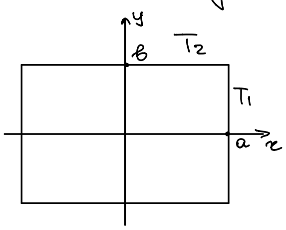
Контрольная работа
по мат. физике

Заболотских Екатерина

группа: 13
вариант: 30

В брусе прямоугольного сечения $[-a \le x \le a]$ $[-b \le y \le b]$ происходит тепловыделение Q_0. Грани $x = \pm a$ поддерживаются при температуре T_1, а грани $y = \pm b$ при температуре T_2. Найти $T(x,y)$



$$\Delta T = -\frac{Q_0}{k}$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = -\frac{Q_0}{k}$$

Граничные условия:

1) $T\big|_{x=a} = T_1$ 2) $\frac{\partial T}{\partial x}\Big|_{x=0} = 0$

3) $T\big|_{y=b} = T_2$ 4) $\frac{\partial T}{\partial y}\Big|_{y=0} = 0$

Вспомогательная задача:

$$\frac{\partial^2 X}{\partial x^2} + \lambda X = 0$$

$$X = C_1 \sin(\sqrt{\lambda} x) + C_2 \cos(\sqrt{\lambda} x)$$
$$X' = C_1 \sqrt{\lambda} \cos(\sqrt{\lambda} x) - C_2 \sqrt{\lambda} \sin(\sqrt{\lambda} x)$$

5) $X\big|_{x=a} = 0$

6) $\frac{\partial X}{\partial x}\Big|_{x=0} = 0$

$$\frac{\partial X}{\partial x}\Big|_{x=0} = 0 \Rightarrow C_1 = 0 \qquad X\big|_{x=a} = 0 \Rightarrow \lambda_n = \left(\frac{\pi(1+2n)}{2a}\right)^2$$

$$X = C \cos(\sqrt{\lambda} x) \qquad \lambda_n = \left(\frac{\pi(1+2n)}{2a}\right)^2$$

$$X_n = \cos\left(\frac{\pi(1+2n)}{2a} x\right)$$

$$T = \sum_{n=1}^{\infty} C_n(y) X_n(x) \qquad C_n = \frac{(X_n(x), T)}{\|X_n\|^2}$$

$$\|X_n\|^2 = \int_0^a X_n^2\, dx = \int_0^a \cos^2\left(\frac{\pi(1+2n)}{2a} x\right) dx = \frac{1}{2}\left(a + \frac{1}{\pi(1+2n)} \sin\left(\pi(1+2n)x\right)\Big|_0^a\right) = \frac{1}{2} a$$

$$(X_n(x), T) = \int_0^a T X_n\, dx = \widetilde{T}$$

$$\int_0^a \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) X_n\, dx = -\frac{Q_0}{k} \int_0^a X_n\, dx$$

$$\int_0^a \frac{\partial^2 T}{\partial x^2} X_n\,dx + \frac{\partial^2 \tilde{T}}{\partial y^2} = -\frac{Q_0}{k}\,\frac{(-1)^n 2a}{\pi(1+2n)}$$

$$\overset{2)\to 0}{\cancel{\frac{\partial T}{\partial x}}}\;\overset{5)\to 0}{\cancel{X_n}}\Big|_0^a - \int_0^a \frac{\partial T}{\partial x} X'_n\,dx + \frac{\partial^2 \tilde{T}}{\partial y^2} = -\frac{Q_0(-1)^n 2a}{k\,\pi(1+2n)}$$

$$-T X'_n\Big|_0^a + \int_0^a T X''_n\,dx + \frac{\partial^2 \tilde{T}}{\partial y^2} = -\frac{Q_0(-1)^n 2a}{k\,\pi(1+2n)}$$

$$-T_1 X'_n(a) - \lambda \tilde{T} + \frac{\partial^2 \tilde{T}}{\partial y^2} = -\frac{Q_0(-1)^n 2a}{k\,\pi(1+2n)}$$

$$\frac{\partial^2 \tilde{T}}{\partial y^2} - \lambda\tilde{T} = -\frac{Q_0(-1)^n 2a}{k\,\pi(1+2n)} - T_1\frac{(-1)^n\pi(1+2n)}{2a}$$

$$\left[X'_n(a) = -\sin\left(\frac{\pi(1+2n)}{2}\right)\cdot\frac{\pi(1+2n)}{2a}\right]$$

$$J\;A = -\frac{Q_0(-1)^n 2a}{k\,\pi(1+2n)} - T_1\frac{(-1)^n\pi(1+2n)}{2a}$$

$$\boxed{\frac{\partial^2 \tilde{T}}{\partial y^2} - \lambda\tilde{T} = A}$$

$$\tilde{T}_{n.ч.} = -\frac{A}{\lambda_n}$$

$$\tilde{T}_{n.o.} = C_1 e^{-\sqrt{\lambda_n}\,y} + C_2 e^{\sqrt{\lambda_n}\,y} = \tilde{C}_1\,ch(\sqrt{\lambda_n}\,y) + \tilde{C}_2\,sh(\sqrt{\lambda_n}\,y)$$

$$\frac{\partial T}{\partial y}\Big|_{y=0} = 0 \Rightarrow \frac{\partial}{\partial y}\int_0^a T X_n\,dx\Big|_{y=0} = 0 \Rightarrow \frac{\partial \tilde{T}}{\partial y}\Big|_{y=0} = 0 \Rightarrow \tilde{C}_2 = 0$$

$$\tilde{T}_{n.o} = \tilde{C}_1\,ch(\sqrt{\lambda_n}\,y)$$

$$T\Big|_{y=b} = T_2 \Rightarrow \int_0^a T X_n\,dx\Big|_{y=b} = T_2\int_0^a X_n\,dx = T_2\,\frac{(-1)^n 2a}{\pi(2n+1)} = \tilde{T}\Big|_{y=b}$$

$$\tilde{T}_n = \tilde{T}_{n.o.} + \tilde{T}_{n.ч.}$$

$$\tilde{T}_n\Big|_{y=b} = C_1\,ch(\sqrt{\lambda_n}\,b) - \frac{A}{\lambda_n} = T_2\,\frac{(-1)^n 2a}{\pi(2n+1)}$$

$$C_1 = \frac{\dfrac{T_2(-1)^n 2a}{\pi(2n+1)} + \dfrac{A}{\lambda_n}}{ch(\sqrt{\lambda_n}\,b)}$$

$$C_1 = \frac{T_2(-1)^n 2a}{\pi(2n+1)\,ch(\sqrt{\lambda_n}\,b)} - \frac{Q_0(-1)^n 2a}{\lambda_n\, ch(\sqrt{\lambda_n}\,b)\, k\pi(1+2n)} - \frac{T_1(-1)^n \pi(1+2n)}{\lambda_n\, 2a\, ch(\sqrt{\lambda_n}\,b)}$$

$$\tilde{T}_n = \frac{T_2(-1)^n 2a}{\pi(2n+1)}\cdot\frac{ch(\sqrt{\lambda_n}\,y)}{ch(\sqrt{\lambda_n}\,b)} - \frac{Q_0(-1)^n 2a}{\lambda_n k\pi(1+2n)}\cdot\frac{ch(\sqrt{\lambda_n}\,y)}{ch(\sqrt{\lambda_n}\,b)} - \frac{T_1(-1)^n \pi(1+2n)}{\lambda_n\cdot 2a}\cdot\frac{ch(\sqrt{\lambda_n}\,y)}{ch(\sqrt{\lambda_n}\,b)}$$

$$+ \frac{Q_0(-1)^n 2a}{k\pi(1+2n)\lambda_n} + \frac{T_1(-1)^n\pi(1+2n)}{2a\ \lambda_n}$$

$$\tilde{T}_n = (-1)^n\left[\frac{ch(\sqrt{\lambda_n}\,y)}{ch(\sqrt{\lambda_n}\,b)}\left(T_2\frac{2a}{\pi(2n+1)} - Q_0\frac{2a}{\lambda_n k\pi(1+2n)} - T_1\frac{\pi(1+2n)}{\lambda_n 2a}\right) + Q_0\frac{2a}{k\pi(1+2n)\lambda_n} + T_1\frac{\pi(1+2n)}{2a\ \lambda_n}\right]$$

$$T = \frac{2}{a}\sum_{n=1}^{\infty}\left((-1)^n\left[\frac{ch(\sqrt{\lambda_n}\,y)}{ch(\sqrt{\lambda_n}\,b)}\left(T_2\frac{2a}{\pi(2n+1)} - Q_0\frac{2a}{\lambda_n k\pi(1+2n)} - T_1\frac{\pi(1+2n)}{\lambda_n 2a}\right) + Q_0\frac{2a}{k\pi(1+2n)\lambda_n} + T_1\frac{\pi(1+2n)}{2a\ \lambda_n}\right]\right)\cos\left(\frac{\pi(1+2n)}{2a}x\right)$$

Улучшение сходимости:

1
$$\sum_{n=1}^{\infty}(-1)^n T_1\frac{\pi(1+2n)}{2a\ \lambda_n}\cos\left(\frac{(2n+1)\pi x}{2a}\right) = \sum_{n=1}^{\infty}(-1)^n T_1\frac{4a^2\pi(1+2n)}{2a\,\pi^2(1+2n)^2}\cos\left(\frac{(2n+1)\pi x}{2a}\right) =$$

$$= \sum_{n=1}^{\infty}(-1)^n T_1\frac{2a}{\pi(1+2n)}\cos\left(\frac{(2n+1)\pi x}{2a}\right) = T_1\frac{2a}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{\cos\frac{(2n+1)\pi x}{2a}}{2n+1} = T_1\frac{2a}{\pi}\frac{\pi}{4}$$

$$= T_1\frac{a}{2}$$

2
$$\sum_{n=1}^{\infty}(-1)^n Q_0\frac{2a}{k\pi(1+2n)\lambda_n}\cos\left(\frac{(2n+1)\pi x}{2a}\right) = \sum_{n=1}^{\infty}(-1)^n Q_0\frac{2a\cdot 4a^2}{k\pi^3(1+2n)^3}\cos\left(\frac{(2n+1)\pi x}{2a}\right) =$$

$$= Q_0\frac{8a^3}{k\pi^3}\sum_{n=1}^{\infty}(-1)^n\frac{\cos\left(\frac{(2n+1)\pi x}{2a}\right)}{(2n+1)^3} = Q_0\frac{8a^3}{k\pi^3}\frac{\pi^3}{32}\left(1-\frac{x^2}{a^2}\right) = Q_0\frac{a^3}{k\,4}\left(1-\frac{x^2}{a^2}\right)$$

Ответ:

$$T=\frac{2}{a}\sum_{n=1}^{\infty}\left[(-1)^n\frac{ch(\sqrt{\lambda_n}\,y)}{ch(\sqrt{\lambda_n}\,b)}\left(T_2\frac{2a}{\pi(2n+1)}-Q_0\frac{8a^3}{k\pi^3(1+2n)^3}-T_1\frac{2a}{\pi(1+2n)}\right)\cdot\cos\left(\frac{\pi(1+2n)}{2a}x\right)\right]$$

$$+T_1\frac{a}{2}+Q_0\frac{a^3}{4k}\left(1-\frac{x^2}{a^2}\right)$$

$$\boxed{=\sum_{n=1}^{\infty}\left[(-1)^n\frac{ch(\sqrt{\lambda_n}\,y)}{ch(\sqrt{\lambda_n}\,b)}\;\frac{\cos\left(\frac{\pi(1+2n)}{2a}x\right)}{\pi(1+2n)}\left(4(T_2-T_1)-8Q_0\frac{a^2}{k\pi^2(1+2n)^2}\right)\right]+ +T_1+Q_0\frac{a^2}{2k}\left(1-\frac{x^2}{a^2}\right)}$$