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A stochastic programming model for dynamic portfolio management with financial derivatives

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ABSTRACT

Stochastic optimization models have been extensively applied to financial portfolios and have proven their effectiveness in asset and asset-liability management. Occasionally, however, they have been applied to dynamic portfolio problems including not only assets traded in secondary markets but also derivative contracts such as options or futures with their dedicated payoff functions. Such extension allows the construction of asymmetric payoffs for hedging or speculative purposes but also leads to several mathematical issues. Derivatives-based nonlinear portfolios in a discrete multistage stochastic programming (MSP) framework can be potentially very beneficial to shape dynamically a portfolio return distribution and attain superior performance. In this article we present a portfolio model with equity options, which extends significantly previous efforts in this area, and analyse the potential of such extension from a modeling and methodological viewpoints. We consider an asset universe and model portfolio set-up including equity, bonds, money market, a volatility-based *exchange-traded-fund* (ETF) and *over-the-counter* (OTC) option contracts on the equity. Relying on this market structure we formulate and analyse, to the best of our knowledge, for the first time, a comprehensive set of optimal option strategies in a discrete framework, including canonical protective puts, covered calls and straddles, as well as more advanced combined strategies based on equity options and the volatility index. The problem formulation relies on a data-driven scenario generation method for asset returns and option prices consistent with arbitrage-free conditions and incomplete market assumptions. The joint inclusion of option contracts and the VIX as asset class in a dynamic portfolio problem extends previous efforts in the domain of volatility-driven optimal policies. By introducing an optimal trade-off problem based on expected wealth and *Conditional Value-at-Risk* (CVaR), we formulate the problem as a stochastic linear program and present an extended set of numerical results across different market phases, to discuss the interplay among asset classes and options, relevant to financial engineers and fund managers. We find that options' portfolios and trading in options strengthen an effective tail risk control, and help shaping portfolios returns' distributions, consistently with an investor's risk attitude. Furthermore the introduction of a volatility index in the asset universe, jointly with equity options, leads to superior risk-adjusted returns, both in- and out-of-sample, as shown in the final case-study.

1. Introduction

Stochastic optimization models have been extensively applied to asset and asset-liability management in the past with contributions dating back to Nielsen and Zenios (1996); Carino and Ziemba (1998); Høyland (1998); Consigli and Dempster (1998); Kouwenberg (2001). The inclusion in dynamic portfolios of derivative contracts, such as options or futures, with their specific payoff functions was initially attempted in continous time markets by Merton et al. (1978); Harrison and Pliska (1981); Brennan and Cao (1996) relying on stochastic control methods to formulate and solve optimal pricing and hedging problems. Key pricing results were then extended to a discrete time set-up mainly through nu-

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merical approaches and stochastic optimization methods (Klaassen, 1998; Consiglio and De Giovanni, 2008; Haarbrücker and Kuhn, 2009; Pflug and Broussev, 2009). In a *multistage stochastic programming* (MSP) framework, optimal portfolio strategies are determined as, and contingent on, scenario tree processes adopted as models of an evolving market uncertainty. The MSP formulation is very popular and practical in several application domains (Bertocchi et al., 2011; Consigli et al., 2016). Contributions involving financial derivatives, however have not been so frequent and mainly fall in these areas:

- Valuation of contingent claims in a discrete setting: King (2002); Blomvall and Lindberg (2003) initially in complete then in incomplete markets (Pennanen and King, 2004; Pınar et al., 2010; Consiglio et al., 2016).
- Hedging and risk control problem solutions for specific problem classes: since (Wu and Sen, 2000; Gondzio et al., 2003), then more recently thanks to Pınar (2013); Barkhagen and Blomvall (2016).
- Under more general model instances and decision criteria optimal dynamic portfolio management models with derivatives have been proposed by Topaloglou et al. (2011); Geyer et al. (2010); Yin and Han (2013).

Recently (Topaloglou et al., 2020) have presented an integrated model for market and currency hedging in international portfolios: the contribution is relevant for all the above-mentioned three perspectives and includes an extended in- and out-of-sample model validation. This research relies and extends results from the first problem class but falls mainly in the last one and through a rather general and comprehensive modeling framework, aims at extending significantly the scope of previously proposed SP-based financial portfolio models with equity options. None of past contributions, even in dynamic frameworks, allowed indeed for option contracts with different maturities and moneyness conditions, nor any effort to derive structured optimal option portfolios' strategies has been previously attempted within a genuine multi-stage framework. Key to this application domain is the formulation of a dynamic portfolio problem in markets which do not allow any arbitrage opportunity nor return generation without any risk exposure. Along this research line (Klaassen, 1998; 2002) tackled effectively the issue of arbitrage-free scenario trees in a stochastic programming ALM problem. Following (Høyland and Wallace, 2001), Klaassen extended the arbitrage-free conditions to allow for moment matching scenario generation. In the same period, King (2002) formulated the contingent claim pricing problem as an optimal portfolio replication problem and opened the way to the solution of the pricing problem under an assumption of market incompleteness with stochastic programming methods: the incomplete market assumption is relevant whenever the asset universe is not sufficient to hedge all underlying risk sources, as in presence of a stochastic volatility or market frictions: see more details on this point in Section 4.2. Still under an incomplete market assumption, Haarbrücker and Kuhn (2009) and Pflug and Broussev (2009) solved an electricity swing option pricing problem. More recently (Consiglio et al., 2016) proposed a parsimonious model for generating arbitrage free scenario trees with an option pricing application to insurance contracts, while (Barkhagen and Blomvall, 2016) considered a hedging probem for an option book using stochastic programming. Topaloglou et al. (2020) presents an arbitrage-free pricing model for currency and equity options as well as quantos (these are stock options with a currency-based payoff) with a novel closed-form pricing scheme specifically for quantos. By extending the investment universe to include equity options in a multiperiod setting, without imposing, a-priori, any specific restriction on the set of eligible option contracts we intend to analyze under pretty general assumptions and, to our knowledge, for the first time:

- The hedging effectiveness and the potential for risk mitigation and performance enhancement of portfolios including equity, volatility, bonds and money market indices together with OTC European equity options in specific US equity market periods.
- The out-of-sample performance from January 2011 to June 2021, of optimal portfolios including options of different maturities, available for trading and terminal exercise.
- The interaction between investment processes with nonlinear payoffs, volatility and investors' risk preferences, as captured by an expected wealth-*Conditional value-at-risk* (CVaR) trade-off.
- The modeling and financial engineering implications of complex derivatives-based option strategies.

In these contexts, previous works on derivatives' portfolios had several limitations. Topaloglou et al. (2011), for instance, even if formulating rigorously an optimal portfolio problem with derivatives, considered a single-stage model in which the options' expiry was forced to coincide with the problem investment horizon. By including both call and put options, though, they extended (Blomvall and Lindberg, 2003) where only call options were considered, in this way allowing more flexible option strategies. In their recent contribution (Topaloglou et al., 2020) extend and generalise significantly their previous results, but still within a 2-stage problem formulation. Yin and Han (2013) were the first to include options in a genuine multistage framework, with option contracts available in each stage but always expiring at the stage immediately following. Such financial and modeling constraint was later on removed by Davari-Ardakani et al. (2016) but only to consider long options' exposure and focusing mainly on a novel scenario generation model. Nowhere options' portfolios including, through options strategies or as specific trading instruments, several types of options were considered, nor options' dynamic trading opportunities, nor the volatility as asset class.

We build on those grounds and analyse in this article the implications of extending the investment universe to equity options and volatility contracts for a portfolio manager with a short (6 month) investment horizon, assumed to revise her strategy monthly. Our ambition is to extend the state-of-the-art in SP-based portfolio management with derivatives by providing a rather general and comprehensive modeling framework for European-style equity options and associated volatility-based option strategies. The followings can be regarded as specific contributions of this article:

- A consistent model extension to incorporate asymmetric payoffs in a discrete decision model and optimize complex derivative strategies within a linear MSP framework, where long as well as short option exposures with several maturities over the investment horizon are considered.
- A fully integrated management of a specific class of derivatives into a portfolio, holistic view as contrasted with overlays. Trading in options at any intermediate time before expiry is included together with the possibility to trade volatility contracts: such possibility becomes today realistic thanks to the *exchange traded funds* (ETF) contracts on the VIX.
- The pricing of equity options in a data-driven scenario method, consistent with an underlying assumption of incomplete markets.

In a previous work, Barro et al. (2019) presented an extended set of results on portfolio selection with an investment universe based on S&P500 subindices and the index itself as reference benchmark. The asymmetric payoffs associated with volatility control or downside risk minimization were introduced relying on a set of *mean-absolute-deviation* (MAD) models. Here we do explicitly consider option contracts in the optimization problem and in this way allow a set of relevant generalizations. The VIX, furthermore is here treated as a potential investment opportunity, rather than as an *early-warning-signal* as was in Barro et al. (2019). We model an optimization problem from the viewpoint of an investor whose decision paradigm considers a trade-off between expected wealth and CVaR. The collected results support the effort both in terms of in-sample model validation as for out-of-sample performance and hedging analysis. We focus on selected equity market periods and then extend the analysis to a decade from January 2011 to June 2021, thus including the recent pandemic period. We find, in particular, that options' strategies are systematically included in optimal dynamic portfolios under different modeling assumptions and risk-reward trade-offs. This evidence remains valid under different asset universe definitions, and in presence of a volatility index as investment opportunity.

The article evolves from Section 2 with the development of a portfolio optimization model where buying and selling decisions on options are allowed, to Section 3 in which the modeling of derivative strategies is considered before summarizing in Section 4 the adopted option pricing methods and scenario generation approach for the asset universe. In Section 5 we present an extended set of results and analyse the overall implications of the proposed extensions. The conclusions complete the article.

2. Optimal investment model with equity options

In a discrete financial market, information is assumed to evolve according to a non recombining scenario tree. We label nodes in the tree as $n \in \mathcal{N}_t$ at time stage t, where every n has a unique ancestor $n- \in \mathcal{N}_{t-1}$ and for $t \leq T-1$ there exists a non-empty set of children nodes $n+ \in \mathcal{N}_{t+1}$ whose cardinality is denoted by $\#n+$. For every $n \in \mathcal{N}_T$ a scenario is a path $\omega_n, \omega_{n-}, \omega_{n--}, \dots, \omega_{n_0}$ where n_0 is the root node. We use $a(n)$ to denote the set of ancestor nodes of n. The decision horizon is discrete and finite, $t \in \mathcal{T}$, $\mathcal{T} := \{t_0, t_1, t_2 \dots, t_J = T\}$. In our application, a 6-month planning horizon, $T = 6$ is considered with monthly steps. The modeling framework, however, can naturally accommodate a variety of decision frequency and planning horizon pairs, yet avoiding the curse-of-dimensionality of MSP models. See, in this respect, the computational results in Section 5 where problems with a scenario tree based on 10 240 scenarios, are solved in seconds of CPU time. For every node $n \in \mathcal{N}_t$ we denote with t_n the time associated with node n. Each node carries a probability of occurrence given by $p(n)$ such that $\sum_{n \in \mathcal{N}_T} p(n) = 1$ and for every non-terminal node $p(n) = \sum_{m \in n+} p(m)$, $\forall n \in \mathcal{N}_t$, $t \leq T-1$. At $t = 0$ there is a unique node n_0 which is the *root*-node and it is labelled as 0. A set of assets in $\mathcal{I}$, $i \in \mathcal{I}$ including equity, bond, money market indices, a volatility index and equity options exposures to complete the investment universe, will be considered in the computational study.

We consider the following random parameters:

- r_n is the risk-free rate in node n for the period from t_n to t_{n+}.
- v_{in} is the price of asset i, in node n and we indicate with $\rho_{i,n} = \frac{v_{in}}{v_{in-}} - 1$ the asset return over the period $t_n - t_{n-}$.
- $O^c_{1n}(j,k)$ is the price in node n of a European call option on the asset $i = 1$ (equity) with strike price equal to k and expiry t_j.
- $O^p_{1n}(j,k)$ is the price in node n of a European put option on asset 1 with strike k and expiry t_j.

and decision variables:

- x^+_{in} nominal amount of asset i purchased in node n,
- x^-_{in} nominal amount of asset i sold in node n,
- x_{in} amount of asset i held in node n,
- $c^l_{1n}(j,k)$ long position in equity call options with strike k and expiry t_j.
- $c^{l+}_{1n}(j,k)$ and $c^{l-}_{1n}(j,k)$ purchases and sales of equity calls, respectively.
- $c^s_{1n}(j,k)$ short positions on equity call options with strike k and expiry t_j
- $c^{s+}_{1n}(j,k)$ and $c^{s-}_{1n}(j,k)$ increments and decrements of short equity call positions, respectively.
- $p^l_{1n}(j,k)$ long positions in equity put options with strike k and expiry t_j.
- $p^{l+}_{1n}(j,k)$ and $p^{l-}_{1n}(j,k)$ purchases and sales of equity puts, respectively.
- $p^s_{1n}(j,k)$ short positions in equity put options with strike k and expiry t_j
- $p^{s+}_{1n}(j,k)$ and $p^{s-}_{1n}(j,k)$ increments and decrements of short equity put positions, respectively.

To ease notation and facilitate a compact model representation we use $\mathcal{C}^h_{1n}(j,k)$ to define a generic option position on the equity with maturity t_j and strike k in node n where $\mathcal{C} = \{c, p\}$, $h = \{l, s\}$ to accommodate the above derivatives transactions.

Specific to the options' trading definition is the distinction between increasing-decreasing long and short option exposures: thanks to the introduced notation we capture in a sufficiently simple way, options trading decisions, hedging policies and protection selling due to short positions. Different model specifications will then support alternative option strategies and associated portfolio dynamics. All decision variables, including options, are assumed $\mathbb{R}_+$-valued: after optimization, actual options' exposures may be determined for given contract size and market convention, by deriving the associated number of call and/or put options to buy or sell. We will not address this operational detail, which however, from a mathematical programming perspective would require the formulation of a mixed-integer stochastic program. Model variables, as shown below, include W_n, the value of the portfolio in node n, C_n the cash generated by transactions involving the call options, P_n similarly for put options. The model specification is completed including policy constraints on the asset portfolio in the form of asset-specific lower L_i and upper U_i portfolio proportions. Options exposures depend instead on the adopted strategy as explained in what follows.

Under these assumptions, the following model details are of interest:

- The exposure on options, either long or short, is determined by the problem solution and may span from specific subperiods to the entire investment horizon.
- Investors' risk profiles are determined by combining asymmetric payoffs of derivatives together with a canonical risk-reward trade-off function in the objective function specification in (1).
- *Bullish* or *bearish* portfolio strategies may rely on derivative contracts and again be employed over specific subperiods of the planning horizon.
- The optimal hedge ratio is determined endogenously by introducing for every contract different strikes and level of moneyness, making optimal partial hedging strategies possible.

2.1. Volatility as an asset class

The introduction in 2014 of an ETF on VIX futures, as expected, has had a significant impact on market practice with remarkable trading activity since its launch, particularly intraday. Relying on such evidence, still within a model portfolio setting (thus based on indices as canonical in fund management), we treat the volatility index as a possible investment opportunity. The VIX has shown almost perfect positive correlation with the ETF over the quotation period and its negative correlation with the S&P500 index (S&P in what follows), is well known and a standard assumption by market agents.

Its introduction in the asset universe leads to several interesting issues and expands the scope of financial innovation under an MSP approach:

- Due to the VIX-S&P negative correlation and thus, the potential indirect hedging role of the ETF on the VIX, would option-based policies be still needed? Indeed, unlike the other indices, the VIX fluctuates in a given range with mean around 20 in normal market conditions: such mean-reverting pattern can also lead to performance protection and hedging opportunities within a multiperiod setting.
- The assumption of a constant equity volatility is ruled out and we need to consider such evidence when pricing the option contracts.
- Option premiums are known to increase in presence of high market volatility: from a portfolio optimization perspective the VIX and the options can thus very well jointly enhance the portfolio performance.
- Low interest rate environments are typically associated with positive equity premia and low volatility, and the contrary in presence of growing or high interest rates: it is then interesting to analyse how optimal policies exploit such evidence.

2.2. Model set-up

We present the mathematical detail of the optimization problem by first sketching its overall structure and then by focusing on the constraints and in particular on the equations devoted to equity options. The objective function is defined through a canonical risk-reward function, whose trade-off is determined by a coefficient $\lambda \in [0,1]$ and, whose risk measure is the terminal CVaR with tolerance α (Rockafellar and Uryasev, 2002). We indicate here the investment opportunities, the call and put option positions with x, c, p, respectively:

$$\max_{x,c,p}(1-\lambda)\mathbb{E}[W_{n\in\mathcal{N}_T}] - \lambda CVaR(\alpha, W_{n\in\mathcal{N}_T}) \tag{1}$$

s.t. for all $t \in \mathcal{T}, n \in \mathcal{N}_t$ and given initial conditions $x_0^0, x_{i,0}, W_0$ (no options in the initial portfolio):

$$W_n = x_n^0 + \sum_{i\in\mathcal{I}} x_{in} v_{1n} + NCP_n + NPP_n \tag{2}$$

$$x_n^0 = x_{n-}^0(1+r_{n-}) + \sum_{i\in\mathcal{I}} x_{in}^- v_{1n}\delta^- - \sum_{i\in\mathcal{I}} x_{in}^+ v_{1n}\delta^+ + C_n + P_n \tag{3}$$

$$x_{i,n} = x_{i,n-} + x_{i,n}^+ - x_{i,n}^- \quad \forall i \in \mathcal{I} \tag{4}$$

$$\mathcal{C}_{1n}^h(j,k) = \mathcal{C}_{1n-}^h(j,k) + \mathcal{C}_{1n}^{h+}(j,k) - \mathcal{C}_{1n}^{h-}(j,k) \quad \forall h, j, k \tag{5}$$

$$L_i W_n \le x_{in} \le U_i W_n, \ \ \forall\ i \in \mathcal{I} \tag{6}$$

$$x_{i,n}^+ = x_{in}^- = C_{1,n}^{h,+}(j,k) = C_{1,n}^{h,-}(j,k) = \mathcal{C}_{1n}^h(j,k) = 0 \quad \forall i \in \mathcal{I}, h \in \mathcal{H}, n \in \mathcal{N}_T \tag{7}$$

We clarify first the overall model structure, the objective function and the specific definition of the wealth equation. Afterwards the derivatives' specific variables C_n and P_n in the cash balance constraint (3) and the option constracts inventory balance Eqs. (5) are considered. The latter in particular may imply pretty different underlying options' exposures and they represent a key modeling contribution of the optimal problem formulation. Through the problem solution optimal options exposures will be determined.

The optimization problem (1) subject to (2)–(7) includes a canonical objective function based on a terminal expected wealth-CVaR trade-off, a wealth equation in (2), a cash balance constraint in (3) accounting for all cash inflows and outflows at every stage, two inventory balance equations: for asset positions in (4) and for derivatives in (5). In (6) we set lower and upper bounds on asset proportions within the current portfolio value and in (7) we rule out possible rebalancing decisions at T.

Consider the wealth Eq. (2): in each node n this is the sum of the cash surplus, the value of the investment portfolio plus the long and minus the short positions in calls and puts. Under this definition *protection buying* increases the value of the portfolio while *protection selling* reduces the wealth value. We have:

$$NCP_n = \sum_{t_n<t_j\le T-1,k}[c_{1n}^l(j,k)O_{1n}^c(j,k) - c_{1n}^s(j,k)O_{1n}^c(j,k)] \tag{8}$$

$$NPP_n = \sum_{t_n<t_j\le T-1,k}[p_{1n}^l(j,k)O_{1n}^p(j,k) - p_{1n}^s(j,k)O_{1n}^p(j,k)]. \tag{9}$$

$\forall n \in \mathcal{N}_t, t \le T-1$, thus excluding options expirying at the final horizon T.

Constraints (4) trace standard stage-by-stage rebalancing decisions on assets $i \in \mathcal{I}$. The policy constraints (6) are canonical in multiperiod portfolio management and define asset-specific lower L_i and upper U_i bounds on asset positions, relative to the current portfolio wealth. Finally due to (7) neither rebalancing nor variations in option exposures or trading on derivatives are allowed at the end of the planning horizon.

Consider now the cash account (3): it will depend on the cash available at the beginning of the period and interests thereof, on assets sellings and buyings and on options. Indeed C_n and P_n for all $n \in \mathcal{N}_t$, $t \in \mathcal{T}$ are generated by the options trading and cash settlements at expiry. Before expiry any option position can be traded at the current price:

- C_n is the monetary amount from cash settlement of call option positions: this includes purchases and sales of long and short call options and cash settlement of expiring options.
- P_n is the monetary amount from cash settlement of put option positions: it includes purchases and sales of put long and short options and cash settlement of expiring options.

We present in Section 5, an extended set of evidences on the benefits generated by the trading as well as the exercise, when in-the-money, of option contracts at maturity. We have:

$$\begin{aligned} C_n = C_n^l + C_n^s = &\sum_{t_n\le t_j\le T,k}[-c_{1n}^{l+}(j,k)O_{1n}^c(j,k) + c_{1n}^{l-}(j,k)O_{1n}^c(j,k)] \\ &+ \sum_{t_{n-}\le t_j\le t_n,k} c_{1n}^l(j,k)\max(0, v_{1n}-k) \\ &+ \sum_{t_n\le t_j\le T,k}[c_{1n}^{s+}(j,k)O_{1n}^c(j,k) - c_{1n}^{s-}(j,k)O_{1n}^c(j,k)] \\ &- \sum_{t_{n-}\le t_j\le t_n,k} c_{1n}^s(j,k)\max(0, v_{1n}-k) \end{aligned} \tag{10}$$

$$\begin{aligned} P_n = P_n^l + P_n^s = &\sum_{t_n\le t_j\le T,k}[-p_{1n}^{l+}(j,k)O_{1n}^p(j,k) + p_{1n}^{l-}(j,k)O_{1n}^p(j,k)] \\ &+ \sum_{t_{n-}\le t_j\le t_n,k} p_{1n}^l(j,k)\max(0, k-v_{1n}) \\ &+ \sum_{t_n\le t_j\le T,k}[p_{1n}^{s+}(j,k)O_{1n}^p(j,k) - p_{1n}^{s-}(j,k)O_{1n}^p(j,k)] \\ &- \sum_{t_{n-}\le t_j\le t_n,k} p_{1n}^s(j,k)\max(0, k-v_{1n}) \end{aligned} \tag{11}$$

where at expiry the value of the options coincides with the payoffs $\max(0, v_{1n}-k)$ for a call and $\max(0, k-v_{1n})$ for a put option. In every node we consider separately the cash flows generated by call and put contracts, taking into account only selling and buying decisions and options maturities, as detailed in (10) and (11). All equations are linear and the asymmetry associated with the options' terminal payoffs is handled through the cash settlement for given option price at expiry.

Upon expiry the contract will be cash-settled so if expiring ITM the payoffs will generate a cash inflow or outflow and in this way affect the cash balance x_n^0. If expiring OTM the associated premiums were already accounted for at the time in which the option position was open and otherwise during their life, before expiry, the options will just be treated as assets or liabilities.

For modeling, as well as for financial consistency, reducing a long option position is different from taking a short position. The term *selling* will be kept distinct from the term *writing*. The same applies to all long and short options on call or put contracts. Throughout the article we consider a long equity investor and assume that none of the assets but the derivatives can go short. Consider the inventory balance equation for option contracts (5), with $\mathcal{C}^h_{1n}(j,k)$ where $\mathcal{C}$ stands for call or put and h reflects long or short positions on asset $i=1$, the equity. In each node $n \in \mathcal{N}_t$ and for every option, we have the following updating on long positions, in nominal terms:

$$c^l_{1n}(j,k) = \begin{cases} c^l_{1n-}(j,k) & \text{if } t_j \le t_n, \forall\ k, \\ c^l_{1n-}(j,k) + c^{l+}_{1n}(j,k) - c^{l-}_{1n}(j,k) & \text{if } t_j > t_n, \forall\ k \end{cases}$$

$$p^l_{1n}(j,k) = \begin{cases} p^l_{1n-}(j,k) & \text{if } t_j \le t_n, \forall\ k, \\ p^l_{1n-}(j,k) + p^{l+}_{1n}(j,k) - p^{l-}_{1n}(j,k) & \text{if } t_j > t_n, \forall\ k \end{cases}$$

And on short positions:

$$c^s_{1n}(j,k) = \begin{cases} c^s_{1n-}(j,k) & \text{if } t_j \le t_n, \forall\ k, \\ c^s_{1n-}(j,k) + c^{s+}_{1n}(j,k) - c^{s-}_{1n}(j,k) & \text{if } t_j > t_n, \forall\ k \end{cases}$$

$$p^s_{1n}(j,k) = \begin{cases} p^s_{1n-}(j,k) & \text{if } t_j \le t_n, \forall\ k, \\ p^s_{1n-}(j,k) + p^{s+}_{1n}(j,k) - p^{s-}_{1n}(j,k) & \text{if } t_j > t_n, \forall\ k \end{cases}$$

.

Following this scheme it becomes possible to adopt strategies involving long and short positions simultaneously on contracts on the same underlying but different maturities and strikes. The protection can increase or decrease over time and the solution of the optimization problem will determine, through the selection of specific option contracts, the optimal degree of protection (depending on the moneyness) and the protection period (relative to the planning horizon).

3. Derivatives strategies

Following the cash balance constraints (3) and the details in C_n and P_n, we devote this section to analyse more in detail the possible combinations of options and their modeling implications from a portfolio management perspective. The popularity of option contracts is well known to depend on their theoretically unlimited return potential particularly under *uncovered* speculative portfolios' exposure and on the effective hedging strategies in case of equity exposures.

By increasing long positions in options an investor will be protecting future buying and selling decisions from price increases (for calls) or decreases (for puts). The final wealth distribution, furthermore, is expected to be positively skewed with a left tail cut. In the presented case study, however, only covered equity option positions will be allowed. By instead increasing short positions, either on call or put contracts, an investor will be offering protection to the market and assume a liability: an effective hedging strategy by the market maker through an underlying continuously adjusted equity position should result in the generation of a risk-free return. Here, however, we won't consider the possibility of short equity positions and thus short put positions will not be considered either.

The proposed modeling approach, though, allows several interesting applications, with relevant, previously unexplored in an MSP set-up, financial engineering implications:

- Simple call- or put-based strategies with equity, bonds and money market, plus volatility, with or without constraints on equity positions: hedging or speculative strategies are allowed in general and the investor may exploit the options' leverage effect.
- Under the same investment universe, but with a lower bound on equity investment, portfolio insurance policies such as a protective put or a covered call may be adopted: in the first case through a long put and negative equity market expectations, while in the second case looking for a profit under stable market expectations.
- More complex strategies based on joint call and put contracts such as straddles, strips and straps depending on expectations on equity prices and volatility.

We are interested in particular on the following strategies based on equity options: (i) protective put, (ii) covered call, (iii) straddle and (iv) strip and strap strategies. Notice however that several other option strategies with even more complex payoffs may be handled in this framework preserving a linear programming formulation. In Section 5 we analyse each such strategy and assess their effectiveness to determine the portfolio evolution and shape the risk exposure, eventually reflected in the wealth probability distribution at the horizon.

3.1. Protective put

This is the classical portfolio insurance strategy based on a long equity position, whose value in specific periods, is protected by a long equity put position: given the nodal equity value upon inception of the strategy, the strike of the option and its maturity will determine the extent of the protection (based on OTM, ATM, ITM options) and the hedging period (depending on the put contract maturity). Equity selling and portfolio rebalancing are always possible and the optimal solution may include rebalancing and protection. Constraints (5) take in this case the following characterization, for all $n \in \mathcal{N}_t, t \in \mathcal{T}$:

$$\begin{aligned} \mathcal{C}^l_{1n} &= \textstyle\sum_{j,k} p^l_{1n}(j,k) \\ \mathcal{C}^l_{1n} &= \mathcal{C}^l_{1n-} + \mathcal{C}^{l+}_{1n} - \mathcal{C}^{l-}_{1n} \\ \textstyle\sum_{j,k} p^l_{1n}(j,k) &\le x_{1n} \\ c^{l,s}_{1n}(j,k) = p^s_{1n}(j,k) &= 0 \quad \forall j,k \end{aligned} \tag{12}$$

3.2. Covered call

A covered call strategy is typically undertaken by an equity investor who, rather than taking a put position at a desirable price, decides to go short a call option and earn the premium: the higher the market volatility the higher the premium for given exercise probability. The investor will in this case loose if the market goes down and the option is exercised but such loss will be compensated by the premium. Portfolio rebalancing will always be possible and the covered call strategy will be activated with given exposure in case of a positive impact on the problem objective value. The maximum profit generated by a covered call strategy will be realized when during the holding period, market volatility will decrease and the option expiring slightly OTM to avoid any loss on the equity position. The following constraints employ the strategy:

$$\begin{aligned} \mathcal{C}^s_{1n} &= \textstyle\sum_{j,k} c^s_{1n}(j,k) \\ \mathcal{C}^s_{1n} &= \mathcal{C}^s_{1n-} + \mathcal{C}^{s+}_{1n} - \mathcal{C}^{s-}_{1n} \end{aligned} \tag{13}$$

$$\sum_{j,k} c^s_{1n}(j,k) \le x_{1n}$$
$$c^l_{1n}(j,k) = p^{s,l}_{1n}(j,k) = 0 \quad \forall j,k$$

3.3. Straddle

A long straddle strategy combines an equity exposure with long ATM call and put contracts: such strategy will thus generate increasing profits in presence of high market volatility and would require the following constraints on the option positions:

$$\begin{aligned} \mathcal{C}^l_{1n} &= \sum_{j,k} p^l_{1n}(j,k) + \sum_{j,k} c^l_{1n}(j,k) \\ \sum_{j,k} p^l_{1n}(j,k) &= \sum_{j,k} c^l_{1n}(j,k) \\ \sum_{j,k} p^l_{1n}(j,k) &\le x_{1n} \\ \mathcal{C}^l_{1n} &= \mathcal{C}^l_{1n-} + \mathcal{C}^{l+}_{1n} - \mathcal{C}^{l-}_{1n} \\ c^s_{1n}(j,k) &= p^s_{1n}(j,k) = 0 \quad \forall j,k \end{aligned} \tag{14}$$

A long straddle would generate a loss in presence of stable market conditions. In case of growing or decreasing equity prices before expiry, on the other hand, this strategy may generate substantial profits.

3.4. Strip and strap

These strategies are very much related to straddles but differ because, still under a volatile equity market expectation, in the case of a strip a negative outlook is judged more likely than a positive one and the opposite holds for a strap: those different expectations will lead in the first case to buy put contracts in exceedance of call contracts, actually the double and in the second to revert the exposure with calls double the puts. A strip strategy would consider:

$$\begin{aligned} \mathcal{C}^l_{1n} &= \sum_{j,k} p^l_{1n}(j,k) + \sum_{j,k} c^l_{1n}(j,k) \\ \sum_{j,k} p^l_{1n}(j,k) &= \left[\sum_{j,k} c^l_{1n}(j,k)\right] \times 2 \\ \sum_{j,k} p^l_{1n}(j,k) &\le x_{1n} \\ \mathcal{C}^l_{1n} &= \mathcal{C}^l_{1n-} + \mathcal{C}^{l+}_{1n} - \mathcal{C}^{l-}_{1n} \\ c^s_{1n}(j,k) &= p^s_{1n}(j,k) = 0 \quad \forall j,k \end{aligned} \tag{15}$$

While for a strap, we would have $\sum_{j,k} c^l_{1n}(j,k) = \left[\sum_{j,k} p^l_{1n}(j,k)\right] \times 2$ and the put position still imposing a constraint on the equity investment.

3.5. Derivatives-based strategy summary

Protective put, covered call and plain straddle or strip and strap portfolios represent popular strategies based on *portfolios of options*. What is new in this setting is that we wish to determine, as a result of the stochastic program solution, an optimal options exposure in terms of combined strategies or employing individual options contracts in specific market phases. We assume investors long on equity: a put contract will protect the equity value in case of possible market drops. The strike of the put and its moneyness will determine the extent of such protection. The covered call, thanks to the call premium revenue, allows a compensation of potential equity losses generated by market drops. Both the protective put and covered call together with an underlying equity position would lead to a maximum potential profit determined by the options strike price. Finally the two at-the-money long positions on calls and puts in a straddle lead to a maximum potential loss given by the sum of the two premiums. Relative to the straddle the strip and strap portfolios introduce an asymmetry in the payoff which reflects different market expectations.

Each such strategy has an initial cost or revenue determined by the option prices and it will generate a profit according to their final payoffs. In terms of portfolio optimization, a perfect hedging strategy, with only equity and options in the portfolio, is bound to generate a sure risk-free return over the planning horizon. On the other hand a fully speculative strategy is known to expose the investor to unlimited profits and losses. When extending the asset universe to include money market and bond indices, a reduction of potential losses may be attained without the investment in derivatives but just by rebalancing the portfolio. When also volatility comes in as asset class, the portfolio risk exposure may be affected substantially. An increasing volatility leads jointly to increasing option premiums and, likely, decreasing equity values making beneficial a long volatility exposure and a reduced equity exposure. On the contrary under a decreasing VIX dynamics, the equity exposure is likely to increase in a bullish market. The VIX as a market signal, rather than its ETF as asset class, is commonly used to assess market expectations, as anticipating forthcoming market adjustments already *priced-in* by the market. Together with recent S&P dynamics, this is the signal that as in Barro et al. (2019) will be adopted to assess the inclusion in the investment universe of one or the other option strategy.

From a methodological perspective, the above stylized observations may be assessed by solving the stochastic program (1) under the constraints (2)–(7) and, depending on the strategy to be evaluated, from (12) to (15) additional conditions.

When implementing an option strategy a key and central issue has to do with the adopted pricing model: in principle as the market evolves and equity volatility fluctuates so will option prices. These are the market premiums quoted in options exchanges as the CBOT or OTC in bilateral agreements. On the other hand market makers will quote option prices according to a risk-neutral principle, or hedging principle which may differ from current market quotations: the market clearing will determine the trading price. In this article we do not distinguish between market and risk-neutral prices but just consider as relevant the fair price determined by the market maker. Indeed the option contracts included in the investment universe will be tailored on the specific case study with monthly rebalancing and 6 month planning horizon. The options actually adopted in the computational part are to be considered synthetic instruments on the S&P.

4. Tree processes and pricing

Consider a price tree process as the one in Fig. 1 (for a 4-stage 16-scenario tree, where a scenario is a unique price path from the root node to a leaf node). The tree structure adopted for pricing a financial instrument, say an option, is the same as the one considered to formulate and solve the optimization problem. The associated scenario probabilities, however, need not be the same.

In what follows, we assume that the maturity of option contracts coincides with one of the stages in the tree. Any option contract, whatever the maturity and strike, will be priced from the root node to expiry.

The introduction of derivatives contracts in a multi stage problem calls for the adoption of a consistent *arbitrage-free* pricing method. Theoretical foundations of no-arbitrage pricing go back as mentioned to Stephen (1976); Harrison and Kreps (1979); Harrison and Pliska (1981); Jacod and Shiryaev (1998). According to the well and longly established fundamental theorem of asset pricing, absence of arbitrage requires the existence of at least one probability measure under which the underlying stock price discounted at the risk-free interest rate will be a martingale, thus constant in expectation. Such measure will be unique if the market is complete. In this article, we take a simple data-driven method to generate price and return scenarios and accordingly in full generality, we do not require the market to be complete.

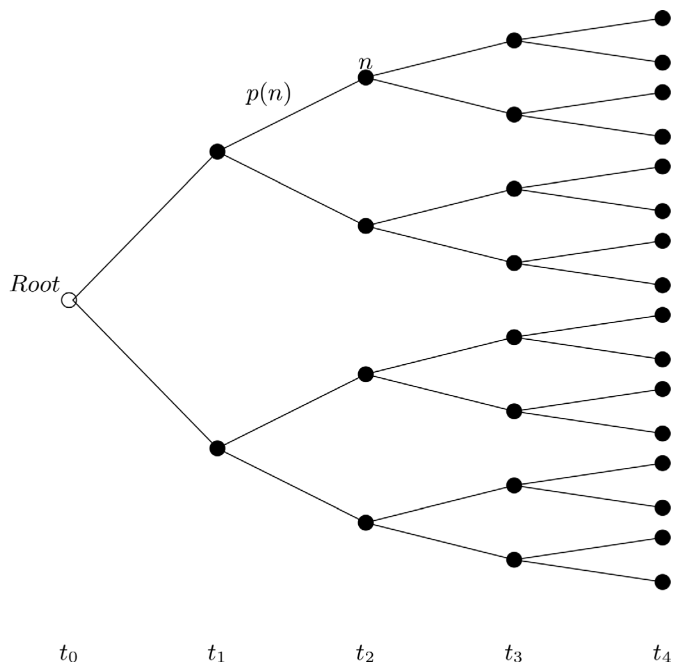

Fig. 1. A scenario and decision tree.

In a discrete, tree-based framework, we rely on a two-steps approach: we first verify the no-arbitrage conditions in the market and then determine the option price. We rely on Klaassen (1997, 2002)'s procedure to verify the arbitrage-free pricing method. We refer to Topaloglou et al. (2008) for a detailed description of the pricing rule applied to a scenario tree price process. More recently (Geyer et al., 2010; Consiglio et al., 2016; Barkhagen and Blomvall, 2016; Topaloglou et al., 2020) presented viable approaches to enforce arbitrage free conditions in a multi-asset portfolio problem and clarified the relationship between branching degree along the tree and financial market equilibrium. They also pointed out that under conditions of incomplete markets, due to lack of a unique replicating portfolio, a more general pricing approach must be taken as clarified in Section 4.2.

4.1. Data-driven arbitrage-free scenario generation

We adopt a data-driven scenario generation approach to determine the asset prices evolution first, verify for absence of arbitrage and then derive a risk-neutral probability measure to determine the option prices, for given equity price dynamics. As opposed to a model-based approach, a data-driven approach relies on a given data history and without any filtering but only through bootstrapping, it generates the sample space of the vector tree price process. To preserve contemporary dependence between time series, we sample simultaneously from each past asset data path. The bootstrapping procedure is based on the generation of random outcomes from a uniform distribution and simultaneous extraction of corresponding realizations from the past history of assets' returns. Once generated the scenario tree is tested for absence of arbitrage according to Klaassen (2002). If the requirement is satisfied the risk neutral probabilities are computed according to Frittelli (2000). The scenario tree generation and arbitrage checking are summarized in Algorithm 1 , while the derivation of the risk neutral probability measure and the option pricing are described below in Algorithm 2 . We recall the notation adopted in the scenario tree. For every $n \in \mathcal{N}_t, t < T$, $\#n+$ defines the cardinality of node n's *children* or descendant nodes, while $q_{n,n+}$ and $p_{n,n+}$ are the conditional risk neutral and empirical prob-

Algorithm 1: Arbitrage-free scenario tree generation (S&P, US 10Y Bond Index, Bill Index and VIX).

1 **Set initial conditions:**
2 Set data history $[t_a, t_b]$ and investment horizon $\mathcal{T} = \{t_0, t_1, .., t_n\}$ with $t_0 \geq t_b$ and $t_n = T$;
3 Upload time series $\mathbf{v}$ and compute historical returns ρ;
4 Set $\mathbf{v}(\mathbf{0})$ root nodal price at time $t_0 = 0$;
5 **Specify scenario tree structure:**
6 For every $t \in \mathcal{T},\ t < T$ and $\forall n \in \mathcal{N}_t$;
7 Set the branching degree $\#n+$ for every t and $n \in \mathcal{N}_t$ resulting in a symmetric scenario tree;
8 Specify the *Nodal Partion Matrix* (NPM) of dimension $\mathcal{N}_T \times \mathcal{T}$: columns for stages and rows for scenarios.;
9 **Generate scenario tree:**
10 **while** $t \leq T-1$ *and for each node* $n \in \mathcal{N}_t$ **do**
11 | Randomly generate $\#n+$ uniform random number $u \in [t_a, t_b]$;
12 | Sample, according to u, from the matrix of historical returns $\rho(\tilde{\mathbf{u}})$;
13 | Specify the NPM-based leaf nodal labels $n \in \mathcal{N}_T$ and scenario labels $n-, n--, \ldots, n_0$;
14 | Compute for every scenario: $\mathbf{v}(\mathbf{n}+) = \mathbf{v}(\mathbf{n})(\mathbf{1} + \rho(\tilde{\mathbf{u}})), \forall n+ \in \mathcal{N}_{t+1}$;
15 **end**
16 **Check for absence of arbitrage:**
17 **while** $t \leq T-1$, *for every* $n \in \mathcal{N}_t$ **do**
18 | Find the dual variables ν_{n+} for which:
19 | $\sum_{n+=1}^{\#n+} \nu_{n+}\,(1 + \rho_{i,n+}) = 1,\ \forall i \in \mathcal{I}$;
20 | **if** $\exists\ \nu_{n+} > 0\ n+ = 1, \ldots, \#n+$ *solution* **then**
21 | | No arbitrage opportunities exist;
22 | **else**
23 | | An arbitrage opportunity is present, Go to Line 9;
24 | **end**
25 **end**

Algorithm 2: Option pricing.

Compute risk neutral probability measure Q:
Set $n_0 = \{1\}$ root node, $n \in \mathcal{N}_t, 0 \leq t < T$, and for every t: $n+ \in (t+1)$ and branching degree $\#n+$;
Set $\mathbf{v_{1,n}}$ be the stock price in every node $n \in \mathcal{N}_t$ for every $t \in \mathcal{T}$;
Set $\rho_{\mathbf{1,n}}$ be the corresponding return process and $R_n = 1 + r_n$;
while $t \leq T-1$, *for each node* $n \in \mathcal{N}_t$ **do**
| Set $p_{n,n+} = \frac{1}{\#n+}$ for the physical conditional probability;
| Solve problem (16);
end
Compute option price:
while $t \leq T, t = t_j$ *and* $n \in \mathcal{N}_{t_j}$ **do**
| Compute for $\mathcal{C} = \{c, p\}$ and any h (no distinction between short and long)the payoff $\mathcal{C}^h_{1n}$;
end
while $t \leq t_{j-1}$ *and for each node* $n \in \mathcal{N}_t$ **do**
| derive by backward recursion

$$\mathcal{C}^h_{1n} = (1 + r_n)^{-1} \sum_{n+ \in \mathcal{N}_{t+1}} q_{n,n+}\mathcal{C}^h_{1n+}. \quad (17)$$

end

Table 1
Arbitrage checking: convergence properties for different number of scenarios and branching degrees. Average acceptance rate (AR) over 40 independent experiments, maximum number of draws registered over the nodes of the tree before accepting the first draw (Max). Results are averages over 40 independent experiments. Total CPU time in seconds (CPU) for each experiment.

# Scenarios	729	4096	6144	8192	10,240	1,000,000
Branching degree	3^6	4^6	$6-4^5$	$8-4^5$	$10-4^5$	10^6
AR	0.8002	0.8002	0.8007	0.8048	0.8082	0.8078
Max	6	8	7	7	8	8
CPU	109.90	303.73	487.56	606.12	950.46	73 324.41

abilities associated with every arc in the subtree originating from n. We use $R_n = 1 + r_n$ for a risk-free compounded return in node n and $\rho_{i,n+} = \frac{v_{i,n+}}{v_{i,n}} - 1$ for one-period return of asset $i \in \mathcal{I}$, with $\mathcal{I} = \{1, \dots, m\}$: according to the asset pricing rule for every subtree and parent node, the assets' expected return must be equal to the risk free interest rate R_n. In the pseudo-code, to simplify notation we use $\mathbf{v}$ for asset price vectors and ρ for associated return vectors.

According to Klaassen (2002), the presence of a strictly positive dual vector ν_{n+} on every subtree is sufficient to guarantee absence of arbitrage of both the first and second type.

The algorithm travels through the tree node-by-node and checks the arbitrage free condition on the dual variables (line 20 of the algorithm), considering the sequence of two-stage subtrees originating from every node until the last stage. At each node the algorithm bootstraps a proposal for assets returns and checks for the absence of arbitrage. In a data-driven model, the algorithm's termination will depend on the tree branching structure and the outcome of the sampling procedure (line 12 of the algorithm). In Table 1 we report results on the number of runs necessary to terminate this recursive procedure for a set of scenario trees with increasing branching degree, spanning a 6 month period and relying on the same data history of 208 monthly returns (June 2002–June 2020). We present computational times and assess the algorithm efficiency by estimating the acceptance rate in each node: this is defined by the ratio of the accepted to the total draws. For every scenario tree with given number of scenarios, we include the acceptance rate averaged over the nodes and we report results on the maximum number of rejections, (line 18 in Algorithm 1), registered before accepting the first draw. All results are averages over 40 independent experiments. We see that the maximum number of necessary draws is rather stable over increasing number of scenarios and branching structure. For computationally tractable problems it stabilizes around 8 and this is confirmed also for a much bigger tree dimension (10^6 scenarios). For illustrative purposes in Fig. 2 we present the average acceptance rate (red line) and the nodal specific acceptance rate computed for a $\{10-4^5\}$ scenario tree structure that corresponds to the size of the scenario trees used in the computational section of this article.

Once an arbitrage free scenario tree for the asset universe is determined, we address the option pricing problem under a risk neutral probability measure.

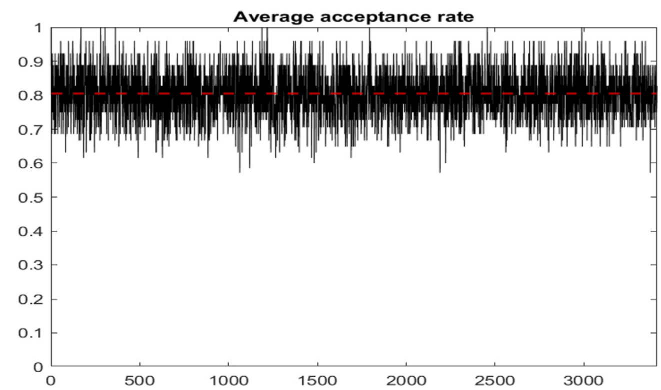

Fig. 2. Average acceptance rate (red dashed line) and node specific acceptance rate computed for $10-4^5$ scenario tree. All results are averages over 40 independent experiments. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

4.2. Risk-neutral pricing: dealing with market incompleteness

The existence of market frictions, such as transaction costs or trading constraints, or in presence of additional sources of uncertainty such as stochastic volatility or a random interest rate intensity, a condition of market incompleteness may arise. See Staum (2008) for a comprehensive review on incomplete markets in finance and possible pricing approaches. Under an assumption of incomplete market, there won't be a unique martingale measure nor a unique self-financed trading strategy able to perfectly replicate the option payoff under any price scenario. In this setting the arbitrage-free condition will be satisfied within a given stock price interval. Jacod and Protter (2017) summarize key pricing results under alternative underlying price processes. In general, given the uncertainty over the pricing measure to adopt, any contingent claim cannot be determined without first introducing a criterion to select a specific risk neutral probability measure. Popular approaches to come around this issue are in Schweitzer (1999) for the minimization of quadratic hedging risk error, and (Carmona, 2008), with references therein, for indifference pricing. Frittelli (2000) developed a financial pricing principle based on a *minimal entropy martingale* (MEM) measure, a concept followed in this article. This measure can be determined by minimizing the relative entropy of the risk neutral measure with respect to the physical measure, a particularly desirable property in a derivatives' portfolio problem. In a discrete setting, furthermore as shown next, this method finds an easy and effective implementation.

Following (Frittelli, 2000), in a finite state incomplete but arbitrage-free market with a random riskfree return, the MEM measure can be found by solving the following optimization problems, for $t \in \mathcal{T},\ t < T$, on all (one-stage-deep) subtrees in the scenario tree and parent nodes n:

$$\min_{q \in \mathbb{R}^{\#n+}, q>0} \left(\sum_{n+=1}^{\#n+} q_{n,n+} \ln\left(\frac{q_{n,n+}}{p_{n,n+}}\right) \right) \tag{16}$$
$$q^T \cdot \mathbf{1} = 1$$
$$q^T \cdot (1 + \rho_{i,n+}) = R_n \quad i = 1, \dots, m$$

where for every $n \in \mathcal{N}_t, t < T$, $\#n+$, $q_{n,n+}$ and $p_{n,n+}$ are the conditional risk neutral and empirical probabilities associated with every arc in the subtree originating from n: according to the asset pricing rule for every subtree and parent node, the assets' expected return must be equal to the risk free interest rate R_n. Notice that

Table 2
Means and standard deviations for 6 month ATM put and call option prices, computed on $10-4^5$ arbitrage free scenario tree, strike price $K = 2363$.

	$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$
Put 6m - mean	76.5684	67.8615	71.6968	74.0275	81.2782	82.4417	83.7332
Put 6m - std.dev.	0	44.9576	70.0919	84.4653	106.8597	120.7577	132.3791
Call 6m - mean	80.2228	88.8555	107.0804	107.5494	109.268	114.1283	118.5068
Call 6m - std.dev.	0	48.7582	90.6608	113.1371	130.9058	148.5491	164.5495

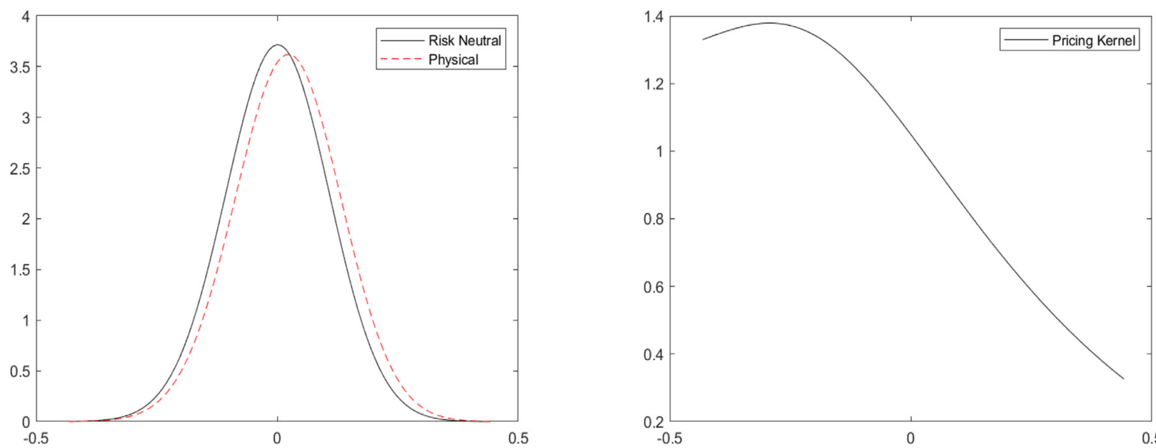

Fig. 3. MEM measure and physical measure on the left-hand-side plot and terminal pricing kernel on the right-hand-side plot.

the system $q^T \cdot (1+\rho_{i,n+}) = R_n$ for every asset i forces the branching degree $\#n+$ to be equal or greater than the m assets in the investment universe. We assume equal conditional probabilities for all branches, and these probabilities are used as starting point for the algorithm. We indicate with t_j an option expiry date.

Given the arbitrage-free scenario tree and risk neutral probability measure the price of a derivative contract can be easily computed by backward recursion, where in each node of the tree $n \in \mathcal{N}_t, t < T$ the entropy price of the contingent claim is computed as the expected value under the minimal entropy measure over the derivative prices in descendant nodes.

For 6-month ATM put and call option prices, we report in Table 2 the mean and standard deviation estimated at every stage $t = 0, 1, \ldots, 6$, from an arbitrage free scenario tree generated acccording to algorithms 1 and 2, with 10 240 scenarios and S&P strike price equal to $K = 2363$. Option prices for 1,3 and 6 month ATM put options can also be computed and they are $put = [35.3696, 53.9619, 76.5684]$, whereas for call options we have $call = [39.0120, 57.6089, 80.2228]$.

Option prices are determined under the risk-neutral measure while the optimization problem is solved under the assumption of equally likely scenarios, thus preserving any risk premium in financial data.

Once all the conditional risk-neutral probabilities are determined along the tree, we can compute the unconditional probabilities on the leaf nodes and compare the associated density functions. In Fig. 3 we compare two smooth probability density functions generated to exemplify the outcome of Algorithm 2: the MEM probability measure and the physical one at the horizon of the planning period and the corresponding pricing kernel that provides information on likelihood ratio of the risk neutral to the physical probability and as such allows the switch between the two.

5. Computational evidence

In this section we present an extended set of results to validate the adopted modeling framework on specific US equity market phases and discuss the most relevant financial evidences. The analysis is based on a data history of monthly observations from January 2002 to June 2021. We take the perspective of a generic US investor with a 6-month planning horizon and a monthly portfolio rebalancing frequency. Every test problem is formulated and solved in the face of uncertainty, with 10 240 scenarios with branching 10^1-4^5 over the 6 stages: at $t = 0$ we generate scenarios relying on the data-driven approach presented in Section 4 and solve the problem to derive the corresponding optimal policy. Every scenario tree is generated with returns sampled from the past history. The asset universe includes the S&P equity index, the Barclays US Aggregate bond index, the Bloomberg 1–3 month Treasury Bill index for the money market and the VIX. Cash is treated as slack, residual investment option. The option strategies rely on *at-the-money* (ATM) European call and put options with 1, 3 and 6 month expiry. Option contracts are treated as synthetic instruments with a maturity consistent with the investment horizon discretization. One option contract is assumed to be written on a unit equity investment and priced according to the arbitrage-free approach in incomplete markets presented in Section 4.2.

Table 3
Statistics for S&P, bond index, money market (MM) monthly returns and VIX. Entire period (Jan 2002–Jun .

	S&P	Bond	MM	VIX
mean	0.0057	0.0036	0.0010	19.5906
std.dev	0.0428	0.0098	0.0012	8.4200
skewness	−0.8558	−0.2640	1.3407	1.8782
kurtosis	2.0905	1.2472	0.8772	4.4644
q(0.01)	−0.1165	−0.0239	0.0000	10.3125
q(0.5)	0.0118	0.0036	0.0005	16.9800
q(0.99)	0.0978	0.0259	0.0043	51.0235

We present in Table 3 descriptive statistics of the asset returns and the VIX over the Jan 2002–June 2021 period.

We assume a unit initial wealth $W_0 = 1$ and, unless otherwise specified, a minimum 60% investment in equity to reflect a risky investor strategy, seeking the maximization of (1) at the 6 month

Table 4
Rolling average of S&P, bond and money market monthly returns and VIX level computed on 6-month windows with monthly data on specific sub-periods.

	S&P		Bond		MM		VIX	
	mean	std.dev	mean	std.dev	mean	std.dev.	mean	std.dev
Period 1								
Jul–Dec 2008	−0.0581	0.0769	−0.0062	0.0116	0.0014	0.0005	39.6917	16.089
Period 2								
Jan–Jun 2017	0.0132	0.0129	0.0037	0.0041	0.0005	0.0002	11.615	0.9682
Period 3								
Jan –Jun 2018	0.0027	0.0339	−0.0027	0.0081	0.0013	0.0002	16.801	2.5735

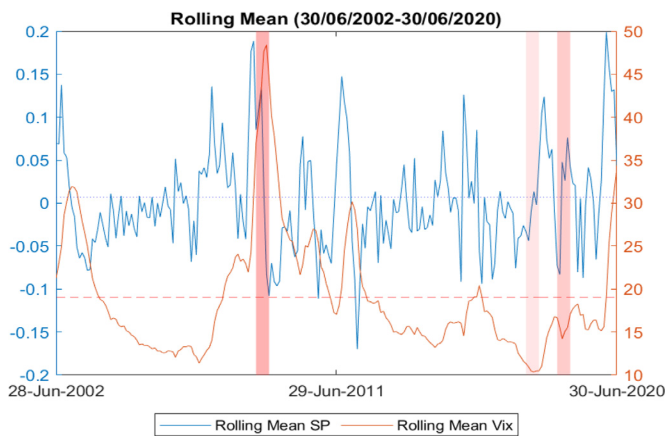

Fig. 4. Rolling S&P average returns, left y axis, VIX level, right y axis, computed on 6-month windows with monthly data (Jan 2002–Jun 2021). Horizontal lines denote long term S&P average return and VIX level, respectively.

horizon. We set $\alpha = 0.95$ in (1) for the CVaR and as λ varies from 1 to 0.5 and to 0 different risk-reward trade-offs are tested.

Over the Jan 2002–Jun 2021 period we identify three periods in the equity market and test the impact of alternative option strategies on the optimal portfolios. Specifically, we focus on the S&P returns' and the VIX mean and standard deviations to identify different market phases. The stylized evidence of the two being negatively correlated over time is confirmed over this extended period as shown in Fig. 4 also with monthly data. Means and volatility dynamics for S&P and VIX are computed on 6-month rolling windows. While the S&P monthly returns' standard deviation provides an historical estimate of monthly volatility, the VIX level captures the 1 month forward (annual) equity volatility implied in the option market. Table 4 describes the three test periods considered and the shaded bands in Fig. 4 help associating the market phases to the identified periods 1, 2 and 3.

Period 1, through fall 2008, is characterized, as well known, by negative equity market returns, a high level of volatility and high VIX values. In period 2 we observe a positive equity market phase and low level of equity volatility as shown by both the S&P data and the VIX. Finally, period 3 is characterized by a stable equity market with moderate, but not negligible market volatility.

In period 1 the VIX level is almost at its 95% quantile of the whole data set, whereas in period 3 the VIX is around its 50% quantile and close to its long-term average. In period 2 the VIX is approximately at its 5% quantile. Following operational evidence, we test the impact on portfolio distributions and optimal asset and derivative exposures of: a protective put strategy in period 1, a covered call strategy in period 2 and a straddle strategy, or variations thereof, in period 3.

By calibrating the risk-reward trade-off in the objective function, through the choice of λ in (1) and relying on equity as well as the VIX, bonds and money market indices, an investor might very well control her risk exposure by rebalancing from risky towards less risky assets.

Ultimately, our interest is on the assessment in- and out-of-sample of downside protection and return generation specifically induced by option exposures as the asset universe is expanded to include an asset reflecting volatility patterns and the trade-off between expected return maximisation and tail risk minimization varies.

5.1. Set-up and experimental design

We develop a set of analyses to validate the model key assumptions and through a sequence of MSP instances, evaluate the optimal policies and wealth distributions under different model assumptions. A selected set of optimal option portfolios is also tested out-of-sample over the 2011–2021 period.

Several option strategies are tested by solving the stochastic problem (1) under the constraints (2)–(7) and, depending on the market phase, by evaluating option strategies based on a protective put (12), or a covered call (13), or a straddle (14), or a strip or strap equations as from Eqs. (15). All option portfolios are thus solutions of an instance of the optimization problem. A set of stochastic programs with call and put contracts treated as generic investment opportunities is also analyzed in detail in what follows.

For computational evidence, Table 5 includes several details on the hardware and software environment, the adopted solution method (Cplex dual simplex algorithm), the problems' sizes, generation and solution times for a representative set of test problems. We see that total generation and solution times vary around a minute for these very large problem instances. The problems dimension is greatly influenced by the number of scenarios over the given stages.

To summarize the collected evidences, we adopt the following convention for every problem instance: $\mathcal{L}(\lambda, \mathcal{C}^h_{j,k}, \mathcal{I}_i, m)$ where: $\lambda \in \{0, 0.5, 1\}$ defines the trade-off in the objective function (1), $\mathcal{C}^h_{j,k}$ the option strategy with $h = 1, 2, 3, 4, 5, 6$ respectively for protective put, covered call, straddle, strip and strap, and with call and put contracts (but no specific strategy), with associated option maturities $t_j = \{1, 3, 6\}$ and moneyness $k = \{ITM, ATM, OTM\}$. For $h = 0$ we have no options involved. $\mathcal{I}_i$ specifies the investment universe where for $i = \{1, 2, 3, 4\}$ we have respectively: equity, equity plus the VIX, all assets, all assets plus the VIX; under any i the investor may opt for a cash deposit. $m = \{1, 2, 3\}$ identifies the test period.

Optimal investment and hedging strategies are presented in terms of current (nodal) portfolio value composition along selected scenarios: we define a *best, median* and *worst* scenario by selecting, after optimization, those scenarios ending at the horizon in correspondence, respectively, to the 99% (best), 50% (median) and 1% (worst) quantiles of the wealth distribution. Portfolio dynamics and terminal wealth distributions are determined by summing the current investment values in each asset class.

Through the solution of an extended set of problems we verify:

Table 5
Generation and solution CPU times in seconds for a selected set of test problems.

	HP ProBook 450G5 Intel(R) Core i5 1.60 GHz,16 GB RAM	Scenario tree 10240 scen.s $10^1 - 4^5$	Solution Cplex 12.6.3.1 Dual simplex	MPS gen CPU time 19.5 (1)
	period 1 $\{i=4, h=1\}$	period 2 $\{i=4, h=2\}$	period 3 $\{i=4, h=3\}$	period 3 $\{i=4, h=6\}$
No. eq.s	249 138	303 743	290 091	262 789
No. var.s	327 636	450 495	368 589	491 448
Non-zeroes	862 650	1 134 905	1 04 500	1 110 654
Pre-solve time (2)	0.36	0.27	0.08	0.33
CPU sol time (3)	52.961	25.156	55.05	38.876
(1) + (2) + (3)	72.82	44.73	74.63	58.71

- The role played by alternative option strategies in optimal dynamic portfolios in every test period: we analyse the terminal wealth distributions and a set of statistical evidences and optimal option exposures for different specifications of the investment universe $i = 1, 2, 3, 4$ and without options $h = 0$. The optimal root-node investment decision is the *here-and-now* (h&n) decision.
- Similar evidences for the case, $h = 6$, in which call and put contracts are jointly available and without any constraint within multi-asset portfolio problems.
- The sensitivity of the optimal wealth distributions and investment policies to alternative risk-reward trade-offs as λ varies in (1). We are also interested to verify the interplay between equity, VIX and option positions under different optimal problems specifications.
- The out-of-sample performance of a set of optimal policies through back testing over the selected test periods as well as the January 2011–June 2021 period.

The scenario generation and pricing procedures have been implemented in Matlab 2019b, MPS instances have been generated and solved using GAMS.

5.2. Optimal option portfolios in different equity market phases

The three periods identified in Table 4 are all half year and as mentioned they reflect different equity market conditions. We thus take an event study approach to verify whether and to which extent optimal dynamic portfolios exploit the introduced option strategies. In this section we consider the case of specific problem instances in each period $m = 1, 2, 3$ and derive a set of observations. In particular:

- For $m = 1$ we solve $\mathcal{L}(1, C^1_{j,k}, \mathcal{I}_i, 1)$: $\lambda = 1$ for CVaR minimization, $h = 1$ for a protective put and under alternative asset universe assumptions, $i = 1, 2, 4$.
- For $m = 2$ we solve $\mathcal{L}(0, C^2_{j,k}, \mathcal{I}_i, 2)$: $\lambda = 0$ for expected wealth maximization, $h = 2$ for a covered call and under alternative asset universes, $i = 1, 2, 4$.
- For $m = 3$ we solve $\mathcal{L}(0, C^3_{j,k}, \mathcal{I}_i, 3)$: $\lambda = 0.5$ for balanced expected wealth-CVaR trade-off, $h = 3, 4, 5$ for a straddle, strip or strap and under alternative asset universes, $i = 1, 2, 4$.

5.2.1. Adverse market conditions, $m = 1$

The second half of 2008 has represented a period of relevant equity market instability globally and specifically in the US market, where in September Lehman Brothers filed for bankruptcy. We test in this semester the most classical portfolio insurance strategy with $\lambda = 1$ in the objective function (1). We present in Fig. 5 the terminal wealth distributions generated by optimal policies when equity and ATM put options, with 1,3,6 month maturities, are considered ($i = 1, h = 1$); when the investment universe is enlarged to include the VIX ($i = 2, h = 1$) and then when all other assets ($i = 4, h = 1$) are considered. The terminal wealth distribution for an equity only portfolio (with cash) is also included.

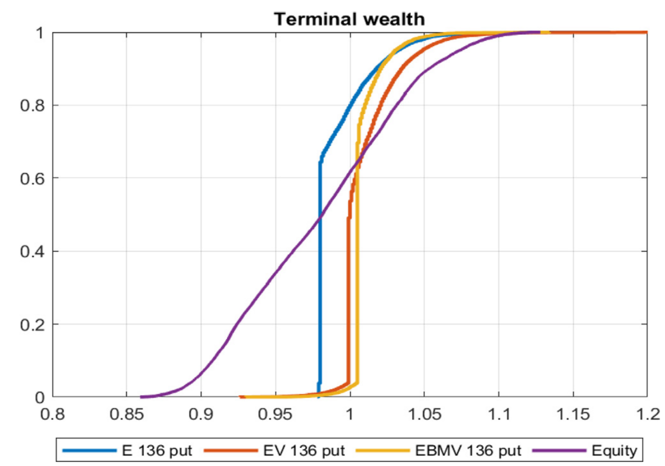

Fig. 5. Protective put strategy $\mathcal{L}(1, C^h_{j,ATM}, \mathcal{I}_i, 1)$, long put options and $i = 1, 2, 4$ with $\lambda = 1$. Terminal wealth distributions for an equity portfolio with no options, $h = 0$ ("Equity" cdf) and with options $h = 1, i = 1, 2, 4$ (cdf's "E 136 put", "EV 136 put" and "EBMV 136 put", respectively).

Table 6
Statistics from solutions $\mathcal{L}(1, C^1_{j,ATM}, \mathcal{I}_i, 1)$, $t_j = \{1, 3, 6\}, i = 1, 2, 4$. Expected terminal wealth $\mathbb{E}(W_T)$ and standard deviation $\sigma(W_T)$, skewness $s(W_T)$, kurtosis $\kappa(W_T)$, Sharpe Ratio $SR = \frac{\mathbb{E}(W_T)-W_0}{\sigma(W_T)}$ and quantiles at the 1%, 50% and 99% over a 6 month investment horizon.

Strategy	Protective put ($h = 1$)		
Investment universe	$i = 1$	$i = 2$	$i = 4$
$\mathbb{E}(W_T)$	0.9901	1.0096	1.0088
$\sigma(W_T)$	0.0191	0.0194	0.011
$s(W_T)$	2.4032	2.4819	3.0669
$\kappa(W_T)$	9.9946	16.3244	22.4172
SR	−0.5175	0.4928	0.798
$q_{0.01}(W_T)$	0.979	0.981	0.989
$q_{0.5}(W_T)$	0.98	1	1.005
$q_{0.99}(W_T)$	1.06	1.078	1.054

Table 6 allows a comparison between terminal wealth statistics generated by optimal policies during this crisis period, as the investment universe increases according to the above scheme.

With specific reference to a multi-asset portfolio with puts ($h = 1, i = 4$), we analyze in Fig. 6 the portfolio composition along the worst, median and best scenarios. From the above first percentile of the terminal portfolio distribution even under the worst scenario the left tail of the distribution is tight and close to 1 with evidence of an effective hedging strategy. Along any scenario a 60% lower bound on the equity must be satisfied. The h&n decision includes equity, bond, VIX and a long 6 month put contract.

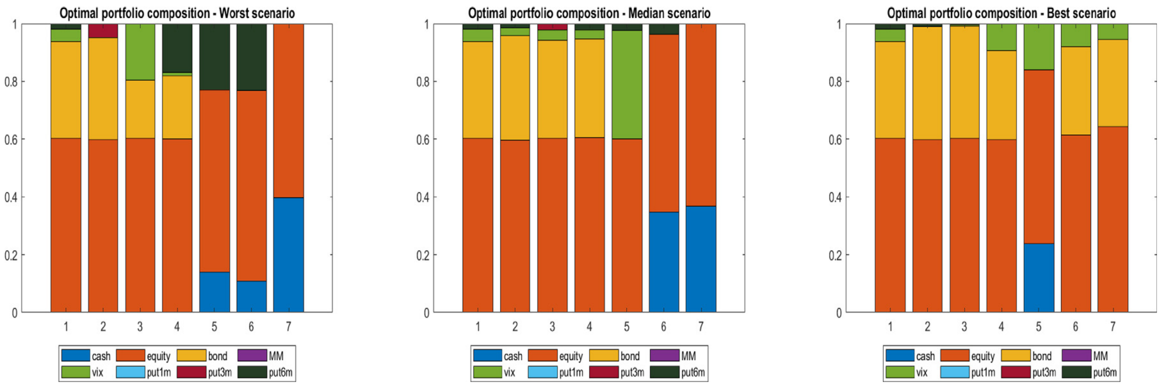

Fig. 6. Protective put $\mathcal{L}(1, \mathcal{C}^1_{j,ATM}, \mathcal{I}_4, 1)$, $t_j = \{1, 3, 6\}$. Portfolio composition along 1% (worst), 50% (median) and 99% (best) quantiles (scenarios) during period 1. $i = 4$, $\lambda = 1$.

The return achieved in the worst scenario on the left tail of the distribution reflects the adopted hedging policy and we see, on the left plot of Fig. 6 that the investment in put contracts is persistent and increasing over the investment horizon specifically thanks to the 6 month puts. When considering the strategies along the median and best scenarios we still observe a role of put contracts over the planning horizon but with a predominant role of bonds and the VIX.

Several remarks can be drawn from the above graphical evidences.

- Under any problem instance a protective put leads to a tight control of the left tail and a good hedging performance.
- The inclusion of a volatility index in the investment universe has a positive impact on the portfolio return distribution and on the portfolio risk-adjusted return. Over the investment horizon volatility exposure through the VIX comes together with long put positions.
- As we extend the invstment universe to include also bonds and money market the terminal wealth distribution improves further and in Fig. 6 we see that scenario-specific investment policies show a good diversification pattern.

5.2.2. Favourable equity market phase with low volatility, $m = 2$

During the first half of 2017 the US equity market experienced a period of relatively low volatility with a positive trend. Under alternative specifications of the investment universe, we test in this case the effects of a covered call strategy; this is characterized by a long equity position together with a short call. The rationale being that the premiums from the call would compensate possible negative equity returns and the selling of the equity portfolio at the strike in case of an expiring ITM call would still be convenient to the portfolio manager. In presence of a volatility index and bonds or money market such rationale would still hold but may be less relevant to the portfolio manager.

In Fig. 7 we display the terminal wealth distributions for a covered call strategy when ATM options are available and the investment universe includes only equity and cash ($i = 1, h = 2$), or the VIX as well ($i = 2, h = 2$) or including all assets ($i = 4, h = 2$). Here $\lambda = 0$ and the investor is assumed to seek an expected terminal wealth maximisation.

We see that short call positions have a positive impact on the mean and the right tail of an equity portfolio distribution and when other assets are considered for investment the benefits on the terminal wealth are relevant, while eventually the left tail worsens sightly. Table 7 shows the terminal portfolio statistics with a positive outcome under any asset universe specification but at the cost of a relatively poor volatility control. The wealth distribution and statistical evidence are consistent with the assumed risk neutral growth objective for $\lambda = 0$.

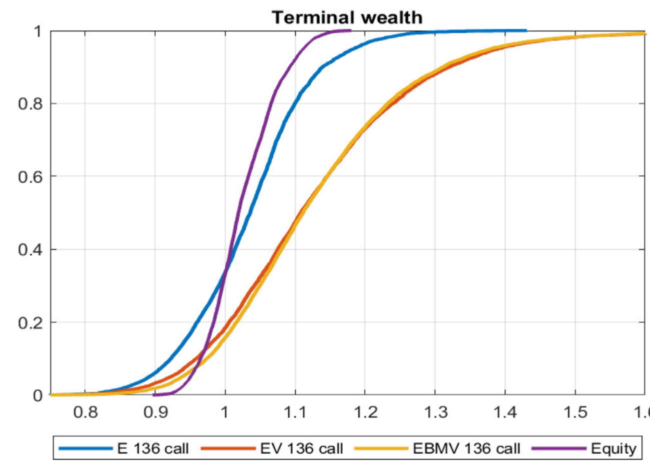

Fig. 7. Covered call strategy $\mathcal{L}(0, \mathcal{C}^2_{j,ATM}, \mathcal{I}_i, 2)$ with $t_j = \{1, 3, 6\}$ short ATM call position and $i = 1, 2, 4$ with $\lambda = 0$. Terminal wealth distributions for an equity portfolio with no options, $h = 0$ ("Equity" cdf), and cdf's associated with the canonical covered call ($h = 1$), plus volatility ($h = 2$) and plus the other assets ($h = 4$), "E 136 call", "EV 136 call" and "EBMV 136 call", respectively.

Table 7
Statistics from problems' solution $\mathcal{L}(0, \mathcal{C}^2_{j,ATM}, \mathcal{I}_i, 2)$, $t_j = \{1, 3, 6\}, i = 1, 2, 4$. Expected terminal wealth $\mathbb{E}(W_T)$ and standard deviation $\sigma(W_T)$, skewness $s(W_T)$, kurtosis $\kappa(W_T)$, Sharpe Ratio $SR = \frac{\mathbb{E}(W_T)-W_0}{\sigma(W_T)}$ and quantiles at the 1%, 50% and 99% over a 6 month investment horizon.

Strategy	Covered call ($h = 2$)		
Investment universe	$i = 1$	$i = 2$	$i = 4$
$\mathbb{E}(W_T)$	1.0337	1.1278	1.1306
$\sigma(W_T)$	0.0876	0.1494	0.1408
$s(W_T)$	0.1796	0.9526	1.0215
$\kappa(W_T)$	3.4999	5.2087	5.3887
SR	0.385	0.8554	0.9272
$q_{0.01}(W_T)$	0.834	0.8499	0.8779
$q_{0.5}(W_T)$	1.036	1.108	1.111
$q_{0.99}(W_T)$	1.2561	1.581	1.5672

Focusing again on a multi-asset investor we show in Fig. 8 the optimal portfolio strategy along three wealth scenarios, from left-hand-side to right-hand-side plots, from worst to best scenarios. Unlike in the previous section, to convey the short call exposure we analyse here the portfolio evolution in terms of assets' nominal exposures rather than in terms of portfolio relative weights. No short calls are included in the h&n portfolio. While there is a

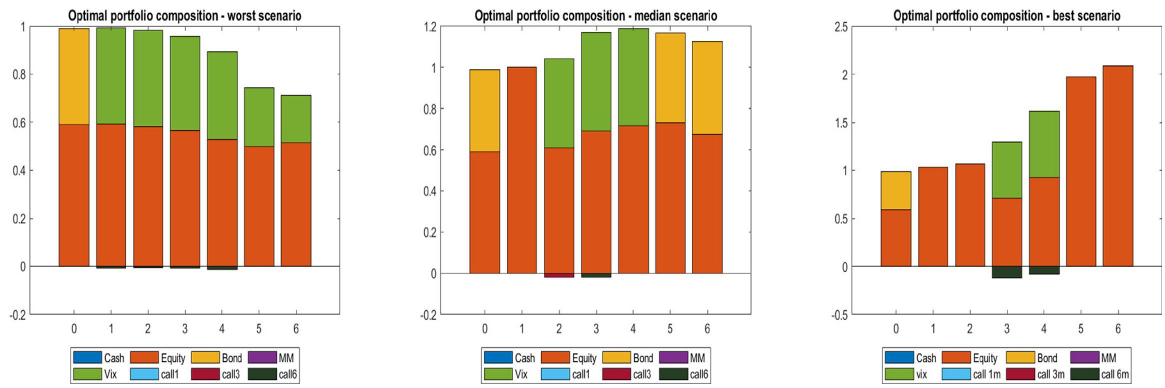

Fig. 8. Covered call $\mathcal{L}(0, \mathcal{C}^2_{j,ATM}, \mathcal{I}_4, 2)$, $t_j = \{1, 3, 6\}$. Portfolio composition along 1% (worst), 50% (median) and 99% (best) quantiles (scenarios) during period 1. $i = 4$, $\lambda = 0$.

call exposure in all intermediate stages and particularly along the best scenario, jointly with the investment in equity and VIX.

A covered call strategy is exploited dynamically with a relevant exposure along the median and the best scenarios, slightly less along the worst case scenario in which nevertheless short calls help improving the portfolio distribution. Period 2 is characterized by a positive equity market and low volatility: volatility exposure leads in the worst scenario to a negative outcome partially compensated by call premiums.

Few concluding remarks on this strategy:

- Short call positions are included in optimal dynamic multi-asset portfolios with a positive impact on portfolio returns but, during a positive market phase, always sold out before expiry.
- In case of an equity portfolio, covered call strategies are employed more extensively over the 6 month investment horizon with a positive effect on the portfolio distribution.

5.2.3. Steady equity market with moderate volatility, $m = 3$

Consider now period 3: this is a very stable equity market with however, non negligible market volatility. We assume a balanced risk-reward trade-off in (1) with $\lambda = 0.5$ and consider the possibility to invest in an ATM straddle or strip or strap: we recall that in these two latter cases, respectively: either 2 long put option contracts for 1 call contract are considered or the opposite. A straddle instead is based on symmetric ATM long call and put positions. The straddle would then generate a loss equal to the sum of the option premiums whenever the S&P remains at the initial price but would generate a profit whenever it moves away from the current value, either left or right with the put protecting the downside and the call the upside. We analyse the terminal wealth distributions and the associated optimal strategies along specific scenarios.

In Fig. 9 we compare the terminal wealth distributions of optimal portfolios with equity and straddle options ($i = 1$), including VIX ($i = 2$), and all assets ($i = 4$), or equity without options ($h = 0$).

Also in this case the option strategies allow an effective control of the left tail of the wealth distribution and a remarkable upside. The most performing strategy is generated by a straddle together with equity and VIX or all assets and VIX. The probability of a loss under any scenario and portfolio composition is negligible while the straddle has a remarkable impact on the portfolio performance and risk-adjusted returns.

For the case in which all assets and the VIX are included in the asset universe we analyse the optimal portfolio strategy leading to those extreme negative and positive quantiles and along the median scenario in Fig. 10. Now the investor is assumed to balance the growth objective with the control of the tail risk. At the root node the h&n decision is almost entirely in equity, a small proportion in the VIX and two long straddle positions with 1 and 3

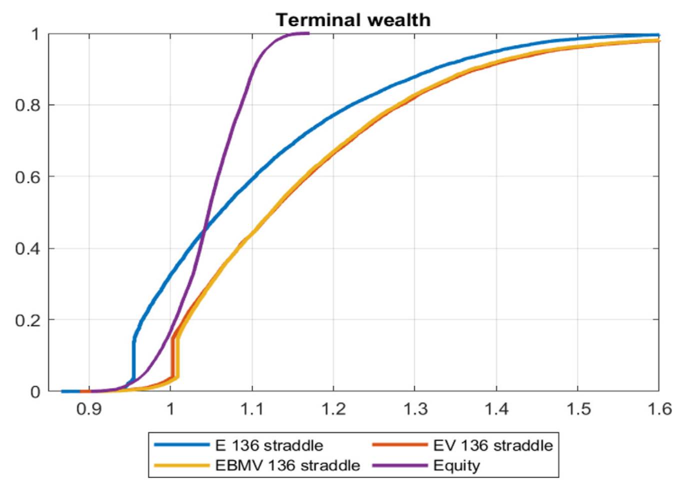

Fig. 9. Straddle strategy $\mathcal{L}(0.5, \mathcal{C}^3_{j,k}, \mathcal{I}_i, 3)$, with $t_j = \{1, 3, 6\}$ ATM straddle positions and $i = 1, 2, 4$ and $\lambda = 0.5$. Terminal wealth distributions for an equity portfolio with no options, $h = 0$ ("Equity" cdf), and cdfs associated with the canonical straddle ($h = 1$), plus volatility ($h = 2$) and plus the other assets ($h = 4$), "E 136 straddle", "EV 136 straddle" and "EBMV 136 straddle", respectively.

month maturities. Straddle contracts together with equity and VIX do characterise the median and best scenarios while on the worst, good though, scenario bond and equity play a predominant role. As for any option, when the straddle expired OTM then we had a cost upon inception and no effect on the portfolio return is accounted for, while when ITM the cash-settlement will generate a profit that will depend on the nodal equity value.

Most of the conclusions and remarks of the previous sections can be confirmed. In this section we have considered the impact of different option strategies on optimal dynamic portfolios under different equity market conditions. Even more than in periods 1 and 2 here optimal strategies are risky and do involve straddle exposures, whose impact, given the adopted scenario tree, on the terminal wealth distribution is very positive. Portfolios are mostly well diversified and volatility as asset class is exploited together with straddle contracts.

In the following section we present a set of results on each test period when allowing free equity investments and analysing the optimal portfolios as the λ varies and the asset universe includes all assets and the VIX.

5.3. Option portfolios comparative results

We develop a comprehensive model assessment by allowing call and put options with different maturity to enter the asset uni-

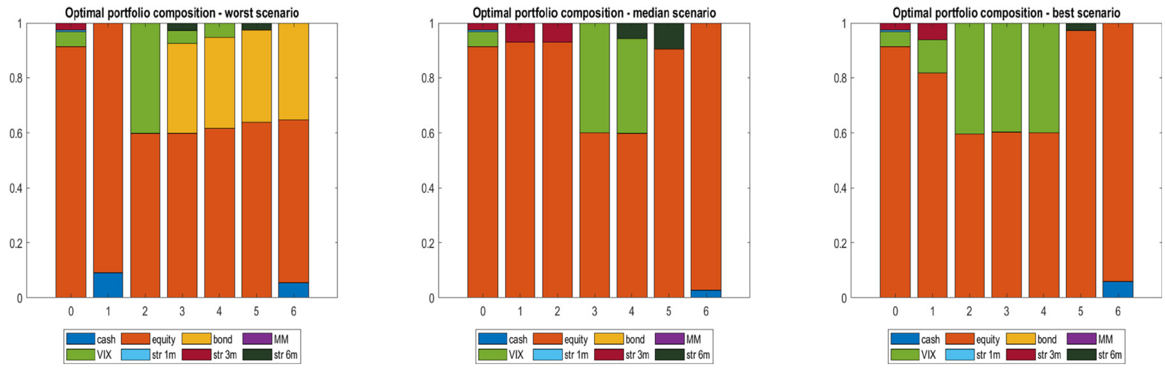

Fig. 10. Straddle $\mathcal{L}(0, \mathcal{C}^2_{j,ATM}, \mathcal{I}_1, 2)$ with $t_j = \{1, 3, 6\}$, with $m = 3$. Optimal *nominal* strategy along three scenarios 1% (worst), 50%(median), 100%(best).

Table 8
Statistics for $\mathcal{L}(0.5, \mathcal{C}^3_{j,ATM}, \mathcal{I}_i, 3)$, $t_j = \{1, 3, 6\}, i = 1, 2, 4$. Expected terminal wealth $\mathbb{E}(W_T)$ and standard deviation $\sigma(W_T)$, skewness $s(W_T)$, kurtosis $\kappa(W_T)$, Sharpe Ratio $SR = \frac{\mathbb{E}(W_T)-W_0}{\sigma(W_T)}$ and quantiles at the 1%, 50% and 99% over a 6 month horizon.

Strategy	Straddle ($h = 3$)			Strip ($h = 4$)	Strap ($h = 5$)
Investment universe	$i = 1$	$i = 2$	$i = 4$	$i = 4$	$i = 4$
$\mathbb{E}(W_T)$	1.1025	1.1641	1.1635	1.1546	1.224
$\sigma(W_T)$	0.1472	0.1689	0.1652	0.1583	0.2334
$s(W_T)$	1.0992	1.9572	2.0376	2.1702	1.5875
$\kappa(W_T)$	3.7096	11.4667	12.1516	13.0282	7.7258
SR	0.6963	0.9719	0.9897	0.9771	0.9597
$q_{0.01}(W_T)$	0.942	0.97	0.976	0.976	0.96
$q_{0.5}(W_T)$	1.061	1.125	1.124	1.114	1.157
$q_{0.99}(W_T)$	1.5331	1.7432	1.7415	1.7292	1.975

verse and be jointly and independently available for hedging or speculative purposes. In each test period and for different risk-reward trade-offs. The asset universe includes equity, bond and money market, $i = 3$, or with the VIX $i = 4$. As before we consider the terminal wealth distributions under the different assumptions to assess the effectiveness of the adopted option strategies, if any, and whether such effect depends on the risk-reward trade-off and the inclusion of the VIX in the investment universe. All results in this section are collected after removing the lower bound on equity positions. We see that under these assumptions, several interesting financial observations can be made and confirm the generality of the adopted modeling framework.

In Table 9 we present a set of statistics for every test problem. We allow $m = 1, 2, 3$ and in each period compute the terminal portfolio statistics generated by optimal strategies over 6 months, always starting with a unit initial wealth $W_0 = 1$ when λ varies. Within each test period, $m = 12, 3$, the same scenario trees with 10 240 scenarios are adopted.

The key differences between the results displayed here above for $m = 1, 2, 3$ and those in Section 5.2, even under the same asset universe structure and λ's, refer to the different treatment of option portfolios: here call and put contracts of any maturity, are treated as investment opportunities while in the previous sections they were structured and constrained as described in models (12), (13), (14) and (15).

We summarize the main relevant evidences resulting from Table 9:

- In each problem instance the introduction of the VIX for $i = 4$ has a positive impact on almost all quantiles of the terminal wealth distributions and independently of m, it leads to improved expected wealths and however increasing volatility: the Sharpe ratios do mostly increase anyhow.
- Under $\lambda = 0$ in anyone of the three periods, the volatility and tail risk controls are less effective and the terminal wealth supports larger than under the other risk-reward trade-offs. As λ increases to 0.5 and to 1, but already in the former case the tail risk at the 1st percentile is very close if not above 1.
- The CVaR minimization problem for $\lambda = 1$ generates consistently a positive risk-adjusted return and very effective tail control. We show below explicitly that already for λ0.5 call and put contracts play a role in the optimal policy even jointly with the VIX.
- When comparing the outcomes in Table 9 with those in the previous sections under the same λ's and in each period $m = 1, 2, 3$, we can see that the results, in terms of terminal wealth quantiles and Sharpe ratios do not differ significantly and in the 2008 period an optimal strategy based on $i = 3$ without the VIX would have led to very good tail risk control and risk-adjusted returns. When $m = 2, \lambda = 0.5$ for $i = 4$ results are pretty similar here and above, while for $m = 3, \lambda = 0$ the portfolio returns' statistics here above are very positive but not as good as those presented in Table 8.

5.3.1. Risk-reward trade-off analysis

Further to the previous remarks, to focus more directly on the optimal portfolio sensitivity to λ, let's focus on the results collected for $m = 3$. Consider the last two columns in Table 9. The portfolio statistics for $i = 3$ and $i = 4$ are associated with the cdf's in Fig. 11 for $h = 6$.

Call and put contracts have an impact on both cases and play a role in determining the optimal portfolio processes. On either plots of Fig. 11 the choice of λ is shown to affect the portfolio returns distributions and for $\lambda = 1$ the control of the left tail is very effective and it is shown below to involve options' positions. As we move towards a maximum expected wealth objective the upside improves with a negative impact on the left tail which is tolerable when $i = 4$.

When considering $\lambda = 0.5$ and still $m = 3$ Fig. 12 displays the wealth processes over the planning horizon when $i = 1$ or $i = 4$.

Table 9
Statistics for different strategies $\mathcal{L}(\lambda, \mathcal{C}^h_{j,ATM}, \mathcal{I}_i, m)$ in periods $m = 1, 2, 3$ with $i = 3, 4$ and for every λ row-wise: the expected terminal wealth $\mathbb{E}(W_T)$ and standard deviation $\sigma(W_T)$, skewness $s(W_T)$, kurtosis $\kappa(W_T)$ and Sharpe Ratio $SR == \frac{\mathbb{E}(W_T)-W_0}{\sigma(W_T)}$, the quantiles at the 1%, 50% and 99% level.

Strategy	No strategy - Free investment in equity options					
	Period $m = 1$		Period $m = 2$		Period $m = 3$	
	$(i = 3)$	$(i = 4)$	$(i = 3)$	$(i = 4)$	$(i = 3)$	$(i = 4)$
$\lambda = \mathbf{0}$						
$\mathbb{E}(W_T)$	1.054	1.1591	1.0795	1.1593	1.0503	1.1513
$\sigma(W_T)$	0.0821	0.1636	0.1312	0.1948	0.0797	0.1601
$s(W_T)$	0.7675	0.509	1.0424	1.3173	0.776	0.5227
$\kappa(W_T)$	4.9261	3.2532	4.7621	6.7847	5.052	3.2649
SR	0.6575	0.9727	0.6057	0.8175	0.6307	0.9447
$q_{0.01}(W_T)$	0.8789	0.843	0.866	0.835	0.877	0.844
$q_{0.50}(W_T)$	1.047	1.144	1.063	1.131	1.043	1.136
$q_{0.99}(W_T)$	1.309	1.5982	1.485	1.796	1.302	1.581
$\lambda = \mathbf{0.5}$						
$\mathbb{E}(W_T)$	1.0124	1.021	1.0571	1.1303	1.0363	1.1144
$\sigma(W_T)$	0.0042	0.0083	0.0728	0.1445	0.0378	0.1216
$s(W_T)$	8.0227	4.1866	1.8872	2.1317	2.4852	1.2259
$\kappa(W_T)$	139.7824	44.7861	7.2337	10.1825	12.6595	4.4447
SR	2.9644	2.5177	0.7838	0.9019	0.9622	0.9406
$q_{0.01}(W_T)$	1.008	1.007	0.988	0.988	0.998	0.956
$q_{0.50}(W_T)$	1.011	1.018	1.031	1.082	1.026	1.076
$q_{0.99}(W_T)$	1.0291	1.053	1.319	1.656	1.19	1.487
$\lambda = \mathbf{1}$						
$\mathbb{E}(W_T)$	1.039	1.1209	1.0078	1.036	1.0118	1.0201
$\sigma(W_T)$	0.0397	0.1251	0.0099	0.0241	0.0041	0.0082
$s(W_T)$	2.4021	1.1991	4.6886	4.0277	6.4911	3.5868
$\kappa(W_T)$	11.8899	4.3443	31.9791	26.9331	115.7215	40.0636
SR	0.9834	0.9662	0.7868	1.4958	2.8613	2.4524
$q_{0.01}(W_T)$	1.001	0.963	0.998	1.002	1.004	1.002
$q_{0.50}(W_T)$	1.028	1.083	1.005	1.027	1.011	1.018
$q_{0.99}(W_T)$	1.1981	1.506	1.056	1.151	1.028	1.051

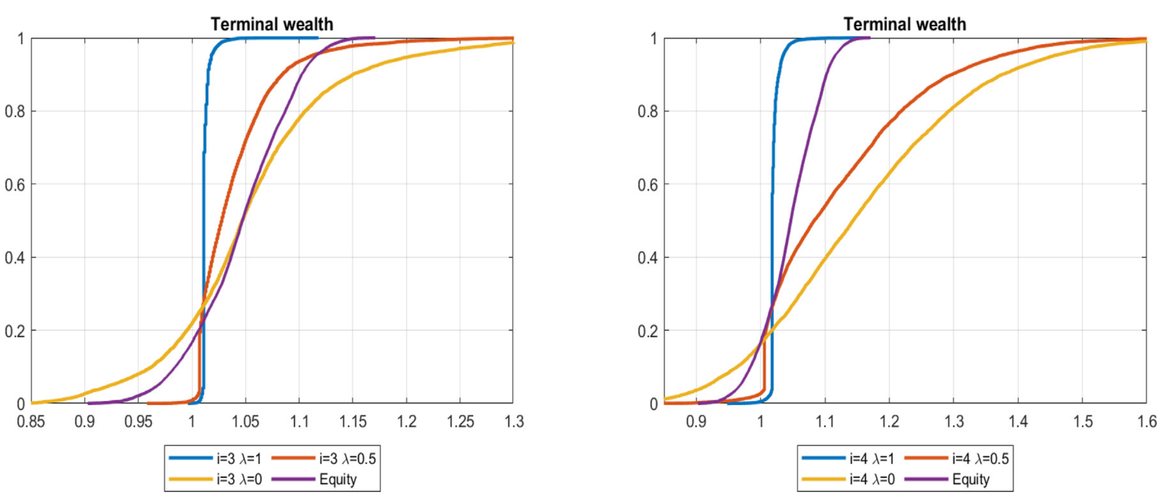

Fig. 11. Terminal wealth distributions with free long call and put investment opportunities in period 3 $\mathcal{L}(0.5, \mathcal{C}^h_{j,ATM}, \mathcal{I}_i, 3)$, $i = 1, h = 0$, $i = 1, h = 6$, $i = 4, h = 6$.

Yet the control over the left tail is effective throughout and the upside when including the VIX and exploiting the options leverage effect is relevant. As λ increases to 1 these optimal wealth processes reduce their dispersion and improve the performance along the worst scenario.

In the following section, we comment on the optimal hedging strategies and volatility-driven policies for $h = 6, i = 3$.

5.3.2. Volatility-based portfolios and hedging policies

Fig. 13 shows the optimal investment strategies, left-hand-side to right-hand-side along the usual worst, median and best scenarios when the asset universe includes all assets, the VIX and the options. The h&n portfolio is the only one under full uncertainty and includes equity, the VIX and 3 month call contracts. Along the three scenarios the optimal portfolios evolve with a good diversification including equity, bonds, the VIX and the options. Along the best scenario, the optimal portfolio includes a large proportion of equity, VIX and 3 and 6 month call and put contracts. The cash settlement of the 6 month call contract at maturity generates the cash position and positive return in the best scenario.

Since the first stages, the investor exploits the opportunity of hedging the equity exposure in a negative market scenario and increases the investment in calls to benefit from a potential positive equity scenario. At the root node and in the second stage, the investor buys call and put contracts, with limited impact in terms of investment value but with high return potential. The option expo-

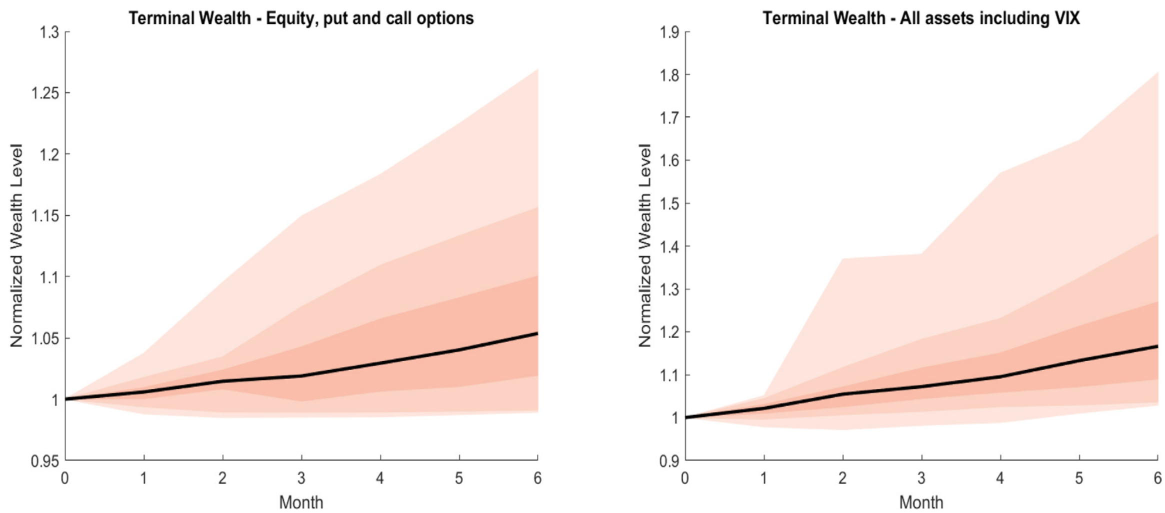

Fig. 12. Optimal wealth evolutions with $h = 6$: long call and put free investment opportunities in period 3 with only equity $i = 1$ or all assets and the VIX $i = 4$: $\mathcal{L}(0.5, \mathcal{C}^6_{j,ATM}, \mathcal{I}_i, 3)$.

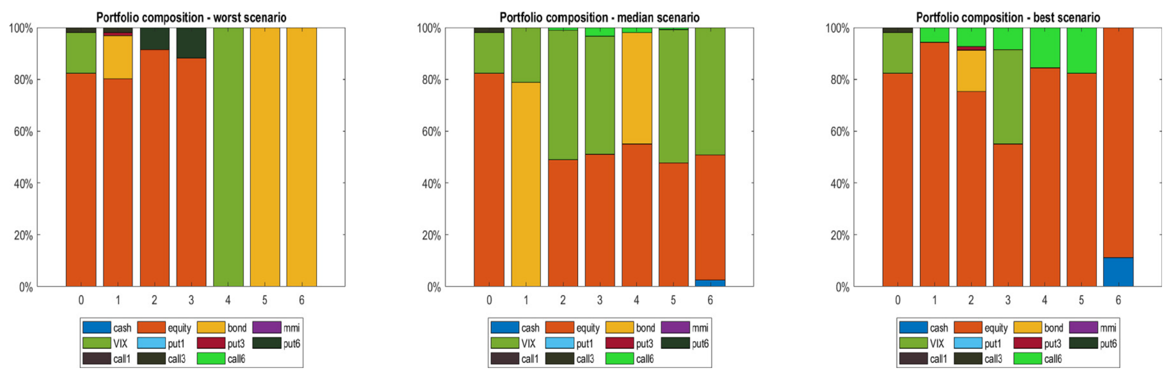

Fig. 13. Optimal portfolio strategies for an investor with long call and puts together with bond, money market index and VIX, over a medium volatility $\mathcal{L}(0.5, \mathcal{C}^6_{j,ATM}, \mathcal{I}_4, 3)$, $t_j = \{1, 3, 6\}$.

sure evolves over time with increasing investment in longer maturity call and puts. The risky portfolio exposure during the first stages of the worst scenario is reduced at the end of the planning horizon when the portfolio concentrates in bonds.

From the collected results we can draw similar conclusions as those in the previous sections. In particular:

- Options are consistently included in optimal portfolios and have a positive impact on the tail risk control also when $\lambda = 0.5$ or 0 and the optimal strategy seeks a growth objective.
- In presence of the VIX as asset class, the investor exploits the negative correlation between S&P and VIX but yet option contracts enter the optimal portfolio.

All the above results refer to portfolios and wealth dynamics associated with the generated problem instances. Once the scenario trees are generated by bootstrapping from the past market history, a given instance of future possible scenario evolutions is given and we have tested a rich set of model specifications and problems formulation leading to optimal portfolio tree processes as those analysed so far. We complete this study by analyzing the out-of-sample evidence, yet considering an event study approach limited to selected test periods.

5.4. Out-of-sample validation

In order to provide additional evidences on the effectiveness of the proposed modeling framework, we present in this section a set of out-of-sample results based on a backtesting procedure to be introduced first. We adopt here an approach inspired by Dempster and Thompson (2002), and aimed at preserving the investment policy time consistency when options with different maturities and spanning the entire investment horizon are considered. Moreover, it accounts for the impact of options' trading before maturity over the planning horizon. Its final purpose remains the evaluation of the portfolio wealth dynamics at realized market prices.

In particular: after the scenario tree has been generated we first select the scenario path closest, according to the Euclidan distance, to the realized equity market price scenario and then we evaluate the associated optimal investment policy (including all assets) at the realized prices.

We refer to that scenario as replicating market scenario or just replicating scenario and to the associated optimal strategy as optimal replicating strategy. The approach deals with the delicate issue induced by the presence of option contracts in those portfolios: indeed those contracts have been introduced as synthetic OTC instruments with an *ad-hoc* maturity set. When present in the optimal portfolio, their impact on realized returns is then evaluated through the cash flows they generate.

The backtesting procedure is first carried out on the three 6-month periods described in Table 4, and then extended to span the 2011–2021 period. We consider a comprehensive portfolio formulation including long put and call options and as asset universe, equity, bond, money market and VIX. In each test period, a specific risk-reward trade-off is considered according to the following pairs: $\lambda = 1$ for $m = 1$, $\lambda = 0$ for $m = 2$ and $\lambda = 0.5$ for $m = 3$.

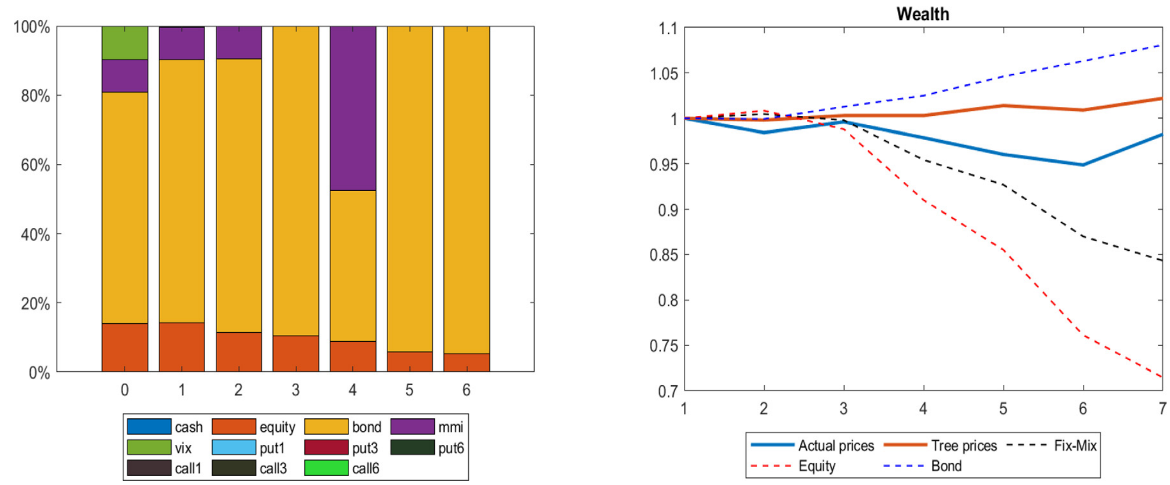

Fig. 14. Optimal portfolio compositions(left-hand-side) and corresponding wealth evolution (right-hand-side) along the replicating scenario for the test problem $\mathcal{L}(1, \mathcal{C}^6_{j,ATM}, \mathcal{I}_4), 1)$, $t_j = \{1, 3, 6\}$ and $m = 1$.

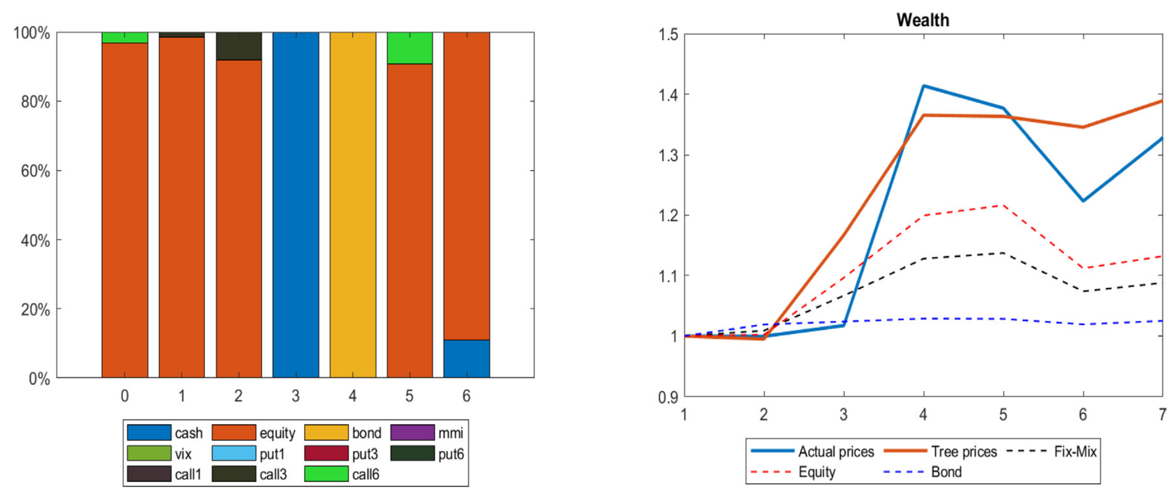

Fig. 15. Optimal portfolio compositions(left-hand-side) and corresponding wealth evolution (right-hand-side) along the replicating scenario for the test problem $\mathcal{L}(0, \mathcal{C}^6_{j,ATM}, \mathcal{I}_4, 2)$, $t_j = \{1, 3, 6\}$ and $m = 2$.

In the more extended out-of-sample analysis a problem instance with $\lambda = 0.5$, a 50% lower bound on the equity investment and a 10% upper bound on the VIX investment are considered. After solving the 6 month MPS problem, the optimal policy to be tested is determined as follows: at time $t = 0$, the h&n decision is taken for period [0,1]; at $t = 1$ we take the point of view of an investor who will rebalance her portfolio along the replicating scenario path. The procedure is then rolled forward till the horizon.

5.4.1. Results over specific subperiods

In Fig. 14 we present on the left-hand-side plot the optimal replicating strategy and on the right-hand-side plot the evolution of the portfolio value evaluated at actual market prices and at tree prices along the same scenario. For comparison three benchmarks associated with an equity only, bond only and fix-mix (equity-bond: 60% – 40%) portfolios are included.

Portfolio composition along the scenario is in line with a negative equity performance registered also by the scenario tree and, given the downside risk minimization objective, leads to limited, and decreasing exposure in equity with bond, MM and VIX dominating the root node decision.

In Fig. 15 the same validation procedure is applied to the solution for period $m = 2$. On the left-hand-side plot the replicating portfolio composition is presented. On the right-hand-side plot: the evolution of the wealth along the scenario at market prices is compared with the optimal wealth along the tree and the same benchmarks as before.

Finally, in Fig. 16 we evaluate the strategy associated with the solution in period $m = 3$.

Portfolio composition along the scenario includes 3 and 6 month call options and trading on them, from an initial consistent exposure to equity, the portfolio switches to a safer cash and bond portfolio composition and back to equity again towards the end of the horizon.

Finally, in Fig. 16 this backtesting approach is applied to period $m = 3$. On the left-hand-side plot the portfolio composition along the replicating scenario and on the right-hand-side the evolution of the optimal portfolio value at market prices, compared with the same tree prices and the three benchmark dynamics.

Portfolio composition in Fig. 16 (left-hand-side) along the replicating scenario is presented: a risky portfolio strategy is adopted together with call options and a switch from S&P to VIX over the 6 months.

5.4.2. Out-of-sample results over 2011–2021

We present here the out-of-sample results from January 2011 to June 2021, thus including the recent 2020–2021 pandemic crisis. The backtesting procedure is extended with a rolling window

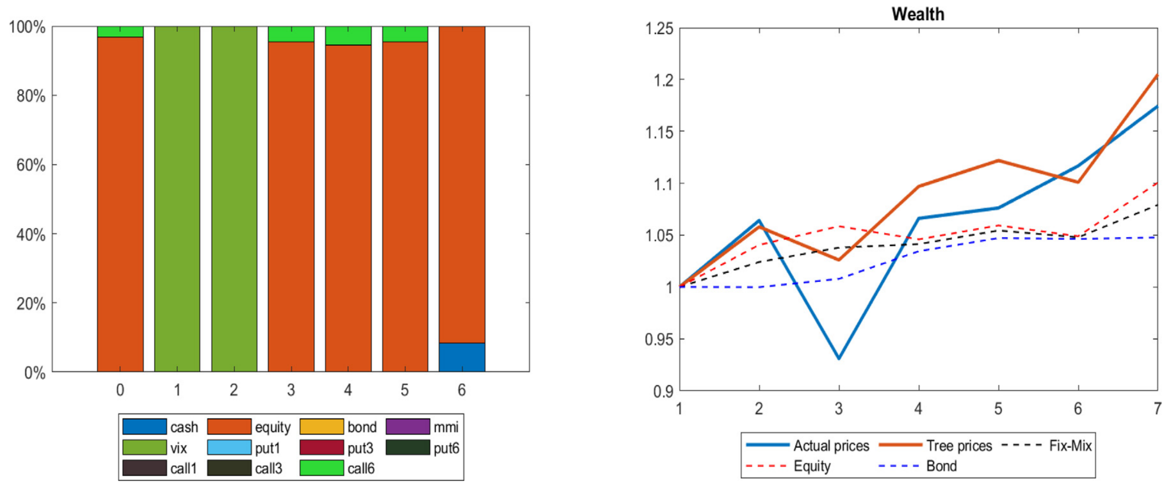

Fig. 16. Optimal portfolio compositions(left-hand-side) and corresponding wealth evolution (right-hand-side) along the replicating scenario for the test problem $\mathcal{L}(0.5, \mathcal{C}^6_{j,ATM}, \mathcal{I}_4, 3)$, $t_j = \{1, 3, 6\}$ and $m = 3$.

Table 10
Statistics on semiannual out-of-sample returns of different portfolio strategies over the Jan 2011–June 2021 period: from left to right optimal portfolios with options: $\mathcal{L}(0.5, 6, 4)$, without options: $\mathcal{L}(0.5, 0, 4)$, fix-mix strategy: equity-bond 60% – 40%, equally weighted portfolio over the 4 asset classes: $1/N$, and only Bond- or only Equity-portfolios. Row-wise: 2011–2021 average ex-post semiannual returns $\mathbb{E}(r_T)$, their standard deviation $\sigma(r_T)$, skewness $s(r_T)$, kurtosis $\kappa(r_T)$, Sharpe Ratio $SR = \frac{\mathbb{E}(r_T)}{\sigma(r_T)}$, minimum and maximum ex-post performance over the testperiod.

	$\mathcal{L}(0.5, 6, 4)$	$\mathcal{L}(0.5, 0, 4)$	Fix-mix	1/N	Bond	Equity
$\mathbb{E}(r_t)$	0.0589	0.0559	0.0460	0.0257	0.0176	0.0585
$\sigma(r_t)$	0.0351	0.0466	0.0417	0.0538	0.0251	0.0605
$s(r_t)$	1.0777	0.5479	−0.2834	−0.3241	−0.2114	−0.3962
$\kappa(r_t)$	3.3172	2.6630	3.0446	2.1880	1.8266	2.8694
SR	1.6792	1.1996	1.1035	0.4775	0.7020	0.9680
Min(r_t)	0.0141	−0.0183	−0.0495	−0.0835	−0.0299	−0.0809
Max(r_t)	0.1499	0.1623	0.1293	0.1156	0.0553	0.1684

scheme, updated every six months. We start with a problem whose initial root decision is at $t = 0 : 1.1.2011$ and move forward with 6 month steps until the last problem, in which $t = 0 : 1.1.2021$ and $T : 30.6.2021$. Every problem instance requires a preliminary scenario tree generation, following the procedure in Section 4.2, the identification of the replicating scenario, the valuation ex-post of the replicating strategy at the realized prices for the assets plus cash. At the 6-month horizon, the portfolio value represents the initial endowment for the following semester. All options have come to natural expiration, either ITM generating a cash flow or OTM with no associated cash flows; thus, no options positions are carried over from one problem to the next. Over the 10.5 years, 21 test problems are solved and the resulting strategies backtested. The problem instance repeatedly solved is, in our convention, $\mathcal{L}(\lambda = 0.5, h = \{0, 6\}, i = 4)$. We consider a portfolio manager with relevant equity exposure that maximizes the objective (1) with $\lambda = 0.5$. By solving the two cases $h = 6$ and $h = 0$, we collect optimal portfolios potentially including freely available long call and put contracts, $h = 6$, or based only on the 4 asset classes (Equity, Bond, MM, VIX), but without options, $h = 0$. Those portfolios are benchmarked against an equity-bond 60 – 40 fix-mix portfolio, a $1/N$ strategy, and two strategies with full investment in equity or bond. We refer to Fig. 4 to capture the different market phases during this extended period.

We present in Table 10 the statistics computed on semiannual out-of-sample returns over the Jan 2011–June 2021, where r_t represents return over a semester with $t \in [1, 21]$.

By focusing specifically on the effects due to options' exposures, we can summarize the main evidences resulting from Table 10:

Table 11
Minimum and maximum differences between semiannual returns of strategies with or without options $\Delta^{\mathcal{L}(0)}_{\mathcal{L}(6)}$ and between returns of strategies with options and Equity $\Delta^{Equity}_{\mathcal{L}(0.5,6,4)}$.

	$\Delta^{\mathcal{L}(0)}_{\mathcal{L}(6)}$	$\Delta^{Equity}_{\mathcal{L}(6)}$
MinΔ	−0.052 (Jan–Jun 2013)	−0.0737 (Jan–Jun 2021)
MaxΔ	0.05 (Jan–June 2019)	0.1182 (Jul–Dec 2015)

- The strategy including options, $h = 6$, replicates the S&P market behavior (Equity), with a comparable average semiannual return over the testperiod, but with consistently lower volatility thus generating on average a higher risk-adjusted return (Sharpe ratio).
- The minimum ex-post return – Min(r_t) – shows that during the 10 years, unlike any other and in particular the one without options, the strategy with options is the only one never leading to a loss in value, thus reflecting a good downside protection.
- Under the given policy constraints, both optimal portfolios with or without options in different subperiods (see below) almost achieved the S&P (Equity) maximum semiannual return recorded between January 2011 and 2021.

In Table 11 we report in particular the minimum and maximum differences between semiannual returns of the strategies with and without options, denoted by $\Delta^{\mathcal{L}(0)}_{\mathcal{L}(6)}$, and against the S&P $\Delta^{Equity}_{\mathcal{L}(6)}$.

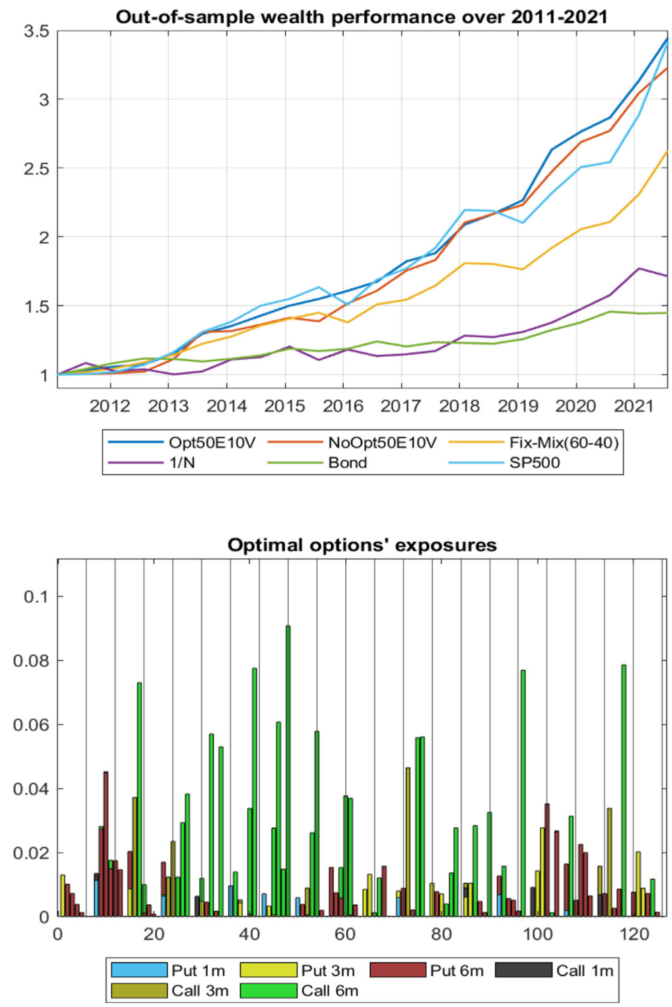

Fig. 17. Top: Optimal wealth evolutions for different strategies over the semesters in the out-of-sample period (Jan 2011–Jun 2021). We denote the wealth trajectories $\mathcal{L}(0.5, \mathcal{C}^6_{j,ATM}, \mathcal{I}_4)$ with Opt50E10V. $\mathcal{L}(0.5, \mathcal{C}^0_{j,ATM}, \mathcal{I}_4)$ with NoOpt50E10V, with Fix-Mix(60 – 40 equity-bond), $1/N$, and Bond and S&P500 for the Equity. Bottom: Optimal option positions for each semester in the out-of-sample period (Jan 2011–Jun 2021). $\mathcal{L}(0.5, \mathcal{C}^6_{j,ATM}, \mathcal{I}_4)$ with long ATM Put and Call with maturity $t_j = 1, 3, 6$.

The maximum positive difference bottom right in the second row of Table 11 refers to the second semester of 2015 when the S&P registered a loss and the option portfolio was protected by 3- and 6- month put options. Bottom left the positive difference was instead motivated by the investment in call options during a positive market evolution. The minimum top right was due to a rallying equity market during the first semester of 2021 that was not fully captured by the strategy with options. Looking at the first column, a maximum difference occurred during the first semester of 2019 thanks to an investment in call options of different maturity. During the first semester of 2013 the negative difference between ex-post returns was primarily due to joint put and call positions during a period of low volatility.

The evolution over time of the out-of-sample returns from different strategies can be inferred from Fig. 17, where the overall wealth dynamics over the 10 and half years are presented together with the associated specific option exposures of the optimal strategy (labelled Opt50E10V). These latter are extracted from Fig. 18 to provide graphical evidence of the specific role played by the options to achieve the above positive performance.

The patterns within each semester can be immediately identified to show that option contracts with different maturities have

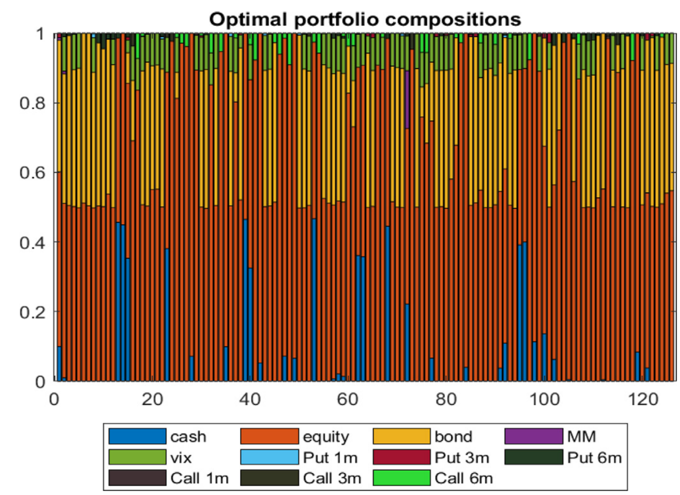

Fig. 18. Monthly replicating portfolio composition in the out-of-sample period (Jan 2011–Jun 2021 in the x-axis span from 0 to month 126 with monthly steps). $\mathcal{L}(0.5, \mathcal{C}^6_{j,ATM}, \mathcal{I}_4)$ with 10% upper bound on VIX and 50% lower bound on equity.

been exploited leading to the returns reported in Table 10. Overall, in 9 out of 21 semesters the strategy with options outperforms the S&P, while in 11 out of 21 semesters it outperforms the strategy without options. The options strategies displayed in the bottom figure also help further explaining the evidence in Table 11.

By focusing on the last year and half of the pandemic period January 2020–June 2021, the bottom plot in Fig. 17 shows that an investment in 3- and 6-month put options protect the portfolio from falling during a negative market phase while in 2021 long 3- and 6-month call options are part of the optimal strategy during a bullish market.

We present as final evidence the evolution of optimal portfolios' composition over the 2011–2021 period along the selected replicating scenarios in Fig. 18, with evidence of the prevailing equity and options (zoomed in the above Fig. 17) exposure with overall relatively well diversified portfolio strategies.

In summary, the 2011–2021 out-of-sample results confirm under the proposed backtesting scheme, the evidences collected in Section 5.4.1 for selected subperiods, and show that:

- Options are consistently included and traded in optimal portfolios with positive ex-post results across different market phases.
- The strategies generated by optimal replicating portfolios including options have consistently provided a positive outcome ex-post, and a significant out-of-sample excess return relative to the 1/N and fix-mix strategies.
- Put options of different maturities have overall provided good hedging results during bearish stock market periods, as in the second semesters of 2015, and 2018.
- Call option positions have been mainly included in the optimal portfolio and traded during rallying market phases, see the first semester of 2016, and in 2021.
- In periods characterized by consistent market uncertainty and high VIX, the simultaneous presence of put and call options have also led to positive ex-post outcome. See, for example, selected months in the second semesters of 2011 and 2021.

6. Conclusion

This article main contribution refers to a comprehensive modeling extension of multistage portfolio management to include equity options and options' portfolios. In the proposed model, under pretty general assumptions and with limited restrictions, optimal investment portfolios and optimal derivatives exposures are determined for an asset universe that includes equity options with several maturities. Besides, the model easily accommodates option strategies such as protective puts, covered calls, straddles or strips, straps, and alternative risk-reward profiles. Furthermore, we analyzed in this context the implications of including, through the VIX, equity volatility as an asset class. In this contribution, we test, to our knowledge for the first time, the impact on optimal dynamic portfolios of equity or multi-asset portfolio managers including options, in selected market phases and then out-of-sample over the 2011–2021 period. Their evolution under alternative risk-reward trade-offs and different volatility regimes has been analysed in depth through an extended computational study based on a stochastic linear programming framework. Section 5 presented results first when specific option portfolios were included in the asset universe and then treating options as stand-alone instruments included in the investment universe. Finally an out-of-sample backtesting set of results has been presented.

We can summarize the main findings of this research:

- Option portfolio and option contracts were shown to be systematically exploited by dynamic optimal portfolios to control downside risk and improve the upside: such evidence has been tested across different test periods assuming a decision space characterized by equity; equity and VIX; equity, bond and money market and finally all these plus the VIX.
- In presence of a risk-averse investor with $\lambda = 1$ and independently of the test period the tail risk control improves in-sample under any of the introduced market assumptions. Both tail and volatility risk control are effective also when $\lambda = 0.5$ and optimal strategies do rely on available option contracts over the investment periods. On the other side when $\lambda = 0$ the optimal portfolio dispersion increases but yet thanks to derivatives' and portfolio dynamic rebalancing terminal in-sample returns and risk-adjusted performances are positive in any period.
- Volatility traders and risky investors are interested in derivative contracts such as equity options and the VIX: we have presented evidence that indeed a positive volatility exposure through the VIX, depending on the agent risk aversion, may be optimal together with equity and equity options positions. These latter will help improving the downside almost under any model specification. We hope in this respect to provide a potentially interesting research perspective on the stream of volatility-driven dynamic portfolio approaches (Dempster et al., 2007; Hill, 2012; Liu et al., 2019).
- In almost every test problems we have seen that optimal options positions were optimal in specific stages over the investment horizons and sometimes brought to expiry (and cash-settled): further, as any decision is taken under residual uncertainty, call and put contract exposures can jointly be optimal, either within straddle or strip or strap strategies or, more generally when considering free calls and puts, sometimes relying on different maturities.
- The out-of-sample evidences are positive both from a specific analysis of three selected subperiods and from a more extended analysis over 21 semesters, starting in January 2011. The optimal portfolios with options overperform the other set of investment strategies. The collected results in this context do depend on the adopted data-driven scenario generation method. Here, under generic assumptions on market incompleteness and assets statistical properties, we have implemented an asset pricing and scenario tree construction approach. This methodological step is relevant for any asset management or asset-liability management model including contingent claims.

The choice on the number of stages and decision points along the horizon made in the computational section of the paper is not binding, neither the scenario tree structure. Other descriptions of the multi-stage decision process increasing or reducing the number of stages and different scenario tree structures accommodate easily.

CRediT authorship contribution statement

Diana Barro: Conceptualization, Methodology, Software, Investigation, Formal analysis, Data curation, Writing – original draft. **Giorgio Consigli:** Conceptualization, Methodology, Validation, Writing – review & editing. **Vivek Varun:** Software, Formal analysis, Data curation, Investigation.

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