JOURNAL OF ALGEBRA 50, 350–379 (1978)

Yoneda Structures on 2-Categories

ROSS STREET

School of Mathematics and Physics, Macquarie University, North Ryde, 2113, Australia

AND

ROBERT WALTERS

Department of Pure Mathematics, University of Sydney, 2006, Australia

Communicated by Saunders MacLane

Received September 31, 1976

INTRODUCTION

Suppose S is the category of sets in some universe, and $\mathscr{C}at$ is a 2-category of categories in some universe which contains the former universe as an element. So S is an object of $\mathscr{C}at$. An object A of $\mathscr{C}at$ is called *admissible* when its homsets $A(a, a')$ are all in S. More generally, an arrow $f\colon A \to B$ in $\mathscr{C}at$ is called *admissible* when the homsets $B(fa, b)$ are all in S. Let $\mathscr{P}A = [A^{\mathrm{op}}, S]$ be the category of contravariant S-valued functors on A, let $yA\colon A \to \mathscr{P}A$ denote the Yoneda embedding which takes a to the contravariant representable functor $A(-, a)$, and let $B(f, 1)\colon B \to \mathscr{P}A$ denote the functor which takes the value $B(fa, b)$ at b, a.

We begin here by generalizing the setting of the above paragraph by taking a 2-category $\mathscr{K}$ in place of $\mathscr{C}at$, a right ideal of admissible arrows in $\mathscr{K}$, and, for admissible A, an admissible arrow $yA\colon A \to \mathscr{P}A$ satisfying three axioms. These axioms are expressed in terms of *liftings* and *extensions* within $\mathscr{K}$. In the case where $\mathscr{K}$ is $\mathscr{C}at$, the concept of "absolute lifting" becomes that of "relative adjoint functor" (Ulmer [26]) and the concept of "extension" becomes that of "Kan extension" (MacLane [20]). For this case, our Axiom 1 amounts to Proposition IV.3.1 in Dubuc [6], or (2.2b) in Gabriel and Ulmer [10]; and our Axiom 2 amounts to 2.5 in Gabriel and Ulmer [10]. To see that Axiom 3 holds notice that application of the Yoneda lemma gives that $\mathscr{P}f$ as defined in Section 2 is isomorphic to $[f^{\mathrm{op}}, S]$.

Observe that all three axioms are facets of the Yoneda lemma (MacLane [20, p. 61]). The relationship between absolute liftings and relative adjunctions works best in the presence of an additional axiom, Axiom 3*, which is, however, not valid in all of the examples.

0021-8693/78/0502–0350$02.00/0

Sections 2 to 6 amount to a development of a large part of category in this generalized framework. Two aspects should be pointed out. The first is the ease with which arguments of ordinary category theory based on homsets can be translated into our language. The second is that many of the proofs throughout the paper are based on the two simple propositions (and their duals) on liftings in Section1; one proposition says that a triangle is a lifting diagram if and only if it becomes a lifting diagram when a lifting diagram is pasted on its right-hand side, while the other characterizes adjunctions in terms of liftings.

It will perhaps be helpful if we outline the interpretation of our results in the case explained in the first paragraph above. Proposition 7 shows that in the presence of the additional Axiom 3* our notion of relative adjunction agrees with that defined in Ulmer [26], and Proposition 8 shows the classical equivalence (Huber [12]) between the homset and the unit–counit definitions of adjunction. If we define a functor to be fully faithful when it induces isomorphisms on homsets, our Corollary 9 shows that this is a concept preserved by representable 2-functors.

For a functor $j\colon A \to B$, the functor $[j^{\mathrm{op}}, S]\colon [B^{\mathrm{op}}, S] \to [A^{\mathrm{op}}, S]$ is cocontinuous, so one expects it to preserve left extensions, in particular, those of Proposition 12. Proposition 13 goes further and gives, under certain admissibility conditions, a formula for a right adjoint to $[j^{\mathrm{op}}, S]$ which means that right Kan extensions of functors into S along j^{op} exist (MacLane [20, p.235]).

We discovered that the notion of *indexed colimit* was the right notion of colimit when working in our generalized setting even though this is not apparent from the particular example $\mathscr{K} = \mathscr{C}at$. It is more instructive to take $\mathscr{K}$ to be the 2-category of additive categories where it is known that tensor products as well as colimits are needed to construct additive left Kan extensions (for example, see Day and Kelly [4]). Tensor products and colimits are both examples of indexed colimits; on the other hand, if the appropriate tensor products and coends exist, we have the formula

$$\operatorname{col}(j, s)\, m = \int^{a} j(m, a) \otimes sa,$$

where j is an Abelian-group-valued additive functor on $M \otimes A^{\mathrm{op}}$ and $S\colon A \to C$ is an additive functor, so that $\operatorname{col}(j, s)$ becomes an additive functor from M to C. For the case where $\mathscr{K}$ is taken to be the 2-category of categories with homs enriched in some closed category, the notion of indexed colimit appears in Auderset [1], Borceux and Kelly [2] (where it is called *mean tensor product*), and Street [24]. The tensor-product-like nature of $\operatorname{col}(j, s)$ can be seen from Proposition 16. Proposition 17 amounts to the higher representation theorem of Day and Kelly [4, p. 185]. Pointwise Kan extensions for hom-enriched categories were defined by Dubuc [6] and our definition agrees with his in that case. Compare our definition with MacLane [20, Corollary 4, p. 241], and our

Proposition 20 with MacLane [20, Corollary 3, p. 235]. Proposition 21 is an adjoint functor theorem and has its analog in Borceux and Kelly [2].

The Eilenberg–Moore category and Kleisli category for a monad on a category are described in MacLane [20, Chap. VI]. Proposition 22 can be found (in the case of ordinary categories) in Linton [18, compare p.13 and p. 41] or, more explicitly, in Street [21, p.166]; it shows that the Eilenberg–Moore algebras for a monad can be regarded as sheaves for a certain generalized topology on the Kleisli category. Proposition 24 amounts to the statement that the Kleisli category is isomorphic to the full subcategory of the Eilenberg–Moore category consisting of the free algebras. The notion of *smallness* in Section 5 is justified by the fact that a category A is equivalent to a category in S when both A and $[A^{\mathrm{op}}, S]$ have homsets in S; this can be deduced from the work of Freyd [9].

The definitions of Section 6 seem to be new even for ordinary categories although related considerations appear in Ulmer [27]. A *total category* is an admissible category A for which the Yoneda functor $yA\colon A \to [A^{\mathrm{op}}, S]$ has a left adjoint. From Proposition 25 we see that this is a strong cocompleteness condition on A. Any suitably cocontinuous functor out of a total category has a right adjoint. In fact, the results of Section 6 have more familiar interpretations in the 2-category of finitely complete categories and left exact functors with, $\mathscr{P}A = [A^{\mathrm{op}}, S]$ as before (see Section 7, Example 3). A total object here is a mild generalization of a Grothendieck S-topos (see [11]). Proposition 26 corresponds then to the fact that the coalgebras for a left exact comonad on a topos yield a topos (see Kock and Wraith [15]). The equivalence of categories at the end of the section is part of the classification theorem of Giraud, namely, that, for a finitely complete small category A and an S-topos C, there is an equivalence of categories between the dual of the category of left exact functors from A to C and the category of geometric morphisms from C to $\mathscr{P}A$ (see Diaconescu [5] for an internalization of this to elementary topoi).

Section 7 discusses the examples indicated in this Introduction in more detail. Contrast the machinery of enriched category theory (Eilenberg and Kelly [7] plus Kelly [13] plus Day and Kelly [4]; or Linton [19]) with the elementary definition of a 2-category together with a Yoneda structure. Yet the setting of a Yoneda structure is strong enough to allow the expression of most of the ideas of enriched category theory. However, examples such as finitely complete categories show that not all Yoneda structures are on 2-categories of hom-enriched categories. The other major example involves categories in a finitely complete, Cartesian-closed category.

There is a logical aspect to this work. Classical logic is intended to be interpreted in a category Set of sets. Lawvere [16, 17] has suggested the existence of a generalized logic meant to be interpreted in a 2-category $\mathscr{C}\!at$ of categories; for example, the notion of quantification appropriate to categories should be Kan extension. From this point of view, the paper "Elementary Cosmoi" [23] was an exploration of the consequences of a generalized comprehension scheme.

In this paper we describe a generalization of that fragment of logic concerned with the "singleton" predicate $A \to [A, 2]$ which is the exponential adjoint of the "equality" predicate "$a = b$": $A \times A \to 2$. The generalization to $\mathscr{C}at$ of equality which is appropriate is "$\hom(a, b)$": $A^{\mathrm{op}} \times A \to \mathrm{Set}$, and so the Yoneda functor $yA\colon A \to [A^{\mathrm{op}}, \mathrm{Set}]$ is the generalization of singleton. The properties of the Yoneda functor which we take as axioms are generalized forms of the substitution properties of equality.

Although this paper follows Street [23], most of the present results (an exception is Proposition 25) were obtained in 1971 and were referred to in the earlier paper. Example 3 in Section 7 is new and we have benefited from the second author's conversations with Julian Cole especially in the relationship to Section 6.

Morphisms of Yoneda structures will be considered elsewhere.

1. Liftings and Extensions

A 2-cell

$$\begin{tikzcd} & A \arrow[dl, "s"'] \arrow[dr, "j"] \arrow[d, phantom, "\overset{\eta}{\Leftarrow}"] & \\ B \arrow[rr, "t"'] & {} & C \end{tikzcd}$$

in a 2-category $\mathscr{K}$ is said to exhibit s as a *left lifting* of j *through* t (or the diagram is said to have *the lifting property*) when pasting η at s determines a bijection between 2-cells

$$\begin{tikzcd} A \arrow[r, bend left=50, "s"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} & A \arrow[dl, "g"'] \arrow[dr, "j"] \arrow[d, phantom, "\Leftarrow"] & \\ B \arrow[rr, "t"'] & {} & C \end{tikzcd}$$

The left lifting is said to be *respected* by a: $X \to A$ when the 2-cell ηa exhibits sa as a left lifting of ja through t. The left lifting is said to be *absolute* when it is respected by all arrows with target A.

Proposition 1. *Suppose the 2-cell η (as above) exhibits s as a left lifting of j through t. A 2-cell*

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "j"] \arrow[d, phantom, "\overset{\xi}{\Leftarrow}"] & \\ D \arrow[r, "u"'] & B \arrow[r, "t"'] & C \end{tikzcd}$$

exhibits f as a left lifting of j through tu if and only if the unique 2-cell

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "s"] & \\ D \arrow[rr, "u"'] & \Leftarrow & B \end{tikzcd}$$

which pastes onto η to yield ξ, exhibits f as a left lifting of s along u. ∎

Propositions 2. *The following conditions on a diagram*

$$\begin{tikzcd} & A \arrow[dl, "s"'] \arrow[dr, "1_A"] & \\ B \arrow[rr, "t"'] & \overset{\eta}{\Leftarrow} & A \end{tikzcd}$$

are equivalent:

(a) *it has the lifting property which is respected by t: $B \to A$*;

(b) *it has the absolute lifting property*;

(c) *there exists a 2-cell*

$$\begin{tikzcd} & B \arrow[dl, "1_B"'] \arrow[dr, "t"] & \\ B & \overset{\varepsilon}{\Leftarrow} & A \arrow[ll, "s"] \end{tikzcd}$$

which pastes onto η at s and pastes onto η at t to yield identity 2-cells. ∎

In the situation of the last proposition we say that s is a *left adjoint* for t with η as *unit* and ϵ as *counit*, and we write $s \dashv t$. The equivalence of (a) and (c) is a dual "formal adjoint-functor" theorem.

We write $\mathscr{K}^{\mathrm{op}}$ for the 2-category obtained from $\mathscr{K}$ on reversing arrows and we write $\mathscr{K}^{\mathrm{co}}$ for the 2-category obtained from $\mathscr{K}$ on reversing 2-cells. Thus:

$$\mathscr{K}^{\mathrm{op}}(A, B) = \mathscr{K}(B, A), \qquad \mathscr{K}^{\mathrm{co}}(A, B) = \mathscr{K}(A, B)^{\mathrm{op}}.$$

Left liftings in $\mathscr{K}^{\mathrm{op}}$ are called *left* (*kan*) *extensions* in $\mathscr{K}$. Left liftings in $\mathscr{K}^{\mathrm{co}}$ are called *right liftings* in $\mathscr{K}$. Left liftings in $\mathscr{K}^{\mathrm{coop}}$ are called *right extensions* in $\mathscr{K}$.

2. The Colax Functor $\mathscr{P}$

Consider a 2-category $\mathscr{K}$ together with a right ideal in the underlying category $|\mathscr{K}|$ of $\mathscr{K}$. The arrows in the ideal are called *admissible*; so, for composable arrows f, g, if g is admissible then gf is admissible. An object A is called *admissible* when its identity arrow $1_A : A \to A$ is admissible; so, any arrow whose target is admissible is admissible. Write $\mathscr{A}d(A, B)$ for the full subcategory of $\mathscr{K}(A, B)$ consisting of the admissible arrows. Let $\mathscr{L}$ denote the full sub-2-category of $\mathscr{K}$ consisting of the admissible objects.

Suppose that, for each admissible object A, an admissible arrow $yA\colon A \to \mathscr{P}A$ is given; and further, suppose that, for each admissible arrow $f\colon A \to B$ with source A, a diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "yA"'] & B \arrow[d, "B(f,1)"] \arrow[dl, phantom, "\overset{\chi^f}{\Rightarrow}"] \\ & PA \end{tikzcd}$$

is given.

Axiom 1. The 2-cell χ^f exhibits $B(f, 1)$ as a left extension of yA along f.

For admissible A, we may as well suppose $A(1, 1) = yA\colon A \to \mathscr{P}A$ and $\chi^1 = 1\colon yA \Rightarrow yA$. Suppose

$$\begin{tikzcd} A \arrow[r, bend left, "f'"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$$

is a 2-cell between admissible arrows. By Axiom 1, there is a unique 2-cell $B(\alpha, 1)$ satisfying the equality

$$\begin{tikzcd} A \arrow[rr, bend left, "f"] \arrow[rr, "f'"'] \arrow[rr, phantom, bend left=20, "\overset{\alpha}{\Rightarrow}"] \arrow[d, "yA"'] & & B \arrow[dll, "B(f',1)"] \arrow[dll, phantom, bend right=10, "\overset{\chi^{f'}}{\Rightarrow}"] \\ PA & & \end{tikzcd} = \begin{tikzcd} A \arrow[rr, "f"] \arrow[d, "yA"'] \arrow[drr, phantom, "\overset{\chi^f}{\Rightarrow}"] & & B \arrow[dll, "B(f,1)"'] \arrow[dll, bend left, "B(f',1)"] \arrow[dll, phantom, bend left=15, "\overset{B(\alpha,1)}{\Rightarrow}"] \\ PA & & \end{tikzcd}$$

For any 2-cell $\begin{tikzcd} X \arrow[r, bend left, "b"] \arrow[r, bend right, "b'"'] \arrow[r, phantom, "\Downarrow\beta"] & B \end{tikzcd}$, we also put

$$\begin{tikzcd} X \arrow[r, bend left, "B(f,b)"] \arrow[r, bend right, "B(f',b')"'] \arrow[r, phantom, "\Downarrow B(\alpha,\beta)"] & PA \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "b"] \arrow[r, bend right, "b'"'] \arrow[r, phantom, "\Downarrow\beta"] & B \arrow[r, bend left, "B(f,1)"] \arrow[r, bend right, "B(f',1)"'] \arrow[r, phantom, "\Downarrow B(\alpha,1)"] & PA \end{tikzcd}$$

This describes a functor

$$B(-, -)\colon \mathscr{A}d(A, B)^{\mathrm{op}} \times \mathscr{K}(X, B) \to \mathscr{K}(X, \mathscr{P}A).$$

Suppose $f\colon A \to B$ is an arrow in L. Then $yB\colon B \to \mathscr{P}B$ is admissible, so that $B(1, f) = (yB)f\colon A \to \mathscr{P}B$ is admissible and we have an arrow $\mathscr{P}f = (\mathscr{P}B)(B(1, f), 1)\colon \mathscr{P}B \to \mathscr{P}A$. For any 2-cell

$$\begin{tikzcd} A \arrow[r, bend left, "f'"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow\alpha"] & B, \end{tikzcd}$$

we write

$$\begin{tikzcd} \mathscr{P}B \arrow[r, bend left, "\mathscr{P}f"] \arrow[r, bend right, "\mathscr{P}f'"'] \arrow[r, phantom, "\Downarrow\mathscr{P}\alpha"] & \mathscr{P}A \end{tikzcd}$$

for the 2-cell $(\mathscr{P}B)(B(1, \alpha), 1)$. This describes a functor

$$\mathscr{P}\colon \mathscr{L}(A, B)^{\mathrm{op}} \to \mathscr{K}(\mathscr{P}B, \mathscr{P}A).$$

Suppose $f\colon A \to B$, $g\colon B \to C$ are arrows such that A, f, gf are admissible. By Proposition 1, there is a unique 2-cell $\chi_f{}^g\colon B(f, 1) \Rightarrow C(gf, g)$ such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "yA"'] & B \arrow[r, "g"] \arrow[d, "B(f,1)"'] \arrow[dl, phantom, "\chi^f"] \arrow[dr, phantom, "\chi_f^g \Rightarrow"] & C \arrow[dl, "C(gf,1)"] \\ & PA & \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[rr, "gf"] \arrow[dr, "yA"'] & \arrow[d, phantom, "\chi^{gf} \Rightarrow"] & C \arrow[dl, "C(gf,1)"] \\ & PA & \end{tikzcd}$$

and moreover, $\chi_f{}^g$ exhibits $C(gf, 1)$ as a left extension of $B(f, 1)$ along g. For any $b\colon X \to B$, put $\chi^g_{f,b} = \chi_f{}^g b\colon B(f, b) \Rightarrow C(gf, gb)$.

Proposition 3. *The 2-cells $\chi^g_{f,b}\colon B(f, b) \Rightarrow C(gf, gb)$ are natural in the subscripts and extraordinary natural in the superscript in so far as they are defined.* ▮

Proposition 4. *For arrows $b\colon X \to B$, $f\colon A \to B$, $g\colon B \to C$, $h\colon C \to D$ such that A, f, gf, hgf are admissible, the following diagram commutes.*

$$\begin{tikzcd} B(f,b) \arrow[r, Rightarrow, "\chi^g_{f,b}"] \arrow[dr, Rightarrow, "\chi^{hg}_{f,b}"'] & C(gf,gb) \arrow[d, Rightarrow, "\chi^h_{gf,gb}"] \\ & D(hgf,hgb) \end{tikzcd}$$

□

For admissible B, the left extension property of χ^{yB} gives a unique 2-cell $\iota_B \colon \mathscr{P}1_B \Rightarrow 1_{\mathscr{P}B}$ such that the following composite is the identity.

$$\begin{tikzcd}
B \arrow[rr, "yB"] \arrow[dr, "yB"'] \arrow[r, phantom, shift right=4, "\chi^{yB}\ \Rightarrow"] & & \mathscr{P}B \arrow[dl, "\mathscr{P}1_B"'] \arrow[dl, bend left=50, "1_{\mathscr{P}B}"] \arrow[dl, phantom, shift left=3, "\overset{\iota_B}{\Rightarrow}"] \\
& \mathscr{P}B &
\end{tikzcd}$$

Given $f \colon A \to B$, $g \colon B \to C$ with A, B, g admissible, the left extension property of χ^{gf} determines a unique 2-cell

$$\begin{tikzcd}
C \arrow[rr, "C(g{,}1)"] \arrow[drr, "C(gf{,}1)"'] \arrow[rr, phantom, shift right=6, "\overset{\theta_{f,g}}{\Rightarrow}"] & & \mathscr{P}B \arrow[d, "\mathscr{P}f"] \\
& & \mathscr{P}A
\end{tikzcd}$$

which, when pasted onto χ^{gf} at $C(gf, 1)$, yields the 2-cell

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "yA"'] \arrow[dr, phantom, "\overset{\chi^{B(1,f)}}{\Longrightarrow}"] & B \arrow[r, "g"] \arrow[d, "yB"] \arrow[dr, phantom, "\overset{\chi^{g}}{\Rightarrow}"] & C \arrow[dl, "C(g{,}1)"] \\
\mathscr{P}A & \mathscr{P}B \arrow[l, "\mathscr{P}f"] &
\end{tikzcd}$$

One sees immediately from the definitions of $\theta_{f,g}$, $\chi_{gf}^{C(g,1)}$, $(\mathscr{P}B)(\chi_{1,f}^{g}, 1)$, that $\theta_{f,g}$ is equal to

$$\mathscr{P}B(\chi_{1,f}^{g}, 1)\, C(g, 1) \cdot \chi_{gf}^{C(g,1)}.$$

It follows then from Proposition 3 that $\theta_{f,g}$ is natural in its subscripts. We conjecture that there is complete coherence for the 2-cells ι_B, $\chi_{f,b}^{g}$; in particular, we do have:

PROPOSITION 5. *Given arrows $f \colon A \to B$, $g \colon B \to C$, $h \colon C \to D$ with A, B, C, h admissible, the following equalities hold:*

$$
\begin{tikzcd}
PB \arrow[r, bend left=50, "1_{PB}"] \arrow[r, phantom, bend left=25, "\overset{\iota_B}{\Rightarrow}"] \arrow[r, "P1_B"'] \arrow[d, "Pf"'] \arrow[d, phantom, shift left=6, "\overset{\theta_{f,yB}}{\Rightarrow}"] & PB \arrow[dl, "Pf"] \\
PA &
\end{tikzcd}
= \text{identity},
\qquad
\begin{tikzcd}
D \arrow[r, "{D(h,1)}"] \arrow[dr, bend right=40, "{D(h,1)}"'] \arrow[dr, phantom, shift right=3, "\overset{\theta_{1,g}}{\Rightarrow}"] & PC \arrow[d, "P1_C"'] \arrow[d, bend left=60, "1_{PC}"] \arrow[d, phantom, shift left=4, "\overset{\iota_C}{\Rightarrow}"] \\
& PC
\end{tikzcd}
= \text{identity},
$$

$$
\begin{tikzcd}
D \arrow[r, "{D(h,1)}"] \arrow[dr, bend right=40, "{D(hgf,1)}"'] \arrow[dr, phantom, shift right=3, "\overset{\theta_{gf,h}}{\Rightarrow}"] & PC \arrow[d, "P(gf)"'] \arrow[r, "Pg"] \arrow[dr, phantom, "\overset{\theta_{f,C(1,g)}}{\Rightarrow}"] & PB \arrow[dl, "Pf"] \\
& PA &
\end{tikzcd}
=
\begin{tikzcd}
D \arrow[r, "{D(h,1)}"] \arrow[dr, "{D(hg,1)}"] \arrow[d, "{D(hgf,1)}"'] \arrow[dr, phantom, shift left=4, "\overset{\theta_{g,h}}{\Rightarrow}"] & PC \arrow[d, "Pg"] \\
PA \arrow[r, phantom, shift left=4, "\overset{\theta_{f,hg}}{\Rightarrow}"] & PB \arrow[l, "Pf"]
\end{tikzcd}
\qquad \square
$$

For $f: A \to B$, $g: B \to C$ in $\mathscr{L}$, put

$$\gamma_{f,g} = \theta_{f,C(1,g)} .$$

COROLLARY 6. *The assignments*

$$A \mapsto \mathscr{P}A, \qquad f \mapsto \mathscr{P}f, \qquad \alpha \mapsto \mathscr{P}\alpha,$$

together with the 2-*cells*

$$\iota_B: \mathscr{P}1_B \Rightarrow 1_{\mathscr{P}B}, \qquad \gamma_{f,g}: \mathscr{P}(gf) \Rightarrow \mathscr{P}f \cdot \mathscr{P}g,$$

determine a colax functor $\mathscr{P}: \mathscr{L}^{\text{coop}} \to \mathscr{K}$. ▮

3. YONEDA STRUCTURES

The data described in Section 2 are said to form a *Yoneda structure* on the 2-category $\mathscr{K}$ when Axiom 1 (see Section 2) and Axioms 2, 3 (see below) are satisfied.

AXIOM 2. The 2-cell χ^f exhibits f as an absolute left lifting of yA through $B(f, 1)$.

AXIOM 3. The 2-cells ι_B and $\theta_{f,g}$ (defined after Proposition 4) are isomorphisms.

Axiom 3 is equivalent to the two statements:

(i) The 2-cell

$$\begin{tikzcd} A \arrow[rr, "yA"] \arrow[dr, "yA"'] & \arrow[d, phantom, "\overset{1_{yA}}{\Rightarrow}"] & \mathscr{P}A \arrow[dl, "1"] \\ & \mathscr{P}A & \end{tikzcd}$$

exhibits $1_{\mathscr{P}A}$ as a left extension of yA along yA.

(ii) The 2-cell

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "yA"'] \arrow[dr, phantom, "\overset{\chi^{B(1,f)}}{\Rightarrow}"] & B \arrow[r, "g"] \arrow[d, "yB"] \arrow[dr, phantom, "\overset{\chi^{g}}{\Rightarrow}"] & C \arrow[dl, "C(g,1)"] \\ \mathscr{P}A & \mathscr{P}B \arrow[l, "\mathscr{P}f"] & \end{tikzcd} \qquad (\mathscr{P}f = \mathscr{P}B(B(1,f),1))$$

exhibits $\mathscr{P}f.C(g, 1)$ as a left extension of yA along gf (where A, B, g are assumed admissible).

From Corollary 6 a consequence of Axiom 3 is that $\mathscr{P}$ is a pseudofunctor.

In many of the examples a further axiom holds:

Axiom 3*. If a 2-cell

$$\begin{tikzcd} B \arrow[r, bend left, "B(f,1)"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & A \end{tikzcd}$$

has the property that when pasted onto χ^f it yields a 2-cell which exhibits f as an absolute left lifting of yA through g then σ is an isomorphism.

We will see (Proposition 11) that, in the presence of Axioms 1 and 2, Axiom 3* implies Axiom 3. Some of the results of the paper have a more satisfactory form when Axiom 3* holds.

Axioms 1 and 2 can be interpreted as bijections between 2-cells

$$(1)\quad \frac{B(f,1) \Rightarrow k}{A(1,1) \Rightarrow kf}, \qquad (2)\quad \frac{fa \Rightarrow b}{A(1,a) \Rightarrow B(f,b)}$$

for arbitrary $k: B \to \mathscr{P}A$, $a: X \to A$, $b: X \to B$, both bijections being obtained by pasting on the 2-cell χ^f.

Suppose $j: A \to C$, $s: A \to B$, $t: B \to C$ are such that A, s, j are admissible. If $B(s, 1)$ is isomorphic to $C(j, t)$ we say that s is a *left adjoint of t relative to j*, and we write $s \dashv_j t$.

Proposition 7. *Suppose $j: A \to C$, $s: A \to B$, $t: B \to C$ are such that A, s, j are admissible. The equality*

$$\begin{tikzcd}[row sep=large] & A \arrow[dl, "s"'] \arrow[dr, "yA"] \arrow[d, phantom, "\chi^{s} \Downarrow"] & \\ B \arrow[rr, bend left, "B(s,1)"] \arrow[rr, bend right, "C(j,t)"'] \arrow[rr, phantom, "\pi \Downarrow"] & {} & PA \end{tikzcd} \quad = \quad \begin{tikzcd}[row sep=large] & A \arrow[dl, "s"'] \arrow[d, "j"] \arrow[dr, "yA"] & \\ B \arrow[r, "t"'] \arrow[ur, phantom, "\overset{\eta}{\Leftarrow}" near end] & C \arrow[r, "C(j,1)"'] \arrow[ur, phantom, "\overset{\chi^{j}}{\Leftarrow}" near start] & PA \end{tikzcd}$$

determines a bijection between 2-cells $\eta: j \Rightarrow ts$ *and 2-cells* $\pi: B(s, 1) \Rightarrow C(j, t)$. *Moreover, if* π *is an isomorphism then the corresponding* η *exhibits* s *as an absolute left lifting of* j *along* t. (*We then call* η *the relative unit of the relative adjunction* $s \dashv_j t$.)

If Axiom 3* *holds then* η *exhibits* s *as an absolute left lifting of* j *along* t *if and only if the corresponding* $\pi: B(s, 1) \Rightarrow C(j, t)$ *is an isomorphism.*

Proof. The bijection is just the composite bijection:

$$\begin{array}{cc} (1) & \dfrac{B(s,1) \Rightarrow C(j,t)}{A(1,1) \Rightarrow C(j,ts)} \\ (2) & \overline{\qquad j \Rightarrow ts \qquad} \end{array}$$

Suppose η and π correspond. If π is an isomorphism then by Axiom 2 the left hand of the equality above has the absolute left lifting property. By Proposition 1 and Axiom 2, the right-hand side of the equality above has the absolute left lifting property if and only if η does.

Now suppose Axiom 3* holds. By Axioms 2 and 3*, the left-hand side has the absolute left lifting property if and only if π is an isomorphism. ∎

PROPOSITION 8. *Suppose* $u: B \to A$, $f: A \to B$ *are such that* A, f *are admissible. A* 2-*cell* $\eta: 1 \dashv uf$ *is a unit for* $f \dashv u$ *if and only if the corresponding* 2-*cell* $\pi: B(f, 1) \Rightarrow A(1, u)$ *is an isomorphism. Further, if* $f \dashv u$ *then for any* X, $a: X \to A$, $b: X \to B$ *with* X, a, fa *admissible* (f *not necessarily admissible*) *we have* $A(a, ub) \simeq B(fa, b)$.

Proof. We have a bijection

$$(2) \quad \frac{\bar{\pi}: A(1, u) \Rightarrow B(f, 1)}{\epsilon: fu \Rightarrow 1},$$

and η, ϵ are unit and counit for $f \dashv u$ if and only if the corresponding π, $\bar{\pi}$ are mutually inverse.

We omit the proof of the second statement. ∎

An arrow $j: A \to B$ with j, A admissible is said to be *fully faithful* if the 2-cell

$\chi^j\colon A(1, 1) \Rightarrow B(j, j)$ is an isomorphism. The next proposition shows that in the presence of Axiom 3* this definition agrees with the representable one.

PROPOSITION 9. *Suppose $j\colon A \to B$ is admissible with A admissible. If j is fully faithful then for all objects X the functor $\mathscr{K}(X, j)\colon \mathscr{K}(X, A) \to \mathscr{K}(X, B)$ is fully faithful.*

If Axiom 3 holds then $\mathscr{K}(X, j)$ is fully faithful for all X if and only if j is fully faithful.*

Proof. The representable definition is equivalent to saying that the identity 2-cell

$$\begin{tikzcd} & A \arrow[dl, "1"'] \arrow[dr, "j"] \arrow[d, phantom, "\overset{1}{\Leftarrow}"] & \\ A \arrow[rr, "j"'] & {} & B \end{tikzcd}$$

has the absolute left lifting property. Applying Proposition 7 gives the result. ∎

The next proposition does not require Axiom 3*.

PROPOSITION 10. *Consider $f\colon A \to B$, $u\colon B \to A$, $\epsilon\colon fu \Rightarrow 1$ with B, u admissible. If ϵ is the counit of an adjunction $f \dashv u$, then the following are equivalent:*

(a) *u fully faithful;*

(b) *for all X, $\mathscr{K}(X, u)$ is fully faithful;*

(c) *ϵ is an isomorphism.*

Proof. (a) $\Rightarrow$ (b) by Proposition 9.

(b) $\Rightarrow$ (c). Both of the following 2-cells

$$\begin{tikzcd} & B \arrow[dl, "1"'] \arrow[dr, "u"] \arrow[d, phantom, "\overset{1}{\Leftarrow}"] & \\ B \arrow[rr, "u"'] & {} & A \end{tikzcd} = \begin{tikzcd} & B \arrow[ddl, "1"'] \arrow[dr, "u"] \arrow[d, phantom, "\overset{\epsilon}{\Leftarrow}"] & & \\ & {} & A \arrow[dll, "f"] \arrow[dr, "1"] \arrow[d, phantom, "\overset{\eta}{\Leftarrow}"] & \\ B \arrow[rrr, "u"'] & & {} & A \end{tikzcd} \qquad \begin{tikzcd} & & & B \arrow[dll, "u"'] \arrow[dd, "u"] \arrow[dl, phantom, "="] \\ & A \arrow[dl, "f"'] \arrow[drr, "1"] \arrow[d, phantom, "\overset{\eta}{\Leftarrow}"] & {} & \\ B \arrow[rrr, "u"'] & {} & & A \end{tikzcd}$$

have the absolute left lifting property. Hence ϵ is an isomorphism.

(c) $\Rightarrow$ (a). The inverse of $\chi^u\colon B(1, 1) \Rightarrow A(u, u)$ is

$$A(u, u) \xRightarrow{\chi^f_{u,u}} B(fu, fu) \xRightarrow{B(\epsilon^{-1}, \epsilon)} B(1, 1). \quad ∎$$

PROPOSITION 11. *Axioms 1, 2, and 3* imply Axiom 3.*

Proof. Refer to the definition of ι_B. The identity 2-cell exhibits yB as an absolute lifting of yB through $1_{\mathscr{P}B}$; so by Axiom 3*, we have that ι_B is an isomorphism.

Refer to the definition of $\theta_{f,g}$. By Axiom 2, χ^{gf} exhibits g as absolute lifting of $yB.f$ through $C(g, 1)$, and $\chi^{B(1,f)}$ exhibits $yB.f$ as absolute lifting of yA through $\mathscr{P}f$. By Proposition 1, pasting $\chi^{B(1,f)}$ and $\chi^{g}f$ at $(yB)f$ yields a 2-cell exhibiting gf as an absolute lifting of yA through $\mathscr{P}f.C(g, 1)$. Now Axiom 3* gives that $\theta_{f,g}$ is an isomorphism. ∎

PROPOSITION 12. *Suppose $j: A \to B$, $k: B \to C$ are such that A, B, k are admissible. The left extension properties of the diagrams*

$$\begin{tikzcd} B \arrow[rr, "yB"] \arrow[dr, "yB"'] \arrow[rr, phantom, shift right=4ex, "\overset{1}{\Rightarrow}"] & & \mathscr{P}B \arrow[dl, "1"] \\ & \mathscr{P}B & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[rr, "k"] \arrow[dr, "yB"'] \arrow[rr, phantom, shift right=4ex, "\overset{\chi^k}{\Rightarrow}"] & & \mathscr{P}C \arrow[dl, "{C(k,1)}"] \\ & \mathscr{P}B & \end{tikzcd}$$

are respected by $\mathscr{P}j$: $\mathscr{P}B \to \mathscr{P}A$.

Proof. By Axiom 3, when the diagrams are composed with $\mathscr{P}j$, the results are isomorphic to the right-hand triangles of the diagrams

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[dr, "yA"'] \arrow[r, phantom, shift right=4ex, "\overset{\chi^j}{\Rightarrow}"] & B \arrow[r, "{B(1,1)}"] \arrow[d, "{B(j,1)}"] \arrow[r, phantom, shift right=4ex, "\overset{\theta_{j,1}}{\Rightarrow}"] & \mathscr{P}B \arrow[dl, "\mathscr{P}j"] \\ & \mathscr{P}A & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "j"] \arrow[dr, "yA"'] \arrow[r, phantom, shift right=4ex, "\overset{\chi^j}{\Rightarrow}"] & B \arrow[r, "k"] \arrow[d, "{B(j,1)}"] \arrow[r, phantom, shift right=4ex, "\overset{\chi^k_j}{\Rightarrow}"] & \mathscr{P}C \arrow[dl, "{C(kj,1)}"] \\ & \mathscr{P}A & \end{tikzcd}$$

The left-hand triangles and the outside triangles in the latter diagrams have the left extension property; by Proposition 1, so do the right-hand triangles. ∎

PROPOSITION 13. *Suppose $j: A \to B$ is such that A, B and $B(j, 1)$: $B \to \mathscr{P}A$ are admissible. Then $\mathscr{P}j$: $\mathscr{P}B \to \mathscr{P}A$ has a right adjoint*

$$\forall j = (\mathscr{P}A)(B(j, 1), 1): \mathscr{P}A \to \mathscr{P}B$$

with unit $1 \Rightarrow \forall j.\mathscr{P}j$ determined by the equality

$$\begin{tikzcd} B \arrow[r, "yB"] \arrow[dr, "yB"'] \arrow[r, phantom, shift right=4ex, "\overset{1}{\Rightarrow}"] & \mathscr{P}B \arrow[r, "\mathscr{P}j"] \arrow[d, "1"] \arrow[r, phantom, shift right=4ex, "\Rightarrow"] & \mathscr{P}A \arrow[dl, "\forall j"] \\ & \mathscr{P}B & \end{tikzcd} = \begin{tikzcd} & \mathscr{P}B \arrow[dr, "\mathscr{P}j"] \arrow[d, phantom, "\Downarrow \theta^{-1}_{j,1}"] & \\ B \arrow[ur, "yB"] \arrow[rr, "{B(j,1)}"] \arrow[dr, "yB"'] \arrow[rr, phantom, shift right=4ex, "\overset{\chi^{B(j,1)}}{\Rightarrow}"] & & \mathscr{P}A \arrow[dl, "\forall j"] \\ & \mathscr{P}B & \end{tikzcd}$$

Proof. The 2-cell stated to be the unit is determined by the equality since $1_{\mathscr{P}B}$ is a left extension of yB along yB. By Proposition 12, the left triangle of the left-hand side of the above equality has the left extension property and this is respected by $\mathscr{P}j$. By Proposition 12, the same is true of the right-hand side of the above equality. So the right triangle on the left-hand side has the left extension property and it is respected by $\mathscr{P}j$ (Proposition 1). By Proposition 2, we have $\mathscr{P}j \dashv \forall j$ with unit as stated. ∎

COROLLARY 14. *If A, $\mathscr{P}A$ are admissible then $y\mathscr{P}A: \mathscr{P}A \to \mathscr{P}\mathscr{P}A$ is a right adjoint for $\mathscr{P}yA$.*

Proof. From the axioms, $(\mathscr{P}A)(yA, 1) \cong 1$. So

$$\forall yA = (\mathscr{P}A)((\mathscr{P}A)(yA, 1), 1) \cong (\mathscr{P}A)(1, 1) = y\mathscr{P}A. \quad ∎$$

4. INDEXED COLIMITS; POINTWISE EXTENSIONS

Suppose A, $s: A \to C$ and M, $j: M \to \mathscr{P}A$ are admissible. A *j-indexed colimit for s* is an admissible arrow $\mathrm{col}(j, s): M \to C$ which is an adjoint of $C(s, 1)$ relative to j; that is

$$\mathrm{col}(j, s) \underset{j}{\dashv} C(s, 1).$$

That is, $C(\mathrm{col}(j, s), 1) \cong (\mathscr{P}A)(j, C(s, 1))$.

If $\mu: j \Rightarrow C(s, 1).\mathrm{col}(j, s)$ is the relative unit for this relative adjunction, then Proposition 7 implies that μ exhibits $\mathrm{col}(j, s)$ as a left lifting of j through s. Such a left lifting is called a *weak j-indexed colimit for s*; this concept is only important insofar as it sometimes implies the existence of a j-indexed colimit.

Suppose $f: C \to D$ is such that fs and $f.\mathrm{col}(j, s)$ are admissible. The j-indexed colimit $\mathrm{col}(j, s)$ is said to be *preserved by f* when $f.\mathrm{col}(j, s)$ is a j-indexed colimit for fs.

PROPOSITION 15. *Any arrow with a right adjoint preserves any (weak) indexed colimits for which this makes sense.*

Proof. Suppose $f \dashv u$. *Then*

$$\begin{aligned} \mathscr{P}A(j, D(fs, 1)) &\cong \mathscr{P}A(j, C(s, u)) \\ &= \mathscr{P}A(j, C(s, 1)).u \\ &\cong C(\mathrm{col}(j, s), 1).u \\ &= C(\mathrm{col}(j, s), u) \\ &\cong D(f\,\mathrm{col}(j, s), 1). \end{aligned}$$

So

$$f\,\mathrm{col}(j, s) \simeq \mathrm{col}(j, fs).$$

We omit the proof that weak j-indexed colimits are preserved by f. ▮

PROPOSITION 16. *Suppose M, A, j: $M \to \mathscr{P}A$, s: $A \to C$ are admissible. If* $\mathrm{col}(j, s)$ *exists and if admissible N, i: $N \to \mathscr{P}M$ are such that* $\mathrm{col}(i, j)$ *exists, then there is an isomorphism*

$$\mathrm{col}(i, \mathrm{col}(j, s)) \cong \mathrm{col}(\mathrm{col}(i, j), s)$$

when either side exists.

Proof.

$$\begin{aligned} C(\mathrm{col}(\mathrm{col}(i, j), s), 1) &\cong \mathscr{P}A(\mathrm{col}(i, j), C(s, 1)) \\ &\cong \mathscr{P}M(i, \mathscr{P}A(j, C(s, 1))) \\ &\cong \mathscr{P}M(i, C(\mathrm{col}(j, s), 1)) \\ &\cong C(\mathrm{col}(i, \mathrm{col}(j, s)), 1). \quad ▮ \end{aligned}$$

Thus $\mathrm{col}(j, s)$ acts like a "tensor product" of j and s. Next we have the following Yoneda-like lemma for "contravariant representables" as indices.

PROPOSITION 17. *For admissible A, f: $A \to B$, X and a: $X \to A$, there is an isomorphism*

$$\mathrm{col}(A(1, a), f) \cong fa.$$

Proof.

$$\begin{aligned} \mathscr{P}A(A(1, a), B(f, 1)) &\cong \mathscr{P}A(A(1, a), 1).B(f, 1) \\ &= \mathscr{P}a.B(f, 1) \\ &\underset{\text{Ax. 3}}{\cong} B(fa, 1) \quad ▮ \end{aligned}$$

PROPOSITION 18. *Suppose A, j: $A \to B$, f: $A \to C$ are admissible. There is a bijection between 2-cells κ and 2-cells τ established by the equality*

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "yA"'] \arrow[dr, "f"] \arrow[dr, phantom, "\overset{\Rightarrow}{\kappa}" above right] \arrow[dr, phantom, "\underset{\Rightarrow}{\chi^f}" below left] & B \arrow[d, "k"] \\ PA & C \arrow[l, "C(f,1)"] \end{tikzcd} = \begin{tikzcd} B \arrow[d, "k"'] \arrow[dr, "B(j,1)"] \arrow[dr, phantom, "\underset{\Leftarrow}{\chi^j}" above right] \arrow[dr, phantom, "\underset{\Leftarrow}{\tau}" below left] & A \arrow[l, "j"'] \arrow[d, "yA"] \\ C \arrow[r, "C(f,1)"'] & PA \end{tikzcd}$$

The 2-cell κ exhibits k as a left extension of f along j if and only if the corresponding τ exhibits k as a left lifting of $B(j, 1)$ through $C(f, 1)$.

Proof. Axioms 1 and 2 give bijections

$$\frac{\dfrac{B(j,1) \Rightarrow C(f,1)\,h}{A(1,1) \Rightarrow C(f, hj)}}{f \Rightarrow hj}$$

When k has the left lifting property the 2-cells at the top are in bijection with 2-cells $k \Rightarrow h$. When k has the left extension property the 2-cells at the bottom are in bijection with 2-cells $k \Rightarrow h$. ∎

In other words, a left extension of f along j is precisely a weak $B(j, 1)$-indexed colimit for f. A 2-cell

$$\begin{tikzcd} A \arrow[rr, "j"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\kappa}{\Rightarrow}"] & B \arrow[dl, "k"] \\ & C & \end{tikzcd}$$

where A, B, f, $B(j, 1)$ are admissible, is said to exhibit admissible k as a *pointwise left extension of f along j* when the 2-cell τ, corresponding to κ as in the above proposition, is a relative unit for $k \dashv_{B(j,1)} C(f, 1)$. So k is a pointwise left extension of f along j precisely when there exists an isomorphism

$$k \cong \mathrm{col}(B(j, 1), f);$$

that is, when there is an isomorphism

$$C(k, 1) \cong (PA)(B(j, 1), C(f, 1)).$$

COROLLARY 19. *Pointwise left extensions are left extensions.* ∎

PROPOSITION 20. *Consider the 2-cell*

$$\begin{tikzcd} A \arrow[rr, "j"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\kappa}{\Rightarrow}"] & B \arrow[dl, "k"] \\ & C & \end{tikzcd}$$

with A, j, f admissible and j fully faithful. If the corresponding 2-cell τ: $B(j, 1) \to C(f, 1).k$ exhibits k as absolute left lifting of $B(j, 1)$ through $C(f, 1)$ then κ is an *isomorphism. Hence, in particular, if k is a pointwise left extension of f along j then κ is an isomorphism.*

Proof. Refer to the equality of Proposition 18. Since τ has the absolute left lifting property and χ^j is an isomorphism, the right-hand side has the absolute left lifting property. So the left-hand side has the absolute left lifting property. But χ^f has the absolute left lifting property too (Axiom 2). ∎

PROPOSITION 21. *Consider A, B admissible and $f: A \to B$. The following three conditions are equivalent*:

(a) *f has a right adjoint u*;

(b) *the weak $B(f, 1)$-indexed colimit of 1_A exists and is preserved by f*;

(c) $\mathrm{col}(B(f, 1), 1_A)$ *exists and is preserved by f.*

Furthermore, in this case

$$u \cong \mathrm{col}(B(f, 1), 1).$$

Proof. (a) $\Rightarrow$ (c). By Proposition 8, (a) implies $B(f, 1) \cong A(1, u)$. So $\mathrm{col}(B(f, 1), 1)$ exists if and only if $\mathrm{col}(A(1, u), 1)$ exists. By Proposition 17, the latter exists and is isomorphic to u. Proposition 15 gives the preservation property.

(c) $\Rightarrow$ (b). Trivial.

(b) $\Rightarrow$ (a). Let u be a weak $B(f, 1)$-indexed colimit for 1_A, so that fu is a weak $B(f, 1)$-indexed colimit for f. Proposition 18 gives that u is a left extension of 1_A along f and fu is a left extension of f along f. By Proposition 2, $f \dashv u$. ∎

5. RELATIONS BETWEEN THE CONSTRUCTIONS OF KLEISLI AND EILENBERG–MOORE

Let (A, s) be a monad in $\mathscr{K}$ in the sense of Street [21]; that is, s is a monoid in the monoidal category $\mathscr{K}(A, A)$ whose tensor product is composition. An *s-algebra* is a diagram

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[dr, "a"'] & A \arrow[d, "s"] \arrow[dl, phantom, "\xi\Downarrow"] \\ & A \end{tikzcd}$$

such that (a, ξ) is an algebra for the monad $\mathscr{K}(X, s)$ on the category $\mathscr{K}(X, A)$ in the sense of Eilenberg and Moore [8]. The *Eilenberg–Moore object* A^s for the monad (A, s) is determined uniquely up to isomorphism as the object for which there is a universal s-algebra:

$$\begin{tikzcd} A^s \arrow[r, "u"] \arrow[dr, "u"'] & A \arrow[d, "s"] \arrow[dl, phantom, "\gamma\Downarrow"] \\ & A \end{tikzcd}$$

The *Kleisli object* A^s for the monad (A, s) is the object for which there is a universal s-opalgebra ($= s$-algebra in $\mathscr{K}^{\mathrm{op}}$):

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "s"'] & A_s \arrow[dl, phantom, "\overset{\omega}{\Rightarrow}"] \\ A \arrow[ur, "h"'] & \end{tikzcd}$$

For reasonable $\mathscr{P}$ (in the examples, $\mathscr{P}$ extends to a 2-functor $\mathscr{K}^{\mathrm{coop}} \to \mathscr{K}$ with a left adjoint) the Kleisli object A^s for (A, s) gives rise to an Eilenberg–Moore object $\mathscr{P}A^s$ for the comonad $(\mathscr{P}A, \mathscr{P}s)$ as required in the next proposition.

Remark. We have been a little vague about what we mean by "universal" in the above. In the case of universal s-algebra, for example, we should mean that (u, γ) induces an isomorphism of categories

$$\mathscr{K}(X, A^s) \cong \mathscr{K}(X, A)^{\mathscr{K}(X,s)}.$$

In the propositions we only prove bijections on objects. The further verifications take care of themselves anyway. ∎

PROPOSITION 22. *Suppose (A, s) is a monad with A admissible. Suppose*

$$\begin{tikzcd} C \arrow[r, "v"] \arrow[dr, "v"'] & PA \arrow[d, "Ps"] \arrow[dl, phantom, "\overset{\nu}{\Rightarrow}"] \\ & PA \end{tikzcd}$$

is the universal $\mathscr{P}s$-coalgebra. An object E is the Eilenberg–Moore object for (A, s) if and only if there is a pullback:

$$\begin{tikzcd} E \arrow[r] \arrow[d] & C \arrow[d, "v"] \\ A \arrow[r, "yA"'] & PA \end{tikzcd}$$

Proof. For any arrow $a: X \to A$, we have bijections

$$\begin{tikzcd} & A \arrow[dd, "s"] \\ X \arrow[ur, "a"] \arrow[dr, "a"'] \arrow[r, phantom, "\xi \Downarrow"] & \\ & A \end{tikzcd} \quad \overset{(?)}{\longleftrightarrow} \quad \begin{tikzcd} X \arrow[r, bend left, "A(1,a)"] \arrow[r, bend right, "A(s,a)"'] \arrow[r, phantom, "\Downarrow"] & PA \end{tikzcd} \quad \overset{\text{Ax. 3}}{\longleftrightarrow} \quad \begin{tikzcd} & PA \arrow[dd, "Ps"] \\ X \arrow[ur, "A(1,a)"] \arrow[dr, "A(1,a)"'] \arrow[r, phantom, "\overset{\theta}{\Rightarrow}"] & \\ & PA \end{tikzcd}$$

One readily sees that (a, ξ) is an s-algebra if and only if $(A(1, a), \theta)$ is a $\mathscr{P}s$-coalgebra. In this case, there exists a unique arrow $x: X \to C$ such that $v \cdot x = A(1, a)$ and $\nu \cdot x = \theta$. So we have a bijection

$$\begin{tikzcd} & A \arrow[dd, "s"] \\ X \arrow[ur, "a"] \arrow[dr, "a"'] \arrow[r, phantom, "\xi \Downarrow"] & \\ & A \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} X \arrow[r, "x"] \arrow[d, "a"'] & C \arrow[d, "v"] \\ A \arrow[r, "yA"'] & PA \end{tikzcd}$$

between such s-algebras and such commutative squares. ▌

An object A is called *small* when it and $\mathscr{P}A$ are admissible. Then the category $\mathscr{K}(A, \mathscr{P}A)$ supports a closed category structure. The internal-hom is given by $[f, g] = (\mathscr{P}A)(f, g)$. The distinguished object is yA. The remaining structure consists of the 2-cells

$$\chi_{f,g}^{\mathscr{P}A(h,1)}: [f, g] \to [[h, f], [h, g]],$$

$$\chi^f: yA \to [f, f],$$

$$\iota_A g: g \leftarrow [yA, g].$$

(Compare Street [23, pp. 156–158].) For any admissible $h: A \to B$ the arrow

$B(h, h)\colon A \to \mathscr{P}A$ supports a natural monoid structure in $\mathscr{K}(A, \mathscr{P}A)$; indeed, the required "multiplication" and "unit" are

$$\chi_{h,h}^{B(h,1)}\colon B(h, h) \to (\mathscr{P}A)(B(h, h), B(h, h)),$$
$$\chi^h\colon yA \to B(h, h).$$

For any object X, let $\mathscr{E}x\mathscr{N}at(h, X)$ ("extraordinary natural transformations") denote the category whose objects are pairs (k, θ) where $k\colon A \to X$ is an admissible arrow and $\theta\colon B(h, h) \to X(k, k)$ is a monoid homomorphism, and whose arrows $\alpha\colon (k, \theta) \to (k', \theta')$ are 2-cells $\alpha\colon k \Rightarrow k'$ such that the following commutes:

$$\begin{tikzcd}
 & X(k,k) \arrow[dr, "X(1{,}\alpha)"] & \\
B(h,h) \arrow[ur, "\theta"] \arrow[dr, "\theta'"'] & & X(k,k') \\
 & X(k',k') \arrow[ur, "X(\alpha{,}1)"'] &
\end{tikzcd}$$

There is a canonical functor

$$\mathscr{A}d(B, X) \to \mathscr{E}x\,\mathscr{N}at(h, X),$$
$$m \to (mh, \chi_{h,h}^m).$$

The arrow $h\colon A \to B$ is called *bijective on objects* when this canonical functor is an isomorphism of categories for all X.

PROPOSITION 23. *In the diagram*

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "u"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & B \arrow[d, "v"] \\
X \arrow[r, "j"'] & Y
\end{tikzcd}$$

suppose A is small, X, h, j are admissible, and α is an isomorphism. If h is bijective on objects and j is fully faithful, then there exist an arrow $w\colon B \to X$ and an isomorphism $\beta\colon jw \Rightarrow v$ unique with the property that $u = wh$ and $\alpha = \beta h$. If α is an identity then so is β.

Proof. $\chi_{u,u}^j$ is an isomorphism. So we have a monoid homomorphism

$$B(h, h) \xrightarrow{\chi_{h,h}^v} Y(vh, vh) \xrightarrow{Y(\alpha,\alpha^{-1})} Y(ju, ju) \xrightarrow{(\chi_{u,u}^j)^{-1}} X(u, u).$$

So there exists a unique $w\colon B \to X$ such that $wh = u$ and the diagram

$$\begin{tikzcd}
B(h,h) \arrow[r, "\chi^{w}_{h,h}"] \arrow[dr, "\chi^{v}_{h,h}"'] & X(u,u) \arrow[r, "\chi^{j}_{u,u}"] & Y(ju,ju) \arrow[d, "Y(1,\alpha)"] \\
& Y(vh,vh) \arrow[r, "Y(\alpha,1)"'] & Y(ju,vh)
\end{tikzcd}$$

commutes. So

$$\alpha\colon (jwh, \chi^{jw}_{h,h}) \to (vh, \chi^{v}_{h,h})$$

is an isomorphism in $\mathscr{E}x\mathscr{N}at(h, Y)$, and hence $\alpha = \beta h$ for a unique isomorphism β, as required. ∎

Remark. Without any size conditions in the diagram of the above proposition with j fully faithful, there are many classes of arrows h for which the conclusion of the proposition can be proved; for example, if $h\colon A \to B$ is a "projection onto a localization" (that is, h is the *coinverter* of a 2-cell

$$\begin{tikzcd} Z \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & A \end{tikzcd}).$$

So if our 2-category admits factorizations of arrows into "bijective on objects" followed by "fully faithful," such h are all bijective on objects. ∎

PROPOSITION 24. *Suppose (A, s) is a monad with A small. Suppose an Eilenberg–Moore object A^s exists and that a left adjoint $f\colon A \to A^s$ for $u\colon A^s \to A$ factors as $A \to^{h} K \to^{j} A^s$ where h is bijective on objects and j is fully faithful. Then K is a kleisli object for (A, s) in $\mathscr{L}$.*

Proof. Since j is fully faithful there is a unique ω such that

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "s"'] \arrow[dr, phantom, "\overset{\omega}{\Rightarrow}" near start] & K \arrow[r, "j"] & A^s \\
A \arrow[ur, "h"'] & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "s"'] \arrow[dr, phantom, "\overset{\varepsilon f}{\Rightarrow}" near start] & A^s \\
A \arrow[ur, "f"'] &
\end{tikzcd}$$

where $\epsilon: fu \Rightarrow 1$ is the counit for $f \dashv u$. Given an s-opalgebra

$$\begin{tikzcd} A \arrow[rr, "k"] \arrow[dr, "s"'] & \arrow[d, phantom, "\overset{\phi}{\Rightarrow}"] & X \arrow[dl, "k"] \\ & A & \end{tikzcd}$$

the composite

$$\bar{\phi} = (A^s(f, f) \cong A(1, s) \xrightarrow{\chi^k_{1,s}} X(k, ks) \xrightarrow{X(1,\phi)} X(k, k))$$

is a monoid homomorphism. For, $\phi.k\eta = 1$ implies

$$\begin{tikzcd} & A^s(f, f) \arrow[dr, "\cong"] & \\ A(1, 1) \arrow[ur, "\chi^f"] \arrow[rr, "A(1, \eta)"'] \arrow[d, "\chi^k"'] & & A(1, s) \arrow[d, "\chi^k_{1,s}"] \\ X(k, k) \arrow[rr, "X(1, k\eta)"] \arrow[drr, "1"'] & & X(k, ks) \arrow[d, "X(1, \phi)"] \\ & & X(k, k) \end{tikzcd}$$

commutes so that $\bar{\phi}$ preserves unit, and $\phi.k\mu = \phi.\phi s$ implies (we leave the diagram to the reader) that $\bar{\phi}$ preserves multiplication. Thus there exists a unique arrow $x: K \to X$ such that $k = xh$ and $\chi^x_{h,h} = \bar{\phi}.\chi^j_{h.h}$. The latter equation transports through the isomorphism

$$\mathscr{K}(h, h) \xrightarrow{\chi^j_{h,h}} A^s(f, f) \cong A(1, s) \qquad \text{to the equation} \quad x\omega = \phi. \quad \blacksquare$$

6. Totality

An arrow $s: A \to C$ is called *total* when A, s are admissible and $C(s, 1): C \to \mathscr{P}A$ has an admissible left adjoint.

Suppose $s: A \to C$ is *total*, and a left adjoint of $C(s, 1)$ is z. From Proposition 8 if j is any admissible arrow $M \to \mathscr{P}A$, with M admissible then

$$\mathscr{P}A(j, C(s, 1)) \cong C(zj, 1).$$

Hence

$$\mathrm{col}(j, s) \text{ exists}; \qquad \mathrm{col}(j, s) = zj.$$

In other words, total arrows have colimits of all indexing types.

Suppose A and $s: A \to C$ are admissible. From Axiom 3 we have that $\mathscr{P}A(yA, 1) \cong 1_{\mathscr{P}A}$; so if $z: \mathscr{P}A \to C$ is a left adjoint for $C(s, 1)$ then the unit of the adjunction exhibits z as an absolute left lifting of $\mathscr{P}A(yA, 1)$ through $C(s, 1)$. Proposition 20 applies, and so there is an isomorphism

$$\begin{tikzcd} A \arrow[rr, "yA"] \arrow[dr, "s"'] & \arrow[d, phantom, "\cong"] & PA \arrow[dl, "z"] \\ & C & \end{tikzcd}$$

If further z, $\mathscr{P}A$ are admissible then by Proposition 8

$$\mathscr{P}A(\mathscr{P}A(yA, 1), C(s, 1)) \cong \mathscr{P}A(1, C(s, 1)) \cong C(z, 1),$$

so z is a pointwise left extension of s along yA.

An object C is called *total* when $1: C \to C$ is total; that is, when C is admissible and $yC: C \to \mathscr{P}C$ has a left adjoint. From the second paragraph of this section and Proposition 21 we see that, when C is total, an arrow $h: C \to D$ in $\mathscr{L}$ has a right adjoint if and only if $D(h, 1)$ is admissible and $h\,\mathrm{col}(D(h, 1), 1)$ is a $D(h, 1)$-indexed colimit for h. This is a very satisfactory adjoint-functor theorem for arrows out of a total object. (The term "absolutely cocomplete" used in Street [23, p. 154] has been replaced here by "total" in view of the present connotations of the word "absolute.")

An arrow $t: A \to C$ is called *kan* when A, C are admissible and $\mathscr{P}t$ has a left adjoint, denoted $\exists t$. From the definition of $\mathscr{P}t$ and the third paragraph of this section (putting $s = C(1, t)$ and $z = \exists t$) we have that $\exists t.yA \cong yC.t$. If further $\mathscr{P}A$, $\exists t$ are admissible then we have that $\exists t$ is precisely a pointwise left extension of $C(1, t)$ along yA.

$$\begin{tikzcd} A \arrow[r, "yA"] \arrow[d, "t"'] \arrow[dr, phantom, "\overset{\cong}{\Rightarrow}"] & PA \arrow[d, "\exists t"] \\ C \arrow[r, "yC"'] & PC \end{tikzcd}$$

PROPOSITION 25. *If C is total and $s: A \to C$ is kan then $s: A \to C$ is total. Consequently,* $\mathrm{col}(j, s)$ *exists for all admissible $j: M \to \mathscr{P}A$, with M admissible.*

Proof. By Axiom 3, we have $C(s, 1) \cong \mathscr{P}s.yC$; but $\mathscr{P}s$, yC are assumed to have left adjoints, and the left adjoint of yC is admissible. ∎

PROPOSITION 26. *Suppose (C, g) is a comonad for which a universal g-coalgebra*

$$\begin{tikzcd} C^g \arrow[r, "u"] \arrow[dr, "u"'] & C \arrow[d, "g"] \arrow[dl, phantom, "\overset{\delta}{\Rightarrow}"] \\ & C \end{tikzcd}$$

exists with u total. Then C^g is total.

Proof. By Axiom 1, the composite

$$C^g(1, 1) \xrightarrow[x^u]{} C(u, u) \xrightarrow[C(1,\delta)]{} C(u, gu)$$

corresponds to a 2-cell $C(u, 1) \to C(u, 1)g$. Since $\mathrm{col}(1, u) \dashv C(u, 1)$, this latter 2-cell corresponds to a 2-cell

$$\begin{tikzcd} \mathscr{P}C^g \arrow[r, "\mathrm{col}(1, u)"] \arrow[dr, "\mathrm{col}(1, u)"'] & C \arrow[d, "g"] \arrow[dl, phantom, "\overset{\bar{\delta}}{\Rightarrow}"] \\ & C \end{tikzcd}$$

The g-coalgebra properties of δ transfer to $\bar{\delta}$, and so $\mathrm{col}(1, u)$, $\bar{\delta}$ induce an arrow $\mathscr{P}C^g \to C^g$ which can be seen to be a left adjoint for yC^g. ▮

PROPOSITION 27. *If $u: A \to C$ is fully faithful, total, and has a left or right adjoint, then A is total.*

Proof. If $f \dashv u$ then $fz \dashv C(u, 1)u$ where $z \dashv C(u, 1)$. But $yA \cong C(u, 1)u$ and so has a left adjoint. So A is total.

If $u \dashv v$, since u is fully faithful it composes with the unit of the adjunction to yield a universal coalgebra

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[dr, "u"'] & C \arrow[d, "g"] \arrow[dl, phantom, "\Rightarrow"] \\ & C \end{tikzcd}$$

for the comonad $g = uv$ on C. So A is total by Proposition 26. ▮

For admissible A and C, Axioms 1 and 2 imply that the functor

$$C(-, 1): \mathscr{L}(A, C)^{\mathrm{op}} \to \mathscr{K}(C, \mathscr{P}A)$$

is fully faithful. Suppose $g\colon C \to \mathscr{P}A$ is an arrow with a left adjoint h with unit λ. Both left- and right-hand triangles of the diagram

$$\begin{tikzcd}
A \arrow[r, "yA"] \arrow[dr, "yA"'] & PA \arrow[r, "h"] \arrow[d, "1"] \arrow[dl, phantom, "="] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & C \arrow[dl, "g"] \\
& PA &
\end{tikzcd}$$

have the left extension property and hence so has outer triangle. So from Axiom 1, $g \simeq C(h.yA, 1)$. Thus if $\mathscr{T}\!ot(A, C)$ denotes the full subcategory of $\mathscr{K}(A, C)$ consisting of the total arrows, and if $\mathscr{A}dj(C, \mathscr{P}A)$ denotes the full subcategory of $\mathscr{K}(C, \mathscr{P}A)$ consisting of those arrows with left adjoints, then $C(\text{—}, 1)$ induces an equivalence of categories

$$\mathscr{T}\!ot(A, C)^{\mathrm{op}} \simeq \mathscr{A}dj(C, \mathscr{P}A).$$

7. Examples

(1) *Hom-Enriched Categories*

Let $\mathscr{V}$ denote a closed category (see Street [23, p. 157]) which is complete with respect to a category $\mathscr{S}et$ of sets. Take $\mathscr{K}$ to be the 2-category of $\mathscr{V}$-categories (see Eilenberg and Kelly [7, pp. 466–467]) whose sets of objects are in $\mathscr{S}et$.

Let V denote any set of objects of $\mathscr{V}$ which is in $\mathscr{S}et$. (Regard V as a full subcategory of $\mathscr{V}$.)

For any object A of $\mathscr{K}$, a *V-attribute F of type A* assigns to each object a of A an object Fa of V, and to each pair of objects a, a' of A an arrow

$$F_{a,a'}\colon Fa \to [A(a', a), Fa']$$

in $\mathscr{V}$, such that the following diagrams commute.

$$\begin{tikzcd}
Fa \arrow[r, "F_{a,a}"] \arrow[dr, "i"'] & {[A(a,a),Fa]} \arrow[d, "{[j,1]}"] \\
& {[I,Fa]}
\end{tikzcd}
\qquad
\begin{tikzcd}
Fc \arrow[rr, "F_{c,b}"] \arrow[d, "F_{c,a}"'] & & {[A(b,c),Fb]} \arrow[d, "{[1,F_{b,a}]}"] \\
{[A(a,c),Fa]} \arrow[d, "L^{A(a,b)}"'] & & {[A(b,c),[A(a,b),Fa]]} \\
{[[A(a,b),A(a,c)],[A(a,b),Fa]]} \arrow[urr, "{[L^a,1]}"'] & &
\end{tikzcd}$$

It is clear that the V-attributes of type A form a set in $\mathscr{S}et$. For V-attributes F, G of type A, let $[F, G]$ denote an inverse limit in v of the diagram:

$$[Fa, Ga] \xrightarrow[{[1, G_{a,b}]}]{} [Fa, [A(b, a), Gb]] \xleftarrow[{[F_{a,b}, 1]}]{} [[A(b, a), Fb], [A(b, a), Gb]]$$

$$\xleftarrow[L^{A(b,a)}]{} [Fb, Gb]$$

where (a, b) runs over the set of pairs of objects of A. It can be shown that a v-category $\mathscr{P}A$ is obtained whose objects are the V-attributes of type A, whose v-valued homs are given by $(\mathscr{P}A)(F, G) = [F, G]$, and whose "identity" and "composition" arrows are induced along the projections $[F, G] \to [Fa, Ga]$ by those of v. So $\mathscr{P}A$ is an object of $\mathscr{K}$.

An arrow $f\colon A \to B$ is called *admissible* when $B(fa, b)$ is in V for all objects a of A and b of B. The admissible objects form a right ideal in $|\mathscr{K}|$. If $f\colon A \to B$ is admissible and b is an object of B, we can define a V-attribute $B(f, b)$ of type A by $B(f, b)a = B(fa, b)$ and

$$B(f, b)_{a,a'} = (B(fa, b) \xrightarrow[L^{fa'}]{} [B(fa', fa), B(fa, b)]$$

$$\xrightarrow[{[f_{a',a}, 1]}]{} [A(a', a), B(fa', b)]).$$

In fact, the arrows $L^{fa}_{b,b'}\colon B(b, b') \to [B(fa, b), B(fa, b')]$ in v induce arrows $B(f, 1)_{b,b'}\colon B(b, b') \to [B(f, b), B(f, b')]$. The latter arrows are the effect on homs of a v-functor $B(f, 1)\colon B \to \mathscr{P}A$ whose value at the object b of B is $B(f, b)$. In particular, when A is admissible, we have a v-functor $yA = A(1, 1)\colon A \to \mathscr{P}A$. Furthermore, the arrows $I \to_j [A(a', a), A(a', a)] \to_{[1, f_{a'a}]} [A(a', a), B(fa', fa)]$ induce arrows $I \to [A(1, a), B(f, fa)]$ which, of course, are arrows $(yA)a = A(1, a) \to B(f, fa)$ in $\mathscr{P}A$. These are the components of a 2-cell

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "yA"'] & \arrow[d, phantom, "\chi^f \Rightarrow"] & B \arrow[dl, "{B(f,1)}"] \\ & PA & \end{tikzcd}$$

in $\mathscr{K}$. Axioms 1, 2, 3 can be verified. Axiom 3* is not usually satisfied.

The Yoneda structure on $\mathscr{K}$ described above is called the *V-attribute structure on v-$\mathscr{C}at$*. It should be noted that $\mathscr{P}A$ is defined for all objects A of $\mathscr{K}$, not just admissible ones, and that $\mathscr{P}$ as obtained in Section 2 extends to a functor $\mathscr{P}\colon \mathscr{K}^{\mathrm{coop}} \to \mathscr{K}$. What is more, this $\mathscr{P}$ has a left 2-adjoint $\mathscr{P}^*\colon \mathscr{K} \to \mathscr{K}^{\mathrm{coop}}$; the objects of the v-category $\mathscr{P}^*A$ are those v-functors $A \to v$ which take objects of A to objects of V. Thus $\mathscr{P}$ preserves indexed limits and $(\mathscr{P}A)^{\mathscr{P}s}$ can be obtained as $\mathscr{P}(A^s)$ in Proposition 22.

The case where v is monoidal rather than closed does not have to be treated separately. Indeed, if v is a complete promonoidal (= premonoidal as in Day [3]) category, it can be regarded as a full promonoidal subcategory of a complete monoidal biclosed category $\hat{v}$ for which the inclusion is continuous (for example, using the convolution structure of Day [3]). Any full subcategory V of v is also a full subcategory of $\hat{v}$; the V-attribute structure on $\hat{v}$-$\mathscr{C}at$ allows us to deal with the appropriately defined V-attribute structure on v-$\mathscr{C}at$. Undoubtedly, to take v to be multilinear (in the sense of Linton [19]) would also give the desired generality (no more!); after all, the structure V inherits from v is multilinear.

(2) *Internal Categories*

Let $\mathscr{E}$ denote a finitely complete, Cartesian-closed category and take $\mathscr{K}$ to be the 2-category $\mathscr{C}at(\mathscr{E})$ of category objects in $\mathscr{E}$ (see Street [22], for example). Let $\mathscr{P}rof(A, B; \mathscr{E})$ denote the category of internal profunctors (see the Appendix of Kock and Wraith [15]) between the objects A, B of $\mathscr{K}$. There is an equivalence of categories

$$\mathscr{P}rof(A, B; \mathscr{E}) \simeq \mathscr{P}rof(1, A^{\mathrm{op}} \times B; \mathscr{E}).$$

An *internal full subcategory of* $\mathscr{E}$ is an object S of $\mathscr{K}$ together with an internal profunctor U from 1 to S which induces a fully faithful functor

$$\mathscr{K}(B, S) \to \mathscr{P}rof(1, B; \mathscr{E})$$

for all B of $\mathscr{K}$. An arrow $F: A \to B$ is called *admissible* (relative to the given internal full subcategory), when the profunctor from 1 to $A^{\mathrm{op}} \times B$ corresponding to the profunctor f/B from A to B is isomorphic to an object in the image of the functor

$$\mathscr{K}(A^{\mathrm{op}} \times B, S) \to \mathscr{P}rof(1, A^{\mathrm{op}} \times B; \mathscr{E}).$$

When such an object $A^{\mathrm{op}} \times B \to S$ exists, let $B(f, 1): B \to [A^{\mathrm{op}}, S]$ denote its exponential adjoint. It is shown elsewhere (Street [25]) that this structure gives a cosmos in the sense of Street [23] with $\mathscr{K}$ as the 2-category and $\mathscr{P}A = [A^{\mathrm{op}}, S]$. Theorems 6 and 7 of Street [23] show that we also have a Yoneda structure on $\mathscr{K}$. Axiom 3* holds in this example.

In particular, when $\mathscr{E}$ is a topos and S is the subobject classifier, the Yoneda structure amounts to one for which the admissible objects are the preordered objects. When $\mathscr{E}$ is a topos with a natural numbers object and S is the internal full subcategory of "finite objects," the Yoneda structure amounts to one for which the admissible objects are the category objects "with finite homs."

(3) *Finitely Complete Categories*

Example (1) can presumably be internalized to a Cartesian closed category $\mathscr{E}$ so as to include Example (2). On the other hand, the intersection of Examples (1) and (2) contains the paradigmatic Yoneda structure where $\mathscr{V} = \mathscr{E} = \mathscr{S}et$, so that $\mathscr{K} = \mathscr{C}at$, and $V = S$ is a universe in $\mathscr{S}et$. Yoneda structures often induce Yoneda structures on 2-categories of algebras for a doctrine (= 2-monad) on the 2-category supporting the original Yoneda structure. We shall present here one simple example of this in the case of the doctrine for finite limits on $\mathscr{C}at$.

Let $\mathscr{L}ex$ denote the sub-2-category of $\mathscr{C}at$ consisting of the finitely complete objects, the left exact functors, and all 2-cells between such functors. The data for the S-attribute structure on $\mathscr{C}at$ all restrict directly to $\mathscr{L}ex$. For admissible A and $f\colon A \to B$ in $\mathscr{L}ex$, the diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "yA"'] & \arrow[d, phantom, "\chi^f \Rightarrow"] & B \arrow[dl, "B(f,1)"] \\ & PA & \end{tikzcd}$$

belongs to $\mathscr{L}ex$. Since $\mathscr{L}ex$ is locally full in $\mathscr{C}at$, Axioms 1 and 2 are immediate for $\mathscr{L}ex$. Observe that a 2-cell

$$\begin{tikzcd} & A \arrow[dl, "s"'] \arrow[dr, "j"] \arrow[d, phantom, "\eta \Leftarrow"] & \\ B \arrow[rr, "t"'] & {} & C \end{tikzcd}$$

has the absolute left lifting property in $\mathscr{L}ex$ if and only if it does in $\mathscr{C}at$. For, suppose it does in $\mathscr{L}ex$. Let L denote the free category with finite limits on 1. An object a of A gives a left exact functor $\bar{a}\colon L \to A$ and so $\eta\bar{a}$ exhibits $s\bar{a}$ as a left lifting of $j\bar{a}$ through t. So sa is a local left adjoint for t at ja. So η has the absolute left lifting property in $\mathscr{C}at$. Axiom 3* is a consequence of this.

We mention an application of Section 6 to this example. Suppose A is a small object of $\mathscr{L}ex$. (When S is a universe this does amount to the condition that the set of all arrows of A should be in S.) By Proposition 27, any reflective subcategory of $[A^{\mathrm{op}}, S]$ for which the reflection is left exact is total. Thus every Grothendieck S-topos (in the sense of [11]) on a finitely complete site is total in $\mathscr{L}ex$. This is part of a theorem of Giraud.

Conversely, one can show that any well-powered total object of $\mathscr{L}ex$ is a cocomplete elementary topos. Further results concerning this example will be given elsewhere.

We could also take S to be the set with two elements in the above. The total objects in the resulting Yoneda structure on $\mathscr{L}ex$ are the complete heyting algebras.

(4) *Compact 2-Categories*

A *compact bicategory* is a bicategory $\mathscr{K}$ in which every arrow has a right adjoint. (Compact monoidal categories have been considered by Kelly [14, Section 102]; this is the case where the bicategory has but one object).

A Yoneda structure is obtained on any compact 2-category $\mathscr{K}$ by taking all arrows to be admissible, $yA: A \to \mathscr{P}A$ to be the identity of A, and $B(f, 1)$ to be a right adjoint for f. Axiom 3* is valid.

Any groupoid can be regarded as a compact 2-category with only identity 2-cells.

(5) *A Trivial Example*

Let $\mathscr{K}$ be any 2-category with a terminal object $*$. An arrow $f: A \to B$ is *admissible* when, for all arrows $a: K \to A$, $b: K \to B$, there is precisely one 2-cell $fa \Rightarrow b$. There is a Yoneda structure on $\mathscr{K}$ for which $\mathscr{P}A = *$ for all (*admissible*) A. Axiom 3* holds.

References

1. C. Auderset, Adjonctions et monades au niveau des 2-categories, *Cahiers Topologie Géom. Différentielle* **15** (1974), 3–20.
2. F. Borceux and G. M. Kelly, A notion of limit for enriched categories, *Bull. Australian Math. Soc.* **12** (1975), 49–72.
3. B. J. Day, "On Closed Categories of Functors," pp. 1–38, Lecture Notes in Mathematics **137**, Springer–Verlag, Berlin, 1970. 1–38.
4. B. J. Day and G. M. Kelly, "Enriched Functor Categories," pp. 178–191, Lecture Notes in Mathematics 106, Springer–Verlag, Berlin, 1969.
5. R. Diaconescu, Change of base for toposes with generators, *J. Pure Appl. Algebra* **6** (1975), 191–218.
6. E. Dubuc, "Kan Extensions in Enriched Category Theory," Lecture Notes in Mathematics 145, Springer–Verlag, Berlin, 1970.
7. S. Eilenberg and G. M. Kelly, Closed categories, *in* "Proceedings Conference on Categorical Algebra at La Jolla," pp. 421–562, Springer–Verlag, Berlin, 1966.
8. S. Eilenberg and J. C. Moore, Adjoint functors and triples, *Illinois J. Math.* **9** (1965), 381–398.
9. P. Freyd, "Several New Concepts," pp. 196–241, Lecture Notes in Mathematics 99, Springer–Verlag, Berlin, 1969.
10. P. Gabriel and F. Ulmer, "Lokal präsentierbare Kategorien," Lecture Notes in Mathematics **221**, Springer–Verlag, Berlin, 1971.
11. A. Grothendieck et J. L. Verdier, "Topos," pp. 299–519, Lecture Notes in Mathematics 269, Springer–Verlag, Berlin, 1972.
12. P. Huber, Standard constructions in Abelian categories, *Math. Ann.* **146** (1962), 321–325.
13. G. M. Kelly, "Adjunction for Enriched Categories," pp. 166–177, Lecture Notes in Mathematics 106, Springer–Verlag, Berlin, 1969.
14. G. M. Kelly, "Many variable functorial calculus. I," pp. 66–105, Lecture Notes in Mathematics 281, Springer–Verlag, 1972.

15. A. Kock and G. C. Wraith, "Elementary Toposes," Aarhus Lecture Notes 30, 1971.
16. F. W. Lawvere, Equality in hyperdoctrines and comprehension schema as an adjoint functor, *in* "Proceedings of Symposia in Pure Mathematics, 17," pp. 1–14, Amer. Math. Soc., Providence, R.I., 1970.
17. F. W. Lawvere, "Metric Spaces, Generalised Logic, and Closed Categories," Preprint from Instituto di Mathematica, Universita di Perguia, 1973.
18. F. E. J. Linton, "An Outline of Functorial Semantics," Lecture Notes in Mathematics 80, pp. 7–52, Springer–Verlag, Berlin, 1969.
19. F. E. J. Linton, The multilinear Yoneda lemmas, pp. 209–229, Lecture Notes in Mathematics 195, Springer–Verlag, Berlin, 1971.
20. S. Mac Lane, Categories for the Working Mathematician, Graduate Texts in Mathematics 5, Springer–Verlag, Berlin, 1971.
21. R. H. Street, The formal theory of monads, *J. Pure Appl. Algebra* 2 (1972), 149–168.
22. R. H. Street, Fibrations and Yoneda's lemma in a 2-category, pp. 104–133, Lecture Notes in Mathematics 420, Springer–Verlag, Berlin, 1974.
23. R. H. Street, Elementary cosmoi, pp. 134–180, Lecture Notes in Mathematics 420, Springer–Verlag, Berlin, 1974.
24. R. H. Street, Limits indexed by category-valued 2-functors, *J. Pure Appl. Algebra* 8 (1976), 149–181.
25. R. H. Street, Cosmoi of internal categories (Preprint, 1976).
26. F. Ulmer, Properties of dense and relative adjoint functors, *J. Algebra* 3 (1968), 77–95.
27. F. Ulmer, The adjoint functor theorem and the Yoneda embedding, *Illinois J. Math.* 15 (1971), 355–361.