

Functors and natural transformations

<i>functors</i>	\rightsquigarrow	<i>category morphisms</i>
<i>natural transformations</i>	\rightsquigarrow	<i>functor morphisms</i>

Functors

A *functor* $\mathbf{F} : \mathbf{K} \rightarrow \mathbf{K}'$ from a category \mathbf{K} to a category \mathbf{K}' consists of:

- a function $\mathbf{F} : |\mathbf{K}| \rightarrow |\mathbf{K}'|$, and
- for all $A, B \in |\mathbf{K}|$, a function $\mathbf{F} : \mathbf{K}(A, B) \rightarrow \mathbf{K}'(\mathbf{F}(A), \mathbf{F}(B))$

such that:

Make explicit categories in which we work at various places here

- \mathbf{F} preserves identities, i.e.,

$$\mathbf{F}(id_A) = id_{\mathbf{F}(A)}$$

for all $A \in |\mathbf{K}|$, and

- \mathbf{F} preserves composition, i.e.,

$$\mathbf{F}(f;g) = \mathbf{F}(f);\mathbf{F}(g)$$

for all $f : A \rightarrow B$ and $g : B \rightarrow C$ in \mathbf{K} .

We really should differentiate between various components of F

Examples

- *identity functors*: $\text{Id}_{\mathbf{K}} : \mathbf{K} \rightarrow \mathbf{K}$, for any category \mathbf{K}
- *inclusions*: $\mathbf{I}_{\mathbf{K} \hookrightarrow \mathbf{K}'} : \mathbf{K} \rightarrow \mathbf{K}'$, for any subcategory \mathbf{K} of \mathbf{K}'
- *constant functors*: $\mathbf{C}_A : \mathbf{K} \rightarrow \mathbf{K}'$, for any categories \mathbf{K}, \mathbf{K}' and $A \in |\mathbf{K}'|$, with $\mathbf{C}_A(f) = \text{id}_A$ for all morphisms f in \mathbf{K}
- *powerset functor*: $\mathbf{P} : \mathbf{Set} \rightarrow \mathbf{Set}$ given by
 - $\mathbf{P}(X) = \{Y \mid Y \subseteq X\}$, for all $X \in |\mathbf{Set}|$
 - $\mathbf{P}(f) : \mathbf{P}(X) \rightarrow \mathbf{P}(X')$ for all $f : X \rightarrow X'$ in \mathbf{Set} ,
 $\mathbf{P}(f)(Y) = \{f(y) \mid y \in Y\}$ for all $Y \subseteq X$
- *contravariant powerset functor*: $\mathbf{P}_{-1} : \mathbf{Set}^{op} \rightarrow \mathbf{Set}$ given by
 - $\mathbf{P}_{-1}(X) = \{Y \mid Y \subseteq X\}$, for all $X \in |\mathbf{Set}|$
 - $\mathbf{P}_{-1}(f) : \mathbf{P}(X') \rightarrow \mathbf{P}(X)$ for all $f : X \rightarrow X'$ in \mathbf{Set} ,
 $\mathbf{P}_{-1}(f)(Y') = \{x \in X \mid f(x) \in Y'\}$ for all $Y' \subseteq X'$

Examples, cont'd.

- *projection functors*: $\pi_1 : \mathbf{K} \times \mathbf{K}' \rightarrow \mathbf{K}$, $\pi_2 : \mathbf{K} \times \mathbf{K}' \rightarrow \mathbf{K}'$
- *list functor*: $\mathbf{List} : \mathbf{Set} \rightarrow \mathbf{Monoid}$, where \mathbf{Monoid} is the category of monoids (as objects) with monoid homomorphisms as morphisms:
 - $\mathbf{List}(X) = \langle X^*, \wedge, \epsilon \rangle$, for all $X \in |\mathbf{Set}|$, where X^* is the set of all finite lists of elements from X , \wedge is the list concatenation, and ϵ is the empty list.
 - $\mathbf{List}(f) : \mathbf{List}(X) \rightarrow \mathbf{List}(X')$ for $f : X \rightarrow X'$ in \mathbf{Set} ,
 $\mathbf{List}(f)(\langle x_1, \dots, x_n \rangle) = \langle f(x_1), \dots, f(x_n) \rangle$ for all $x_1, \dots, x_n \in X$
- *totalisation functor*: $\mathbf{Tot} : \mathbf{Pfn} \rightarrow \mathbf{Set}_*$, where \mathbf{Set}_* is the subcategory of \mathbf{Set} of sets with a distinguished element $*$ and $*$ -preserving functions
 - $\mathbf{Tot}(X) = X \uplus \{*\}$
 - $\mathbf{Tot}(f)(x) = \begin{cases} f(x) & \text{if it is defined} \\ * & \text{otherwise} \end{cases}$

Define \mathbf{Set}_* as the category of algebras

Examples, cont'd.

- *carrier set functors*: $|-| : \mathbf{Alg}(\Sigma) \rightarrow \mathbf{Set}^S$, for any algebraic signature $\Sigma = \langle S, \Omega \rangle$, yielding the algebra carriers and homomorphisms as functions between them
- *reduct functors*: $-|_{\sigma} : \mathbf{Alg}(\Sigma') \rightarrow \mathbf{Alg}(\Sigma)$, for any signature morphism $\sigma : \Sigma \rightarrow \Sigma'$, as defined earlier
- *term algebra functors*: $\mathbf{T}_{\Sigma} : \mathbf{Set} \rightarrow \mathbf{Alg}(\Sigma)$ for all (single-sorted) algebraic signatures $\Sigma \in |\mathbf{AlgSig}|$

Generalise to many-sorted signatures

 - $\mathbf{T}_{\Sigma}(X) = T_{\Sigma}(X)$ for all $X \in |\mathbf{Set}|$
 - $\mathbf{T}_{\Sigma}(f) = f^{\#} : T_{\Sigma}(X) \rightarrow T_{\Sigma}(X')$ for all functions $f : X \rightarrow X'$
- *diagonal functors*: $\Delta_{\mathbf{K}}^G : \mathbf{K} \rightarrow \mathbf{Diag}_{\mathbf{K}}^G$ for any graph G with nodes $N = |G|_{nodes}$ and edges $E = |G|_{edges}$, and category \mathbf{K}
 - $\Delta_{\mathbf{K}}^G(A) = D^A$, where D^A is the “constant” diagram, with $D_n^A = A$ for all $n \in N$ and $D_e^A = id_A$ for all $e \in E$
 - $\Delta_{\mathbf{K}}^G(f) = \mu^f : D^A \rightarrow D^B$, for all $f : A \rightarrow B$, where $\mu_n^f = f$ for all $n \in N$

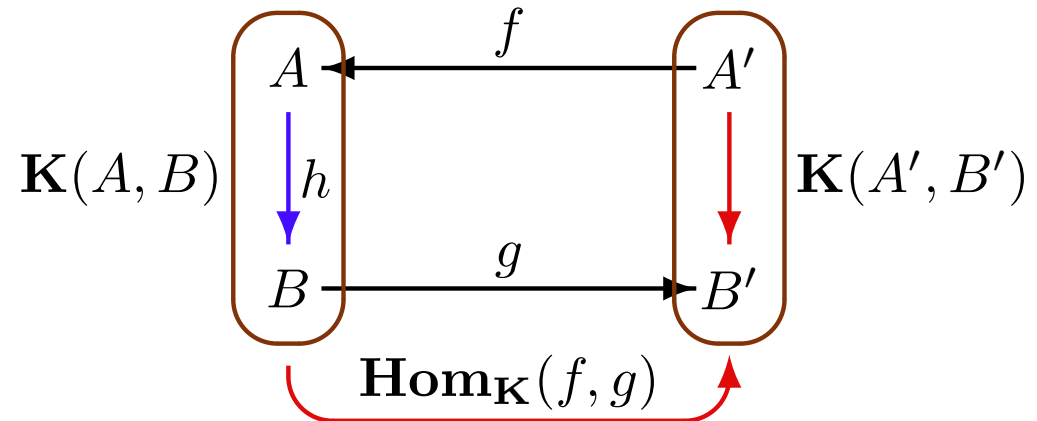
Hom-functors

Given a *locally small* category \mathbf{K} , define

$$\mathbf{Hom}_{\mathbf{K}} : \mathbf{K}^{op} \times \mathbf{K} \rightarrow \mathbf{Set}$$

a binary *hom-functor*, contravariant on the first argument and covariant on the second argument, as follows:

- $\mathbf{Hom}_{\mathbf{K}}(\langle A, B \rangle) = \mathbf{K}(A, B)$, for all $\langle A, B \rangle \in |\mathbf{K}^{op} \times \mathbf{K}|$, i.e., $A, B \in |\mathbf{K}|$
- $\mathbf{Hom}_{\mathbf{K}}(\langle f, g \rangle) : \mathbf{K}(A, B) \rightarrow \mathbf{K}(A', B')$, for $\langle f, g \rangle : \langle A, B \rangle \rightarrow \langle A', B' \rangle$ in $\mathbf{K}^{op} \times \mathbf{K}$, i.e., $f : A' \rightarrow A$ and $g : B \rightarrow B'$ in \mathbf{K} , as a function given by $\mathbf{Hom}_{\mathbf{K}}(\langle f, g \rangle)(h) = f;g;h$.



Also: $\mathbf{Hom}_{\mathbf{K}}(A, -) : \mathbf{K} \rightarrow \mathbf{Set}$
 $\mathbf{Hom}_{\mathbf{K}}(-, B) : \mathbf{K}^{op} \rightarrow \mathbf{Set}$

Functors preserve...

- Check whether functors preserve:
 - monomorphisms
 - epimorphisms
 - (co)retractions
 - isomorphisms
 - (co)cones
 - (co)limits
 - ...
- A functor is (finitely) continuous if it preserves all existing (finite) limits.
Which of the above functors are (finitely) continuous?

Dualise!

Functors compose...

Given two functors $\mathbf{F} : \mathbf{K} \rightarrow \mathbf{K}'$ and $\mathbf{G} : \mathbf{K}' \rightarrow \mathbf{K}''$, their *composition* $\mathbf{F};\mathbf{G} : \mathbf{K} \rightarrow \mathbf{K}''$ is defined as expected:

- $(\mathbf{F};\mathbf{G})(A) = \mathbf{G}(\mathbf{F}(A))$ for all $A \in |\mathbf{K}|$
- $(\mathbf{F};\mathbf{G})(f) = \mathbf{G}(\mathbf{F}(f))$ for all $f : A \rightarrow B$ in \mathbf{K}

\mathbf{Cat} , the category of (sm)all categories

- objects: (sm)all categories
- morphisms: functors between them
- composition: as above

Characterise isomorphisms in \mathbf{Cat}

Define products, terminal objects, equalisers and pullback in \mathbf{Cat}

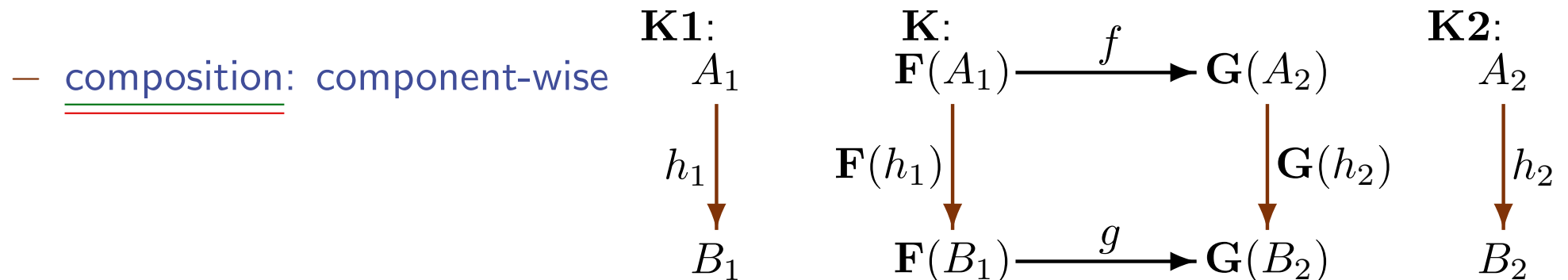
Try to define their duals

Comma categories

Given two functors with a common target, $\mathbf{F} : \mathbf{K1} \rightarrow \mathbf{K}$ and $\mathbf{G} : \mathbf{K2} \rightarrow \mathbf{K}$, define their *comma category*

(\mathbf{F}, \mathbf{G})

- objects: triples $\langle A_1, f : \mathbf{F}(A_1) \rightarrow \mathbf{G}(A_2), A_2 \rangle$, where $A_1 \in |\mathbf{K1}|$, $A_2 \in |\mathbf{K2}|$, and $f : \mathbf{F}(A_1) \rightarrow \mathbf{G}(A_2)$ in \mathbf{K}
- morphisms: a morphism in (\mathbf{F}, \mathbf{G}) is any pair $\langle h_1, h_2 \rangle : \langle A_1, f : \mathbf{F}(A_1) \rightarrow \mathbf{G}(A_2), A_2 \rangle \rightarrow \langle B_1, g : \mathbf{F}(B_1) \rightarrow \mathbf{G}(B_2), B_2 \rangle$, where $h_1 : A_1 \rightarrow B_1$ in $\mathbf{K1}$, $h_2 : A_2 \rightarrow B_2$ in $\mathbf{K2}$, and $\mathbf{F}(h_1);g = f;\mathbf{G}(h_2)$ in \mathbf{K} .



Examples

- The category of graphs as a comma category:

$$\mathbf{Graph} = (\mathbf{Id}_{\mathbf{Set}}, \mathbf{CP})$$

where $\mathbf{CP} : \mathbf{Set} \rightarrow \mathbf{Set}$ is the (Cartesian) product functor ($\mathbf{CP}(X) = X \times X$ and $\mathbf{CP}(f)(\langle x, x' \rangle) = \langle f(x), f(x') \rangle$). **Hint:** write objects of this category as $\langle E, \langle source, target \rangle : E \rightarrow N \times N, N \rangle$

- The category of algebraic signatures as a comma category:

$$\mathbf{AlgSig} = (\mathbf{Id}_{\mathbf{Set}}, (-)^+)$$

where $(-)^+ : \mathbf{Set} \rightarrow \mathbf{Set}$ is the non-empty list functor ($(X)^+$ is the set of all non-empty lists of elements from X , $(f)^+(\langle x_1, \dots, x_n \rangle) = \langle f(x_1), \dots, f(x_n) \rangle$). **Hint:** write objects of this category as $\langle \Omega, \langle arity, sort \rangle : \Omega \rightarrow S^+, S \rangle$

Define \mathbf{K}^{\rightarrow} , $\mathbf{K} \downarrow A$ as comma categories. The same for $\mathbf{Alg}(\Sigma)$.

Cocompleteness of comma categories

Fact: *If $\mathbf{K1}$ and $\mathbf{K2}$ are (finitely) cocomplete categories, $\mathbf{F} : \mathbf{K1} \rightarrow \mathbf{K}$ is a (finitely) cocontinuous functor, and $\mathbf{G} : \mathbf{K2} \rightarrow \mathbf{K}$ is a functor then the comma category (\mathbf{F}, \mathbf{G}) is (finitely) cocomplete.*

Proof (idea):

Construct coproducts and coequalisers in (\mathbf{F}, \mathbf{G}) , using the corresponding constructions in $\mathbf{K1}$ and $\mathbf{K2}$, and cocontinuity of \mathbf{F} .

*State and prove the dual fact,
concerning completeness of comma categories*

Coproducts:

$$\begin{array}{ccccc}
 & A_1 & \mathbf{F}(A_1) & \xrightarrow{f} & \mathbf{G}(A_2) & A_2 \\
 & \downarrow \iota_{A_1} & \downarrow \mathbf{F}(\iota_{A_1}) & & \downarrow \mathbf{G}(\iota_{A_2}) & \downarrow \iota_{A_2} \\
 A_1 + B_1 & & \mathbf{F}(A_1 + B_1) & \xrightarrow{\quad} & \mathbf{G}(A_2 + B_2) & A_2 + B_2 \\
 & \uparrow \iota_{B_1} & \uparrow \mathbf{F}(\iota_{B_1}) & & \uparrow \mathbf{G}(\iota_{B_2}) & \uparrow \iota_{B_2} \\
 & B_1 & \mathbf{F}(B_1) & \xrightarrow{g} & \mathbf{G}(B_2) & B_2
 \end{array}$$

Coequalisers:

$$\begin{array}{ccccc}
 A_1 & \mathbf{F}(A_1) & \xrightarrow{f} & \mathbf{G}(A_2) & A_2 \\
 \downarrow h_1 \quad \downarrow h'_1 & \downarrow \mathbf{F}(h_1) \quad \downarrow \mathbf{F}(h'_1) & & \downarrow \mathbf{G}(h_2) \quad \downarrow \mathbf{G}(h'_2) & \downarrow h_2 \quad \downarrow h'_2 \\
 B_1 & \mathbf{F}(B_1) & \xrightarrow{g} & \mathbf{G}(B_2) & B_2 \\
 \downarrow c_1 & \downarrow \mathbf{F}(c_1) & & \downarrow \mathbf{G}(c_2) & \downarrow c_2 \\
 C_1 & \mathbf{F}(C_1) & \xrightarrow{\quad} & \mathbf{G}(C_2) & C_2
 \end{array}$$

Indexed categories

An *indexed category* is a functor

$$\mathcal{C} : \mathbf{Ind}^{op} \rightarrow \mathbf{Cat}$$

Standard example: $\mathbf{Alg} : \mathbf{AlgSig}^{op} \rightarrow \mathbf{Cat}$

The Grothendieck construction: Given $\mathcal{C} : \mathbf{Ind}^{op} \rightarrow \mathbf{Cat}$, define a category $\mathbf{Flat}(\mathcal{C})$:

- objects: $\langle i, A \rangle$ for all $i \in |\mathbf{Ind}|$, $A \in |\mathcal{C}(i)|$
- morphisms: a morphism from $\langle i, A \rangle$ to $\langle j, B \rangle$, $\langle \sigma, f \rangle : \langle i, A \rangle \rightarrow \langle j, B \rangle$, consists of a morphism $\sigma : i \rightarrow j$ in \mathbf{Ind} and a morphism $f : A \rightarrow \mathcal{C}(\sigma)(B)$ in $\mathcal{C}(i)$
- composition: given $\langle \sigma, f \rangle : \langle i, A \rangle \rightarrow \langle i', A' \rangle$ and $\langle \sigma', f' \rangle : \langle i', A' \rangle \rightarrow \langle i'', A'' \rangle$, their composition in $\mathbf{Flat}(\mathcal{C})$, $\langle \sigma, f \rangle; \langle \sigma', f' \rangle : \langle i, A \rangle \rightarrow \langle i'', A'' \rangle$, is given by

$$\langle \sigma, f \rangle; \langle \sigma', f' \rangle = \langle \sigma; \sigma', f; \mathcal{C}(\sigma)(f') \rangle$$

Fact: If \mathbf{Ind} is complete, $\mathcal{C}(i)$ are complete for all $i \in |\mathbf{Ind}|$, and $\mathcal{C}(\sigma)$ are continuous for all $\sigma : i \rightarrow j$ in \mathbf{Ind} , then $\mathbf{Flat}(\mathcal{C})$ is complete.

Try to formulate and prove a theorem concerning cocompleteness of $\mathbf{Flat}(\mathcal{C})$

Natural transformations

Given two parallel functors $\mathbf{F}, \mathbf{G} : \mathbf{K} \rightarrow \mathbf{K}'$, a *natural transformation* from \mathbf{F} to \mathbf{G}

$$\tau : \mathbf{F} \rightarrow \mathbf{G}$$

is a family $\tau = \langle \tau_A : \mathbf{F}(A) \rightarrow \mathbf{G}(A) \rangle_{A \in |\mathbf{K}|}$ of \mathbf{K}' -morphisms such that for all

$f : A \rightarrow B$ in \mathbf{K} (with $A, B \in |\mathbf{K}|$), $\tau_A; \mathbf{G}(f) = \mathbf{F}(f); \tau_B$

Then, τ is a *natural isomorphism* if for all $A \in |\mathbf{K}|$, τ_A is an isomorphism.

$$\begin{array}{ccc}
 \mathbf{K}: & & \mathbf{K}': \\
 A & & \mathbf{F}(A) \xrightarrow{\tau_A} \mathbf{G}(A) \\
 \downarrow f & & \downarrow \mathbf{F}(f) \quad \downarrow \mathbf{G}(f) \\
 B & & \mathbf{F}(B) \xrightarrow{\tau_B} \mathbf{G}(B)
 \end{array}$$

Examples

- *identity transformations*: $id_{\mathbf{F}} : \mathbf{F} \rightarrow \mathbf{F}$, where $\mathbf{F} : \mathbf{K} \rightarrow \mathbf{K}'$, for all objects $A \in |\mathbf{K}|$, $(id_{\mathbf{F}})_A = id_A : \mathbf{F}(A) \rightarrow \mathbf{F}(A)$
- *singleton functions*: $sing : \mathbf{Id}_{\mathbf{Set}} \rightarrow \mathbf{P} (: \mathbf{Set} \rightarrow \mathbf{Set})$, where for all $X \in |\mathbf{Set}|$, $sing_X : X \rightarrow \mathbf{P}(X)$ is a function defined by $sing_X(x) = \{x\}$ for $x \in X$
- *singleton-list functions*: $sing^{\mathbf{List}} : \mathbf{Id}_{\mathbf{Set}} \rightarrow |\mathbf{List}| (: \mathbf{Set} \rightarrow \mathbf{Set})$, where $|\mathbf{List}| = \mathbf{List}; |-| : \mathbf{Set}(\rightarrow \mathbf{Monoid}) \rightarrow \mathbf{Set}$, and for all $X \in |\mathbf{Set}|$, $sing_X^{\mathbf{List}} : X \rightarrow X^*$ is a function defined by $sing_X^{\mathbf{List}}(x) = \langle x \rangle$ for $x \in X$
- *append functions*: $append : |\mathbf{List}|; \mathbf{CP} \rightarrow |\mathbf{List}| (: \mathbf{Set} \rightarrow \mathbf{Set})$, where for all $X \in |\mathbf{Set}|$, $append_X : (X^* \times X^*) \rightarrow X^*$ is the usual append function (list concatenation) polymorphic functions between algebraic types

Polymorphic functions

Work out the following generalisation of the last two examples:

- for each algebraic type scheme $\forall \alpha_1 \dots \alpha_n \cdot T$, built in **Standard ML** using at least products and algebraic data types (no function types though), define the corresponding functor $\llbracket T \rrbracket : \mathbf{Set}^n \rightarrow \mathbf{Set}$
- argue that in a representative subset of **Standard ML**, for each polymorphic expression $E : \forall \alpha_1 \dots \alpha_n \cdot T \rightarrow T'$ its semantics is a natural transformation $\llbracket E \rrbracket : \llbracket T \rrbracket \rightarrow \llbracket T' \rrbracket$

Theorems for free!
(see Wadler 89)

Yoneda lemma

Given a locally small category \mathbf{K} , functor $\mathbf{F} : \mathbf{K} \rightarrow \mathbf{Set}$ and object $A \in |\mathbf{K}|$:

$$\text{Nat}(\mathbf{Hom}_{\mathbf{K}}(A, -), \mathbf{F}) \cong \mathbf{F}(A)$$

natural transformations from $\mathbf{Hom}_{\mathbf{K}}(A, -)$ to \mathbf{F} , between functors from \mathbf{K} to \mathbf{Set} , are given exactly by the elements of the set $\mathbf{F}(A)$

EXERCISES:

- Dualise: for $\mathbf{G} : \mathbf{K}^{op} \rightarrow \mathbf{Set}$,

$$\text{Nat}(\mathbf{Hom}_{\mathbf{K}}(-, A), \mathbf{G}) \cong \mathbf{G}(A)$$

- Characterise all natural transformations from $\mathbf{Hom}_{\mathbf{K}}(A, -)$ to $\mathbf{Hom}_{\mathbf{K}}(B, -)$, for all objects $A, B \in |\mathbf{K}|$.

Proof

- For $a \in \mathbf{F}(A)$, define $\tau^a : \mathbf{Hom}_{\mathbf{K}}(A, -) \rightarrow \mathbf{F}$, as the family of functions $\tau_B^a : \mathbf{K}(A, B) \rightarrow \mathbf{F}(B)$ given by $\tau_B^a(f) = \mathbf{F}(f)(a)$ for $f : A \rightarrow B$ in \mathbf{K} .

This is a natural transformation, since for $g : B \rightarrow C$ and then $f : A \rightarrow B$,

$$\mathbf{F}(g)(\tau_B^a(f)) = \mathbf{F}(g)(\mathbf{F}(f)(a))$$

$$= \mathbf{F}(f;g)(a) = \tau_C^a(f;g)$$

$$= \tau_C^a(\mathbf{Hom}_{\mathbf{K}}(A, g)(f))$$

Then $\tau_A^a(id_A) = a$, and so for distinct $a, a' \in \mathbf{F}(A)$, τ^a and $\tau^{a'}$ differ.

- If $\tau : \mathbf{Hom}_{\mathbf{K}}(A, -) \rightarrow \mathbf{F}$ is a natural transformation then $\tau = \tau^a$, where we put $a = \tau_A(id_A)$, since for $B \in |\mathbf{K}|$ and $f : A \rightarrow B$, $\tau_B(f) = \mathbf{F}(f)(\tau_A(id_A))$ by naturality of τ :

K:

B

$g \downarrow$

C

Set:

$$\mathbf{K}(A, B) \xrightarrow{\tau_B^a} \mathbf{F}(B)$$

$$\begin{array}{ccc} \downarrow (-);g = \mathbf{Hom}_{\mathbf{K}}(A, g) & & \downarrow \mathbf{F}(g) \\ \mathbf{K}(A, C) & \xrightarrow{\tau_C^a} & \mathbf{F}(C) \end{array}$$

A

$f \downarrow$

B

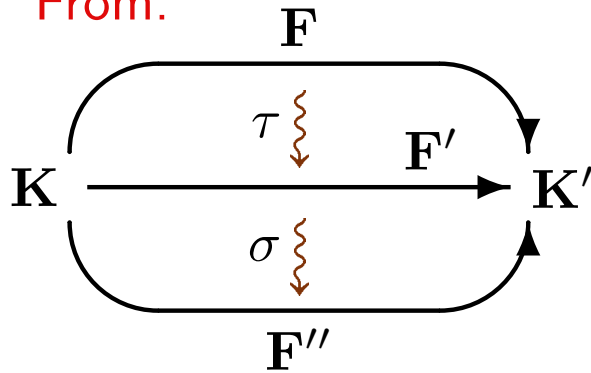
$$\mathbf{K}(A, A) \xrightarrow{\tau_A} \mathbf{F}(A)$$

$$\begin{array}{ccc} \downarrow (-);f = \mathbf{Hom}_{\mathbf{K}}(A, f) & & \downarrow \mathbf{F}(f) \\ \mathbf{K}(A, B) & \xrightarrow{\tau_B} & \mathbf{F}(B) \end{array}$$

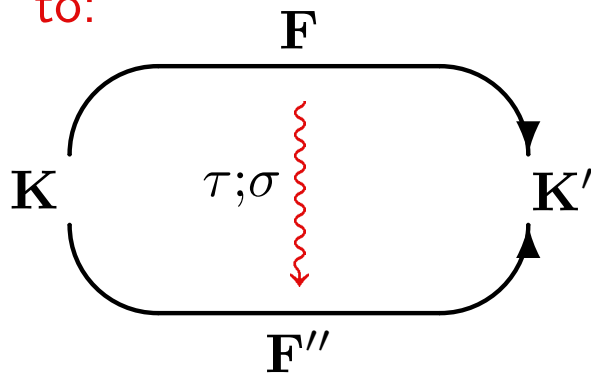
Compositions

vertical composition:

From:

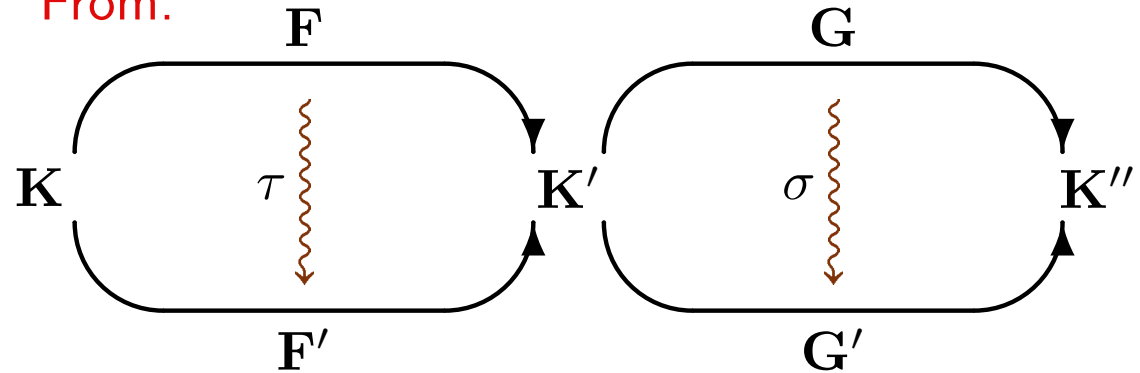


to:

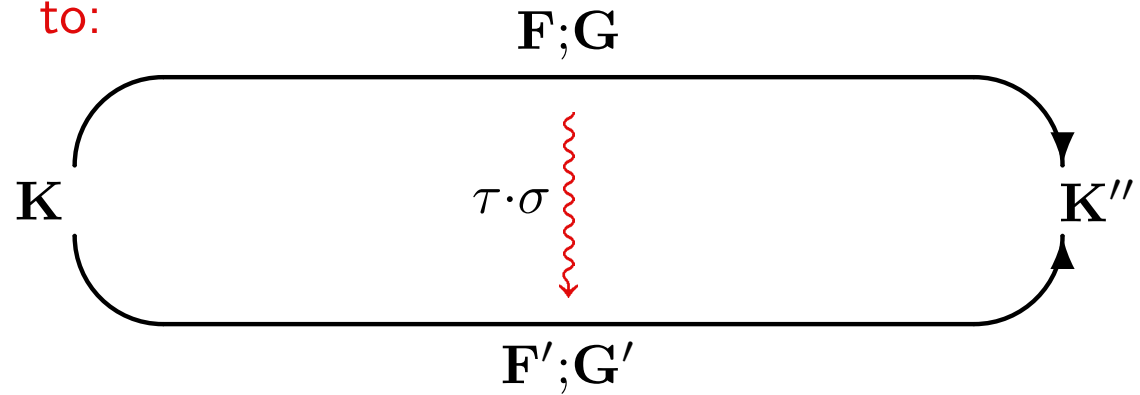


horizontal composition:

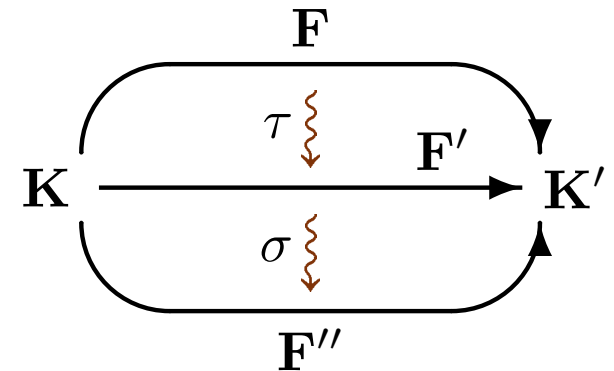
From:



to:



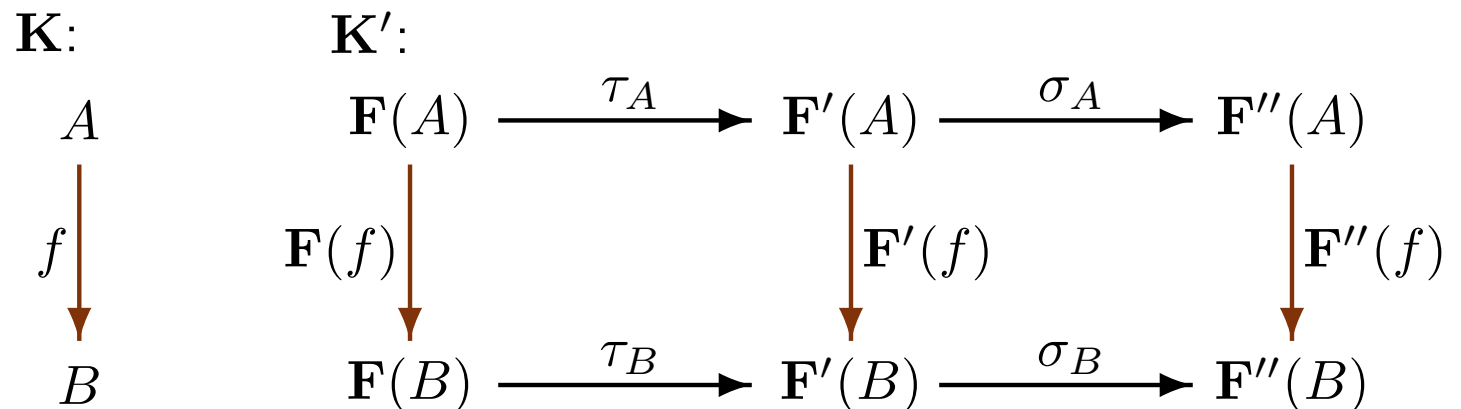
Vertical composition



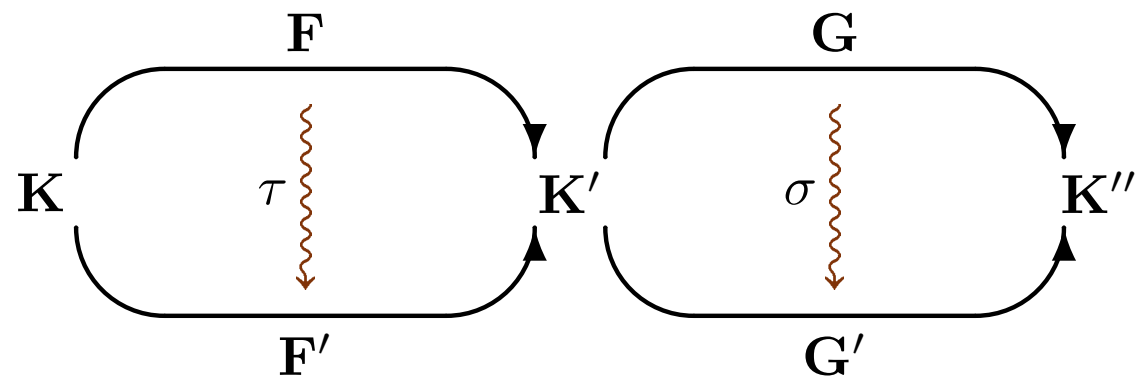
The *vertical composition* of natural transformations $\tau : \mathbf{F} \rightarrow \mathbf{F}'$ and $\sigma : \mathbf{F}' \rightarrow \mathbf{F}''$ between parallel functors $\mathbf{F}, \mathbf{F}', \mathbf{F}'' : \mathbf{K} \rightarrow \mathbf{K}'$

$$\tau; \sigma : \mathbf{F} \rightarrow \mathbf{F}''$$

is a natural transformation given by $(\tau; \sigma)_A = \tau_A; \sigma_A$ for all $A \in |\mathbf{K}|$.



Horizontal composition



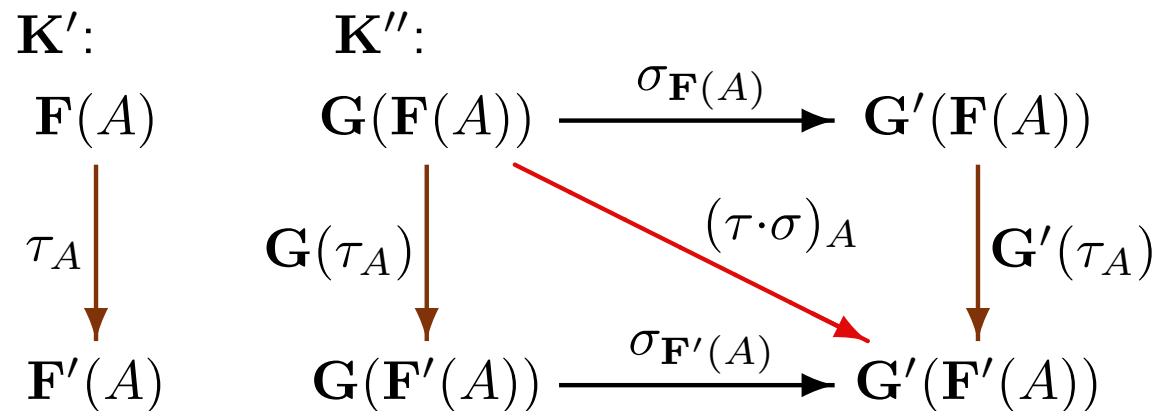
The *horizontal composition* of natural transformations $\tau : \mathbf{F} \rightarrow \mathbf{F}'$ and $\sigma : \mathbf{G} \rightarrow \mathbf{G}'$ between composable pairs of parallel functors $\mathbf{F}, \mathbf{F}' : \mathbf{K} \rightarrow \mathbf{K}'$, $\mathbf{G}, \mathbf{G}' : \mathbf{K}' \rightarrow \mathbf{K}''$

$$\tau \cdot \sigma : \mathbf{F}; \mathbf{G} \rightarrow \mathbf{F}'; \mathbf{G}'$$

is a natural transformation given by $(\tau \cdot \sigma)_A = \mathbf{G}(\tau_A); \sigma_{\mathbf{F}'(A)} = \sigma_{\mathbf{F}(A)}; \mathbf{G}'(\tau_A)$ for all $A \in |\mathbf{K}|$.

Multiplication by functor:

- $\tau \cdot \mathbf{G} = \tau \cdot id_{\mathbf{G}} : \mathbf{F}; \mathbf{G} \rightarrow \mathbf{F}'; \mathbf{G}$,
i.e., $(\tau \cdot \mathbf{G})_A = \mathbf{G}(\tau_A)$
- $\mathbf{F} \cdot \sigma = id_{\mathbf{F}} \cdot \sigma : \mathbf{F}; \mathbf{G} \rightarrow \mathbf{F}; \mathbf{G}'$,
i.e., $(\mathbf{F} \cdot \sigma)_A = \sigma_{\mathbf{F}(A)}$



Show that indeed, $\tau \cdot \sigma$ is a natural transformation

Functor categories

Given two categories \mathbf{K}, \mathbf{K}' , define the *category of functors from \mathbf{K}' to \mathbf{K}* , $\mathbf{K}^{\mathbf{K}'}$, as follows:

- objects: functors from \mathbf{K}' to \mathbf{K}
- morphisms: natural transformations between them
- composition: vertical composition of the natural transformations

Exercises:

- View the category of S -sorted sets, \mathbf{Set}^S , as a functor category
- Show how any functor $\mathbf{F} : \mathbf{K}'' \rightarrow \mathbf{K}'$ induces a functor $(\mathbf{F}; -) : \mathbf{K}^{\mathbf{K}'} \rightarrow \mathbf{K}^{\mathbf{K}''}$
- Check whether $\mathbf{K}^{\mathbf{K}'}$ is (finitely) (co)complete whenever \mathbf{K} is so.
- Check when $(\mathbf{F}; -) : \mathbf{K}^{\mathbf{K}'} \rightarrow \mathbf{K}^{\mathbf{K}''}$ is (finitely) (co)continuous, for a given functor $\mathbf{F} : \mathbf{K}'' \rightarrow \mathbf{K}'$

Diagrams as functors

Each diagram D over graph G in category \mathbf{K} yields a functor $\mathbf{F}_D : \mathbf{Path}(G) \rightarrow \mathbf{K}$ given by:

- $\mathbf{F}_D(n) = D_n$, for all nodes $n \in |G|_{nodes}$
- $\mathbf{F}_D(n_0 e_1 n_1 \dots n_{k-1} e_k n_k) = D_{e_1}; \dots; D_{e_k}$, for paths $n_0 e_1 n_1 \dots n_{k-1} e_k n_k$ in G

Moreover:

- for distinct diagrams D and D' of shape G , \mathbf{F}_D and $\mathbf{F}_{D'}$ are different
- all functors from $\mathbf{Path}(G)$ to \mathbf{K} are given by diagrams over G

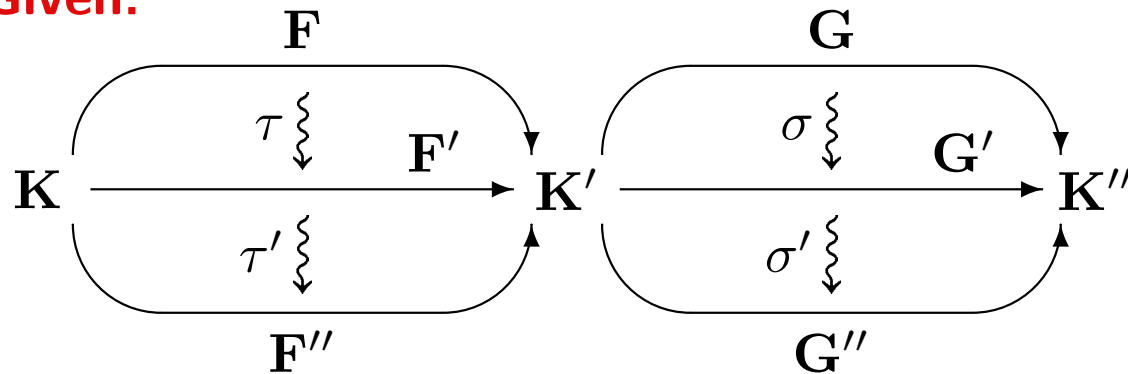
Diagram morphisms $\mu : D \rightarrow D'$ between diagrams of the same shape G are exactly natural transformations $\mu : \mathbf{F}_D \rightarrow \mathbf{F}_{D'}$.

$$\mathbf{Diag}_{\mathbf{K}}^G \cong \mathbf{K}^{\mathbf{Path}(G)}$$

Diagrams are functors from small (shape) categories

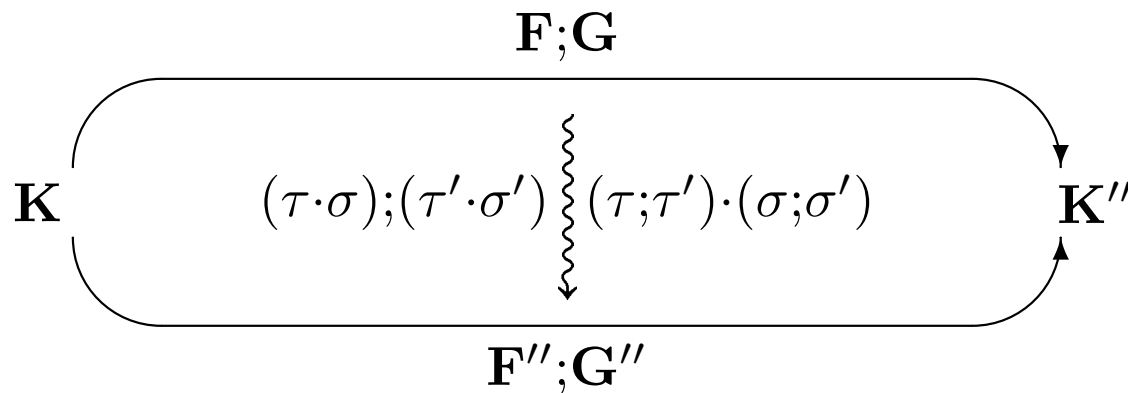
Double law

Given:



then:

$$(\tau \cdot \sigma); (\tau' \cdot \sigma') = (\tau; \tau') \cdot (\sigma; \sigma')$$



This holds in **Cat**, which is a paradigmatic example of a two-category.

A category **K** is a *two-category* when for all objects $A, B \in |\mathbf{K}|$, $\mathbf{K}(A, B)$ is again a category, with *1-morphisms* (the usual **K**-morphisms) as objects and *2-morphisms* between them. Those 2-morphisms compose vertically (in the categories $\mathbf{K}(A, B)$) and horizontally, subject to the double law as stated here.

In two-category **Cat**, we have $\mathbf{Cat}(\mathbf{K}', \mathbf{K}) = \mathbf{K}^{\mathbf{K}'}$.

Equivalence of categories

- Two categories \mathbf{K} and \mathbf{K}' are *isomorphic* if there are functors $\mathbf{F} : \mathbf{K} \rightarrow \mathbf{K}'$ and $\mathbf{G} : \mathbf{K}' \rightarrow \mathbf{K}$ such that $\mathbf{F};\mathbf{G} = \text{Id}_{\mathbf{K}}$ and $\mathbf{G};\mathbf{F} = \text{Id}_{\mathbf{K}'}$.
- Two categories \mathbf{K} and \mathbf{K}' are *equivalent* if there are functors $\mathbf{F} : \mathbf{K} \rightarrow \mathbf{K}'$ and $\mathbf{G} : \mathbf{K}' \rightarrow \mathbf{K}$ and natural isomorphisms $\eta : \text{Id}_{\mathbf{K}} \rightarrow \mathbf{F};\mathbf{G}$ and $\epsilon : \mathbf{G};\mathbf{F} \rightarrow \text{Id}_{\mathbf{K}'}$.
- A category is *skeletal* if any two isomorphic objects are identical.
- A *skeleton* of a category is any of its maximal skeletal subcategory.

Fact: *Two categories are equivalent iff they have isomorphic skeletons.*

All “categorical” properties are preserved under equivalence of categories