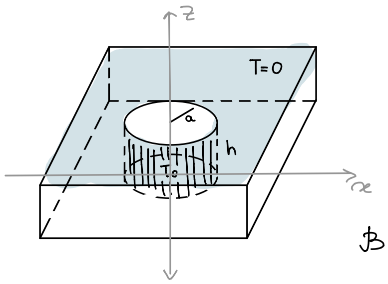
115 Стенки цилиндрического, просверленного в неограниченной пластине толщиной h, поддерживают при температуре T_0. Найти стационарное распределение температуры в пластине, если её грани имеют нулевую температуру.



$$\Delta T = 0 \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2} = 0$$

1) $T|_{r=a} = T_0$ 2) $T|_{r=\infty} < \infty$

3) $T|_{z=0} = 0$ 4) $T|_{z=h} = 0$

Вспомогательная задача:

$$Z'' + \lambda Z = 0 \qquad Z = C_1\cos(\sqrt{\lambda}z) + C_2\sin(\sqrt{\lambda}z)$$

5) $Z|_{z=0} = 0$ $\qquad \lambda_n = \left(\frac{\pi n}{h}\right)^2$, $n \in \mathbb{N}$

6) $Z|_{z=h} = 0$ $\qquad Z_n = \sin\left(\frac{\pi n}{h}z\right)$

$$T = \sum_{n=1}^{\infty} Z_n(z)\,C_n(r) \qquad C_n = \frac{(T, Z_n)}{\|Z_n\|^2} = \frac{\tilde{T}}{\|Z_n\|^2} = \frac{\tilde{T}}{\frac{1}{2}h}$$

$$\|Z_n\|^2 = \int_0^h \sin^2\left(\frac{\pi n}{h}z\right)dz = \frac{1}{2}\left(h - \sin\left(\frac{2\pi n}{h}z\right)\cdot\frac{h}{2\pi n}\Big|_0^h\right) = \frac{1}{2}h$$

$$(T, Z_n) = \int_0^h T Z_n\,dz$$

$$\int_0^h \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right) Z_n\,dz = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\underbrace{\int_0^h T Z_n\,dz}_{\tilde{T}}\right) + \int_0^h \frac{\partial^2 T}{\partial z^2} Z_n\,dz = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{T}}{\partial r}\right) + \overset{5)6)\to 0}{\cancel{\frac{\partial T}{\partial z} Z_n}}\Big|_0^h - \int_0^h \frac{\partial T}{\partial z} Z_n'\,dz = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{T}}{\partial r}\right) - \overset{3)4)\to 0}{\cancel{T}}\, Z_n'\Big|_0^h + \int_0^h T Z_n''\,dz = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{T}}{\partial r}\right) - \lambda_n \tilde{T} = 0$$

$$\tilde{T}_n = C_1 J_0(i\sqrt{\lambda_n}\,r) + C_2 Y_0(i\sqrt{\lambda_n}\,r)$$

$$T\big|_{r=\infty} < \infty \Rightarrow \tilde{T}\big|_{r=\infty} < \infty \Rightarrow C_1 = 0 \Rightarrow \tilde{T} = C_1\, Y_0(i\sqrt{\lambda}\, r)$$

$$T\big|_{r=a} = T_0 \Rightarrow \tilde{T}\big|_{r=a} = T_0 \int_0^h Z_n\, dz = T_0 \left(-\cos\left(\frac{\pi n}{h} z\right) \cdot \frac{h}{\pi n} \Big|_0^h \right) =$$

$$= T_0 \left(\frac{h}{\pi n} \left(1 - (-1)^n\right) \right), \quad n \in \mathbb{N}$$

$$\Rightarrow C_2 = \frac{T_0 \left(\frac{(1-(-1)^n)}{\pi n} \cdot h \right)}{Y_0(i\sqrt{\lambda_n}\, a)}$$

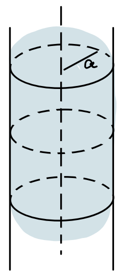
$$\Rightarrow \tilde{T}_n = T_0 \left(\frac{(1-(-1)^n)}{\pi n} h \right) \frac{Y_0(i\sqrt{\lambda_n}\, r)}{Y_0(i\sqrt{\lambda_n}\, a)}$$

$$T(r,z) = \sum_{n=1}^{\infty} \frac{T_0 \left(\frac{(1-(-1)^n)}{\pi n} \cancel{h} \right) \frac{Y_0(i\sqrt{\lambda_n}\, r)}{Y_0(i\sqrt{\lambda_n}\, a)}}{\frac{1}{2} \cancel{h}} \sin\left(\frac{\pi n}{h} z\right) =$$

$$= \frac{2T_0}{\pi} \sum_{n=1}^{\infty} \frac{(1-(-1)^n)}{n} \frac{Y_0\left(i \frac{\pi n}{h} r\right)}{Y_0\left(i \frac{\pi n}{h} a\right)} \sin\left(\frac{\pi n}{h} z\right) =$$

$$= \frac{4T_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n+1} \frac{K_0\left(\frac{2n+1}{h} \pi r\right)}{K_0\left(\frac{2n+1}{h} \pi a\right)} \sin^2\left(\frac{n+1}{n} \pi z\right)$$

126 б) Найти распр-е температуры $T(r,\tau)$ в цилиндре радиуса a, температура поверхности которого изменяется по закону $T|_{r=a} = A\sin\omega\tau$. Начальная температура цилиндра равна 0.



$$\Delta T - \frac{\partial T}{\partial \tau} = 0 \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) - \frac{\partial T}{\partial \tau} = 0$$

1) $T|_{r=a} = A\sin\omega t$ 3) $T|_{\tau=0} = 0$

2) $T|_{r=0} < \infty$

Вспомогательная задача

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \lambda R = 0 \qquad R = C_1 J_0(\sqrt{\lambda_n}\, r) + C_2 Y_0(\sqrt{\lambda_n}\, r)$$

4) $R|_{r=0} < \infty \Rightarrow C_2 = 0$

5) $R|_{r=a} = 0 \Rightarrow \lambda_n = \left(\frac{\alpha_n}{a}\right)^2 \qquad R_n = J_0\left(\frac{\alpha_n}{a} r\right)$

$\lambda = 0 \Rightarrow R = C_1 \ln(x) + C_2 \Rightarrow \begin{matrix} C_1 = 0 \\ C_2 = 0 \end{matrix}$ – трив.

$$T = \sum_{n=1}^{\infty} C_n(\tau) R_n(r) \qquad C_n = \frac{(R_n, T)}{\|R_n\|^2} = \frac{\tilde{T}}{\frac{a^2}{2} J_1^2(\alpha_n)}$$

$$\|R_n\|^2 = \int_0^a r J_0^2\left(\frac{\alpha_n}{a} r\right) dr = \frac{1}{2}\left[r^2\left(J_0'\left(\frac{\alpha_n}{a} r\right)\right)^2 + r^2\left(\cancelto{0}{J_0\left(\frac{\alpha_n}{a} r\right)}\right)^2\right]\Bigg|_0^a = \frac{a^2}{2} J_1^2(\alpha_n)$$

$$\int_0^a \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) - \frac{\partial T}{\partial \tau}\right] R_n \cdot r\, dr = 0$$

$$\int_0^a \frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) R_n\, dr - \frac{\partial}{\partial \tau}\tilde{T} = 0$$

$$r\frac{\partial T}{\partial r} \overset{5)}{\cancelto{0}{R_n}}\Big|_0^a - \int_0^a r\frac{\partial T}{\partial r} R_n'\, dr - \frac{\partial \tilde{T}}{\partial \tau} = 0$$

$$-T\cdot r R_n'\Big|_0^a + \int_0^a T\,(r R_n')'\, dr - \frac{\partial \tilde{T}}{\partial \tau} = 0$$

$$-A\sin(\omega t)\cdot a\, J_1(\alpha_n)\cdot\frac{\alpha_n}{a} - \lambda_n \tilde{T}_n - \frac{\partial \tilde{T}_n}{\partial \tau} = \underset{\text{прив. вр.}}{0}$$

$$\lambda_n \tilde{T}_n + \frac{\partial \tilde{T}_n}{\partial \tau} = -\alpha_n\cdot J_1(\alpha_n)\cdot A\sin(\omega t) \qquad \tau = \frac{kt}{c\rho} \Rightarrow t = \frac{c\rho}{k}\tau$$

$$\lambda_n \tilde{T}_n + \frac{\partial \tilde{T}_n}{\partial \tau} = -\alpha_n J_1(\alpha_n)\cdot A\sin\left(\omega\frac{c\rho}{k}\tau\right)$$

$$\tilde{T}_{n.r.} = C e^{-\lambda_n \tau}$$

$$\tilde{T}_{n.o.} = \frac{\lambda_n \gamma_n A J_1(\gamma_n)}{\lambda_n^2 + \left(\frac{\omega c \rho}{k}\right)^2} \cdot \sin\left(\frac{\omega c \rho}{k}\tau\right) + \frac{\gamma_n J_1(\gamma_n) A \frac{\omega c \rho}{k}}{\lambda_n^2 + \left(\frac{\omega c \rho}{k}\right)^2} \cdot \cos\left(\frac{\omega c \rho}{k}\tau\right)$$

$$T\Big|_{\tau=0} = 0 \Rightarrow \tilde{T}\Big|_{\tau=0} = 0 \Rightarrow C = \frac{A\gamma_n J_1(\gamma_n)\frac{\omega c \rho}{k}}{\lambda_n^2 + \left(\frac{\omega c \rho}{k}\right)^2}$$

$$\longrightarrow T = \sum_{n=1}^{\infty}\left[\frac{\lambda_n \gamma_n A J_1(\gamma_n)}{\lambda_n^2 + \left(\frac{\omega c \rho}{k}\right)^2} \cdot \sin\left(\frac{\omega c \rho}{k}\tau\right) + \frac{\gamma_n J_1(\gamma_n) A \frac{\omega c \rho}{k}}{\lambda_n^2 + \left(\frac{\omega c \rho}{k}\right)^2} \cdot \cos\left(\frac{\omega c \rho}{k}\tau\right) + \frac{A\gamma_n J_1(\gamma_n)\frac{\omega c \rho}{k}}{\lambda_n^2 + \left(\frac{\omega c \rho}{k}\right)^2} e^{-\lambda_n \tau}\right] \cdot \frac{J_0\left(\frac{\gamma_n}{a} r\right)}{\frac{a^2}{2} J_1^2(\gamma_n)}$$

$$\sum_{n=1}^{\infty} \frac{2}{a^2 J_1^2(\gamma_n)} \cdot \frac{A\lambda_n \gamma_n J_1(\gamma_n)}{\lambda_n^2 + \left(\frac{\omega c \rho}{k}\right)^2} \cdot \sin\left(\frac{\omega c \rho}{k}\tau\right) \cdot J_0\left(\frac{\gamma_n}{a} r\right) =$$

$$= \frac{2A\sin\left(\frac{\omega c \rho}{k}\tau\right)}{a^2} \sum_{n=1}^{\infty} \frac{\lambda_n \gamma_n}{J_1(\gamma_n)\left(\lambda_n^2 + \left(\frac{\omega c \rho}{k}\right)^2\right)} J_0\left(\frac{\gamma_n}{a} r\right) =$$

$$\lambda_n \overset{\uparrow}{=} \left(\frac{\gamma_n}{a}\right)^2$$

$$= \frac{2A\sin\left(\frac{\omega c \rho}{k}\tau\right)}{a^2} \sum_{n=1}^{\infty} \frac{\gamma_n^3 \cdot J_0\left(\frac{\gamma_n}{a} r\right)}{\cancel{a^2} J_1(\gamma_n)\left(\frac{\gamma_n^4}{a^4} + \left(\frac{\omega c \rho}{k}\right)^2\right)} =$$

$$\frac{\omega c \rho}{k} = \theta$$

$$= \frac{2A\sin\left(\frac{\omega c \rho}{k}\tau\right)}{\cancel{a^2}} \sum_{n=1}^{\infty} \frac{\gamma_n^3 J_0\left(\frac{\gamma_n}{a} r\right) \cancel{a^4}^{\,2}}{J_1(\gamma_n)\left(\gamma_n^4 + \left(\frac{\omega c \rho}{k}\right)^2 \cdot a^4\right)} = 2A\sin\left(\frac{\omega c \rho}{k}\tau\right) \sum_{n=1}^{\infty} \frac{(\gamma_n^4 + \theta^2 a^4 - \theta^2 a^4) J_0\left(\frac{\gamma_n}{a} r\right)}{J_1(\gamma_n)(\gamma_n^4 + \theta^2 a^4)\gamma_n} =$$

$$= 2A\sin(\theta\tau) \sum_{n=1}^{\infty} \cancel{\frac{J_0\left(\frac{\gamma_n}{a} r\right)}{J_1(\gamma_n)\gamma_n}}^{\frac{1}{2}} - \frac{\theta^2 a^4 J_0\left(\frac{\gamma_n}{a} r\right)}{\gamma_n J_1(\gamma_n)(\gamma_n^4 + \theta^2 a^4)} = A\sin(\theta\tau) - \sum_{n=1}^{\infty} \frac{2A\theta^2 a^4 J_0\left(\frac{\gamma_n}{a} r\right)}{J_1(\gamma_n)(\gamma_n^4 + \theta^2 a^4)\gamma_n} \sin(\theta\tau)$$

$$\longrightarrow T = A\left[\sin\theta\tau + 2\theta a^2 \cdot \sum_{n=1}^{\infty} \frac{\gamma_n J_0\left(\frac{\gamma_n}{a} r\right)}{(\gamma_n^4 + a^4\theta^2) J_1(\gamma_n)} \cdot \left(e^{-\frac{\gamma_n^2}{a^2}\tau} - \cos(\theta\tau) - \frac{a^2\theta}{\gamma^2}\sin(\theta\tau)\right)\right]$$