Adjunctions

Recall:

Term algebras

Fact: *For any S-sorted set X of variables, Σ-algebra A and valuation $v : X \to |A|$, there is a unique Σ-homomorphism $v^{\#} : T_\Sigma(X) \to A$ that extends v, so that*

$$\boxed{id_{X \hookrightarrow |T_\Sigma(X)|};v^{\#} = v}$$

$$\begin{tikzcd}
 & X \arrow[rr, "id_{X \hookrightarrow |T_\Sigma(X)|}"] \arrow[drr, "v"'] & & {|T_\Sigma(X)|} \arrow[d, "|v^{\#}|"] & & T_\Sigma(X) \arrow[d, "v^{\#}", "\exists!"'] & \\
\mathbf{Set}^S & & & {|A|} & & A & \mathbf{Alg}(\Sigma)
\end{tikzcd}$$

Free objects

Consider any functor $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$

Definition: *Given an object $A \in |\mathbf{K}|$, a free object over A w.r.t. $\mathbf{G}$ is a $\mathbf{K}'$-object $A' \in |\mathbf{K}'|$ together with a $\mathbf{K}$-morphism $\eta_A : A \to \mathbf{G}(A')$ (called unit morphism) such that given any $\mathbf{K}'$-object $B' \in |\mathbf{K}'|$ with $\mathbf{K}$-morphism $f : A \to \mathbf{G}(B')$, for a unique $\mathbf{K}'$-morphism $f^\# : A' \to B'$ we have*

$$\boxed{\eta_A;\mathbf{G}(f^\#) = f}$$

Paradigmatic example:

Term algebra $T_\Sigma(X)$ with unit $id_{X \hookrightarrow |T_\Sigma(X)|} : X \to |T_\Sigma(X)|$ is free over $X \in |\mathbf{Set}^S|$ w.r.t. the carrier functor $|\_| : \mathbf{Alg}(\Sigma) \to \mathbf{Set}^S$

$$\begin{tikzcd}
 & \mathbf{K} & \mathbf{K}' \arrow[l, "\mathbf{G}"'] \\
A \arrow[r, "\eta_A"] \arrow[dr, "f"'] & \mathbf{G}(A') \arrow[d, "\mathbf{G}(f^\#)"] & A' \arrow[d, "f^\#", "\exists!"'] \\
 & \mathbf{G}(B') & B'
\end{tikzcd}$$

Examples

- Consider inclusion $i : \mathbf{Int} \hookrightarrow \mathbf{Real}$, viewing $\mathbf{Int}$ and $\mathbf{Real}$ as (thin) categories, and i as a functor between them. For any real $r \in \mathbf{Real}$, the ceiling of r, $\lceil r \rceil \in \mathbf{Int}$ is free over r w.r.t. i.

  What about free objects w.r.t. the inclusion of rationals into reals?

- For any set $X \in |\mathbf{Set}|$, the "free monoid" $\mathbf{List}(X) = \langle X^*, \widehat{\ }, \epsilon \rangle$ is free over X w.r.t. $|\_| : \mathbf{Monoid} \to \mathbf{Set}$.

- For any graph $G \in |\mathbf{Graph}|$, the category of its paths, $\mathbf{Path}(G) \in |\mathbf{Cat}|$, is free over G w.r.t. the graph functor $G : \mathbf{Cat} \to \mathbf{Graph}$.

- Discrete topologies, completion of metric spaces, free groups, ideal completion of partial orders, ideal completion of free partial algebras, ...

Makes precise these and other similar examples

Indicate unit morphisms!

Free equational models

- Recall: for any algebraic signature $\Sigma = \langle S, \Omega \rangle$, term algebra $T_\Sigma(X)$ is free over $X \in |\mathbf{Set}^S|$ w.r.t. the carrier functor $|\_| : \mathbf{Alg}(\Sigma) \to \mathbf{Set}^S$.
- For any set of Σ-equations Φ, for any set $X \in |\mathbf{Set}^S|$, there exist a model $\mathbf{F}_\Phi(X) \in \mathit{Mod}(\Phi)$ that is free over X w.r.t. the carrier functor $|\_| : \mathbf{Mod}(\langle \Sigma, \Phi \rangle) \to \mathbf{Set}^S$, where $\mathbf{Mod}(\langle \Sigma, \Phi \rangle)$ is the full subcategory of $\mathbf{Alg}(\Sigma)$ given by the models of Φ.
- For any algebraic signature morphism $\sigma : \Sigma \to \Sigma'$, for any Σ-algebra $A \in |\mathbf{Alg}(\Sigma)|$, there exist a Σ'-algebra $\mathbf{F}_\sigma(A) \in |\mathbf{Alg}(\Sigma')|$ that is free over A w.r.t. the reduct functor $\_|_\sigma : \mathbf{Alg}(\Sigma') \to \mathbf{Alg}(\Sigma)$.
- For any equational specification morphism $\sigma : \langle \Sigma, \Phi \rangle \to \langle \Sigma', \Phi' \rangle$, for any model $A \in \mathit{Mod}(\Phi)$, there exist a model $\mathbf{F}_\sigma(A) \in \mathit{Mod}(\Phi')$ that is free over A w.r.t. the reduct functor $\_|_\sigma : \mathbf{Mod}(\langle \Sigma', \Phi' \rangle) \to \mathbf{Mod}(\langle \Sigma, \Phi \rangle)$.

Prove the above.

Facts

Consider a functor $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$, and object $A \in |\mathbf{K}|$, and an object $A' \in |\mathbf{K}'|$ free over A w.r.t. $\mathbf{G}$ with unit $\eta_A : A \to \mathbf{G}(A')$.

- A free objects over A w.r.t. $\mathbf{G}$ the initial objects in the comma category $(\mathbf{C}_A, \mathbf{G})$, where $\mathbf{C}_A : \mathbf{1} \to \mathbf{K}$ is the constant functor.
- A free object over A w.r.t. $\mathbf{G}$, if exists, is unique up to isomorphism.
- The function $(\_)^{\#} : \mathbf{K}(A, \mathbf{G}(B')) \to \mathbf{K}'(A', B')$ is bijective for each $B' \in |\mathbf{K}'|$.
- For any morphisms $g_1, g_2 : A' \to B'$ in $\mathbf{K}'$, $g_1 = g_2$ iff $\eta_A;\mathbf{G}(g_1) = \eta_A;\mathbf{G}(g_2)$.

Colimits as free objects

Fact: *In a category $\mathbf{K}$, given a diagram D of shape $G(D)$, the colimit of D in $\mathbf{K}$ is a free object over D w.r.t. the diagonal functor $\Delta_{\mathbf{K}}^{G(D)} : \mathbf{K} \to \mathbf{Diag}_{\mathbf{K}}^{G(D)}$.*

Spell this out for initial objects, coproducts, coequalisers, and pushouts

Left adjoints

Consider a functor $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$.

Fact: *Assume that for each object $A \in |\mathbf{K}|$ there is a free object over A w.r.t. $\mathbf{G}$, say $\mathbf{F}(A) \in |\mathbf{K}'|$ is free over A with unit $\eta_A : A \to \mathbf{G}(\mathbf{F}(A))$. Then the mapping:*

- $(A \in |\mathbf{K}|) \mapsto (\mathbf{F}(A) \in |\mathbf{K}'|)$
- $(f : A \to B) \mapsto ((f;\eta_B)^{\#} : \mathbf{F}(A) \to \mathbf{F}(B))$

form a functor $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$. Moreover, $\eta : \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$ is a natural transformation.

$$
\begin{tikzcd}
 & \mathbf{K} & \mathbf{K}' \arrow[l, "\mathbf{G}"'] \\
A \arrow[r, "\eta_A"] \arrow[d, "f"'] & \mathbf{G}(\mathbf{F}(A)) \arrow[d, "\mathbf{G}(\mathbf{F}(f))"] & \mathbf{F}(A) \arrow[d, "{\mathbf{F}(f) = (f;\eta_B)^{\#}}"] \\
B \arrow[r, "\eta_B"] & \mathbf{G}(\mathbf{F}(B)) & \mathbf{F}(B)
\end{tikzcd}
$$

Proof

$\mathbf{F}$ preserves identities:

$$\mathbf{F}(id_A) = (id_A;\eta_A)^{\#} = id_{\mathbf{F}(A)}$$

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, "id_A"'] & \mathbf{G}(\mathbf{F}(A)) \arrow[d, "{\mathbf{G}(id_{\mathbf{F}(A)}) = id_{\mathbf{G}(\mathbf{F}(A))}}"] & \mathbf{F}(A) \arrow[d, "id_{\mathbf{F}(A)}"] \\
A \arrow[r, "\eta_A"] & \mathbf{G}(\mathbf{F}(A)) & \mathbf{F}(A)
\end{tikzcd}$$

$\mathbf{F}$ preserves composition:

$$\mathbf{F}(f;g) = (f;g;\eta_C)^{\#} = \mathbf{F}(f);\mathbf{F}(g)$$

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, "f"'] & \mathbf{G}(\mathbf{F}(A)) \arrow[d, "\mathbf{G}(\mathbf{F}(f))"] \arrow[dd, bend left=60, "{\mathbf{G}(\mathbf{F}(f);\mathbf{F}(g)) = \mathbf{G}(\mathbf{F}(f));\mathbf{G}(\mathbf{F}(g))}"] & \mathbf{F}(A) \arrow[d, "\mathbf{F}(f)"] \arrow[dd, bend left=60, "\mathbf{F}(f);\mathbf{F}(g)"] \\
B \arrow[r, "\eta_B"] \arrow[d, "g"'] & \mathbf{G}(\mathbf{F}(B)) \arrow[d, "\mathbf{G}(\mathbf{F}(g))"] & \mathbf{F}(B) \arrow[d, "\mathbf{F}(g)"] \\
C \arrow[r, "\eta_C"] & \mathbf{G}(\mathbf{F}(C)) & \mathbf{F}(C)
\end{tikzcd}$$

Left adjoints

Definition: *A functor* $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ *is left adjoint to (a functor)* $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$ *with unit (natural transformation)* $\eta : \mathbf{Id_K} \to \mathbf{F};\mathbf{G}$ *if for all objects* $A \in |\mathbf{K}|$, $\mathbf{F}(A) \in |\mathbf{K}'|$ *is free over* A *with unit morphism* $\eta_A : A \to \mathbf{G}(\mathbf{F}(A))$.

Examples

- The term-algebra functor $T_\Sigma : \mathbf{Set}^S \to \mathbf{Alg}(\Sigma)$ is left adjoint to the carrier functor $|\_| : \mathbf{Alg}(\Sigma) \to \mathbf{Set}^S$, for any algebraic signature $\Sigma = \langle S, \Omega \rangle$.
- The ceiling $\lceil \_ \rceil : \mathbf{Real} \to \mathbf{Int}$ is left adjoint to the inclusion $i : \mathbf{Int} \hookrightarrow \mathbf{Real}$ of integers into reals.
- The path-category functor $\mathbf{Path} : \mathbf{Graph} \to \mathbf{Cat}$ is left adjoint to the graph functor $G : \mathbf{Cat} \to \mathbf{Graph}$.
- ... other examples given by the examples of free objects above ...

Uniqueness of left adjoints

Fact: *A left adjoint to any functor* $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$, *if exists, is determined uniquely up to a natural isomorphism: if* $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ *and* $\mathbf{F}' : \mathbf{K} \to \mathbf{K}'$ *are left adjoint to* $\mathbf{G}$ *with units* $\eta : \mathbf{Id_K} \to \mathbf{F};\mathbf{G}$ *and* $\eta' : \mathbf{Id_K} \to \mathbf{F}';\mathbf{G}$, *respectively, then there exists a natural isomorphism* $\tau : \mathbf{F} \to \mathbf{F}'$ *such that* $\eta;(\tau{\cdot}\mathbf{G}) = \eta'$.

$$
\begin{tikzcd}
 & \mathbf{G}(\mathbf{F}(A)) \arrow[dd, "{\mathbf{G}(\tau_A) = (\tau{\cdot}\mathbf{G})_A}"] \\
A \arrow[ur, "\eta_A"] \arrow[dr, "\eta'_A"'] & \\
 & \mathbf{G}(\mathbf{F}'(A))
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{F}(A) \arrow[dd, "\tau_A"] \\
\\
\mathbf{F}'(A)
\end{tikzcd}
$$

Proof: For each $A \in |\mathbf{K}|$, $\tau_A = (\eta'_A)^{\#}$.

Put also $\tau_A^{-1} = (\eta_A)^{\#'}$.

Then show:

- $\tau_A;\tau_A^{-1} = id_{\mathbf{F}(A)}$ and $\tau_A^{-1};\tau_A = id_{\mathbf{F}'(A)}$
- $\tau : \mathbf{F} \to \mathbf{F}'$ is indeed a natural transformation

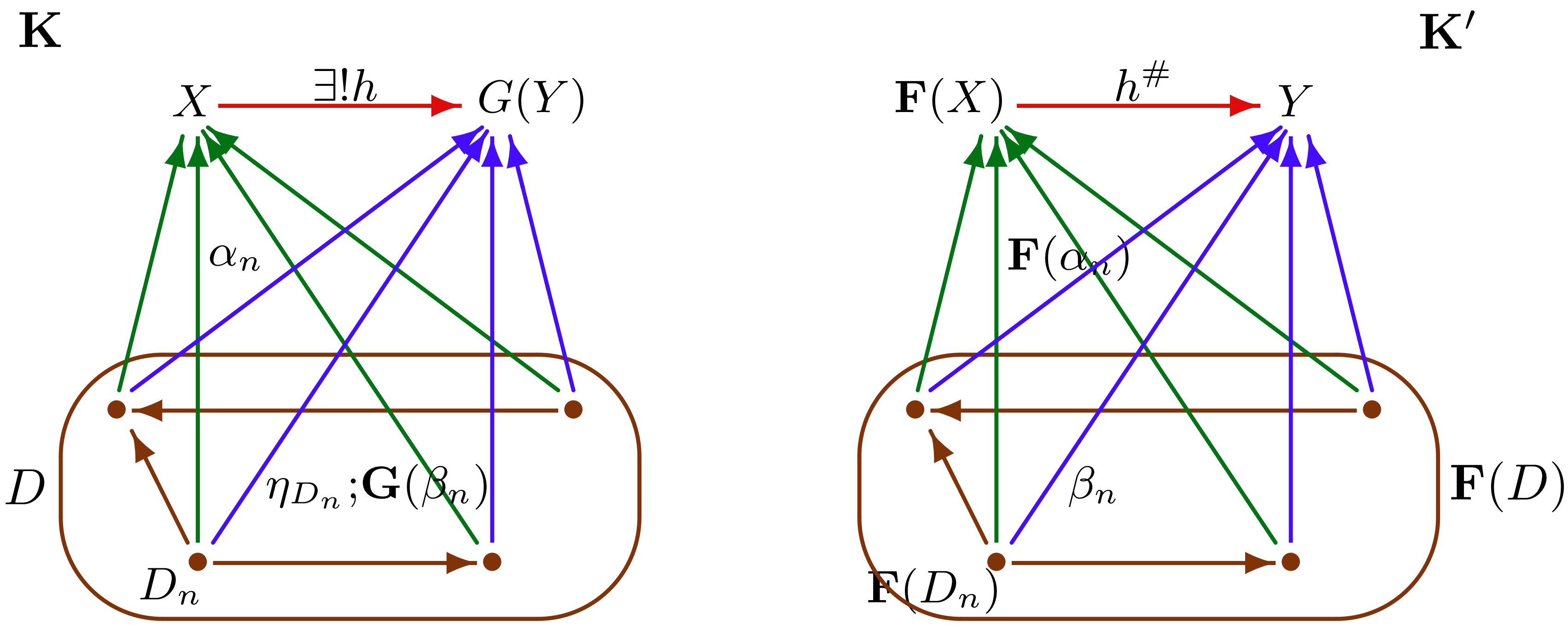
- For $f : A \to B$, $\mathbf{F}(f) = (f;\eta_B)^{\#}$.
- For $g_1, g_2 : \mathbf{F}(A) \to \bullet$, if $\eta_A;\mathbf{G}(g_1) = \eta_A;\mathbf{G}(g_2)$ then $g_1 = g_2$.

Left adjoints and colimits

Let $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ be left adjoint to $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$ with unit $\eta : \mathbf{Id_K} \to \mathbf{F};\mathbf{G}$.

Fact: $\mathbf{F}$ *is cocontinuous (preserves colimits).*

Proof:

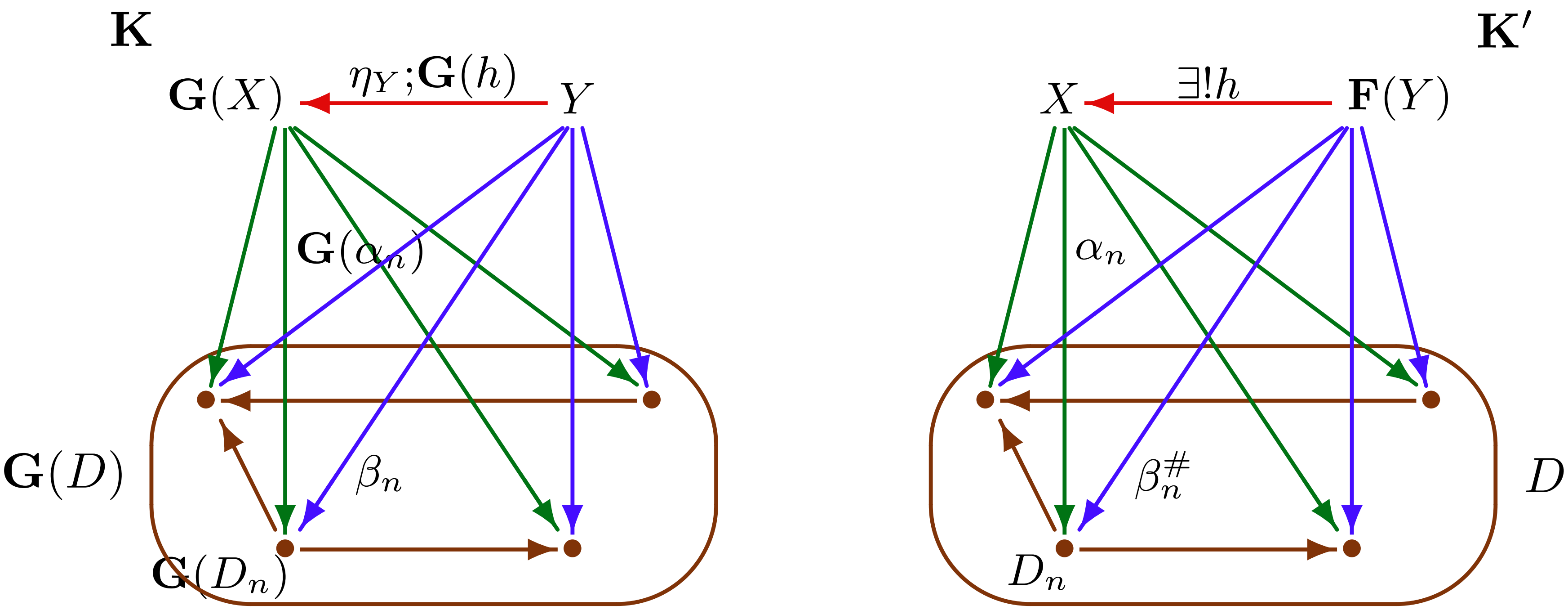


Left adjoints and limits

Let $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ be left adjoint to $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$ with unit $\eta : \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$.

Fact: $\mathbf{G}$ *is continuous (preserves limits).*

Proof:



Existence of left adjoints

Fact: *Let $\mathbf{K}'$ be a locally small complete category. Then a functor $\mathbf{G} : \mathbf{K} \to \mathbf{K}'$ has a left adjoint iff*

1. *$\mathbf{G}$ is continuous, and*
2. *for each $A \in |\mathbf{K}'|$ there exists a set $\{f_i : A \to \mathbf{G}(X_i) \mid i \in \mathcal{I}\}$ (of objects $X_i \in |\mathbf{K}|$ with morphisms $f_i : A \to \mathbf{G}(X_i)$, $i \in \mathcal{I}$) such that for each $B \in |\mathbf{K}'|$ and $h : A \to \mathbf{G}(B)$, for some $f : X_i \to B$, $i \in \mathcal{I}$, we have $h = f_i;f$.*

Proof:

"$\Rightarrow$": Let $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ be left adjoint to $\mathbf{G}$ with unit $\eta : \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$. Then 1 follows by the previous fact, and for 2 just put $\mathcal{I} = \{*\}$, $X_* = \mathbf{F}(A)$, and $f_* = \eta_A : A \to \mathbf{G}(\mathbf{F}(A))$

"$\Leftarrow$": It is enough to show that for each $A \in |\mathbf{K}'|$ the comma category $(\mathbf{C}_A, \mathbf{G})$ has an initial object. Under our assumptions, $(\mathbf{C}_A, \mathbf{G})$ is complete. The rest follows by the next fact.

On the existence of initial objects

Fact: *A locally small complete category* $\mathbf{K}$ *has an initial object if there exists a set of objects* $\mathcal{I} \subseteq |\mathbf{K}|$ *such that for all* $B \in |\mathbf{K}|$*, for some* $X \in \mathcal{I}$ *there is* $f : X \to B$*.*

Proof: Let $P \in |\mathbf{K}|$ be a product of $\mathcal{I}$, with projections $p_X : P \to X$ for $X \in \mathcal{I}$. Let $e : E \to P$ be an "equaliser" (limit) of all morphisms in $\mathbf{K}(P, P)$. Then E is initial in $\mathbf{K}$, since for any $B \in |\mathbf{K}|$:

- $e;p_X;f : E \to B$, where $f : X \to B$ for some $X \in \mathcal{I}$.
- Given $g_1, g_2 : E \to B$, take their equaliser $e' : E' \to E$. As in the previous item, we have $h : P \to E'$. Then $h;e;e' : P \to P$, and by the construction of $e : E \to P$, $e;h;e';e = e;id_P = id_E;e$. Now, since e is mono, $e;h;e' = id_E$, and so e' is a mono retraction, hence an isomorphism, which proves $g_1 = g_2$.

Cofree objects

Consider any functor $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$

Definition: *Given an object $A' \in |\mathbf{K}'|$, a cofree object under A' w.r.t. $\mathbf{F}$ is a $\mathbf{K}$-object $A \in |\mathbf{K}|$ together with a $\mathbf{K}$-morphism $\varepsilon_{A'} : \mathbf{F}(A) \to A'$ (called counit morphism) such that given any $\mathbf{K}$-object $B \in |\mathbf{K}|$ with $\mathbf{K}'$-morphism $g : \mathbf{F}(B) \to A'$, for a unique $\mathbf{K}$-morphism $g^{\#} : B \to A$ we have*

$$\boxed{\mathbf{G}(g^{\#});\varepsilon_{A'} = g}$$

$$\mathbf{K} \xrightarrow{\mathbf{F}} \mathbf{K}'$$

$$\begin{tikzcd}
A & \mathbf{F}(A) \arrow[r, "\varepsilon_{A'}"] & A' \\
B \arrow[u, "\exists! \; g^{\#}"] & \mathbf{F}(B) \arrow[u, "\mathbf{F}(g^{\#})"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Paradigmatic example:

Function spaces, coming soon

Examples

- Consider inclusion $i : \mathbf{Int} \hookrightarrow \mathbf{Real}$, viewing $\mathbf{Int}$ and $\mathbf{Real}$ as (thin) categories, and i as a functor between them. For any real $r \in \mathbf{Real}$, the floor of r, $\lfloor r \rfloor \in \mathbf{Int}$ is cofree under r w.r.t. i.

  What about cofree objects w.r.t. the inclusion of rationals into reals?

- Fix a set $X \in |\mathbf{Set}|$. Consider functor $\mathbf{F}_X : \mathbf{Set} \to \mathbf{Set}$ defined by:
  - for any set $A \in |\mathbf{Set}|$, $\mathbf{F}_X(A) = A \times X$
  - for any function $f : A \to B$, $\mathbf{F}_X(f) : A \times X \to B \times X$ is a function given by $\mathbf{F}_X(f)(\langle a, x \rangle) = \langle f(a), x \rangle$.

  Then for any set $A \in |\mathbf{Set}|$, the powerset $A^X \in |\mathbf{Set}|$ (i.e., the set of all functions from X to A) is a cofree objects under A w.r.t. $\mathbf{F}_X$. The counit morphism $\varepsilon_A : \mathbf{F}_X(A^X) = A^X \times X \to A$ is the evaluation function: $\varepsilon_A(\langle f, x \rangle) = f(x)$.

  A generalisation to deal with exponential objects will (not) be discussed later

Facts

Dual to those for free objects: Consider a functor $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$, object $A' \in |\mathbf{K}'|$, and an object $A \in |\mathbf{K}|$ cofree under A' w.r.t. $\mathbf{F}$ with counit $\varepsilon_{A'} : \mathbf{F}(A) \to A'$.

- Cofree objects under A' w.r.t. $\mathbf{F}$ are the terminal objects in the comma category $(\mathbf{F}, \mathbf{C}_{A'})$, where $\mathbf{C}_{A'} : \mathbf{1} \to \mathbf{K}'$ is the constant functor.
- A cofree object under A' w.r.t. $\mathbf{F}$, if exists, is unique up to isomorphism.
- The function $(\_)^{\#} : \mathbf{K}'(\mathbf{F}(B), A') \to \mathbf{K}(B, A)$ is bijective for each $B \in |\mathbf{K}|$.
- For any morphisms $g_1, g_2 : B \to A$ in $\mathbf{K}$, $g_1 = g_2$ iff $\mathbf{F}(g_1);\varepsilon_{A'} = \mathbf{F}(g_2);\varepsilon_{A'}$.

Limits as cofree objects

Fact: *In a category $\mathbf{K}$, given a diagram D of shape $G(D)$, the limit of D in $\mathbf{K}$ is a cofree object under D w.r.t. the diagonal functor $\Delta_{\mathbf{K}}^{G(D)} : \mathbf{K} \to \mathbf{Diag}_{\mathbf{K}}^{G(D)}$.*

Spell this out for terminal objects, products, equalisers, and pullbacks

Right adjoints

Consider a functor $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$.

Fact: *Assume that for each object $A' \in |\mathbf{K}'|$ there is a cofree object under A' w.r.t. $\mathbf{F}$, say $\mathbf{G}(A') \in |\mathbf{K}|$ is cofree under A' with counit $\varepsilon_{A'} : \mathbf{F}(\mathbf{G}(A')) \to A'$. Then the mapping:*

- $(A' \in |\mathbf{K}'|) \mapsto (\mathbf{G}(A') \in |\mathbf{K}|)$
- $(g : B' \to A') \mapsto ((\varepsilon_{B'};g)^{\#} : \mathbf{G}(B') \to \mathbf{G}(A'))$

form a functor $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$. Moreover, $\varepsilon : \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$ is a natural transformation.

$$\mathbf{K} \xrightarrow{\mathbf{G}} \mathbf{K}'$$

$$\begin{tikzcd}
\mathbf{G}(A') & \mathbf{F}(\mathbf{G}(A')) \arrow[r, "\varepsilon_{A'}"] & A' \\
\mathbf{G}(B') \arrow[u, "{\mathbf{G}(g) = (\varepsilon_{B'};g)^{\#}}"] & \mathbf{F}(\mathbf{G}(B')) \arrow[u, "\mathbf{F}(\mathbf{G}(g))"] \arrow[r, "\varepsilon_{B'}"] & B' \arrow[u, "g"']
\end{tikzcd}$$

Right adjoints

Definition: *A functor $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$ is right adjoint to (a functor) $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ with counit (natural transformation) $\varepsilon : \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$ if for all objects $A' \in |\mathbf{K}'|$, $\mathbf{G}(A') \in |\mathbf{K}|$ is cofree under A' with counit morphism $\varepsilon_{A'} : \mathbf{F}(\mathbf{G}(A')) \to A'$.*

Fact: *A right adjoint to any functor $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$, if exists, is determined uniquely up to a natural isomorphism: if $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$ and $\mathbf{G}' : \mathbf{K}' \to \mathbf{K}$ are right adjoint to $\mathbf{F}$ with counits $\varepsilon : \mathbf{G};\mathbf{F}$ and $\varepsilon' : \mathbf{G}';\mathbf{F}$, respectively, then there exists a natural isomorphism $\tau : \mathbf{G} \to \mathbf{G}'$ such that $(\tau{\cdot}\mathbf{F});\varepsilon' = \varepsilon$.*

Fact: *Let $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$ be right adjoint to $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ with counit $\varepsilon : \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$. Then $\mathbf{G}$ is continuous (preserves limits) and $\mathbf{F}$ is cocontinuous (preserves colimits).*

From left adjoints to adjunctions

Fact: *Let* $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ *be left adjoint to* $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$ *with unit* $\eta : \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$*.*
Then there is a natural transformation $\varepsilon : \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$ *such that:*

- $(\mathbf{G}\cdot\eta);(\varepsilon\cdot\mathbf{G}) = id_{\mathbf{G}}$

$$\begin{tikzcd}
\mathbf{G}(A') \arrow[rr, "\eta_{\mathbf{G}(A')}"] \arrow[drr, "id_{\mathbf{G}(A')}"'] & & \mathbf{G}(\mathbf{F}(\mathbf{G}(A'))) \arrow[d, "\mathbf{G}(\varepsilon_{A'})"] & \mathbf{F}(\mathbf{G}(A')) \arrow[d, "\varepsilon_{A'}"] \\
& & \mathbf{G}(A') & A'
\end{tikzcd}$$

- $(\eta\cdot\mathbf{F});(\mathbf{F}\cdot\varepsilon) = id_{\mathbf{F}}$

$$\begin{tikzcd}
\mathbf{G}(\mathbf{F}(A)) & \mathbf{F}(\mathbf{G}(\mathbf{F}(A))) \arrow[rr, "\varepsilon_{\mathbf{F}(A)}"] & & \mathbf{F}(A) \\
A \arrow[u, "\eta_A"'] & \mathbf{F}(A) \arrow[u, "\mathbf{F}(\eta_A)"] \arrow[urr, "id_{\mathbf{F}(A)}"'] & &
\end{tikzcd}$$

Proof (idea):
Put $\varepsilon_{A'} = (id_{\mathbf{G}(A')})^{\#}$.

From right adjoints to adjunctions

Fact: *Let* $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$ *be right adjoint to* $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ *with counit* $\varepsilon : \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$. *Then there is a natural transformation* $\eta : \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$ *such that:*

- $(\mathbf{G}{\cdot}\eta);(\varepsilon{\cdot}\mathbf{G}) = id_{\mathbf{G}}$

$$\begin{tikzcd} \mathbf{G}(A') \arrow[r, "\eta_{\mathbf{G}(A')}"] \arrow[dr, "id_{\mathbf{G}(A')}"'] & \mathbf{G}(\mathbf{F}(\mathbf{G}(A'))) \arrow[d, "\mathbf{G}(\varepsilon_{A'})"] & \mathbf{F}(\mathbf{G}(A')) \arrow[d, "\varepsilon_{A'}"] \\ & \mathbf{G}(A') & A' \end{tikzcd}$$

- $(\eta{\cdot}\mathbf{F});(\mathbf{F}{\cdot}\varepsilon) = id_{\mathbf{F}}$

$$\begin{tikzcd} \mathbf{G}(\mathbf{F}(A)) & \mathbf{F}(\mathbf{G}(\mathbf{F}(A))) \arrow[r, "\varepsilon_{\mathbf{F}(A)}"] & \mathbf{F}(A) \\ A \arrow[u, "\eta_A"] & \mathbf{F}(A) \arrow[u, "\mathbf{F}(\eta_A)"] \arrow[ur, "id_{\mathbf{F}(A)}"'] & \end{tikzcd}$$

Proof (idea):
Put $\eta_A = (id_{\mathbf{F}(A)})^{\#}$.

From adjunctions to left and right adjoints

Fact: *Consider two functors* $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ *and* $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$ *with natural transformations* $\eta : \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$ *and* $\varepsilon : \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$ *such that:*

- $(\mathbf{G}\cdot\eta);(\varepsilon\cdot\mathbf{G}) = id_{\mathbf{G}}$
- $(\eta\cdot\mathbf{F});(\mathbf{F}\cdot\varepsilon) = id_{\mathbf{F}}$

Then:

- $\mathbf{F}$ *is left adjoint to* $\mathbf{G}$ *with unit* η*.*
- $\mathbf{G}$ *is right adjoint to* $\mathbf{F}$ *with counit* ε*.*

Proof: For $A \in |\mathbf{K}|$, $B' \in |\mathbf{K}'|$ and $f : A \to \mathbf{G}(B')$, define $f^{\#} = \mathbf{F}(f);\varepsilon_{B'}$. Then $f^{\#} : \mathbf{F}(A) \to B'$ satisfies $\eta_A;\mathbf{G}(f^{\#}) = f$ and is the only such morphism in $\mathbf{K}'(\mathbf{F}(A), B')$. This proves that $\mathbf{F}(A)$ is free over A with unit η_A, and so indeed, $\mathbf{F}$ is left adjoint to $\mathbf{G}$ with unit η.

The proof that $\mathbf{G}$ is right adjoint to $\mathbf{F}$ with counit ε is similar.

Adjunctions

Definition: *An adjunction between categories* $\mathbf{K}$ *and* $\mathbf{K}'$ *is*

$$\boxed{\langle \mathbf{F}, \mathbf{G}, \eta, \varepsilon \rangle}$$

where $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ *and* $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$ *are functors, and* $\eta : \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$ *and* $\varepsilon : \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$ *natural transformations such that:*

- $(\mathbf{G}{\cdot}\eta);(\varepsilon{\cdot}\mathbf{G}) = id_{\mathbf{G}}$
- $(\eta{\cdot}\mathbf{F});(\mathbf{F}{\cdot}\varepsilon) = id_{\mathbf{F}}$

Equivalently, such an adjunction may be given by:

- Functor $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$ and all $A \in |\mathbf{K}|$, a free object over A w.r.t. $\mathbf{G}$.
- Functor $\mathbf{G} : \mathbf{K}' \to \mathbf{K}$ and its left adjoint.
- Functor $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ and all $A' \in |\mathbf{K}'|$, a cofree object under A' w.r.t. $\mathbf{F}$.
- Functor $\mathbf{F} : \mathbf{K} \to \mathbf{K}'$ and its right adjoint.