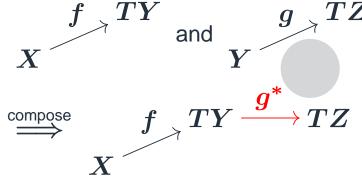
(Eilenberg-Moore) algebras for Arrows

 \mathbb{C} : category of types and pure functions. $\mathbb{C} = \mathbf{Sets}, \mathbf{Cpo}$, etc.

- **Monad** $T:\mathbb{C}\to\mathbb{C}$ is a functor equipped with:
 - \square unit $X {\longrightarrow} TX$ for each X
 - \square extension $(X \xrightarrow{f} TY) \xrightarrow{(-)^*} (TX \xrightarrow{f^*} TY)$
- Useful as an interface for computations $\emph{with structured output}, or$ computations $\emph{with effect} \ X \longrightarrow TY$. We can...
 - embed pure functionsdue to unit



compose computationsdue to extension



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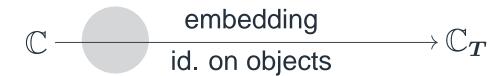
Comparing strong monad vs. Arrow Arrows, like Monads, are monoids [Heunen&Jacobs.MFPS'06]

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A monad T gives rise to a *Kleisli category*.



base category cat. of *pure functions*

Kleisli category

cat. of computations with str. output

$$rac{X \longrightarrow Y ext{ in } \mathbb{C}_T}{X \longrightarrow TY ext{ in } \mathbb{C}}$$

- Embedding is due to unit.
- lacksquare \mathbb{C}_T is a category.
 - In particular, composition of arrows is due to extension $(-)^*$.

Strong monad

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Comparing strong monad vs. Arrow

Arrows, like Monads, are monoids [He-unen&Jacobs,MFPS'06]

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Assume \mathbb{C} has \times .

Strong monad = monad + compatibility with ×

lacksquare Defn. A strong monad T is a monad with **strength**

$$(TX) imes Y {\longrightarrow} T(X imes Y)$$

Allows us to add **environments** $(-) \times Z$:

add env.
$$f \times Z \xrightarrow{(TY) \times Z} \stackrel{\operatorname{st}_{Y,Z}}{\longrightarrow} T(Y \times Z)$$
 $X \times Z$ Computation from $X \times Z$ to $Y \times Z$

Use of comonads, (monad + comonad + distr. law)

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Similarly,

- lacksquare A $egin{array}{c} {\sf comonad} \ M \ {\sf M} \ {\sf is} \ {\sf an} \ {\sf interface} \ {\sf for} \ {\sf computation} \ {\sf with} \ {\it structured} \ {\it input} \ MX \longrightarrow Y \ {\sf M} \ {\sf interface} \ {\sf input} \ {\sf M} \ {\sf input} \ {\sf inpu$
 - ☐ Brookes & Geva '92, Kieburtz '99, Uustalu & Vene '05
 - $oxed{\mathbb{C}} \longrightarrow \mathbb{C}_{oldsymbol{M}}$

base category cat. of *pure functions*

co-Kleisli category

cat. of computations with str. input

$$\frac{X \longrightarrow Y \text{ in } \mathbb{C}_M}{\overline{MX} \longrightarrow Y \text{ in } \mathbb{C}}$$

Use of comonads, (monad + comonad + distr. law)

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- lacksquare Monad T + comonad M + distributive law $MT \stackrel{\lambda}{\Rightarrow} TM$ is an interface for computation with $structured\ input\ \&\ output\ MX \longrightarrow TY$
 - ☐ Uustalu & Vene, '05
 - $oxed{\Box} \qquad \mathbb{C} \xrightarrow{\mathsf{embedding}} \mathbb{C}_{T,M,oldsymbol{\lambda}}$

base category cat. of *pure functions*

bi-Kleisli category

cat. of *computations with str. I/O*

$$rac{X \longrightarrow Y ext{ in } \mathbb{C}_{T,M,\lambda}}{MX \longrightarrow TY ext{ in } \mathbb{C}}$$

Arrows [Hughes'00]

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An Arrow on $\mathbb C$ is...

- lacksquare a bifunctor $A(-,+):\mathbb{C}^{^{\mathrm{op}}} imes\mathbb{C} o\mathrm{Sets},$
- equipped with

$$\operatorname{arr}_{X,Y} : \operatorname{Hom}_{\mathbb{C}}(X,Y) o A(X,Y)$$

$$>\!\!>_{X,Y,Z}$$
 : $A(X,Y) \times A(Y,Z) \rightarrow A(X,Z)$

$$\operatorname{first}_{X,Y,Z} : A(X,Y) \to A(X \times Z, Y \times Z)$$

with coherence conditions

$$(a \ggg b) \ggg c = a \ggg (b \ggg c),$$

$$\operatorname{arr}(g \circ f) = \operatorname{arr}(f) \ggg \operatorname{arr}(g),$$

$$\operatorname{arr} \operatorname{id} \ggg a = a = a \ggg \operatorname{arr} \operatorname{id},$$

$$\operatorname{first}(a) \ggg \operatorname{arr}(\pi_1) = \operatorname{arr}(\pi_1) \ggg a,$$

$$\operatorname{first}(a) \ggg \operatorname{arr}(\operatorname{id} \times f) = \operatorname{arr}(\operatorname{id} \times f) \ggg \operatorname{first}(a),$$

$$\operatorname{first}(a) \ggg \operatorname{arr}(\alpha) = \operatorname{arr}(\alpha) \ggg \operatorname{first}(\operatorname{first}(a)),$$

$$\operatorname{first}(\operatorname{arr}(f)) = \operatorname{arr}(f \times \operatorname{id})$$

$$\operatorname{first}(a \ggg b) = \operatorname{first}(a) \ggg \operatorname{first}(b)$$

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An $m{Arrow}$ on $\mathbb C$ is...

- a bifunctor $A(-,+):\mathbb{C}^{^{\mathrm{op}}} imes\mathbb{C} o\mathrm{Sets},$ A(X,Y): set of structured computations from X to Y
- equipped with

$$\operatorname{arr}_{X,Y}$$
 : $\operatorname{Hom}_{\mathbb{C}}(X,Y) o A(X,Y)$

embeds pure functions

$$>\!\!>_{X,Y,Z}$$
: $A(X,Y) \times A(Y,Z) \rightarrow A(X,Z)$

composes structured computations

$$\operatorname{first}_{X,Y,Z} \; : \; A(X,Y) \to A(X \times Z, Y \times Z)$$

allows for handling environment

Straightforward to **enrich** the whole setting, by replacing \mathbf{Sets} by symmetric monoidal $\mathbb V$

Arrows generalize (co)monads

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lacksquare Strong monad T gives rise to an Arrow A_T by

$$A_T(X,Y) = \operatorname{Hom}_{\mathbb{C}}(X,TY)$$

lacksquare Comonad M gives rise to an Arrow A_M by

$$A_M(X,Y) = \operatorname{Hom}_{\mathbb{C}}(MX,Y)$$

 $\blacksquare \quad \mathsf{Monad} \ T + \mathsf{comonad} \ M + \mathsf{distributive} \ \mathsf{law} \ MT \overset{\lambda}{\Rightarrow} TM \\ \mathsf{gives} \ \mathsf{rise} \ \mathsf{to} \ \mathsf{an} \ \mathsf{Arrow} \ A_{T,M,\lambda} \ \mathsf{by}$

$$A_{T,M,\lambda}(X,Y) = \operatorname{Hom}_{\mathbb{C}}(MX,TY)$$

Comparing strong monad vs. Arrow

	Strong monad $oldsymbol{T}$	Arrow $A(-,+)$
structured computation from $oldsymbol{X}$ to $oldsymbol{Y}$	$oldsymbol{X} o oldsymbol{T} oldsymbol{Y}$ in $\mathbb C$	$a\in A(X,Y)$
pure function is embedded due to	unit ${igcap \eta_Y \ Y}$	$\operatorname{Hom}_{\mathbb{C}}(X,Y) \stackrel{\operatorname{arr}}{\longrightarrow} A(X,Y)$
composition of computation due to	$(X \stackrel{f}{ ightarrow} TY)$ extension $(TX \stackrel{f^*}{ ightarrow} TY)$	$A(X,Y) imes A(Y,Z) \ \downarrow > > \ A(X,Z)$
compatible with × (i.e. environment) due to	$(TX) imes Y$ strength $\int_{}^{}\!$	$A(X,Y) \ ext{ first } \ A(X imes Z,Y imes Z)$

Arrows, like Monads, are monoids [Heunen&Jacobs,MFPS'06]

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This similarity can be put more formally:

- Monad with η , $(-)^*$
 - = **monoid** in functor category $[\mathbb{C}, \mathbb{C}]$
- Strong monad with η , $(-)^*$, st
 - = *monoid* in $[\mathbb{C},\mathbb{C}]$
 - + (compatibility with ×)
- Arrow with arr, >>>, first
 - = monoid in $[\mathbb{C}^{^{\mathrm{op}}} \times \mathbb{C}, \mathbf{Sets}]$
 - + (compatibility with ×)

 $[\mathbb{C},\mathbb{C}]$: monoidal

tensor: $F\otimes G=F\circ G$

unit: id

 $[\mathbb{C}^{^{\mathrm{op}}} \times \mathbb{C}, \mathbf{Sets}]$: monoidal

tensor: like composition

unit: $\mathbf{Hom}_{\mathbb{C}}$

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Obvious defi nition:

Kleisli category \mathbb{C}_A for A is

$$rac{X \stackrel{a}{\longrightarrow} Y \quad ext{in } \mathbb{C}_A}{a \in A(X,Y)}$$

obj.
$$rac{X\in\mathbb{C}_A}{X\in\mathbb{C}}$$

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Kleisli category \mathbb{C}_A for A is

arrow
$$\dfrac{X\overset{a}{\longrightarrow} Y \quad \text{in } \mathbb{C}_A}{a\in A(X,Y)}$$

obj.
$$rac{X \in \mathbb{C}_A}{X \in \mathbb{C}}$$

- Inclusion functor
- $X \stackrel{f}{\longrightarrow} Y \stackrel{\longmapsto}{\longmapsto} rac{\operatorname{arr}(f) \in A(X,Y)}{\overset{\operatorname{arr}(f)}{\longrightarrow} Y}$

 \mathbb{C}_A is a category. In particular, composition is due to >>>>.

$$\mathsf{Cf}. \ \mathsf{Hom}_{\mathbb{C}}(X,Y) \ \downarrow^{\mathsf{arr}} \ A(X,Y)$$

$$A(X,Y) \times A(Y,Z)$$
 $\downarrow \gg$
 $A(X,Z)$

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Let's look at \mathbb{C}_{A} ...

- \blacksquare \mathbb{C}_A is **pre-monoidal** (which is, almost monoidal)
 - \square Instead of a bifunctor $\otimes: \mathbb{C}_A \times \mathbb{C}_A \to \mathbb{C}_A$, we have two functors for each Z

$$(-) \ltimes Z: \;\; \mathbb{C}_A o \mathbb{C}_A$$

$$Z imes (-): \;\; \mathbb{C}_A o \mathbb{C}_A$$

given by

$$egin{array}{ccccc} X & \stackrel{a}{\longrightarrow} Y & \operatorname{in} \mathbb{C}_A \ X imes Z & \stackrel{}{\longrightarrow} Y imes Z & \operatorname{in} \mathbb{C}_A \ & \operatorname{first}(a) & & & & & & \\ & a lepha Z & & & & & & & \end{array}$$

Cf.
$$A(X,Y)$$
 $igcup_{ ext{first}} A(X imes Z,Y imes Z)$

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Let's look at \mathbb{C}_{A} ...

- \blacksquare \mathbb{C}_A is **pre-monoidal** (which is, almost monoidal)
 - ☐ But not quite monoidal. If it were,

$$egin{aligned} egin{aligned} egin{aligned\\ egin{aligned} egi$$

should commute,

which is unlikely.

E.g. A(X,Y) : "state-based computations from X to Y"

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In fact, $\mathbb{C} \to \mathbb{C}_A$ is a *Freyd category*.

A Freyd category on \mathbb{C} is: [Power, Robinson, Thielecke]

$$\mathbb{C} \xrightarrow{\text{id. on objects}} \mathbb{K} \qquad ,$$
 with fi nite products
$$\qquad \qquad \text{pre-monoidal}$$

preserving appropriate structures.

1-1 correspondence between Arrows and Freyd categories: [Heunen&Jacobs, MFPS'06]

$$\{ ext{Arrows on }\mathbb{C}\} \stackrel{\cong}{\longrightarrow} \{ ext{Freyd categories on }\mathbb{C}\}$$
 $A(-,+) \longmapsto (\mathbb{C} o \mathbb{C}_A)$
 $\operatorname{Hom}_{\mathbb{K}}(J-,J+) \longleftarrow (\mathbb{C} \stackrel{J}{\rightarrow} \mathbb{K})$

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Why "Kleisli"?

Eilenberg-Moore construction as right 2-adjoint

Kleisli as left 2-adjoint

Arrows on Freyd categories

Kleisli for Arrows, as left 2-adjoint

Four 2-functors, three 2-adjunctions

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Why is this "Kleisli"?

$$\{ ext{Arrows on }\mathbb{C}\} egin{array}{c}\cong & \cong & \text{Freyd categories on }\mathbb{C}\} \ A dash & \longrightarrow & (\mathbb{C}
ightarrow \mathbb{C}_A) \end{array}$$

- For Arrows induced by (co)monads, "Kleisli" (for Arrows) coincides with usual Kleisli (for monads).
- 2-categorical characterization. Details now

$$Freyd$$
 op $Arr(Freyd)$, just like "Kleisli"

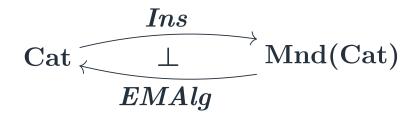
$$\overbrace{\operatorname{Cat}
olimits}{Ins}$$
 $\operatorname{Cat}
olimits \longrightarrow \operatorname{Mnd}(\operatorname{Cat}_*) \quad \text{for monads [Street'72]}$
 $\operatorname{Kleisli} \quad \bigcirc$

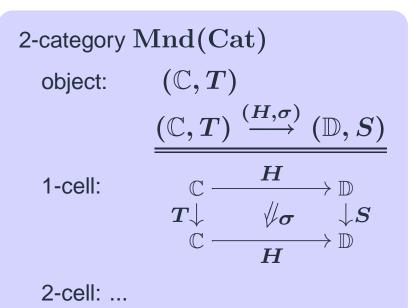
Eilenberg-Moore construction as right 2-adjoint

2-categorical characterization [Street'72] of

- Eilenberg-Moore construction of algebras
- for monads

is presented as a showcase.



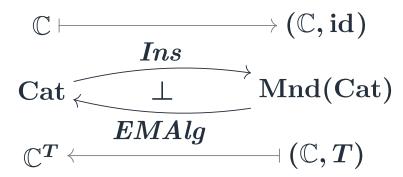


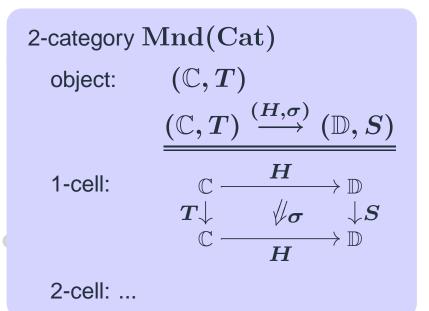
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Eilenberg-Moore construction as right 2-adjoint

2-categorical characterization [Street'72] of

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is presented as a showcase.

$$egin{array}{c} \mathbb{C} & \longmapsto & (\mathbb{C}, \mathrm{id}) \ \hline Ins \ & \coprod & & \operatorname{Mnd}(\operatorname{Cat}) \ \hline EMAlg \ & \mathbb{C}^T & \longleftarrow & (\mathbb{C}, T) \ \hline \end{array}$$

2-category $\operatorname{Mnd}(\operatorname{Cat})$ object: (\mathbb{C},T) $\underbrace{(\mathbb{C},T) \overset{(H,\sigma)}{\longrightarrow} (\mathbb{D},S)}$ 1-cell: $\mathbb{C} \overset{H}{\longrightarrow} \mathbb{D}$ $\mathbb{C} \overset{H}{\longrightarrow} \mathbb{D}$ $\mathbb{C} \overset{\mathcal{L}}{\longrightarrow} \mathbb{D}$ 2-cell: ...

obj. in
$$\mathbb{C}^T \iff 1 \to \mathbb{C}^T$$
 in $\operatorname{Cat} \iff (1,\operatorname{id}) \to (\mathbb{C},T)$ in $\operatorname{Mnd}(\operatorname{Cat})$ $\iff \operatorname{id} \downarrow \underset{X}{\overset{}{\downarrow}} \underset{\mathbb{C}}{\overset{}{\longrightarrow}} \mathbb{C} \iff \underset{X}{\overset{\mathsf{adj.}}{\downarrow}} \underset{\mathbb{C}}{\overset{\mathsf{adj.}}{\longrightarrow}} \mathbb{C}$

Kleisli as left 2-adjoint

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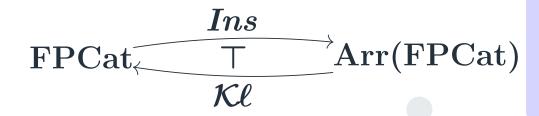
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Similarly, [Street'72]



Can we do the same for Arrows?



2-category Arr(FPCat)

object: (\mathbb{C},A)

1-cell: ...

2-cell: ...

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Why "Kleisli"?

Eilenberg-Moore construction as right 2-adjoint

Kleisli as left 2-adjoint

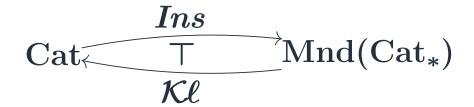
Arrows on Freyd categories

Kleisli for Arrows, as left 2-adjoint

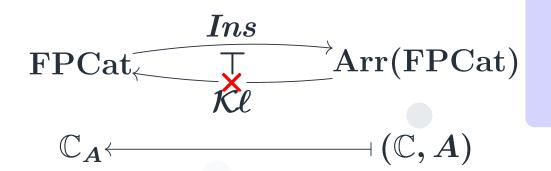
Four 2-functors, three 2-adjunctions

(Eilenberg-Moore) algebras for Arrows

Similarly, [Street'72]



Can we do the same for Arrows? No.



pre-monoidal, not with finite products!

2-category Arr(FPCat)

object: (\mathbb{C},A)

1-cell: ...

2-cell: ...

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(Eilenberg-Moore) algebras for Arrows We extend notion of Arrows:

on FPCat \Longrightarrow on Freyd

- Arrow A on $\mathbb{C} \in \mathbf{FPCat}$:
 - $A:\mathbb{C}^{^{\mathrm{op}}} imes\mathbb{C} o\mathrm{Sets},\qquad ext{with arr,>>>>, first}$

- with coherence conditions
- Arrow A on $(\mathbb{C} \xrightarrow{J} \mathbb{K}) \in \text{Freyd}$:
 - $\Box \ \ A: \mathbb{K}^{^{\mathrm{op}}} \times \mathbb{K} \to \mathrm{Sets}, \qquad \text{with arr,} >>>, \mathrm{first}$

with similar coherence conditions Especially, first is compatible with Cartesian structure carried from $\mathbb C$ to $\mathbb K$

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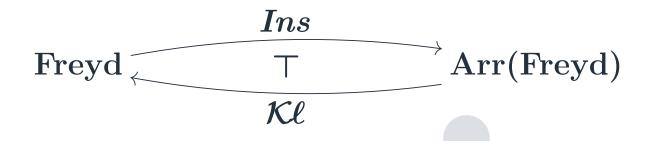
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Then indeed $\mathcal{K}\ell\dashv Ins$ for Arrows.



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Then indeed $\mathcal{K}\ell\dashv Ins$ for Arrows.

$$(\mathbb{C} \xrightarrow{J} \mathbb{K}) \longmapsto (\mathbb{C} \xrightarrow{J} \mathbb{K}, \operatorname{Hom}_{\mathbb{K}})$$
 $Freyd \longleftarrow T \longrightarrow \operatorname{Arr}(\operatorname{Freyd})$
 $(\mathbb{C} \xrightarrow{J} \mathbb{K} \xrightarrow{J_A} \mathbb{K}_A) \longleftarrow (\mathbb{C} \xrightarrow{J} \mathbb{K}, A)$

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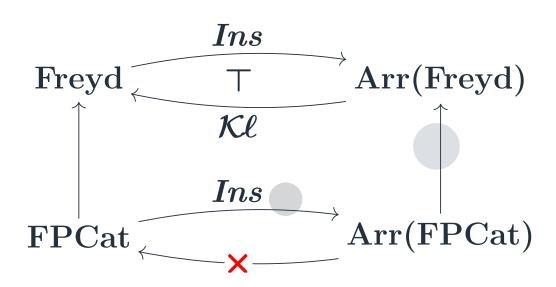
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Freyd

Arr(Freyd)

$$(\mathbb{C} \stackrel{J}{
ightarrow} \mathbb{K}) \stackrel{U}{\longleftarrow} (\mathbb{C} \stackrel{J}{
ightarrow} \mathbb{K}, \; A)$$

$$(\mathbb{C} \stackrel{J}{
ightarrow} \mathbb{K}) \stackrel{Ins}{ o} (\mathbb{C} \stackrel{J}{
ightarrow} \mathbb{K}, \; \operatorname{\mathbf{Hom}}_{\mathbb{K}})$$

$$(\mathbb{C} \stackrel{J}{ o} \mathbb{K} \stackrel{J_A}{ o} \mathbb{K}_A) \stackrel{\mathcal{K}\ell}{ o} (\mathbb{C} \stackrel{J}{ o} \mathbb{K}, \stackrel{\pmb{A}}{ o})$$

$$(\mathbb{C} \stackrel{J}{
ightarrow} \mathbb{K}) dash \stackrel{Arr}{\longrightarrow} (\mathbb{C} \stackrel{\mathrm{id}}{
ightarrow} \mathbb{C}, \ \operatorname{\mathbf{Hom}}_{\mathbb{K}}(J-,J+))$$

- \blacksquare Ins is a full embedding.
 - \Box Freyd is a full reflective 2-subcategory of Arr(Freyd).

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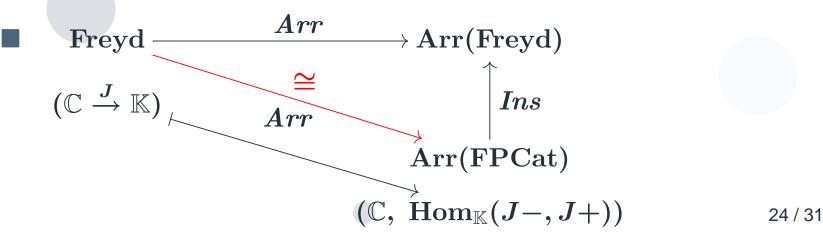
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Freyd Arr(Freyd)

$$(\mathbb{C} \stackrel{J}{
ightarrow} \mathbb{K}) \stackrel{ar{U}}{ o} (\mathbb{C} \stackrel{J}{
ightarrow} \mathbb{K}, \ A)$$

$$(\mathbb{C} \stackrel{J}{
ightarrow} \mathbb{K}) \stackrel{Ins}{ o} (\mathbb{C} \stackrel{J}{
ightarrow} \mathbb{K}, \; \operatorname{\mathbf{Hom}}_{\mathbb{K}})$$

$$(\mathbb{C} \stackrel{J}{ o} \mathbb{K} \stackrel{J_A}{ o} \mathbb{K}_A) \stackrel{\mathcal{K}\ell}{ o} (\mathbb{C} \stackrel{J}{ o} \mathbb{K}, \stackrel{\pmb{A}}{ o})$$



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Conclusions & future work

For a monad T, consider the following notion of "algebras"...

lacksquare Defn. An T-algebra is a natural transformation



in $[\mathbb{C},\mathbb{C}]$, compatible with η , $(-)^*$

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Conclusions & future work

For a monad T, consider the following notion of "algebras"...

lacksquare Defn. An T-algebra is a natural transformation



in $[\mathbb{C},\mathbb{C}]$, compatible with η , $(-)^*$

We aim at generalizing this to Arrows

Recall:

- □ Monad
 - = monoid in $[\mathbb{C},\mathbb{C}]$
- ☐ Arrow
 - = monoid in $[\mathbb{C}^{^{op}} \times \mathbb{C}, \mathbf{Sets}]$ + (compatibility with \times)

 $[\mathbb{C},\mathbb{C}]$: monoidal

tensor: $F \otimes G = F \circ G$

unit: id

 $[\mathbb{C}^{^{\mathrm{op}}} imes \mathbb{C}, \mathbf{Sets}]$: monoidal

tensor: like composition

unit: $\mathbf{Hom}_{\mathbb{C}}$

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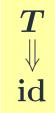
lacktriangle Defn. An algebra for Arrow A is a natural transformation

in
$$[\mathbb{C}^{^{\mathrm{op}}} imes \mathbb{C},\, \mathbf{Sets}],$$

compatible with arr, >>>, first.

- Justification?
 - $\ \ \square$ For A_T induced by a monad T,

this is the same as algebras



- \square Also for comonads, (monad + comonad + distr. law).
- ☐ Proof: non-trivial computation!

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Conclusions & future work

- Justifi cation? (ctn'd)
 - ☐ Characterized as a left-inverse of Kleisli inclusion.[Envisaged by John Power]

$$\left\{egin{align*}{c} A(-,+) \ A ext{-algebras} \end{array}
ight. egin{align*}{c} A(-,+) \ A ext{-algebras} \end{array}
ight. \cong \left\{egin{align*}{c} ext{left inverse of} \ \mathbb{C} \longrightarrow \mathbb{C}_A \end{array}
ight.
ight.$$

 \square Also the case for (co)monads. For a monad T,

$$\left\{egin{array}{ccc} T \ T \ \end{array}
ight. & \left\{egin{array}{ccc} ext{left inverse of} \ \mathbb{C} \longrightarrow \mathbb{C}_T \end{array}
ight.
ight\}$$

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$$\left\{egin{align*}{c} A(-,+) \ A ext{-algebras} \end{array}
ight. egin{align*}{c} A(-,+) \ Hom_{\mathbb{C}}(-,+) \end{array}
ight\} \hspace{0.5cm} \cong \hspace{0.5cm} \left\{egin{align*}{c} \operatorname{left\ inverse\ of} \ \mathbb{C} \longrightarrow \mathbb{C}_{A} \end{array}
ight\}$$

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$$\left\{egin{array}{ccc} T \ T ext{-algebras} & \left\|\sigma
ight. \ & \mathrm{id} \end{array}
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ight\}$$

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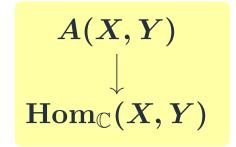
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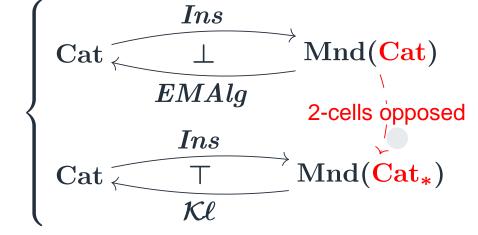
Specifi c object(s) as a carrier?

E.g.



- ☐ Doesn't work: meaning of "compatibility with >>>>" is not clear
- As a 2-categorical dual of Kleisli?

☐ For monads,



□ Does not work for

 $Freyd \qquad op \qquad Arr(Freyd) \ \mathcal{K}\ell$

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Conclusions & future work

- Is the question "What is an algebra for Arrows?" reasonable?
 - ☐ If not, why?
 - □ Examples of Arrow-algebras?

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Conclusions & future work

- Arrows generalize (co)monads: they are *monoids*.
- Kleisli for Arrows:

$$\{ ext{Arrows on }\mathbb{C}\} egin{array}{c}\cong & \cong & \text{Freyd cat. on }\mathbb{C}\} \ A & \longmapsto & (\mathbb{C} \to \mathbb{C}_A) \end{array}$$

- □ 2-categorical characterization
- Eilenberg-Moore algebras for Arrows:

$$A(-,+)$$
 , just like T for monads . $\mathbf{Hom}_{\mathbb{C}}(-,+)$

☐ Examples?

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- □ 2-categorical characterization
- Eilenberg-Moore algebras for Arrows:

$$A(-,+)$$
 , just like T for monads . $\mathbf{Hom}_{\mathbb{C}}(-,+)$

☐ Examples?

Thank you for your attention!

Contact: http://www.cs.ru.nl/~ichiro