Monads

category theory for
effects in typed functional programming

monoids: a categorical generalisation
algebras: a categorical generalisation

Monads

A *monad* in a category $\mathbf{K}$ is a triple:

$$\boxed{\langle \mathbf{T} \colon \mathbf{K} \to \mathbf{K}, \eta \colon \mathbf{Id}_{\mathbf{K}} \to \mathbf{T}, \mu \colon \mathbf{T};\mathbf{T} \to \mathbf{T} \rangle}$$

such that for each $X \in |\mathbf{K}|$

- $\eta_{\mathbf{T}(X)};\mu_X = id_{\mathbf{T}(X)} = \mathbf{T}(\eta_X);\mu_X$

$$\begin{tikzcd}
\mathbf{T} \arrow[r, "\mathbf{T}\cdot\eta"] \arrow[dr, "id_{\mathbf{T}}"'] & \mathbf{T};\mathbf{T} \arrow[d, "\mu"] & \mathbf{T} \arrow[l, "\eta\cdot\mathbf{T}"'] \arrow[dl, "id_{\mathbf{T}}"] \\
& \mathbf{T} &
\end{tikzcd}$$

- $\mu_{\mathbf{T}(X)};\mu_X = \mathbf{T}(\mu_X);\mu_X$

$$\begin{tikzcd}
\mathbf{T};\mathbf{T};\mathbf{T} \arrow[r, "\mathbf{T}\cdot\mu"] \arrow[d, "\mu\cdot\mathbf{T}"'] & \mathbf{T};\mathbf{T} \arrow[d, "\mu"] \\
\mathbf{T};\mathbf{T} \arrow[r, "\mu"] & \mathbf{T}
\end{tikzcd}$$

Trivial examples

- *Identity* monad
- *Terminal* monad
- Monads in partial orders: *closure operators*
- ...

Simple Examples

Examples of monads in $\mathbf{Set}$

- *Partiality* monad:
  - $\mathbf{P}(X) = X + \{\perp\}$;
  - $\eta^{\mathbf{P}}_X(x) = x$;
  - $\mu^{\mathbf{P}}_X(x) = x$ for $x \in X$, $\mu^{\mathbf{P}}_X(x) = \perp$ for $x \notin X$.
- *Exceptions* monad;
  - $\mathcal{E}(X) = X + E$;
  - $\eta^{\mathcal{E}}_X(x) = x$;
  - $\mu^{\mathcal{E}}_X(x) = x$ for $x \in X$, $\mu^{\mathcal{E}}_X(e) = e$ for $e \in E$.
- *Nondeterminism* monad:
  - $\mathcal{P}(X) = 2^X$;
  - $\eta^{\mathcal{P}}_X(x) = \{x\}$;
  - $\mu^{\mathcal{P}}_X(U) = \bigcup U$ for $U \in 2^{2^X}$.

Typical examples

Examples of monads in $\mathbf{Set}$

- *List* monad:
  - $\mathcal{L}(X) = X^*$;
  - $\eta^{\mathcal{L}}_X(x) = \langle x \rangle$;
  - $\mu^{\mathcal{L}}_X(\langle l_1, \ldots, l_n \rangle) = append(l_1, \ldots append(l_{n-1}, l_n) \ldots)$.
- *Term* monad:
  - $\mathcal{T}_\Sigma(X) = T_\Sigma(X)$;
  - $\eta^{\mathcal{T}_\Sigma}_X(x) = x$;
  - $\mu^{\mathcal{T}_\Sigma}_X(t) = t[id_{T_\Sigma(X)}]$ for $t \in T_\Sigma(T_\Sigma(X))$.

Difficult(?) examples

Examples of monads in $\mathbf{Set}$

- *Side-effects* monad:
  - $\mathcal{S}(X) = (X \times S)^S$;
  - $\eta^{\mathcal{S}}_X(x)(s) = \langle x, s \rangle$;
  - $\mu^{\mathcal{S}}_X(f)(s) = g(s')$ where $f(s) = \langle g, s' \rangle$, for $f \in ((X \times S)^S \times S)^S$.
- *Continuation* monad:
  - $\mathcal{K}(X) = A^{(A^X)}$;
  - $\eta^{\mathcal{K}}_X(x)(k) = k(x)$;
  - $\mu^{\mathcal{K}}_X(f)(k) = f(\lambda g \in A^{(A^X)} \cdot g(k))$, for $f \in A^{(A^{(A^{(A^X)})})}$.

Instead of more examples

Adjunctions give rise to monads

Fact: *For any adjunction* $\langle \mathbf{F}, \mathbf{G}, \eta, \varepsilon \rangle \colon \mathbf{K} \to \mathbf{K}'$, *we have the monad:*

$$\boxed{\langle \mathbf{T} \colon \mathbf{K} \to \mathbf{K}, \eta^{\mathbf{T}} \colon \mathbf{Id}_{\mathbf{K}} \to \mathbf{T}, \mu^{\mathbf{T}} \colon \mathbf{T};\mathbf{T} \to \mathbf{T} \rangle}$$

given by:

- $\mathbf{T} = \mathbf{F};\mathbf{G}$
- $\eta^{\mathbf{T}} = \eta \colon \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$
- $\mu^{\mathbf{T}} = \mathbf{F}{\cdot}\varepsilon{\cdot}\mathbf{G} \colon \mathbf{F};(\mathbf{G};\mathbf{F});\mathbf{G} \to \mathbf{F};\mathbf{G}$
  (i.e. $\mu^{\mathbf{T}}_X = \mathbf{G}(\varepsilon_{\mathbf{F}(X)}) \colon \mathbf{G}(\mathbf{F}(\mathbf{G}(\mathbf{F}(X)))) \to \mathbf{G}(\mathbf{F}(X))$*)*

Proof

unit laws:

- $(\mathbf{G}{\cdot}\eta);(\varepsilon{\cdot}\mathbf{G}) = id_{\mathbf{G}}$ implies $(\mathbf{F}{\cdot}(\mathbf{G}{\cdot}\eta));(\mathbf{F}{\cdot}\varepsilon{\cdot}\mathbf{G}) = id_{\mathbf{F};\mathbf{G}}$
- $(\eta{\cdot}\mathbf{F});(\mathbf{F}{\cdot}\varepsilon) = id_{\mathbf{F}}$ implies $((\eta{\cdot}\mathbf{F}){\cdot}\mathbf{G});(\mathbf{F}{\cdot}\varepsilon{\cdot}\mathbf{G}) = id_{\mathbf{F};\mathbf{G}}$

$$\begin{tikzcd}
\mathbf{F};\mathbf{G} \arrow[r, "(\mathbf{F};\mathbf{G}){\cdot}\eta"] \arrow[dr, "id_{\mathbf{F};\mathbf{G}}"'] & \mathbf{F};\mathbf{G};\mathbf{F};\mathbf{G} \arrow[d, "\mathbf{F}{\cdot}\varepsilon{\cdot}\mathbf{G}"] & \mathbf{F};\mathbf{G} \arrow[l, "\eta{\cdot}(\mathbf{F};\mathbf{G})"'] \arrow[dl, "id_{\mathbf{F};\mathbf{G}}"] \\
& \mathbf{F};\mathbf{G} &
\end{tikzcd}$$

Proof cntd.

associativity:

$$\begin{tikzcd}
\mathbf{F};\mathbf{G};\mathbf{F};\mathbf{G};\mathbf{F};\mathbf{G} \arrow[r, "(\mathbf{F};\mathbf{G})\cdot(\mathbf{F}\cdot\varepsilon\cdot\mathbf{G})"] \arrow[d, "(\mathbf{F}\cdot\varepsilon\cdot\mathbf{G})\cdot(\mathbf{F};\mathbf{G})"'] & \mathbf{F};\mathbf{G};\mathbf{F};\mathbf{G} \arrow[d, "\mathbf{F}\cdot\varepsilon\cdot\mathbf{G}"] \\
\mathbf{F};\mathbf{G};\mathbf{F};\mathbf{G} \arrow[r, "\mathbf{F}\cdot\varepsilon\cdot\mathbf{G}"] & \mathbf{F};\mathbf{G}
\end{tikzcd}$$

Follows by the commutativity of the diagrams below:

$$\begin{tikzcd}
\mathbf{G};\mathbf{F};\mathbf{G};\mathbf{F} \arrow[r, "\mathbf{G}\cdot(\mathbf{F}\cdot\varepsilon)"] \arrow[d, "(\varepsilon\cdot\mathbf{G})\cdot\mathbf{F}"'] & \mathbf{G};\mathbf{F} \arrow[d, "\varepsilon"] \\
\mathbf{G};\mathbf{F} \arrow[r, "\varepsilon"] & \mathbf{Id_K}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{F}(\mathbf{G}(\mathbf{F}(\mathbf{G}(X)))) \arrow[r, "\varepsilon_{\mathbf{F}(\mathbf{G}(X))}"] \arrow[d, "\mathbf{F}(\mathbf{G}(\varepsilon_X))"'] & \mathbf{F}(\mathbf{G}(X)) \arrow[d, "\varepsilon_X"] \\
\mathbf{F}(\mathbf{G}(X)) \arrow[r, "\varepsilon_X"] & X
\end{tikzcd}$$

Eilenberg-Moore '65

Algebras

Check this out for the term monad

Given a monad $\langle \mathbf{T}, \eta, \mu \rangle$ in $\mathbf{K}$:

The category $\boxed{\mathbf{Alg}(\mathbf{T})}$ of $\mathbf{T}$-algebras and $\mathbf{T}$-homomorphisms:

- $\mathbf{T}$-*algebras*:
  $\langle A \in |\mathbf{K}|, a\colon \mathbf{T}(A) \to A \rangle$ such that $\mathbf{T}(a);a = \mu_A;a$ and $\eta_A;a = id_A$
- $\mathbf{T}$-*homomorphism* from $\langle A, a\colon \mathbf{T}(A) \to A \rangle$ to $\langle B, b\colon \mathbf{T}(B) \to B \rangle$:
  $h\colon A \to B$ such that $\mathbf{T}(h);b = a;h$

$$\begin{tikzcd} \mathbf{T}(\mathbf{T}(A)) \arrow[r, "\mathbf{T}(a)"] \arrow[d, "\mu_A"'] & \mathbf{T}(A) \arrow[d, "a"] \\ \mathbf{T}(A) \arrow[r, "a"] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "id_A"'] & \mathbf{T}(A) \arrow[d, "a"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} \mathbf{T}(A) \arrow[r, "\mathbf{T}(h)"] \arrow[d, "a"'] & \mathbf{T}(B) \arrow[d, "b"] \\ A \arrow[r, "h"] & B \end{tikzcd}$$

Monadic adjunction

Let $\mathbf{G}^{\mathbf{T}} \colon \mathbf{Alg}(\mathbf{T}) \to \mathbf{K}$ be the obvious projection: $\mathbf{G}^{\mathbf{T}}(\langle A, a\rangle) = A$, ...

For $X \in |\mathbf{K}|$, $\mathbf{F}^{\mathbf{T}}(X) = \langle \mathbf{T}(X), \mu_X \colon \mathbf{T}(\mathbf{T}(X)) \to \mathbf{T}(X)\rangle$ with unit $\eta_X \colon X \to \mathbf{G}^{\mathbf{T}}(\mathbf{F}^{\mathbf{T}}(X))$ is free over X w.r.t. $\mathbf{G}$:

$$
\begin{tikzcd}
X \arrow[rr, "\eta_X"] \arrow[dr, "f"'] & & \mathbf{T}(X) \arrow[d, "\mathbf{T}(f)"] \\
& A \arrow[r, "\eta_A"] \arrow[dr, "id_A"'] & \mathbf{T}(A) \arrow[d, "a"] \\
& & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{T}(\mathbf{T}(X)) \arrow[r, "\mu_X"] \arrow[d, "\mathbf{T}(\mathbf{T}(f))"'] & \mathbf{T}(X) \arrow[d, "\mathbf{T}(f)"] \arrow[dd, bend left=80, "{\exists!\ f^{\#} = \mathbf{T}(f);a}"] \\
\mathbf{T}(\mathbf{T}(A)) \arrow[r, "\mu_A"] \arrow[d, "\mathbf{T}(a)"'] & \mathbf{T}(A) \arrow[d, "a"] \\
\mathbf{T}(A) \arrow[r, "a"] & A
\end{tikzcd}
$$

All monads arise from adjunctions

Given a monad $\langle \mathbf{T}, \eta, \mu \rangle$ in $\mathbf{K}$ we have an adjunction

$$\boxed{\langle \mathbf{F}^{\mathbf{T}}, \mathbf{G}^{\mathbf{T}}, \eta, \varepsilon^{\mathbf{T}} \rangle \colon \mathbf{K} \to \mathbf{Alg}(\mathbf{T})}$$

$\varepsilon_{\langle A,a \rangle} \colon \mathbf{F}^{\mathbf{T}}(\mathbf{G}^{\mathbf{T}}(\langle A, a \rangle)) \to \langle A, a \rangle$ is $a \colon \mathbf{T}(A) \to A$.

Fact: $\langle \mathbf{T}, \eta, \mu \rangle$ *is the monad associated with* $\langle \mathbf{F}^{\mathbf{T}}, \mathbf{G}^{\mathbf{T}}, \eta, \varepsilon^{\mathbf{T}} \rangle$.

Fact: *Given an adjunction* $\langle \mathbf{F}, \mathbf{G}, \eta, \varepsilon \rangle \colon \mathbf{K} \to \mathbf{K}'$, *let* $\langle \mathbf{T} = \mathbf{F};\mathbf{G}, \eta, \mu^{\mathbf{T}} = \mathbf{F}{\cdot}\varepsilon{\cdot}\mathbf{G} \rangle$ *be the monad it yields, and then let* $\langle \mathbf{F}^{\mathbf{T}}, \mathbf{G}^{\mathbf{T}}, \eta, \varepsilon^{\mathbf{T}} \rangle \colon \mathbf{K} \to \mathbf{Alg}(\mathbf{T})$ *be the adjunction for* $\mathbf{T}$. *Then there is a unique comparison functor* $\Phi \colon \mathbf{K}' \to \mathbf{Alg}(\mathbf{T})$ *such that* $\Phi;\mathbf{G}^{\mathbf{T}} = \mathbf{G}$ *and* $\mathbf{F};\Phi = \mathbf{F}^{\mathbf{T}}$.

$\Phi(A') = \langle \mathbf{G}(A'), \mathbf{G}(\varepsilon_{A'}) \colon \mathbf{G}(\mathbf{F}(\mathbf{G}(A'))) \to \mathbf{G}(A') \rangle$

Free algebras

Given a monad $\langle \mathbf{T}, \eta, \mu \rangle$ in $\mathbf{K}$:

View $\mathbf{Kl}(\mathbf{T})$ as the image of $\mathbf{F}^{\mathbf{T}}$ in $\mathbf{Alg}(\mathbf{T})$

The *Kleisli category* $\boxed{\mathbf{Kl}(\mathbf{T})}$ for $\mathbf{T}$:

- objects: $|\mathbf{Kl}(\mathbf{T})| = |\mathbf{K}|$
- morphisms: $f\colon X \to Y$ in $\mathbf{Kl}(\mathbf{T})$ are morphisms $f\colon X \to \mathbf{T}(Y)$ in $\mathbf{K}$
- composition: given $f\colon X \to Y$, $g\colon Y \to Z$ in $\mathbf{Kl}(\mathbf{T})$, $f;g\colon X \to Z$ in $\mathbf{Kl}(\mathbf{K})$ is $f;\mathbf{T}(g);\mu_Y\colon X \to \mathbf{T}(Z)$ in $\mathbf{K}$.

$$X \xrightarrow{f} \mathbf{T}(Y) \xrightarrow{\mathbf{T}(g)} \mathbf{T}(\mathbf{T}(Z)) \xrightarrow{\mu_Y} \mathbf{T}(Z)$$

Again: there is an adjunction $\langle \mathbf{F}^{\mathbf{T}}, \mathbf{G}^{\mathbf{T}}, \eta, \varepsilon^{\mathbf{T}} \rangle\colon \mathbf{K} \to \mathbf{Kl}(\mathbf{T})$ which gives rise to the monad $\langle \mathbf{T}, \eta, \mu \rangle$, and for any adjunction $\langle \mathbf{F}, \mathbf{G}, \eta, \varepsilon \rangle\colon \mathbf{K} \to \mathbf{K}'$ which also gives rise to this monad, we have a comparison functor $\Psi\colon \mathbf{K}' \to \mathbf{Kl}(\mathbf{T})$ such that $\Psi;\mathbf{G}^{\mathbf{T}} = \mathbf{G}$ and $\mathbf{F};\Psi = \mathbf{F}^{\mathbf{T}}$.

Triples

A *triple* in $\mathbf{K}$:

$$\boxed{\langle T, \eta, (\_)^* \rangle}$$

where

- $T \colon |\mathbf{K}| \to |\mathbf{K}|$,
- $\eta_A \colon A \to T(A)$ for all $A \in |\mathbf{K}|$,
- $f^* \colon T(A) \to T(B)$ for all $f \colon A \to T(B)$

are such that

- $\eta_A^* = id_{T(A)}$ for all $A \in |\mathbf{K}|$
- $\eta_A;f^* = f$ for all $f \colon A \to T(B)$
- $f^*;g^* = (f;g^*)^*$ for all $f \colon A \to T(A)$,

Triples and monads are the same concepts

Triples as monads, monads as triples

Given a monad $\langle \mathbf{T}, \eta, \mu \rangle$ in $\mathbf{K}$, put:

- $T(A) = \mathbf{T}(A)$ for $A \in |\mathbf{K}|$,
- $\eta_A = \eta_A : A \to T(A)$ for $A \in |\mathbf{K}|$,
- $f^* = \mathbf{T}(f);\mu_A : T(A) \to T(B)$ for $f : A \to T(B)$.

This yields a triple $\langle T, \eta, (\_)^* \rangle$.

"Triple" the monads given as examples

Given a triple $\langle T, \eta, (\_)^* \rangle$ in $\mathbf{K}$, put:

- $\mathbf{T}(A) = T(A)$ for $A \in |\mathbf{K}|$, and $\mathbf{T}(f) = (f;\eta_B)^*$ for $f : A \to B$,
- $\eta_A = \eta_A : A \to T(A)$ for $A \in |\mathbf{K}|$,
- $\mu_A = id^*_{T(A)} : T(T(A)) \to T(A)$ for $A \in |\mathbf{K}|$,

This yields a monad $\langle T, \eta, \mu \rangle$.

Further monadic concepts

- Most importantly:

  $$\boxed{\textit{Functional programming with effects}}$$

  Given a triple $\langle T, \eta, (\_)^* \rangle$:

  – `return` $\_\_ : \alpha \to T\alpha$ is η_α

  – $\_\_$ `>>=` $\_\_ : T\alpha \to (\alpha \to T\beta) \to T\beta$ is given by x `>>=` $f = f^*(x)$

  – `do`-notation from $\_\_$ `>>=` $\_\_$ and λ-notation

- *Strong* (context-preserving) monads
- Monad *composition* and *distributivity laws* for monads
- Monad *transformers*
- . . .