WHAT IS A STACK?

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There are several approaches to defining stacks. I will take the approach in which a stack is viewed as the 2-categorical analogue of a sheaf[1]. From this perspective stack theory can be viewed as a natural arena for the 2-categorical analogue of geometry, just as sheaf theory been understood as a natural arena for geometry. Aside from having a pleasing geometric interpretation, the 2-categorical viewpoint has the benefits of being the most general and of leading the way towards higher stacks (as in [To]).

I will state the definition of a stack immediately, and then work towards understanding the definition by recalling some concepts from category theory and giving some examples.

Recall that a presheaf on a site S with values in a category $\mathcal{C}$ is nothing more than a functor

$$F : S^{op} \to \mathcal{C}.$$

A sheaf is a presheaf satisfying the sheaf axiom, which says that for every cover $\{U_i \to X\}_{i \in I}$ in S, $F(X)$ is the equalizer of the diagram:

$$\prod_{i \in I} F(U_i) \rightrightarrows \prod_{(i,j) \in I \times I} F(U_{ij}), \tag{0.1}$$

where $U_{ij} = U_i \times_X U_j$ denotes the fibered product.

Now let $\mathcal{C}$ be a 2-category, and view a site S as a 2-category with only identity 2-arrows. Then a **stack** on S is a 2-functor

$$F : S^{op} \to \mathcal{C}$$

satisfying the stack axioms. The stack axioms are encoded in the following statement: for every cover $\{U_i \to X\}_{i \in I}$ in S, $F(X)$ is an equalizer, or more correctly a 2-categorical limit, of the diagram:

$$\prod F(U_i) \rightrightarrows \prod F(U_{ij}) \mathrel{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}} \prod F(U_{ijk}) \ . \tag{0.2}$$

Now I will write down the definitions of site, equalizer and limit, 2-category, 2-functor, and 2-categorical limit, which are all used in the definition of a stack. In the section after that there will be geometric examples.

1.–Definitions

Sites. A site is a category with just enough extra structure for the sheaf axiom to be phrased. More precisely, a site $\mathcal{S} = (S, J)$ is a category S equipped with a collection J of covers of objects of S. A **cover** of an object X is a family of arrows $\{U_i \to X\}$ of S. The collection J must satisfy:

(1) Every isomorphism $\{Y \to X\}$ is a cover.
(2) A cover of a cover is a cover, that is, if $\{U_i \to X\}$ is a cover and $\{V_{j_i} \to U_i\}$ are covers for each i then the composite $\{V_{i_j} \to U_i \to X\}$ is also a cover.
(3) Pullbacks of covers are covers, that is, if $Y \to X$ is an arrow in S and $\{U_i \to X\}$ is a cover then $\{U_i \times_X Y \to Y\}$ is a cover as well [2].

C.D. partially supported by NSF grant DMS-0703718.

[1] This is the approach taken, for instance, in the final chapters of [KS].

[2] We assume that for any arrow $Y \to X$ and any member U_i of a cover $\{U_i \to X\}$, the fibered product $Y \times_X U_i$ exists. This requirement can be circumvented by using the language of covering sieves, and does not result in any loss of generality. We also assume all sites are **subcanonical**, meaning that the Yoneda functors $\mathrm{Hom}_S(\ , X) : S \to \mathrm{Sets}$ are not just presheaves, but sheaves. All of the sites mentioned below are subcanonical.

When the collection of covers is understood we refer to the site and its underlying category interchangeably. For convenience we write $U_{ij} \equiv U_i \times_X U_j$ and refer to this as the intersection of U_i and U_j. This notation still treats the indices as an ordered pair, so that $U_{ij} \neq U_{ji}$.

Typical examples of sites are Top (the category of topological spaces with open covers), Man (the category of smooth manifolds and open covers), and $\mathrm{Aff}_{\daleth}$ (the site of affine $\daleth$-schemes with fpqc covers). Also, for any site S and object X of S, there is the site of objects over X, written $S|_X$, and the site $\mathrm{Cov}(X)$ of all members of covers of X (i.e. all "open subsets" of X). $\mathrm{Cov}(X)$ is the site one implicitly refers to when speaking of a "sheaf on X."

Equalizers and limits. Let $\mathcal{C}$ be a category and $f, g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. An **equalizer** of f and g is an object $E \in \mathcal{C}$ and an arrow $E \xrightarrow{\alpha} A$ such that:

(1) $f \circ \alpha = g \circ \alpha$
(2) Any other arrow $E' \xrightarrow{\alpha'} A$ satisfying $f \circ \alpha' = g \circ \alpha'$ determines an unique arrow $E' \xrightarrow{h} E$ satisfying
$$(E' \xrightarrow{\alpha'} A) \;=\; (E' \xrightarrow{h} E \xrightarrow{\alpha} A).$$

An equalizer, if it exists, is unique up to unique isomorphism, which makes us comfortable saying "the" equalizer instead of "an" equalizer. Analogous usage of the word "the" for limits and 2-limits will be in force without mention.

In Sets, the equalizer of two functions $f, g : A \to B$ is the subset
$$\{a \in A \mid f(a) = g(a)\ \} \subset A.$$
Thus for a presheaf of sets F, the sheaf axiom (0.1) states that for any cover $\{U_i \to X\}$,
$$F(X) \simeq \{\ f = (f_i) \in \prod F(U_i) \mid \rho_1 f = \rho_2 f\ \}$$
where ρ_1 is induced by omitting the second factor of an intersection $U_{ij} \to U_i$, and ρ_2 corresponds to omitting the first factor $U_{ij} \to U_j$. (For example $(\rho_1 f)_{ij} = f_i|_{U_{ij}}$.)

The notions of site and equalizer are all one needs to define sheaves. For the rest of this section we describe limits and express the sheaf axiom in terms of limits, in a couple of ways. The main purpose for doing this is just to develop notation that will be used for the stack axiom.

A **limit** of a functor $F : \mathcal{I} \to \mathcal{C}$, is an object $c \in \mathcal{C}$ (often denoted $\lim_{\mathcal{I}} F$) together with an arrow $c \xrightarrow{\alpha_x} F(x)$ for each $x \in \mathcal{I}$, such that

(1) For each $\gamma \in \mathrm{Hom}_{\mathcal{I}}(x, y)$ the following diagram is commutative:
$$\begin{tikzcd} c \arrow[r, "\alpha_x"] \arrow[dr, "\alpha_y"'] & F(x) \arrow[d, "F(\gamma)"] \\ & F(y) \end{tikzcd}$$
(2) Any other such data $\{c', c' \xrightarrow{\alpha'_x} F(x)\}$ determines an unique arrow $c' \to c$ making all possible composite diagrams commute.

The image of F (that is, all objects of the form $F(x)$ and all arrows of the form $F(x) \xrightarrow{F(\gamma)} F(y)$) forms what is called a diagram in $\mathcal{C}$, and in this case a **limit of a diagram** is a synonym for the limit of the functor.

An equalizer is a special case of a limit, indeed let $\Delta_2 := ([1] \overset{a,b}{\rightrightarrows} [2])$ be the category with two objects and two non-identity arrows, then the equalizer of two arrows $f, g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is the same as $\lim_{\Delta_2} F$, where $F : \Delta_2 \to \mathcal{C}$ is defined by:
$$F([1]) = A, \quad F([2]) = B, \quad F(a) = f, \quad F(b) = g.$$

In particular, the sheaf axiom can interpreted as a limit. For a presheaf $F : S^{op} \to \mathcal{C}$ and a cover $\mathfrak{U} = \{U_i \to X\}$, define $F_{\mathfrak{U}} : \Delta_2 \to \mathcal{C}$ by

$$F_{\mathfrak{U}}([1]) = \prod F(U_i), \quad F_{\mathfrak{U}}([2]) = \prod F(U_{ij}), \quad F_{\mathfrak{U}}(a) = \rho_1, \quad F_{\mathfrak{U}}(b) = \rho_2.$$

Then F is a sheaf if and only if for all such covers

$$F(X) \simeq \lim_{\Delta_2} F_{\mathfrak{U}}.$$

There is a "bigger" limit one can form from the presheaf F. Let Δ be the category whose objects are $\{[1], [2], [3], \dots\}$ and whose arrows are

$$\mathrm{Hom}_{\Delta}([\ell], [k]) := \text{Order preserving injections } \{1, \dots, \ell\} \to \{1, \dots, k\}.$$

so that there are k-choose-ℓ arrows from $[\ell]$ to $[k]$. Composition is given by composition of functions (note then that distinct pairs of arrows can compose to the same arrow).

Let Δ_n denote the full subcategory whose objects are $\{[1], \dots, [n]\}$. Then $F_{\mathfrak{U}}$ extends to $F_{\mathfrak{U}} : \Delta_n \to \mathcal{C}$ and to $F_{\mathfrak{U}} : \Delta \to \mathcal{C}$, by the formulas

$$[k] \mapsto F_{\mathfrak{U}}([k]) := \prod F(U_{i_1 \dots i_k})$$

$$\mathrm{Hom}_{\Delta}([k-1], [k]) \ni (j^{th} \text{ arrow }) \longmapsto \rho_{1 \dots j-1 j+1 \dots k} \in \mathrm{Hom}_{\mathcal{C}}(F_{\mathfrak{U}}([k-1]), F_{\mathfrak{U}}([k]))$$

where $\rho_{1 \dots j-1 j+1 \dots k}$ comes from the arrows $U_{i_1 \dots i_k} \to U_{i_1 \dots i_{j-1} i_{j+1} \dots i_k}$ omitting the j^{th} factor of a k-fold intersection.

By virtue of F being a functor, the arrows

$$\lim_{\Delta} F_{\mathfrak{U}} \to \cdots \to \lim_{\Delta_3} F_{\mathfrak{U}} \to \lim_{\Delta_2} F_{\mathfrak{U}}$$

are all isomorphisms, so the sheaf axiom is also equivalent to $F(X) \simeq \lim_{\Delta} F_{\mathfrak{U}}$. This phrasing of the axiom is in some sense the most natural, and translates directly into a stack axiom when $\mathcal{C}$ is replaced by a 2-category.

2.–Definitions

2-Categories. A 2-category $\mathcal{C}$ is the following data:

(1) A collection of **objects** $A, B, \dots$ of $\mathcal{C}$.
(2) A category $\mathcal{C}(A, B)$ for every pair of objects.
(3) An associative composition functor $\circ : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \to \mathcal{C}(A, C)$ for every triplet of objects.

Objects of $\mathcal{C}(A, B)$ are called **1-arrows** while morphisms in $\mathcal{C}(A, B)$ are called **2-arrows**. These must satisfy the following properties:

- For each A, the category $\mathcal{C}(A; A)$ has an identity object which acts as a unit for the composition functor.
- The interchange law holds, that is, for composeable 2-arrows $\alpha, \beta \in \mathrm{Mor}\,\mathcal{C}(B, C)$ and $\eta, \tau \in \mathrm{Mor}\,\mathcal{C}(A, B)$, we have $(\alpha \cdot \beta) \circ (\eta \cdot \tau) \equiv (\alpha \circ \eta) \cdot (\beta \circ \tau)$.

The standard example to keep in mind is the 2-category **CAT** in which objects are small categories, 1-arrows are functors, and 2-arrows are natural transformations. Thus $\mathrm{CAT}(A, B)$ is the category of functors from A to B and natural transformations between them. In this case the interchange law is a relation automatically satisfied by natural transformations between composeable functors.

Remark 2.1. What we have defined here is sometimes referred to as a *strict* 2-category, in contrast with the notion of a weak 2-category (also called a bicategory). We are content to restrict to strict 2-categories because they are what arise in stack theory, though in any case every weak 2-category is equivalent (via a weak 2-functor) to a strict 2-category. On the other hand, in our setup, 2-functors between strict 2-categories need not be strict.

2-Functors. A 2-functor $F : \mathcal{C} \to \mathcal{D}$ between 2-categories is an assignment

$$\mathrm{Objects}(\mathcal{C}) \ni A \longmapsto FA \in \mathrm{Objects}(\mathcal{D})$$

together with functors

$$\mathcal{C}(A,B) \xrightarrow{F_{A,B}} \mathcal{D}(FA,FB)$$

that preserve identity objects and intertwine the compositions of $\mathcal{C}$ and $\mathcal{D}$ up to coherent natural transformations.

Note that every category can be viewed as a 2-category with only identity 2-arrows, so it makes sense to speak of a 2-functor $\mathcal{C} \to \mathcal{D}$ even when $\mathcal{C}$ or $\mathcal{D}$ is only a 1-category.

Now we are ready to understand the definition of a stack:

Stacks. Let S be a site and let $\mathcal{C}$ be a 2-category. A **stack** is a 2-functor $F : S^{op} \to \mathcal{C}$ satisfying the stack axiom, which says that for every object X of S and every cover $\{U_i \to X\}$, $F(X)$ is an equalizer, or more correctly a 2-categorical limit, of the diagram:

$$\prod F(U_i) \rightrightarrows \prod F(U_{ij}) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \prod F(U_{ijk}) \ . \tag{2.1}$$

To finish, we just need to define 2-categorical limits.

Categorical concepts in 2-categories. A general prescription for giving the 2-categoric analogue of a categorical concept (that is, a concept phrased in terms of objects and arrows) is to replace arrows with 1-arrows and equality signs with invertible 2-arrows.

Performing the replacements turns a commutative diagram into a **2-commutative diagram**, which is by definition a diagram of objects and 1-arrows which commutes up to an invertible 2-arrow. For example the 2-commutative square:

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] \arrow[ur, Rightarrow, "\eta"] & Z \end{tikzcd}$$

corresponds to the equation $\eta : f \circ i \Rightarrow g \circ j$, where η is an invertible 2-arrow.

2-Limits. To obtain the definition of a 2-limit, we follow the prescription, that is we take the commutative diagrams which define a limit and replace them with 2-commutative diagrams. Actually we will restrict to the slightly simpler limit of a 2-functor whose domain is a 1-category, which is all that is needed for stacks.

Let $F : \mathcal{I} \to \mathcal{C}$ be a 2-functor, and suppose that $\mathcal{I}$ is a 1-category. A **2-limit** of F, is an object $c \in \mathcal{C}$ together with a 1-arrow $c \xrightarrow{\alpha_x} F(x)$ for each $x \in \mathcal{I}$, and a 2-arrow α_τ for each $\tau \in \mathrm{Hom}_{\mathcal{I}}(x,y)$ such that:

(1) All of the diagrams are 2-commutative:

$$\begin{tikzcd} c \arrow[r, "\alpha_x"] \arrow[d, equal] & F(x) \arrow[d, "F(\tau)"] \arrow[dl, Rightarrow, "\alpha_\tau"'] \\ c \arrow[r, "\alpha_y"'] & F(y) \end{tikzcd}$$

(2) For each composition of arrows $x \xrightarrow{\tau} y \xrightarrow{\eta} z$ in $\mathcal{I}$ we have $\alpha_\eta \alpha_\tau = \alpha_{\eta\tau}$.

(3) Any other such data

$$c' \xrightarrow{\alpha'_x} F(x), \quad F(\tau)\alpha'_x \xRightarrow{\alpha'_\tau} \alpha'_y$$

determines an essentially unique 1-arrow $c' \to c$ making all possible composite diagrams 2-commute.

It makes sense to refer to a 2-limit as a 2-limit of a diagram, where the diagram is just the image of the 2-functor.

Thus for a 2-functor $F : \mathcal{S}^{op} \to \mathcal{C}$ and cover $\mathfrak{U} = \{U_i \to X\}$, the diagram

$$\prod F(U_i) \underset{\rho_2}{\overset{\rho_1}{\rightrightarrows}} \prod F(U_{ij}) \underset{\rho_{23}}{\overset{\rho_{12},\,\rho_{13}}{\rightrightarrows\!\!\to}} \prod F(U_{ijk}) \,. \tag{2.2}$$

is the image of $F_{\mathfrak{U}} : \Delta_3 \to \mathcal{C}$. Its 2-limit should be an object $A_{\mathfrak{U}} \in \mathcal{C}$, together with 1-arrows

$$A_{\mathfrak{U}} \xrightarrow{a} \prod F(U_i), \quad A_{\mathfrak{U}} \xrightarrow{b} \prod F(U_{ij}), \quad A_{\mathfrak{U}} \xrightarrow{c} \prod F(U_{ijk})$$

and invertible 2-arrows

$$\rho_1 a \Rightarrow b, \quad \rho_2 a \Rightarrow b, \quad \rho_{12} b \Rightarrow c, \quad \rho_{13} b \Rightarrow c, \quad \rho_{23} b \Rightarrow c. \tag{2.3}$$

Actually, one should also specify 2-arrows to c such as $\rho_{12}\rho_1 a \Rightarrow c$ but to do so is redundant because necessarily, $(\rho_{12}\rho_1 a \Rightarrow c) = (\rho_{12}(\rho_1 a) \Rightarrow \rho_{12} b \Rightarrow c)$.

Because F is a 2-functor, $\lim_\Delta F_{\mathfrak{U}} \simeq \lim_{\Delta_3} F_{\mathfrak{U}}$, which leads to the aesthetically pleasing statement of the stack axioms, that $F(X) \simeq \lim_\Delta F_{\mathfrak{U}}$ for all X and $\mathfrak{U}$.

For a general 2-category $\mathcal{C}$ we have said all that can be said about $A_{\mathfrak{U}}$. But for $\mathcal{C} = \mathrm{CAT}$ the 2-limit exists and can be constructed as follows.

- The objects of $A_{\mathfrak{U}}$ are:

  $$\{ \ (P = (P_i) \in \prod F(U_i) \ , \rho_2 P \overset{\phi_P}{\to} \rho_1 P \ ) \mid \rho_{12}(\phi_P) \circ \rho_{23}(\phi_P) = \rho_{13}(\phi_P) \ \}$$

  The defining equation $\rho_{12}(\phi) \circ \rho_{23}(\phi) = \rho_{13}(\phi)$ only makes sense because of the equalities

  $$\rho_{12}\rho_1 = \rho_{13}\rho_1, \quad \rho_{23}\rho_1 = \rho_{12}\rho_2, \quad \rho_{13}\rho_2 = \rho_{23}\rho_2$$

  (which are a consequence of the relations in Δ).
- The morphisms in $A_{\mathfrak{U}}$ are:

  $$\mathrm{Hom}_{A_{\mathfrak{U}}}((P, \phi_P), (Q, \phi_Q)) = \{ \ P \xrightarrow{\alpha} Q \mid \phi_Q \rho_2(\alpha) = \rho_1(\alpha)\phi_P \ \}.$$

- The arrow $A_{\mathfrak{U}} \xrightarrow{a} \prod F(U_i)$ is just the projection $(P, \rho_2 P \xrightarrow{\phi} \rho_1 P) \mapsto P$.
- The arrow $A_{\mathfrak{U}} \xrightarrow{b} \prod F(U_{ij})$ is $b := \rho_1 a$, and similarly $c := \rho_{13}\rho_1 a = \rho_{13} b$.
- The arrows from Equation (2.3) are somewhat obvious: $\rho_1 a \Rightarrow b$ is the identity, $\rho_2 a \Rightarrow b$ is the assignment $(P, \rho_2 P \overset{\phi}{\to} \rho_1 P\ ) \mapsto \phi$, which the reader may verify is a natural transformation, and $\rho_{12} b = \rho_{12}\rho_1 a \Rightarrow \rho_{13}\rho_1 a = c$ is the identity, $\rho_{13} b = \rho_{13}\rho_1 a \Rightarrow c$ is also the identity, and $\rho_{23} b = \rho_{23}\rho_1 a = \rho_{12}\rho_2 a \Rightarrow \rho_{12}\rho_1 a = c$ is ρ_{12} applied to the arrow $\rho_2 a \Rightarrow \rho_1 a$.
- The fact that the composite $\rho_{13}\rho_2 a \Rightarrow \rho_{23} b \Rightarrow c$ equals $\rho_{13}\rho_2 a \Rightarrow c$ is equivalent to the conditions $\rho_{12}(\phi) \circ \rho_{23}(\phi) = \rho_{13}(\phi)$.
- Finally, any other such data $(A', a', b', \rho_1 a' \overset{\eta}{\Rightarrow} b', \dots)$ determines an arrow $A' \to A_{\mathfrak{U}}$ by the formulas:

  $$A' \ni x \mapsto (a'(x), \eta(x)) \in A_{\mathfrak{U}} \quad \mathrm{Mor}\, A' \ni \gamma \mapsto a'(\gamma) \in \mathrm{Mor}\, A_{\mathfrak{U}}.$$

We leave it to the reader to check that the remaining details about the fact that this is indeed the 2-limit.

This category, $A_{\mathfrak{U}}$, is called descent data in the algebraic geometry literature. The functor $F(X) \to A_{\mathfrak{U}}$ is an equivalence of categories if and only if it is fully faithful and essentially surjective, and taking these two conditions separately reproduces the more common phrasing of the stack axioms (for groupoid or category valued stacks) which are:

(1) ($F(X) \to A_{\mathfrak{U}}$ is essentially surjective) Given a collection of objects $P_i \in F(U_i)$ on a cover $\{U_i \to X\}$ and isomorphisms $P_j|_{U_{ij}} \xrightarrow{\phi_{ij}} P_i|_{U_{ij}}$ satisfying $\phi_{ij}\phi_{jk} = \phi_{ik}$ on triple intersections U_{ijk}, there exists a "glued together" object $P \in F(X)$ and isomorphisms $P|_{U_i} \xrightarrow{s_i} P_i$ satisfying $s_i = \phi_{ij} s_j$.

(2) ($F(X) \to A_{\mathfrak{U}}$ is fully faithful) Given a collection of arrows $\gamma_i \in \operatorname{Mor} F(U_i)$ such that $\gamma_i|_{U_{ij}} = \gamma_j|_{U_{ij}}$, there exists a unique "glued together" arrow $\gamma \in \operatorname{Mor} F(X)$ such that $\gamma|_{U_i} = \gamma_i$ for all i.

3. Stackification and examples

In this section S denotes a subcanonical site, meaning that for every $X \in S$ the Yoneda functor $\underline{X} := \operatorname{Hom}_S(\ ,X) : S \to \text{Sets}$ is a sheaf.

3.1. **Sheafification and stackification.** One way to come about examples of sheaves is to start with a presheaf $F : S^{op} \to \mathcal{C}$, and define a new functor (which is automatically a sheaf) by the formula:

$$F^{sh} : S^{op} \to \mathcal{C}, \qquad F^{sh}(X) := \operatorname{colim}_{\mathfrak{U} \in \operatorname{Cov} X} \lim_{\Delta} F_{\mathfrak{U}}. \tag{3.1}$$

Recall that for a cover $\mathfrak{U} = \{U_i \to X\}$, $\lim_\Delta F_{\mathfrak{U}}$ is simply the equalizer of $\prod F(U_i) \rightrightarrows \prod F(U_{ij})$. To form the colimit over $\operatorname{Cov} X$ we are viewing the covers of X as a set directed by refinement. F^{sh} is of course called the **sheafification** of F; sheafification has nice properties which can be encoded by saying that it is a functor and that it is left adjoint to the inclusion $\text{Sheaves}(S) \hookrightarrow \text{Presheaves}(S)$.

One obtains examples of stacks by an analogous operation. Given a 2-category $\mathcal{C}$ and a 2-functor $F : S^{op} \to \mathcal{C}$, the **stackification** F^{st} of F is the stack defined by Equation (3.1), where now the limit and colimit are interpreted in the 2-categorical sense.

Stackification can be counterintuitive, because while a sheaf with values in a 1-category is already a special case of a stack, (that is the sheaf and stack axioms coincide on a 1-category), a sheaf $F : S^{op} \to \mathcal{C}$ with values in a 2-category is dramatically changed by stackification. To give an explicit example, consider the sheaf

$$\underline{G} = \operatorname{Hom}_S(\ ,G) : S \to \text{Groups} \subset \text{Groupoids}$$

for G a group object in S. It is very different from its stackification, which we denote

$$[G] : S \to \text{Groupoids}$$

(the notations $\mathscr{B}G$, Prin_G and Tors_G are also common). Let us see precisely what $[G](X)$ is.

For a fixed cover $\mathfrak{U} = \{U_i \to X\}$, $\lim_{\Delta_3} \underline{G}_{\mathfrak{U}} = A_{\mathfrak{U}}$ (defined in the previous section) is the groupoid whose objects are pairs

$$(P_i) \in \prod \operatorname{Hom}_S(U_i, G), \quad (P_i|_{U_{ij}} \xrightarrow{\phi_{ji}} P_j|_{U_{ij}})$$

satisfying $\phi_{ij}\phi_{jk} = \phi_{ik}$ on U_{ijk}. Note that ϕ_{ij} can be viewed as left multiplication by maps $\phi_{ij} \in \operatorname{Hom}_S(U_{ij}, G)$. An arrow of $A_{\mathfrak{U}}$ from (ϕ_{ij}) to (ψ_{ij}) is a collection $(\eta_i) \in \prod \operatorname{Hom}_S(U_i, G)$ satisfying $\eta_i \phi_{ij} = \psi_{ij}\eta_j$.

Then an object of the colimit $[G](X) = \operatorname{colim}_{\mathfrak{U}} A_{\mathfrak{U}}$ is represented by an object of $A_{\mathfrak{U}}$ for some cover $\mathfrak{U}$, and two representatives are equivalent if there is a common refinement of covers upon which they are isomorphic (i.e. connected by an arrow). An arrow of $[G](X)$ is represented by an arrow of $A_{\mathfrak{U}}$ for some cover $\mathfrak{U}$, and two arrows are equivalent if there is a common refinement of covers upon which the two arrows are *equal*.

The full subcategory $A^0_{\mathfrak{U}}$ of $A_{\mathfrak{U}}$ whose objects are pairs $((P_i), \phi_{ij})$ with $P_i = e_{U_i} \in \operatorname{Hom}(U_i, G)$ (i.e. the unit map) is equivalent to $A_{\mathfrak{U}}$, and $[G](X)$ is equivalent to the colimit over these subcategories, so whenever convenient we assume $P_i = e_{U_i}$ and refer to an object simply as (ϕ_{ij}). (For the groupoid minded reader, this gives a simpler characterization than $A_{\mathfrak{U}}$, because $A^0_{\mathfrak{U}}$ is easily seen as equivalent to the functor category $\operatorname{Fun}(\check{\mathfrak{U}}, G)$ where $\check{\mathfrak{U}}$ denotes the Čech groupoid of the cover $\mathfrak{U}$.)

For the rest of this section we will illustrate several of the different interpretations of stacks by explaining the different notations $[G]$, Prin_G, Tors_G, and $\mathscr{B}G$.

Specializing to $S = Top$, an object (ϕ_{ij}) of $A_{\mathfrak{U}}$ can be viewed as transition functions defining a topological principal G-bundle. The bundle is:

$$P(\phi) := \coprod U_i \times G/\sim \phi_{ij}$$

It is essentially the definition of a principal bundle that the assignment $(\phi_{ij}) \mapsto P(\phi)$ induces an equivalence of groupoids, for each X,

$$[G](X) \to \mathrm{Prin}_G(X)$$

where $\mathrm{Prin}_G(X)$ denotes the groupoid of principal G-bundles and bundle isomorphisms on X. In fact this gives an equivalence of 2-functors $[G] \simeq \mathrm{Prin}_G(-)$. (Thus the assignment $X \mapsto \mathrm{Prin}_G(X)$ is a stack).

This explains the nomenclature Prin_G. And of course G-torsor is the algebraic geometry name for a principal bundle, which explains the usage of Tors_G. The term $\mathscr{B}G$ is used because, when we view $X \in S$ as a stack the 2-Yoneda lemma provides an isomorphism:

$$\mathrm{Hom}_{\mathrm{Stacks}}(X, [G]) \simeq [G](X).$$

In other words every map of stacks from X to $[G]$ corresponds to a unique principal G-bundle on X, so that $[G] = \mathscr{B}G$ can rightfully be referred to as the classifying stack for G-bundles.

Everything we have said about the stack $[G]$ associated to a group object in S can also be said for a *groupoid* object $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$ in S. In this case the principal $\mathcal{G}$-bundle $P(\phi)$ associated to $(\phi_{ij}) \in \prod \mathrm{Hom}_S(U_{ij}, \mathcal{G})$ is defined as:

$$P(\phi) = \coprod U_i \times_{\phi_{ii}, \mathcal{G}_0, r} \mathcal{G}_1/\phi_{ij}.$$

Thus we have the stack $[\mathcal{G}]$ associated to a groupoid and $[\mathcal{G}] \simeq \mathrm{Prin}_{\mathcal{G}}$.

There is an equivalence relation (called Morita equivalence) on groupoids in S, such that two groupoids $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent if and only if there is an equivalence $[\mathcal{G}] \simeq [\mathcal{H}]$. This explains the bracket notation $[-]$ that we use; the stack $[\mathcal{G}]$ can be interpreted as an equivalence class of groupoids.

3.2. **Example- Gerbes.** Let $\mathcal{G}$ and $\mathcal{M}$ be groupoids, and let A be an abelian group. Suppose A acts on $\mathcal{G}$, acting trivially on $\mathcal{G}_0$, and in such a way that $\mathcal{G}_1 \to \mathcal{G}_1/A \simeq \mathcal{M}_1$ is both a principal A-bundle in S and a functor. Then the stack $[\mathcal{G}]$ is called an A-gerbe over $[\mathcal{M}]$.

For example $[A]$ is an A-gerbe over a point, and a central extension of groups $A \to G \to H$ makes $[G]$ an A-gerbe over $[H]$. Most gerbes can be presented via a 2-cocycle construction:

Given $\sigma : \mathcal{G}_2 \to A$ satisfying $\sigma(g_2, g_3)\sigma(g_1, g_2g_3) = \sigma(g_1, g_2g_3)\sigma(g_1, g_2)$, the space $A \times \mathcal{G}_1$ with structure maps

$$s(a, g) := s(g), \quad r(a, g) := r(g), \quad (a_1, g_1) \circ (a_2, g_2) := (a_1a_2\sigma(g_1, g_2), g_1g_2)$$

defines a groupoid which we denote $\mathcal{G}(\sigma)$, and $[\mathcal{G}(\sigma)]$ is an A-gerbe over $[\mathcal{G}]$.

There are two examples of this construction which relate to important objects of study in noncommutative geometry. (Here we assume $S = \mathrm{Top}$.) The first is when $\mathcal{G}$ is the group $\mathbb{Z}^n$ and $A = U(1)$, in which case a 2-cocycle $\sigma : (\mathbb{Z}^n)^2 \to U(1)$ determines a gerbe $[\mathbb{Z}^n(\sigma)]$ which is very closely related to the n-dimensional noncommutative torus $C^*(\mathbb{Z}^n; \sigma)$.

The second example relating to noncommutative geometry is the continuous trace algebra $A(X, H)$ with spectrum X and Dixmier-Douady invariant $H \in H^3(X, \mathbb{Z})$. Under the connecting homomorphism in Čech cohomology

$$H^3(X; \mathbb{Z}) \xrightarrow{\delta} H^2(X; U(1))$$

the image of H is represented by a cover $\mathfrak{U} := \{U_i \to X\}$ and a 2-cocycle $\sigma = \delta H : \check{\mathfrak{U}}_2 \to U(1)$. This induces a $U(1)$-gerbe $[\check{\mathfrak{U}}(\sigma)]$ over X. ($\check{\mathfrak{U}}$ denotes the Čech groupoid of the cover.) The C^*-algebra $A(X, H)$ is Morita equivalent to $C^*(\widehat{\mathfrak{U}}; \sigma)$, so the gerbe and the continuous trace algebra are closely related.

3.3. Stacks that aren't in groupoids. The most common target 2-category for a stack is Groupoids. However, the assignment $X \mapsto \text{Sheaves}(X)$ is a stack in categories. Also $X \mapsto \text{Vect}(X)$ lands in categories. Another important target is the 2-category of rings, bimodules, and isomorphisms of bimodules, as well as its counterpart for appropriate topological rings.

3.4. Representable stacks. To talk about representable stacks one needs the notion of a morphism of stacks. A **morphism** between two stacks $F, G : S \to \mathcal{C}$ is, from the enriched category perspective, simply a monoidal natural transformation $\eta : F \Rightarrow G$. This is the data of a 1-arrow $F(U) \xrightarrow{\eta_U} G(U)$ for each $U \in S$, as well as a 2-arrow η_f for each $f \in \text{Hom}_S(V, U)$ making the following diagram 2-commutative:

$$\begin{tikzcd} F(U) \arrow[r, "\eta_U"] \arrow[d, "F(f)"'] & G(U) \arrow[d, "G(f)"] \\ F(V) \arrow[r, "\eta_V"'] \arrow[ur, Rightarrow, "\eta_f"] & G(V). \end{tikzcd}$$

If the 2-category $\mathcal{C}$ has a notion of epimorphism or monomorphism for 1-arrows, then it is natural to say the morphism η is an **epimorphism** or **monomorphism** if it is so pointwise, meaning that for each "point" $U \in S$, η_U is epi or mono. A morphism of stacks which is both an epimorphism and a monomorphism is called an **equivalence**.

For example, in CAT the monomorphisms are the fully faithful functors and the epimorphisms are the essential surjections.

Recall that in general a functor $F : \mathcal{C} \to \text{Sets}$ is called representable if it is equivalent to the Yoneda functor $\text{Hom}_{\mathcal{C}}(\ , X)$ for some object X of $\mathcal{C}$. So a representable sheaf is just a sheaf of the form $\text{Hom}_S(\ , X)$, or its sheafification if the site S is not subcanonical. Similarly, a stack of groupoids is called **representable** if it is equivalent to the stackification $[\mathcal{G}]$ of the groupoid valued functor $\text{Hom}_S(\ , \mathcal{G})$, where $\mathcal{G}$ is a groupoid object in S.

It can be taken as a definition, that two groupoid objects $\mathcal{G}$ and $\mathcal{H}$ in a site S are **Morita equivalent** if and only if there exists an equivalence of stacks $[\mathcal{G}] \simeq [\mathcal{H}]$. This definition agrees with Morita equivalence of topological groupoids and Lie groupoids, as is shown, for example, in [BeXu] and [Met].

Here is a very minimal bibliography:

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