Symbolic Function

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Symbolic Math Toolbox

Perform symbolic math computations

- Symbolic Math Toolbox™ provides functions for solving, plotting, and manipulating symbolic math equations. The toolbox provides functions in common mathematical areas such as calculus, linear algebra, algebraic and ordinary differential equations, equation simplification, and equation manipulation.
- Symbolic Math Toolbox lets you analytically perform differentiation, integration, simplification, transforms, and equation solving.

- $\sin(\pi) = 0$
- $\sin(3.14159) \approx 0$

How to Solve a Mathematical Equation

$$f(x) = 2x$$

$$f(a) = 2a$$

$$f(10) = ?$$

$$f(8) = ?$$

$$f(6) = ?$$

What if ?

$$f(x) = 2e^{3x}x^3\log(x)$$

```
% method 1
2*10
2*8
2*6
```

```
% method 2
x = 10;
2*x
```

```
% method 3
x = 10
compf(x)
```

```
function fx = compf(x1)
fx = x1*2;
end
```

Create Symbolic Variables and Expression

- syms – Create symbolic variables in MATLAB
- You can write a equation using a symbolic variable.

```
syms a b c x

eq1 = a*x + b
eq2 = a*x^2 + b*x + c
```

eq1 = $b + a\,x$

eq2 = $a\,x^2 + b\,x + c$

Built-in Function: expand

expand

Expand expressions and simplify inputs of functions by using identities

Syntax

```
expand(S)
expand(S,Name,Value)
```

Description

`expand(S)` multiplies all parentheses in `S`, and simplifies inputs to functions such as `cos(x + y)` by applying standard identities.

```
syms x
y1 = (x-2)*(x-4)
y2 = expand(y1)
```

y1 = $(x-2)\,(x-4)$

y2 = $x^2 - 6\,x + 8$

Built-in Function: simplify

simplify

R2019a

Algebraic simplification

collapse all in page

Syntax

```
S = simplify(expr)
S = simplify(expr,Name,Value)
```

Description

example

`S = simplify(expr)` performs algebraic simplification of `expr`. If `expr` is a symbolic vector or matrix, this function simplifies each element of `expr`.

`S = simplify(expr,Name,Value)` performs algebraic simpli[…] specified by one or more `Name,Value` pair arguments.

```
syms x y

y1 = (1-x^2)/(1+x)
y2 = expand(y1)
y3 = simplify(y1)
```

y1 =

$$-\frac{x^2-1}{x+1}$$

y2 =

$$\frac{1}{x+1}-\frac{x^2}{x+1}$$

y3 = $1-x$

Built-in Function: subs

subs

R2019a

Symbolic substitution

collapse all in page

Syntax

```
subs(s,old,new)
subs(s,new)
subs(s)
```

```
syms x
y1 = (x-2)*(x-3)
y2 = expand(y1)
y3 = subs(y1, x , 4)
```

y1 = $(x-2)\ (x-3)$

y2 = $x^2 - 5\,x + 6$

y3 = 2

Description

`subs(s,old,new)` returns a copy of `s`, replacing all occurrences of `old` with `new`, and then evaluates `s`. example

`subs(s,new)` returns a copy of `s`, replacing all occurrences of the default variable in `s` with `new`, and then evaluates `s`. The default variable is defined by `symvar`. example

Use of Numeric and Symbolic Variables

```
y1 = (x1-2)*(x1-3)
```

```
Undefined function or variable 'x1'.

Did you mean:
```

```
x1 = 3;
y1 = (x1-2)*(x1-3)
```

```
y1 = 0
```

Numeric variables

```
syms x1
y1 = (x1-2)*(x1-3)
```

y1 = $(x_1 - 2)\ (x_1 - 3)$

```
syms x1
y1 = (x1-2)*(x1-3)
subs(y1, x1, 3)
```

y1 = $(x_1 - 2)\ (x_1 - 3)$

ans = 0

Symbolic variables

Built-in Function: int

int

Definite and indefinite integrals

co

Syntax

```
int(expr,var)
int(expr,var,a,b)
int( __ ,Name,Value)
```

Description

`int(expr,var)` computes the indefinite integral of `expr` with respect to the symbolic scalar variable `var`. Specifying the variable `var` is optional. If you do not specify it, `int` uses the default variable determined by `symvar`. If `expr` is a constant, then the default variable is x.

`int(expr,var,a,b)` computes the definite integral of `expr` with respect to `var` from `a` to `b`. If you do not specify it, `int` uses the default variable determined by `symvar`. If `expr` is a constant, then the default variable is x.

`int(expr,var,[a b])` is equivalent to `int(expr,var,a,b)`.

`int( __ ,Name,Value)` specifies additional options using one or more `Name,Value` pair arguments. For example, `'IgnoreAnalyticConstraints',true` specifies that `int` applies additional simplifications to the integrand.

Example: Simple Integration 1

Examples: Evaluate the following indefinite integrals

① $\int x^3 \cos(x^4+2)\,dx$

Aside: $u = x^4+2$ $\quad \frac{du}{dx} = 4x^3 \Rightarrow \frac{1}{4}du = x^3\,dx$

$= \int \cos u \left(\frac{1}{4}du\right)$

$= \frac{1}{4}\int \cos u\,du = \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^4+2) + C$

Lecture note: Integration by Substitution (Courtesy of Brenda Lee)

```
syms x
expr = x^3 * cos(x^4+2);
result = int(expr, x)
```

```
result =
```

$$\frac{\sin(x^4+2)}{4}$$

Example: Simple Integration 2

④ $\int e^{x} \sin x \, dx$

Aside: $u = e^{x}$, $du = e^{x} dx$ | $dv = \sin x \, dx$, $v = -\cos x$

$$= -e^{x}\cos x + \underbrace{\int \cos x \, e^{x} dx}_{\text{IBP again!}}$$

→ Aside: $u = e^{x}$, $du = e^{x} dx$ | $dv = \cos x \, dx$, $v = \sin x$

$$= -e^{x}\cos x + \left[e^{x}\sin x - \int \sin x \, e^{x} dx \right]$$

$$= e^{x}\sin x - e^{x}\cos x - \underbrace{\int e^{x}\sin x \, dx}_{\text{another time? NO!! Group together!}}$$

$\Downarrow$

$$\int e^{x}\sin x \, dx = e^{x}\sin x - e^{x}\cos x - \int e^{x}\sin x \, dx$$

$$2\int e^{x}\sin x \, dx = e^{x}\sin x - e^{x}\cos x$$

$$\int e^{x}\sin x \, dx = \frac{e^{x}}{2}(\sin x - \cos x) + C$$

```
syms x
expr = exp(x)*sin(x);
result = int(expr, x)
```

result =

$$-\frac{e^{x}\left(\cos(x) - \sin(x)\right)}{2}$$

Lecture note: Integration by Parts (Courtesy of Brenda Lee)

Example: Simple Integration 3

Example 3: Find $\int \sin^4 x \, dx$

$$\int \sin^4 x \, dx = \int (\sin^2 x)^2 dx = \int \left(\frac{1 - \cos 2x}{2} \right)^2 dx$$

$$= \frac{1}{4} \int (1 - 2\cos 2x + \cos^2 2x) \, dx$$ ← need to get rid of $\cos^2(2x)$

$$\cos^2(2x) = \frac{1}{2}(1 + \cos 4x)$$

$$= \frac{1}{4} \int \left(1 - 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4} \int \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4} \left[\frac{3}{2}x - \sin 2x + \frac{1}{8} \sin 4x \right] + C$$

Lecture note: Trigonometric Integrals (Courtesy of Brenda Lee)

```
syms x
expr = sin(x)^4;
result = int(expr, x)
```

```
result =
```

$$\frac{3\,x}{8} - \frac{\sin(2\,x)}{4} + \frac{\sin(4\,x)}{32}$$

Example: Simple Integration 4

$$① \int_0^4 \sqrt{2x+1}\,dx$$

Aside: $\underline{u = 2x+1}$ | $\frac{du}{dx} = 2$, $\frac{1}{2}du = \underline{dx}$ | $u(0) = 2(0)+1 = \underline{1}$, $u(4) = 2(4)+1 = \underline{9}$

$$= \int_1^9 \sqrt{u}\left(\tfrac{1}{2}du\right)$$

$$= \frac{1}{2}\int_1^9 u^{1/2}\,du$$

$$= \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_1^9 = \frac{1}{3}\left[9^{3/2} - 1^{3/2}\right] = \frac{1}{3}(27-1) = \underline{\underline{\frac{26}{3}}}$$

```
syms x
y1 = sqrt(2*x + 1)
y1_int = int(y1)

y_def1 = int(y1, x, 0, 4)
y_def2 = subs(y1_int, x, 4) - subs(y1_int, x, 0)
```

y1 = $\sqrt{2x+1}$

y1_int =

$$\frac{(2x+1)^{3/2}}{3}$$

y_def1 =

$$\frac{26}{3}$$

y_def2 =

$$\frac{26}{3}$$

Lecture note: Substitution Method (Courtesy of Brenda Lee)

Example in MATLAB Grader

Write a script to create a variable named:

(a) 'int_val1', which contains the solution to the following indefinite integral.

$$\int \frac{\ln(x)}{\sqrt{x}}\,dx$$

(b) 'int_val2', which contains the solution to the following as a symbolic variable.

$$\int_1^3 \frac{\ln(x)}{\sqrt{x}}\,dx$$

(c) 'int_val3', which contains the solution to part (b) as a double-type number.

(From MATH118 PP1)

Built-in Function: diff

diff

Differentiate symbolic expression or function

Syntax

```
diff(F)
diff(F,var)
diff(F,n)
diff(F,var,n)
diff(F,var1,...,varN)
```

Description

`diff(F)` differentiates `F` with respect to the variable determined by `symvar(F,1)`.

`diff(F,var)` differentiates `F` with respect to the variable `var`.

`diff(F,n)` computes the `n`th derivative of `F` with respect to the variable determined by `symvar`.

`diff(F,var,n)` computes the `n`th derivative of `F` with respect to the variable `var`.

`diff(F,var1,...,varN)` differentiates `F` with respect to the variables `var1,...,varN`.

Example: Simple Differentiation 1

④ $\frac{d}{dx} e^x x^{-3} = e^x(x^{-3}) + e^x(-3x^{-4})$

$$= \frac{xe^x}{x^4} - \frac{3e^x}{x^4}$$

$$= \frac{e^x(x-3)}{x^4}$$

```
syms x
expr = exp(x) * x^(-3);
result = diff(expr)
```

result =

$$\frac{e^x}{x^3} - \frac{3\,e^x}{x^4}$$

③ $f(x) = \frac{x^3}{1-x^2}$

$$f'(x) = \frac{3x^2(1-x^2) - x^3(-2x)}{(1-x^2)^2}$$

$$= \frac{3x^2 - 3x^4 + 2x^4}{(1-x^2)^2}$$

$$= \frac{3x^2 - x^4}{(1-x^2)^2}$$

```
syms x
expr = x^3./(1-x^2);
result = diff(expr, x);
simplify(result)
```

ans =

$$-\frac{x^2\,(x^2-3)}{(x^2-1)^2}$$

Lecture note: Introduction to Differentiation (Courtesy of Brenda Lee)

Example: Simple Differentiation 2

② $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$

$$\ln y = \ln\left(x^{3/4}\sqrt{x^2+1}\right) - \ln(3x+2)^5$$

$$\ln y = \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

$$\frac{d}{dx}(\ln y) = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\,\frac{2x}{x^2+1} - \frac{5(3)}{3x+2}$$

$$\frac{1}{y}y' = \frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}$$

$$y' = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

Example: Simple Differentiation 2 (Continue)

```
syms x
expr_num = x^(3/4)*sqrt(x^2+1);
expr_den = (3*x + 2)^5;
expr = expr_num/expr_den;
result = diff(expr);
result_v1 = simplify(result)

result_v2 = x^(3/4)*sqrt(x^2+1)/(3*x + 2)^5;
result_v2 = result_v2*(3/(4*x) + x/(x^2+1)-15/(3*x+2))

x = 10;
double(subs(result_v1))
double(subs(result_v2))
```

```
result_v1 =
```

$$-\frac{39\,x^3 - 14\,x^2 + 51\,x - 6}{4\,x^{1/4}\,(3\,x+2)^6\,\sqrt{x^2+1}}$$

```
result_v2 =
```

$$\frac{x^{3/4}\,\sqrt{x^2+1}\,\left(\frac{x}{x^2+1} - \frac{15}{3\,x+2} + \frac{3}{4\,x}\right)}{(3\,x+2)^5}$$

```
ans = -4.9642e-07
ans = -4.9642e-07
```

Example in MATLAB Grader

Write a script to create a variable named:

(a) 'diff_val1', which contains the value of the derivative shown below:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{x^3}^{e^x} \csc(3t^2+1)\,dt$$

(b) 'diff_val2', which contains the value of the derivative shown in (a) when x = 5.

(c) 'diff_val3', which contains your answer from part (b) in expanded form.

(From MATH118 Tutorial 1)

Example: Matrix Operation

```
syms a11 a12 a13 a21 a22 a23 a31 a32 a33 a41 a42 a43
syms b11 b12 b13 b14 b21 b22 b23 b24 b31 b32 b33 b34
syms r1 r2 r3 r4
syms c1 c2 c3 c4

A = [a11 a12 a13; a21 a22 a23; a31 a32 a33; a41 a42 a43]
B = [b11 b12 b13 b14; b21 b22 b23 b24; b31 b32 b33 b34]
r = [r1 r2 r3 r4]
c = [c1; c2; c3; c4]
```

A =

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$$

B =

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{pmatrix}$$

r = $\begin{pmatrix} r_1 & r_2 & r_3 & r_4 \end{pmatrix}$

c =

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$$

Example: Matrix Operation (Continue)

val_a = $c_1 r_1 + c_2 r_2 + c_3 r_3$

vec_b = $\begin{pmatrix} c_1 r_1 & c_2 r_2 & c_3 r_3 & c_4 r_4 \end{pmatrix}$

vec_c =

$$\begin{pmatrix} a_{11} r_1 + a_{21} r_2 + a_{31} r_3 + a_{41} r_4 \\ a_{12} r_1 + a_{22} r_2 + a_{32} r_3 + a_{42} r_4 \\ a_{13} r_1 + a_{23} r_2 + a_{33} r_3 + a_{43} r_4 \end{pmatrix}$$

mat_d =

$$\begin{pmatrix} a_{11} + c_1 & a_{12} + c_1 & a_{13} + c_1 \\ a_{21} + c_2 & a_{22} + c_2 & a_{23} + c_2 \\ a_{31} + c_3 & a_{32} + c_3 & a_{33} + c_3 \\ a_{41} + c_4 & a_{42} + c_4 & a_{43} + c_4 \end{pmatrix}$$

mat_e =

$$\begin{pmatrix} b_{11} r_1 & b_{12} r_2 & b_{13} r_3 & b_{14} r_4 \\ b_{21} r_1 & b_{22} r_2 & b_{23} r_3 & b_{24} r_4 \\ b_{31} r_1 & b_{32} r_2 & b_{33} r_3 & b_{34} r_4 \end{pmatrix}$$

mat_f =

$$\begin{pmatrix} a_{11} c_1 + a_{12} c_2 + a_{13} c_3 \\ a_{21} c_1 + a_{22} c_2 + a_{23} c_3 \\ a_{31} c_1 + a_{32} c_2 + a_{33} c_3 \\ a_{41} c_1 + a_{42} c_2 + a_{43} c_3 \end{pmatrix}$$

mat_g =

$$\begin{pmatrix} a_{11} b_{11} + a_{12} b_{21} & a_{11} b_{12} + a_{12} b_{22} \\ a_{21} b_{11} + a_{22} b_{21} & a_{21} b_{12} + a_{22} b_{22} \end{pmatrix}$$

```
val_a = r(1:3)*c(1:3)
vec_b = r.*transpose(c)
vec_c = transpose(r * A)
mat_d = A + [c c c]
mat_e = B .* [r ; r ; r]
mat_f = A * c(1:3)
mat_g = A(1:2,1:2) * B(1:2, 1:2)
```

Example: det (Matrix Operation: Determinant)

det

Determinant of symbolic matrix

Syntax

```
B = det(A)
B = det(A,'Algorithm','minor-expansion')
```

Description

`B = det(A)` returns the determinant of the square matrix

`B = det(A,'Algorithm','minor-expansion')` uses th

3.1 What is a Determinant?

Let's start by defining a determinant simply as a computed value for a square matrix that involves all of its elements. Typically we start by defining the determinant computation for a 2×2 matrix. The matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

has a determinant defined as

$$\det(A) = |A| = a_{11}a_{22} - a_{21}a_{12} = ad - cb$$

```
syms a b c d
A = [a b; c d];
det_A = det(A)
```

det_A = $a\,d - b\,c$

Example: How to Validate the Properties of Determinants

3.4 Properties of Determinants

If A & B are $n \times n$ matrices, and k is a scalar, then the following properties hold:

Matrix Multiplication

$$\det(AB) = \det(A)\det(B)$$

Scalar Multiplication

$$\det(kA) = k^n \det(A)$$

Transpose of Matrix

$$\det(A^T) = \det(A)$$

Example: Properties of Determinants (Continue)

```
syms a11 a12 a21 a22
syms b11 b12 b21 b22
syms k

A = [a11 a12; a21 a22];
B = [b11 b12; b21 b22];

det_AB = simplify(det(A*B))
det_A_B = simplify(det(A)*det(B))

det_kA = simplify(det(k*A))
det_k2_A = simplify(k^2*det(A))

det_AT = det(transpose(A))
det_A = det(A)
```

det_AB = $(a_{11}\, a_{22} - a_{12}\, a_{21})\ (b_{11}\, b_{22} - b_{12}\, b_{21})$

det_A_B = $(a_{11}\, a_{22} - a_{12}\, a_{21})\ (b_{11}\, b_{22} - b_{12}\, b_{21})$

det_kA = $k^2\ (a_{11}\, a_{22} - a_{12}\, a_{21})$

det_k2_A = $k^2\ (a_{11}\, a_{22} - a_{12}\, a_{21})$

det_AT = $a_{11}\, a_{22} - a_{12}\, a_{21}$

det_A = $a_{11}\, a_{22} - a_{12}\, a_{21}$

Example in MATLAB Grader

Write a script that creates a variable named:

(a) 'det_val1', which creates a symbolic expression for the determinant of the matrix shown below, where a, b and c are any real numbers.

$$\begin{bmatrix} 2 & -5 & 7 \\ a & 3 & 9 \\ -8 & b & c \end{bmatrix}$$

(b) 'det_val2', which is the value of the expression created in part (a) when a = 3, b = 6 and c = -2. 'det_val2' is a symbolic expression.

(c) 'det_va3', which contains the solution to part (b) as a double-type number.

Slide Credits and References

- Stormy Attaway, 2018, Matlab: A Practical Introduction to Programming and Problem Solving, 5th edition
- Lecture slides for “Matlab: A Practical Introduction to Programming and Problem Solving”
- Holly Moore, 2018, MATLAB for Engineers, 5th edition