VE230 RC slides Week 1

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Overview

The first week only contains the content about the vectors. Thus, we would cover two things in this recitation class:

- Vector review
- Coordinates

Vectors

dot product:

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|cos\theta_{\vec{A}\vec{B}}$$

- Commutative: $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
- Distributive: $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$
- Not associative: $\vec{A} \cdot (\vec{B} \cdot \vec{C}) \neq (\vec{A} \cdot \vec{B}) \cdot \vec{C}$
  e.g. $(\vec{a_x} \cdot \vec{a_y}) \cdot \vec{a_z} \neq \vec{a_x} \cdot (\vec{a_y} \cdot \vec{a_z})$
- For the three edges A, B, C in a triangle,
  $C^2 = A^2 + B^2 - 2ABcos(\theta_{A,B})$

Vectors

cross product:

$$\vec{A} \times \vec{B} = \vec{a_n}||\vec{A}||\vec{B}|sin\theta_{\vec{A}\vec{B}}|$$

- The cross product is always perpendicular to both $\vec{A}, \vec{B}$, the direction follows right hand rule.
- Not Commutative: $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$ (have opposite directions).
- Distributive: $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$
- Not associative: $\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$
  e.g. $\vec{a_x} \times (\vec{a_x} \times \vec{a_y}) = \vec{a_x} \times \vec{a_z} = -\vec{a_y}$,
  $(\vec{a_x} \times \vec{a_x}) \times \vec{a_y} = 0 \neq -\vec{a_y}$

Vectors

Some useful rules:

- $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = Volume$
- BAC-CAB: $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

Review Exercises

P.2-1 Given three vectors $\vec{A}, \vec{B}$ and $\vec{C}$ as follows,
$\vec{A} = \vec{a_x} + \vec{a_y}2 - \vec{a_z}3$,
$\vec{B} = -\vec{a_y}4 + \vec{a_z}$,
$\vec{C} = \vec{a_x}5 - \vec{a_z}2$,
find

a) $\vec{a_A}$: note a_A represents the unit vector of $\vec{A}$.

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find

a) $\vec{a_A}$: note a_A represents the unit vector of $\vec{A}$.
We can obtain that

$$a_A = \frac{1}{\sqrt{1^2 + 2^2 + 3^2}} \vec{A}.$$

Review Exercises

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find

b) b_c:the component of $\vec{A}$ in the direction of $\vec{C}$

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find

b) b_c:the component of $\vec{A}$ in the direction of $\vec{C}$
We can obtain that

$$b_c = \frac{\vec{A} \cdot \vec{C}}{|\vec{C}|} = \frac{11}{\sqrt{29}}$$

Review Exercises

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c) $\vec{A} \times \vec{C}$ (1. use properties; 2. calculate by matrix)

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find

c) $\vec{A} \times \vec{C}$ (1. use properties; 2. calculate by matrix)

- $\vec{a_x} \times \vec{a_x} = 0, \vec{a_x} \times \vec{a_y} = \vec{a_z}, \ldots$
- direct product by matrix

Answer is

$$\vec{A} \times \vec{C} = -4\vec{a_x} - 13\vec{a_y} + 10\vec{a_z}.$$

Review Exercises

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$\vec{B} = -\vec{a_y}4 + \vec{a_z}$,
$\vec{C} = \vec{a_x}5 - \vec{a_z}2$,
find

d) prove $\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$

Review Exercises

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find

e) $(\vec{A} \times \vec{B}) \times \vec{C}$ and $\vec{A} \times (\vec{B} \times \vec{C})$ (convert the form to the standard BAC-CAB)

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find

e) $(\vec{A} \times \vec{B}) \times \vec{C}$ and $\vec{A} \times (\vec{B} \times \vec{C})$ (convert the form to the standard BAC-CAB)

- $(\vec{A} \times \vec{B}) \times \vec{C} = -\vec{C} \times (\vec{A} \times \vec{B})$
- $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

Review Exercises

P.2-2

Given

$$\vec{A} = \vec{a_x} - \vec{a_y}2 + \vec{a_x}3$$

,

$$\vec{B} = \vec{a_x} + \vec{a_y} - \vec{a_z}2$$

, find the expression for a unit vector $\vec{C}$ that is perpendicular to both $\vec{A}$ and $\vec{B}$.

Review Exercises

P.2-2

Given

$$\vec{A} = \vec{a_x} - \vec{a_y}2 + \vec{a_x}3$$

,

$$\vec{B} = \vec{a_x} + \vec{a_y} - \vec{a_z}2$$

, find the expression for a unit vector $\vec{C}$ that is perpendicular to both $\vec{A}$ and $\vec{B}$.

$$\vec{C} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$$

Review Exercises

P.2-4

Show that, if $\vec{A} \cdot \vec{B} = \vec{A} \cdot \vec{C}$ and $\vec{A} \times \vec{B} = \vec{A} \times \vec{C}$, where $\vec{A}$ is not a null vector, then $\vec{B} = \vec{C}$.

Review Exercises

P.2-4

Show that, if $\vec{A} \cdot \vec{B} = \vec{A} \cdot \vec{C}$ and $\vec{A} \times \vec{B} = \vec{A} \times \vec{C}$, where $\vec{A}$ is not a null vector, then $\vec{B} = \vec{C}$.

- $\vec{A} \cdot \vec{B} = \vec{A} \cdot \vec{C} \rightarrow \vec{A} \perp (\vec{B} - \vec{C})$
- $\vec{A} \times \vec{B} = \vec{A} \times \vec{C} \rightarrow \vec{A} \parallel (\vec{B} - \vec{C})$

Since A is not null, it is obvious that $\vec{B} - \vec{C} = \vec{0}$

Coordinates

Three basis (u_1, u_2, u_3): number of linearly independent basis = dimension of the space. For the three types of coordinates we discuss, u_i is orthogonal to each other.
For arbitrary vector $\vec{A}$:

$$\vec{A} = \vec{a_{u1}} A_{u1} + \vec{a_{u2}} A_{u2} + \vec{a_{u3}} A_{u3}$$

,
Norm of $\vec{A}$:

$$|\vec{A}| = \sqrt{A_{u1}^2 + A_{u2}^2 + A_{u3}^2}$$

For a differential length dl,

$$dl = \vec{a_{u1}}(h_1 du_1) + \vec{a_{u2}}(h_2 du_2) + \vec{a_{u3}}(h_3 du_3)$$

h_i is called metric coefficient.

Coordinates

differential volume:

$$dv = h_1 h_2 h_3 du_1 du_2 du_3$$

differential area vector with a direction normal to the surface,

$$d\vec{s} = \vec{a_n} ds$$

differential area ds_1 normal to the unit vector $\vec{a_{u1}}$.

Cartesian Coordinates

- $$(u_1, u_2, u_3) = (x, y, z)$$
- Right hand rule:
  $$\vec{a_x} \times \vec{a_y} = \vec{a_z}$$
- $$\vec{A} = \vec{a_x} A_x + \vec{a_y} A_y + \vec{a_z} A_z$$
  , where $\vec{a_i}$ is the basis for i-axis.

Cartesian Coordinates

- dot product and cross product:

$$\vec{a_x} \cdot \vec{a_x} = 1, \vec{a_x} \times \vec{a_y} = \vec{a_z}.$$

- differential length:

$$d\vec{l} = \vec{a_x}dx + \vec{a_y}dy + \vec{a_z}dz \tag{1}$$

- differential area:

$$ds_x = dydz$$

, as $h_1 = h_2 = h_3 = 1$,
(ds_x is the surface perpendicular to the x-axis, the forms for other surfaces follow the same pattern).

- differential volume:

$$dv = dxdydz$$

Cylindrical Coordinate

- $$(u_1, u_2, u_3) = (r, \phi, z)$$

  Claim: as a_r can change its direction in the x-y plane, vectors in x-y plane could be represented simply by $\vec{a_r}$. Thus, all vectors in cylindrical coordinate could be represented by $\vec{a_r}$ and $\vec{a_z}$.
- Right hand rule:
  $$\vec{a_r} \times \vec{a_\phi} = \vec{a_z}$$
- $$\vec{A} = \vec{a_r}Ar + \vec{a_\phi}A_\phi + \vec{a_z}A_z$$
- differential length:
  $$d\vec{l} = \vec{a_r}dr + \vec{a_\phi}rd\phi + \vec{a_z}dz \tag{2}$$

  , as $h_1 = 1, h_2 = r, h_3 = 1$

Cylindrical Coordinate

- differential area:

$$ds_r = rd\phi dz$$

- differential volume:

$$dv = rdrd\phi dz$$

- From cylindrical coordinate to Cartesian coordinate: represent A_x by the quantities in cylindrical coordinate.The same applies to A_y.

Cylindrical Coordinate

Conversion of quantities between Cartesian coordinate and Cylindrical coordinate:

1.

$$\begin{cases} x = rcos\phi \\ y = rsin\phi \\ z = z \end{cases}$$

2.

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \phi = arctan\frac{y}{x} \\ z = z \end{cases}$$

You can try to write the conversion between dx, dy, dz and $dr, d\phi, dz$.

Cylindrical Coordinate

The conversion between dx, dy, dz and $dr, d\phi, dz$:

$$\begin{array}{cc} \begin{bmatrix} A_r \cos\phi - A_\phi \sin\phi \\ A_r \sin\phi + A_\phi \cos\phi \\ A_z \end{bmatrix} & \begin{matrix} A_x \\ A_y \\ A_z \end{matrix} \end{array}$$

Spherical Coordinate

- Figure for Spherical Coordinate. Notice the position of ϕ, θ.
- $$(u_1, u_2, u_3) = (R, \theta, \phi)$$
- Right hand rule:
$$\vec{a_R} \times \vec{\theta} = \vec{\phi}$$
- $$\vec{A} = \vec{a_R} A_R + \vec{a_\theta} A_\theta + \vec{a_\phi} A_\phi$$
- differential length:
$$d\vec{l} = \vec{a_R} dR + \vec{a_\theta} R d\theta + \vec{a_\phi} R sin\theta d\phi \tag{3}$$
, as $h_1 = 1, h_2 = R, h_3 = R sin\theta$.
- differential area:
$$ds_R = R^2 sin\theta d\theta d\phi$$

Spherical Coordinate

- differential volume:

$$dv = R^2 sin\theta dR d\theta d\phi$$

- conversion of quantities between Cartesian coordinate and Spherical coordinate:

  1.

$$\begin{cases} x = Rsin\theta cos\phi \\ y = Rsin\theta sin\phi \\ z = Rcos\theta \end{cases}$$

  2.

$$\begin{cases} R = \sqrt{x^2 + y^2 + z^2} \\ \theta = arctan\frac{\sqrt{x^2+y^2}}{z} \\ \phi = arctan\frac{y}{x} \end{cases}$$

- From Spherical coordinate to Cartesian coordinate: represent A_x by the quantities in Spherical coordinate; write the formula in the form of matrix. (Similar to cylindrical coordinate).

Review Exercises

P.2-17

A field is expressed in spherical coordinates by $\vec{E} = \vec{a_R}(25/R^2)$.

b) Find the angle that $\vec{E}$ makes with the vector $\vec{B} = \vec{a_x}2 - \vec{a_y}2 + \vec{a_z}$ at point $P(-3, 4, -5)$.

Review Exercises

P.2-17

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b) Find the angle that $\vec{E}$ makes with the vector $\vec{B} = \vec{a_x}2 - \vec{a_y}2 + \vec{a_z}$ at point $P(-3, 4, -5)$.

Answer

$$\cos(\theta) = \frac{\vec{E} \cdot \vec{B}}{|\vec{E}| \cdot |\vec{B}|}.$$

Review Exercises

Express the base vectors $\vec{a_x}, \vec{a_y}, \vec{a_z}$ of a Cartesian coordinate in spherical coordinate system.

Review Exercises

Express the base vectors $\vec{a_x}, \vec{a_y}, \vec{a_z}$ of a Cartesian coordinate in spherical coordinate system.

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & -\sin\phi & \cos\theta\cos\phi \\ \sin\theta\sin\phi & \cos\phi & \cos\theta\sin\phi \\ \cos\theta & 0 & -\sin\theta \end{bmatrix} \begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix}$$

Review Exercises

P.2-19

Determine the values of the following products of base vectors.

a) $\vec{a_x} \cdot \vec{a_\phi}$

c) $\vec{a_r} \times \vec{a_x}$

Review Exercises

P.2-19

Determine the values of the following products of base vectors.

a) $\vec{a_x} \cdot \vec{a_\phi}$

c) $\vec{a_r} \times \vec{a_x}$

Answer

a) $\vec{a_x} \cdot \vec{a_\phi} = \cos(\frac{\pi}{2} + \phi)$

c) $\vec{a_r} \times \vec{a_x} = -\vec{a_z} \sin(\phi)$

Further plans

In the future RC classes, I will review some problems in the homework. You can turn in your assignment directly in my RC classes. Though the assignment might not be graded in details and not counted into the final grade, we still hope you can come to the RC and submit your assignments during the RC, which will not take so much time. The assignments will give you a basic review about the course material, which will be so beneficial for your exam and quiz.