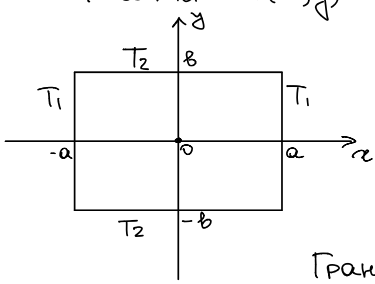
Контрольная работа
по мат. физике

Заболотских Екатерина

группа : 13
вариант: 30

В брусе прямоугольного сечения $[-a \le x \le a]$ $[-b \le y \le b]$ происходит тепловыделение $Q_0$. Грани $x = \pm a$ поддерживаются при температуре $T_1$, а грани $y = \pm b$ при температуре $T_2$.
Найти $T(x,y)$




$$\Delta T = \frac{Q_0}{k}$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = -\frac{Q_0}{k}$$

Граничные условия:

$$T\big|_{x=a} = T_1 \qquad T\big|_{x=0} < \infty$$

$$T\big|_{y=b} = T_2 \qquad T\big|_{y=0} < \infty$$

Вспомогательная задача:

$$\varphi'' + \lambda \varphi = 0$$

$$\varphi'\big|_{x=0} = 0 \qquad \varphi\big|_{x=a} = 0$$

$$\lambda_n = \frac{(2n+1)^2 \pi^2}{4a^2} \qquad \varphi_n = \cos\left(\frac{(2n+1)\pi}{2a} x\right)$$

Метод Гринберга:

$$T(x,y) = \sum_{n=1}^{\infty} C_n(y) \varphi_n(x)$$

$$C_n(y) = \frac{\int_0^a T(x,y) \varphi_n(x)\,dx}{\int_0^a \sin^2\left(\frac{\pi n}{a} x\right) dx} = \frac{\tilde{T}(y)}{\frac{1}{2} a}$$

$$\int_0^a \frac{\partial^2 T}{\partial x^2}\varphi_n\,dx + \frac{\partial^2 \tilde{T}_n}{\partial y^2} = -\frac{Q_0}{k}\int_0^a \varphi_n\,dx$$

$$-\frac{Q_0}{k}\int_0^a \cos\left(\frac{2n+1}{2a}\pi x\right)dx = -\frac{Q_0}{k}\frac{2a}{(2n+1)\pi}\sin\left(\frac{(2n+1)\pi}{2a}x\right)\Big|_0^a = -\frac{Q_0}{k}\frac{2a}{(2n+1)\pi}(-1)^n$$

$$\int_0^a \frac{\partial^2 T}{\partial x^2}\varphi_n\,dx = \cancel{\frac{\partial T}{\partial x}\varphi_n\Big|_0^a} - \int_0^a \frac{\partial T}{\partial x}\varphi_n'\,dx = -T\varphi_n'\Big|_0^a + \int_0^a T\varphi_n''\,dx =$$

$$= +T_1\frac{(2n+1)\pi}{2a}\sin\left(\frac{(2n+1)\pi}{2a}a\right) - T(0)\cancel{\frac{(2n+1)\pi}{2a}}\sin\left(\frac{(2n+1)\pi}{2a}a\right)$$

$$\frac{\partial^2\tilde{T}}{\partial y^2} - \lambda\tilde{T}_n = -\frac{Q_0}{k}\frac{2a}{(2n+1)\pi}(-1)^n - T_1\frac{(2n+1)\pi}{2a}(-1)^n$$

$$\tilde{T}_n\Big|_{y=b} = \int_0^a T_2\cos\left(\frac{2n+1}{2a}\pi x\right)dx\Big|_{y=b} = T_2\frac{2a}{(2n+1)\pi}\sin\left(\frac{(2n+1)\pi}{2a}x\right)\Big|_{y=b} =$$

$$= \frac{2aT_2}{(2n+1)\pi}(-1)^n \qquad \tilde{T}_n' = \sqrt{\lambda}Ae^{\sqrt{\lambda}y} - \sqrt{\lambda}Be^{-\sqrt{\lambda}y}$$

$$\frac{\partial\tilde{T}_n}{\partial y}\Big|_{y=0} = 0 = \sqrt{\lambda}A - \sqrt{\lambda}B \Rightarrow A = B$$

$$\tilde{T}_n^{\text{частное}} = C \Rightarrow -\lambda C = -\frac{Q_0}{k}\frac{2a}{(2n+1)\pi}(-1) - T_1\frac{(2n+1)}{2n}(-1)^n$$

$$C = \frac{4a^2}{(2n+1)^2\pi^2}\frac{Q_0}{k}\frac{2a}{(2n+1)\pi}(-1)^n + T_1\frac{4a}{(2n+1)^2\pi^2}\frac{(2n+1)\pi}{2a}(-1)^n$$

$$\tilde{T}_n^{\text{частн.}} = \frac{Q_0}{k}\frac{8a^3}{(2n+1)^3\pi^3}(-1)^n + T_1\frac{2a}{(2n+1)\pi}(-1)^n$$

$$\tilde{T}_n\Big|_{y=b} = \frac{2aT_2}{(2n+1)\pi}(-1)^n = A\left(e^{\sqrt{\lambda}b} + e^{-\sqrt{\lambda}b}\right) + \tilde{T}_n^{\text{ч.}}$$

$$A = -\frac{Q_0\,4a^3}{ch(\sqrt{\lambda}b)k(2n+1)^3\pi^3}(-1)^n - \frac{1}{ch\sqrt{\lambda}b}\frac{a(-1)^n}{(2n+1)^2\pi^2}(T_2 - T_1)$$

$$\Rightarrow \tilde{T}_n = -\left(\frac{Q_0 \cdot 4a^3 \cdot (-1)^n}{k\,ch(\sqrt{\lambda}b)(2n+1)^3\pi^3} + \frac{1}{ch\sqrt{\lambda}b}\frac{a(-1)^n}{(2n+1)^2\pi^2}(T_2 - T_1)\right)\left(e^{\sqrt{\lambda}y} - e^{-\sqrt{\lambda}y}\right) +$$

$$+ \frac{Q_0 8a^3}{k(2n+1)^3\pi^3}(-1)^n + T_1\frac{2a}{(2n+1)\pi}(-1)^n$$

$$C_n = \frac{2\cdot\tilde{T}_n}{a} = -\left(\frac{Q_0 8a^2(-1)^n}{k\,ch(\sqrt{\lambda}b)(2n+1)^3\pi^3} + \frac{2}{ch\sqrt{\lambda}b}\frac{(-1)^n}{(2n+1)^2\pi^2}(T_2 - T_1)\right)\cdot$$

$$\cdot\left(e^{\sqrt{\lambda}y} - e^{-\sqrt{\lambda}y}\right) + \frac{Q_0}{k}\frac{16a^2}{(2n+1)^3\pi^3}(-1)^n + T_1\frac{4}{(2n+1)\pi}(-1)^n$$

$$T_n = \sum_{n=1}^{\infty}\left[-\frac{Q_0 8a^2(-1)^n}{k\,ch(\sqrt{\lambda}b)(2n+1)^3\pi^3} - \frac{2}{ch(\sqrt{\lambda}b)}\frac{(-1)^n}{(2n+1)^2\pi^2}(T_2 - T_1)\right)\cdot 2ch\sqrt{\lambda}y$$

$$+ \frac{Q_0}{k}\frac{16a^2\quad(-1)^n}{(2n+1)^3\pi^3} + T_1\frac{4}{(2n+1)\pi}(-1)^n\Bigg]\cos\left(\frac{(2n+1)\pi x}{2a}\right)$$

Улучшение сходимости:

$$\sum_{n=1}^{\infty} T_1\frac{4}{(2n+1)\pi}(-1)^n\cos\left(\frac{(2n+1)\pi}{2a}x\right) = \frac{4T_1}{\cancel{\pi}}\cdot\frac{\cancel{\pi}}{\cancel{4}} = T_1$$

$$\sum_{n=1}^{\infty}\frac{Q_0}{k}\frac{16a^2}{(2n+1)^3\pi^3}(-1)^n\cos\left(\frac{(2n+1)\pi}{2a}x\right) = \frac{Q_0}{\pi^3 k}16a^2\cdot\frac{\pi^3}{32}\left(1 - \frac{x^2}{a^2}\right) =$$

$$= \frac{Q_0 a^2}{2k}\left(1 - \frac{x^2}{a^2}\right)$$