7. $f(x) = \sin \alpha x \qquad \alpha > 0$

$$y'' + \lambda y = 0 \qquad x \in [0; l]$$

$y(0) = 0$ $\qquad y = C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x$

$y(l) = 0$ $\qquad y(0) = C_1 = 0$

$\qquad y(l) = C_2 \sin \sqrt{\lambda}\, l = 0$

$C_2 \neq 0$: $\quad \sin \sqrt{\lambda}\, l = 0$

$\sqrt{\lambda}\, l = \pm \pi n, \quad n \in \mathbb{N}$

$$\lambda_n = \frac{\pi^2 n^2}{l^2} \qquad \tilde{y}_k = C_k \underbrace{\sin\left(\frac{\pi n}{l} x\right)}_{y_k}$$

$$f(x) = \sum_{k=0}^{\infty} C_k y_k \qquad C_k = \frac{(y_k, f)}{\|y_k\|^2}$$

$$(y_k, f) = \int_0^l \sin \alpha x \cdot y_k \, dx = -\frac{1}{\lambda_n} \int_0^l \sin \alpha x \, y_k'' dx = -\frac{1}{\lambda_n} \left(\sin \alpha n \, y_k' \Big|_0^l - \right.$$

$$\left. - \alpha \int_0^l \cos \alpha x \, y_k' \, dx \right) = \frac{-1}{\lambda_n} \left(\sin \alpha l \cdot y_k'(l) - \alpha \left(\cos \alpha x \, \cancel{y_k} \right)\Big|_0^l + \alpha \underbrace{\int_0^l \sin \alpha x \, y_n \, dx}_{(y_k, f)} \right)$$

$$(y_k, f) = \frac{-1}{\lambda_n} \left(\sin \alpha l \, y_k'(l) + \alpha (y_k, f) \right)$$

$$(y_k, f) = \frac{-\alpha}{\lambda_n} (y_k, f) - \frac{1}{\lambda_n} \sin(\alpha l) y_k'(l)$$

$$(y_k, f)\left(1 + \frac{\alpha}{\lambda_n}\right) = -\frac{1}{\lambda_n} \sin(\alpha l) y_k'(l)$$

$$(y_k, f) = \frac{-1}{\lambda_n + \alpha} \sin(\alpha l) y_k'(l)$$

$$(y_k, f) = -\frac{1}{\lambda_n + \alpha} \left(\sin \alpha l \; y_k'(l) \right) = \frac{-1}{\frac{\pi^2 n^2}{l^2} + \alpha} \cdot \frac{\pi n}{l} \cdot \left(\sin(\alpha l) \cos\left(\frac{\pi n}{l} l\right) \right) =$$

$$= \frac{-\pi n l}{\pi^2 n^2 + l^2 \alpha} \left(\sin(\alpha l)(-1)^{n+} \right) = \frac{(-1)^{n+2} \pi n l}{\pi^2 n^2 + l^2 \alpha} \sin(\alpha l)$$

$$\|y_k\|^2 = \int_0^l \sin^2\left(\frac{\pi n}{l} x\right) dx = \int_0^l \frac{1 - \cos\left(\frac{2\pi n}{l} x\right)}{2} dx = \frac{1}{2}\left(l - \frac{l}{2\pi n} \left(\sin\left(\frac{2\pi n}{l} x\right) \Big|_0^l \right) \right) = \frac{l}{2}$$

$$C_n = \frac{\frac{(-1)^{n+2} \pi n l}{\pi^2 n^2 + l^2 \alpha} \sin(\alpha l) \cdot 2}{l} = 2 \cdot \frac{(-1)^{n+2} \pi n}{\pi^2 n^2 + l^2 \alpha} \sin(\alpha l)$$

Ответ: $f(x) = \sum\limits_{n=1}^{\infty} 2 \frac{(-1)^{n+2} \pi n}{\pi^2 n^2 + l^2 \alpha} \sin(\alpha l) \sin\left(\frac{\pi n}{l} x\right)$