VE230 RC slides Week 3

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Overview

Content

- Electrostatics in Free Space
- Coulomb's Law
- Gauss's Law and Application
- Electric Potential

Electrostatics in Free Space

Static electric charges (source) in free space $\rightarrow$ electric field

Electric field intensity

$$\mathbf{E} = \lim_{q \to 0} \frac{\mathbf{F}}{q} \quad (\mathbf{V}/\mathrm{m})$$

Fundamental Postulates of Electrostatics

- Differential form:

$$\vec{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \quad (divergence)$$

$$\vec{\nabla}\times\mathbf{E} = 0 \quad (curl)$$

  where ρ is the volume charge density of free charges (C/m^3), ϵ_0 is the permittivity of free space, a universal constant.

- Integral form:

$$\oint_S \mathbf{E}\cdot d\mathbf{s} = \frac{Q}{\epsilon_0}$$

$$\oint_C \mathbf{E}\cdot d\ell = 0$$

  where Q is the total charge contained in volume V bounded by surface S. Also, the scalar line integral of the static electric field intensity around any closed path vanishes.

E is **not solenoidal** (unless $\rho = 0$), but **irrotational**

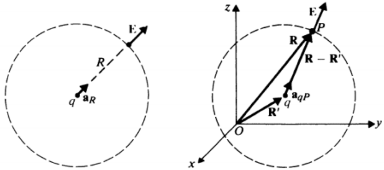
Electric Field due to a System of Discrete Charges

- **a single point charge (charge on the origin):**

$$\mathbf{E} = \mathbf{a}_R E_R = \mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2} \quad (\mathbf{V/m})$$

- **a single point charge (charge is not on the origin):**

$$\mathbf{E}_p = \frac{q\left(\mathbf{R} - \mathbf{R}'\right)}{4\pi\epsilon_0 \left|\mathbf{R} - \mathbf{R}'\right|^3} \quad (\mathbf{V/m})$$



(a) Point charge at the origin. (b) Point charge not at the origin.

FIGURE 3–2
Electric field iFIGURE due to a point charge.

Electric Field due to a System of Discrete Charges

- **a single point charge (charge is not on the origin):** When a point charge q_2 is placed in the field of another point charge q_1 at the origin, a force F_{12} is experienced by q_2 due to the electric field intensity $\vec{E_{12}}$ of q_1 at q_2. Then we have:

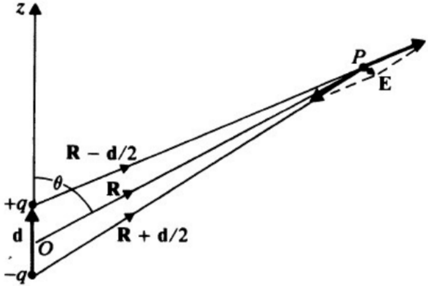
$$\vec{F_{12}} = q_2 \vec{E_{12}} = \vec{a_R} \frac{q_1 q_2}{4\pi\epsilon_0 R^2}$$

- **several point charges:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{q_k \left(\mathbf{R} - \mathbf{R}_k'\right)}{\left|\mathbf{R} - \mathbf{R}_k'\right|^3}$$

Electric Dipole

- **Electric Field**



Electric Dipole

- **Electric Field**

  general expression:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\frac{\mathbf{R}-\frac{\mathbf{d}}{2}}{\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^3} - \frac{\mathbf{R}+\frac{\mathbf{d}}{2}}{\left|\mathbf{R}+\frac{\mathbf{d}}{2}\right|^3}\right\}$$

  if $d << R$:

$$\mathbf{E} \cong \frac{q}{4\pi\epsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\mathbf{R} - \mathbf{d}\right]$$

- **Electric Dipole Moment**

  Definition:

$$\mathbf{p} = q\mathbf{d}$$

  ,where q is the charge, vector $\mathbf{d}$ goes from $-q$ to $+q$.

$$\mathbf{p} = \mathbf{a}_z p = p\left(\mathbf{a}_R \cos\theta - \mathbf{a}_\theta \sin\theta\right)$$

$$\mathbf{R}\cdot\mathbf{p} = Rp\cos\theta$$

Electric Dipole

- **Electric Field:** (spherical coordinate)

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 R^3}\left(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta\right) \quad (\mathrm{V/m})$$

Electric Field due to a Continuous Distribution of Charge

- **General Differential Element:**

$$d\mathbf{E} = \mathbf{a}_R \frac{\rho dv'}{4\pi\epsilon_0 R^2},$$

where dv' is the differential volume element.

- **Line Charge:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{L'} \mathbf{a}_R \frac{\rho_\ell}{R^2} d\ell' \quad (\mathbf{V/m})$$

- **Surface Charge:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{\mathbf{S}'} \mathbf{a}_R \frac{\rho_s}{R^2} ds' \quad (\mathbf{V}/\mathrm{m})$$

- **Volume Charge:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{V'} \mathbf{a}_R \frac{\rho}{R^2} dv' \quad (\mathbf{V/m})$$

Gauss's Law and Application

Definition

The total outward flux of the E-field over any closed surface in free space is equal to **the total charge enclosed in the surface** divided by ϵ_0. (Note that we can choose arbitrary surface S for our convenience.)

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}$$

Application

- **Conditions for Maxwell's Integral Equations:**
  There is **a high degree of symmetry** in the charge distribution or in the electrical field (i.e., spherically symmetric, planar, line charge, etc.

Gauss's Law Application

Example:

Determine the electric field intensity of an infinitely long, straight, line charge of a uniform density ρ_s in air.

Gauss's Law Application

Example:

Determine the electric field intensity of an infinitely long, straight, line charge of a uniform density ρ_s in air.

Answer

$$E \cdot 2\pi \cdot r \cdot l = \frac{\rho \cdot l}{\epsilon_0}$$

Several Useful Models

Note: The charge distribution should be **uniform**.

different models	E(magnitude)
infinitely long, line charge	$E = \frac{\rho_\ell}{2\pi r \epsilon_0}$
infinite planar charge	$E = \frac{\rho_s}{2\epsilon_0}$
uniform spherical surface charge with radius R	$\begin{cases} E = 0 (r < R) \\ E = \frac{Q}{4\pi r^2 \epsilon_0} (r > R) \end{cases}$
uniform sphere charge with radius R	$\begin{cases} E = \frac{Qr}{4\pi R^3} (r < R) \\ E = \frac{Q}{4\pi r^2 \epsilon_0} (r > R) \end{cases}$
infinitely long, cylindrical charge with radius R	$\begin{cases} E = \frac{\rho_v r}{2\epsilon_0} (r < R) \\ E = \frac{\rho_v R^2}{2 r \epsilon_0} (r > R) \end{cases}$

Electric Potential

- **Expression:**

$$\mathbf{E} = -\nabla V$$

  the reason for the negative sign: consistent with the convention that in going against the $\mathbf{E}$ field, the electric potential V increases.

- **Electric Potential Difference:**

$$V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot dl$$

- **Electric Potential due to a Charge Distribution**

$$V = \frac{q}{4\pi\epsilon_0 R}$$

Electric Potential

- **Line Charge:**

$$V = \frac{1}{4\pi\epsilon_0} \int_{L'} \frac{\rho_l}{R} dl' \quad (V)$$

- **Surface Charge:**

$$V = \frac{1}{4\pi\epsilon_0} \int_{\mathbf{S}'} \frac{\rho_s}{R} ds' \quad (V)$$

- **Volume Charge:**

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho}{R} dv' \quad (V)$$

Assignment

P.3-22

The polarization in a dielectric cube of side L centered at the origin is given by $\mathbf{P} = P_0(\mathbf{a_x}x + \mathbf{a_y}y + \mathbf{a_z}z)$.

a) Determine the surface and volume bound-charge densities

b) Show that the total bound charge is zero.

Assignment

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a) Determine the surface and volume bound-charge densities

b) Show that the total bound charge is zero.

a) $\rho_{ps} = \bar{P} \cdot \bar{a}_n = P_0 \frac{L}{2}$ on all six faces of the cube.

$\rho_p = -\bar{\nabla} \cdot \bar{P} = -3P_0$.

b) $Q_s = 6L^2 \rho_{ps} = 3P_0 L^3$, $\quad Q_v = L^3 \rho_p = -3P_0 L^3$.

Total bound charge $= Q_s + Q_v = 0$.

Assignment

P.3-23

Determine the electric field intensity at the center of a small spherical cavity cut out of a large block of dielectric in which a polarization $\mathbf{P}$ exists.

Assignment

P.3-23

Determine the electric field intensity at the center of a small spherical cavity cut out of a large block of dielectric in which a polarization **P** exists.

3-19 Assume $\bar{P} = \bar{a}_z P$. Surface charge denity $\rho_{ps} = \bar{P}\cdot\bar{a}_n$

$$= (\bar{a}_z P)\cdot(-\bar{a}_R)$$

$$= -P\cos\theta.$$

The z-component of the electric field intensity due to a ring of ρ_{ps} contained in width $R\,d\theta$ at θ is

$$dE_z = \frac{P\cos\theta}{4\pi\epsilon_0 R^2}(2\pi R\sin\theta)(R\,d\theta)\cos\theta$$

$$= \frac{P}{2\epsilon_0}\cos^2\theta\sin\theta\,d\theta.$$

At the center of the cavity: $\bar{E} = \bar{a}_z E_z = \bar{a}_z \frac{P}{2\epsilon_0}\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta = \frac{\bar{P}}{3\epsilon_0}.$

Assignment

P.3-25

Assume that the z=0 plane separates two lossless dielectric regions with $\epsilon_{r1} = 2$ and $\epsilon_{r2} = 3$. If we know that $\boldsymbol{E_1}$ in region 1 is $\boldsymbol{a_x}2y - \boldsymbol{a_y}3x + \boldsymbol{a_z}(5+z)$, what do we also know about $\boldsymbol{E_2}$ and $\boldsymbol{D_2}$ in region 2? Can we determine $\boldsymbol{E_2}$ and $\boldsymbol{D_2}$ at any point in region 2? Explain.

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P.3-21 At the z=0 plane : $\bar{E}_1 = \bar{a}_x 2y - \bar{a}_y 3x + \bar{a}_z 5$.

$$\bar{E}_{1t}(z=0) = \bar{E}_{2t}(z=0) = \bar{a}_x 2y - \bar{a}_y 3x .$$

$$\bar{D}_{1n}(z=0) = \bar{D}_{2n}(z=0) \longrightarrow 2\bar{E}_{1n}(z=0) = 3\bar{E}_{2n}(z=0)$$

$$\longrightarrow \bar{E}_{2n}(z=0) = \frac{2}{3}(\bar{a}_z 5) = \bar{a}_z \frac{10}{3} .$$

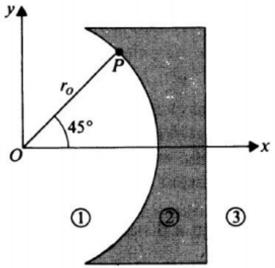
$$\therefore \bar{E}_2(z=0) = \bar{a}_x 2y - \bar{a}_y 3x + \bar{a}_z \frac{10}{3} ,$$

$$\bar{D}_2(z=0) = (\bar{a}_x 6y - \bar{a}_y 9x + \bar{a}_z 10)\epsilon_0 .$$

Assignment

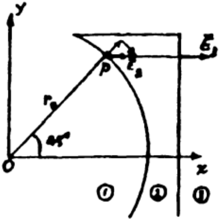
P.3-28

Dielectric lenses can be used to collimate electromagnetic fields. In Fig.1 the left surface of the lens is that of a circular cylinder, and the right surface is a plane. If $\boldsymbol{E_1}$ at point $P(r_0, 45°, z)$ in region 1 is $\boldsymbol{a_r}5 - \boldsymbol{a_\phi}3$, what must be the dielectric constant of the lens in order that $\boldsymbol{E_3}$ in region 3 is parallel to the x-axis?



Assignment

P. 3-24



Assume $\bar{E}_2 = \bar{a}_r E_{2r} + \bar{a}_\phi E_{2\phi}$

B.C.: $\bar{a}_n \times \bar{E}_1 = \bar{a}_n \times \bar{E}_2 \longrightarrow E_{2\phi} = -$

For $\bar{E}_3$, and hence $\bar{E}_2$, to be parallel to the x-axis,

$$E_{2\phi} = -E_{2r} \longrightarrow E_{2r} = 3.$$

B.C.: $\bar{a}_n \cdot \bar{D}_1 = \bar{a}_n \cdot \bar{D}_2 \longrightarrow 5 = 3$

$$\therefore \ \epsilon_{r2} = 5/3.$$

Assignment

P.3-32

The radius of the core and the inner radius of the outer conductor of a very long coaxial transmission line are r_i and r_o, respectively. The space between the conductors is filled with two coaxial layers of dielectrics. The dielectric constants of the dielectrics are ϵ_{r_1} for $r_i < r < b$ and ϵ_{r2} for $b < r < r_o$. Determine its capacitance per unit length.

Assignment

P.3-32

The radius of the core and the inner radius of the outer conductor of a very long coaxial transmission line are r_i and r_o, respectively. The space between the conductors is filled with two coaxial layers of dielectrics. The dielectric constants of the dielectrics are ϵ_{r_1} for $r_i < r < b$ and ϵ_{r2} for $b < r < r_o$. Determine its capacitance per unit length.

P.3-30 $\bar{D} = \bar{a}_r \dfrac{\rho_\ell}{2\pi r}$. $\bar{E}_1 = \bar{a}_r \dfrac{\rho_\ell}{2\pi\epsilon_0\epsilon_{r1} r}$, $r_i<r<b$; $\bar{E}_2 = \bar{a}_r \dfrac{\rho_\ell}{2\pi\epsilon_0\epsilon_{r2} r}$, $b<r<r_o$

$$V = -\int_{r_o}^{r_i} \bar{E}\cdot d\bar{r} = \frac{\rho_\ell}{2\pi\epsilon_0}\left[\frac{1}{\epsilon_{r1}}\ln\left(\frac{b}{r_i}\right) + \frac{1}{\epsilon_{r2}}\ln\left(\frac{r_o}{b}\right)\right].$$

$$C = \frac{\rho_\ell}{V} = \frac{2\pi\epsilon_0}{\frac{1}{\epsilon_{r1}}\ln\left(\frac{b}{r_i}\right) + \frac{1}{\epsilon_{r2}}\ln\left(\frac{r_o}{b}\right)} \quad (F/m).$$