Partial Differential Equation Notes

Kasper Rosenkrands

24.09.2019

Contents

Introduction

This book will include notes and exercises as well as mock exams from the Partial Differential Equations course at Aalborg University.

1 Course Exercises

1.1 Exercise 1.4

Linear PDE, rewrite to normal form

$$\nabla u(x)\begin{bmatrix}-x_2\\ x_1\end{bmatrix} - u(x) = 0$$

thus

$$A(x) = \begin{bmatrix}-x_2\\ x_1\end{bmatrix}, \quad b(x) = -1, \quad f(x) = 0.$$

We then solve $\partial_t \Phi(t,x) = A(\Phi(t,x))$,

$$\begin{aligned}\varphi_1'(t) &= -\varphi_2(t), \quad \varphi_1(0) = x_1\\ \varphi_2'(t) &= -\varphi_1(t), \quad \varphi_2(0) = x_2.\end{aligned}$$

Thus $\Phi(t,x) = e^{t\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}} = \begin{bmatrix}\cos(t) & -\sin(t)\\ \sin(t) & \cos(t)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}x_1\cos(t) - x_2\sin(t)\\ x_1\sin(t) + x_2\cos(t)\end{bmatrix}$. Parameterizing the hypersurface gives $\gamma(q) = \begin{bmatrix}q\\ 0\end{bmatrix}$. By Lemma 1.8

$$X(t,q) = \Phi(t,\gamma(q)) = \begin{bmatrix}q\sin(t)\\ q\cos(t)\end{bmatrix}.$$

Inverting the function gives

$$x_1 = \cos(t) \;\Leftrightarrow\; \frac{x_1}{q} = \cos(t) \;\Leftrightarrow\; t = \cos^{-1}\left(\frac{x_1}{q}\right).$$

Now we have found t which we can use to find q as

$$\begin{aligned}x_2 &= q\sin(t)\\ &= q\sin\left(\cos^{-1}\left(\frac{x_1}{q}\right)\right)\\ &= q\sqrt{1 - \frac{x_1^2}{q^2}}\\ x_2^2 &= q^2 - x_1^2\\ q^2 &= x_1^2 + x_2^2\\ q &= \sqrt{x_1^2 + x_2^2}\\ &= \|x\|.\end{aligned}$$

By Theorem 1.9

$$\begin{aligned}Z(\psi) &= g(\|x\|)e^{-\int_0^{\cos^-1(x_1/\|x\|)} -1 \text{ d}s}\\ &= g(\|x\|)e^{\cos^-1(x_1/\|x\|)}.\end{aligned}$$

Not quite the answer but this is as close as i can get.

1.2 Exercise 1.8

The exercise is to solve the following Cauchy-problem

$$u(x)\partial_1 u(x) + \partial_2 u(x) = 1, \qquad x \in \mathbb{R}^2,\ u(q,q) = 0,\ q \in \mathbb{R} \tag{1}$$

with

$$\Sigma = \{(q,q) \mid q \in \mathbb{R}\} \tag{2}$$

1.2.1 Transform to normal form

The normal form is given by

$$u'(x)A(x) + b(x)u(x) = f(x), \qquad x \in \Omega \tag{3}$$

were $u'(x)$ is given by

$$u'(x) = [\partial_1 u(x)\ \ \ldots\ \ \partial_d u(x)] \tag{4}$$

We can write our equation as

$$[\partial_1 u(x)\ \partial_2 u(x)] \begin{bmatrix} u(x) \\ 1 \end{bmatrix} = 1 \tag{5}$$

i.e. our $A(x)$ and $b(x)$ becomes

$$A(x) = \begin{bmatrix} u(x) \\ 1 \end{bmatrix}, \qquad b(x) = 1 \tag{6}$$

which in turn gives us that (3) becomes

$$\nabla u(x) A(x,z) = 1 \tag{7}$$

1.2.2 Parameterizing the hypersurface

The initial hypersurface Σ by the following

$$\gamma(q) := \begin{bmatrix} q \\ q \end{bmatrix}, \qquad q \in \mathbb{R}.$$

Taking the derivative of the parameterization yields

$$\gamma'(q) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The tangent space of Σ becomes

$$T_{\gamma(q)}\Sigma = \left\{ \begin{bmatrix} s \\ s \end{bmatrix} \;\middle|\; s \in \mathbb{R} \right\}.$$

The condition that $A\Big(\gamma(q), \phi(\gamma(q))\Big) \in T_{\gamma(q)}\Sigma$ is always satisfied as $A\Big(\gamma(q), \phi(\gamma(q))\Big)$ is given by the following

$$A\Big(\gamma(q), \phi(\gamma(q))\Big) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The Cauchy problem is non-characteristic in everything point contained in Σ.

1.2.3 System of equations

We get the following system of equations

$$\begin{aligned}
\partial_t x_1(t,q) &= Z(t,q) & , \ x_1(0,q) = q \quad (8)\\
\partial_t x_2(t,q) &= 1 & , \ x_2(0,q) = q \quad (9)\\
\partial_t Z(t,q) &= 1 & , \ Z(0,q) = 0 \quad (10)
\end{aligned}$$

From (10) we quickly see that

$$Z(t,q) = t. \tag{11}$$

From (9) we coclude that

$$x_2(t,q) = q + t.$$

Considering (8) and (11) we see that

$$\begin{aligned}
x_1(t,q) &= q + \int_0^t s \ ds & (12)\\
&= q + \frac{1}{2}t^2. & (13)
\end{aligned}$$

1.2.4 Finding t and q

We will start by finding t, we have the following expressions

$$x_1 = q + \frac{1}{2}t^2, \qquad x_2 = q + t.$$

We can start by isolating q is the second equation

$$q = x_2 - t,$$

inserting this in the first equation gives the following

$$\begin{aligned}
x_1 &= q + \frac{1}{2}t^2 & (14)\\
&= x_2 - t + \frac{1}{2}t^2 & (15)
\end{aligned}$$

Rewriting (15) gives

$$x_1 - x_2 + \frac{1}{2} = \frac{1}{2}\underbrace{\left(t^2 - 2t + 1\right)}_{(t-1)^2} \tag{16}$$

Multiplying both sides of (16) by 2 gives

$$\begin{aligned}
(t-1)^2 &= 2x_1 - 2x_2 + 1 & (17)\\
t - 1 &= \pm\sqrt{2x_1 - 2x_2 + 1} & (18)\\
t &= 1 \pm \sqrt{2x_1 - 2x_2 + 1} & (19)
\end{aligned}$$

1.3 Exercise 1.10

Find a solution for the Cauchy-problem

$$\begin{aligned}
\partial_1 u(x) + \partial_2 u(x) &= u(x)^2\\
u(x_1, 0) &= g(x_1)
\end{aligned}$$

Solution: This is a kvasilinear PDE, we will start by bringing it to normal form

$$A(x, u(x)) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad b(x, u(x)) = u(x)^2.$$

From Lemma 1.15

$$\begin{aligned}
\partial_t X_1(t, q) &= 1, & X_1(0, q) &= q \\
\partial_t X_2(t, q) &= 1, & X_2(0, q) &= 0 \\
\partial_t Z(t, q) &= Z(t, q)^2, & Z(0, q) &= g(q).
\end{aligned}$$

We solve the system

$$\begin{aligned}
Z(t, q) &= \frac{1}{\frac{1}{g(q)} - t} \\
X_1(t, q) &= q + t \\
X_2(t, q) &= t.
\end{aligned}$$

We can now invert t and q

$$x_1 = q + t, \quad x_2 = t$$
$$t = x_2, \quad x_1 = q + x_2 \iff q = x_1 - x_2.$$

From theorem 1.16

$$u(x) = Z(\psi(x)) = \frac{1}{\frac{1}{g(x_1 - x_2)} - x_2}.$$

2 Exam Exercises

2.1 Mock exam 1

2.1.1 Exercise 1

Let

$$\Omega := (0, \infty)^2.$$

Calculate the solution to the following Cauchy-Problem,

$$\begin{aligned}
&x_1 \partial_1 u(x) + x_2 \partial_2 u(x) + x_1 x_2 u(x) = 0, \quad x \in \Omega; \\
&u(q, q^2) = 5, \quad q > 0.
\end{aligned}$$

Solution: As the exercise does not ask if the problem is non-characteristic, we suppose that it is. The first step is to bring the problem to normal form. The normal form is given by

$$\nabla u(x) A(x) + u(x) b(x) = f(x).$$

In our case

$$A(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b(x) = x_1 x_2 \quad \text{and} \quad f(x) = 0.$$

Next we want to parameterize the hypersurface, which becomes

$$\gamma(q) := \begin{bmatrix} q \\ q^2 \end{bmatrix}.$$

When then seek

$$\Phi(t, x),$$

which we find as the solution to

$$\partial\Phi(t, x) = A(\Phi(t, x)).$$

As we in our case have $d = 2$ the ODE can be written as the following system

$$\begin{aligned}
\varphi_1'(t) &= A(\Phi(t))_1, \quad \varphi_1(0) = x_1 \\
\varphi_2'(t) &= A(\Phi(t))_2, \quad \varphi_2(0) = x_2,
\end{aligned}$$

by the definiton of $A(x)$ the system becomes

$$\begin{aligned}
\varphi_1'(t) &= \varphi(t)_1, \quad \varphi_1(0) = x_1 \\
\varphi_2'(t) &= \varphi(t)_2, \quad \varphi_2(0) = x_2.
\end{aligned}$$

By solving the system we get the following $\Phi(t, x)$

$$\Phi(t, x) = \begin{bmatrix} \varphi_1(t, x) \\ \varphi_2(t, x) \end{bmatrix} = \begin{bmatrix} x_1 e^t \\ x_2 e^t \end{bmatrix}.$$

By Lemma 1.8 we have $X(t, q) = \Phi(t, \gamma(q))$, thus

$$X(t, q) = \Phi(t, \gamma(q)) = \begin{bmatrix} qe^t \\ q^2 e^t \end{bmatrix}.$$

As we seek values for t and q we can now invert the functions

$$\begin{aligned}
x_1 &= qe^t \\
x_2 &= q^2 e^t
\end{aligned}$$

Isolating q in the first equation gives

$$x_1 = qe^t \Leftrightarrow e^{-t} x_1 = q,$$

Inserting this in the second equation yields

$$x_2 = \left(e^{-t} x_1\right)^2 e^t = e^{-2t} x_1^2 e^t = e^{-t} x_1^2 = \frac{1}{e^t} x_1^2,$$

this in turn gives

$$\begin{aligned}
x_2 = \frac{1}{e^t} x_1^2 \quad &\Leftrightarrow \quad e^t = \frac{x_1^2}{x_2} \\
\Leftrightarrow \quad \log\left(e^t\right) = \log\left(\frac{x_1^2}{x_2}\right) \quad &\Leftrightarrow \quad t = \log\left(x_1^2\right) - \log\left(x_2\right).
\end{aligned}$$

The expression we have now found for t can be inserted into the expression for x_1 to get q

$$q = x_1 e^{-\left(\log\left(x_1^2\right) - \log(x_2)\right)} = x_1 e^{\log(x_2) - \log\left(x_1^2\right)} = x_1 \frac{x_2}{x_1^2} = \frac{x_2}{x_1}.$$

To sum up we have now found both t and q as

$$t = \log\left(x_1^2\right) - \log\left(x_2\right), \quad q = \frac{x_2}{x_1}.$$

We can now insert in Theorem 1.9 what we have found. The theorem states that

$$Z(t, q) = \phi(\gamma(q)) e^{-\int_0^t b(X(s,q))ds} + \int_0^t e^{-\int_s^t b(X(r,q))dr} f(X(s, q))ds,$$

with $\phi(\gamma(q))$ being $Z(0,q)$ i.e. the given initial value. Inserting what we now, especially that $f(x) = 0$, yields

$$\begin{aligned}
Z(\log\left(x_1^2\right) - \log\left(x_2\right), \frac{x_2}{x_1}) &= 5e^{-q^3 \int_0^{\log\left(x_1^2\right)-\log(x_2)} e^{2s} ds} \\
&= 5e^{-q^3 \left[\frac{1}{2}e^{2s}\right]_0^{\log\left(x_1^2\right)-\log(x_2)}} \\
&= 5e^{-q^3 \frac{1}{2}\left(e^{2\left(\log\left(x_1^2\right)-\log(x_2)\right)} - e^0\right)} \\
&= 5e^{-q^3 \frac{1}{2}\left(e^{2\log\left(\frac{x_1^2}{x_2}\right)} - e^0\right)} \\
&= 5e^{-q^3 \frac{1}{2}\left(e^{\log\left(\frac{x_1^4}{x_2^2}\right)} - e^0\right)} \\
&= 5e^{-q^3 \frac{1}{2}\left(\frac{x_1^4}{x_2^2} - 1\right)} \\
&= 5e^{-\frac{x_2^3}{x_1^3} \frac{1}{2}\left(\frac{x_1^4}{x_2^2} - 1\right)} \\
&= 5e^{-\frac{1}{2}\left(\frac{x_1^4 x_2^3}{x_1^3 x_2^2} - \frac{x_2^3}{x_1^3}\right)} \\
&= 5e^{-\left(x_1 x_2 - x_2^3/x_1^3\right)/2}.
\end{aligned}$$

Which in turns implies that

$$u(x) = 5e^{-\left(x_1 x_2 - x_2^3/x_1^3\right)/2}, \quad x \in \Omega.$$

2.1.2 Exercise 2

Let

$$\Omega := \left\{ x \in \mathbb{R}^2 \;\middle|\; |x_1 - x_2| < \frac{\pi}{2} \right\}$$

Calculate the solution for the Cauchy problem given by

$$\begin{aligned}
\frac{2}{1+u(x)^2}\partial_1 u(x) + \frac{1}{1+u(x)^2}\partial_2 u(x) &= 1, \quad x \in \Omega \\
u(q,q) &= 0, \quad q \in \mathbb{R}.
\end{aligned}$$

Solution We start by rewriting to normal form

$$\begin{aligned}
\frac{2}{1+u(x)^2}\partial_1 u(x) + \frac{1}{1+u(x)^2}\partial_2 u(x) &= 1 \\
u'(x)A(x,u(x)) &= b(x,u(x))
\end{aligned}$$

for

$$u'(x) = \left[\partial_1 u(x)\ \partial_2 u(x)\right], \quad A(x,u(x)) = \begin{bmatrix} \dfrac{2}{1+u(x)^2} \\ \dfrac{1}{1+u(x)^2} \end{bmatrix}, \quad b(x,u(x)) = 1.$$

From Lemma 1.15 we need to solve the following system, notice that the initial condition originate from $u(q,q) = 0$.

$$\begin{aligned}
\partial_t X_1(t,q) &= A(X(t,q), Z(t,q))_1, & X_1(0,q) &= q \\
\partial_t X_2(t,q) &= A(X(t,q), Z(t,q))_2, & X_2(0,q) &= q \\
\partial_t Z(t,q) &= b(X(t,q), Z(t,q)), & Z(0,q) &= 0.
\end{aligned}$$

Inserting what we have found so far yields the following system

$$\begin{aligned}
\partial_t X_1(t,q) &= \frac{2}{1+Z(t,q)^2}, & X_1(0,q) &= q \\
\partial_t X_2(t,q) &= \frac{1}{1+Z(t,q)^2}, & X_2(0,q) &= q \\
\partial_t Z(t,q) &= 1, & Z(0,q) &= 0.
\end{aligned}$$

As $Z(t,q)$ is present in the first to equation, we will start by solving this, then we will insert this in the other two equations and then solve these

$$\begin{aligned}
Z(t,q) &= t \\
X_1(t,q) &= 2\tan^{-1}(t) + q \\
X_2(t,q) &= \tan^{-1}(t) + q.
\end{aligned}$$

We now need to isolate t and q in the to last equations, i.e. 2 equations with 2 unknowns.

$$x_2 = \tan^{-1}(t) + q \;\Leftrightarrow\; x_2 - q = \tan^{-1}(1) \;\Leftrightarrow\; \tan(x_2 - q) = t.$$

$$x_1 = 2\tan^{-1}(t) + q \;\Leftrightarrow\; x_1 = 2(x_2 - q) + q \;\Leftrightarrow\; x_1 = 2x_2 - q \;\Leftrightarrow\; q = 2x_2 - x_1.$$

$$t = \tan(x_2 - q) = \tan(x_2 - (2x_2 - x_1)) = \tan(x_1 - x_2).$$

As suggested by Theorem 1.16 we have

$$u(x) = Z(\Psi(x)) = \tan(x_1 - x_2).$$

2.1.3 Exercise 3

Solve the following initial-value-problem

$$\begin{aligned}
(\partial_t^2 - \partial_x^2)u(t,x) = 0, \quad t, x \in \mathbb{R} \\
u(0,x) = \cos(x), \quad \partial_t u(0,x) = \cos^3(x), \quad x \in \mathbb{R}.
\end{aligned}$$

Hint: $\cos^3(y) = \cos(y) - \sin^2(y)\cos(y)$

Solution: Because we have $d = 1$ we can use d'Alemberts formula, recognizing that problem is in fact a wave equation.

Remember that $u(0,x) = g(x)$ and that $\partial_t u(0,x) = h(x)$, i.e.

$$g(x) = \cos(x), \quad h(x) = \cos^3(x).$$

D'Alemberts formula is given by

$$u(t,x) = \frac{1}{2}\left\{g(x+t) - g(x-t)\right\} + \frac{1}{2}\int_{x-t}^{x+t} h(y)\mathrm{d}y.$$

We insert what we know and calculate

$$\begin{aligned}
u(t,x) &= \frac{1}{2}\left\{\cos(x+t) - \cos(x-t)\right\} + \frac{1}{2}\int_{x-t}^{x+t} \cos^3(y)\mathrm{d}y \\
&= \frac{1}{2}\left\{\cos(x+t) - \cos(x-t)\right\} + \frac{1}{2}\int_{x-t}^{x+t} \cos(y) - \sin^2(y)\cos(y)\mathrm{d}y \\
&= \frac{1}{2}\left\{\cos(x+t) - \cos(x-t)\right\} + \frac{1}{2}\int_{x-t}^{x+t} \cos(y)\mathrm{d}y - \frac{1}{2}\int_{x-t}^{x+t} \sin^2(y)\cos(y)\mathrm{d}y \\
&= \frac{1}{2}\left\{\cos(x+t) - \cos(x-t)\right\} + \frac{1}{2}[\sin(y)]_{x-t}^{x+t} - \frac{1}{2}\int_{x-t}^{x+t} \sin^2(y)\cos(y)\mathrm{d}y \\
&= \frac{1}{2}\left\{\cos(x+t) - \cos(x-t) + \sin(x+t) - \sin(x-t)\right\} - \frac{1}{2}\int_{x-t}^{x+t} \sin^2(y)\cos(y)\mathrm{d}y,
\end{aligned}$$

what remains is to calculate the last antiderivative, which we will do below using substitution

$$u = \sin(x), \quad \frac{\mathrm{d}u}{\mathrm{d}y} = \cos(y) \;\Leftrightarrow\; \mathrm{d}u = \cos(y)\mathrm{d}y.$$

We can now insert this into the antiderivative remembering that we also need to substitute the bounds, i.e. applying the function u on them

$$\frac{1}{2}\int_{x-t}^{x+t} \sin^2(y)\cos(y)\mathrm{d}y \;\Leftrightarrow\; \frac{1}{2}\int_{x-t}^{x+t} \sin^2(y)\cos(y)\mathrm{d}y$$
$$\Leftrightarrow\; \frac{1}{2}\int_{\sin(x-t)}^{\sin(x+t)} u^2\mathrm{d}u \;\Leftrightarrow\; \frac{1}{6}[u^3]_{\sin(x-t)}^{\sin(x+t)} \;\Leftrightarrow\; \frac{1}{6}\left(\sin^3(x+t) - \sin^3(x-t)\right).$$

Inserting what we just found, we have expressed the solution explicitly as

$$u(t,x) = \frac{1}{2}\left\{\cos(x+t) - \cos(x-t) + \sin(x+t) - \sin(x-t)\right\} - \frac{1}{6}\left(\sin^3(x+t) - \sin^3(x-t)\right).$$

2.1.4 Exercise 4

Solve the following intial-value/border-value-problem,

$$\left(\partial_t - \frac{1}{2}\partial_x^2\right)u(t,x) = 0, \quad t > 0,\ x \in (0,\pi);$$
$$u(0,x) = 4\sin(3x)\cos(3x), \quad x \in (0,\pi);$$
$$u(t,0) = u(t,\pi) = 0, \quad t > 0.$$

Hint: $\sin(2y) = 2\sin(y)\cos(y),\ y \in \mathbb{R}$.

Solution: As the problem is on the form of a heat equation with homogenous Dirichlet bound-conditions, the solution according to theorem 4.10 given by

$$u(t,x) = \sum_{n=1}^{\infty} e^{-tn^2\pi^2/2\ell^2} B_n \sin(n\pi x/\ell)\mathrm{d}y.$$

We define B_n as

$$B_n := \frac{2}{\ell}\int_0^{\ell} g(x)\sin(n\pi x/\ell)\mathrm{d}x, \quad n \in \mathbb{N},$$

in our case $\ell = \pi$ and $g(x) = 4\sin(3x)\sin(3x)$, we see that according to the hint

$$\sin(2y) = 2\sin(y)\cos(y) \;\Leftrightarrow\; 2\sin(6y) = 4\sin(3y)\cos(3y).$$

We can see that $g(x) = 2\sin(6x)$, we insert what we know into the definition of B_n to get

$$B_n = \frac{2}{\pi}\int_0^{\pi} 2\sin(6x)\sin(nx)\mathrm{d}x, \quad n \in \mathbb{N}.$$

According to theorem 4.10 the solution to the problem is given by

$$u(t,x) = \sum_{n=1}^{\infty} e^{-tn^2/2}\frac{2}{\pi}\int_0^{\pi} 2\sin(6x)\sin(nx)\mathrm{d}x\sin(nx).$$

Furthermore, according to remark C.3 there is only one n for which the term in the sum is not equal to zero, namely $n = 6$, as $n = m$. Thus the solution can be simplified to the following

$$u(t,x) = 2e^{-t6^2/2}\sin(6x) = 2e^{-18t}\sin(6x).$$

2.2 Mock exam 2

2.2.1 Exercise 1

Let $\Omega = \{x \in \mathbb{R}^2 \mid x_2 > x_1\}$, find the solution to the following Cauchy-problem

$$\begin{aligned}
&\partial_1 u(x) + \partial_2 u(x) + x_1 u(x) = e^{-x_2^2/2}, \quad x \in \Omega \\
&u(0,q) = 1, \quad q > 0.
\end{aligned}$$

Solution: The first step is to bring the problem to normal form i.e.

$$\begin{aligned}
u'(x)A(x) + u(x)b(x) &= f(x) \\
\begin{bmatrix} \partial_1 u(x) & \partial_2 u(x) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + u(x)x_1 &= e^{-x_2^2/2}
\end{aligned}$$

this implies that

$$A(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad b(x) = x_1, \quad f(x) = e^{-x_2^2/2}$$

Parameterizing the hypersurface gives $\gamma(q) = \begin{bmatrix} 0 \\ q \end{bmatrix}$. We then seek $\Phi(t,x)$ which we find as the solution to $\partial_t \Phi(t,x) = A(\Phi(t,x))$. As we in our case have $d = 2$, the ODE becomes a system of differential equations given as

$$\begin{aligned}
\varphi_1'(t) = A(\Phi(t))_1, \quad \varphi_1(0) = x_1 \\
\varphi_2'(t) = A(\Phi(t))_2, \quad \varphi_2(0) = x_2.
\end{aligned}$$

The system in our case becomes

$$\begin{aligned}
\varphi_1'(t) = 1, \quad \varphi_1(0) = x_1 \\
\varphi_2'(t) = 1, \quad \varphi_2(0) = x_2,
\end{aligned}$$

by solving the system we get

$$\Phi(t) = \begin{bmatrix} x_1 + t \\ x_2 + t \end{bmatrix},$$

and by Lemma 1.8 we have

$$X(t,q) = \Phi(t, \gamma(q)) = \begin{bmatrix} t \\ q + t \end{bmatrix}.$$

Thus seeking values for t and q we invert the function

$$\begin{aligned}
x_1 &= t \\
x_2 &= q + t.
\end{aligned}$$

This implies that

$$\begin{aligned}
t &= x_1 \\
q &= x_2 - X_1.
\end{aligned}$$

We can now insert this into the formula from theorem 1.9

$$\begin{aligned}
Z(t,q) &= \phi(\gamma(q)) e^{\int_0^t b(X(s,q))\mathrm{d}s} + \int_0^t e^{-\int_s^t b(X(r,q))\mathrm{d}r} f(X(s,q))\mathrm{d}s \\
&= e^{-\int_0^{x_1} s \ \mathrm{d}s} + \int_0^{x_1} e^{-\int_s^{x_1} r \ \mathrm{d}r} e^{-(q+s)^2/2}\mathrm{d}s \\
&= e^{-\int_0^{x_1} s \ \mathrm{d}s} + \int_0^{x_1} e^{-\int_s^{x_1} r \ \mathrm{d}r} e^{-((x_2-x_1)+s)^2/2}\mathrm{d}s
\end{aligned}$$

We can start by solving the following integrals

$$
\begin{aligned}
-\int_0^{x_1} s\ \mathrm{d}s &= -\left[\frac{1}{2}s^2\right]_0^{x_1} = -\frac{1}{2}x_1^2 \\
-\int_s^{x_1} r\ \mathrm{d}r &= -\left[\frac{1}{2}r^2\right]_s^{x_1} = -\frac{1}{2}\left(x_1^2 - s^2\right).
\end{aligned}
$$

Inserting these gives

$$
\begin{aligned}
Z(t,q) &= e^{-x_1^2/2} + \int_0^{x_1} e^{-\frac{1}{2}\left(x_1^2-s^2\right)-((x_2-x_1)^2+s^2+2(x_2-x_1)s)/2}\mathrm{d}s \\
&= e^{-x_1^2/2} + \int_0^{x_1} e^{-\left(x_1^2-s^2\right)/2-((x_2-x_1)^2+s^2+2(x_2-x_1)s)/2}\mathrm{d}s \\
&= e^{-x_1^2/2} + \int_0^{x_1} e^{(-x_1^2+s^2-(x_2-x_1)^2-s^2-2(x_2-x_1)s)/2}\mathrm{d}s \\
&= e^{-x_1^2/2} + e^{(-x_1^2-x_2^2-x_1^2+2x_2x_1)/2}\int_0^{x_1} e^{-(x_2-x_1)s}\mathrm{d}s \\
&= e^{-x_1^2/2} + e^{-x_1^2+x_2x_1-x_2^2/2}\left[-\frac{1}{x_2-x_1}e^{-(x_2-x_1)s}\right]_0^{x_1} \\
&= e^{-x_1^2/2} + e^{-x_1^2+x_2x_1-x_2^2/2}\left(-\frac{e^{-(x_2-x_1)x_1}}{x_2-x_1} + \frac{1}{x_2-x_1}\right) \\
&= e^{-x_1^2/2} + e^{-x_1^2+x_2x_1-x_2^2/2}\left(-\frac{e^{-(x_2-x_1)x_1}}{x_2-x_1} + \frac{1}{x_2-x_1}\right) \\
&= e^{-x_1^2/2} + \frac{e^{-x_1^2+x_2x_1-x_2^2/2} - e^{-x_1^2+x_2x_1-x_2^2/2-x_2x_1+x_1^2}}{x_2-x_1} \\
&= e^{-x_1^2/2} + \frac{e^{-x_1^2+x_2x_1-x_2^2/2} - e^{-x_2^2/2}}{x_2-x_1}.
\end{aligned}
$$

2.2.2 Exercise 2

Let $q_0 > 1$. Calculate the solution of the following Cauchy-problem in a small neighbourhood of $(1, q_0)$,

$$
\begin{aligned}
x_1\partial_2 u(x) - x_2\partial_1 u(x) &= 0 \\
u(1,q) &= \cos(q), \quad q \in \mathbb{R}.
\end{aligned}
$$

Hint: If we use the definition from the notes, then the length of the vector $X(t,q)$ not of t.

Solution: This is a linear PDE, so the first thing is to bring it to normal form

$$
\nabla u(x)\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} + u(x)\cdot 0 = 0,
$$

thus

$$
A(x) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}, \quad b(x) = 0, \quad f(x) = 0.
$$

Next we find $\Phi(t,x)$ as $\partial_t\Phi(t,x) = A(\Phi(t,x))$ yields the following system

$$
\begin{aligned}
\varphi_1'(t) &= -\varphi_2(t), & \varphi_1(0) &= x_1 \\
\varphi_2'(t) &= \varphi_1(t), & \varphi_2(0) &= x_2.
\end{aligned}
$$

This is a system of differential equations, which can be expressed in matrix notation as

$$
\begin{bmatrix} \varphi_1'(t) \\ \varphi_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \end{bmatrix}.
$$

By Putzer's algorithm the solution is

$$
\begin{aligned}
\begin{bmatrix}\varphi_1(t)\\ \varphi_2(t)\end{bmatrix} &= e^{t\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\\
&= \begin{bmatrix}\cos(t) & -\sin(t)\\ \sin(t) & \cos(t)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\\
&= \begin{bmatrix}x_1\cos(t) - x_2\sin(t)\\ x_1\sin(t) + x_2\cos(t)\end{bmatrix}
\end{aligned}
$$

By lemma 1.8

$$
X(t,q) = \begin{bmatrix}\cos(t) & -\sin(t)\\ \sin(t) & \cos(t)\end{bmatrix}\begin{bmatrix}1\\ q\end{bmatrix}
$$

and as the hint suggest the length of the legnth of this vector does not depend on t, so

$$
\begin{aligned}
\sqrt{x_1^2 + x_2^2} &= \sqrt{1+q^2}\\
x_1^2 + x_2^2 &= 1 + q^2\\
q &= \sqrt{x_1^2 + x_2^2 - 1}.
\end{aligned}
$$

Thus as $Z(t,q) = \cos(q)$ and $u(x) = Z(\psi(x))$ it implies that

$$
u(x) = \cos\left(\sqrt{x_1^2 + x_2^2 - 1}\right).
$$

2.2.3 Exercise 3

solve the follwoing initial value problem

$$
\begin{aligned}
\left(\partial_t^2 - \partial_x^2\right) u(t,x) &= \sin(x-t), \quad t,x \in \mathbb{R}\\
u(0,x) = \cos(x), \quad \partial_t u(0,x) &= 0, \quad x \in \mathbb{R}.
\end{aligned}
$$

Solution: We are dealing with an initial value problem for a wave equation as defined in definition 2.3. To solve the problem we look to Lemma 2.4.

We need to first solve

$$
\begin{aligned}
\left(\partial_t^2 - \partial_x^2\right) v(t,x) &= 0, \quad t,x \in \mathbb{R}\\
v(0,x) = \cos(x), \quad \partial_t v(0,x) &= 0, \quad x \in \mathbb{R},
\end{aligned}
$$

this is done using (2.28), also known as d'Alamberts formula

$$
\begin{aligned}
v(t,x) &= \frac{1}{2}\left\{\cos(x+t) + \cos(x-t)\right\} + \frac{1}{2}\int_{x-t}^{x+t} 0 \, \mathrm{d}y\\
&= \frac{1}{2}\left\{\cos(x+t) + \cos(x-t)\right\}.
\end{aligned}
$$

Then we need to solve

$$
\begin{aligned}
\left(\partial_t^2 - \partial_x^2\right) w(t,x) &= \sin(x-t), \quad t,x \in \mathbb{R}\\
w(0,x) &= \partial_t w(0,x) = 0,
\end{aligned}
$$

according to remark 2.26 the solution is given by

$$
\begin{aligned}
w(t,x) &= \frac{1}{2}\left(\int_0^t \int_{x-(t-s)}^{x+t-s} \sin(y-s)\ \mathrm{d}y\ \mathrm{d}s\right)\\
&= \frac{1}{2}\left(\int_0^t \left[-\cos(s-y)\right]_{x-(t-s)}^{x+t-s}\ \mathrm{d}s\right)\\
&= \frac{1}{2}\left(\int_0^t \cos(s-x+t-s)-\cos(s-x-t+s)\ \mathrm{d}s\right)
\end{aligned}
$$

we split the integral and solve the right most by substitution as follows

$$
u = 2s-x-t, \quad \frac{\mathrm{d}u}{\mathrm{d}s} = 2 \ \Leftrightarrow\ \mathrm{d}u = 2\ \mathrm{d}s\ \Leftrightarrow\ \frac{1}{2}\ \mathrm{d}u = \mathrm{d}s,
$$

this gives

$$
\begin{aligned}
&= \frac{1}{2}\left(\int_0^t \cos(x+t)\ \mathrm{d}s - \frac{1}{2}\int_{-x-t}^{t-x}\cos(u)\ \mathrm{d}u\right)\\
&= \frac{1}{2}\left(\left[\cos(x+t)s\right]_0^t - \frac{1}{2}\left[\sin(u)\right]_{-x-t}^{t-x}\right)\\
&= \frac{1}{2}\left(\cos(x+t)t - \frac{1}{2}\left(\sin(t-x)-\sin(-x-t)\right)\right)\\
&= \frac{t}{2}\cos(x+t) - \frac{1}{4}\left(\sin(t-x)-\sin(-x-t)\right)\\
&= \frac{t}{2}\cos(x+t) + \frac{1}{4}\left(\sin(t-x)-\sin(x+t)\right).
\end{aligned}
$$

Now that we have found both solution we to add these to together to achive the solution for the original problem, i.e. $u = v + w$

$$
u(t,x) = \frac{1}{2}\left\{\cos(x+t)+\cos(x-t)\right\} + \frac{t}{2}\cos(x+t) + \frac{1}{4}\left(\sin(t-x)-\sin(x+t)\right).
$$

2.2.4 Exercise 4

Solve the following mixed initial-value/boundary-value-problem

$$
\begin{aligned}
\left(\partial_t - \frac{1}{2}\partial_x^2\right)u(t,x) = 0, &\quad t>0,\ x\in(0,\pi)\\
u(0,x) = \cos^2(7x), &\quad x\in(0,\pi)\\
\partial_x u(t,0) = \partial_x u(t,\pi) = 0, &\quad t>0.
\end{aligned}
$$

Hint: $\cos^2(y) = (1+\cos(2y)/2)/2, \quad y\in\mathbb{R}$.

Solution: Heat equation on an interval with homogenous Neumann conditions. we remember the defintion of A_n

$$
A_n := \frac{2}{\ell}\int_0^\ell g(x)\cos(n\pi x/\ell)\ \mathrm{d}x, \quad n\in\mathbb{N}_{\nvdash}.
$$

Theorem 4.11 states that the solution is given by

$$
u(t,x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} e^{-tn^2\pi^2/2\ell^2} A_n \cos(n\pi x/\ell).
$$

In our case $\ell = \pi$ and $g(x) = \cos^2(7x)$, we start by calculating A_0

$$
\begin{aligned}
A_0 &= \frac{2}{\pi}\int_0^\pi \cos^2(7x)\ \mathrm{d}x \\
&= \frac{2}{\pi}\int_0^\pi (1+\cos(14x)/2)/2\ \mathrm{d}x \\
&= \frac{2}{\pi}\left(\int_0^\pi \frac{1}{2}\ \mathrm{d}x + \frac{1}{4}\int_0^\pi \cos(14x)\ \mathrm{d}x\right)
\end{aligned}
$$

we substitute in the right most integral using

$$
u = 14x, \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 14 \ \Leftrightarrow\ \mathrm{d}u = 14\ \mathrm{d}x \ \Leftrightarrow\ \frac{1}{14}\ \mathrm{d}u = \mathrm{d}x,
$$

giving us

$$
\begin{aligned}
&= \frac{2}{\pi}\left(\left[\frac{1}{2}x\right]_0^\pi + \frac{1}{4}\left[\sin(u)\right]_0^{14\pi}\right) \\
&= \frac{2}{\pi}\frac{\pi}{2} = 1.
\end{aligned}
$$

Inserting in the solution

$$
\begin{aligned}
u(t,x) &= \frac{1}{2} + \sum_{n=1}^\infty e^{tn^2/2}\frac{2}{\pi}\left(\int_0^\pi \left(1+\cos(14x)/2\right)/2\cos(nx)\ \mathrm{d}x\right)\cos(nx) \\
&= \frac{1}{2} + \sum_{n=1}^\infty e^{tn^2/2}\frac{2}{\pi}\left(\int_0^\pi \frac{1}{2}\ \mathrm{d}x\frac{1}{4}\int_0^\pi \cos(14x)\cos(nx)\ \mathrm{d}x\right)\cos(nx) \\
&= \frac{1}{2} + \sum_{n=1}^\infty e^{tn^2/2}\frac{2}{\pi}\int_0^\pi \frac{1}{2}\ \mathrm{d}x + \frac{1}{2\pi}\int_0^\pi \cos(14x)\cos(nx)\ \mathrm{d}x\cos(nx) \\
&= \frac{1}{2} + \sum_{n=1}^\infty e^{tn^2/2}\frac{2}{\pi}\int_0^\pi \frac{1}{2}\ \mathrm{d}x + \frac{1}{4\pi}\int_{-\pi}^\pi \cos(14x)\cos(nx)\ \mathrm{d}x\cos(nx) \\
&= \frac{1}{2} + e^{-t14^2/2}\cos(14x)/4 \\
&= \frac{1}{2} + \frac{1}{4}e^{-98t}\cos(14x).
\end{aligned}
$$

The disappearance of the sum follows form remark C.4 on page 167.

2.2.5 Exercise 5

Find a solution of the following initial-value-problem

$$
\begin{aligned}
\left(\partial_t - \frac{1}{2}\partial_x^2\right)u(t,x) &= t, \quad t>0,\ x\in\mathbb{R} \\
u(0,x) &= 2+3x, \quad x\in\mathbb{R}.
\end{aligned}
$$

Solution: As it is an initial-value-problem of the form of a wave equation (inhomogenous). From corollary

3.13 the solution is given by

$$
\begin{aligned}
u(t,x) &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t}(2+3y)\ \mathrm{d}y + \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(t-s)}} e^{-(y-x)^2/2(t-s)} s\ \mathrm{d}y\ \mathrm{d}s \\
&= 2\int_{\mathbb{R}} \mathit{density}\ \mathrm{d}y + 3\int_{\mathbb{R}} \mathit{density}\ y\ \mathrm{d}y + \int_0^t \int_{\mathbb{R}} \mathit{density}_2\ s\ \mathrm{d}y\ \mathrm{d}s \\
&= 2 + 3x + \int_0^t s\ \mathrm{d}s \\
&= 2 + 3x + [s^2/2]_0^t \\
&= 2 + 3x + t^2/2.
\end{aligned}
$$

The calculations follows as the integral of a density over $\mathbb{R}$ is equal to 1.

2.2.6 Exercise 6

Let $u : \mathbb{R}^3 \to \mathbb{R}$ be given by

$$u(x) := \left(x_1^2 - x_2^2\right) x_3 + 4x_1x_3 - x_2, \quad x \in \mathbb{R}^3.$$

Determine the mean value of u over a ball in $\mathbb{R}^3$ with radius 23876517 and center $(1,2,3)$.

Solution: As u(x) is continuous it satisfies the mean value characteristic and therefore takes its mean value independet of the radius, by definition 5.4.

$$
\begin{aligned}
u(1,2,3) &= (1^2 - 2^2)3 + 4 \cdot 1 \cdot 3 - 2 \\
&= (1-4)3 + 12 - 2 \\
&= 3 - 12 + 12 - 2 \\
&= 1.
\end{aligned}
$$