Economics 500

Autumn 2021 **Assignment 2** **Due: Oct. 18**

1. Suppose that an individual's preference over three goods can be represented by the following utility function:

$$u(x_1, x_2, x_3) = {x_1}^{\alpha_1}{x_2}^{\alpha_2}{x_3}^{\alpha_3}.$$

(a) Without loss of generality, we may assume $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Explain why?

(b) What are the Kuhn-Tucker conditions for the corresponding UMP?

(c) In solving the Walrasian demands, do we need to worry about the non-negative (feasibility) constraints and why?

(d) Solve for the individual's Walrasian demand functions and indirect utility function.

1.a) we can assume $\alpha_1 + \alpha_2 + \alpha_3$=1 because the MRS is preserved during any positive monotonic transformation.

MRS:
$$\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = \frac{x_2 \cancel{x_2^{\alpha_2}} \cdot \cancel{x_3^{\alpha_3}} \cdot \alpha_1 \cancel{x_1^{(\alpha_1-1)}}}{\cancel{x_1^{\alpha_1}} \cdot \cancel{x_3^{\alpha_3}} \cdot \alpha_2 \cancel{x_2^{(\alpha_2-1)}} x_1} = \frac{\alpha_1 x_2}{\alpha_2 x_1}$$

$$\frac{\partial u/\partial x_2}{\partial u/\partial x_3} = \frac{\cancel{x_1^{\alpha_1}} \cdot \cancel{x_3^{\alpha_3}} \cdot \alpha_2 \cancel{x_2^{(\alpha_2-1)}} x_3}{\cancel{x_1^{\alpha_1}} \cdot \cancel{x_2^{\alpha_2}} \cdot \alpha_3 \cancel{x_3^{(\alpha_3-1)}} x_2} = \frac{\alpha_2 x_3}{\alpha_3 x_2}$$

$$\frac{\partial u/\partial x_1}{\partial u/\partial x_3} = \frac{\cancel{x_2^{\alpha_2}} \cdot \cancel{x_3^{\alpha_3}} \cdot \alpha_1 \cancel{x_1^{(\alpha_1-1)}} x_3}{\cancel{x_1^{\alpha_1}} \cdot \cancel{x_2^{\alpha_2}} \cdot \alpha_3 \cancel{x_3^{(\alpha_3-1)}} x_1} = \frac{\alpha_1 x_3}{\alpha_3 x_1}$$

transformation:
$$f(u(x_1,x_2,x_3)) = x_1^{\frac{\alpha_1}{\alpha_1+\alpha_2+\alpha_3}} + x_2^{\frac{\alpha_2}{\alpha_1+\alpha_2+\alpha_3}} + x_3^{\frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3}}$$

$$\frac{\alpha_1+\alpha_2+\alpha_3}{\alpha_1+\alpha_2+\alpha_3} = 1$$

b) $\mathcal{L} = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} - \lambda(w$

K-T conditions:

feasibility: ${x_1}^{\alpha_1}{x_2}^{\alpha_2}{x_3}^{\alpha_3}$ at optimal values is possible

$x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}$ ln() transformation…

$$\alpha_1 \ln(x_1) + \alpha_2 \ln(x_2) + \alpha_3 \ln(x_3) = 0$$

Cannot take the ln() of 0, therefore $x_1, x_2, x_3 > 0$.
Since ln() of any number is always positive, to get zero, $\min\{\alpha_1, \alpha_2, \alpha_3\} < 0$.

complimentary slackness: $\lambda(x_1, x_2, x_3) = 0$

$$\mathcal{L} = x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3} + \lambda(w - (x_1p_1 + x_2p_2 + x_3p_3)) + \mu_1x_1 + \mu_2x_2 + \mu_3x_3$$

for $\lambda \geq 0$, $\mu_1, \mu_2, \mu_3 \geq 0$

① $\frac{\partial \mathcal{L}}{\partial x_1} = \frac{x_1^{\alpha_1-1}}{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} - p_1\lambda + \mu_1$

② $\frac{\partial \mathcal{L}}{\partial x_2} = x_1^{\alpha_1} \frac{x_2^{\alpha_2-1}}{\alpha_2} x_3^{\alpha_3} - p_2\lambda + \mu_2$

③ $\frac{\partial \mathcal{L}}{\partial x_3} = x_1^{\alpha_1} x_2^{\alpha_2} \frac{x_3^{\alpha_3-1}}{\alpha_3} - p_3\lambda + \mu_3$

④ $\frac{\partial \mathcal{L}}{\partial \lambda} = w - (x_1p_1 + x_2p_2 + x_3p_3)$

⑤ $\frac{\partial \mathcal{L}}{\partial \mu_1} = x_1$

⑥ $\frac{\partial \mathcal{L}}{\partial \mu_2} = x_2$

⑦ $\frac{\partial \mathcal{L}}{\partial \mu_3} = x_3$

$\therefore$

c) no. because if $x1, x2,$ or $x3 = 0,$ *then* $u = 0$, but since $u > 0$ is also possible, having 0 of any good is never an optimal solution ($x1, x2,$ and $x3$ must be strictly positive at the optimum). Thus, the feasibility constraints do not come into play when considering the optimum.

Set the constraints equal to zero.

d) ① $\lambda = \frac{\cancel{x_1^{\alpha_1-1}}\,\cancel{x_2^{\alpha_2}}\,\cancel{x_3^{\alpha_3}}}{p_1\alpha_1 \, x_1} = \frac{x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3-1}}{p_3\alpha_3 \, x_3} \Longrightarrow p_1\alpha_1x_1 = p_3 \times$

$p_1\alpha_1x_1 = p_2\alpha_2x_2$

$x_1p_1 = \frac{x_2p_2\alpha_2}{\alpha_1}$

② $\lambda = \frac{\cancel{x_1^{\alpha_1}}\,\cancel{x_2^{\alpha_2-1}}\,\cancel{x_3^{\alpha_3}}}{p_2\alpha_2 \, x_2} = \frac{x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3-1}}{p_3\alpha_3 \, x_3} \Longrightarrow p_3\alpha_3x_3 = p_2\alpha_2x_2$

$\frac{p_3\alpha_3x_3}{\alpha_2} = p_2x$

③ $\lambda = \frac{x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3-1}}{p_3\alpha_3}$

$\lambda = \left(\frac{\alpha_1}{p_1}\right)^{\alpha_1} + \left(\frac{\alpha_2}{p_2}\right)^{\alpha_2} + \left(\frac{\alpha_3}{p_3}\right)^{-3}$

④ $w = x_1p_1 + x_2p_2 + x_3p_3$

$w = \frac{x_2p_2\alpha_2}{\alpha_1} + \frac{p_3\alpha_3x_3}{\alpha_2} + \frac{p_1\alpha_1x_1}{\alpha_3}$

$w = \frac{x_2p_2\alpha_2^2 + x_3p_3\alpha_3^2 + x_1p_1\alpha_1^2}{\cancel{\alpha_1 + \alpha_2 + \alpha_3} \; 1}$

$w = x_2p_2\alpha_2^2 + x_3p_3\alpha_3^2 + x_1p_1\alpha_1^2$

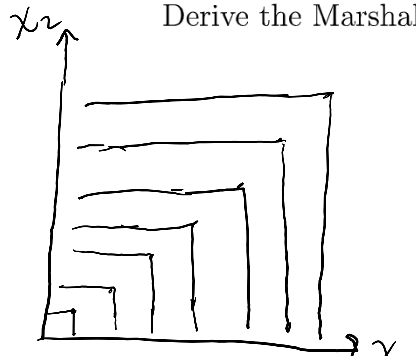
$x_1 = \frac{\alpha_1 w}{p_1}, \quad x_2 = \frac{\alpha_2 w}{p_2}, \quad x_3 = \frac{\alpha_3 w}{p_3}$ demand

$u(x_1, x_2, x_3) = \left(\frac{\alpha_1 w}{p_1}\right)^{\alpha_1}\left(\frac{\alpha_2 w}{p_2}\right)^{\alpha_2}\left(\frac{\alpha_3 w}{p_3}\right)^{\alpha_3}$ indirect utility fxn.

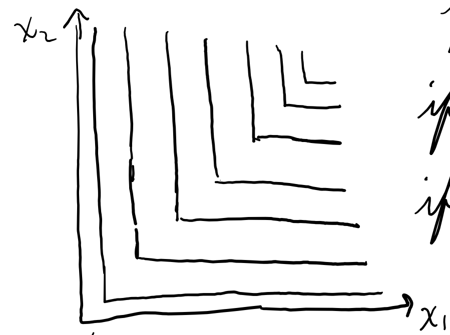
2. A consumer of two goods faces positive prices (p_1 and p_2) and has a positive income (w). His utility function is

$$u(x_1, x_2) = \max\{\alpha x_1, \alpha x_2\} + \min\{x_1, x_2\}, \qquad \text{where } 0 < \alpha < 1.$$

Derive the Marshallian demand functions.



min



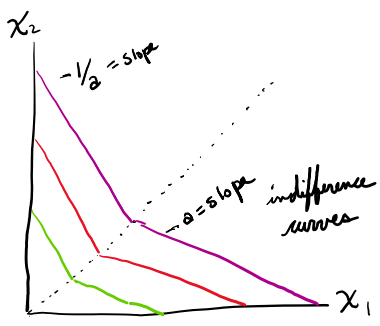
max

if $x_1 = 0$ $\quad \alpha x_2 + 0 \Rightarrow u = \alpha x_2$

if $x_2 = 0$ $\quad \alpha x_1 + 0 \Rightarrow u = \alpha x_1$

if $x_1, x_2 \neq 0$ $\quad u = \max(\alpha x_1, \alpha x_2) + \min(x_1, x_2)$

$w = p_1 x_1 + p_2 x_2 \Leftarrow$ budget constraint



can re-write as

$u = \alpha x_1 + x_2$ if $x_1 > x_2$

$u = \alpha x_2 + x_1$ if $x_2 \geq x_1$

* if $x_1 = x_2$, then

$w = p_1 x_1 + p_2 x_1$

$w = (p_1 + p_2) x_1$

$\frac{w}{p_1 + p_2} = x_1 = x_2$ when $a < \frac{p_1}{p_2} < \frac{1}{a}$

when $\frac{p_1}{p_2} \leq a$

if $x_2 = 0$ then $\longrightarrow$ $x_2 = 0$, $\frac{w}{p_1} = x_1$

when $\frac{p_1}{p_2} \geq \frac{1}{a}$

if $x_1 = 0$ then $\longrightarrow$ $x_1 = 0$; $\frac{w}{p_2} = x_2$

3. Consider the following utility function

$$u(x,y) = \begin{cases} \dfrac{xy}{x+y} & \text{if } (x,y) \geq (0,0) \text{ and } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Derive the Walrasian demand functions for good x and good y.

(b) What is the indirect utility function?

(c) Verify the homogenity and Walra's law of the demand functions and the indirect utility function.

$$\frac{xy}{x+y} + (w - p_x x - p_y y)\lambda = \mathcal{L}$$

$$xy(x+y)^{-1} + \lambda(w - p_x x - p_y y) = \mathcal{L}$$

$$\frac{\partial \mathcal{L}}{\partial x} : -xy(x+y)^{-2} + (x+y)^{-1} y - \lambda p_x = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} : -xy(x+y)^{-2} + (x+y)^{-1} x - \lambda p_y = 0$$

$$\frac{y}{x+y} = \lambda p_x + \frac{xy}{(x+y)^2}$$

$$\frac{y}{x+y} - \frac{xy}{(x+y)^2} = \lambda$$

$$\frac{x}{x+y} = \lambda p_y + \frac{xy}{(x+y^2)}$$

$$\frac{x}{x+y} - \frac{xy}{(x+y)^2} = \lambda$$

$$\frac{y}{x+y} - \cancel{\frac{xy}{(x+y)^2}} = \cancel{\lambda}$$

$$- \frac{x}{x+y} - \cancel{\frac{xy}{(x+y)^2}} = \cancel{\lambda}$$

$$\frac{y - x}{x+y} = 0$$

$$y = x$$

$$w = p_x x + p_y y$$

$$w = p_x x + p_y x$$

$$w = x(p_x + p_y)$$

$$\boxed{\frac{w}{p_x + p_y} = x = y}$$

if $(x,y) \geq (0,0)$ & $(x,y) \neq (0,0)$

otherwise 0

b) $$u(x,y) = \frac{\left(\frac{w}{p_x + p_y}\right)^{\cancel{2}}}{\frac{2\cancel{w}}{\cancel{p_x} + p_y}} = \boxed{\frac{w}{2(p_x + p_y)}}$$

c) homogeneity: $\lambda u(x,y) = u(\lambda x, \lambda y)$

$$\frac{\lambda xy}{x+y} = \frac{\lambda x \cdot \lambda y}{\lambda x + \lambda y}$$

$$= \frac{\lambda^2 (xy)}{\lambda (x+y)}$$

$$= \frac{\lambda xy}{x+y}$$

Walras: $p \cdot x = w$ for all $x \in x(p,w)$

$$x = \frac{w}{p_x + p_y} \qquad y = \frac{w}{p_x + p_y}$$

$$p_y y = \frac{p_y w}{p_x + p_y}$$

$$p_x x = \frac{p_x w}{p_x + p_y}$$

$$x \cancel{p_x} (p_x + p_y) = \cancel{p_x} w \qquad \cancel{p_y} y (p_x + p_y) = \cancel{p_y} w$$

$$\boxed{(p_x + p_y) x = w} \qquad \boxed{y(p_x + p_y) = w}$$