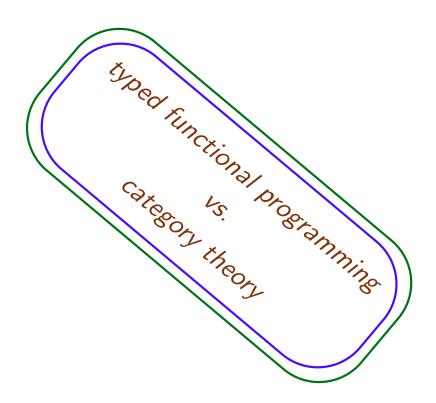
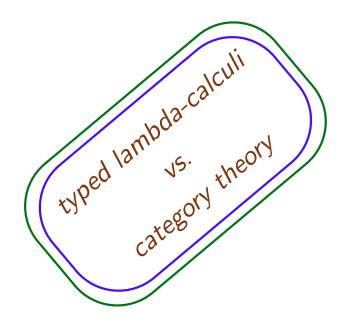
Cartesian closed categories CCC





Cartesian categories

Definition: A category **K** is Cartesian if it comes equipped with finite products.

Equivalently:

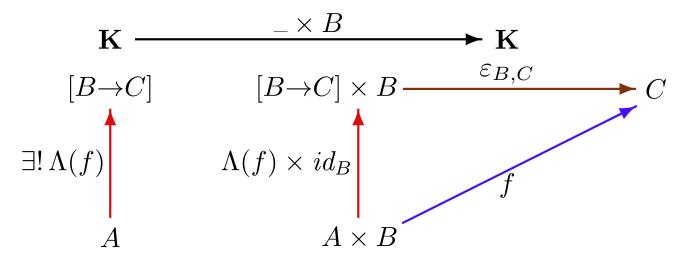
- $1 \in |\mathbf{K}|$ a terminal object
- $A \times B$ a product of A and B, for every $A, B \in |\mathbf{K}|$

Examples: Set, Pfn, Cpo, semilattices, Cat, $T_{\Sigma,\Phi}^{op}$, ...

Recall the definitions of these categories and the constructions of products in each of them

Cartesian closed categories

Definition: A Cartesian category \mathbf{K} is closed if for all $B, C \in |K|$ we indicate $[B \rightarrow C] \in |\mathbf{K}|$ and $\varepsilon_{B,C} \colon [B \rightarrow C] \times B \rightarrow C$ such that for all $A \in |\mathbf{K}|$ and $f \colon A \times B \rightarrow C$ there is a unique $\Lambda(f) \colon A \rightarrow [B \rightarrow C]$ satisfying $(\Lambda(f) \times id_B); \varepsilon_{B,C} = f$



Examples: Set, Cpo, Cat, ...

Heyting semilattices ($b \Leftarrow c$ is such that for all a, $a \land b \leq c$ iff $a \leq (b \Leftarrow c)$

Non-examples: Pfn, $T_{\Sigma,\Phi}^{op}$.

Summing up

A category \mathbf{K} is a Cartesian closed category (CCC) if:

- $C: \mathbf{K} \to \mathbf{1}$ has a right adjoint $C_1: \mathbf{1} \to \mathbf{K}$, yielding $1 \in |K|$.
- $\Delta \colon \mathbf{K} \to \mathbf{K} \times \mathbf{K}$ has a right adjoint $-\times -: \mathbf{K} \times \mathbf{K} \to \mathbf{K}$ with counit given by $\pi_{A,B} \colon A \times B \to A$ and $\pi'_{A,B} \colon A \times B \to B$ for $A,B \in |K|$.
- for each $B \in |K|$, $_ \times B \colon \mathbf{K} \to \mathbf{K}$ has right adjoint $[B \to _] \colon \mathbf{K} \to \mathbf{K}$, with counit given by $\varepsilon_{B,C} \colon [B \to C] \times B \to C$, for $C \in |\mathbf{K}|$.

Spelling this out

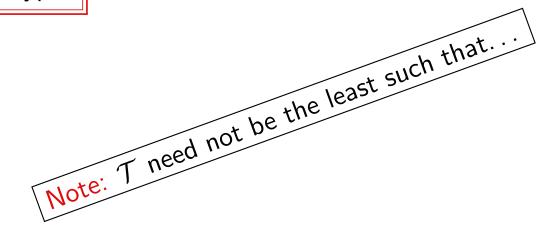
- $1 \in |\mathbf{K}|$
 - for $A \in |K|$: $\langle \rangle_A : A \to 1$ such that $\langle \rangle_A = f$ for all $f : A \to 1$.
- for $A, B \in |\mathbf{K}|$, $A \times B \in |\mathbf{K}|$, $\pi_{A,B} \colon A \times B \to A$, $\pi'_{A,B} \colon A \times B \to B$:
 - for $C \in |\mathbf{K}|$, for $f: C \to A$, $g: C \to B$: $\langle f, g \rangle : C \to A \times B$ such that
 - $-\langle f,g\rangle;\pi_{A,B}=f \text{ and } \langle f,g\rangle;\pi'_{A,B}=g$
 - for $h: C \to A \times B$, $h = \langle h; \pi_{A,B}, h; \pi'_{A,B} \rangle$
- for $B, C \in |\mathbf{K}|$, $[B \rightarrow C] \in |\mathbf{K}|$, $\varepsilon_{B,C} \colon [B \rightarrow C] \times B \rightarrow C$:
 - for $A \in |\mathbf{K}|$, for $f: A \times B \to C$: $\Lambda(f): A \to [B \to C]$ such that
 - $(\Lambda(f) \times id_B); \varepsilon_{B,C} = f$
 - for $h: A \to [B \to C]$, $\Lambda((h \times id_B); \varepsilon_{B,C}) = h$.

Typed λ -calculus with products

Types

The set \mathcal{T} of types $\tau \in \mathcal{T}$ is such that

- $1 \in \mathcal{T}$
- $\tau \times \tau' \in \mathcal{T}$, for all $\tau, \tau' \in \mathcal{T}$
- $\tau \rightarrow \tau' \in \mathcal{T}$, for all $\tau, \tau' \in \mathcal{T}$



Contexts

Contexts Γ are of the form:

• $x_1:\tau_1,\ldots,x_n:\tau_n$, where $n\geq 0,\ x_1,\ldots,x_n$ are distinct variables, and $\tau_1,\ldots,\tau_n\in\mathcal{T}$

Typed terms in contexts

 $\Gamma \vdash t \colon \tau$

Typing/formation rules coming next

Omitting the usual definitions, like:

- free variables FV(M),
- substitution M[N/x], etc.

Same for the usual simple properties, like:

- weakening context extension;
- subject reduction;
- uniqueness of types;
- removing unused variables from contexts;
 etc

Typing rules

$$\frac{x_1:\tau_1,\ldots,x_n:\tau_n \vdash x_i:\tau_i}{x_1:\tau_1,\ldots,x_n:\tau_n \vdash M:\tau}$$

$$\frac{x_1:\tau_1,\ldots,x_n:\tau_n \vdash M:\tau}{x_1:\tau_1,\ldots,x_{i-1}:\tau_{i-1},x_{i+1}:\tau_{i+1},\ldots,x_n:\tau_n \vdash \lambda x_i:\tau_i.M:\tau_i\to\tau}$$

$$\frac{\Gamma \vdash M:\tau\to\tau' \qquad \Gamma \vdash N:\tau}{\Gamma \vdash MN:\tau'}$$

$$\frac{\Gamma \vdash M:\tau \qquad \Gamma \vdash N:\tau'}{\Gamma \vdash \langle \rangle:1} \qquad \frac{\Gamma \vdash M:\tau \qquad \Gamma \vdash N:\tau'}{\Gamma \vdash \langle M,N\rangle:\tau\times\tau'}$$

$$\frac{\Gamma \vdash \pi_{\tau,\tau'}:\tau\times\tau'\to\tau}{\Gamma \vdash \pi_{\tau,\tau'}':\tau\times\tau'\to\tau'}$$

Semantics

Let \mathbf{K} be an arbitrary but fixed CCC.

- Types denote objects, $[\![\tau]\!] \in |\mathbf{K}|$, satisfying:
 - [1] = 1
 - $\llbracket \tau \times \tau' \rrbracket = \llbracket \tau \rrbracket \times \llbracket \tau' \rrbracket$
 - $\llbracket \tau \rightarrow \tau' \rrbracket = \llbracket \tau \rrbracket \rightarrow \llbracket \tau' \rrbracket \rrbracket$
- So do contexts:

$$- [x_1:\tau_1,\ldots,x_n:\tau_n] = [\tau_1] \times \ldots \times [\tau_n]$$

• Terms denote morphisms:

$$\llbracket M \rrbracket \colon \llbracket \Gamma \rrbracket \to \llbracket \tau \rrbracket$$

defined by induction on the derivation of $\Gamma \vdash M : \tau$ (coming next).

Semantics of λ **-terms**

- $\llbracket x_i \rrbracket = \pi_i \colon \llbracket \Gamma \rrbracket \to \llbracket \tau_i \rrbracket$ for $\Gamma = x_1 \colon \tau_1, \dots, x_n \colon \tau_n$, where π_i is the obvious projection.
- $\boxed{ \llbracket \lambda x_i : \tau_i.M \rrbracket = \Lambda(\rho; \llbracket M \rrbracket) : \llbracket \Gamma' \rrbracket \to \llbracket \llbracket \tau_i \rrbracket \to \llbracket \tau \rrbracket \rrbracket } \text{ for } \Gamma = x_1 : \tau_1, \ldots, x_n : \tau_n,$ $\Gamma' = x_1 : \tau_1, \ldots, x_{i-1} : \tau_{i-1}, x_{i+1} : \tau_{i+1}, \ldots, x_n : \tau_n, \text{ and } \Gamma \vdash M : \tau, \text{ where }$ $\rho \colon \llbracket \Gamma' \rrbracket \times \llbracket \tau_i \rrbracket \to \llbracket \Gamma \rrbracket \text{ is the obvious isomorphism.}$
- $\boxed{ \llbracket MN \rrbracket = \langle \llbracket M \rrbracket, \llbracket N \rrbracket \rangle; \varepsilon_{\llbracket \tau \rrbracket, \llbracket \tau' \rrbracket} \colon \llbracket \Gamma \rrbracket \to \llbracket \tau' \rrbracket } \text{ for } \Gamma \vdash M \colon \tau \to \tau' \text{ and } \Gamma \vdash N \colon \tau.$
- $\quad \llbracket \langle \rangle \rrbracket = \langle \rangle_{\llbracket \Gamma \rrbracket} \colon \llbracket \Gamma \rrbracket \to 1$
- $\quad \boxed{ \llbracket \langle M, N \rangle \rrbracket = \langle \llbracket M \rrbracket, \llbracket N \rrbracket \rangle \colon \llbracket \Gamma \rrbracket \to \llbracket \tau \rrbracket \times \llbracket \tau' \rrbracket } \text{ for } \Gamma \vdash M \colon \tau \text{ and } \Gamma \vdash N \colon \tau'.$
- $\left[\!\left[\pi_{\tau,\tau'}\right]\!\right] = \Lambda(\pi'_{\llbracket\Gamma\rrbracket,\llbracket\tau\rrbracket\times\llbracket\tau'\rrbracket};\pi_{\llbracket\tau\rrbracket,\llbracket\tau'\rrbracket}) \colon \left[\!\left[\Gamma\right]\!\right] \to \left[\!\left[\tau\right]\!\right] \times \left[\!\left[\tau'\right]\!\right] \to \left[\!\left[\tau\right]\!\right] \times \left[\!\left[\tau'\right]\!\right] = \Lambda(\pi'_{\llbracket\Gamma\rrbracket,\llbracket\tau'\rrbracket};\pi'_{\llbracket\tau\rrbracket,\llbracket\tau'\rrbracket}) \colon \left[\!\left[\Gamma\right]\!\right] \to \left[\!\left[\tau\right]\!\right] \times \left[\!\left[\tau'\right]\!\right] \to \left[\!\left[\tau'\right]\!\right]$

Equational β , η -calculus

Judgements:

$$\Gamma \vdash M = N \colon \tau$$

for $\tau \in \mathcal{T}$, $\Gamma \vdash M : \tau$, $\Gamma \vdash N : \tau$

Axioms:

- (β) $\Gamma \vdash (\lambda x : \tau . M) N = M[N/x] : \tau'$, for $\Gamma \vdash \lambda x : \tau . M : \tau \rightarrow \tau'$, $\Gamma \vdash N : \tau$
- $(\eta) \ \Gamma \vdash \lambda x : \tau . Mx = M : \tau \rightarrow \tau', \text{ for } \Gamma \vdash M : \tau \rightarrow \tau', \ x \not\in dom(\Gamma)$
 - $\Gamma \vdash M = \langle \rangle : 1$, for $\gamma \vdash M : 1$
 - $\Gamma \vdash \pi_{\tau,\tau'}\langle M, N \rangle = M : \tau$ and $\Gamma \vdash \pi'_{\tau,\tau'}\langle M, N \rangle = N : \tau'$, for $\Gamma \vdash M : \tau$, $\Gamma \vdash N : \tau'$
 - $\Gamma \vdash M = \langle \pi_{\tau,\tau'}M, \pi'_{\tau,\tau'}M \rangle \colon \tau \times \tau'$, for $\Gamma \vdash M \colon \tau \times \tau'$

Rules: reflexivity, symmetry, transitivity, congruence.

Soundness

Given $\Gamma \vdash M : \tau$ and $\Gamma \vdash N : \tau$

if
$$\Gamma dash M = N \colon au$$
 then $\llbracket M
rbracket = \llbracket N
rbracket$

Proof: :-)

Just check that the axioms and rules of the equational β , η -calculus are sound w.r.t. the semantics in any CCC. For example:

 $(\eta) \ \text{ for } \Gamma \vdash M \colon \tau \to \tau', \ x \not\in dom(\Gamma), \ \text{given the isomorphism } \rho \colon \llbracket \Gamma \rrbracket \times \llbracket \tau \rrbracket \to \llbracket \Gamma, x \colon \tau \rrbracket \colon \\ \llbracket \lambda x \colon \tau . M x \rrbracket = \Lambda(\rho \colon \llbracket M x \rrbracket) = \Lambda(\rho \colon (\langle \llbracket M \rrbracket, \llbracket x \rrbracket \rangle ; \varepsilon_{\llbracket \tau \rrbracket, \llbracket \tau' \rrbracket})) = \\ \Lambda(\langle \rho \colon \llbracket M \rrbracket, \rho \colon \llbracket x \rrbracket \rangle ; \varepsilon_{\llbracket \tau \rrbracket, \llbracket \tau' \rrbracket}) = \Lambda((\llbracket M \rrbracket \times id_{\llbracket \tau \rrbracket}) ; \varepsilon_{\llbracket \tau \rrbracket, \llbracket \tau' \rrbracket}) = \llbracket M \rrbracket.$

Warning: The "real work" is in the proof of soundess for (β) , where induction on the structure of terms is needed.

Completeness

Given
$$\Gamma \vdash M : \tau$$
 and $\Gamma \vdash N : \tau$,

if in every CCC,
$$\llbracket M
rbracket = \llbracket N
rbracket$$
 then $\Gamma dash M = N \colon au$

Proof: It is enough to prove this for terms in the empty context.

Define a CCC λ :

- Category **\(\lambda**:
 - objects are all types: $|\lambda| = \mathcal{T}$
 - morphisms are λ-terms modulo equality: $λ(τ,τ') = {M | ⊢ M : τ → τ'}/≈$, where M ≈ N iff ⊢ M = N : τ' → τ
 - composition: $[M]_{\approx}$; $[N]_{\approx} = [\lambda x : \tau . N(Mx)]_{\approx}$, for $\vdash M : \tau \to \tau'$, $\vdash N : \tau' \to \tau''$
 - identities: $id_{\tau} = [\lambda x : \tau . x]_{\approx}$.

• Products in λ :

- terminal object $1 \in |\lambda|$, with $\langle \rangle_{\tau} = [\lambda x : \tau . \langle \rangle]_{\approx}$
- binary product $\tau \times \tau'$, with $\pi_{\tau,\tau'} = [\pi_{\tau,\tau'}]_{\approx}$, $\pi'_{\tau,\tau'} = [\pi'_{\tau,\tau'}]_{\approx}$ and pairing $\langle [M]_{\approx}, [N]_{\approx} \rangle = [\langle M, N \rangle]_{\approx}$ for $\vdash M \colon \tau$, $\vdash N \colon \tau'$.
- Exponent in λ : $\tau \to \tau'$, with $\varepsilon_{\tau,\tau'} = [\lambda x : (\tau \to \tau') \times \tau . (\pi_{\tau \to \tau',\tau} x) (\pi'_{\tau \to \tau',\tau} x)]_{\approx}$ and $\Lambda([M]_{\approx}) = [\lambda x : \tau . \lambda y : \tau' . M\langle x,y\rangle]_{\approx}$, for $\vdash M : \tau \times \tau' \to \tau''$.

Now: if in every CCC, $[\![M]\!] = [\![N]\!]$, then this holds in particular in λ , and so $\Gamma \vdash M = N \colon \tau$.

To wrap this up: add constants of arbitrary types

SUMMING UP:

CCCs coincide with λ -calculi