VE230 RC slides Week 5

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Today's contents:

1. Revisit of Boundary Value Problems
2. Current Density and Ohm's Law
3. Electromotive Force and Kirchhoff's Voltage Law
4. Equation of Continuity and Kirchhoff's Current Law

Revisit of Boundary Value Problems

Basic form

Laplace Equation:

$$\nabla^2 V = 0.$$

And some of the other boundary conditions.

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And some of the other boundary conditions.

Solution: (three situations and three formats)

- $X = Ae^{-k_1 x} + Be^{k_1 x}, \quad Y = Ce^{-k_2 x} + De^{k_2 x}$
- $X = A\sin(k_1 x) + B\cos(k_1 x), \quad Y = C\sin(k_2 x) + D\cos(k_2 x)$
- $X = a + bx, \quad Y = c + dx$

How to solve???

1. First, write out the expansion of the Laplace Equation, i.e.,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

2. According to the boundary conditions given, judge whether V is independent of the variable.
3. If there is only one variable left, then the solution is the third format, i.e., $X = a + bx, \quad Y = c + dx$. Then use the boundary conditions to find the parameter of the solution.
4. If there is more than one variable involved, then tried the first format and second format to find if they can satisfy the boundary conditions. If one of the formats can be found that satisfies the boundary conditions, then stop, because of the uniqueness theorem.

Example

P.4–23 Two infinite insulated conducting planes maintained at potentials 0 and V_0 form a wedge-shaped configuration, as shown in Fig. 4–24. Determine the potential distributions for the regions: (a) $0 < \phi < \alpha$, and (b) $\alpha < \phi < 2\pi$.

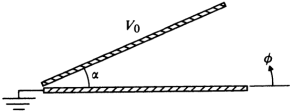

FIGURE 4–24
Two infinite insulated conducting planes maintained at constant potentials (Problem P.4–23).

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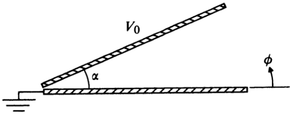


FIGURE 4–24
Two infinite insulated conducting planes maintained at constant potentials (Problem P.4–23).

P.4-23 Solution: $V(\phi) = A_0\phi + B_0$.

a) B.C. ①: $V(0) = 0 \longrightarrow B_0 = 0$.
B.C. ②: $V(\alpha) = V_0 = A_0\alpha \longrightarrow A_0 = \frac{V_0}{\alpha}$. $\Big\}$ $\therefore V(\phi) = \frac{V_0}{\alpha}\phi$, $0 \le \phi \le \alpha$.

b) B.C. ①: $V(\alpha) = V_0 = A_1\alpha + B_1$
B.C. ②: $V(2\pi) = 0 = 2\pi A_1 + B$ $\Big\} \longrightarrow A_1 = -\frac{V_0}{2\pi-\alpha}$, $B_1 = \frac{2\pi V_0}{2\pi-\alpha}$.

$\therefore V(\phi) = \frac{V_0}{2\pi-\alpha}(2\pi-\phi)$, $\alpha \le \phi \le 2\pi$.

Types of electric currents caused by the motion of free charges:

1. **conduction currents**: drift motion of conduction electrons and/or holes in conductors/semiconductors.
2. electrolytic currents: migration of positive and negative ions.
3. convenction currents: motion of electrons and/or ions in a vacuum.

Current Density and Ohm's Law

$$I = \int_S \vec{J} \cdot d\vec{s} \quad (A)$$

where $\vec{J}$ is the volume current density or current density, defined by

$$\vec{J} = Nq\vec{u} \quad (A/m^2)$$

,

where N is the number of charge carriers per unit volume, each of charges q moves with a velocity $\vec{u}$.
Since Nq is the free charge per unit volume, by $\rho = Nq$, we have:

$$\vec{J} = \rho\vec{u} \quad (A/m^2)$$

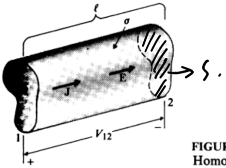
Current Density and Ohm's Law

For conduction currents,

$$\vec{J} = \sigma \vec{E} \quad (A/m^2)$$

where $\sigma = \rho_e \mu_e$ is conductivity, a macroscopic constitutive parameter of the medium. $\rho_e = -Ne$ is the charge density of the drifting electrons and is negative. $\vec{u} = -\mu_e \vec{E} \quad (m/s)$ where μ_e is the electron mobility measured in $(m^2/V \cdot s)$.
Materials where $\vec{J} = \sigma \vec{E} \quad (A/m^2)$ holds are called ohmic media. The form can be referred as the point form of Ohm's law.
Derivation of voltage-current relationship of a piece of homogeneous material by the point form of Ohm's law.

Current Density and Ohm's Law



σ, S

FIGURE 5–3
Homogeneous conductor with a constant cross section.

Thus, the resistance is defined as

$$R = \frac{l}{\sigma S} \quad (\Omega)$$

where l is the length of the homogeneous conductor, S is the area of the uniform cross section.

Current Density and Ohm's Law

The conductance G (reciprocal of resistance), is defined by

$$G = \frac{1}{R} = \sigma \frac{S}{l} \quad (S)$$

1. Resistance in series:

$$R_{sr} = R_1 + R_2$$

2. Resistance in parallel:

$$\frac{1}{R_{||}} = \frac{1}{R_1} + \frac{1}{R_2}$$

, or

$$G_{||} = G_1 + G_2$$

Electromotive Force and Kirchhoff's Voltage Law

A steady current cannot be maintained in the same direction in a closed circuit by an electrostatic field, which is:

$$\oint_C \frac{1}{\sigma}\vec{J} \cdot d\vec{l} = 0$$

Kirchhoff's voltage law: around a closed path in an electric circuit, the algebraic sum of the emf's (voltage rises) is equal to the algebraic sum of the voltage drops across the resistance, which is:

$$\sum_j V_j = \sum_k R_k I_k \quad (V)$$

Equation of Continuity and Kirchhoff's Current Law

Equation of continuity:

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \quad (A/m^3)$$

,
where ρ is the volume charge density.
For steady currents, as $\partial\rho/\partial t = 0$, $\nabla \cdot \vec{J} = 0$. By integral, we have Kirchhoff's current law, stating that the algebraic sum of all the currents flowing out of a junction in an electric circuit is zero:

$$\sum_j I_j = 0$$

Equation of Continuity and Kirchhoff's Current Law

For a simple medium conductor, the volume charge density ρ can be expressed as:

$$\rho = \rho_0 e^{-(\rho/\epsilon)t} \quad (C/m^3)$$

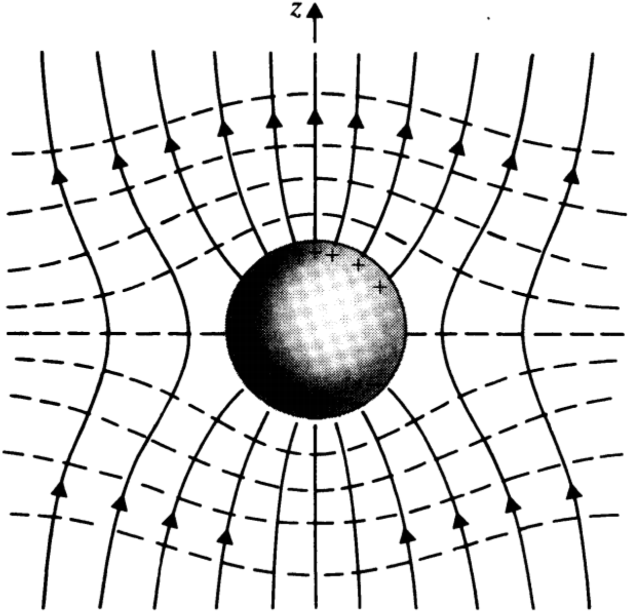
where ρ_0 is the initial charge density at $t = 0$. The equation implies that the charge density at a given location will decrease with time exponentially.
Relaxation time: an initial charge density ρ_0 will decay to $1/e$ or 36.8% of its original value:

$$\tau = \frac{\epsilon}{\sigma} \quad (s)$$

Assignment Part

P.4–28 Rework Example 4–10, assuming that $V(b, \theta) = V_0$ in Eq. (4–155a).

EXAMPLE 4–10 An uncharged conducting sphere of radius b is placed in an initially uniform electric field $\mathbf{E}_0 = \mathbf{a}_z E_0$. Determine (a) the potential distribution $V(R, \theta)$, and (b) the electric field intensity $\mathbf{E}(R, \theta)$ after the introduction of the sphere.


z
+
+
+
+

$$V_n(R, \theta) = [A_n R^n + B_n R^{-(n+1)}]P_n(\cos\theta). \qquad (4\text{–}154)$$

a) To determine the potential distribution $V(R, \theta)$ for $R \geq b$, we note the following boundary conditions:

$$V(b, \theta) = 0^\dagger \qquad (4\text{–}155a)$$

$$V(R, \theta) = -E_0 z = -E_0 R\cos\theta, \qquad \text{for } R \gg b. \qquad (4\text{–}155b)$$

Equation (4–155b) is a statement that the original $\mathbf{E}_0$ is not disturbed at points very far away from the sphere. By using Eq. (4–154) we write the general solution

as

$$V(R, \theta) = \sum_{n=0}^{\infty} [A_n R^n + B_n R^{-(n+1)}]P_n(\cos\theta), \qquad R \geq b. \qquad (4\text{–}156)$$

However, in view of Eq. (4–155b), all A_n except A_1 must vanish, and $A_1 = -E_0$. We have, from Eq. (4–156) and Table 4–2,

$$V(R, \theta) = -E_0 R P_1(\cos\theta) + \sum_{n=0}^{\infty} B_n R^{-(n+1)} P_n(\cos\theta)$$

$$= B_0 R^{-1} + (B_1 R^{-2} - E_0 R)\cos\theta + \sum_{n=2}^{\infty} B_n R^{-(n+1)} P_n(\cos\theta), \qquad R \geq b. \qquad (4\text{–}157)$$

P. 4-28 Starting from Eq. (4-157) and applying the b.c. $V(b,\theta)=V_0$:

$$V_0 = \frac{B_0}{b} + \left(\frac{B_1}{b^2} - E_0 b\right)\cos\theta - \sum_{n=2}^{\infty} B_n b^{-(n+1)} P_n(\cos\theta), \quad R \geq b.$$

$$\longrightarrow B_0 = bV_0, \quad B_1 = E_0 b^3; \quad B_n = 0 \text{ for } n \geq 2.$$

$$\therefore V(R,\theta) = \frac{b}{R}V_0 - E_0\left[1-\left(\frac{b}{R}\right)^3\right]R\cos\theta, \quad R \geq b.$$

$$\bar{E}(R,\theta) = \bar{a}_R\left\{\frac{bV_0}{R^2} + E_0\left[1+2\left(\frac{b}{R}\right)^3\right]\cos\theta\right\} - \bar{a}_\theta E_0\left[1-\left(\frac{b}{R}\right)^3\right]R\sin\theta, \ R \geq b.$$

$$\rho_s = \epsilon_0 E_R\Big|_{R=b} = \epsilon_0\frac{V_0}{b} + 3\epsilon_0 E_0\cos\theta.$$

P.5–6 Lightning strikes a lossy dielectric sphere—$\epsilon = 1.2\,\epsilon_0$, $\sigma = 10$ (S/m)—of radius 0.1 (m) at time $t = 0$, depositing uniformly in the sphere a total charge 1 (mC). Determine, for all t,

a) the electric field intensity both inside and outside the sphere,

b) the current density in the sphere.

P.5–7 Refer to Problem P.5–6.

a) Calculate the time it takes for the charge density in the sphere to diminish to 1% of its initial value.

b) Calculate the change in the electrostatic energy stored in the sphere as the charge density diminishes from the initial value to 1% of its value. What happens to this energy?

c) Determine the electrostatic energy stored in the space outside the sphere. Does this energy change with time?

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P. 5-7 a) $e^{-(\sigma/\epsilon)t} = \frac{\rho}{\rho_0} = 0.01 \longrightarrow t = \frac{\ln 100}{(\sigma/\epsilon)} = 4.88\times 10^{-12}\,(s) = 4.88\,(ps).$

b) $W_i = \frac{\epsilon}{2}\int_{V'} E_i^2\, dv' = \frac{2\pi\rho_0 b^3}{45\epsilon} e^{-2(\sigma/\epsilon)t} = (W_i)_0 \left[e^{-(\sigma/\epsilon)t}\right]^2.$

$\therefore \frac{W_i}{(W_i)_0} = \left[e^{-(\sigma/\epsilon)t}\right]^2 = 0.01^2 = 10^{-4}.$ Energy dissipated as heat loss.

c) Electrostatic energy stored outside the sphere $W_o = \frac{\epsilon_0}{2}\int_b^\infty E_o^2\, 4\pi R^2 dR = \frac{Q_0^2}{8\pi\epsilon_0 b} = 45\ (kJ)$

— Constant.

P.5–10 The space between two parallel conducting plates each having an area S is filled with an inhomogeneous ohmic medium whose conductivity varies linearly from σ_1 at one plate ($y = 0$) to σ_2 at the other plate ($y = d$). A d-c voltage V_0 is applied across the plates as in Fig. 5–11. Determine

a) the total resistance between the plates,

b) the surface charge densities on the plates,

c) the volume charge density and the total amount of charge between the plates.

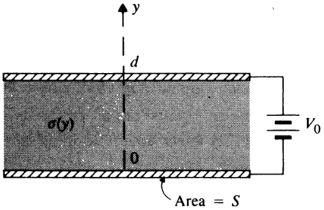
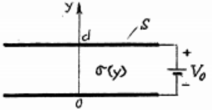


FIGURE 5–11
Inhomogeneous ohmic medium with conductivity $\sigma(y)$ (Problem P.5–10).

P. 5-10



$$\sigma(y) = \sigma_1 + (\sigma_2 - \sigma_1)\frac{y}{d}$$

a) Neglecting fringing effect and assuming a current density

$$\bar{J} = -\bar{a}_y J_0 \longrightarrow \bar{E} = \frac{\bar{J}}{\sigma} = -\bar{a}_y \frac{J_0}{\sigma(y)}.$$

$$V_0 = -\int_0^d \bar{E}\cdot\bar{a}_y\,dy = \int_0^d \frac{J_0\,dy}{\sigma_1 + (\sigma_2-\sigma_1)\frac{y}{d}} = \frac{J_0 d}{\sigma_2-\sigma_1}\ln\frac{\sigma_2}{\sigma_1}.$$

$$R = \frac{V_0}{I} = \frac{V_0}{J_0 S} = \frac{d}{(\sigma_2-\sigma_1)S}\ln\frac{\sigma_2}{\sigma_1}.$$

b) $(\rho_s)_u = \epsilon_0 E_y(d) = \frac{\epsilon_0 J_0}{\sigma_2} = \frac{\epsilon_0(\sigma_2-\sigma_1)V_0}{\sigma_2 d \ln(\sigma_2/\sigma_1)}$ on upper plate,

$(\rho_s)_l = -\epsilon_0 E_y(0) = -\frac{\epsilon_0 J_0}{\sigma_1} = -\frac{\epsilon_0(\sigma_2-\sigma_1)V_0}{\sigma_1 d \ln(\sigma_2/\sigma_1)}$ on lower plate.

c) $\rho = \bar{\nabla}\cdot\bar{D} = \frac{d}{dy}(\epsilon_0 E) = -\epsilon_0 J_0 \frac{d}{dy}\left[\frac{1}{\sigma_1 + (\sigma_2-\sigma_1)y/d}\right] = \epsilon_0 J_0 \frac{(\sigma_2-\sigma_1)/d}{[\sigma_1 + (\sigma_2-\sigma_1)y/d]^2}.$

P.5–16 Determine the resistance between two concentric spherical surfaces of radii R_1 and R_2 $(R_1 < R_2)$, assuming that a material of conductivity $\sigma = \sigma_0(1 + k/R)$ fills the space between them. (*Note:* Laplace's equation for V does not apply here.)

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P.5-16 Assume a current I between the spherical surfaces.

$$\bar{J} = \bar{a}_R \frac{I}{4\pi R^2} = \sigma \bar{E}.$$

$$V_0 = -\int_{R_2}^{R_1} \bar{E} \cdot d\bar{R} = \int_{R_1}^{R_2} \frac{I\,dR}{4\pi\sigma R^2} = \frac{I}{4\pi\sigma_0} \int_{R_1}^{R_2} \frac{dR}{R^2(1+k/R)}$$

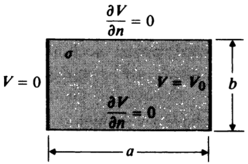
$$= \frac{I}{4\pi\sigma_0} \int_{R_1}^{R_2} \frac{1}{k}\left(\frac{1}{R} - \frac{1}{R+k}\right) dR = \frac{I}{4\pi\sigma_0 k} \ln \frac{R_2(R_1+k)}{R_1(R_2+k)}.$$

$$R = \frac{V_0}{I} = \frac{1}{4\pi\sigma_0 k} \ln \frac{R_2(R_1+k)}{R_1(R_2+k)}.$$

P.5–22 Assume a rectangular conducting sheet of conductivity σ, width a, and height b. A potential difference V_0 is applied to the side edges, as shown in Fig. 5–14. Find

a) the potential distribution,

b) the current density everywhere within the sheet. (*Hint:* Solve Laplace's equation in Cartesian coordinates subject to appropriate boundary conditions.)



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A conducting sheet (Problem P.5–22).

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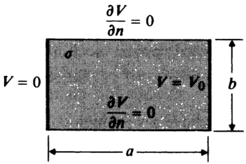

FIGURE 5–14
A conducting sheet (Problem P.5–22).

P.5-22 Specified boundary conditions can be satisfied by solutions of Laplace's equation with zero separation constants: $k_x = k_y = 0$. $X(x) = A_0 x + B_0$; $Y(y) = C_0 y + D_0$.

$B_0 = C_0 = 0$. $V(x) = A_0 D_0 x$

a) At $x = a$, $V(a) = V_0 = A_0 D_0 a \longrightarrow A_0 D_0 = \frac{V_0}{a}$

$$\therefore V = \frac{V_0}{a} x$$

b) $\bar{E} = -\bar{\nabla} V = -\bar{a}_x \frac{V_0}{a} \longrightarrow \bar{J} = \sigma \bar{E} = -\bar{a}_x \frac{\sigma V_0}{a}$.