Project 1

Finite difference simulation of 2D waves

1. The matematical problem

We are tasked with numerically solving the two-dimensional, standard, linear wave equation, with damping

$$\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) + f(x,y,t),$$

in a rectangular spatial domain

$$\Omega = [0, L_x] \times [0, L_y].$$

We use the homogenous Neumann condition

$$\frac{\partial u}{\partial n} = u = 0.$$

The initial conditions are

$$u(x,y,0) = I(x,y)$$

$$\frac{\partial}{\partial t}u(x,y,0) = u_t(x,y,0) = V(x,y).$$

2. Discretization

We want to discretize the wave equation so that we can solve it numerically. To clarify some notation we will use $u(x_i, y_j, t_n) = u^n_{i,j}$, where $x_i = i\Delta x$, $y_j = j\Delta y$ and $t_n = n\Delta t$. We will use ghostpoints to solve the boundary problems, and so our indices range lie in the ranges $i \in [1, \dots, N_x]$, $j \in [1, \dots, N_y]$ and $n \in [0, \dots, N_t]$. This means that we use ghostpoints $i = 0, i = N_x + 1$ and $j = 0, j = N_y + 1$.

Left hand side

The discretization of the time domain is achieved by using both the forward and backwards taylor expansion on $u(x, y, t)$. This gives us

$$u^{n+1}_{i,j} = u^n_{i,j} + \Delta t * [u_t]^n_{i,j} + \frac{\Delta t^2}{2}[u_{tt}]^n_{i,j} + O(\Delta t^3)$$

$$u^{n-1}_{i,j} = u^n_{i,j} - \Delta t * [u_t]^n_{i,j} + \frac{\Delta t^2}{2}[u_{tt}]^n_{i,j} + O(\Delta t^3).$$

Adding the equations above gives us

$$u^{n+1}_{i,j} + u^{n-1}_{i,j} = 2u^n_{i,j} + \Delta t[u_{tt}]^n_{i,j},$$

which solved for $[u_{tt}]^n_{i,j}$ gives us a discretization of $\frac{\partial^2 u}{\partial t^2}$

$$[u_t t]^n_{i,j} = \frac{u^{n+1}_{i,j} - 2u^n_{i,j} + u^{n-1}_{i,j}}{\Delta t^2}.$$

Instead substracting them gives us

$$u^{n+1}_{i,j} - u^{n-1}_{i,j} = 2\Delta t[u_t]^n_{i,j},$$

which solved for $[u_t]^n_{i,j}$ gives us a discretization of $\frac{\partial u}{\partial t}$

$$[u_t]^n_{i,j} = \frac{u^{n+1}_{i,j} - u^{n-1}_{i,j}}{2\Delta t}.$$

Right hand side

Discretization of the spatial domain is achieved by defining

$$\phi \equiv q(x,y)u_x,$$

and

$$\phi_x \equiv \frac{\partial \phi}{\partial x},$$

and to first discretize the outer derivative. We want to use the centered difference scheme to discretize our derivatives. By taylor expansion this gives us

$$\phi_{i+\frac{1}{2},j} = \phi_{i,j} + \frac{\Delta x}{2}[\phi_x]_{i,j}$$

$$\phi_{i-\frac{1}{2},j} = \phi_{i,j} - \frac{\Delta x}{2}[\phi_x]_{i,j}.$$

Substracting the equations above gives us

$$[\phi_x]_{i,j} = \frac{\phi_{i+\frac{1}{2},j} - \phi_{i-\frac{1}{2},j}}{\Delta x}.$$

We then discretize $\phi_{i+\frac{1}{2},j}$ and $\phi_{i-\frac{1}{2},j}$

$$\phi_{i+\frac{1}{2},j} = q_{i+\frac{1}{2},j}[u_x]_{i+\frac{1}{2},j} = q_{i+\frac{1}{2},j}\frac{u_{i+1,j} - u_{i,j}}{\Delta x},$$

$$\phi_{i-\frac{1}{2},j} = q_{i-\frac{1}{2},j}[u_x]_{i-\frac{1}{2},j} = q_{i-\frac{1}{2},j}\frac{u_{i,j} - u_{i-1,j}}{\Delta x},$$

and insert them in the expression for $[\phi_x]_{i,j}$to get

$$\phi_x = \left[\frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right)\right]^n_{i,j} = \frac{1}{\Delta x^2}\left(q_{i+\frac{1}{2},j}(u^n_{i+1,j} - u^n_{i,j}) - q_{i-\frac{1}{2},j}(u^n_{i,j} - u^n_{i-1,j})\right).$$

Now, using the arithmic mean approximation for $q_{i+\frac{1}{2},j}$ and $q_{i-\frac{1}{2},j}$

$$q_{i+\frac{1}{2},j} \approx \frac{1}{2}(q_{i,j} + q_{i+1,j}),$$

$$q_{i-\frac{1}{2},j} \approx \frac{1}{2}(q_{i,j} + q_{i-1,j}),$$

we get

$$\phi_x = \frac{1}{2\Delta x^2}\left((q_{i,j} + q_{i+1,j})(u^n_{i+1,j} - u^n_{i,j}) - (q_{i,j} + q_{i-1,j})(u^n_{i,j} - u^n_{i-1,j})\right).$$

The exact same deduction applies to $\phi_y = \partial\phi/\partial y$, only doing the steps for the j index. This gives us

$$\phi_y = \left[\frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right)\right]^n_{i,j} = \frac{1}{2\Delta y^2}\left((q_{i,j} + q_{i,j+1})(u^n_{i,j+1} - u^n_{i,j}) - (q_{i,j} + q_{i,j-1})(u^n_{i,j} - u^n_{i,j-1})\right).$$

Complete discretization of general scheme

This gives us a complete discretization of the original equation

$$\frac{u^{n+1}_{i,j} - 2u^n_{i,j} + u^{n-1}_{i,j}}{\Delta t^2} + b\frac{u^{n+1}_{i,j} - u^{n-1}_{i,j}}{2\Delta t} = \frac{1}{2\Delta x^2}\left((q_{i,j} + q_{i+1,j})(u^n_{i+1,j} - u^n_{i,j}) - (q_{i,j} + q_{i-1,j})(u^n_{i,j} - u^n_{i-1,j})\right) + \frac{1}{2\Delta y^2}\left((q_{i,j} + q_{i,j+1})(u^n_{i,j+1} - u^n_{i,j}) - (q_{i,j}\right.$$

Defining some constants

$$B \equiv \frac{b}{2\Delta t}, E \equiv \frac{1}{\Delta t^2},$$

and solving the equation above for $u^{n+1}_{i,j}$ gives us

$$u^{n+1}_{i,j} = \frac{1}{E+B}\left(\frac{1}{2\Delta x^2}((q_{i,j} + q_{i+1,j})(u^n_{i+1,j} - u^n_{i,j}) - (q_{i,j} + q_{i-1,j})(u^n_{i,j} - u^n_{i-1,j})) + \frac{1}{2\Delta y^2}((q_{i,j} + q_{i,j+1})(u^n_{i,j+1} - u^n_{i,j}) - (q_{i,j} + q_{i,j-1})(u^n_{i,j} - u^n_{i,j-1})) + \right.$$

Boundary conditions

Using the homogenous Neumann condition we get

$$\frac{\partial u(x,y,t)}{\partial n} = \vec{n}\nabla u = 0.$$

We use a centered difference scheme to discretize for both directions

$$\frac{\partial u^n_{i,j}}{\partial x} = \frac{u^n_{i+1,j} - u^n_{i-1,j}}{2\Delta x},$$

$$\frac{\partial u^n_{i,j}}{\partial y} = \frac{u^n_{i,j+1} - u^n_{i,j-1}}{2\Delta y}.$$

At the boundary $i = 1$ we then get

$$u^n_{1,j} = \frac{u^n_{2,j} - u^n_{0,j}}{2\Delta x} = 0$$

$$=> u^n_{2,j} = u^n_{0,j}.$$

It is now trivial to see that we get

$$u^n_{N_x+1,j} = u^n_{N_x-1,j}, \qquad \text{for } i = N_x,$$

$$u^n_{i,2} = u^n_{i,0}, \qquad \text{for } j = 1,$$

$$u^n_{i,N_y+1} = u^n_{i,N_y-1}, \qquad \text{for } j = N_y.$$

This means that our ghost points take the same values as the ones on the edges of our interior spatial mesh.

Imposing boundary conditions on q

The function $q(x, y)$ does not follow the Neumann Condition, so we have to explicitly find the values of q for the ghost points. Again we use both forward and backwards Taylor expansion as a means to achieve this.

$$q_{i+1,j} = q_{i,j} + (q_x)_{i,j}\Delta x + O(\Delta x)^2,$$

$$q_{i-1,j} = q_{i,j} - (q_x)_{i,j}\Delta x + O(\Delta x)^2.$$

Adding the equations give us

$$q_{i+1,j} + q_{i-1,j} = 2q_{i,j},$$

and the same goes for the y-coordinate

$$q_{i,j+1} + q_{i,j-1} = 2q_{i,j}.$$

With this we can now replace all the ghost points values of q with values for q inside the interior mesh

$$q_{0,j} = 2q_{1,j} - q_{2,j} \qquad q_{N_x+1,j} = 2q_{N_x,j} - q_{N_x-1,j}$$

$$q_{i,0} = 2q_{i,1} - q_{i,2} \qquad q_{i,N_y+1} = 2q_{i,N_y} - q_{i,N_y-1}$$

The modified scheme for the first step

From the initial conditions we get

$$u^0_{i,j} = I_{i,j},$$

$$[u_t]^0_{i,j} = V_{i,j}.$$

Once again, using a centered diffenece scheme on u_t we get

$$[u_t]^0_{i,j} = \frac{u^1_{i,j} - u^{-1}_{i,j}}{2\Delta t} = V_{i,j}$$

$$=> u^{-1}_{i,j} = u^1_{i,j} - 2\Delta t V_{i,j},$$

which we can use to eliminate $u^{-1}_{i,j}$ from the equation solving the first step, $u^1_{i,j}$. This gives us

$$u^1_{i,j} = \frac{1}{E+B}\left(\frac{1}{2\Delta x^2}((q_{i,j} + q_{i+1,j})(u^0_{i+1,j} - u^0_{i,j}) - (q_{i,j} + q_{i-1,j})(u^0_{i,j} - u^0_{i-1,j})) + \frac{1}{2\Delta y^2}((q_{i,j} + q_{i,j+1})(u^0_{i,j+1} - u^0_{i,j}) - (q_{i,j} + q_{i,j-1})(u^0_{i,j} - u^0_{i,j-1})) + 2\right.$$

$$=>$$

$$u^1_{i,j} = u^0_{i,j} + \frac{1}{2E}\left(\frac{1}{2\Delta x^2}((q_{i,j} + q_{i+1,j})(u^0_{i+1,j} - u^0_{i,j}) - (q_{i,j} + q_{i-1,j})(u^0_{i,j} - u^0_{i-1,j})) + \frac{1}{2\Delta y^2}((q_{i,j} + q_{i,j+1})(u^0_{i,j+1} - u^0_{i,j}) - (q_{i,j} + q_{i,j-1})(u^0_{i,j} - u^0_{i,j-1}))\right.$$

where

$$B \equiv \frac{b}{2\Delta t}, E \equiv \frac{1}{\Delta t^2}.$$

3. Implementation of the general class solver

First is an implementation of the class solver without vectorization:

In [1]:

```python
import numpy as np
import matplotlib.pyplot as plt


class Wave2D():
    def __init__(self, b, T, Lx, Ly, I, V, qq, Nx, Ny, f):
        """ Initializing class variables and functions  """

        self.b = b
        self.Nx = Nx
        self.Lx = Lx
        self.Ly = Ly
        self.Ny = Ny
        self.x = np.linspace(0,self.Lx,self.Nx)
        self.y = np.linspace(0,self.Ly,self.Ny)
        self.X, self.Y = np.meshgrid(self.x,self.y)
        self.X = self.X.T
        self.X = self.Y.T
        self.dx = self.x[1] - self.x[0]
        self.dy = self.y[1] - self.y[0]

        # Making functions sent to init availible for all methods in class
        self.f = lambda x,y,t: f(x,y,t)
        self.I = lambda x,y: I(x,y)
        self.V = lambda x,y: V(x,y)
        self.qq = lambda x,y: qq(x,y)
        self.make_q()

        self.T = T
        self.Nt = int(round(self.T/self.dt))

        #Defining arrays to hold solutions
        self.u_new = np.zeros((self.Nx+2, self.Ny+2))
        self.u = np.zeros((self.Nx+2, self.Ny+2))
        self.u_old = np.zeros((self.Nx+2, self.Ny+2))

        #Variables for simplyfying the mathematical expressions
        self.dt2 = self.dt*self.dt
        self.E = 1./self.dt2
        self.B = self.b/(2*self.dt)
        self.Cy = 1./(2*self.dy*self.dy)
        self.Cx = 1./(2*self.dx*self.dx)


        # Variable for holding error norm
        self.linf_norm = 0

    def stability(self):
        """ Sets dt after stability criteria"""

        maximum_velocity = np.max(self.q[1:self.Nx+1,1:self.Ny+1])
        beta_factor = 0.9
        self.dt = beta_factor*(1./np.sqrt(maximum_velocity))*(1/np.sqrt(1/self.dx**2 + 1/self.dy**2))

    def set_initial_conditions(self):
        """ Setting initial values and solving first modified step """

        self.set_initial_u_old()
        self.first_step()

    def set_initial_u_old(self):
        """ Initializes the first u^(n-1) on the inner mesh points/boundary
        and calls a method to initialize ghost cells as well"""

        for i in range(1, self.Nx +1):
            for j in range(1, self.Ny +1):
                self.u_old[i,j] = self.I(self.x[i-1], self.y[j-1])

        self.updating_ghost_cells(self.u_old)

    def updating_ghost_cells(self,uu):
        """ Method for updating ghost cells """


        for i in range(1, self.Nx +1):
            uu[i,0] = uu[i,2]
            uu[i,self.Ny+1] = uu[i,self.Ny-1]


        for j in range(1, self.Ny +1):
            uu[0,j] = uu[2,j]
            uu[self.Nx+1,j] = uu[self.Nx-1,j]

    def make_q(self):
        """ Fills a matrix with values from the q(x,y) function """
        self.q = np.zeros((self.Nx+2,self.Ny+2))


        for i in range(1, self.Nx+1):
            for j in range(1, self.Ny+1):
                self.q[i,j] = self.qq(self.x[i-1], self.y[j-1])

        for i in range(1,self.Nx+1):
            self.q[i,0] = 2*self.q[i,1] - self.q[i,2]
            self.q[i,self.Ny +1] = 2*self.q[i,self.Ny] - self.q[i,self.Ny-1]


        for j in range(1,self.Ny+1):
            self.q[0,j] = 2*self.q[1,j] - self.q[2,j]
            self.q[self.Nx+1,j] = 2*self.q[self.Nx,j] - self.q[self.Nx-1,j]

        self.stability()

    def first_step(self):
        """ Calculates the first modified step and calls method
        to update ghost cells """

        q = self.q
        u_old = self.u_old


        for i in range(1, self.Nx+1):
            for j in range(1, self.Ny+1):

                self.u[i,j] = u_old[i,j] + (1/(2*self.E))*(self.Cx*((q[i,j] + q[i+1,j]) * (u_old[i+1,j]
 - u_old[i,j]) - (q[i,j] + q[i-1,j]) * (u_old[i,j] - u_old[i-1,j])) + self.Cy*((q[i,j] + q[i,j+1]) * (u
_old[i,j+1] - u_old[i,j]) - (q[i,j] + q[i,j-1]) * (u_old[i,j] - u_old[i,j-1])) + 2*self.dt*self.V(self.
x[i-1], self.y[j-1])*(self.E - self.B) + self.f(self.x[i-1], self.y[j-1],0))

        self.updating_ghost_cells(self.u)

    def advance_general_scheme(self):
        """ The general scheme for advancing the solution"""

        q = self.q
        u = self.u

        for i in range(1, self.Nx+1):
            for j in range(1, self.Ny+1):

                self.u_new[i,j] = (1/(self.E+self.B))*(self.Cx*((q[i,j] + q[i+1,j])*(u[i+1,j] - u[i,j])
 - (q[i,j] + q[i-1,j]) * (u[i,j] - u[i-1,j])) + self.Cy*((q[i,j] + q[i,j+1]) * (u[i,j+1] - u[i,j]) - (q
[i,j] + q[i,j-1]) * (u[i,j] - u[i,j-1])) + 2*self.E*u[i,j] - (self.E - self.B) * self.u_old[i,j] + self
.f(self.x[i-1], self.y[j-1], self.t))

        self.updating_ghost_cells(self.u_new)

    def swap(self):
        """ Swaps u variables for each time step"""
        self.u_old, self.u, self.u_new = self.u, self.u_new, self.u_old

    def time_evolution(self):
        """ Method for progressing the time evolution of the solution. """

        self.t = self.dt
        while self.t <= self.T:
            self.advance_general_scheme()
            self.swap()
            self.t += self.dt

    def plot(self, X_ax, Y_ax , title):
        """ Plots final solution in the X,Y-plane with amplitude of wave as contour"""

        plt.contourf(self.X.T,self.Y.T, self.u[1:self.Nx+1,1:self.Ny+1])
        plt.title(title)
        plt.xlabel(X_ax)
        plt.ylabel(Y_ax)
        plt.colorbar()
        plt.figure()




    def true_error(self,analytical):
        """ Calculates the linf norm"""

        analytical_values = np.zeros([self.Nx,self.Ny])
        analytic = lambda x,y,t: analytical(x,y,t)

        for i in range(0,self.Nx):
            for j in range(0,self.Ny):
                analytical_values[i,j] = analytic(self.x[i],self.y[j],self.t)

        computed_error = analytical_values - self.u[1:self.Nx+1, 1:self.Ny+1]
        self.linf_norm = np.max(np.abs(computed_error))
```

Below is the same implementation of the class, only now vectorized where possible. This is the class we will use furter on since it is much more efficient.

In [2]:

```python
import numpy as np
import matplotlib.pyplot as plt


class Wave2D():
    def __init__(self, b, T, Lx, Ly, I, V, qq, Nx, Ny, f):
        """ Initializing class variables  """


        self.b = b
        self.Nx = Nx
        self.Lx = Lx
        self.Ly = Ly
        self.Ny = Ny
        self.x = np.linspace(0,self.Lx,self.Nx)
        self.y = np.linspace(0,self.Ly,self.Ny)
        self.X, self.Y = np.meshgrid(self.x,self.y)
        self.X = self.X.T
        self.X = self.Y.T
        self.dx = self.x[1] - self.x[0]
        self.dy = self.y[1] - self.y[0]

        # Making functions sent to init availible for all methods in class
        self.f = lambda x,y,t: f(x,y,t)
        self.I = lambda x,y: I(x,y)
        self.V = lambda x,y: V(x,y)
        self.qq = lambda x,y: qq(x,y)
        self.make_q()

        self.T = T
        self.Nt = int(round(self.T/self.dt))

        #Defining arrays to hold solutions
        self.u_new = np.zeros((self.Nx+2, self.Ny+2))
        self.u = np.zeros((self.Nx+2, self.Ny+2))
        self.u_old = np.zeros((self.Nx+2, self.Ny+2))

        #Variables for simplyfying the mathematical expressions
        self.dt2 = self.dt*self.dt
        self.E = 1./self.dt2
        self.B = self.b/(2*self.dt)
        self.Cy = 1./(2*self.dy*self.dy)
        self.Cx = 1./(2*self.dx*self.dx)


        # Variable for holding error norm
        self.linf_norm = 0

    def stability(self):
        """ Sets dt after stability criteria"""

        maximum_velocity = np.max(self.q[1:self.Nx+1,1:self.Ny+1])
        beta_factor = 0.9
        self.dt = beta_factor*(1./np.sqrt(maximum_velocity))*(1/np.sqrt(1/self.dx**2 + 1/self.dy**2))

    def set_initial_conditions(self):
        """ Setting initial values and solving first modified step """

        self.set_initial_u_old()
        self.first_step()

    def set_initial_u_old(self):
        """ Initializes the first u^(n-1) on the inner mesh points/boundary
        and calls a method to initialize ghost cells as well"""

        for i in range(1, self.Nx +1):
            for j in range(1, self.Ny +1):
                self.u_old[i,j] = self.I(self.x[i-1], self.y[j-1])

        self.updating_ghost_cells(self.u_old)

    def updating_ghost_cells(self,uu):
        """ Method for updating ghost cells """

        uu[1:-1,0] = uu[1:-1,2]
        uu[1:-1,self.Ny+1] = uu[1:-1,self.Ny-1]
        uu[0,1:-1] = uu[2,1:-1]
        uu[self.Nx+1,1:-1] = uu[self.Nx-1,1:-1]

    def make_q(self):
        """ Fills a matrix with values from the q(x,y) function """
        self.q = np.zeros((self.Nx+2,self.Ny+2))

        self.q[1:-1, 1:-1] = self.qq(self.X, self.Y)
        self.q[1:-1, 0] = 2*self.q[1:-1, 1] - self.q[1:-1, 2]
        self.q[1:-1, self.Ny +1] = 2*self.q[1:-1, self.Ny] - self.q[1:-1, self.Ny-1]
        self.q[0, 1:-1] = 2*self.q[1, 1:-1] - self.q[2, 1:-1]
        self.q[self.Nx+1, 1:-1] = 2*self.q[self.Nx, 1:-1] - self.q[self.Nx-1, 1:-1]

        self.stability()

    def first_step(self):
        """ Calculates the first modified step and calls method
        to update ghost cells """

        q = self.q
        u_old = self.u_old

        self.u[1:-1,1:-1] = u_old[1:-1,1:-1] + (1/(2*self.E))*(self.Cx*((q[1:-1,1:-1] + q[2:,1:-1]) * (
u_old[2:,1:-1] - u_old[1:-1,1:-1]) - (q[1:-1,1:-1] + q[0:-2,1:-1]) * (u_old[1:-1,1:-1] - u_old[0:-2,1:-
1])) + self.Cy*((q[1:-1,1:-1] + q[1:-1,2:]) * (u_old[1:-1,2:] - u_old[1:-1,1:-1]) - (q[1:-1,1:-1] + q[1
:-1,0:-2]) * (u_old[1:-1,1:-1] - u_old[1:-1,0:-2])) + 2*self.dt*self.V(self.X, self.Y)*(self.E - self.B
) + self.f(self.X, self.Y,0))

        self.updating_ghost_cells(self.u)

    def advance_general_scheme(self):
        """ The general scheme for advancing the solution"""

        q = self.q
        u = self.u

        self.u_new[1:-1,1:-1] = (1/(self.E+self.B))*(self.Cx*((q[1:-1,1:-1] + q[2:,1:-1])*(u[2:,1:-1] -
 u[1:-1,1:-1]) - (q[1:-1,1:-1] + q[0:-2,1:-1]) * (u[1:-1,1:-1] - u[0:-2,1:-1])) + self.Cy*((q[1:-1,1:-1
] + q[1:-1,2:]) * (u[1:-1,2:] - u[1:-1,1:-1]) - (q[1:-1,1:-1] + q[1:-1,0:-2]) * (u[1:-1,1:-1] - u[1:-1,
0:-2])) + 2*self.E*u[1:-1,1:-1] - (self.E - self.B) * self.u_old[1:-1,1:-1] + self.f(self.X, self.Y, se
lf.t))

        self.updating_ghost_cells(self.u_new)

    def swap(self):
        """ Swaps u variables for each time step"""
        self.u_old, self.u, self.u_new = self.u, self.u_new, self.u_old

    def time_evolution(self):
        """ Method for progressing the time evolution of the solution.
            Also calls method to calculate true error from analytical solution"""

        self.t = self.dt
        while self.t <= self.T:
            self.advance_general_scheme()
            self.swap()
            self.t += self.dt

    def plot(self, X_ax, Y_ax , title):
        """ Plots final solution in the X,Y-plane """

        plt.contourf(self.X.T,self.Y.T, self.u[1:self.Nx+1,1:self.Ny+1])
        plt.title(title)
        plt.xlabel(X_ax)
        plt.ylabel(Y_ax)
        plt.colorbar()
        plt.figure()




    def true_error(self,analytical):

        analytical_values = np.zeros([self.Nx,self.Ny])
        analytic = lambda x,y,t: analytical(x,y,t)

        for i in range(0,self.Nx):
            for j in range(0,self.Ny):
                analytical_values[i,j] = analytic(self.x[i],self.y[j],self.t)

        computed_error = analytical_values - self.u[1:self.Nx+1, 1:self.Ny+1]
        self.linf_norm = np.max(np.abs(computed_error))
```

4. Verification

Constant Solution

For a constant solution $u(x, y, t) = U$, the wave equation reduces to

$$f(x,y,t) = 0.$$

From the initial conditions we get

$$u(x,y,0) = I(x,y) = U,$$

$$u_t(x,y,0) = V(x,y) = 0.$$

For simplicity we use $q(x, y) = 1$. The only parameter left to fit is then b, which has to take the value $b = 0$ since a constant solution can't have dampening.

If we insert the constant solution U along with

$$f(x,y,t) = 0$$

into the discrete equation we get

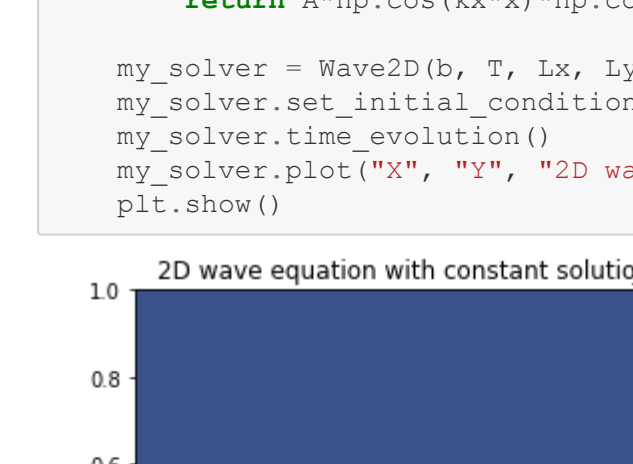
$$U = \frac{1}{E+B}\left(\frac{1}{2\Delta x^2}((q_{i,j} + q_{i+1,j})(U - U) - (q_{i,j} + q_{i-1,j})(U - U)) + \frac{1}{2\Delta y^2}((q_{i,j} + q_{i,j+1})(U - U) - (q_{i,j} + q_{i,j-1})(U - U)) + E2U - (E - B)U\right)$$

$$= \frac{1}{E+B}(E2U - EU + BU) = U,$$

which shows that the constant solution is also an exact solution of the discrete equations. The cell below shows a test case for the constant solution case:

In [3]:

```python
    b = 0
    T = 1
    Nx = 10
    Ny = 10
    Lx = 1
    Ly = 1

    def I(x,y):
        return 1

    def V(x,y):
        return 0

    def q(x,y):
        return 1

    def f(x,y,t):
        return 0

    def analytical_solution(x,y,t):
        A = 1
        mx = 1
        my = 1
        w = 1

        kx = (mx*np.pi/Lx)
        ky = (my*np.pi/Ly)

        return A*np.cos(kx*x)*np.cos(ky*y)*np.cos(w*t)

    my_solver = Wave2D(b, T, Lx, Ly, I, V, q, Nx, Ny, f)
    my_solver.set_initial_conditions()
    my_solver.time_evolution()
    my_solver.plot("X", "Y", "2D wave equation with constant solution")
    plt.show()
```




```
<Figure size 432x288 with 0 Axes>
```

5. Standing, undampned waves

Δt is defined by Δx and Δy in our class as

$$\Delta t = \beta\frac{1}{\sqrt{C^2(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2})}}.$$

As long as $\Delta x = \Delta y$, which we use in our code, one can see that $\Delta t \propto \Delta x$ and so Δx can be interperated as the common discretization parameter h.

With a standing wave solution

$$u_e(x,y,t) = A\cos(k_x x)\cos(k_y y)\cos(\omega t), \qquad k_x = \frac{mx\pi}{L_x}, \qquad k_y = \frac{m\pi}{L_y},$$

we get initial conditions

$$u_e(x,y,0) = I(x) = A\cos(k_x x)\cos(k_y y),$$

$$\frac{\partial}{\partial t}u_e(x,y,0) = V(x) = -\omega A\cos(k_x x)\cos(k_y y)sin(\omega * 0) = 0.$$

We know that the two-dimensional, standard, linear wave equation, without dampening and with a constant wave velocity C can be expressed as

$$\frac{\partial^2 u}{\partial t^2} = C^2\nabla^2 u$$

An approptiate ω is found by solving this equation using the standing wave solution

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 u_e,$$

$$\frac{\partial^2 u}{\partial x^2} = -k_x^2 u_e,$$

$$\frac{\partial^2 u}{\partial y^2} = -k_y^2 u_e,$$

which gives us

$$\omega = C^2\sqrt{k_x^2 + k_y^2}$$

We choose to use the maximum error (the largest distance from the analytical expression) at the last timestep N_t to calculate the error norm, which is to say we use the error norm

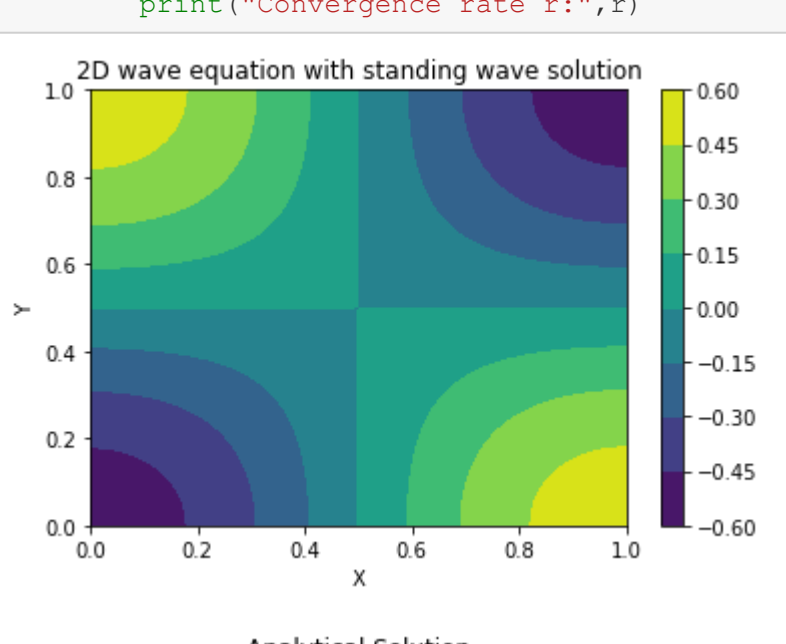
$$E = ||e^{N_t}_{i,j}||_{l^\infty} = \max_i \max_j |e^{N_t}_{i,j}|.$$

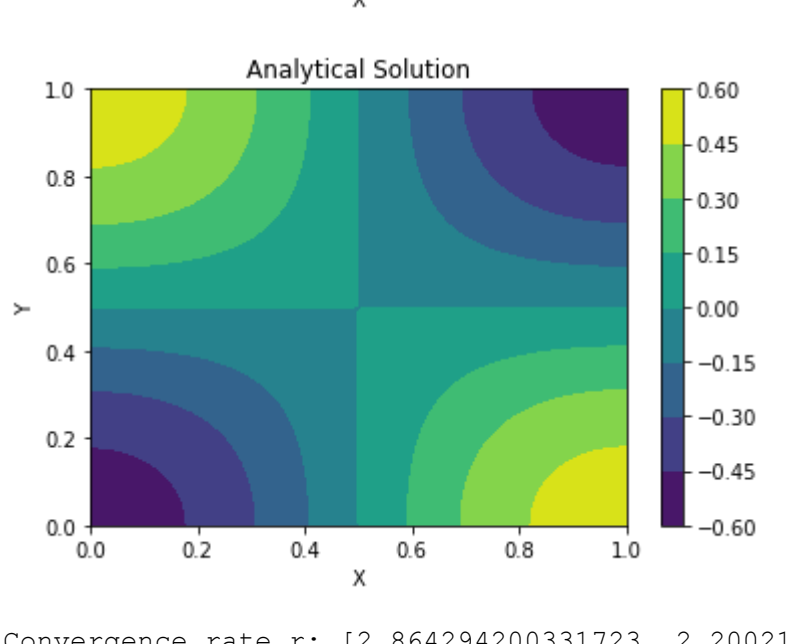
We calculate the error norm using a set of $N_x = N_y$ values (as our h is $\Delta x = \Delta y$ which are decided by $N_x = N_y$) and use these to compute a sequence of r values by the relation

$$r = \frac{\log(\frac{E_{i+1}}{E_i})}{\log(\frac{\Delta x_{i+1}}{\Delta x_i})}.$$

A test of the standing undampned wave is shown below, which plots both the analytical solution and the calculated u for the last time step as well as prints the convergence rate r.

```
In [4]:
b = 0
T = 1
Lx = 1
Ly = 1

A = 2
mx = 1
my = 1
kx = (mx*np.pi/Lx)
ky = (my*np.pi/Ly)
w = np.sqrt((mx*np.pi/Lx)**2 + (my*np.pi/Ly)**2)*1    #As long as q =1 is constant

def q(x,y):
    return 1

def f(x,y,t):
    return 0

def I(x,y):
    return A*np.cos(kx*x)*np.cos(ky*y)

def V(x,y):
    return 0

def analytical_solution(x,y,t):
    return A*np.cos(kx*x)*np.cos(ky*y)*np.cos(w*t)

N = [2**i for i in range(1,7)]
l_inf =[]
delta_x = []

for n in N:

    Nx = n
    Ny = n

    my_solver = Wave2D(b, T, Lx, Ly, I, V, q, Nx, Ny, f)
    my_solver.set_initial_conditions()
    my_solver.time_evolution()
    my_solver.true_error(analytical_solution)
    l_inf.append(my_solver.linf_norm)
    delta_x.append(my_solver.dx)

    if n == N[-1]:
        t = my_solver.t
        my_solver.plot("X", "Y", "2D wave equation with standing wave solution")
        x = my_solver.x
        y = my_solver.y
        X,Y = np.meshgrid(x,y)
        plt.contourf(X,Y, analytical_solution(X,Y,t))
        plt.title("Analytical Solution")
        plt.xlabel("X")
        plt.ylabel("Y")
        plt.colorbar()
        plt.show()


r = []
for i in range(len(l_inf)-1):
    r.append(np.log(l_inf[i+1]/l_inf[i])/np.log(delta_x[i+1]/delta_x[i]))

print("Convergence rate r:",r)
```






```
Convergence rate r: [2.864294200331723, 2.200213492936658, 1.9669160761097773, 2.036798952046302, 2.0186919288748695]
```

Waves with damping and variable wave velocity

We have a possible solution for the general 2D wave equation problem with damping and variable wave velocity

$$u_e(x, y, t) = (A\cos(\omega t) + B\sin(\omega t))e^{-dt}\cos(xk_x)\cos(yk_y),$$

and want to determine a suitable $f(x, y, t)$ along with some constants such that it in fact is a solution. To simplify the mathematical calculations, we can with confidence put $B = 0$ since the $\sin(\omega t)$ is just a factor contributing to the shift in phase. We choose a non constant $q(x, y) = q^x(x)q^y(y)$ where $q^x(x) = 1$ and $q^y(y) = y$ so that our $q(x, y) = y$. This can be intereprated as a wave moving in on a shore, where the wave velocity in the direction toward the shore (for us the y direction) increases because of shallower water while the wave velocity along the shore (for us the x direction) stays constant. To further simplify the calculations we define $X \equiv cos(xk_x)$ and $Y \equiv cos(yk_y)$, since these have derivatives wich are easy to keep track of. This means that we now have

$$u_e(x, y, t) = A\cos(\omega t)e^{-dt}XY,$$

$$u_t = (-\omega A\sin(\omega t)e^{-dt} - de^{-dt}A\cos(\omega t))XY,$$

$$u_{tt} = AXYe^{-dt}((d^2 - \omega^2)\cos(\omega t) + 2d\omega\sin(\omega t)),$$

$$u_x = A\cos(\omega t)e^{-dt}X_xY,$$

$$u_y = A\cos(\omega t)e^{-dt}XY_y.$$

We use the two latter to find $(q(x, y)u_x)_x$ and $(q(x, y)u_y)_y$:

$$(yu_x)_x = yu_{xx} = A\cos(\omega t)e^{-dt}X_{xx}Y = -k_x^2yA\cos(\omega t)e^{-dt}XY,$$

$$(yu_y)_y = u_y + yu_{yy} = -k_yA\cos(\omega t)e^{-dt}Xsin(yk_y) - k_y^2yA\cos(\omega t)e^{-dt}XY.$$

We can now insert these into general 2D wave equation problem

$$AXYe^{-dt}((d^2 - \omega^2)\cos(\omega t) + 2d\omega\sin(\omega t)) + b(-\omega A\sin(\omega t)e^{-dt} - de^{-dt}A\cos(\omega t))XY =$$

$$-k_x^2yA\cos(\omega t)e^{-dt}XY - k_yA\cos(\omega t)e^{-dt}Xsin(yk_y) - k_y^2yA\cos(\omega t)e^{-dt}XY + f.$$

Solving for f gives us

$$f = AXYe^{-dt}\cos(\omega t)\left[d^2 - \omega^2 + \omega(2d - b)\tan(\omega t) - db + yk_x^2 + k_y\tan(yk_y) + yk_y^2\right] =$$

$$u_e\left[d^2 - \omega^2 + \omega(2d - b)\tan(\omega t) - db + yk_x^2 + k_y\tan(yk_y) + yk_y^2\right].$$

We define $K^2 \equiv k_x^2 + k_y^2$ and choose $b = 2d$ which gives us

$$f = u_e\left[yK^2 + k_y\tan(yk_y) - d^2 - \omega^2\right],$$

and by also choosing $\omega = d$ we can further simplify the expression as

$$f = u_e\left[yK^2 + k_y\tan(yk_y) - 2\omega^2\right].$$

Keeping in mind that we have choosen $b = 2d$ and $\omega = d$ we find $I(x, y)$ and $V(x, y)$ by using the initial conditions

$$u(x, y, 0) = I(x, y) = A\cos(\omega * 0)e^{-\omega * 0}\cos(xk_x)\cos(yk_y) = A\cos(xk_x)\cos(yk_y),$$

$$u_t(x, y, 0) = V(x, y) = (-\omega A\cos(\omega * 0)e^{-\omega * 0} - \omega A\sin(\omega * 0)e^{-\omega * 0})\cos(xk_x)\cos(yk_y) = -\omega A\cos(xk_x)\cos(yk_y).$$

In total we now have the following expressions to solve using our solver class

$$u_e(x, y, t) = A\cos(\omega t)e^{-dt}\cos(xk_x)\cos(yk_y),$$

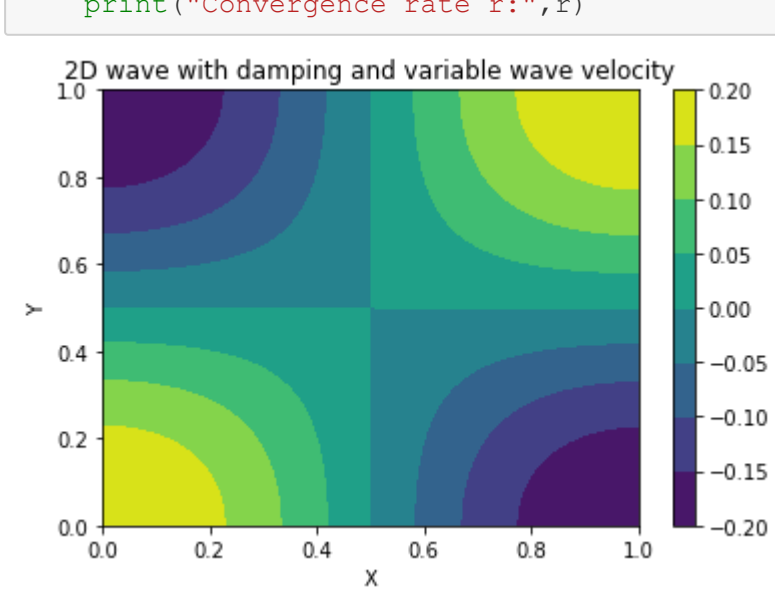
$$f(x, y, t) = u_e\left[yK^2 + k_y\tan(yk_y) - 2\omega^2\right],$$

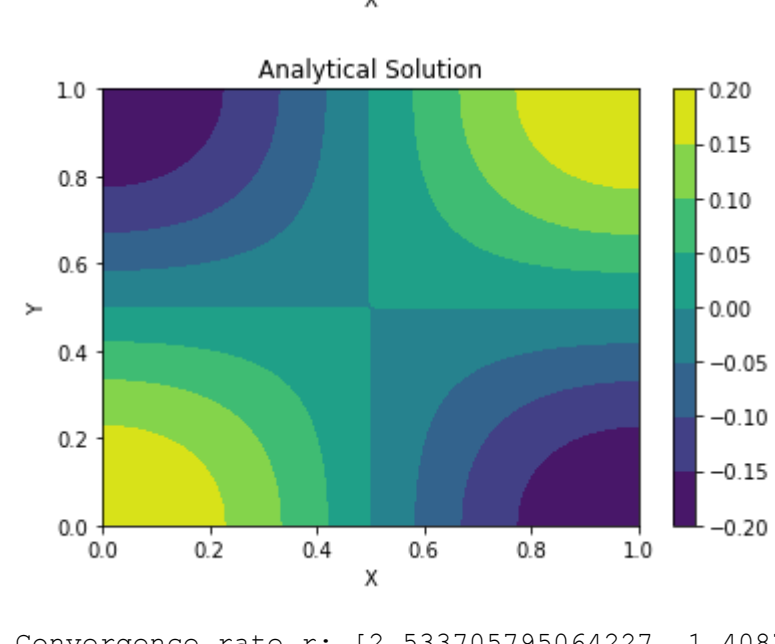
$$I(x, y) = A\cos(xk_x)\cos(yk_y),$$

$$V(x, y) = -\omega A\cos(xk_x)\cos(yk_y).$$

A test of the implemented wave with dampening and variable wave velocity is shown below.

```
In [5]:
A = 1
w = 1
b = 2*w

T = 1
Lx = 1
Ly = 1

mx = 1
my = 1
kx = (mx*np.pi/Lx)
ky = (my*np.pi/Ly)
K2 = kx**2 + ky**2

def q(x,y):
    return y

def analytical_solution(x,y,t):
    return A*np.cos(w*t)*np.exp(-w*t)*np.cos(kx*x)*np.cos(ky*y)

def I(x,y):
    return A*np.cos(kx*x)*np.cos(ky*y)

def V(x,y):
    return -w*A*np.cos(kx*x)*np.cos(ky*y)

def f(x,y,t):
    return A*np.cos(w*t)*np.exp(-w*t)*np.cos(kx*x)*np.cos(ky*y)*(K2*y + ky*np.tan(y*ky)-2*w*w)

N = [2**i for i in range(1,7)]
l_inf =[]
delta_x = []

for n in N:

    Nx = n
    Ny = n

    my_solver = Wave2D(b, T, Lx, Ly, I, V, q, Nx, Ny, f)
    my_solver.set_initial_conditions()
    my_solver.time_evolution()
    my_solver.true_error(analytical_solution)
    l_inf.append(my_solver.linf_norm)
    delta_x.append(my_solver.dx)


    if n == N[-1]:
        t = my_solver.t
        x = my_solver.x
        y = my_solver.y
        X,Y = np.meshgrid(x,y)
        my_solver.plot("X", "Y", "2D wave with damping and variable wave velocity")
        plt.contourf(X,Y, analytical_solution(X,Y,t))
        plt.title("Analytical Solution")
        plt.xlabel("X")
        plt.ylabel("Y")
        plt.colorbar()
        plt.show()

r = []
for i in range(len(l_inf)-1):
    r.append(np.log(l_inf[i+1]/l_inf[i])/np.log(delta_x[i+1]/delta_x[i]))

print("Convergence rate r:",r)
```






```
Convergence rate r: [2.533705795064227, 1.4087291014092072, 1.9678273793000063, 1.980274832221538, 1.9869229222776332]
```

Investigating a physical problem

Below are the implementations of three different underwater hills. For each hill a series of plots are shown that represent the evolution of the wave for different times.

```
In [6]:
T = 1
Lx = 1
Ly = 1
g = 9.81

Im = 0.5 #Starts with peak in the middle of the plot for Im = 0.5
I0 = 1.
Ia = 1
Is = 0.1
H0 = 10
B0 = 0.
Ba = 9
Bmx = 0.5
Bmy = 0.5
Bs = 0.1
b = 1  # Now a scaling parameter

def Gaussian_2D(x,y):
    BigB = B0 + Ba*np.exp(-((x-Bmx)/Bs)**2 - ((y-Bmy)/(b*Bs))**2)
    BigH = H0 - BigB
    return g*BigH

def Cosine_hat(x,y):
    Bs = 0.1 + np.sqrt(Lx**2 + Ly**2)  # Make sure Bs is greater than the restriction
    BigB = B0 + Ba*np.cos(np.pi*(x-Bmx)/(2*Bs))*np.cos(np.pi*(y-Bmy)/(2*Bs))
    BigH = H0 - BigB
    return g*BigH


def Box(x,y):
    Bmx = Lx
    Bmy = Lx
    Bs = Lx

    #Works as long as Lx = Ly and b >= 1, since then we are always inside the rectangle

    BigB = B0 + Ba
    BigH = H0 - BigB
    return g*BigH


def I(x,y):
    return I0 + Ia*np.exp(-((x-Im)/Is)**2)

def V(x,y):
    return 0

def f(x,y,t):
    return 0

n = 50

Nx = n
Ny = n
```

2D Gaussian Hill

```
In [7]: T = [i*0.001 for i in range(1,31)]

        for t in T:
            my_solver = Wave2D(b, t, Lx, Ly, I, V,Gaussian_2D, Nx, Ny, f)
            my_solver.set_initial_conditions()
            my_solver.time_evolution()
            my_solver.plot("X", "Y", " 2D Gaussian hill for T = %f" % t)

        plt.show()
```

```
/Users/kasparagasvaer/opt/anaconda3/lib/python3.7/site-packages/ipykernel_launcher.py:135: RuntimeWarning: More than 20 figures have been opened. Figures created through the pyplot interface (`matplotlib.pyplot.figure`) are retained until explicitly closed and may consume too much memory. (To control this warning, see the rcParam `figure.max_open_warning`).
```


```
&lt;Figure size 432x288 with 0 Axes&gt;
```

Cosine hat hill

```
In [8]: T = [i*0.01 for i in range(1,21)]

        for t in T:
            my_solver = Wave2D(b, t, Lx, Ly, I, V,Cosine_hat, Nx, Ny, f)
            my_solver.set_initial_conditions()
            my_solver.time_evolution()
            my_solver.plot("X", "Y", " Cosine hat hill for T = %f" % t)

        plt.show()
```

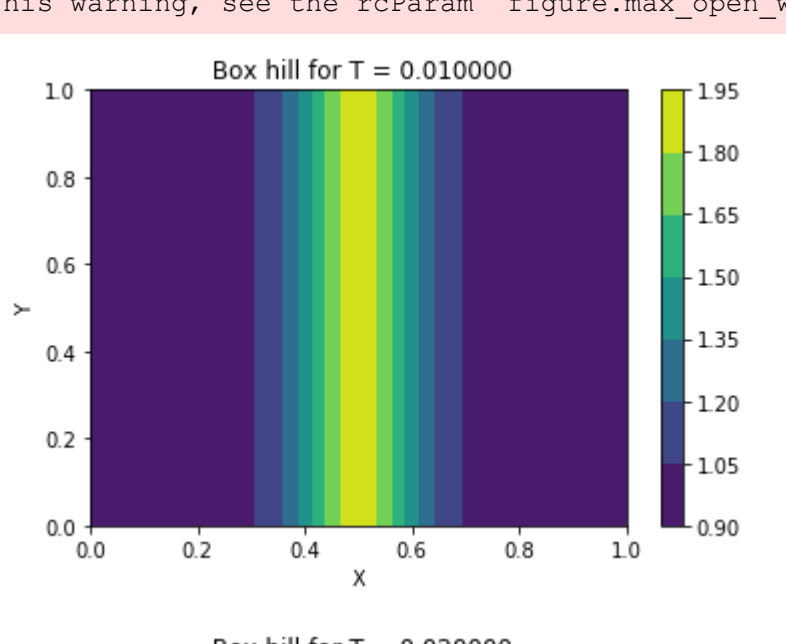
```
/Users/kasparagasvaer/opt/anaconda3/lib/python3.7/site-packages/ipykernel_launcher.py:135: RuntimeWarning: More than 20 figures have been opened. Figures created through the pyplot interface (`matplotlib.pyplot.figure`) are retained until explicitly closed and may consume too much memory. (To control this warning, see the rcParam `figure.max_open_warning`).
```

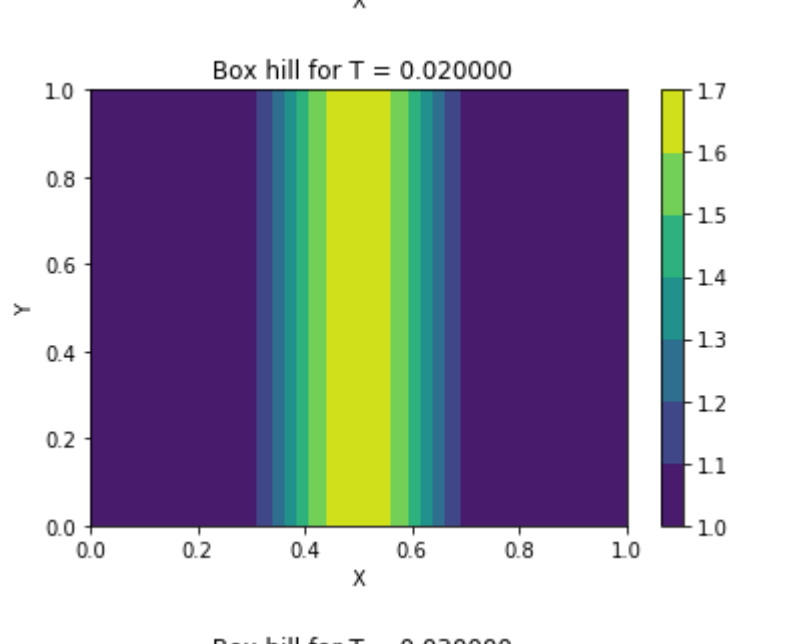

```
&lt;Figure size 432x288 with 0 Axes&gt;
```

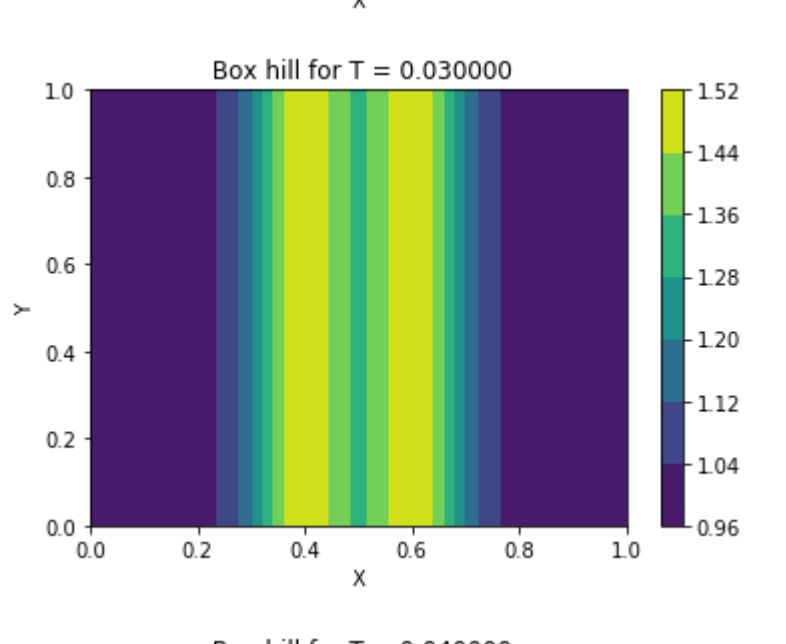
Box

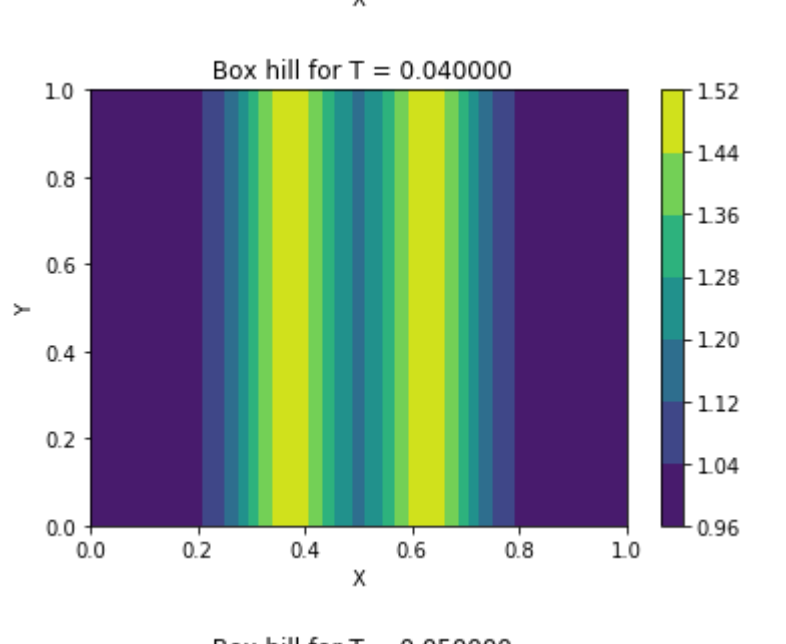
In [9]:
```python
T = [i*0.01 for i in range(1,31)]

for t in T:
    my_solver = Wave2D(b, t, Lx, Ly, I, V,Box, Nx, Ny, f)
    my_solver.set_initial_conditions()
    my_solver.time_evolution()
    my_solver.plot("X", "Y", " Box hill for T = %f" % t)

plt.show()
```

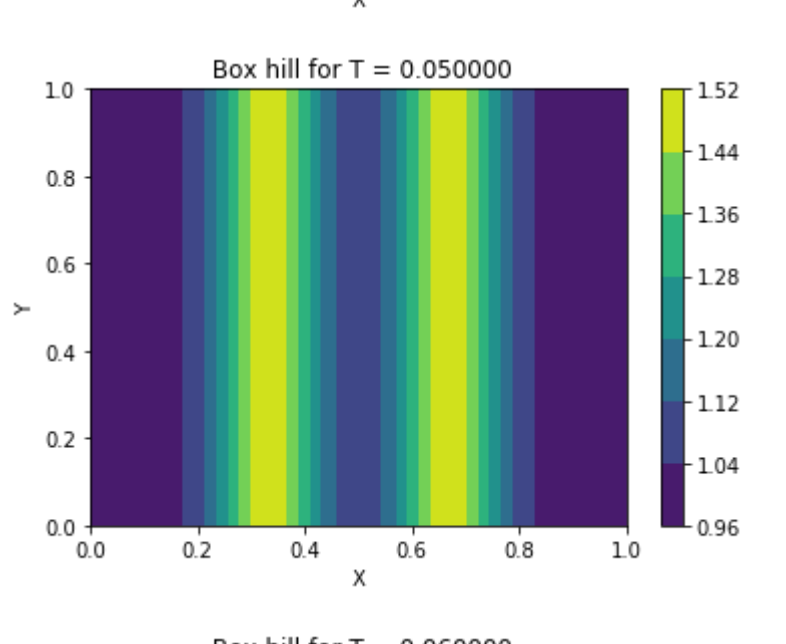
```
/Users/kasparagasvaer/opt/anaconda3/lib/python3.7/site-packages/ipykernel_launcher.py:135: RuntimeWarning: More than 20 figures have been opened. Figures created through the pyplot interface (`matplotlib.pyplot.figure`) are retained until explicitly closed and may consume too much memory. (To control this warning, see the rcParam `figure.max_open_warning`).
```

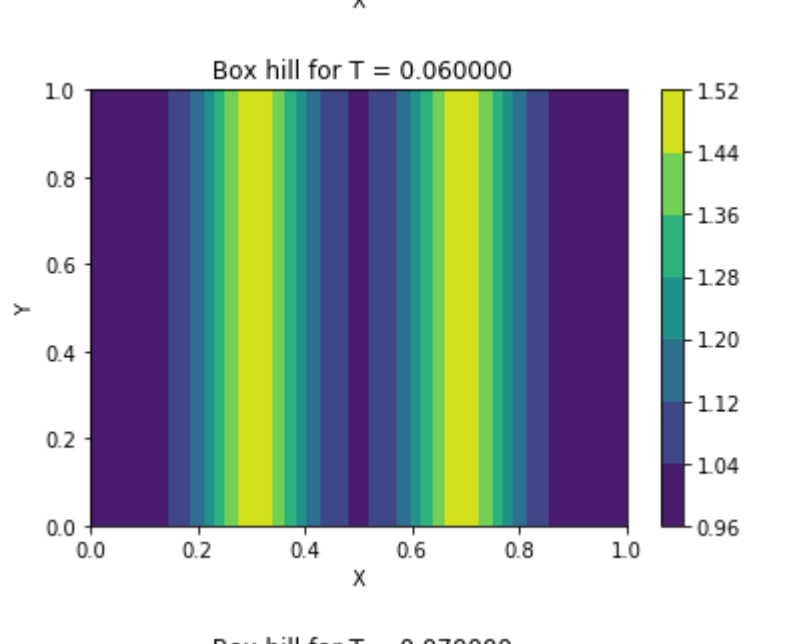


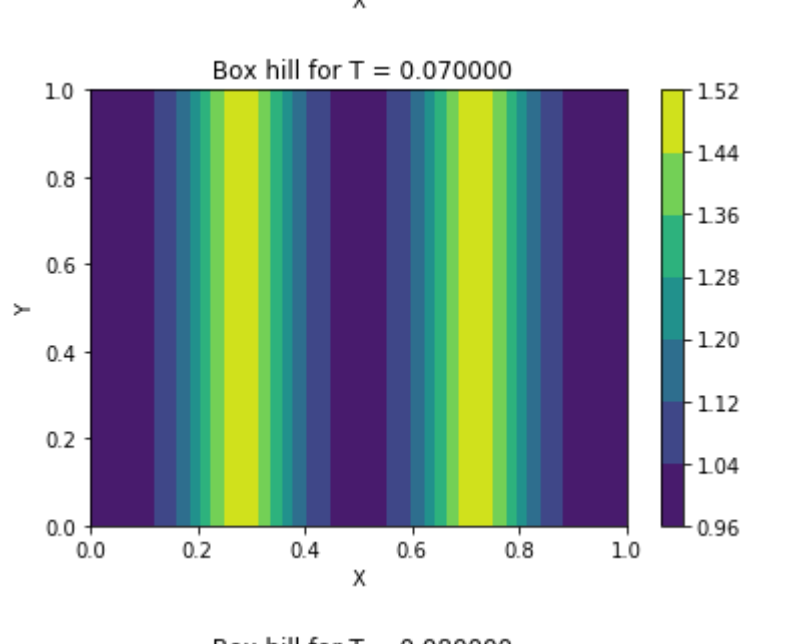


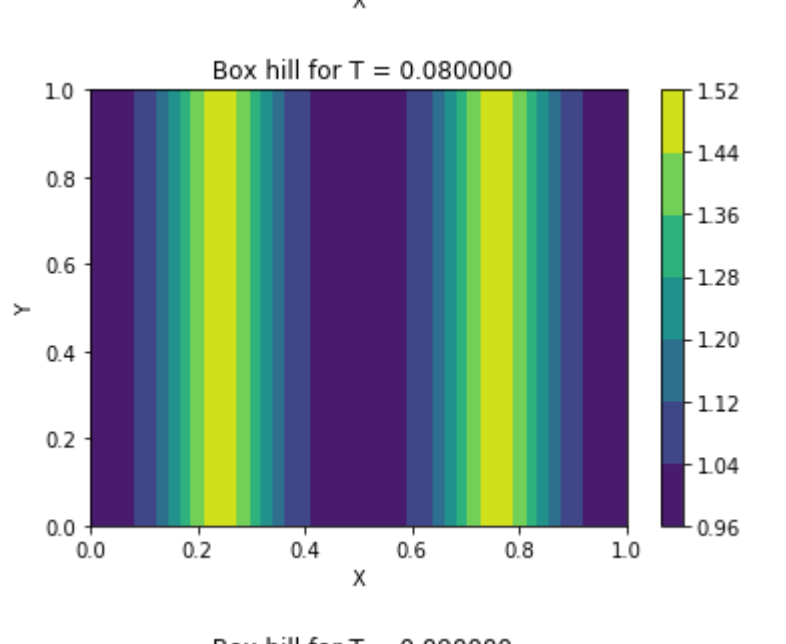


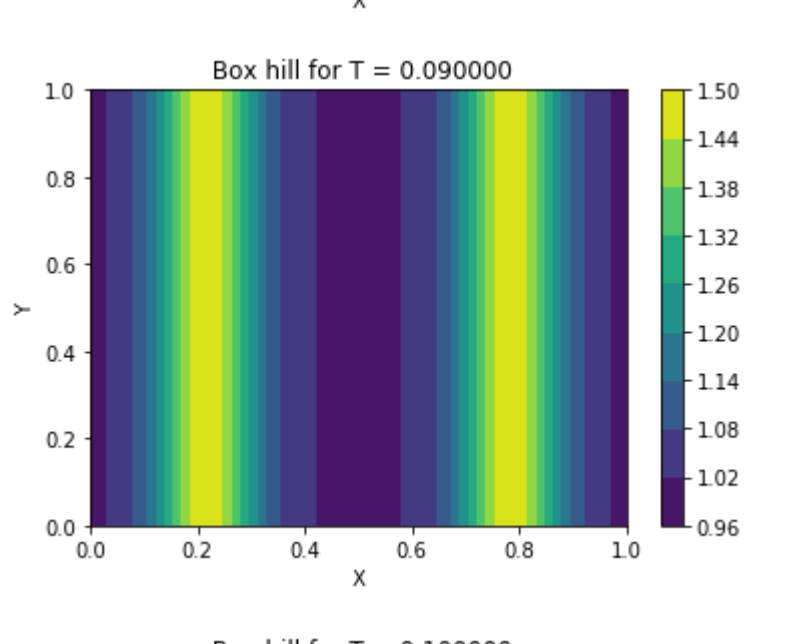


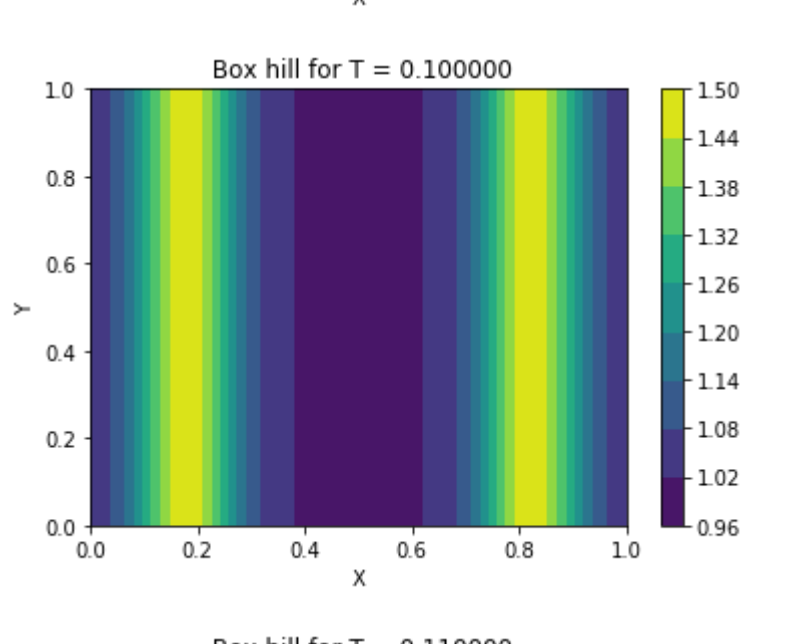


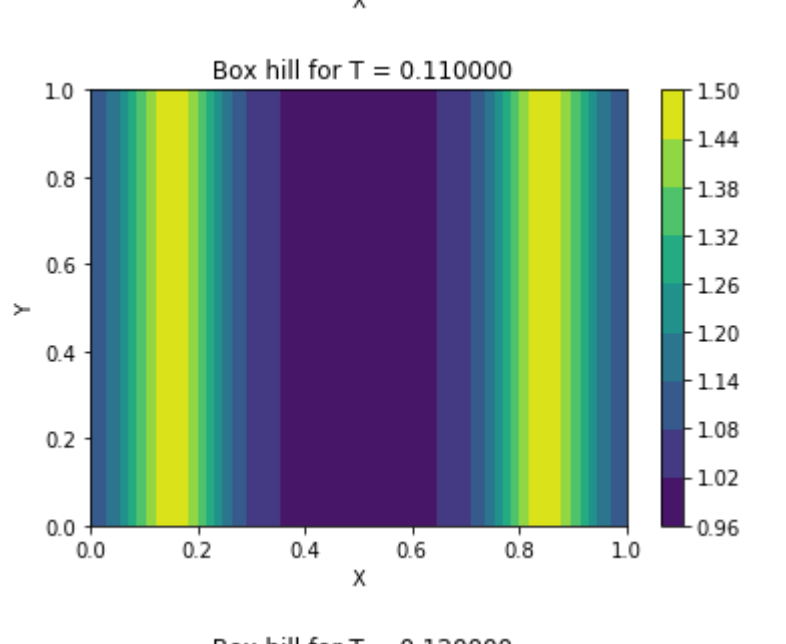


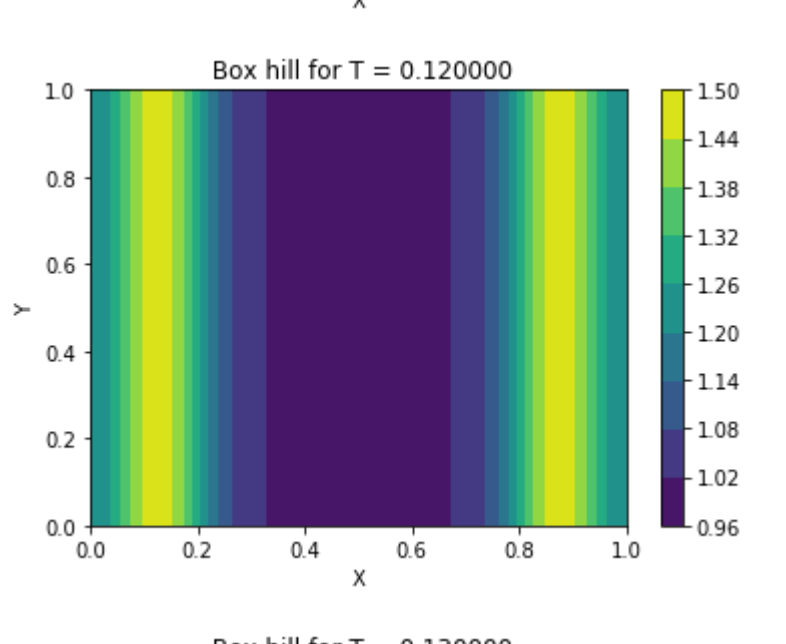


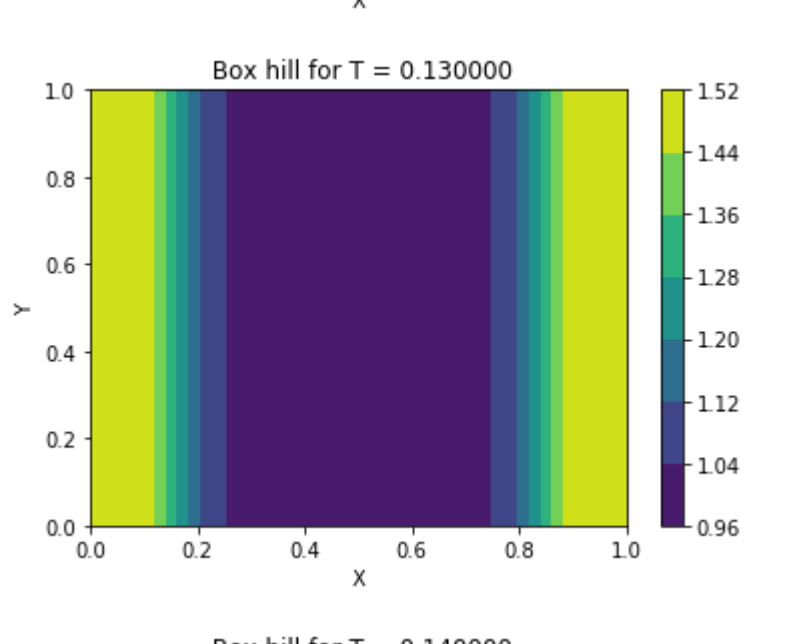


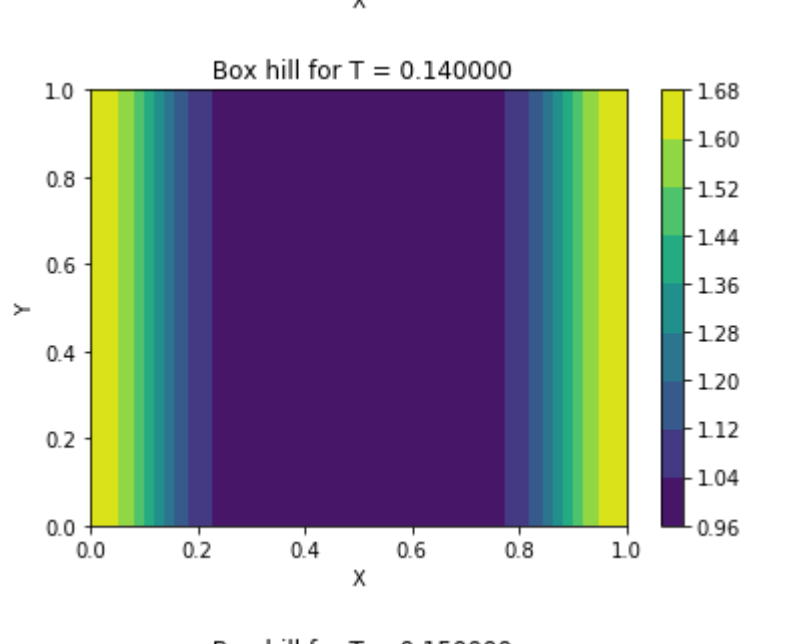


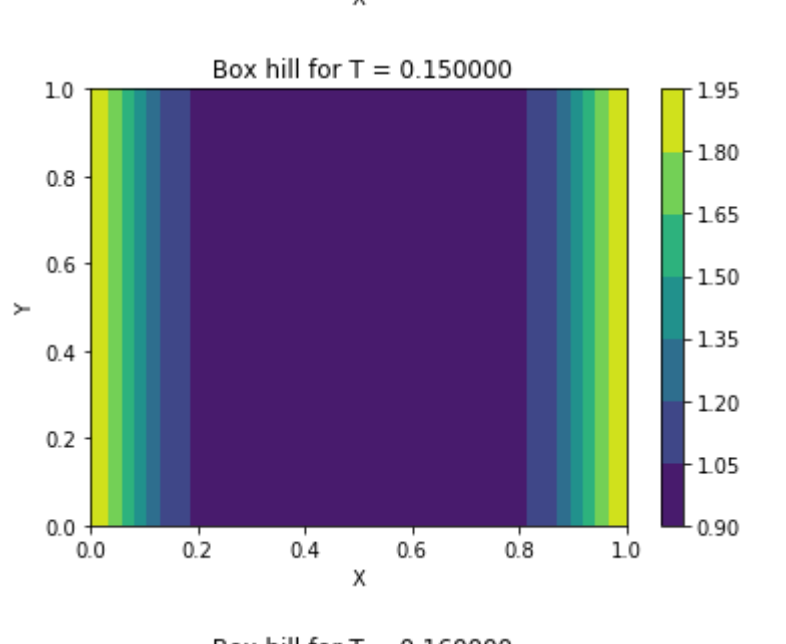


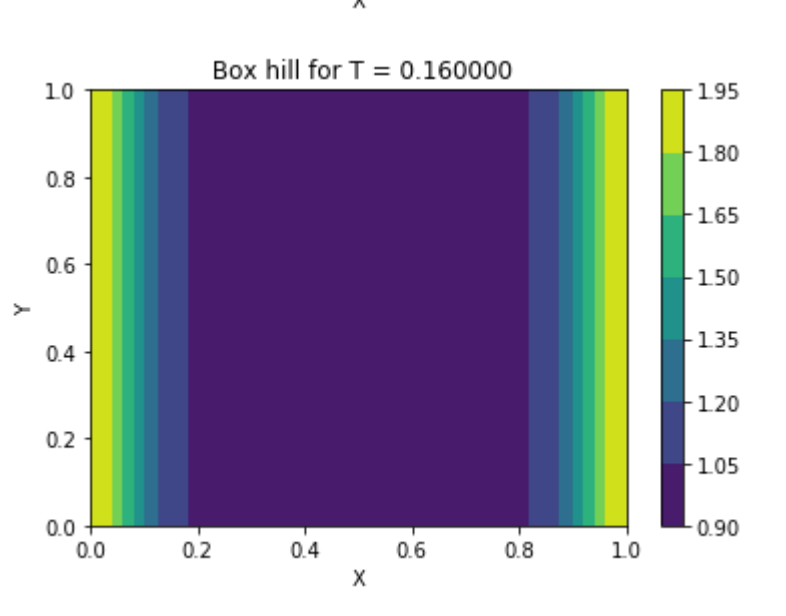


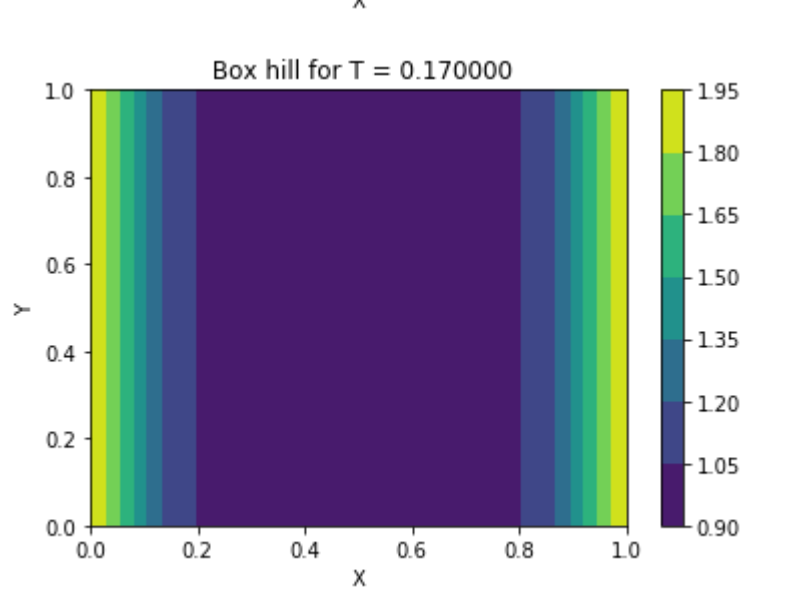


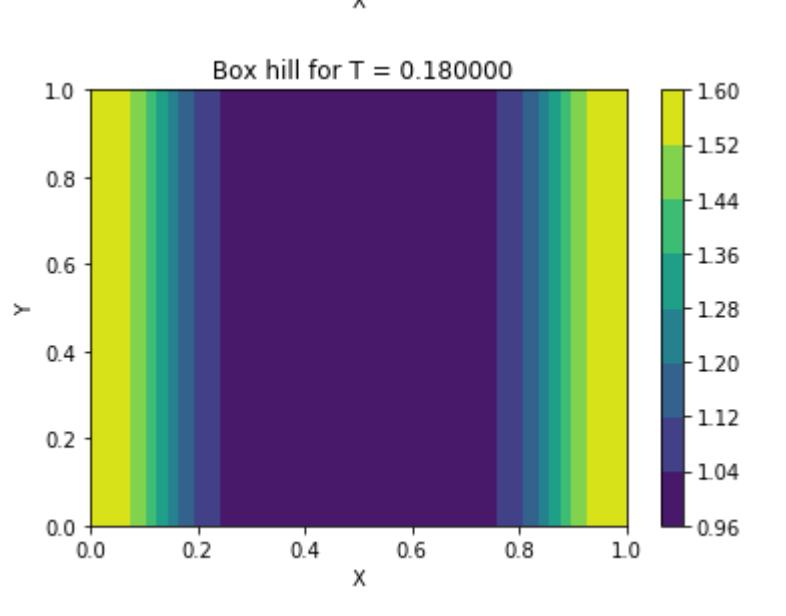


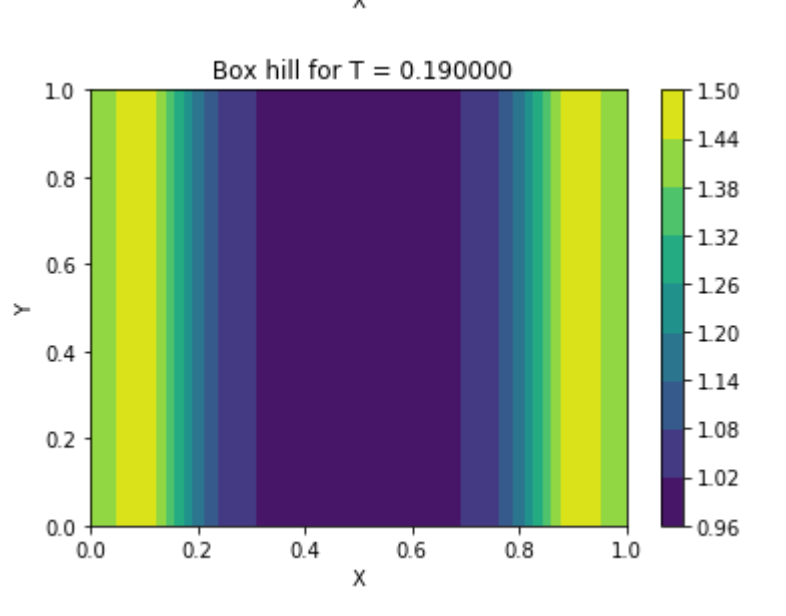


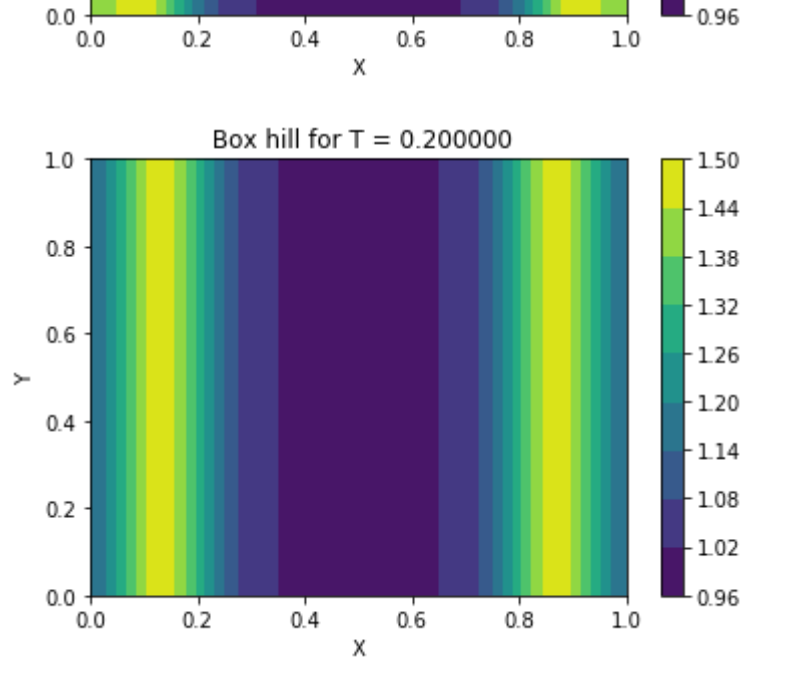


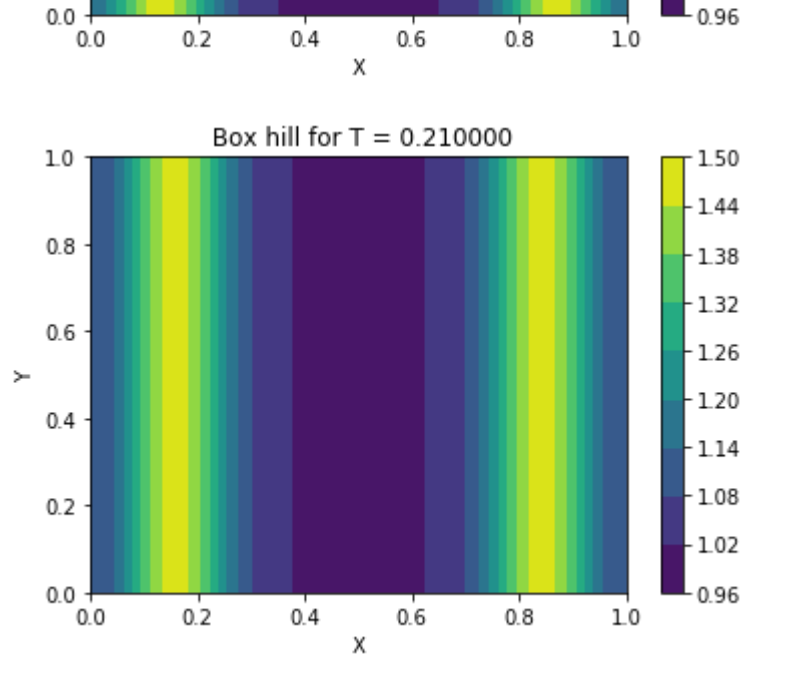


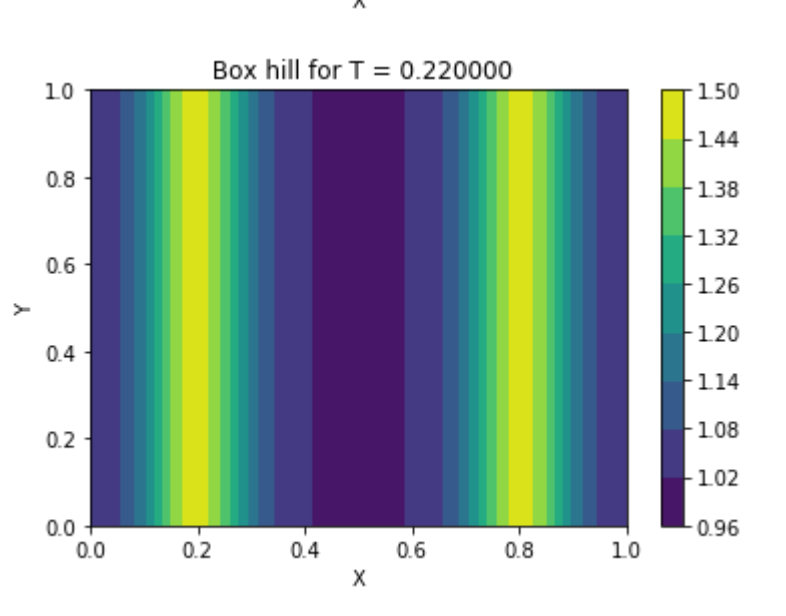


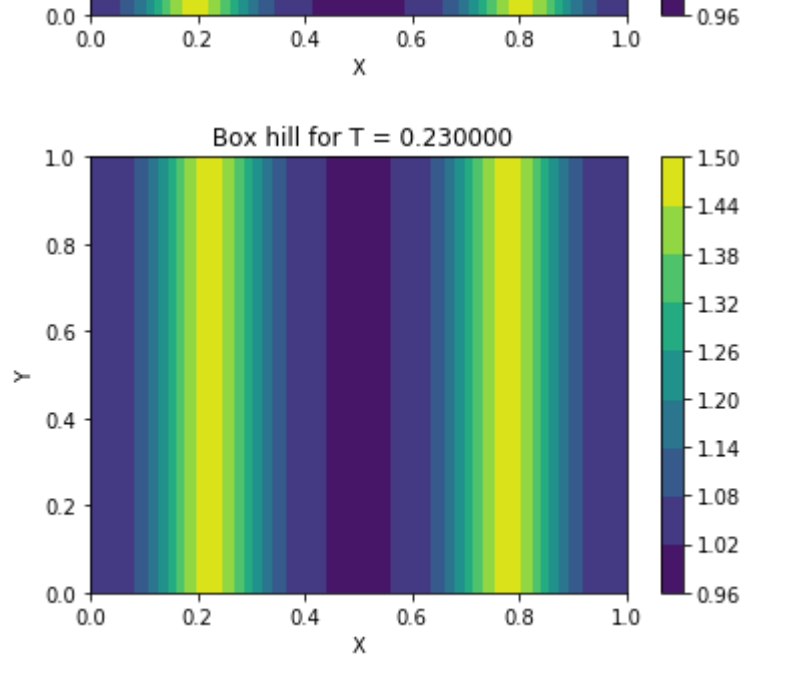


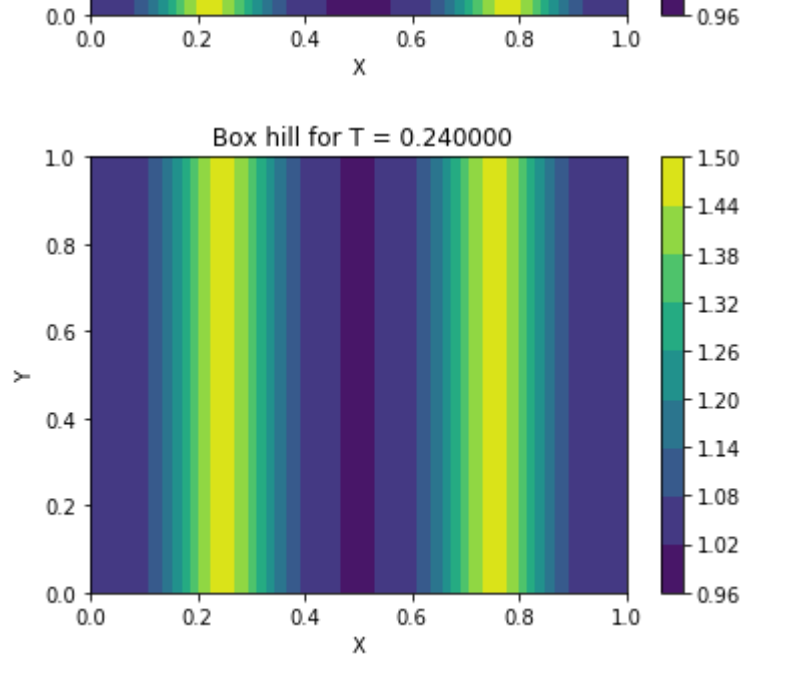


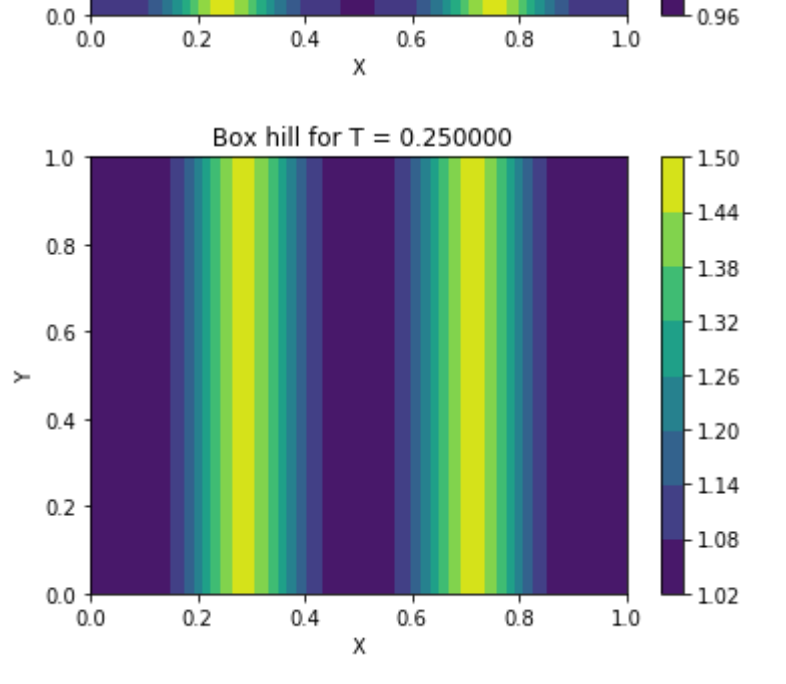


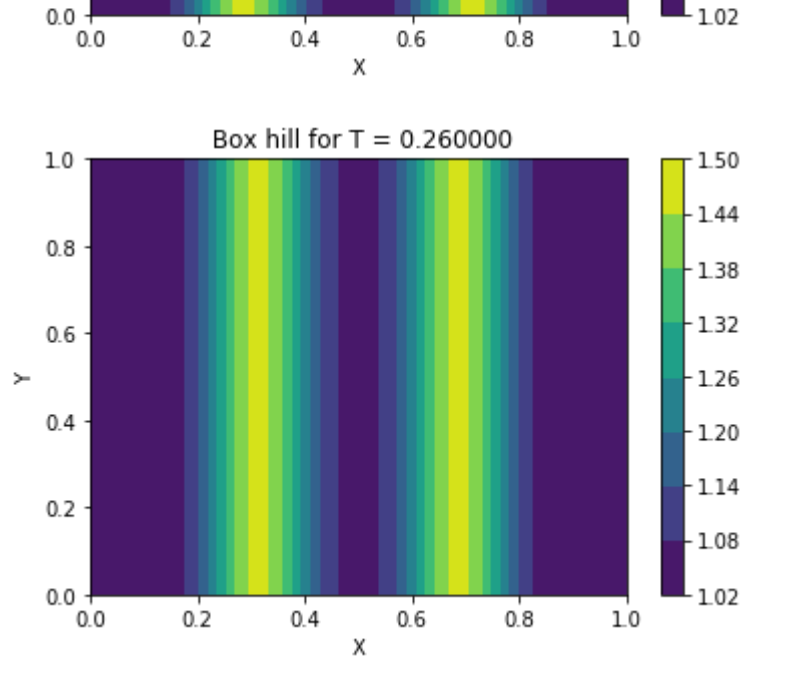


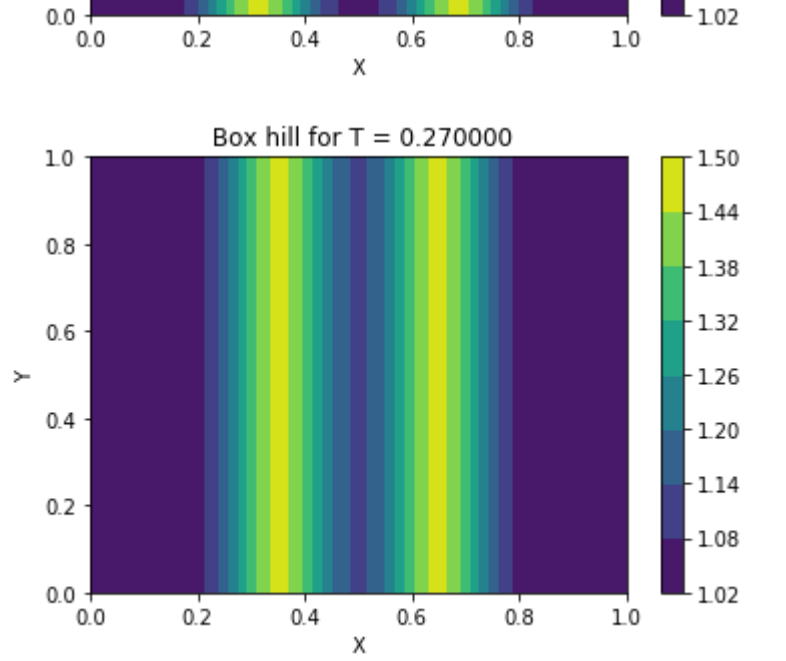


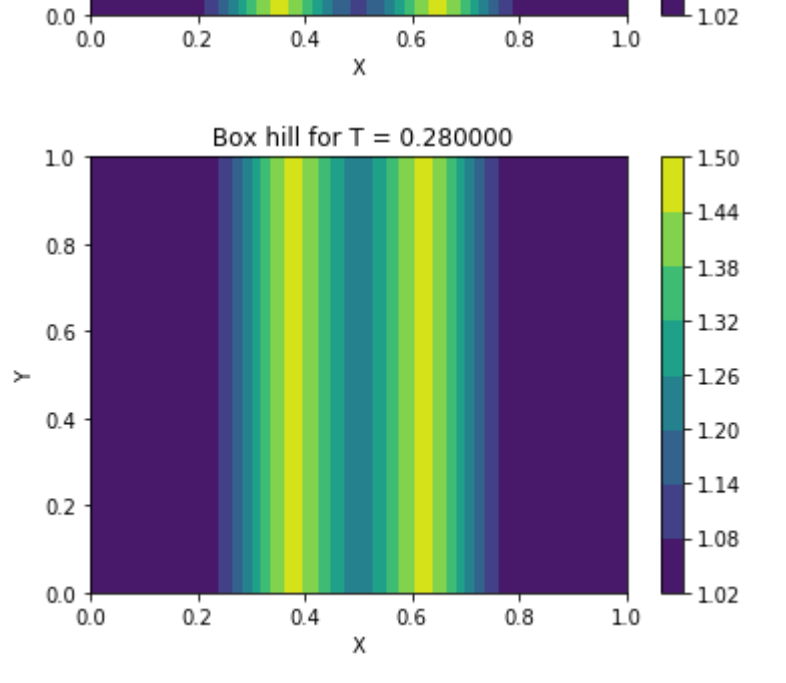


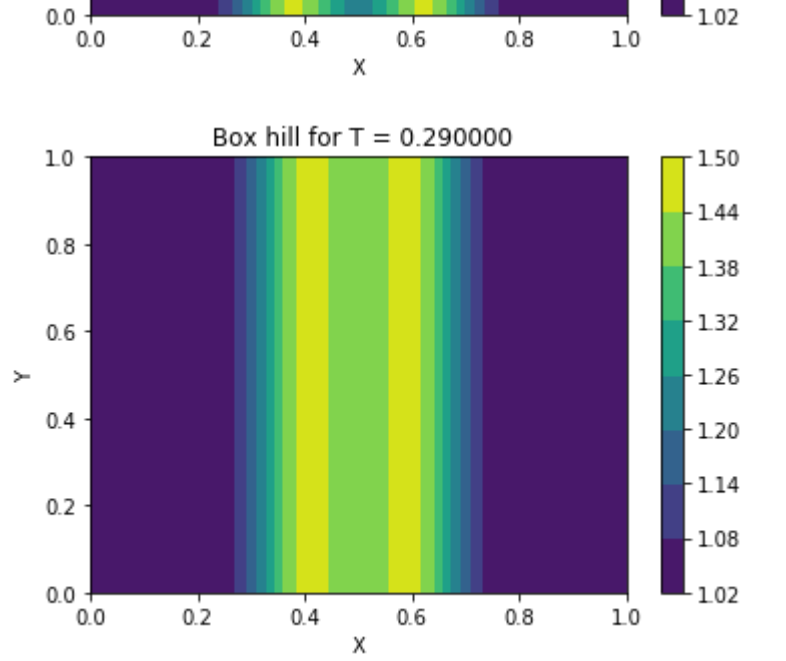


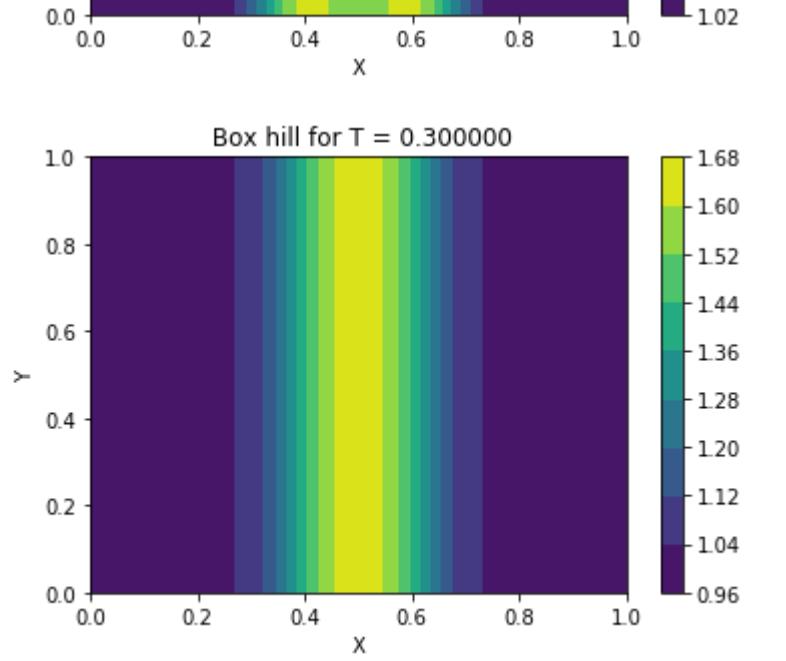










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