

Vectors and Matrices

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Remind: Matrix, Row Vector, and Column Vector

1.4 Algebraic Procedures to Solve Linear Systems

1.4.1 Matrices and Systems of Equations

A matrix is an ordered rectangular array of numbers. A general matrix with m rows and n columns has the following structure:

$$\begin{array}{c} \text{column, } j \\ \begin{array}{cccccc} & 1 & 2 & 3 & \cdots & n \\ \text{row, } i & \begin{array}{c} 1 \\ 2 \\ \vdots \\ m \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{bmatrix} & \end{array} \end{array} \Bigg]_{m \times n}$$

where a_{ij} is each number in the array indexed by its row position i and column position j . Given their rectangular structure, matrices are ideal for storing systems of equations.

Matrix, Row Vector, and Column Vector

- A **matrix** is used to store a set of values of the same type; every value is stored in an **element**.
- MATLAB stands for “**matrix laboratory**”
- A matrix looks like a table; it has both rows and columns
- A matrix with m rows and n columns is called $m \times n$; these are called its **dimensions**;
- A **vector** is a special case of a matrix in which one of the dimensions is 1
- The term **array** is frequently used in MATLAB to refer generically to a matrix or a vector
 - a row vector with n elements is $1 \times n$, e.g. 1×4 :
 - a column vector with m elements is $m \times 1$, e.g. 3×1 :
- A **scalar** is an even more special case; it is 1×1 , or in other words, just a single value

Creating Row Vectors

- Direct method: put the values you want in square brackets, separated by either **commas** or **spaces**

```
>> v = [1 2 3 4]
```

```
v =
```

```
1 2 3 4
```

```
>> v = [1,2,3,4]
```

```
v =
```

```
1 2 3 4
```

- Colon operator: iterates through values in the form ***first:step:last*** e.g. 5:3:14 returns vector [5 8 11 14]
 - If no step is specified, the default is 1 so for example 2:4 creates the vector [2 3 4]
 - Can go in reverse e.g. 4:-1:1 creates [4 3 2 1]
 - Will not go beyond last e.g., 1:2:6 creates [1 3 5]

Example: Creating Row Vectors

```
% row vector
row_vec1 = [1 2 3 4]; % sparated by space
row_vec2 = [1  2  3  4]; % separated by multiple spaces
row_vec3 = [1, 2, 3, 4]; % separated by commas
row_vec4 = 1:4; % use a colon operator
row_vec5 = [1:4]; % okay with putting square brackets
```

```
% identical
```

```
row_vec1
row_vec2
row_vec3
row_vec4
row_vec5
```

```
row_vec1 = 1x4
    1     2     3     4
```

```
row_vec2 = 1x4
    1     2     3     4
```

```
row_vec3 = 1x4
    1     2     3     4
```

```
row_vec4 = 1x4
    1     2     3     4
```

```
row_vec5 = 1x4
    1     2     3     4
```

The **transpose** of a matrix is the operation of flipping the rows to columns and vice versa across the diagonal of the matrix. For example,

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3 \times 2} \quad \text{has transpose} \quad A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2 \times 3}$$

where the power of T indicates the transpose of a matrix. Note that in the example, the dimensions of the matrix changes from 3×2 to 2×3 (with the same total number of elements) and diagonal elements remain the same.

Creating Column Vectors

- A column vector is an $m \times 1$ vector
- Direct method: can create by separating values in square brackets with semicolons
e.g. `[4 ; 7 ; 2]`
- You cannot directly create a column vector using methods such as the colon operator, but you can create a row vector and then **transpose** it to get a column vector using the transpose operator `'` e.g., `[4 7 2]'`

Example: Creating Column Vectors

```
col_vec1 = [5;4;3];  
col_vec2 = 5:-1:3; % not a column vector  
col_vec3 = transpose(5:-1:3);  
col_vec4 = col_vec2';  
  
% col_vec1 is the same with col_vec3 and col_vec4  
% col_vec1 is NOT the same with col_vec2  
col_vec1  
col_vec2  
col_vec3  
col_vec4
```

```
col_vec1 = 3×1  
    5  
    4  
    3  
  
col_vec2 = 1×3  
    5    4    3  
  
col_vec3 = 3×1  
    5  
    4  
    3  
  
col_vec4 = 3×1  
    5  
    4  
    3
```


Referring to Elements

- The elements in a vector are numbered sequentially; each element number is called the *index*, or *subscript* and are shown above the elements here:

1	2	3	4	5
5	33	11	-4	2

- Refer to an element using its *index* or *subscript* in parentheses, e.g. `vec(4)` is the 4th element of a vector `vec` (assuming it has at least 4 elements)
- Can also refer to a subset of a vector by using an *index vector* which is a vector of indices e.g. `vec([2 5])` refers to the 2nd and 5th elements of `vec`; `vec([1:4])` refers to the first 4 elements

The index in MATLAB starts from 1

Modifying Vectors

- Elements in a vector can be changed e.g.

`vec(3) = 11`

- A vector can be extended by referring to elements that do not yet exist; if there is a gap between the end of the vector and the new specified element(s), zeros are filled in, e.g.

```
>> vec = [3 9];
```

```
>> vec(4:6) = [33 2 7]
```

```
vec =
```

```
3 9 0 33 2 7
```

Extending vectors is not recommended

Refereeing: Error

Assigning : Okay

```
vec = [1 2 37];
```

```
vec
```

```
vec(1) = 5;
```

```
vec
```

```
vec(1:3) = [6 7 8];|
```

```
vec
```

```
% vec(6) % error due to refer unassigned value
```

```
vec = 1x3
```

```
1 2 37
```

```
vec = 1x3
```

```
5 2 37
```

```
vec = 1x3
```

```
6 7 8
```

Concatenation

- Vectors can be created by joining together existing vectors, or adding elements to existing vectors
- This is called *concatenation*
- For example:

```
>> v = 2:5;
```

```
>> x = [33 11 2];
```

```
>> w = [v x]
```

```
w =
```

```
    2    3    4    5   33   11    2
```

```
>> newv = [v 44]
```

```
newv =
```

```
    2    3    4    5   44
```

Example: Concatenation

```
vec = [1 2 3 7];  
vec1 = [vec 8 7]; % adding elements  
vec2 = [vec [8 7]]; % adding a row vector  
vec3 = [vec [8;7]']; % adding a column vector after transposing  
vec4 = [vec';8;7]';  
% adding a column vector to the column vector (vec1')  
% and transpose an entire column vector  
vec5 = vec;  
vec5(5:6) = [8 7]; % valid code but not recommended  
  
vec6 = cat(2, vec, [8 7]); % doc cat  
% 1: column, 2: row, 3: depth  
  
% identical  
vec1  
vec2  
vec3  
vec4  
vec5
```

```
vec1 = 1x6  
      1      2      3      7      8      7  
  
vec2 = 1x6  
      1      2      3      7      8      7  
  
vec3 = 1x6  
      1      2      3      7      8      7  
  
vec4 = 1x6  
      1      2      3      7      8      7  
  
vec5 = 1x6  
      1      2      3      7      8      7
```

Creating a Matrix

- Separate values within rows with blanks or commas, and separate the rows with semicolons
- Can use any method to get values in each row (any method to create a row vector, including colon operator)

```
>> mat = [1:3; 6 11 -2]
```

```
mat =
```

```
1    2    3
```

```
6   11   -2
```

- *There must ALWAYS be the same number of values in every row!!*

Example: Creating Matrices

```
mat1 = [1 2 3;4 5 6;7 8 9]; % direct assignment (concatenate rows)
mat2 = [1:3;4:6;7:9]; % use a colon operator
mat3 = [[1;4;7] [2;5;8] [3;6;9]]; % concatenate columns
mat4 = reshape(1:9, 3, 3)'; % will learn
```

% identical

mat1
mat2
mat3
mat4

mat1 = 3×3

1	2	3
4	5	6
7	8	9

mat2 = 3×3

1	2	3
4	5	6
7	8	9

mat3 = 3×3

1	2	3
4	5	6
7	8	9

mat4 = 3×3

1	2	3
4	5	6
7	8	9

Functions that create matrices

There are many built-in functions to create matrices

- **rand(n)** creates an nxn matrix of random reals
- **rand(n,m)** create an nxm matrix of random reals
- **randi([range],n,m)** creates an nxm matrix of random integers in the specified range
- **zeros(n)** creates an nxn matrix of all zeros
- **zeros(n,m)** creates an nxm matrix of all zeros
- **ones(n)** creates an nxn matrix of all ones
- **ones(n,m)** creates an nxm matrix of all ones

Note: there is no twos function – or thirteens – just **zeros** and **ones**!

Example: Functions to Create Matrices

```
mat1_0 = [0 0 0; 0 0 0; 0 0 0];
```

```
mat2_0 = zeros(3,3);
```

```
mat1_0
```

```
mat2_0
```

```
mat1_1 = [1 1 1; 1 1 1; 1 1 1];
```

```
mat2_1 = ones(3,3);
```

```
mat1_1
```

```
mat2_1
```

```
rand(3,3) % random matrix where its elements are real number with
```

```
mat1_random = randi([1 10], 3, 3); % random integer
```

```
mat1_random
```

```
mat1_0 = 3x3
```

```
0     0     0
0     0     0
0     0     0
```

```
mat2_0 = 3x3
```

```
0     0     0
0     0     0
0     0     0
```

```
mat1_1 = 3x3
```

```
1     1     1
1     1     1
1     1     1
```

```
mat2_1 = 3x3
```

```
1     1     1
1     1     1
1     1     1
```

```
ans = 3x3
```

```
0.7922    0.0357    0.6787
0.9595    0.8491    0.7577
0.6557    0.9340    0.7431
```

```
mat1_random = 3x3
```

```
4     8     1
7     1     1
2     3     9
```


Matrix Elements

- To refer to an element in a matrix, you use the matrix variable name followed by the index of the row, and then the index of the column, in parentheses
- ALWAYS refer to the row first, column second
- This is called subscripted indexing
- Can also refer to any subset of a matrix
 - To refer to the entire mth row: `mat(m,:)`
- To refer to the entire nth column: `mat(:,n)`
- To refer to the last row or column use **end**, e.g. `mat(end,m)` is the mth value in the last row
- Can modify an element or subset of a matrix in an assignment statement

Modifying Matrices

- An individual element in a matrix can be modified by assigning a new value to it
- Entire rows and columns can also be modified
- Any subset of a matrix can be modified, as long as what is being assigned has the same dimensions as the subset being modified
- Exception to this: a scalar can be assigned to any size subset; the same scalar is assigned to every element in the subset

Example: Modifying Matrix

```
mat0 = zeros(5,5);
row_vec = 1:5;
col_vec = transpose(6:10);
mat_sub = ones(2,3);

% initialize all mat# with mat0 (a 5x5 zero matrix)
mat01 = mat0; mat02 = mat0; mat03 = mat0;
mat04 = mat0; mat05 = mat0; mat06 = mat0;
mat07 = mat0; mat08 = mat0; mat09 = mat0;
mat10 = mat0;

% evaluate your resulting matrix
mat01(2,:) = row_vec;
mat02(:,3) = col_vec;
mat03(1:2,1:3) = mat_sub; % elements in the first two rows
mat04(2:4,1:2) = mat_sub';
mat05(1:4,1) = col_vec(1:4);
mat06(1:4,1) = row_vec(1:4)'; % first four elements in the first column
mat07(2:end, 1) = col_vec(2:end);
mat08(2:end, end) = row_vec(2:end)';
% 2nd element to the last elements in the last column

mat09(1:4, 1:2) = 3;
mat10(1,:) = 5; % elements of all columns in the first row
```

mat01 = 5x5

0	0	0	0	0
1	2	3	4	5
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

mat02 = 5x5

0	0	6	0	0
0	0	7	0	0
0	0	8	0	0
0	0	9	0	0
0	0	10	0	0

mat03 = 5x5

1	1	1	0	0
1	1	1	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

mat04 = 5x5

0	0	0	0	0
1	1	0	0	0
1	1	0	0	0
1	1	0	0	0
0	0	0	0	0

mat05 = 5x5

6	0	0	0	0
7	0	0	0	0
8	0	0	0	0
9	0	0	0	0
0	0	0	0	0

mat06 = 5x5

1	0	0	0	0
2	0	0	0	0
3	0	0	0	0
4	0	0	0	0
0	0	0	0	0

mat07 = 5x5

0	0	0	0	0
7	0	0	0	0
8	0	0	0	0
9	0	0	0	0
10	0	0	0	0

mat08 = 5x5

0	0	0	0	0
0	0	0	0	2
0	0	0	0	3
0	0	0	0	4
0	0	0	0	5

mat09 = 5x5

3	3	0	0	0
3	3	0	0	0
3	3	0	0	0
3	3	0	0	0
0	0	0	0	0

mat10 = 5x5

5	5	5	5	5
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

Matrix Dimension

- There are several functions to determine the dimensions of a vector or matrix:
 - **size** returns the # of rows and columns for a vector or matrix
 - Important: capture both of these values in an assignment statement
`[r c] = size(mat)`
 - **numel** returns the total # of elements in a vector or matrix
 - **length**(vec) returns the # of elements in a vector
 - **length**(mat) returns the larger dimension (row or column) for a matrix
- Very important to be general in programming: do not assume that you know the dimensions of a vector or matrix – use **length** or **size** to find out!

Example: Matrix Dimension

```
mat1 = rand(3,5);  
[sc1, sr1] = size(mat1); % use 'size' function  
n_elem1 = sc1 * sr1;  
  
sc2 = size(mat1,1); % doc size  
sr2 = size(mat1,2); % 1 is a column, 2 is a row direction  
n_elem2 = numel(mat1);  
  
sc3 = numel(mat1(:,1)); % return a size of a column in mat1  
sr3 = numel(mat1(1,:)); % return a size of a row in mat1  
n_elem3 = numel(mat1); % return the total # of elements in mat1  
  
% they are identical  
[sc1 sc2 sc3]  
[sr1 sr2 sr3]  
[n_elem1 n_elem2 n_elem3]
```

```
ans = 1×3  
      3      3      3
```

```
ans = 1×3  
      5      5      5
```

```
ans = 1×3  
     15     15     15
```

Linear indexing: only using one index into a matrix (MATLAB will unwind it column-by-column)

Very important

reshape

```
% introduce 'reshape': doc reshape|
vec0 = 1:10;
mat0 = reshape(vec0, 2, 5);
mat1 = reshape(vec0, 2, []); % same with mat0.
% The second dimension is determined based on the # of element

% mat_error = reshape(vec0, 3,5); % error: dimension mismatch

mat0
mat1
```

```
mat0 = 2x5
     1     3     5     7     9
     2     4     6     8    10

mat1 = 2x5
     1     3     5     7     9
     2     4     6     8    10
```

Example: Linear Indexing

```
% linear indexing
mat00 = zeros(4,4);
row_vec = 1:4;
col_vec = transpose(5:8);
mat_sub = [1 2 3; 4 5 6];

% initialize all mat# with mat00 (a 5x5 zero m
mat01 = mat00; mat02 = mat00; mat03 = mat00;
mat04 = mat00; mat05 = mat00; mat06 = mat00;
mat07 = mat00; mat08 = mat00;

% You should understand how they work
mat01(1:4) = row_vec;
mat02(1:4) = col_vec;
mat03(1:5) = [col_vec; 10];
mat04(1:6) = mat_sub;
mat05(1:6) = mat_sub'; |
% mat_error(1:7) = mat_sub; % because of mismatching # of elements
mat06(8:11) = row_vec;
mat07([1 3 5 6]) = col_vec;
mat08([6 5 3 1]) = col_vec;

mat09 = reshape(1:16, 4, 4);
mat09(16:-1:1) = 1:16;

mat10 = reshape(1:16, 4, 4);
mat10(16:-1:1) = mat10(:); % this's cool!
```

```
mat00 = 4x4
    0    0    0    0
    0    0    0    0
    0    0    0    0
    0    0    0    0

row_vec = 1x4
    1    2    3    4

col_vec = 4x1
     5
     6
     7
     8

mat_sub = 2x3
     1     2     3
     4     5     6
```

```
mat01 = 4x4
     1     0     0     0
     2     0     0     0
     3     0     0     0
     4     0     0     0
```

```
mat02 = 4x4
     5     0     0     0
     6     0     0     0
     7     0     0     0
     8     0     0     0
```

```
mat03 = 4x4
     5    10     0     0
     6     0     0     0
     7     0     0     0
     8     0     0     0
```

```
mat04 = 4x4
     1     3     0     0
     4     6     0     0
     2     0     0     0
     5     0     0     0
```

```
mat05 = 4x4
     1     5     0     0
     2     6     0     0
     3     0     0     0
     4     0     0     0
```

```
mat06 = 4x4
     0     0     2     0
     0     0     3     0
     0     0     4     0
     0     1     0     0
```

```
mat07 = 4x4
     5     7     0     0
     0     8     0     0
     6     0     0     0
     0     0     0     0
```

```
mat08 = 4x4
     8     6     0     0
     0     5     0     0
     7     0     0     0
     0     0     0     0
```

```
mat09 = 4x4
    16    12     8     4
    15    11     7     3
    14    10     6     2
    13     9     5     1
```

```
mat10 = 4x4
    16    12     8     4
    15    11     7     3
    14    10     6     2
    13     9     5     1
```

Empty Vectors

- An *empty vector* is a vector with no elements; an empty vector can be created using square brackets with nothing inside []
- to delete an element from a vector, assign an empty vector to that element
- delete an entire row or column from a matrix by assigning []
 - Note: cannot delete an individual element from a matrix

```
vec1 = 1:5;  
vec1(end) = []; % remove the last element (become 1x4)  
vec1(2) = []; % remove the second element (become 1x3)  
  
% vec1(5) % errors  
  
mat1 = reshape(1:16, 4, 4);  
mat1(:,3) = []; % remove the 3rd column in  
mat1(2, :) = [];  
% mat1(2, 1) = [] % error
```

```
vec1 = 1x3  
      1      3      4  
  
mat1 = 3x3  
      1      5     13  
      3      7     15  
      4      8     16
```


3D Matrices

- A three dimensional matrix has dimensions $m \times n \times p$
- Can create using built-in functions, e.g. the following creates a $3 \times 5 \times 2$ matrix of random integers; there are 2 layers, each of which is a 3×5 matrix

```
>> randi([0 50], 3,5,2)
```

```
ans(:,:,1) =
```

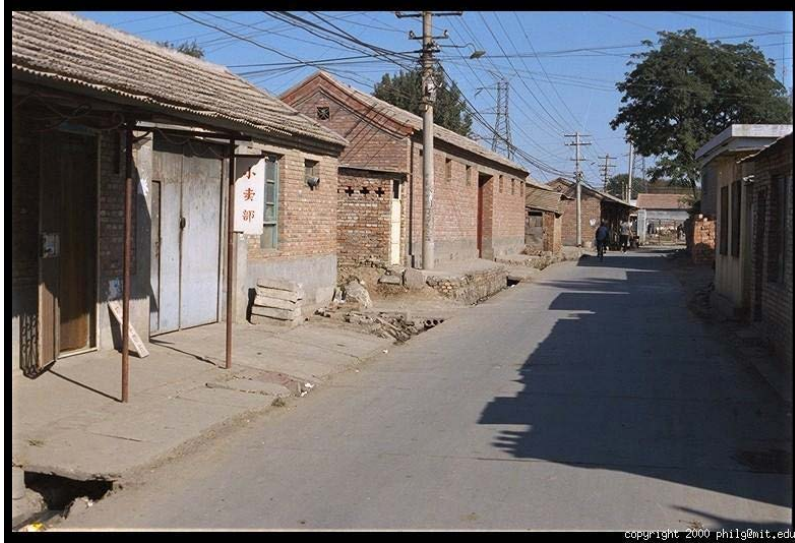
```
36  34   6  17  38
38  33  25  29  13
14   8  48  11  25
```

```
ans(:,:,2) =
```

```
35  27  13  41  17
45   7  42  12  10
48   7  12  47  12
```

Challenging ! But very useful !!

(Off-Topic) Color Image



R

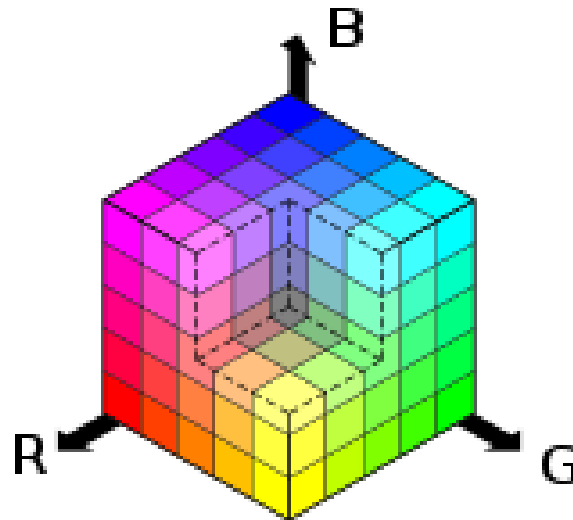


G



B

0	3	2	5	4	7	6	9	8
3	0	1	2	3	4	5	6	7
2	1	0	3	2	5	4	7	6
5	2	3	0	1	2	3	4	5
4	3	2	1	0	3	2	5	4
7	4	5	2	3	0	1	2	3
6	5	4	3	2	1	0	3	2
9	6	7	4	5	2	3	0	1
8	7	6	5	4	3	2	1	0



Example: 3D Matrix

```
mat3D = reshape(1:120, 8, 5, 3);
```

```
mat3D(:,:,1)
```

```
mat3D(:,:,2)
```

```
mat3D(:,:,3)
```

mat3D(:,:,1)

```
ans = 8x5
```

1	9	17	25	33
2	10	18	26	34
3	11	19	27	35
4	12	20	28	36
5	13	21	29	37
6	14	22	30	38
7	15	23	31	39
8	16	24	32	40

```
ans = 8x5
```

41	49	57	65	73
42	50	58	66	74
43	51	59	67	75
44	52	60	68	76
45	53	61	69	77
46	54	62	70	78
47	55	63	71	79
48	56	64	72	80

mat3D(:,:,2)

```
ans = 8x5
```

81	89	97	105	113
82	90	98	106	114
83	91	99	107	115
84	92	100	108	116
85	93	101	109	117
86	94	102	110	118
87	95	103	111	119
88	96	104	112	120

mat3D(:,:,3)

```
ans = 8x5
```

81	89	97	105	113
82	90	98	106	114
83	91	99	107	115
			08	116
			09	117
41	49	57	65	73
42	50	58	66	74
43	51	59	67	75
			12	120

```
ans = 8x5
```

41	49	57	65	73
42	50	58	66	74
43	51	59	67	75
			76	
			77	
			78	
			79	
			80	

```
ans = 8x5
```

1	9	17	25	33
2	10	18	26	34
3	11	19	27	35
4	12	20	28	36
5	13	21	29	37
6	14	22	30	38
7	15	23	31	39
8	16	24	32	40

Example: 3D Matrix (Continue)

`mat3D(:,:,3)`

```
ans = 8x5
    81    89    97   105   113
    82    90    98   106   114
    83    91    99   107   115
    84    92   100   108   116
    85    93   101   109   117
    86    94   102   110   118
    87    95   103   111   119
    88    96   104   112   120
```

```
mat1 = mat3D(:,1,1);
mat2 = mat3D(1,1,:);
mat3 = mat3D(1,:,1);
```

Column vector

```
mat1 = 8x1
     1
     2
     3
     4
     5
     6
     7
     8
```

8 x 1

```
ans = 8x5
    41    49    57    65    73
    42    50    58    66    74
    43    51    59    67    75
    44    52    60    68    76
    45    53    61    69    77
    46    54    62    70    78
    47    55    63    71    79
    48    56    64    72    80
```

`mat3D(:,:,2)`

3D vector

```
mat2 =
mat2(:,:,1) =
     1

mat2(:,:,2) =
```

1 x 1 x 3

```
ans = 8x5
     1     9    17    25    33
     2    10    18    26    34
     3    11    19    27    35
     4    12    20    28    36
     5    13    21    29    37
     6    14    22    30    38
     7    15    23    31    39
     8    16    24    32    40
```

`mat3D(:,:,1)`

Row vector

```
mat2(:,:,3) =
    81
mat3 = 1x5
     1     9    17    25    33
```

1 x 5

Example: 3D Matrix (Continue)

`mat3D(:,:,3)`

```
ans = 8x5
    81    89    97   105   113
    82    90    98   106   114
    83    91    99   107   115
    84    92   100   108   116
    85    93   101   109   117
    86    94   102   110   118
    87    95   103   111   119
    88    96   104   112   120
```

```
ans = 8x5
    41    49    57    65    73
    42    50    58    66    74
    43    51    59    67    75
    44    52    60    68    76
    45    53    61    69    77
    46    54    62    70    78
    47    55    63    71    79
    48    56    64    72    80
```

```
ans = 8x5
     1     9    17    25    33
     2    10    18    26    34
     3    11    19    27    35
     4    12    20    28    36
     5    13    21    29    37
     6    14    22    30    38
     7    15    23    31    39
     8    16    24    32    40
```

`mat3D(:,:,1)`

```
mat4 = mat3D(2,:,:);
mat5 = mat3D(:,2,:);
```

`mat4 =` **1 x 5 x 3**

```
mat4(:,:,1) =
```

```
     2    10    18    26    34
```

```
mat4(:,:,2) =
```

```
    42    50    58    66    74
```

```
mat4(:,:,3) =
```

```
    82    90    98   106   114
```

`mat5 =` **8 x 1 x 3**

```
mat5(:,:,1) =
```

```
     9
    10
    11
    12
    13
    14
    15
    16
```

```
mat5(:,:,2) =
```

```
    49
    50
    51
    52
    53
    54
    55
    56
```

```
mat5(:,:,3) =
```

```
    89
    90
    91
    92
    93
    94
    95
    96
```

Arrays as function arguments

- Entire arrays (vectors or matrices) can be passed as arguments to functions; this is very powerful!
- The result will have the same dimensions as the input
- For example:

```
>> vec = randi([-5 5], 1, 4)
```

```
vec =
```

```
    -3     0     5     1
```

```
>> av = abs(vec)
```

```
av =
```

```
     3     0     5     1
```

Scalar Operations

- Numerical operations can be performed on every element in a vector or matrix
- For example, **Scalar multiplication**: multiply every element by a scalar

```
>> [4 0 11] * 3
```

```
ans =
```

```
12  0  33
```

- Another example: scalar addition; add a scalar to every element

```
>> zeros(1,3) + 5
```

```
ans =
```

```
5  5  5
```

```
mat1 = ones(3,3);  
mat2 = mat1 + 3; % add 3 to all elements in mat1  
mat3 = mat1 + ones(3,3)*3; % it is the same with the above
```

```
mat4 = zeros(size(mat1));  
mat4(:) = 4; % modify all elements to 4
```

```
mat2 = 3x3  
4     4     4  
4     4     4  
4     4     4
```

```
mat3 = 3x3  
4     4     4  
4     4     4  
4     4     4
```

```
mat4 = 3x3  
4     4     4  
4     4     4  
4     4     4
```

Array Operations

- **Array operations** on two matrices A and B:
 - these are applied term-by-term, or element-by-element
 - **this means the matrices must have the same dimensions (NO! after R2016)**
 - In MATLAB:
 - matrix addition: $A + B$
 - matrix subtraction: $A - B$ or $B - A$
 - For operations that are based on multiplication (multiplication, division, and exponentiation), a dot must be placed in front of the operator
 - array multiplication: $A .* B$
 - array division: $A ./ B$, $A .\ B$
 - array exponentiation $A .^ 2$
 - **matrix multiplication: NOT an array operation**

They are totally different!

Example: Array Operations

```
matA = ones(3,3)*6;  
matB = ones(3,3)*2;
```

```
mat1 = matA + matB;  
mat2 = matB*4;  
mat3 = matA - matB;  
mat4 = matA.^2;
```

```
matC = ones(4,4);  
matC(1:2,1:2) = 2;  
  
matD = ones(4,4)*12;
```

```
mat5 = matC + matD;  
mat6 = matC - matD;  
mat7 = matC.*matD;  
mat8 = matD./matC;
```

matA = 3x3

6	6	6
6	6	6
6	6	6

matB = 3x3

2	2	2
2	2	2
2	2	2

matC = 4x4

2	2	1	1
2	2	1	1
1	1	1	1
1	1	1	1

matD = 4x4

12	12	12	12
12	12	12	12
12	12	12	12
12	12	12	12

mat1 = 3x3

8	8	8
8	8	8
8	8	8

mat2 = 3x3

8	8	8
8	8	8
8	8	8

mat3 = 3x3

4	4	4
4	4	4
4	4	4

mat4 = 3x3

36	36	36
36	36	36
36	36	36

mat5 = 4x4

14	14	13	13
14	14	13	13
13	13	13	13
13	13	13	13

mat6 = 4x4

-10	-10	-11	-11
-10	-10	-11	-11
-11	-11	-11	-11
-11	-11	-11	-11

mat7 = 4x4

24	24	12	12
24	24	12	12
12	12	12	12
12	12	12	12

mat8 = 4x4

6	6	12	12
6	6	12	12
12	12	12	12
12	12	12	12

- **false** equivalent to `logical(0)`
- **true** equivalent to `logical(1)`
- **false** and **true** are also functions that create matrices of all **false** or **true** values
- As of R2016a, this can also be done with **ones** and **zeros**, e.g.
`logzer = ones(1,5, 'logical')`

Logical Vectors

- Using relational operators on a vector or matrix results in a **logical** vector or matrix

```
>> vec = [44 3 2 9 11 6];
```

```
>> logv = vec > 6
```

```
logv =
```

```
1  0  0  1  1  0
```

- Can use this to index into a vector or matrix (only if the index vector is the type **logical**)

```
>> vec(logv)
```

```
ans =
```

```
44  9  11
```

Logical Indexing!!!

Example: Logical Vectors

```
vec1 = 2:11;
```

```
logi_vec = ones(1,10, 'logical'); % all true  
logi_vec([3 5 7]) = false; % assign false logical values
```

```
vec2 = vec1(logi_vec); % access all values except values at 3, 5, and 7 locations  
vec3 = vec1(~logi_vec); % access values at 3, 5, and 7 locations
```

```
logi_equal_5 = (vec1==5);  
vec4 = vec1(logi_equal_5); % values of 5
```

```
logi_larger_5 = (vec1>5);  
vec5 = vec1(logi_larger_5); % values more than 5
```

```
logi_smaller_5 = (vec1<5);  
vec6 = vec1(logi_smaller_5); % values less than 5
```

vec1

2	3	4	5	6	7	8	9	10	11
---	---	---	---	---	---	---	---	----	----

logi_vec

1	1	0	1	0	1	0	1	1	1
---	---	---	---	---	---	---	---	---	---

Q1: Find numbers less than or equal to 10 in 'vec1'

Q2: Find numbers not equal to 5 in 'vec1'

Q3: Find numbers larger than 3 and less than 7.

```
vec1 =  
    2    3    4    5    6    7    8    9   10   11  
vec2 =  
    2    3    5    7    9   10   11  
vec3 =  
    4    6    8  
vec4 = 5  
vec5 =  
    6    7    8    9   10   11  
vec6 =  
    2    3    4  
mat1 =  
    2    4    6    8   10  
    3    5    7    9   11  
vec7 = 5  
vec8 =  
    6    7    8    9   10   11  
vec9 =  
    2    3    4
```

Example: Logical Vectors (Continue)

vec1

2	3	4	5	6	7	8	9	10	11
---	---	---	---	---	---	---	---	----	----

vec1>3

0	0	1	1	1	1	1	1	1	1
---	---	---	---	---	---	---	---	---	---

vec1<7

1	1	1	1	1	0	0	0	0	0
---	---	---	---	---	---	---	---	---	---

logi_3_7 = (vec1>3) && (vec1<7)

0	0	1	1	1	0	0	0	0	0
---	---	---	---	---	---	---	---	---	---

vec1 (logi_3_7)

4	5	6
---	---	---

Logical Built-in Functions

- **any** returns true if anything in the input argument is true
- **all** returns true only if everything in the input argument is true
- **find** finds locations and returns indices

```
>> vec
```

```
vec =
```

```
44  3  2  9 11  6
```

```
>> find(vec>6)
```

```
ans =
```

```
1  4  5
```

Example: Logical Vectors

How to check if there is a certain number in a vector? Let's say 5 in 'vec1'.

```
vec1 = 2:11;  
tmp1 = (vec1 == 5);  
is_there_five_v1 = logical(sum(tmp1));  
  
is_there_five_v2 = any(vec1==5);  
is_there_five_v3 = ~all(vec1~=5);
```

```
is_there_five_v1 = logical  
1  
is_there_five_v2 = logical  
1  
is_there_five_v3 = logical  
1
```

vec1

2	3	4	5	6	7	8	9	10	11
---	---	---	---	---	---	---	---	----	----

tmp1 = (vec1 == 5)

0	0	0	1	0	0	0	0	0	0
---	---	---	---	---	---	---	---	---	---

any(tmp1) => true
any(tmp1) => false
find(tmp1) => 4

Test 'is_there_five_v2' and 'is_there_five_v3'.

Element-wise operators

- `|` and `&` are used for matrices; go through element-by-element and return logical 1 or 0
- `||` and `&&` are used for scalars

Logical: 0 is false otherwise true

Example: Logical AND and OR

```
mat1 = reshape(0:11, 3, 4);  
mat2 = mat1;  
mat2(1,3) = 0;  
  
logi_mat1 = (mat1 | mat2)  
logi_mat2 = logical(logical(mat1) + logical(mat2))  
  
logi_mat3 = (mat1 & mat2)  
logi_mat4 = logical(mat1 .* mat2)
```

0	3	6	9
1	4	7	10
2	5	8	11

mat1

0	3	0	9
1	4	7	10
2	5	8	11

mat2

```
logi_mat1 = 3x4 logical array  
    0     1     1     1  
    1     1     1     1  
    1     1     1     1
```

```
logi_mat2 = 3x4 logical array  
    0     1     1     1  
    1     1     1     1  
    1     1     1     1
```

```
logi_mat3 = 3x4 logical array  
    0     1     0     1  
    1     1     1     1  
    1     1     1     1
```

```
logi_mat4 = 3x4 logical array  
    0     1     0     1  
    1     1     1     1  
    1     1     1     1
```

Matrix Multiplication: Dimensions

- **Matrix multiplication** is NOT an array operation
 - it does NOT mean multiplying term by term
- In MATLAB, the multiplication **operator** `*` performs matrix multiplication
- In order to be able to multiply a matrix A by a matrix B, the number of columns of A must be the same as the number of rows of B
- If the matrix A has dimensions $m \times n$, that means that matrix B must have dimensions $n \times \text{something}$; we'll call it p
 - In mathematical notation, $[A]_{m \times n} [B]_{n \times p}$
 - We say that the **inner dimensions** must be the same
- The resulting matrix C has the same number of rows as A and the same number of columns as B
 - in other words, the **outer dimensions** $m \times p$
 - In mathematical notation, $[A]_{m \times n} [B]_{n \times p} = [C]_{m \times p}$.
 - This only defines the size of C; it does not explain how to calculate the values

2.3.4 Matrix Multiplication

2.3.4.1 Matrix times a Vector

A linear system of equations with coefficient matrix A , variable vector \vec{x} and constant term vector \vec{b} , can be expressed as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{matrix} \text{matrix} & \text{vector} & \text{vector} \\ (m \times n) & (n \times 1) & = (m \times 1) \end{matrix}$$

Compatibility – the number of columns in the matrix **must** equal the number of rows in the vector.

Example: Matrix Times a Vector

```
matA = reshape(1:9,3,3);  
vecA = [1;2;3];  
  
vec1 = matA*vecA; % matrix times a vector  
  
vec2 = zeros(3,1);  
vec2(1) = matA(1,:)*vecA; % row vector x column vector  
vec2(2) = matA(2,:)*vecA;  
vec2(3) = matA(3,:)*vecA;  
  
vec3 = zeros(3,1);  
vec3(1) = sum(matA(1,:)'.*vecA); % element-wise operation (vector, vector)  
vec3(2) = sum(matA(2,:)'.*vecA);  
vec3(3) = sum(matA(3,:)'.*vecA);  
  
vec4 = zeros(3,1);  
vec4(1) = dot(matA(1,:)', vecA); % dot product  
vec4(2) = dot(matA(2,:)', vecA);  
vec4(3) = dot(matA(3,:)', vecA);  
  
vec5 = sum(matA.*[vecA';vecA';vecA'],2); % element-wise operation (matrix, matrix)
```

```
vec1 =  
      30  
      36  
      42  
  
vec2 =  
      30  
      36  
      42  
  
vec3 =  
      30  
      36  
      42  
  
vec4 =  
      30  
      36  
      42  
  
vec5 =  
      30  
      36  
      42
```

Example: Matrix Times a Vector (Symbolic Expression)

```
syms a11 a12 a13 a21 a22 a23 a31 a32 a33
syms x1 x2 x3
A = [a11 a12 a13; a21 a22 a23; a31 a32 a33];
x = [x1;x2;x3];
```

```
vec1_syms = A*x;
```

```
vec2_syms(1,1) = A(1,:)*x;
```

```
vec2_syms(2,1) = A(2,:)*x;
```

```
vec2_syms(3,1) = A(3,:)*x;
```

```
vec3_syms(1,1) = sum(transpose(A(1,:)).*x);
```

```
vec3_syms(2,1) = sum(transpose(A(2,:)).*x);
```

```
vec3_syms(3,1) = sum(transpose(A(3,:)).*x);
```

```
vec5_syms = sum(A .* transpose([x x x]), 2);
```

vec1_syms =

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}$$

vec2_syms =

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}$$

vec3_syms =

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}$$

vec5_syms =

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}$$

2.3.4.2 Matrix times a Matrix

Matrix multiplication is an extension of matrix and vector multiplication.

Consider the product of an $m \times n$ matrix A , and an $n \times p$ matrix B :

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ b_{31} & b_{32} & \cdots & b_{3p} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$

$$\begin{matrix} \text{matrix} & \text{matrix} & \text{matrix} \\ (m \times n) & (n \times p) & = (m \times p) \end{matrix}$$

Compatibility – the number of columns in the first matrix **must** equal the number of rows in the second matrix.

Example: Matrix Times a Matrix

```
matA = reshape(1:9,3,3);  
matB = reshape(10:18,3,3);  
  
mat1 = matA*matB; % matrix times a matrix  
  
mat2 = zeros(3,3);  
mat2(:,1) = matA*matB(:,1); % matrix x column vector  
mat2(:,2) = matA*matB(:,2);  
mat2(:,3) = matA*matB(:,3);  
  
mat3 = zeros(3,3);  
mat3(1,:) = matA(1,:)*matB(:,1); % row vector x column vector  
mat3(1,2) = matA(1,:)*matB(:,2);  
mat3(1,3) = matA(1,:)*matB(:,3);  
  
mat3(2,1) = matA(2,:)*matB(:,1);  
mat3(2,2) = matA(2,:)*matB(:,2);  
mat3(2,3) = matA(2,:)*matB(:,3);  
  
mat3(3,1) = matA(3,:)*matB(:,1);  
mat3(3,2) = matA(3,:)*matB(:,2);  
mat3(3,3) = matA(3,:)*matB(:,3);  
  
mat4 = transpose(matB'* matA'); % AB = (B^T A^T)^T
```

```
mat1 = 3x3  
    138    174    210  
    171    216    261  
    204    258    312
```

```
mat2 = 3x3  
    138    174    210  
    171    216    261  
    204    258    312
```

```
mat3 = 3x3  
    138    174    210  
    171    216    261  
    204    258    312
```

```
mat4 = 3x3  
    138    174    210  
    171    216    261  
    204    258    312
```

Common Pitfalls

- Attempting to create a matrix that does not have the same number of values in each row
- Confusing matrix multiplication and array multiplication. Array operations, including multiplication, division, and exponentiation, are performed term by term (so the arrays must have the same size); the operators are `.*`, `./`, `.\`, and `.^`.
- Attempting to use `||` or `&&` with arrays. Always use `|` and `&` when working with arrays; `||` and `&&` are only used with scalars.

- If possible, try not to extend vectors or matrices, as it is not very efficient.
- To be general, never assume that the dimensions of any array (vector or matrix) are known. Instead, use the function **numel** to determine the number of elements in a vector, and the function **size** for a matrix:

```
len = numel(vec);
```

```
[r, c] = size(mat);
```

- Use **true** instead of **logical(1)** and **false** instead of **logical(0)**, especially when creating vectors or matrices.

Slide Credits and References

- Stormy Attaway, 2018, Matlab: A Practical Introduction to Programming and Problem Solving, 5th edition
- Lecture slides for “Matlab: A Practical Introduction to Programming and Problem Solving”
- Holly Moore, 2018, MATLAB for Engineers, 5th edition