VE230 RC slides Week 6

han.fang

July 18, 2021

Materials to cover

Today's content

- Fundamental Postulates
- Vector Magnetic Potential
- Magnetization and Equivalent Current Densitites
- Magnetic Field Intensity, Relative Permeability and Magnetic circuit
- Boundary Conditions for Magnetostatic Fields

Fundamental Postulates

differential form	integral form	Comment
$\nabla \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot d\vec{s} = 0$	$\vec{B}$ is solenoidal, Conservation of magnetic flux: no isolated magnetic charges, no magnetic flow source, flux lines always close upon themselves
$\nabla \times \vec{B} = \mu_0 \vec{J}$	$\oint_C \vec{B} \cdot dl = \mu_0 I$	Ampere's circuital law

where μ_0 is the permeability of free space, $\mu_0 = 4\pi \times 10^{-7} \quad H/m$.

Fundamental Postulates

Because the divergence of the curl of any vector field is zero,

$$\nabla \cdot \vec{J} = \frac{\nabla \cdot (\nabla \times \vec{B})}{\mu_0} = 0$$

which is consistent with the formula

$$\nabla \cdot \vec{J} = \frac{\partial \rho}{\partial t} = 0$$

for steady current.

Example

An infinitely long, straight conductor with a circular cross section of radius b carries a steady current I. Determine the magnetic flux density both inside and outside the conductor.

Example

An infinitely long, straight conductor with a circular cross section of radius b carries a steady current I. Determine the magnetic flux density both inside and outside the conductor.

Answer

Ampere's circuital law: $\oint_C \vec{B} \cdot dl = \mu_0 I$.

Vector Magnetic Potential

As $\nabla \cdot \vec{B} = 0$, $\vec{B}$ is solenoidal, thus could be expressed as:

$$\vec{B} = \nabla \times \vec{A} \quad (T) \tag{1}$$

, where $\vec{A}$ is called the vector magnetic potential.

Magnetic flux Φ:

$$\Phi = \int_S \vec{B} \cdot d\vec{s} = \oint_C \vec{A} \cdot dl$$

For Eq 1, by doing Laplacian transformation and assume $\nabla \cdot \vec{A} = 0$, vector Poisson's equation:

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

The solution is then

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\vec{J}}{\vec{R}} dv' \tag{2}$$

Vector Magnetic Potential

For a thin wire with cross-sectional area S, $dv' = Sdl'$, current flow is entirely along the wire, we then have

$$\vec{J}dv' = JSdl' = Idl'$$

Thus, Eq 2 becomes:

$$\vec{A} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{dl'}{R}$$

Based on this form and properties of differentiation, we can get Biot-Savart law:

$$\vec{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\vec{l'} \times \vec{a_R}}{R^2}$$

Vector Magnetic Potential

The formula for Biot-Savart law could also be written as:

$$\vec{B} = \oint_{C'} d\vec{B}$$

and

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\left(\frac{d\vec{l'} \times \vec{a_R}}{R^2}\right) = \frac{\mu_0 I}{4\pi}\left(\frac{d\vec{l'} \times \vec{R}}{R^3}\right)$$

Comment: Biot-Savart law is more difficult to apply than Ampere's circuital law, but Ampere's circuital law cannot be used to determine $\vec{B}$ from I in a circuit if a closed path cannot be found where $\vec{B}$ has a constant magnitude.

Scalar magnetic potential

If a region is current free, i.e. $\vec{J} = 0$,

$$\nabla \times \vec{B} = 0$$

,
thus $\vec{B}$ can be expressed as the gradient of a scalar field.
Assume

$$\vec{B} = -\mu_0 \nabla V_m \tag{3}$$

, where V_m is called the scalar magnetic potential, the negative sign is conventional, μ_0 is the permeability of free space.
Thus, between two points P_1, P_2,

$$V_{m2} - V_{m1} = -\int_{P_1}^{P_2} \frac{1}{\mu_0} \vec{B} \cdot dl$$

Scalar magnetic potential

If there were magnetic charges with a volume density ρ_m in a volume V', we could find V_m from:

$$V_m = \frac{1}{4\pi}\int_{V'}\frac{\rho_m}{R}dv'$$

Then we could obtain $\vec{B}$ by Eq 3. Note that this is only a mathematical model, isolated magnetic charges have never been found.

For a bar magnet the fictitous magnetic charges $+q_m, -q_m$ assumed to be separated by d (magnetic dipole), the scalar magnetic potential V_m is given by:

$$V_m = \frac{\vec{m}\cdot\vec{a_R}}{4\pi R^2}$$

Magnetization and Equivalent Current Densitites

Define magnetization vector, $\vec{M}$, as

$$\vec{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \vec{m_k}}{\Delta v}$$

, which is the volume density of magnetic dipole moment,

- The effect of magnetization is vector is equivalent to both
  - a volume current density:
  $$\vec{J_m} = \nabla \times \vec{M}$$
  - a surface current density:
  $$\vec{J_{ms}} = \vec{M} \times \vec{a_n}$$
- Then we can determine A by:
$$\vec{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\nabla' \times \vec{M}}{R} dv' + \frac{\mu_0}{4\pi} \oint_{S'} \frac{\vec{M} \times \vec{a_n'}}{R} ds'$$
- Then we could obtain $\vec{B}$ from $\vec{A}$.

Equivalent Magnetization Charge Densities

In current-free region, a manetized body may be replaced by

1. an equivalent/fictitous magnetization surface charge density

$$\rho_{ms} = \vec{M} \cdot \vec{a_n}$$

2. an equivalent/fictitous magnetization volume charge density

$$\rho_m = -\nabla \cdot \vec{M}$$

Magnetic Field Intensity, Relative Permeability and Magnetic circuit

Considering the effect of both the internal dipole moment and the induced magnetic moment in a magnetic material, the magnetic field intensity $\vec{H}$ is defined as:

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

$$\nabla \times \vec{H} = \vec{J}$$

directly relates the magnetic field intensity with the density of free charge.

The integral form of which is then,

$$\oint_C \vec{H} \cdot dl = I$$

Magnetic Field Intensity, Relative Permeability and Magnetic circuit

It is another form of Ampere's circuital law: the circulation of the magnetic field intensity around any closed path is equal to the free current flowing through the surface bounded by the path.
If the closed path C is chosen to enclose N turns of a winding carrying a current I that excites a magnetic circuit, we have

$$\oint_C \vec{H} \cdot dl = NI = V_m$$

,
V_m is analogous to electromotive force (emf) and is called magnetomotive force (mmf).

Magnetic Field Intensity, Relative Permeability and Magnetic circuit

If we define:

$$\vec{M} = \chi_m H$$

,

where χ_m is magnetic susceptibility, Then

$$\vec{B} = \mu_0(1 + \chi_m)\vec{H} = \mu_0\mu_r\vec{H} = \mu\vec{H}$$

$$\vec{H} = \frac{1}{\mu}\vec{B}$$

$$\mu_r = 1 + \chi_m = \frac{\mu}{\mu_0}$$

where μ_r is the relative permeability of the medium. $\mu = \mu_r\mu_0$ is the absolute permeability/permeability.

Magnetic Field Intensity, Relative Permeability and Magnetic circuit

If $\vec{B}$ is approximately constant over the core cross section,

$$R_f = \frac{l_f}{\mu S}$$

,
where $l_f = 2\pi r_o - l_g$ is the length of the ferromagnetic core.

$$R_g = \frac{l_g}{\mu_0 S}$$

Both the reluctance R_f of the ferromagnetic core and R_g of the air gap have the same formula.

Magnetic Field Intensity, Relative Permeability and Magnetic circuit

We have the analogous quantities:

Magnetic Circuits	Electric Circuits
mmf, $\mathscr{V}_m\ (=NI)$	emf, $\mathscr{V}$
magnetic flux, Φ	electric current, I
reluctance, $\mathscr{R}$	resistance, R
permeability, μ	conductivity, σ

And the analysis of the magnetic circuit is similar to the electric circuits.

Magnetic Field Intensity, Relative Permeability and Magnetic circuit

Similar to Kirchhoff's law,

$$\sum_j N_j I_j = \sum_k R_k \Phi_k$$

,

around a closed path in a magnetic circuit, the algebraic sum of ampere-turns is equal to the algebraic sum of the products of the reluctances and fluxes.

$$\sum_j \Phi_j = 0$$

,

the algebraic sum of all the magnetic fluxes flowing out of a junction in a magnetic circuit is zero.

Behavior of Magnetic Materials

1. Diamagnetic, if $\mu_r \lesssim 1$ (χ_m is a very small negative number)
2. Paramagnetic, if $\mu_r \gtrsim 1$ (χ_m is a very small positive number)
3. Ferromagnetic, if $\mu_r >> 1$ (χ_m is a large positive number)

Boundary Conditions for Magnetostatic Fields

1. The normal component of $\vec{B}$ is continuous across an interface,

$$B_{1n} = B_2$$

, which is

$$\mu_1 H_{1n} = \mu_2 H_{2n}$$

,

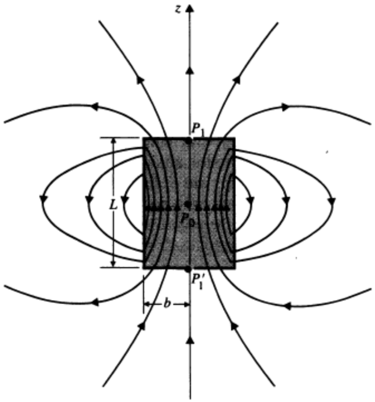
2. The tangential component of the $\vec{H}$ field is discontinuous across an interface where a free surface current exists.

$$a_{n2} \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s$$

When the conductivities of both media are finite, currents are defined by volume current densities and free surface currents do not exist on the interface. Hence $\vec{J}_s = 0$, the tangential component of $\vec{H}$ is continuous across the boundary of almost all physical media; it is discontinuous only when an interface with an ideal perfect conductor or a superconductor assumed.

Boundary Conditions for Magnetostatic Fields

The magnetic flux lines both inside and outside a cylindrical bar magnet having a uniform axial magnetization $\vec{M} = \vec{a_z} M_0$:



Boundary Conditions for Magnetostatic Fields

Notice here, $\vec{H} = \vec{B}/\mu_0 - \vec{M}$, $\vec{H} = \vec{B}/\mu_0$ outside the magnet as $\vec{M} = 0$ outside , and $\vec{M}$ is a constant vector inside the magnet. For current-free regions, $\vec{B} = -\mu\nabla V_m$, thus,

$$\nabla^2 V_m = 0$$

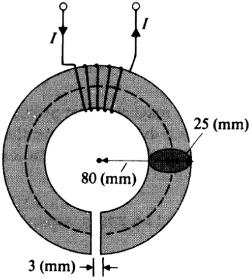
, similar to the Lapace's equation. Similar to what we have discussed in Chapter 4, we could expect the similar approach here to solve boundary-value problems.

P1 A toroidal iron core of relative permeability 3000 has a mean radius $R = 80(\text{ mm})$ and a circular cross section with radius $b = 25(\text{ mm})$. An air gap $\ell_g = 3(\text{ mm})$ exists, and a current I flows in a 500 -turn winding to produce a magnetic flux of $10^{-5}(\text{ Wb})$. (See Fig. 1.) Neglecting flux leakage and using mean path length, find

a) the reluctances of the air gap and of the iron core,

b) $\mathbf{B}_g$ and $\mathbf{H}_g$ in the air gap, and $\mathbf{B}_c$ and $\mathbf{H}_c$ in the iron core,

c) the required current I.



P.6-27 a) $\mathcal{R}_g = \frac{\ell_g}{\mu_0 S} = \frac{3\times10^{-3}}{4\pi\times10^{-7}\times(\pi\times0.025)^2} = 1.21\times10^6\ (H^{-1})$,

$\mathcal{R}_c = \frac{2\pi\times0.08 - 0.003}{3000\times(4\pi\times10^{-7})\times(\pi\times0.025)^2} = 6.75\times10^4 (H^{-1})$.

b) $\bar{B}_g = \bar{B}_c = \bar{a}_\phi \frac{10^{-5}}{\pi\times0.025^2} = \bar{a}_\phi\, 5.09\times10^{-3}\ (T)$,

$\bar{H}_g = \frac{1}{\mu_0}\bar{B}_g = \bar{a}_\phi \frac{5.09\times10^{-3}}{4\pi\times10^{-7}} = \bar{a}_\phi\, 4.05\times10^3\ (A/m)$,

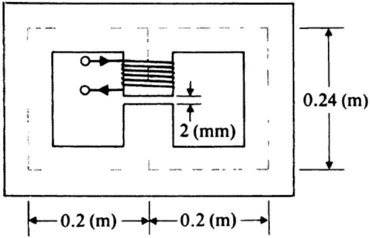
$\bar{H}_c = \frac{1}{\mu_0\mu_r}\bar{B}_c = \bar{a}_\phi \frac{4.05\times10^3}{3000} = \bar{a}_\phi\, 1.35\ \ (A/m)$.

c) $NI = \Phi(\mathcal{R}_c + \mathcal{R}_g)$, $I = \frac{1}{N}\Phi(\mathcal{R}_c+\mathcal{R}_g) = 0.0256\,(A) = 25.6\,(mA)$.

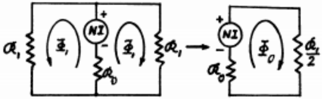
P2 Consider the magnetic circuit in Fig. 6 – 45. A current of 3 (A) flows through 200 turns of wire on the center leg. Assuming the core to have a constant cross-sectional area of 10^{-3} (m^2) and a relative permeability of 5000:

a) Determine the magnetic flux in each leg.

b) Determine the magnetic field intensity in each leg of the core and in the air gap.



P. 6–28 Magnetic circuit:



$\Phi_1 = \Phi_0/2$

$$\frac{1}{\mu_0 S} = \frac{1}{(4\pi 10^{-7}) \times 10^{-3}} = 7.95 \times 10^8$$

Neglecting leakage flux and assuming constant flux density over S:

$$\mathcal{R}_0 = \frac{0.002}{\mu_0 S} - \frac{0.24 - 0.002}{\mu_0 \mu_r S} = 1.60 \times 10^6 \, (H^{-1}),$$

$$\mathcal{R}_1 = \frac{0.24 + 2 \times 0.2}{\mu_0 \mu_r S} = 0.102 \times 10^6 \, (H^{-1}).$$

a) $\Phi_0 = \dfrac{NI}{\mathcal{R}_0 + \mathcal{R}_1/2} = 3.63 \times 10^{-4} \, (Wb)$; $\quad \Phi_1 = \dfrac{\Phi_0}{2} = 1.82 \times 10^{-4} \, (Wb).$

b) $H_1 = \dfrac{\Phi_1}{\mu_0 \mu_r S} = 28.9 \, (A/m),$

$(H_0)_g = \dfrac{1}{\mu_0 S} \Phi_0 = 28.9 \times 10^4 \, (A/m)$ in air gap,

$(H_0)_c = (H_0)_g / \mu_r = 57.8 \, (A/m).$

P3 What boundary conditions must the scalar magnetic potential V_m satisfy at an interface between two different magnetic media?

P3 What boundary conditions must the scalar magnetic potential V_m satisfy at an interface between two different magnetic media?

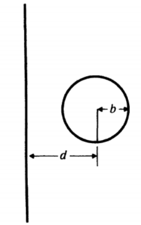
P. 6-31 $\bar{H}_1 = -\bar{\nabla} V_{m1}$, $\bar{H}_2 = -\bar{\nabla} V_{m2}$.

Boundary conditions: $\mu_1 H_{1n} = \mu_2 H_{2n} \longrightarrow \mu_1 \frac{\partial V_{m1}}{\partial n} = \mu_2 \frac{\partial V_{m2}}{\partial n}$,

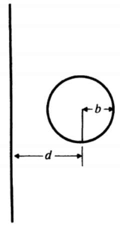
$H_{1t} = H_{2t} \longrightarrow V_{m1} = V_{m2}$

(assuming absence of current)

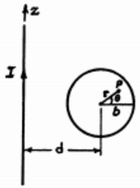
P4 Determine the mutual inductance between a very long, straight wire and a conducting circular loop, as shown in Fig. 3.



P4 Determine the mutual inductance between a very long, straight wire and a conducting circular loop, as shown in Fig. 3.



P. 6-39



Assume a current I.

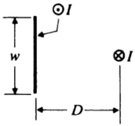
B at $P(r,\theta)$ is $\bar{a}_\phi \dfrac{\mu_0 I}{2\pi(d+r\cos\theta)}$.

$$\Lambda_{12} = \frac{\mu_0 I}{2\pi}\int_0^b\int_0^{2\pi}\frac{r\,dr\,d\theta}{d+r\cos\theta}$$

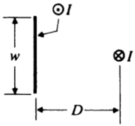
$$= \frac{\mu_0 I}{2\pi}\int_0^b \frac{2\pi r\,dr}{\sqrt{d^2-r^2}} = \mu_0 I\left(d-\sqrt{d^2-b^2}\right).$$

$$L_{12} = \mu_0\left(d-\sqrt{d^2-b^2}\right).$$

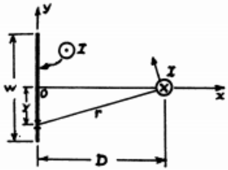
P5 The cross section of a long thin metal strip and a parallel wire is shown in Fig. 4. Equal and opposite currents I flow in the conductors. Find the force per unit length on the conductors.



P5 The cross section of a long thin metal strip and a parallel wire is shown in Fig. 4. Equal and opposite currents I flow in the conductors. Find the force per unit length on the conductors.



P.6-43 Magnetic field intensity at the wire due to the current $dI = \frac{I}{w}dy$ in an elemental dy is



$$|d\bar{H}| = \frac{dI}{2\pi r} = \frac{I\,dy}{2\pi w\sqrt{D^2+y^2}}.$$

Symmetry $\longrightarrow$ H at the wire has only a y-component.

$$\bar{H} = \bar{a}_y \int (dH)\cdot\left(\frac{D}{r}\right) = \bar{a}_y 2\int_0^{w/2} \frac{ID\,dy}{2\pi w(D^2+y^2)}$$

$$= \bar{a}_y \frac{I}{\pi w}\tan^{-1}\left(\frac{w}{2D}\right).$$

$$\bar{f}' = \bar{I}\times\bar{B} = (-\bar{a}_z I)\times(\mu_0\bar{H}) = \bar{a}_x \frac{\mu_0 I^2}{\pi w}\tan^{-1}\left(\frac{w}{2D}\right) \quad (\mathrm{N/m}).$$

P6 One end of a long air-core coaxial transmission line having an inner conductor of radius a and an outer conductor of inner radius b is short-circuited by a thin, tight-fitting conducting washer. Find the magnitude and the direction of the magnetic force on the washer when a current I flows in the line.

P6 One end of a long air-core coaxial transmission line having an inner conductor of radius a and an outer conductor of inner radius b is short-circuited by a thin, tight-fitting conducting washer. Find the magnitude and the direction of the magnetic force on the washer when a current I flows in the line.

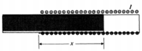
P.6-48 Let x-axis be the center line of the coaxial cable.
The magnetic energy stored in a section of length x is

$$W_m = \frac{1}{2} L I^2.$$

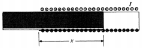
$$L = \frac{\Phi}{I} = \frac{x}{I}\int_a^b B_\phi \, dr = \frac{x}{I}\int_a^b \frac{\mu_0 I}{2\pi r} dr = \frac{\mu_0 x}{2\pi} \ln\frac{b}{a}.$$

$$\bar{F}_x = \bar{a}_x \frac{\partial W_m}{\partial x} = \bar{a}_x \left(\frac{I^2}{2}\right)\frac{\partial L}{\partial x} = \bar{a}_x \frac{\mu_0 I^2}{4\pi} \ln\frac{b}{a}.$$

P7 A current I flows in a long solenoid with n closely wound coil-turns per unit length. The cross-sectional area of its iron core, which has permeability μ, is S. Determine the force acting on the core if it is withdrawn to the position shown in Fig. 5.



P7 A current I flows in a long solenoid with n closely wound coil-turns per unit length. The cross-sectional area of its iron core, which has permeability μ, is S. Determine the force acting on the core if it is withdrawn to the position shown in Fig. 5.



P.6-53 $W_m = \frac{1}{2}\int \mu H^2 dv$

Assume a virtual displacement, Δx, of the iron core.

$$W_m(x+\Delta x) = W_m(x) + \frac{1}{2}\int_{S\Delta x} (\mu - \mu_0) H^2 dv$$

$$= W_m(x) + \frac{1}{2}\mu_0(\mu_r - 1) n^2 I^2 S\Delta x.$$

$$(F_I)_x = \frac{\partial W_m}{\partial x} = \frac{\mu_0}{2}(\mu_r - 1) n^2 I^2 S,$$ in the direction of increasing x.