VE230 RC slides Week 2

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Course Overview

Course overview

- Integral
- Fields
- Assignment 2

Integral

Line integrals

When integrated along a certain differential length, use equations we introduced last class to convert differential length to integrable quantities with regard to different coordinates. For example, in Cartesian coordinates, when integrating on $d\vec{l}$, you should convert $d\vec{l}$ into the form

$$d\vec{l} = \vec{a_x}dx + \vec{a_y}dy + \vec{a_z}dz.$$

Integral

A small example

Suppose $\vec{F} = \vec{a_x} xy - \vec{a_y} 2x$, calculate its integral along $y = x^2$ in the range [0,1].

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Answer

$$
\begin{aligned}
W &= \int \vec{F} \cdot d\vec{l} \\
&= \int (\vec{a_x}xy - \vec{a_y}2x) \cdot (\vec{a_x}dx + \vec{a_y}dy) \\
&= \int xydx - 2xdy \\
&= \int_0^1 (x * x^2 dx - 2x * 2xdx) = -\frac{13}{12}
\end{aligned}
$$

Integral

Flux: the integrals of the vector field rush out the surface.

$$\int_S \mathbf{A} \cdot d\mathbf{s},$$

Flux of vector field **A** flowing through the area S

$$d\mathbf{s} = \mathbf{a}_n \, ds$$

1. If S is a closed surface → $\mathbf{a}_n$ is in the outward direction
2. If S is an open surface → $\mathbf{a}_n$ is decided by right-hand rule (thumb)

Scalar field and vector field

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Connection

The vector field can be the gradient of a scalar field.

Gradient of a scalar field

- $$\nabla V = \vec{a_n}\frac{dV}{dn}$$
- ∇V at certain point is a vector.
- $$dV = (\nabla V)\cdot d\vec{l}$$
  : the space rate of increase of V in the $\vec{a_l}$ direction is equal to the projection (the component) of the gradient of V in that direction.
- $$\nabla V = \vec{a_{u}}_1\frac{\partial V}{h_1\partial u_1} + \vec{a_{u}}_2\frac{\partial V}{h_2\partial u_2} + \vec{a_{u}}_3\frac{\partial V}{h_3\partial u_3}$$
  , when V is taken off:
  $$\nabla \equiv \vec{a_{u}}_1\frac{\partial}{h_1\partial u_1} + \vec{a_{u}}_2\frac{\partial}{h_2\partial u_2} + \vec{a_{u}}_3\frac{\partial}{h_3\partial u_3}$$

Divergence of a vector field

- divergence of a vector field $\vec{A}$ at a point $div\vec{A}$ as the net outward flux of $\vec{A}$ per unit volume as the volume about the point tends to zero:

$$div\vec{A} = \lim_{\Delta v \to 0} \frac{\oint\limits_S \vec{A}\,\mathrm{d}s}{\Delta v} \tag{1}$$

- source: net positive divergence; sink: net negative divergence. zero divergence: no source/sink.
- $div\vec{A}$ at certain point is a scalar.

Divergence of a vector field

- For Cartesian coordinate,

$$div\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

- $$\nabla \cdot \vec{A} \equiv div\vec{A}$$

- $$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3}[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_1 h_3 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)]$$

Divergence Theorem

- $$\int_V \nabla \cdot \vec{A} dv = \oint_S \vec{A} \cdot d\vec{s}$$
  , the volume integral of the divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume.
- A most famous theorem in the physics - Gauss's law can be deduced by divergence theorem very easily. Thus, the divergence theorem will be quite fundamental in this course. Make sure you understand it.

Curl of a vector field

- $$curl\vec{A} \equiv \nabla \times \vec{A} = \lim_{\Delta s \to 0} \frac{1}{\Delta s}[\vec{a_n} \oint_C \vec{A} \cdot d\vec{l}]_{max}$$

  : the curl of a vector field $\vec{A}$, denoted by $curl\vec{A}$ or $\nabla \times \vec{A}$, is a vector whose magnitude is the maximum net circulation of $\vec{A}$ per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented to make the net circulation maximum. (Right hand rule defines the positive normal to an area).
- $\nabla \times \vec{A}$ in a general coordinate:

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \vec{a_{u1}} h_1 & \vec{a_{u2}} h_2 & \vec{a_{u3}} h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

- curl-free vector field ($\nabla \times \vec{A} = 0$): **irrotational** or **conservative field**, like the gravitation potential field.

Stokes's Theorem

- $$\int_S (\nabla \times \vec{A}) d\vec{s} = \oint_C \vec{A} \cdot d\vec{l}$$

  : the surface integral of the curl of a vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface.

Other Identities

I

- $$\nabla \times (\nabla V) \equiv 0$$

  : the curl of the gradient of any scalar field is identically zero.
- Another interpretation: If a vector field is curl-free, it can be expressed as the gradient of a scalar field.
- Since a curl-free vector field is irrotational or conservative, an irrotational/conservative vector field can always be expressed as the gradient of a scalar field.

II

- $$\nabla \cdot (\nabla \times \vec{A}) \equiv 0$$

  : the divergence of the curl of any vector field is identically zero.
- Another interpretation: if a vector field is divergenceless, it can be expressed as the curl of another vector field.
- Divergenceless field is called solenoidal field, which will be further discussed in later classes.

Other Identities

Laplacian in Cartesian Coordinates

$$\nabla^2 V = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

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Helmholtz's Theorem

Any vector in 3D can be decomposable into a sum of the following vector fields. A vector field (vector point function) is determined to within an additive constant if both its divergence and its curl are specified everywhere.

$$F = -\nabla V + \nabla \times A.$$

Other useful vector properties

$$\nabla(\psi\phi) = \phi\nabla\psi + \psi\nabla\phi$$

$$\nabla\cdot(\psi\mathbf{A}) = \psi\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla\psi$$

$$\nabla\times(\psi\mathbf{A}) = \psi(\nabla\times\mathbf{A}) + \nabla\psi\times\mathbf{A}$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0$$

$$\nabla\times(\nabla\psi) = \mathbf{0}$$

$$\nabla\cdot(\nabla\psi) = \nabla^2\psi \text{ (scalar Laplacian)}$$

$$\nabla(\nabla\cdot\mathbf{A}) - \nabla\times(\nabla\times\mathbf{A}) = \nabla^2\mathbf{A} \text{ (vector Laplacian)}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A})$$

About the Assignments

I hope you don't just copy the answer I showed, but instead tried to understand the problem. In order to make sure the quality of other 2 RCs, the slides I submit on canvas will be without the answer. But you don't need to take the screen shot, since the answer will be released after the deadline. Also, for those students who just want to copy the answer, it is a waste of time, since the assignments will not be counted into the final.

Assignments

P.3-8

A line Charge of uniform density ρ_l in free space forms a semicircle of radius b. Determine the magnitude and direction of the electric field intensity at the center of the semicircle.

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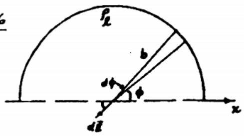
-6



$$dE_y = -\frac{\rho_\ell (b\,d\phi)}{4\pi\epsilon_0 b^2}\sin\phi,$$

$$\bar{E} = \bar{a}_y E_y = -\bar{a}_y \frac{\rho_\ell}{4\pi\epsilon_0 b}\int_0^{\pi} \sin\phi\, d\phi$$

$$= -\bar{a}_y \frac{\rho_\ell}{2\pi\epsilon_0 b}.$$

Assignments

P.3-9

Three uniform line charges– ρ_{l_1}, ρ_{l_2}, and ρ_{l_3}, each of length L–form an equilateral triangle. Assuming that $\rho_{l_1} = 2\rho_{l_2} = 2\rho_{l_3}$, determine the electric field intensity at the center of the triangle.

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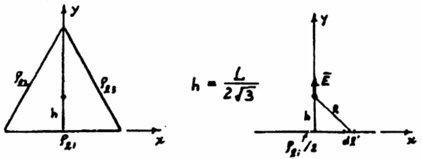
-7



$\bar{E}$ at the center of triangle would be zero if all three line charges were of the same charge density. The problem is equivalent to that of a single line charge of density $\rho_{l1}/2$. By symmetry, there will only be a y-component.

$$\bar{E} = \bar{a}_y E_y = \bar{a}_y \int_{-L/2}^{L/2} \frac{(\rho_{l1}/2)\, d\ell'}{4\pi\epsilon_0 R^2}\left(\frac{h}{R}\right) = \bar{a}_y \int_{-L/2}^{L/2} \frac{\rho_{l1}\, h\, d\ell'}{8\pi\epsilon_0 (h^2+\ell'^2)^{3/2}}$$

$$= \bar{a}_y \frac{3\rho_{l1}}{4\pi\epsilon_0 L} = \bar{a}_y \frac{3\rho_{l2}}{2\pi\epsilon_0 L}.$$

Assignments

P.3-12

Two infinitely long coaxial cylindrical surfaces, $r = a$ and $r = b(b > a)$, carry surface charge densities ρ_{sa} and ρ_{sb}, respectively.

a) Determine **E** everywhere.

b) What must be the relation between a and b in order that **E** vanishes for $r > b$?

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P.3-10 Cylindrical symmetry: $\bar{E} = \bar{a}_r E_r$. Apply Gauss's law.

a) $r < a$, $E_r = 0$; $a < r < b$, $E_r = a\rho_{sa}/\epsilon_0 r$;

$r > b$, $E_r = (a\rho_{sa} + b\rho_{sb})/\epsilon_0 r$.

b) $b/a = -\rho_{sa}/\rho_{sb}$.

Assignments

P.3-13

Determine the work done in carrying $a - 2(\mu C)$ charge from P_1(2,1,-1) to $P_2(8, 2, -1)$ in the field $\boldsymbol{E} = \boldsymbol{a_x} y + \boldsymbol{a_y} x$.

a) along the parabola $x = 2y^2$

b) along the straight line joining P_1 and P_2.

Assignments

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a) along the parabola $x=2y^2$

b) along the straight line joining P_1 and P_2.

P3-13 $W=-q\int \vec{E}\cdot d\vec{l} = -q(y\,dx + x\,dy)$

(a) $x=2y^2$, $dx=4y\,dy$, $W=-q\int_1^2 4y^2dy + 2y^2dy = -14q = 28\,\mu J$

(b) $x=6y-4$ $dx=6\,dy$

$W=-q\int_1^2 (12y-4)\,dy = -14q = 28\,\mu J$

Assignments

P.3-16

A finite line charge of length L carrying uniform line charge density ρ_l is coincident with the x-axis.

a) Determine V in the plane bisecting the line charge.

b) Determine **E** on the bisecting plane from ρ_l directly by applying Coulomb's law.

c) Check the answer in part(b) with $-\nabla V$.

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3-16 (a) $V = 2\int_0^{\frac{L}{2}} \frac{\rho_L}{4\pi\varepsilon_0 R}\,dx = 2\int_0^{\frac{L}{2}} \frac{\rho_L}{4\pi\varepsilon_0\sqrt{x^2+y^2}}\,dx$

$= \frac{\rho_L}{2\pi\varepsilon_0}\left[\ln\left(x+\sqrt{x^2+y^2}\right)\right]_0^{\frac{L}{2}} = \frac{\rho_L}{2\pi\varepsilon_0}\left[\ln\left(\frac{L}{2}+\sqrt{\frac{L^2}{4}+y^2}\right) - \ln y\right]$

(b) $\vec{E} = \vec{a}_y E_y = \int_{-\frac{L}{2}}^{\frac{L}{2}} \vec{a}_y \frac{\rho_L \cdot y\,dx}{4\pi\varepsilon_0 R^3} = \frac{\rho_L\, y}{2\pi\varepsilon_0}\vec{a}_y\int_0^{\frac{L}{2}} \frac{1}{\left(\sqrt{x^2+y^2}\right)^3}\,dx = \frac{\rho_L}{2\pi\varepsilon_0 y}\cdot\frac{L}{\sqrt{L^2+4y^2}}\cdot\vec{a}_y$

(c) $\vec{E} = -\vec{a}_y\frac{\partial V}{\partial y} = -\vec{a}_y\frac{\partial}{\partial y}\left[\frac{\rho_L}{2\pi\varepsilon_0}\left(\ln\left(\frac{L}{2}+\sqrt{\frac{L^2}{4}+y^2}\right)-\ln y\right)\right] = \vec{a}_y\frac{\rho_L}{2\pi\varepsilon_0 y}\cdot\frac{L}{\sqrt{L^2+4y^2}}$

Same as (b)

Assignments

P.3-19

A charge Q is distributed uniformly over the wall of a circular tube of radius b and height h. Determine V and $\mathbf{E}$ on its axis.

a) at a point outside the tube, then

b) at a point inside the tube.

Assignments

P.3-19

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a) at a point outside the tube, then

b) at a point inside the tube.

-19 $\rho_s = \frac{Q}{2\pi b h}$ $\quad R = b$ $\quad V = \frac{\rho_b}{4\pi\varepsilon_0}\int_0^h\int_0^{2\pi} \frac{1}{\sqrt{b^2+(z-z')^2}}\, d\varphi\, dz'$

$$= \frac{\rho_b}{2\varepsilon_0} \ln\left(\frac{z+\sqrt{b^2+z^2}}{z-h+\sqrt{b^2+(z-h)^2}}\right) \quad \rho = \frac{Q}{2\pi b h}$$

(a) $\vec{E} = -\nabla V = -\vec{a}_z \frac{\partial V}{\partial z} = \vec{a}_z \frac{\rho_b}{2\varepsilon_0}\left(\frac{1}{\sqrt{b^2+(z-h)^2}} - \frac{1}{\sqrt{b^2+z^2}}\right)$

(b) $V = \frac{\rho b}{2\varepsilon_0} \ln \frac{z+\sqrt{b^2+z^2}}{\sqrt{b^2+(z-h)^2}+z-h}$

$$\vec{E} = \vec{a}_z \frac{b\rho_s}{2\varepsilon_0}\left(\frac{1}{\sqrt{b^2+(z-h)^2}} - \frac{1}{\sqrt{b^2+z^2}}\right)$$