Пример N 1

Рассмотрим сглаженный винеровский процесс.

$$X(t) = \frac{1}{t}\int_0^t w(\tau)\,d\tau$$

Оценить интенсивность пересечений его траект. с пост. ур-м α.

Решение:

$Y(t) = \int_0^t w(\tau)\,d\tau$ Нужно исследовать выбросы Y за уровень $a(t) = \alpha t$

По общим правилам: $\bar{m}_a(t_1, t_2) = \int_{t_1}^{t_2} \mu_a(t)\,dt$

$$\mu_a(t) = \int_{-\infty}^{\infty} |v - \dot{a}(t)|\, f_{y\dot{y}}(a(t), v; t)\,dv$$

$$\mu_a(t) = \int_{-\infty}^{\infty} |v - \alpha|\, f_{y\dot{y}}(\alpha t, v; t)\,dv$$

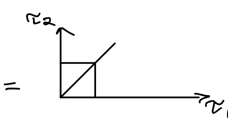
$$\begin{cases} Y(t) = \int_0^t w(\tau)\,d\tau \\ \dot{Y}(t) = W(t) = V(t) \end{cases}$$

$\bar{y}(t) = 0 \quad \bar{v}(t) = 0$

Получим закон распределения (Y, V) — система нормальных величин.
Вычислим дисперсии

$$D[V(t)] = t$$

$$D[Y(t)] = D\left[\int_0^t W(\tau)\,d\tau\right] = \int_0^t\int_0^t \min(\tau_1, \tau_2)\,d\tau_1 d\tau_2 = \frac{t^3}{3}$$



$$k_{yv}(t) = M[Y(t)V(t)] = M\left[\int_0^t w(\tau)\,d\tau \; w(t)\right] = \int_0^t \min(\tau, t)\,d\tau = \int_0^t \tau\,d\tau = \frac{t^2}{2}$$

Коэфф. корреляции $r = r_{yv} = \frac{k_{yv}}{\sigma_y \sigma_v} = \frac{\frac{t^2}{2}}{\sqrt{t}\sqrt{\frac{t^3}{3}}} = \frac{\sqrt{3}}{2}$

$$f_{y\dot{y}}(y, v; t) = \frac{1}{2\pi\sigma_y\sigma_{\dot{y}}\sqrt{1-r^2}}\, e^{\frac{-1}{2(1-r^2)}\left[\frac{y^2}{\sigma_y^2} + \frac{v^2}{\sigma_v^2} - \frac{2r y v}{\sigma_y \sigma_v}\right]} = \frac{\sqrt{3}}{\pi t^2}\, e^{-\left(\frac{6y^2}{t^3} - \frac{6yv}{t^2} + \frac{2v^2}{t}\right)}$$

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}}\, e^{\frac{-1}{2(1-r^2)}\left[\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2rxy}{\sigma_x\sigma_y}\right]}$$

$$\mu_a(t) = \int_{-\infty}^{\infty} |v - \alpha|\, \frac{\sqrt{3}}{\pi t^2}\, e^{-\left(\frac{6y^2}{t^3} - \frac{6yv}{t^2} + \frac{2v^2}{t}\right)}\,dv$$

Делаем замену переменной $z = v - \alpha$

$$\mu_a(t) = \frac{\sqrt{3}}{\pi t^2}\int_{-\infty}^{\infty} |z|\, e^{-\frac{(z^2 - \alpha z + \alpha^2)}{t}}\,dz = \frac{\sqrt{3}}{\pi t^2}\Big[\underbrace{\int_0^{\infty} z\, e^{-\frac{(z^2 - \alpha z + \alpha^2)}{t}}\,dz}_{1} + \underbrace{\int_0^{\infty} z\, e^{-\frac{(z^2 + \alpha z + \alpha^2)}{t}}\,dz}_{2}$$

В интеграле ① замена $z = \frac{s_1}{2}\sqrt{t} - \frac{\alpha}{2}$

В интеграле ② замена $z = \frac{s_2}{2}\sqrt{t} + \frac{\alpha}{2}$

$$m_a(t) = \frac{\sqrt{3}}{\pi t^2}\frac{t}{4}\left[\int_{\frac{\alpha}{\sqrt{t}}}^{\infty} e^{-\frac{s_1^2}{2}}\left(s_1 - \frac{\alpha}{\sqrt{t}}\right)ds_1 + \int_{-\frac{\alpha}{\sqrt{t}}}^{\infty} e^{-\frac{s_2^2}{2}}\left(s_2 + \frac{\alpha}{\sqrt{t}}\right)ds_2\right] e^{-\frac{3}{2}\frac{\alpha^2}{t}} =$$

$$= \frac{\sqrt{3}}{4\pi t^2}\left[2e^{-\frac{\alpha^2}{2t}} + \frac{\alpha}{\sqrt{t}}\left(\int_{-\frac{\alpha}{\sqrt{t}}}^{\infty} e^{-\frac{s^2}{2}}ds + \int_{\frac{\alpha}{\sqrt{t}}}^{\infty} e^{-\frac{s^2}{2}}ds\right)e^{-\frac{\alpha^2}{2t}}\right] =$$

$$= \frac{\sqrt{3}}{4\pi t^2}\left[2e^{-\frac{\alpha^2}{2t}} + \frac{\alpha}{\sqrt{t}}\int_{-\frac{\alpha}{\sqrt{t}}}^{\frac{\alpha}{\sqrt{t}}} e^{-\frac{s^2}{2}}ds\; e^{-\frac{3\alpha^2}{2t}}\right]$$

$$= \frac{\sqrt{3}}{2\pi t}\left[e^{-\frac{2\alpha^2}{t}} + \frac{\alpha}{\sqrt{t}}\int_{0}^{\frac{\alpha}{\sqrt{t}}} e^{-\frac{s^2}{2}}ds\; e^{-\frac{3\alpha^2}{2t}}\right] = \frac{\sqrt{3}}{2\pi t}e^{-\frac{2\alpha^2}{t}}\left[1 + e^{\frac{\alpha^2}{2t}}\frac{\alpha}{\sqrt{t}}\int_{0}^{\frac{\alpha}{\sqrt{t}}} e^{-\frac{s^2}{2}}ds\right]$$

$\alpha = 0 \quad \Rightarrow m_a = \frac{\sqrt{3}}{2\pi t}$

$\bar{m}_a(t_1, t_2) = \frac{\sqrt{3}}{2\pi}\ln\left(\frac{t_2}{t_1}\right)$

$X = \int_0^t$