VE230 Mid Review Part I: Vector Analysis

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Overview

The vector analysis contains four parts:

- Vector review
- Coordinates
- Integral
- Fields

Vectors

dot product:

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|cos\theta_{\vec{A}\vec{B}}$$

- Commutative: $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
- Distributive: $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$
- Not associative: $\vec{A} \cdot (\vec{B} \cdot \vec{C}) \neq (\vec{A} \cdot \vec{B}) \cdot \vec{C}$
  e.g. $(\vec{a_x} \cdot \vec{a_y}) \cdot \vec{a_z} \neq \vec{a_x} \cdot (\vec{a_y} \cdot \vec{a_z})$
- For the three edges A, B, C in a triangle,
  $C^2 = A^2 + B^2 - 2ABcos(\theta_{A,B})$

Vectors

cross product:

$$\vec{A} \times \vec{B} = \vec{a_n}||\vec{A}||\vec{B}|sin\theta_{\vec{A}\vec{B}}|$$

- The cross product is always perpendicular to both $\vec{A}, \vec{B}$, the direction follows right hand rule.
- Not Commutative: $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$ (have opposite directions).
- Distributive: $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$
- Not associative: $\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$
  e.g. $\vec{a_x} \times (\vec{a_x} \times \vec{a_y}) = \vec{a_x} \times \vec{a_z} = -\vec{a_y}$,
  $(\vec{a_x} \times \vec{a_x}) \times \vec{a_y} = 0 \neq -\vec{a_y}$

Vectors

Some useful rules:

- $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = Volume$
- BAC-CAB: $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

Coordinates

Three basis (u_1, u_2, u_3): number of linearly independent basis = dimension of the space. For the three types of coordinates we discuss, u_i is orthogonal to each other.
For arbitrary vector $\vec{A}$:

$$\vec{A} = \vec{a_{u1}} A_{u1} + \vec{a_{u2}} A_{u2} + \vec{a_{u3}} A_{u3}$$

,
Norm of $\vec{A}$:

$$|\vec{A}| = \sqrt{A_{u1}^2 + A_{u2}^2 + A_{u3}^2}$$

For a differential length dl,

$$dl = \vec{a_{u1}}(h_1 du_1) + \vec{a_{u2}}(h_2 du_2) + \vec{a_{u3}}(h_3 du_3)$$

h_i is called metric coefficient.

Coordinates

differential volume:

$$dv = h_1 h_2 h_3 du_1 du_2 du_3$$

differential area vector with a direction normal to the surface,

$$d\vec{s} = \vec{a_n} ds$$

differential area ds_1 normal to the unit vector $\vec{a_{u1}}$.

Cartesian Coordinates

- $$(u_1, u_2, u_3) = (x, y, z)$$
- Right hand rule:
$$\vec{a_x} \times \vec{a_y} = \vec{a_z}$$
- $$\vec{A} = \vec{a_x} A_x + \vec{a_y} A_y + \vec{a_z} A_z$$
, where $\vec{a_i}$ is the basis for i-axis.

Cartesian Coordinates

- dot product and cross product:

$$\vec{a_x} \cdot \vec{a_x} = 1, \vec{a_x} \times \vec{a_y} = \vec{a_z}.$$

- differential length:

$$\vec{dl} = \vec{a_x}dx + \vec{a_y}dy + \vec{a_z}dz \tag{1}$$

- differential area:

$$ds_x = dydz$$

, as $h_1 = h_2 = h_3 = 1$,
(ds_x is the surface perpendicular to the x-axis, the forms for other surfaces follow the same pattern).

- differential volume:

$$dv = dxdydz$$

Cylindrical Coordinate

- $$(u_1, u_2, u_3) = (r, \phi, z)$$

  Claim: as a_r can change its direction in the x-y plane, vectors in x-y plane could be represented simply by $\vec{a_r}$. Thus, all vectors in cylindrical coordinate could be represented by $\vec{a_r}$ and $\vec{a_z}$.
- Right hand rule:

  $$\vec{a_r} \times \vec{a_\phi} = \vec{a_z}$$

- $$\vec{A} = \vec{a_r} Ar + \vec{a_\phi} A_\phi + \vec{a_z} A_z$$

- differential length:

  $$d\vec{l} = \vec{a_r} dr + \vec{a_\phi} r d\phi + \vec{a_z} dz \tag{2}$$

  , as $h_1 = 1, h_2 = r, h_3 = 1$

Cylindrical Coordinate

- differential area:

$$ds_r = r d\phi dz$$

- differential volume:

$$dv = r dr d\phi dz$$

- From cylindrical coordinate to Cartesian coordinate: represent A_x by the quantities in cylindrical coordinate.The same applies to A_y.

Cylindrical Coordinate

Conversion of quantities between Cartesian coordinate and Cylindrical coordinate:

1.
$$\begin{cases} x = rcos\phi \\ y = rsin\phi \\ z = z \end{cases}$$

2.
$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \phi = arctan\frac{y}{x} \\ z = z \end{cases}$$

You can try to write the conversion between dx, dy, dz and $dr, d\phi, dz$.

Cylindrical Coordinate

The conversion between dx, dy, dz and $dr, d\phi, dz$:

$$\begin{array}{cc} \left[\begin{matrix} A_r \cos\phi - A_\phi \sin\phi \\ A_r \sin\phi + A_\phi \cos\phi \\ A_z \end{matrix}\right] & \begin{matrix} A_x \\ A_y \\ A_z \end{matrix} \end{array}$$

Spherical Coordinate

- Figure for Spherical Coordinate. Notice the position of ϕ, θ.
- $$(u_1, u_2, u_3) = (R, \theta, \phi)$$
- Right hand rule:
$$\vec{a_R} \times \vec{\theta} = \vec{\phi}$$
- $$\vec{A} = \vec{a_R} A_R + \vec{a_\theta} A_\theta + \vec{a_\phi} A_\phi$$
- differential length:
$$\vec{dl} = \vec{a_R} dR + \vec{a_\theta} R d\theta + \vec{a_\phi} R sin\theta d\phi \tag{3}$$
, as $h_1 = 1, h_2 = R, h_3 = R sin\theta$.
- differential area:
$$ds_R = R^2 sin\theta d\theta d\phi$$

Spherical Coordinate

- differential volume:

$$dv = R^2 sin\theta dR d\theta d\phi$$

- conversion of quantities between Cartesian coordinate and Spherical coordinate:
  1. $$\begin{cases} x = Rsin\theta cos\phi \\ y = Rsin\theta sin\phi \\ z = Rcos\theta \end{cases}$$
  2. $$\begin{cases} R = \sqrt{x^2 + y^2 + z^2} \\ \theta = arctan\frac{\sqrt{x^2+y^2}}{z} \\ \phi = arctan\frac{y}{x} \end{cases}$$
- From Spherical coordinate to Cartesian coordinate: represent A_x by the quantities in Spherical coordinate; write the formula in the form of matrix. (Similar to cylindrical coordinate).

Integral

Line integrals

When integrated along a certain differential length, use equations we introduced last class to convert differential length to integrable quantities with regard to different coordinates. For example, in Cartesian coordinates, when integrating on $d\vec{l}$, you should convert $d\vec{l}$ into the form

$$d\vec{l} = \vec{a_x}dx + \vec{a_y}dy + \vec{a_z}dz.$$

Integral

A small example

Suppose $\vec{F} = \vec{a_x}xy - \vec{a_y}2x$, calculate its integral along $y = x^2$ in the range [0,1].

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Suppose $\vec{F} = \vec{a_x}xy - \vec{a_y}2x$, calculate its integral along $y = x^2$ in the range [0,1].

Answer

$$
\begin{aligned}
W &= \int \vec{F} \cdot d\vec{l} \\
&= \int (\vec{a_x}xy - \vec{a_y}2x) \cdot (\vec{a_x}dx + \vec{a_y}dy) \\
&= \int xydx - 2xdy \\
&= \int_0^1 (x * x^2 dx - 2x * 2xdx) = -\frac{13}{12}
\end{aligned}
$$

Integral

Flux: the integrals of the vector field rush out the surface.

$$\int_S \mathbf{A} \cdot d\mathbf{s},$$

Flux of vector field **A** flowing through the area S

$$d\mathbf{s} = \mathbf{a}_n\, ds$$

1. If S is a closed surface → $\mathbf{a}_n$ is in the outward direction
2. If S is an open surface → $\mathbf{a}_n$ is decided by right-hand rule (thumb)

Scalar field and vector field

Scalar field

The intensity of the field is a scalar value, for example, gravitation potential field.

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Connection

The vector field can be the gradient of a scalar field. But it has some conditions: what conditions?

Gradient of a scalar field

- $$\nabla V = \vec{a_n}\frac{dV}{dn}$$
- ∇V at certain point is a vector. Think about how can we represent ∇ singly?
- $$dV = (\nabla V)\cdot d\vec{l}$$
  : the space rate of increase of V in the $\vec{a_l}$ direction is equal to the projection (the component) of the gradient of V in that direction.
- $$\nabla V = \vec{a_{u1}}\frac{\partial V}{h_1\partial u_1} + \vec{a_{u2}}\frac{\partial V}{h_2\partial u_2} + \vec{a_{u3}}\frac{\partial V}{h_3\partial u_3}$$
  , when V is taken off:
  $$\nabla \equiv \vec{a_{u1}}\frac{\partial}{h_1\partial u_1} + \vec{a_{u2}}\frac{\partial}{h_2\partial u_2} + \vec{a_{u3}}\frac{\partial}{h_3\partial u_3}$$

Divergence of a vector field

- divergence of a vector field $\vec{A}$ at a point $div\vec{A}$ as the net outward flux of $\vec{A}$ per unit volume as the volume about the point tends to zero:

$$div\vec{A} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A}\,\mathrm{d}s}{\Delta v} \tag{4}$$

- source: net positive divergence; sink: net negative divergence. zero divergence: no source/sink.
- $div\vec{A}$ at certain point is a scalar.

Divergence of a vector field

- For Cartesian coordinate,

$$div\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

-

$$\nabla \cdot \vec{A} \equiv div\vec{A}$$

-

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3}[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_1 h_3 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)]$$

Divergence Theorem

- $$\int_V \nabla \cdot \vec{A} dv = \oint_S \vec{A} \cdot d\vec{s}$$
  , the volume integral of the divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume.
- A most famous theorem in the physics - Gauss's law can be deduced by divergence theorem very easily. Thus, the divergence theorem will be quite fundamental in this course. Make sure you understand it.

Curl of a vector field

- $$curl\vec{A} \equiv \nabla \times \vec{A} = \lim_{\Delta s \to 0} \frac{1}{\Delta s}[\vec{a_n} \oint_C \vec{A} \cdot d\vec{l}]_{max}$$

  : the curl of a vector field $\vec{A}$, denoted by $curl\vec{A}$ or $\nabla \times \vec{A}$, is a vector whose magnitude is the maximum net circulation of $\vec{A}$ per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented to make the net circulation maximum. (Right hand rule defines the positive normal to an area).
- $\nabla \times \vec{A}$ in a general coordinate:

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \vec{a_{u1}} h_1 & \vec{a_{u2}} h_2 & \vec{a_{u3}} h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

- curl-free vector field ($\nabla \times \vec{A} = 0$): **irrotational** or **conservative field**, like the gravitation potential field.

Stokes's Theorem

- $$\int_S (\nabla \times \vec{A}) d\vec{s} = \oint_C \vec{A} \cdot d\vec{l}$$

  : the surface integral of the curl of a vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface.

Other Identities

I

- $$\nabla \times (\nabla V) \equiv 0$$

  : the curl of the gradient of any scalar field is identically zero.
- Another interpretation: If a vector field is curl-free, it can be expressed as the gradient of a scalar field.
- Since a curl-free vector field is irrotational or conservative, an irrotational/conservative vector field can always be expressed as the gradient of a scalar field.

II

- $$\nabla \cdot (\nabla \times \vec{A}) \equiv 0$$

  : the divergence of the curl of any vector field is identically zero.
- Another interpretation: if a vector field is divergenceless, it can be expressed as the curl of another vector field.
- Divergenceless field is called solenoidal field, which will be further discussed in later classes.

Other Identities

Laplacian in Cartesian Coordinates

$$\nabla^2 V = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Other Identities

Laplacian in Cartesian Coordinates

$$\nabla^2 V = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Helmholtz's Theorem

Any vector in 3D can be decomposable into a sum of the following vector fields. A vector field (vector point function) is determined to within an additive constant if both its divergence and its curl are specified everywhere.

$$F = -\nabla V + \nabla \times A.$$

Other useful vector properties

$$\nabla(\psi\phi) = \phi\nabla\psi + \psi\nabla\phi$$

$$\nabla\cdot(\psi\mathbf{A}) = \psi\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla\psi$$

$$\nabla\times(\psi\mathbf{A}) = \psi(\nabla\times\mathbf{A}) + \nabla\psi\times\mathbf{A}$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0$$
$$\nabla\times(\nabla\psi) = \mathbf{0}$$

$$\nabla\cdot(\nabla\psi) = \nabla^2\psi \text{ (scalar Laplacian)}$$
$$\nabla(\nabla\cdot\mathbf{A}) - \nabla\times(\nabla\times\mathbf{A}) = \nabla^2\mathbf{A} \text{ (vector Laplacian)}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A})$$