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Fast imitation simulation of the general model of the redundant system with recovery

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FACULTY OF COMPUTER SCIENCE AND CYBERNETICS
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Graduate qualification master's thesis

FAST IMITATION SIMULATION OF THE GENERAL MODEL OF THE REDUNDANT SYSTEM WITH RECOVERY

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Contents

1 Introduction

Historically, the problem of duplication with recovery was considered in the work of Epstein and Hosford. Approaches to the theory of death and reproduction processes can be used to consider this problem.

Some devices may be out of service. In order to maintain the operating mode continuously, there is an identical backup device, which immediately comes into operation as soon as the main device fails. The device that failed, immediately begin to recover. After recovery, it completely restores its properties. In the standby state, the device does not deteriorate. The duration of trouble-free operation of the device is subject to distribution $F(x) = 1 - e^{-\lambda x}$. The recovery time is random and has a distribution $G(x) = 1 - e^{-\mu x}$. The question is, how is the time of failure of the duplicate system distributed, if the failure of the system occurs when both devices are out of order?

In the simplest case of one basic element the problem can be solved analytically. The solution of this problem is considered in Section 3. This problem can be generalized in the case of many basic elements and, accordingly, many spare elements. To solve duplication problems in the general case, the method of simulation can be used. The idea of this method is to simulate a real system by generating random variables of the duration of trouble-free operation and repair time.

The efficiency of the simulation method can be increased by building special distribution functions at each step of the system simulation, which generates the time of continuous operation of devices and finding the conditional probabilities of failure of the main element on the basis of which system reliability is assessed.

This is the essence of the method of rapid simulation. At each step of the method, we get not just the result of a single element, but the probability of system failure at this step. Due to this, the required accuracy of the overall reliability of the system can be achieved in fewer iterations.

2 Duplicate with recovery (unloaded reserve).

We can specify three states of the system: E_0, E_1 and E_2 - in the system there are $0, 1$ and 2 failed devices. Probabilities of transitions from one state to another over time h as follows:

$$P\{E_0(t) \to E_1(t+h)\} = \lambda h + o(h),$$

$$P\{E_1(t) \to E_0(t+h)\} = \nu h + o(h),$$

$$P\{E_1(t) \to E_2(t+h)\} = \lambda h + o(h),$$

$$P\{E_2(t) \to E_1(t+h)\} = o(h),$$

Thus, in the studied process of death and reproduction $\lambda_0 = \lambda_1 = \lambda, \mu_1 = \nu$ i $\mu_2 = 0$; λ_k and μ_k are equal to 0. Let us recall the well-known differential equations of the processes of death and reproduction.

$$P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t) \tag{2.1}$$

$k \geq 1$

$$P_k'(t) = (-\lambda_k + \mu_k)P_k(t) + \lambda_{k-1}P_{k-1}(t) + \mu_{k+1}P_{k+1}(t) \tag{2.2}$$

Then the equations(4.1) and (4.2) for this problem take the following form:

$$P_0'(t) = -\lambda P_0(t) + \nu P_1(t)$$

$$P_1'(t) = -(\lambda + \nu)P_1(t) + \lambda P_0(t)$$

$$P_2'(t) = -\lambda P_1(t)$$

Initial conditions of the problem: $P_0(0) = 1, P_1(0) = 0, P_2(0) = 0.$

Substitution $P_0(t)$ from the second equation of the system into the first leads to the following equation for $P_1(t)$:

$$P_1''(t) + (2\lambda + \nu)P_1'(t) + \lambda^2 P_1(t) = 0.$$

Its solution, which satisfies the initial condition, has the form

$$P_1(t) = Ce^{-(\lambda+\frac{\nu}{2})t}[e^{\sqrt{2\nu+\frac{\nu^2}{4}}t} - e^{-\sqrt{\lambda\nu+\frac{\nu^2}{4}}t}]$$

Now we find that

$$P_0(t) = Ce^{-(\lambda+\frac{\nu}{2})t}[(\frac{\nu}{2} + \sqrt{\lambda\nu + \frac{\nu^2}{4}})e^{\sqrt{\lambda\nu+\frac{\nu^2}{4}}t} + (\sqrt{\lambda\nu + \frac{\nu^2}{4}} - \frac{\nu}{2})e^{-\sqrt{\lambda\nu+\frac{\nu^2}{4}}t}]$$

and

$$C = \frac{\lambda}{\sqrt{4\lambda\nu + \nu^2}}.$$

The desired probability of failure-free operation of a duplicated system, as it is easy to see, is equal to

$$R(t) = P_0(t) + P_1(t) = e^{-(\lambda+\frac{\nu}{2})t}[\frac{t}{2}\sqrt{4\lambda\nu + \nu^2} + \frac{2\lambda + \nu}{\sqrt{4\lambda\nu + \nu^2}}\frac{t}{2}\sqrt{4\lambda\nu + \nu^2}].$$

For $\nu = 0$ we obtain a system without recovery. Well from the last equality, in this particular case

$$R(t) = e^{-\lambda t}(1 + \lambda t)$$

The average duration of non-failure operation of a redundant system is equal to

$$\int_0^{\inf} R(t)\,dt = \frac{2\lambda + \nu}{\lambda^2} = \frac{2}{\lambda} + \frac{\nu}{\lambda^2}.$$

The first term in this sum is the average duration of the redundant system's uptime when there is no recovery.The second term is the addition to the uptime that recovery provides.The larger ν, i.e., the higher theintensity of recovery, the greater the effect of recovery. Usually ν is much larger than λ, i.e. recovery proceeds faster than the period of uptime ends, thus the efficiency of recovery is high.

3 Immitational modeling

The duplicate system S with one main, one spare elements and one repair place is considered. Let, the element in the reserve does not refuse.

ξ - time to failure of the main element.

t_0 - the initial time of the system.

T_ν - time to system failure, where

$$\nu = \min(k \geq 1 : \xi_k < \nu_k)$$

$$T_\nu = \sum_{k=0}^{\nu} \xi_k$$

3.1 Algorithm

Objects at time T_k:

$$\begin{aligned} A_1^k &= \{z_1^k, \tau_1^k\}, \\ A_2^k &= \{z_2^k\}, \\ A_3^k &= \{z_3^k\}, \\ A_4^k &= \{z_4^k, \tau_4^k\}, \end{aligned}$$

Then

$\Theta_k = \min(\tau_1^k, \tau_4^k)$,

$T_{k+1} = T_k + \Theta_k$

Objects at time T_{k+1}:

a)Failure of the main element at the time T_{k+1}

$$A_1^{k+1} = \{z_1^{k+1}, \tau_1^{k+1}\} = \begin{cases} (1, \xi_k) & z_2^k = 1 \\ (0, M) & z_2^k = 0 \end{cases}$$

$$A_2^{k+1} = \{z_2^{k+1} = 0\},$$

$$A_3^{k+1} = \{z_3^{k+1}\} =, \begin{cases} 1 & z_4^k = 1 \\ 0 & z_4^k = 0 \end{cases}$$

$$A_4^{k+1} = \{z_4^{k+1}, \tau_4^{k+1}\} =, \begin{cases} (1, \tau_4^k - \Theta_k) & z_4^k = 1 \\ (1, \nu_k) & z_4^k = 0 \end{cases}$$

b) Completion of repairs at the time T_{k+1}

$$A_1^{k+1} = \{z_1^{k+1}, \tau_1^{k+1}\} = \begin{cases} (1, \tau_1^k - \Theta_k) & z_1^k = 1 \\ (1, \xi_k) & z_1^k = 0 \end{cases}$$

$$A_2^{k+1} = \{z_2^{k+1}\} = \begin{cases} 1 & z_1^k = 1 \\ 0 & z_1^k = 0 \end{cases}$$

$$A_3^{k+1} = \{z_3^{k+1} = 0\},$$

$$A_4^{k+1} = \{z_4^{k+1}, \tau_4^{k+1}\} =, \begin{cases} (1, \nu_k) & z_3^k = 1 \\ (0, M) & z_3^k = 0 \end{cases}$$

4 Fast immitational modeling

4.1 Duplicate system

4.1.1 Formulation of the problem

The system S with one main and one reserve elements is considered. In addition, the system has one repair place. We believe that the element in the reserve does not fail.

Denote by τ - the time to failure of the main element, $\tau; \tau_1, \tau_2, ..$ - independent copies of τ, $P(\tau < t) = F(t)$ - d.f.;
η - recovery(repair) time, $P(\eta < t) = G(t)$ - d.f.
$\eta_1, \eta_2, ..$ independent copies of η
System S fails when both elements are faulty;
τ_c — time to the failure of the system S. That is $\tau_c = \tau_0 + \sum_{i=1}^{\nu} \tau_i, \nu = min(k : \tau_k < \eta_k)$.

The idea of the method of rapid simulation is to combine some probability formulas, such as the formula of total probability with direct simulation (Monte Carlo method).
Therefore, the moment of failure of S occurs when the repair time η_ν exceeds the operating time of the element τ_ν.

We will be interested in the reliability of the S on interval $(0, t_z) : P_c(t_z) = P(\tau_c > t_z)$. But it will be more convenient for us to calculate the value $Q_c(t_z) = 1 - P_c(t_z)$, - probability of failure of S.

4.1.2 Algorithm

For a sufficiently large N the external cycle is constructed by $j = 1, 2, .., N$, to find the value $\widehat{Q}(j)$ (some rough estimate $Q_c(t_z)$). Then at large N estimate

$$\widehat{Q}_c(t_z) = \frac{I}{N} \sum_{j=1}^{N} \widehat{Q}(j) \approx Q_c(t_z)$$

For a fixed j we build an internal loop on $k = 0, 1, 2, ...$

<u>Step $0, (k = 0)$</u>

1. We calculate $P_0 = P(\tau < t_z) = F(t_z)$
2. We simulate τ_0 with d.f. $\widehat{F}_0(x)$:

$$P(\widetilde{\tau}_0 < x) = P(\tau < x/\tau < \Delta_0), \Delta_0 = t_z - 0 = t_z$$

$$\widetilde{F}_0(x) = \begin{cases} \frac{F(x)}{F(\Delta_0)} & x < \Delta_0 \\ 1 & x \geq \Delta_0 \end{cases}$$

3. Put $\tau_0 = 0 + \widetilde{\tau}_0$.

<u>Step $1, (k = 1)$</u>

1. We calculate

$$P_1 = P(\tau + \widetilde{\tau}_0 < t_z) = F(\Delta_1), \Delta_1 = t_z - T_0$$

2. We simulate $\widetilde{\tau}_1$ with d.f. $\widetilde{F}_1(x) = P(\tau < x/\tau < \Delta_1)$:

$$\widetilde{F}_1(x) = \begin{cases} \frac{F(x)}{F(\Delta_1)} & x < \Delta_1 \\ 1 & x \geq \Delta_1 \end{cases}$$

3. $T_1 = T_0 + \widetilde{\tau}_1$

4. We check the condition

$$\widetilde{\tau}_1 > \eta_1 \tag{4.1}$$

If (4.1) is satisfied, then we go to the next step of the algorithm ($k \geq 2$).

Otherwise we have a system failure and calculate

$$\widetilde{Q}(j) = P_0 \times P_1$$

and move on to $(j+1)$.

Let k steps be done (transition from k to $k+1$)

We have calculated values

$$T_0, T_1, .., T_k \quad \text{and} \quad P_0, P_1, .., P_k$$

1. We calculate $P_{k+1} = F(\Delta_{k+1})$, де $\Delta_{k+1} = t_z - T_k$.

2. We simulate random variable $\widetilde{\tau}_{k+1}$ with distribution function $\widetilde{F}_{k+1}(x) = P(\tau < x/\tau < \Delta_{k+1})$:

$$\widetilde{F}_{k+1}(x) = \begin{cases} \frac{F(x)}{F(\Delta_{k+1})} & x < \Delta_{k+1} \\ 1 & x \geq \Delta_{k+1} \end{cases}$$

3. $T_{k+1} = T_k + \widetilde{\tau}_{k+1}$

4. We check the condition

$$\widetilde{\tau}_{k+1} > \eta_{k+1} \tag{4.2}$$

If (4.2) is satisfied, then go to step $(k+2)$. Otherwise we have a case of system failure.

We calculate

$$\widehat{Q}(j) = \prod_{i=1}^{k+1} P_i$$

and we leave a cycle on k, and in the external cycle we move to $(j+1)$.

Remark

Simulating of random variable $\widetilde{\tau}_k$. We know that $\widetilde{F}_k(\widetilde{\tau}_k) = u$, where u - uniformly distributed random variable on the interval $[0, 1]$. Then

$$\widetilde{F}_k(\widetilde{\tau}_k) = \frac{F(\widetilde{\tau}_k)}{F(\Delta_k)} = u$$

$$\Rightarrow F(\widetilde{\tau}_k) = uF(\Delta_k)$$

$$\Rightarrow \widetilde{\tau}_k = F^{-1}(uF(\Delta_k))$$

For example, $F(x) = 1 - e^{-\lambda x} \Rightarrow \widetilde{\tau}_k \equiv -\frac{1}{\lambda} \ln(1 - u + ue^{-\lambda \Delta_k})$. Random variable u simulated be generator *Rand*.

4.2 Redundant system with recovery.

4.2.1 Problem statement

System S consists of the main (working) element ($m = 1$), n spare elements ($n \geq 1$, cold reserve) and l repair units ($1 \leq l \leq n+m$).
Let τ - the time before the failure of the main element, d.f. $\tau P(\tau < t) = F(t)$; η - recovery time, d.f. $\eta P(\eta < t) = G(t), (\tau_k), (\eta_k)$ independent copies of a random variable τ and η respectively.
Failure of the system S means that all $(n+1)$ elements are defective.

$$\tau_c \text{ - time till failure of the system } S,$$

$\mathrm{P}_c(t) = P(\tau_c \geq t)$ - reliability S,

$\mathrm{Q}_c(t) = 1 - P_c(t)$ -the probability of failure S on the interval $(0, t)$.
We are interested in the value of $Q_c(t_z)$ for some fixed interval $(0, t_z)$.

Main objects

(are considered in some special moments of time T_k)

Main element
$A_1 = \{\xi_1\}$,

where ξ_1 -residual operating time of the main element

Reserve

$A_2 = \{\nu_2\}$,

where ν_2 - number of reserve elements, $\nu_2 \in (0, 1, 2, .., n)$

Queue

$A_3 = \{\nu_3\}$,

where ν_3 - length of queue, $\nu_3 \in (0, 1, 2, .., n + 1 - l)$

Repair

$A_4 = \{\nu_4, \bar{\eta}_1, .., \bar{\eta}_{\nu_4}\}$,

where ν_4 - number of occupied repair units, $\nu_4 \in (0, 1, 2, .., l)$

$\bar{\eta}_i$ - residual repair time of the i-th unit;

$\bar{\eta}_{min} = \min_{1 \leq i \leq \nu_4} \bar{\eta}_i$ - minimum residual repair time.

4.2.2 Algorithm

For a sufficiently large number N бwe construct an external cycle of $j = 1, 2, .., N$. And for each such trajectory with number j there is some conditional probability $\widehat{Q}(j)$ (rough estimate $Q_c(t_z)$).

Then for $N \to \infty$

$$\widehat{Q}_c(t_z) = \frac{1}{N} \sum_{j=1}^{N} \widehat{Q}(j) \approx Q_c(t_z) \tag{4.3}$$

The inner cycle of k is the passage of special points $T_0 = 0 < T_1 < T_2 < .. <$

$T_\nu < t_z$, where the point $T_k, k \geq 1$ - it is either the moment of failure of the main element, or the moment of the end of repair by any repair unit (T_ν - the moment of failure of the system S on the j-th trajectory).

At moments T_k some probability P_k is calculated, and after the end of a cycle on k we find

$$\widehat{Q}_j = \prod_{i=1}^{\nu-1} P_k,$$

where $\widehat{Q}_j$ - this is the conditional probability of failure of the system S at fixed values (η_i) for i-th trajectory.
(Of course ν and P_k depend on j, more precisely on realizations of random numbers on the j-th iteration).
We assume that we have already created all the necessary random number sensors $(\eta_k), (\tau_k)$ and so on. We describe in detail the passage of the internal cycle on k.

<u>Step 0 (statements at $T_0 = 0$)</u>

1. We calculate $P_0 = P(\tau < t_z) = F(t_z)$

2. We model a random variable $\widetilde{\tau}_0$ with d.f. $\widetilde{F}_0(x)$:

$$\widetilde{F}_0(x) = \begin{cases} \frac{F(x)}{F(\Delta_0)}, & x < \Delta_0 \\ 1, & x \geq \Delta_0, \text{ де } \Delta_0 = t_z - T_0 = t_3 \end{cases}$$

3. Objects at T_0:

$$A_1^0 = \{\xi_1^0 = \widetilde{\tau}_0\},$$
$$A_2^0 = \{\nu_2^0 = n\},$$
$$A_3^0 = \{\nu_3^0 = 0\},$$
$$A_4^0 = \{\nu_4^0 = 0\}$$

<u>Step 1 (statements at T_1)</u>

1. $T_1 = T_0 + \xi_1^0 = \xi_1^0$

2. For $\Delta_1 = t_z = T_1$ we find $P_1 = P(\tau + \tau_1 < t_z) = P(\tau < \Delta_1) = F(\Delta_1)$

3. We model a random variable $\widetilde{\tau}_1$ з ф.р. $\widetilde{F}_1(x)$:

$$\widetilde{F}_1(x) = \begin{cases} \frac{F(x)}{F(\Delta_1)}, & 0 < x < \Delta_1 \\ 1, & x \geq \Delta_1 \end{cases}$$

4. Objects at T_1:

$$A_1^1 = \{\xi_1^1 = \widetilde{\tau}_1\},$$
$$A_2^1 = \{\nu_2^1 = n - 1\},$$
$$A_3^1 = \{\nu_3^1 = 0\},$$
$$A_4^1 = \{\nu_4^1 = \bar{\eta}_1 = \eta_1\}$$

Note that in steps $k = 0$ and $k = 1$ failure of the system S is impossible.

Step $k \geq 2$ (statements at T_k)

We have $T_0, T_1, ..., T_{k-1}$ and $P_0, P_1, .., P_{k-1}$

Let statements at T_{k-1} be:

$$A_1^{k-1} = \{\xi_1^{k-1}\},$$

$$A_2^{k-1} = \{\nu_2^{k-1}\},$$

$$A_3^{k-1} = \{\nu_3^{k-1}\},$$

$$A_4^{k-1} = \{\nu_4^{k-1}, \bar{\xi}_1^{k-1}, .., \bar{\xi}_{\nu_4}^{k-1}, \bar{\xi}_{min}^{k-1} = \min_{1 \leq i \leq \nu_4^{k-1}}\},$$

$$\Theta^k = min(\xi_1^{k-1}, \eta_{min}^{k-1})$$

Let

$$T_k = T_{k-1} + \Theta^k \tag{4.4}$$

There are two possible cases:

a) $\Theta^k = \xi_1^{k-1}$ (at T_k failure of the main element)

b) $\Theta^k = \eta_{min}^{k-1}$ (at T_k a work unit will complete the repair)

Consider the case a)

a2) We check the condition

$$\nu_2^{k-1} = 0 \text{(whether there is a reserve)} \tag{4.5}$$

a3) If condition (4.5) is satisfied, then at T_k from (4.4) the system S will fail, let

$$\widehat{Q}(j) = \prod_{i=0}^{k-1} P_i$$

and we leave the loop on k.

a4) If (4.5) is not satisfied, then

(i) we calculate

$$P_k = F(\Delta_k),$$

$$\Delta_k = t_z - T_k,$$

(ii) we model a random variable $\widetilde{\tau}^k .. \widetilde{F}_k(x)$:

$$\widetilde{F}_k(x) = \begin{cases} \frac{F(x)}{F(\Delta_k)}, & 0 < x < \Delta_k \\ 1, & x \geq \Delta_k \end{cases}$$

(iii) we find statements at T_k:

$$A_1^k = \{\xi_1^k = \widetilde{\tau}^k\},$$

$$A_2^k = \{\nu_2^k = \nu_2^{k-1} - 1\},$$

$$A_3^k = \begin{cases} \nu_3^k = \nu_3^{k-1} + 1, & \text{if } \nu_4^{k-1} = l, \\ 0, & \text{if } \nu_4^{k-1} \le l \end{cases}$$

$$A_4^k = \begin{cases} \nu_4^k, \bar{\eta}_1^{k-1} - \Theta^k, .., \bar{\eta}_{\nu_4}^{k-1} - \Theta^k, & \text{if } \nu_4^{k-1} = l, \\ \nu_4^k + 1, \bar{\eta}_1^{k-1} - \Theta^k, .., \bar{\eta}_{\nu_4}^{k-1} - \Theta^k, & \text{if } \nu_4^{k-1} \le l \end{cases}$$

<u>Go to step $(k+1)$</u>

Case b)

b2) Let $\bar{\eta}_{min}^{k-1} = \bar{\eta}_r^{k-1}$ (r-th the repair unit has completed the repair)

Then

(i) $P_k = 1$

(ii) we find statements at T_k:

$$A_1^k = \{(\xi_1^{k-1} - \Theta^k)\},$$

$$A_2^k = \{(\nu_2^k = \nu_2^{k-1} + 1)\},$$

$$A_3^k = \{(\nu_3^k = max(0, \nu_3^{k-1} - 1))\},$$

$$A_4^k = \begin{cases} \nu_4^k = \nu_4^{k-1}, \bar{\eta}_1 - \Theta^k, .., \bar{\eta}_{r-1}^{k-1} - \Theta^k, r\eta_k, \bar{\eta}_{r+1}^{k-1} - \Theta^k, .., \bar{\eta}_{\nu_4}^{k-1} - \Theta^k, & \nu_3^{k-1} > 0, \\ \nu_4^k = \nu_4^{k-1} - 1, \bar{\eta}_1^{k-1} - \Theta^k, .., \bar{\eta}_{r-1}^{k-1} - \Theta^k, \bar{\eta}_{r+1}^{k-1} - \Theta^k, .., \bar{\eta}_{\nu_4}^{k-1}, & \nu_3^{k-1} = 0 \end{cases}$$

5 Fast imitational modeling of the system with many elements

5.1 Problem statement

The redundant system with recovery which contains $n_1, n_2, ..n_s$ working elements of s types is considered [exponential distribution τ, parameters $\lambda_1, \lambda_2, ..\lambda_s$], $m_1, m_2, ..m_s$ - the number of spare elements of each type, r - number of repair units (η - repair time is arbitrary, a) $\eta = d = const$, b) $\eta =$ exponential with parameter μ, $\lambda = \frac{\mu}{10}$).

The system fails when for one of its types of elements the working element fails, and all spare elements of the set type in repair.

Let $T_\nu -$ time to system failure, then

$$P_\nu(t_z) = P(T_\nu > t_z) \text{ - system reliability}$$

With the help of fast imitational modeling find $P_\nu(t_z)$. (fast imitational modeling: on $j-$th external iteration we find $\widehat{Q}(j)$, the conditional probability of the event $T_\nu > t_z$ for fixed modeled η_i, $j = \overline{1, N}, N \approx 10^5 - 10^6$)

5.2 Algorithm

As in the case of a simple system with one main, one spare element and one repair unit, the process of calculating the probability of failure of the system S пis to simulate the operating time before failure of each individual main element, operating time of involved repair units and finding the probability of failure of one element in the period from the intermediate moment T_k to the final moment t_z.

The following description of the algorithm uses the following notation: queue - repair queue, and $\Delta_k = t_z - T_k$.

At the beginning of the cycle of estimating the probability of failure of the system, the resulting probability $\widehat{Q}(j)$ is initialized to a value of 1.

$$\widehat{Q}(j) = 1.$$

At each step of the cycle, the state of the system is considered at intervals T_k, each of which corresponds either to the failure of some basic element, or to the completion of the repair by some repair unit.

At every moment T_k, if this moment is the moment of failure of the main element of a class, it is checked whether there is at least one spare element of this class. If the spare element is missing, we assume that the system has failed.

If a spare is available, there are two possible cases:
1. There is at least one free repair unit. In this case, the repair time ν of the main distribution element is generated

$$F(x) = 1 - e^{-\mu x},$$

(μ - the distribution parameter of the corresponding class of elements)
the probability of failure of one of the main elements in the period up to t_z is calculated:

$$P_k = 1 - e^{-(\sum \lambda)\Delta_k},$$

(the sum is taken for all elements of all classes, and for each element λ of the corresponding class is taken)
and the resulting probability is multiplied by the calculated probability of failure of one of the elements

$$\widehat{Q}(j) = \widehat{Q}(j) * P_k,$$

the operating time τ of the reserve element as the main one is simulated from distribution

$$\widetilde{F}(x) = \begin{cases} \frac{F(x)}{F(\Delta_k)}, & 0 < x < \Delta_k \\ 1, & x \geq \Delta_k \end{cases}$$

The number of spare elements of this class is reduced by one:

$$n_i = n_i - 1, \quad i - \text{номер класу}, i = \overline{1, s}.$$

The number of free repair units is also reduced by one:

$$r = r - 1,$$

2. There are no free repair units. In this case, the main element is queued for repairs

$$queue \to \{i\} \quad || \quad queue,$$

(i - the class number of the main element to be repaired),
If the time T_k is the moment of completion of the repair of an element of a class, the number of spare elements of this class is increased by one

$$n_i = n_i + 1.$$

the number of free repair units also increases by one

$$r = r + 1.$$

If the queue for repairs is not empty, the first element of it falls into repair. The item is removed from the queue

$$queue \quad || \quad \{i\} \to queue,$$

(i - the class number of the main element to be repaired),
the repair time ν of the main distribution element is generated

$$F(x) = 1 - e^{-\mu x},$$

(μ - the distribution parameter of the corresponding class of elements)
the number of free repair units is reduced by one:

$$r = r - 1.$$

In the event of a system failure due to the absence of a spare element of the required type, a separate cycle of probability estimation is completed. The probability estimate is the resulting cycle probability

$$\widehat{Q}(j) = \prod_{i=0}^{k-1} P_i.$$

Finally, the reliability of the system is calculated as the average value of the resulting probabilities obtained in a series of evaluation cycles:

$$\widehat{Q}_c(t_z) = \frac{1}{N} \sum_{j=1}^{N} \widehat{Q}(j) \approx Q_c(t_z).$$

Implementation

The algorithm is implemented in the library in C ++ together with the known method of conventional simulation and the function of calculating the duplicate system according to the existing analytical formula of section 2. Software implementation of the algorithm allows modeling a system with any number of classes repair time. According to the analytical formula, the reliability of the system can be calculated in the simple case of one main element, one spare element and one repair unit. The library can be imported into the Jupyter scientific computing system.

5.3 Results

1. Table 1 shows examples of algorithms for normal and fast modeling of a simple with parameters $\lambda = 1.0, \mu = 5.0, t = 5.0$; (Theoretical value is 0.507359)

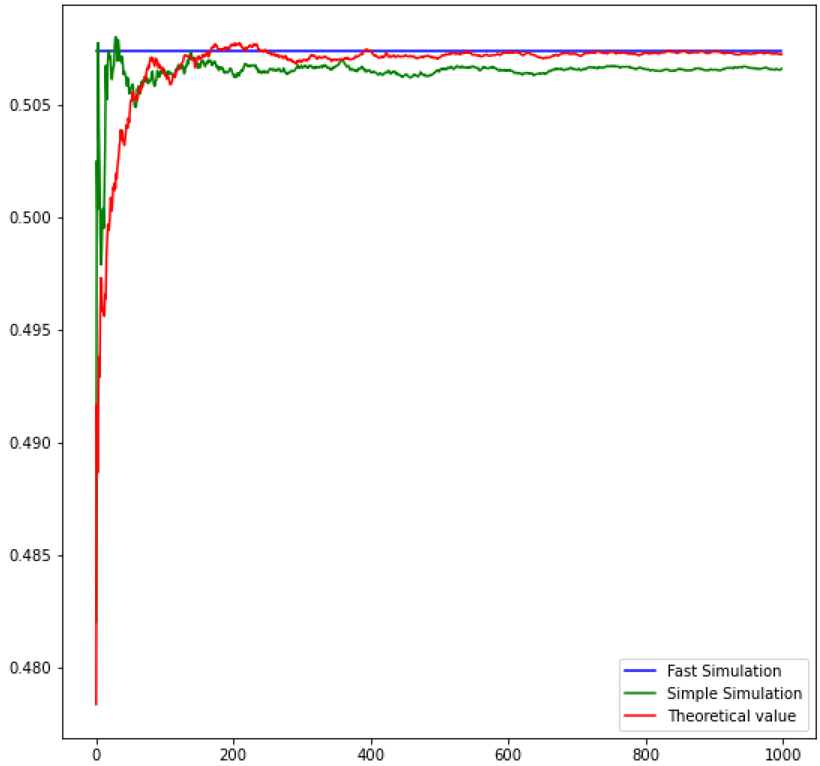
Tab. 1

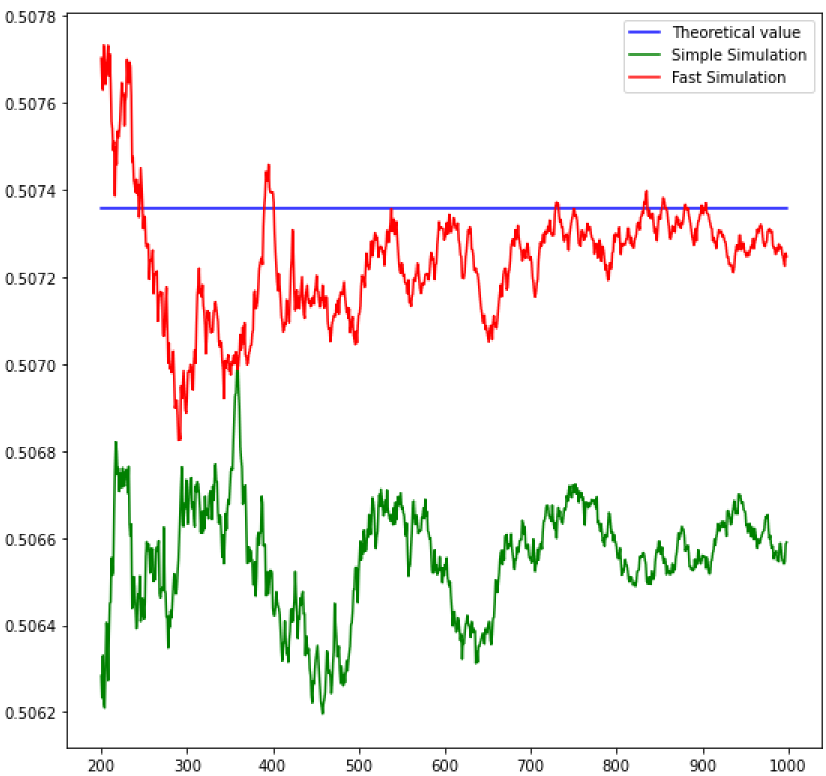
Usual Simulation	Fast Simulation
0.50953	0.502224
0.50977	0.509022
0.50829	0.508088
0.50789	0.505641
0.5071	0.509664
0.50767	0.503931
0.51108	0.506086
0.50602	0.50946
0.50661	0.511239
0.50699	0.508602
avg: 0.508095	avg: 0.507575

Algorithms give a similar estimate, but the method of the fast simulation of the same accuracy was achieved in 10 times fewer cycles. In this case, a tenfold reduction in the number of cycles leads to a corresponding reduction in real-time calculations.

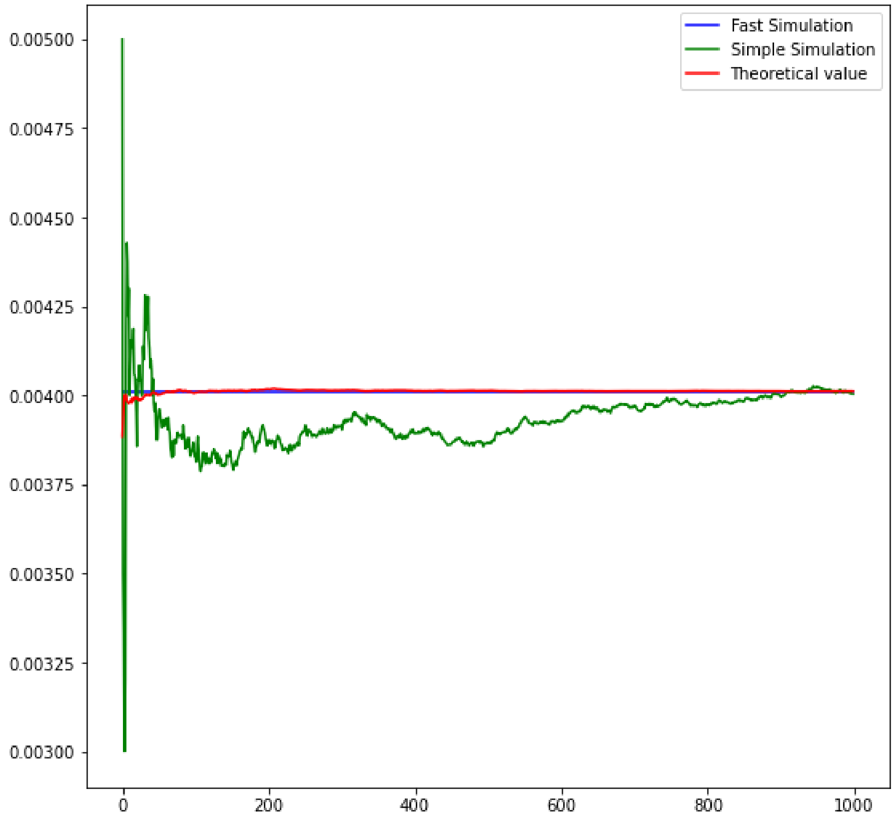
Graphs

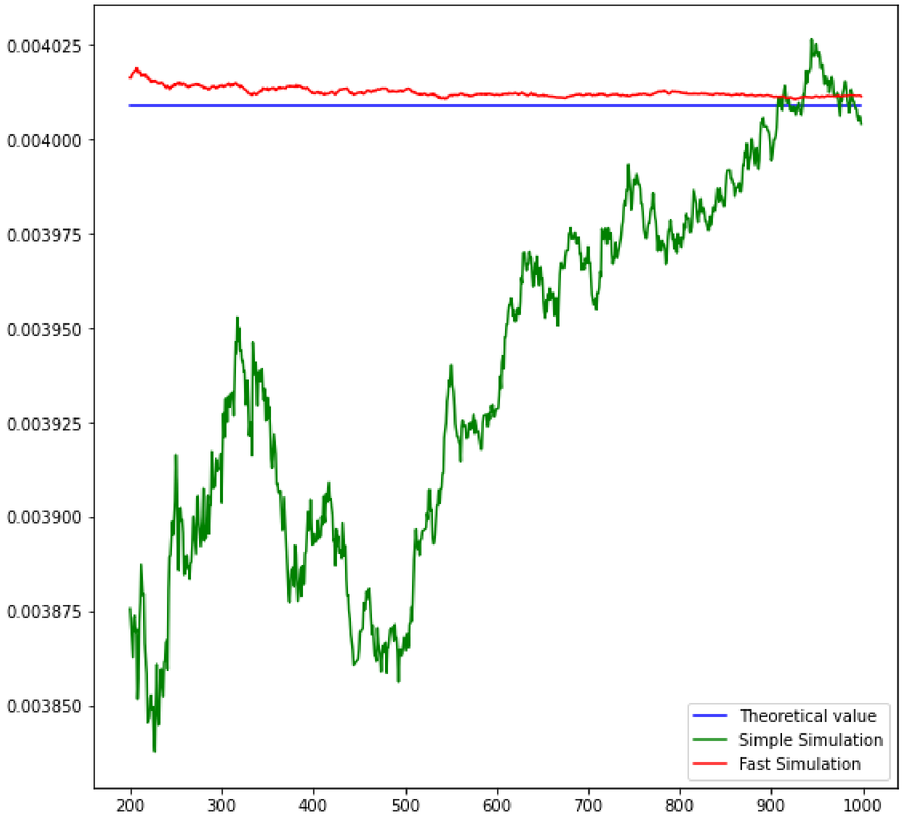
1. $\lambda = 1.0, \mu = 5.0, t = 5.0$



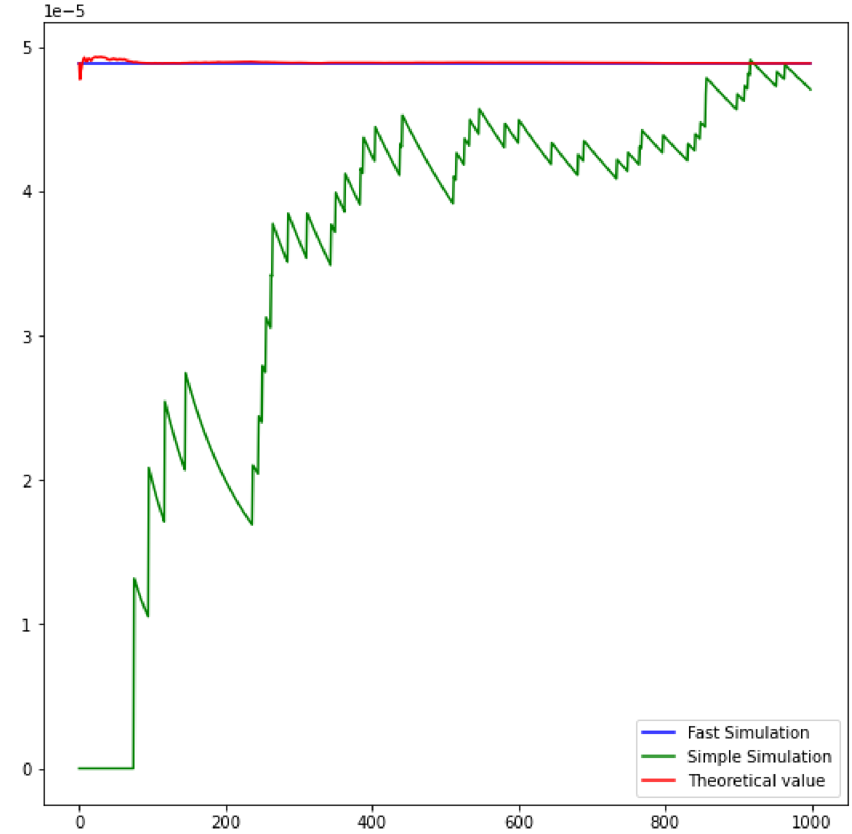


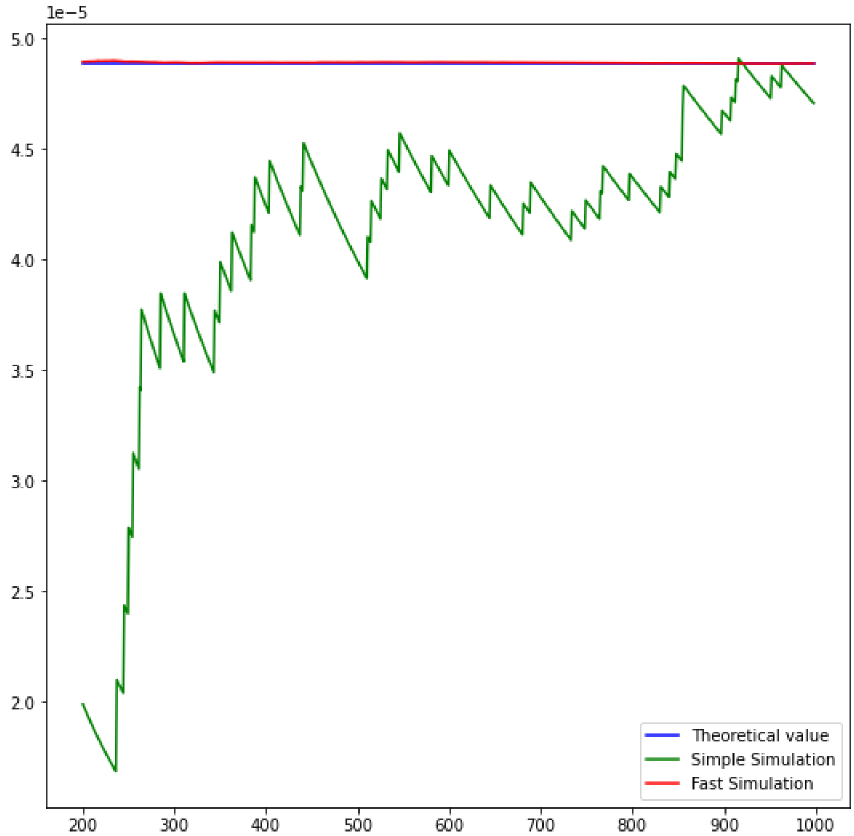
2. $\lambda = 1.0, \mu = 5.0, t = 0.1$



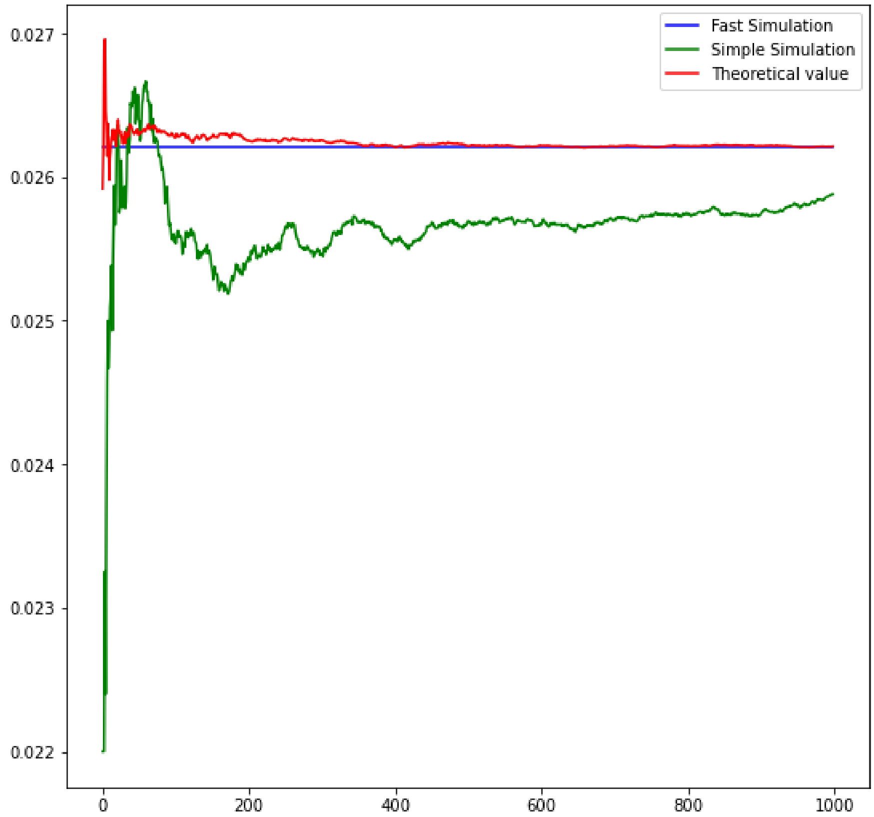


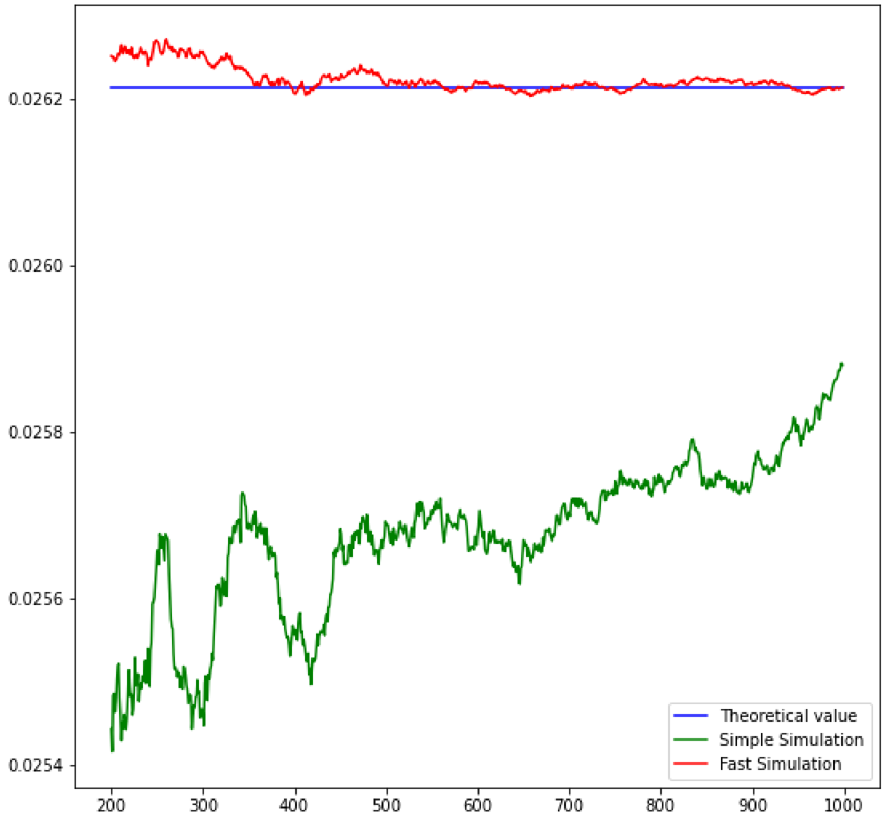
3. $\lambda = 1.0, \mu = 5.0, t = 0.01$



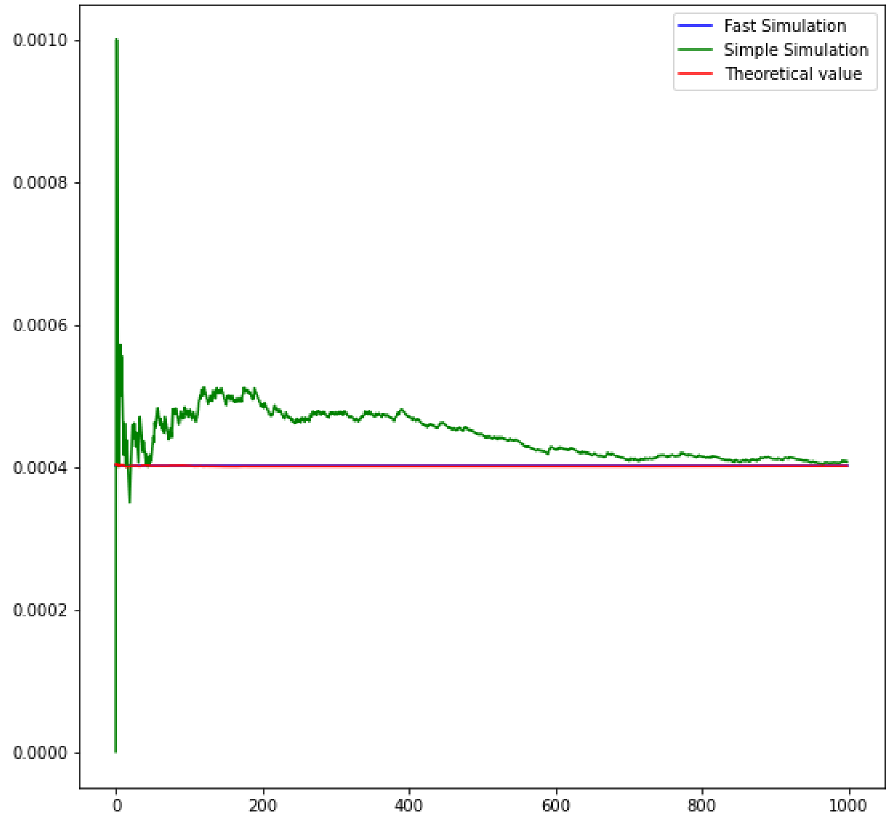


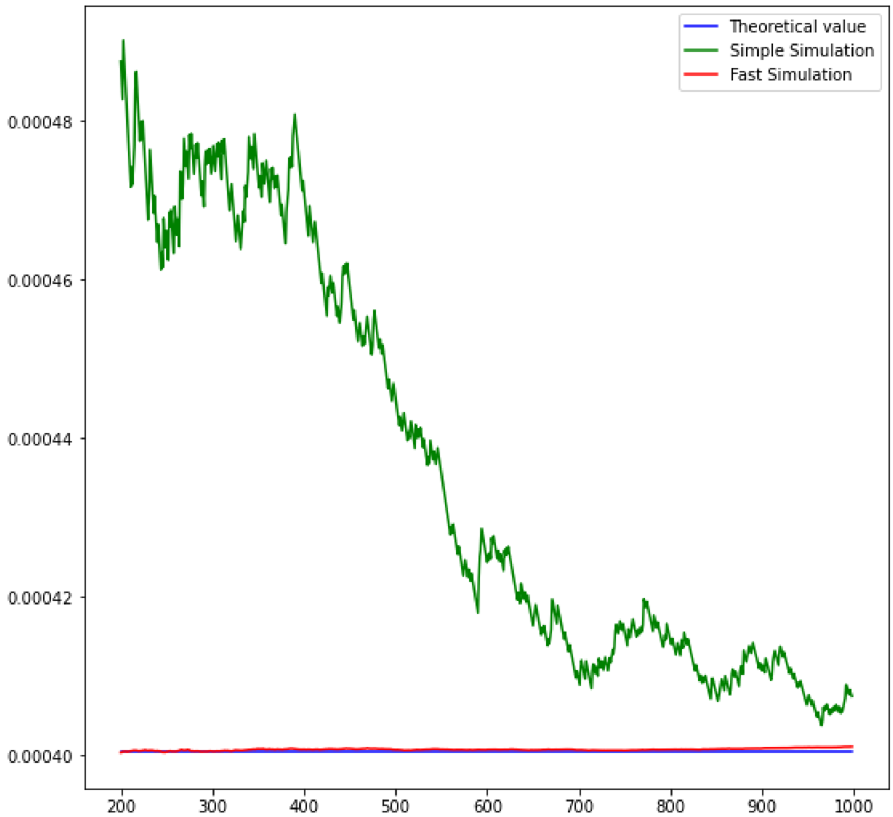
3. $\lambda = 4.0, \mu = 40.0, t = 0.1$





4. $\lambda = 3.0, \mu = 30.0, t = 0.01$





6 Conclusions

The method of fast simulation allows to estimate the reliability of the system in fewer computational cycles. This significantly reduces the time to calculate the probability of the system failure.

The efficiency of the method is determined by the fact that at each step the conditional probability of the event $P(T_\nu > t_z)$ is calculated, where T_ν is the system failure time, while in the usual method only a single result is obtained on the failure of the system, or on the successful completion of the study interval $[0, t_z]$.

Experiments show that for some values of parameters, with decreasing time, the usual method does not give a stable exact approximation of the theoretical value, while the method of fast modeling reaches a value very close to the theoretical for a small number of iterations.

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