

Equations for HOC

1 Predictive equations

$$\frac{\partial \bar{u}}{\partial t} = -\bar{w} \frac{\partial \bar{u}}{\partial z} - f(v_g - \bar{v}) - \frac{\partial}{\partial z} \overline{u'w'} \quad (1)$$

$$\frac{\partial \bar{v}}{\partial t} = -\bar{w} \frac{\partial \bar{v}}{\partial z} + f(u_g - \bar{u}) - \frac{\partial}{\partial z} \overline{v'w'} \quad (2)$$

$$\frac{\partial \bar{q}_t}{\partial t} = -\bar{w} \frac{\partial \bar{q}_t}{\partial z} - \frac{\partial}{\partial z} \overline{w'q'_t} + \left. \frac{\partial \bar{q}_t}{\partial t} \right|_{\text{ls}} \quad (3)$$

$$\frac{\partial \bar{\theta}_l}{\partial t} = -\bar{w} \frac{\partial \bar{\theta}_l}{\partial z} - \frac{\partial}{\partial z} \overline{w'\theta'_l} + \bar{R} + \left. \frac{\partial \bar{\theta}_l}{\partial t} \right|_{\text{ls}} \quad (4)$$

$$\begin{aligned} \frac{\partial \overline{w'^2}}{\partial t} = & -\bar{w} \frac{\partial \overline{w'^2}}{\partial z} - \frac{\partial \overline{w'^3}}{\partial z} - 2\overline{w'^2} \frac{\partial \bar{w}}{\partial z} + \frac{2g}{\theta_0} \overline{w'\theta'_v} \\ & - \frac{C_4}{\tau} \left(\overline{w'^2} - \frac{2}{3} \bar{e} \right) - C_5 \left(-2\overline{w'^2} \frac{\partial \bar{w}}{\partial z} + \frac{2g}{\theta_0} \overline{w'\theta'_v} \right) + \frac{2}{3} C_5 \left(\frac{g}{\theta_0} \overline{w'\theta'_v} - \overline{u'w'} \frac{\partial \bar{u}}{\partial z} - \overline{v'w'} \frac{\partial \bar{v}}{\partial z} \right) \\ & - \frac{C_1}{\tau} \overline{w'^2} + \nu_1 \nabla_z^2 \overline{w'^2} \end{aligned} \quad (5)$$

$$\frac{\partial \overline{q_t'^2}}{\partial t} = -\bar{w} \frac{\partial \overline{q_t'^2}}{\partial z} - \frac{\partial \overline{w'q_t'^2}}{\partial z} - 2\overline{w'q'_t} \frac{\partial \bar{q}_t}{\partial z} - \frac{C_2}{\tau} \overline{q_t'^2} + \nu_2 \nabla_z^2 \overline{q_t'^2} \quad (6)$$

$$\frac{\partial \overline{\theta_l'^2}}{\partial t} = -\bar{w} \frac{\partial \overline{\theta_l'^2}}{\partial z} - \frac{\partial \overline{w'\theta_l'^2}}{\partial z} - 2\overline{w'\theta'_l} \frac{\partial \bar{\theta}_l}{\partial z} - \frac{C_2}{\tau} \overline{\theta_l'^2} + \nu_2 \nabla_z^2 \overline{\theta_l'^2} \quad (7)$$

$$\frac{\partial \overline{q'_t \theta'_l}}{\partial t} = -\bar{w} \frac{\partial \overline{q'_t \theta'_l}}{\partial z} - \frac{\partial \overline{w'q'_t \theta'_l}}{\partial z} - \overline{w'q'_t} \frac{\partial \bar{\theta}_l}{\partial z} - \overline{w'\theta'_l} \frac{\partial \bar{q}_t}{\partial z} - \frac{C_2}{\tau} \overline{q'_t \theta'_l} + \nu_2 \nabla_z^2 \overline{q'_t \theta'_l} \quad (8)$$

$$\begin{aligned} \frac{\partial \overline{w'q'_t}}{\partial t} = & -\bar{w} \frac{\partial \overline{w'q'_t}}{\partial z} - \frac{\partial \overline{w'^2 q'_t}}{\partial z} - \overline{w'^2} \frac{\partial \bar{q}_t}{\partial z} - \overline{w'q'_t} \frac{\partial \bar{w}}{\partial z} + \frac{g}{\theta_0} \overline{q'_t \theta'_v} \\ & - \frac{C_6}{\tau} \overline{w'q'_t} - C_7 \left(-\overline{w'q'_t} \frac{\partial \bar{w}}{\partial z} + \frac{g}{\theta_0} \overline{q'_t \theta'_v} \right) + \nu_6 \nabla_z^2 \overline{w'q'_t} \end{aligned} \quad (9)$$

$$\begin{aligned} \frac{\partial \overline{w'\theta'_l}}{\partial t} = & -\bar{w} \frac{\partial \overline{w'\theta'_l}}{\partial z} - \frac{\partial \overline{w'^2 \theta'_l}}{\partial z} - \overline{w'^2} \frac{\partial \bar{\theta}_l}{\partial z} - \overline{w'\theta'_l} \frac{\partial \bar{w}}{\partial z} + \frac{g}{\theta_0} \overline{\theta'_l \theta'_v} \\ & - \frac{C_6}{\tau} \overline{w'\theta'_l} - C_7 \left(-\overline{w'\theta'_l} \frac{\partial \bar{w}}{\partial z} + \frac{g}{\theta_0} \overline{\theta'_l \theta'_v} \right) + \nu_6 \nabla_z^2 \overline{w'\theta'_l} \end{aligned} \quad (10)$$

$$\begin{aligned} \frac{\partial \overline{w'^3}}{\partial t} = & -\bar{w} \frac{\partial \overline{w'^3}}{\partial z} - \frac{\partial \overline{w'^4}}{\partial z} + 3\overline{w'^2} \frac{\partial \overline{w'^2}}{\partial z} - 3\overline{w'^3} \frac{\partial \bar{w}}{\partial z} + \frac{3g}{\theta_0} \overline{w'^2 \theta'_v} \\ & - \frac{C_8}{\tau} (C_{8b} Sk w^4 + 1) \overline{w'^3} - C_{11} \left(-3\overline{w'^3} \frac{\partial \bar{w}}{\partial z} + \frac{3g}{\theta_0} \overline{w'^2 \theta'_v} \right) + (K_w + \nu_8) \nabla_z^2 \overline{w'^3} \end{aligned} \quad (11)$$

\bar{R} is the radiative heating rate, f the Coriolis parameter and u_g , v_g the geostrophic winds. $\left. \frac{\partial \bar{q}_t}{\partial t} \right|_{\text{ls}}$ and $\left. \frac{\partial \bar{\theta}_t}{\partial t} \right|_{\text{ls}}$ are large-scale moisture and temperature forcings. g is the gravity, ρ_0 and θ_0 the reference density and potential temperature.

Because the model does not predict any higher-order moments of the horizontal winds, we assume that the turbulence kinetic energy, \bar{e} , is proportional to the vertical velocity variance $\overline{w'^2}$:

$$\bar{e} = \frac{3}{2} \overline{w'^2}. \quad (12)$$

The time scale τ is:

$$\tau = \begin{cases} \frac{L}{\sqrt{\bar{e}}}; & L/\sqrt{\bar{e}} \leq \tau_{\max} \\ \tau_{\max}; & L/\sqrt{\bar{e}} > \tau_{\max} \end{cases}. \quad (13)$$

Additionally, τ is set to a minimum value τ_{\min} whenever $\overline{w'^2} \leq 0.005 \text{ m}^2 \text{ s}^{-2}$.

The eddy diffusivity coefficient K_w is

$$K_w = 0.22 L \bar{e}^{1/2}. \quad (14)$$

The momentum fluxes are closed using a down gradient approach:

$$\overline{u'w'} = -K_m \frac{\partial \bar{u}}{\partial z} \quad (15a)$$

$$\overline{v'w'} = -K_m \frac{\partial \bar{v}}{\partial z} \quad (15b)$$

where the turbulent-transfer coefficient K_m is given by:

$$K_m = c_K L \bar{e}^{1/2} \quad (16)$$

with $c_K = 0.548$ as in Duynkerke and Driedonks (1987).

The specific values of the constants C_i and ν_i are as follows: $C_1 = 1.7$; $C_2 = 1.04$; $C_4 = 4.5$; $C_5 = 0$; $C_6 = 4.85$; $C_7 = 0.8$; $C_8 = 2.73$; $C_{11} = 0.2$; $\nu_1 = \nu_2 = \nu_8 = 20 \text{ (m}^2/\text{s)}$; and $\nu_6 = 30 \text{ (m}^2/\text{s)}$.

2 PDF closure

Details of the PDF closure can be found in Larson and Golaz (2005), hereafter referred to as LG. We only briefly summarize key aspects here.

2.1 Transport terms

The transport terms appearing in Eqs (1)-(11) are closed as follows. First, we define $c_{w\theta_l}$ and c_{wq_t} as in Eqs (LG15) and (LG16):

$$c_{w\theta_l} = \frac{\overline{w'\theta'_l}}{\sqrt{\overline{w'^2}}\sqrt{\overline{\theta'^2_l}}} \quad (17)$$

$$c_{wq_t} = \frac{\overline{w'q'_t}}{\sqrt{\overline{w'^2}}\sqrt{\overline{q'^2_t}}} \quad (18)$$

The width of the individual w plumes is given by (LG37):

$$\tilde{\sigma}_w^2 = \gamma [1 - \max(c_{w\theta_l}^2, c_{wq_t}^2)] \quad (19)$$

We define the following quantities in order to simplify the notation:

$$a_1 = \frac{1}{(1 - \tilde{\sigma}_w^2)} \quad (20)$$

$$a_2 = \frac{1}{(1 - \tilde{\sigma}_w^2)^2} \quad (21)$$

$$a_3 = 3\tilde{\sigma}_w^4 + 6(1 - \tilde{\sigma}_w^2)\tilde{\sigma}_w^2 + (1 - \tilde{\sigma}_w^2)^2 - \frac{3}{2} \quad (22)$$

The turbulence moment $\overline{w'^4}$ is given by (LG40):

$$\overline{w'^4} = \overline{w'^2}^2 \left(a_3 + \frac{3}{2} \right) + a_1 \frac{\overline{w'^3}^2}{\overline{w'^2}} \quad (23)$$

The flux transport terms are given by (LG42):

$$\overline{w'^2\theta'_l} = a_1 \frac{\overline{w'^3}}{\overline{w'^2}} \overline{w'\theta'_l} \quad (24)$$

$$\overline{w'^2q'_t} = a_1 \frac{\overline{w'^3}}{\overline{w'^2}} \overline{w'q'_t} \quad (25)$$

The variance transport terms follow (LG46):

$$\overline{w'\theta_l'^2} = \frac{1}{3}\beta a_1 \frac{\overline{w'^3}}{\overline{w'^2}} \overline{\theta_l'^2} + \left(1 - \frac{1}{3}\beta\right) a_2 \frac{\overline{w'^3}}{\overline{w'^2}^2} \overline{w'\theta_l'^2} \quad (26)$$

$$\overline{w'q_t'^2} = \frac{1}{3}\beta a_1 \frac{\overline{w'^3}}{\overline{w'^2}} \overline{q_t'^2} + \left(1 - \frac{1}{3}\beta\right) a_2 \frac{\overline{w'^3}}{\overline{w'^2}^2} \overline{w'q_t'^2} \quad (27)$$

Finally, the covariance term is obtained substituting (LG56) into (LG48):

$$\overline{w'q_t'\theta_l'} = \frac{1}{3}\beta a_1 \frac{\overline{w'^3}}{\overline{w'^2}} \overline{q_t'\theta_l'} + \left(1 - \frac{1}{3}\beta\right) a_2 \frac{\overline{w'^3}}{\overline{w'^2}^2} \overline{w'q_t'} \overline{w'\theta_l'} \quad (28)$$

2.2 Buoyancy terms

There are more unclosed terms involving θ_v . They are $\overline{w'\theta_v'}$, $\overline{q_t'\theta_v'}$, $\overline{\theta_l'\theta_v'}$, and $\overline{w'^2\theta_v'}$ and can be written as:

$$\overline{\chi'\theta_v'} = \underbrace{\overline{\chi'\theta_l'}}_{\equiv A (\approx 200K)} + \frac{1 - \epsilon_0}{\epsilon_0} \theta_0 \underbrace{\overline{\chi'q_t'}}_{\equiv B (\approx 2000K)} + \left(\frac{L_v}{c_p} \left(\frac{p_0}{p} \right)^{R_d/c_p} - \frac{1}{\epsilon_0} \theta_0 \right) \overline{\chi'q_l'}, \quad (29)$$

where χ' represents w' , q_t' , θ_l' or w'^2 . Here $\epsilon_0 = R_d/R_v$, R_d is the gas constant of dry air, R_v is the gas constant of water vapor, L_v is the latent heat of vaporization, c_p is the heat capacity of air, and p_0 is a reference pressure. The correlations involving liquid water ($\overline{\chi'q_l'}$) can be computed for the given family of PDFs (see next section).

3 Cloud properties

The cloud properties, such as cloud fraction, mean liquid water and correlations involving liquid water ($\overline{\chi'q_l'}$) are obtained from the PDF. To do so, a certain number of properties are computed for each Gaussian ($i = 1, 2$):

$$T_{li} = \theta_{li} \left(\frac{p}{p_0} \right)^{R_d/c_p} \quad (30)$$

$$q_{si} = \frac{R_d}{R_v} \frac{e_s(T_{li})}{p - [1 - (R_d/R_v)]e_s(T_{li})} \quad (31)$$

$$\beta_i = \frac{R_d}{R_v} \left(\frac{L}{R_d T_{li}} \right) \left(\frac{L}{c_p T_{li}} \right) \quad (32)$$

$$s_i = q_{ti} - q_{si} \frac{1 + \beta_i q_{ti}}{1 + \beta_i q_{si}} \quad (33)$$

$$c_{q_{ti}} = \frac{1}{1 + \beta_i q_{si}} \quad (34)$$

$$c_{\theta_{li}} = \frac{1 + \beta_i q_{ti}}{[1 + \beta_i q_{si}]^2} \frac{c_p}{L} \beta_i q_{si} \left(\frac{p}{p_0} \right)^{R_d/c_p} \quad (35)$$

$$\sigma_{si}^2 = c_{\theta_{li}}^2 \sigma_{\theta_{li}}^2 + c_{q_{ti}}^2 \sigma_{q_{ti}}^2 - 2c_{\theta_{li}} \sigma_{\theta_{li}} c_{q_{ti}} \sigma_{q_{ti}} r_{q_{ti} \theta_{li}} \quad (36)$$

$$C_i = \frac{1}{2} \left[1 + \operatorname{erf} \left(\frac{s_i}{\sqrt{2} \sigma_{si}} \right) \right] \quad (37)$$

$$q_{li} = s_i C_i + \frac{\sigma_{si}}{\sqrt{2\pi}} \exp \left[-\frac{1}{2} \left(\frac{s_i}{\sigma_{si}} \right)^2 \right] \quad (38)$$

where C_i and q_{li} are the cloud fractions and liquid water of each individual Gaussian.

The layer-averaged cloud properties are given by:

$$\overline{C} = aC_1 + (1 - a)C_2 \quad (39)$$

$$\overline{q_l} = aq_{l1} + (1 - a)q_{l2} \quad (40)$$

$$\overline{w'q'_l} = a(w_1 - \bar{w})q_{l1} + (1 - a)(w_2 - \bar{w})q_{l2} \quad (41)$$

$$\overline{w'^2 q'_l} = a \left((w_1 - \bar{w})^2 + \sigma_{w1}^2 \right) q_{l1} + (1 - a) \left((w_2 - \bar{w})^2 + \sigma_{w2}^2 \right) q_{l2} - \overline{w'^2} (aq_{l1} + (1 - a)q_{l2}) \quad (42)$$

$$\begin{aligned} \overline{\theta'_l q'_l} = & a \left[(\theta_{l1} - \bar{\theta}_l) q_{l1} - C_1 (c_{\theta_{l1}} \sigma_{\theta_{l1}}^2 - r_{q_{t1} \theta_{l1}} c_{q_{t1}} \sigma_{q_{t1}} \sigma_{\theta_{l1}}) \right] \\ & + (1 - a) \left[(\theta_{l2} - \bar{\theta}_l) q_{l2} - C_2 (c_{\theta_{l2}} \sigma_{\theta_{l2}}^2 - r_{q_{t2} \theta_{l2}} c_{q_{t2}} \sigma_{q_{t2}} \sigma_{\theta_{l2}}) \right] \end{aligned} \quad (43)$$

$$\begin{aligned} \overline{q'_t q'_l} = & a \left[(q_{t1} - \bar{q}_t) q_{l1} + C_1 (c_{q_{t1}} \sigma_{q_{t1}}^2 - r_{q_{t1} \theta_{l1}} c_{\theta_{l1}} \sigma_{q_{t1}} \sigma_{\theta_{l1}}) \right] \\ & + (1 - a) \left[(q_{t2} - \bar{q}_t) q_{l2} + C_2 (c_{q_{t2}} \sigma_{q_{t2}}^2 - r_{q_{t2} \theta_{l2}} c_{\theta_{l2}} \sigma_{q_{t2}} \sigma_{\theta_{l2}}) \right] \end{aligned} \quad (44)$$

4 Steady-state solutions for the variances

4.1 $\overline{q_t'^2}$ and $\overline{\theta_l'^2}$

Start with (6), substitute (27), assume steady-state and rearrange:

$$\begin{aligned} & \frac{C_2}{\tau} \overline{q_t'^2} + \bar{w} \frac{\partial \overline{q_t'^2}}{\partial z} + \frac{1}{3} \beta \frac{\partial}{\partial z} \left(a_1 \frac{\overline{w'^3}}{\overline{w'^2}} \overline{q_t'^2} \right) - \nu_2 \nabla_z^2 \overline{q_t'^2} \\ &= - \left(1 - \frac{1}{3} \beta \right) \frac{\partial}{\partial z} \left(a_2 \frac{\overline{w'^3}}{\overline{w'^2}^2} \overline{w' q_t'^2} \right) - 2 \overline{w' q_t'} \frac{\partial \bar{q}_t}{\partial z} \end{aligned} \quad (45)$$

The goal is to recast (45) so that $\overline{q_t'^2}$ can be computed using a tridiagonal solver:

$$\mathbf{a}(\mathbf{k}) \mathbf{qtp2}(\mathbf{k} - 1) + \mathbf{b}(\mathbf{k}) \mathbf{qtp2}(\mathbf{k}) + \mathbf{c}(\mathbf{k}) \mathbf{qtp2}(\mathbf{k} + 1) = \mathbf{d}(\mathbf{k}) \quad (46)$$

We now compute the contributions of each term in (45) to $\mathbf{a}(\mathbf{k})$, $\mathbf{b}(\mathbf{k})$, $\mathbf{c}(\mathbf{k})$, and $\mathbf{d}(\mathbf{k})$.

4.1.1 Term 1

$$\mathbf{b}(\mathbf{k}) = \mathbf{b}(\mathbf{k}) + \frac{C_2}{\mathbf{taum}(\mathbf{k})} \quad (47)$$

4.1.2 Term 2

$$\begin{aligned} & \left. \bar{w} \frac{\partial \overline{q_t'^2}}{\partial z} \right|_{\mathbf{zm}(\mathbf{k})} \\ &= \frac{\mathbf{wmm}(\mathbf{k})}{\mathbf{dzm}(\mathbf{k})} \left(\frac{1}{2} (\mathbf{qtp2}(\mathbf{k}) + \mathbf{qtp2}(\mathbf{k} + 1)) - \frac{1}{2} (\mathbf{qtp2}(\mathbf{k} - 1) + \mathbf{qtp2}(\mathbf{k})) \right) \\ &= \frac{\mathbf{wmm}(\mathbf{k})}{2\mathbf{dzm}(\mathbf{k})} \mathbf{qtp2}(\mathbf{k} + 1) - \frac{\mathbf{wmm}(\mathbf{k})}{2\mathbf{dzm}(\mathbf{k})} \mathbf{qtp2}(\mathbf{k} - 1) \end{aligned} \quad (48)$$

Separating out the contributions:

$$\begin{aligned} \mathbf{a}(\mathbf{k}) &= \mathbf{a}(\mathbf{k}) - \frac{\mathbf{wmm}(\mathbf{k})}{2\mathbf{dzm}(\mathbf{k})} \\ \mathbf{c}(\mathbf{k}) &= \mathbf{c}(\mathbf{k}) + \frac{\mathbf{wmm}(\mathbf{k})}{2\mathbf{dzm}(\mathbf{k})} \end{aligned} \quad (49)$$

4.1.3 Term 3

$$\begin{aligned}
& \frac{1}{3}\beta \frac{\partial}{\partial z} \left(a_1 \frac{\overline{w'^3}}{\overline{w'^2}} \overline{q_t'^2} \right) \Big|_{\mathbf{zm}(\mathbf{k})} \\
&= \frac{\beta}{6\mathbf{dzm}(\mathbf{k})} \left[\frac{(\mathbf{a1m}(\mathbf{k}) + \mathbf{a1m}(\mathbf{k} + 1)) \mathbf{wp3}(\mathbf{k} + 1) (\mathbf{qtp2}(\mathbf{k}) + \mathbf{qtp2}(\mathbf{k} + 1))}{\max(\mathbf{wp2}(\mathbf{k}) + \mathbf{wp2}(\mathbf{k} + 1), 2\epsilon)} \right. \\
&\quad \left. - \frac{(\mathbf{a1m}(\mathbf{k} - 1) + \mathbf{a1m}(\mathbf{k})) \mathbf{wp3}(\mathbf{k}) (\mathbf{qtp2}(\mathbf{k} - 1) + \mathbf{qtp2}(\mathbf{k}))}{\max(\mathbf{wp2}(\mathbf{k} - 1) + \mathbf{wp2}(\mathbf{k}), 2\epsilon)} \right]
\end{aligned} \tag{50}$$

Separating out the contributions:

$$\begin{aligned}
\mathbf{a}(\mathbf{k}) &= \mathbf{a}(\mathbf{k}) - \frac{\beta}{6\mathbf{dzm}(\mathbf{k})} \frac{(\mathbf{a1m}(\mathbf{k} - 1) + \mathbf{a1m}(\mathbf{k})) \mathbf{wp3}(\mathbf{k})}{\max(\mathbf{wp2}(\mathbf{k} - 1) + \mathbf{wp2}(\mathbf{k}), 2\epsilon)} \\
\mathbf{b}(\mathbf{k}) &= \mathbf{b}(\mathbf{k}) + \frac{\beta}{6\mathbf{dzm}(\mathbf{k})} \left(\frac{(\mathbf{a1m}(\mathbf{k}) + \mathbf{a1m}(\mathbf{k} + 1)) \mathbf{wp3}(\mathbf{k} + 1)}{\max(\mathbf{wp2}(\mathbf{k}) + \mathbf{wp2}(\mathbf{k} + 1), 2\epsilon)} - \frac{(\mathbf{a1m}(\mathbf{k} - 1) + \mathbf{a1m}(\mathbf{k})) \mathbf{wp3}(\mathbf{k})}{\max(\mathbf{wp2}(\mathbf{k} - 1) + \mathbf{wp2}(\mathbf{k}), 2\epsilon)} \right) \\
\mathbf{c}(\mathbf{k}) &= \mathbf{c}(\mathbf{k}) + \frac{\beta}{6\mathbf{dzm}(\mathbf{k})} \frac{(\mathbf{a1m}(\mathbf{k}) + \mathbf{a1m}(\mathbf{k} + 1)) \mathbf{wp3}(\mathbf{k} + 1)}{\max(\mathbf{wp2}(\mathbf{k}) + \mathbf{wp2}(\mathbf{k} + 1), 2\epsilon)}
\end{aligned} \tag{51}$$

4.1.4 Term 4

$$-\nu_2 \nabla_z^2 \overline{q_t'^2} \Big|_{\mathbf{zm}(\mathbf{k})} = \frac{\nu_2}{\mathbf{dzm}(\mathbf{k})} \left(\frac{\mathbf{qtp2}(\mathbf{k} + 1) - \mathbf{qtp2}(\mathbf{k})}{\mathbf{dzt}(\mathbf{k} + 1)} - \frac{\mathbf{qtp2}(\mathbf{k}) - \mathbf{qtp2}(\mathbf{k} - 1)}{\mathbf{dzt}(\mathbf{k})} \right) \tag{52}$$

Separating out the contributions:

$$\begin{aligned}
\mathbf{a}(\mathbf{k}) &= \mathbf{a}(\mathbf{k}) - \frac{\nu_2}{\mathbf{dzm}(\mathbf{k})\mathbf{dzt}(\mathbf{k})} \\
\mathbf{b}(\mathbf{k}) &= \mathbf{b}(\mathbf{k}) + \frac{\nu_2}{\mathbf{dzm}(\mathbf{k})} \left(\frac{1}{\mathbf{dzt}(\mathbf{k} + 1)} + \frac{1}{\mathbf{dzt}(\mathbf{k})} \right) \\
\mathbf{c}(\mathbf{k}) &= \mathbf{c}(\mathbf{k}) - \frac{\nu_2}{\mathbf{dzm}(\mathbf{k})\mathbf{dzt}(\mathbf{k} + 1)}
\end{aligned} \tag{53}$$

4.1.5 Term 5

$$\begin{aligned}
& - \left(1 - \frac{1}{3}\beta \right) \frac{\partial}{\partial z} \left(a_2 \frac{\overline{w'^3}}{\overline{w'^2}} \overline{w'q_t'^2} \right) \Big|_{\mathbf{zm}(\mathbf{k})} \\
&= -\frac{1 - \frac{1}{3}\beta}{2\mathbf{dzm}(\mathbf{k})} \left[\frac{(\mathbf{a2m}(\mathbf{k}) + \mathbf{a2m}(\mathbf{k} + 1)) \mathbf{wp3}(\mathbf{k} + 1) (\mathbf{wpqtp}(\mathbf{k}) + \mathbf{wpqtp}(\mathbf{k} + 1))^2}{\max(\mathbf{wp2}(\mathbf{k}) + \mathbf{wp2}(\mathbf{k} + 1), 2\epsilon)^2} \right. \\
&\quad \left. - \frac{(\mathbf{a2m}(\mathbf{k} - 1) + \mathbf{a2m}(\mathbf{k})) \mathbf{wp3}(\mathbf{k}) (\mathbf{wpqtp}(\mathbf{k} - 1) + \mathbf{wpqtp}(\mathbf{k}))^2}{\max(\mathbf{wp2}(\mathbf{k} - 1) + \mathbf{wp2}(\mathbf{k}), 2\epsilon)^2} \right]
\end{aligned} \tag{54}$$

Separating out the contributions:

$$\begin{aligned}
& d(k) \\
= & d(k) - \frac{1 - \frac{1}{3}\beta}{2dzm(k)} \left[\frac{(a2m(k) + a2m(k+1)) wp3(k+1) (wpqtp(k) + wpqtp(k+1))^2}{\max(wp2(k) + wp2(k+1), 2\epsilon)^2} \right. \\
& \left. - \frac{(a2m(k-1) + a2m(k)) wp3(k) (wpqtp(k-1) + wpqtp(k))^2}{\max(wp2(k-1) + wp2(k), 2\epsilon)^2} \right]
\end{aligned} \tag{55}$$

4.1.6 Term 6

$$-2 \overline{w'q'_t} \frac{\partial \bar{q}_t}{\partial z} \bigg|_{zm(k)} = -2wpqtp(k) \frac{qtm(k+1) - qtm(k)}{dzm(k)} \tag{56}$$

Separating out the contributions:

$$d(k) = d(k) - 2wpqtp(k) \frac{qtm(k+1) - qtm(k)}{dzm(k)} \tag{57}$$

4.2 $\overline{q'_t\theta'_l}$

Start with (8), substitue (28), assume steady-state and rearrange:

$$\begin{aligned}
& \frac{C_2}{\tau} \overline{q'_t\theta'_l} + \bar{w} \frac{\partial \overline{q'_t\theta'_l}}{\partial z} + \frac{1}{3}\beta \frac{\partial}{\partial z} \left(a_1 \frac{\overline{w'^3}}{\overline{w'^2}} \overline{q'_t\theta'_l} \right) - \nu_2 \nabla_z^2 \overline{q'_t\theta'_l} \\
= & - \left(1 - \frac{1}{3}\beta \right) \frac{\partial}{\partial z} \left(a_2 \frac{\overline{w'^3}}{\overline{w'^2}} \overline{w'q'_t w'\theta'_l} \right) - \overline{w'q'_t} \frac{\partial \bar{\theta}_l}{\partial z} - \overline{w'\theta'_l} \frac{\partial \bar{q}_t}{\partial z}
\end{aligned} \tag{58}$$

As for the variances, the goal is to recast (58) so that $\overline{q'_t\theta'_l}$ can be computed using a tridiagonal solver:

$$a(k) qtpthlp(k-1) + b(k) qtpthlp(k) + c(k) qtpthlp(k+1) = d(k) \tag{59}$$

We now compute the contributions of each term in (58) to $a(k)$, $b(k)$, $c(k)$, and $d(k)$.

4.2.1 Term 1

$$b(k) = b(k) + \frac{C_2}{\tau_{aum}(k)} \tag{60}$$

4.2.2 Term 2

$$\begin{aligned}
& \bar{w} \frac{\partial \overline{q'_t \theta'_l}}{\partial z} \Big|_{\mathbf{zm}(\mathbf{k})} \\
&= \frac{\mathbf{wmm}(\mathbf{k})}{\mathbf{dzm}(\mathbf{k})} \left(\frac{1}{2} (\mathbf{qtpthlp}(\mathbf{k}) + \mathbf{qtpthlp}(\mathbf{k} + 1)) - \frac{1}{2} (\mathbf{qtpthlp}(\mathbf{k} - 1) + \mathbf{qtpthlp}(\mathbf{k})) \right) \\
&= \frac{\mathbf{wmm}(\mathbf{k})}{2\mathbf{dzm}(\mathbf{k})} \mathbf{qtpthlp}(\mathbf{k} + 1) - \frac{\mathbf{wmm}(\mathbf{k})}{2\mathbf{dzm}(\mathbf{k})} \mathbf{qtpthlp}(\mathbf{k} - 1)
\end{aligned} \tag{61}$$

Separating out the contributions:

$$\begin{aligned}
\mathbf{a}(\mathbf{k}) &= \mathbf{a}(\mathbf{k}) - \frac{\mathbf{wmm}(\mathbf{k})}{2\mathbf{dzm}(\mathbf{k})} \\
\mathbf{c}(\mathbf{k}) &= \mathbf{c}(\mathbf{k}) + \frac{\mathbf{wmm}(\mathbf{k})}{2\mathbf{dzm}(\mathbf{k})}
\end{aligned} \tag{62}$$

4.2.3 Term 3

$$\begin{aligned}
& \frac{1}{3} \beta \frac{\partial}{\partial z} \left(a_1 \frac{\overline{w'^3}}{\overline{w'^2}} \overline{q'_t \theta'_l} \right) \Big|_{\mathbf{zm}(\mathbf{k})} \\
&= \frac{\beta}{6\mathbf{dzm}(\mathbf{k})} \left[\frac{(\mathbf{a1m}(\mathbf{k}) + \mathbf{a1m}(\mathbf{k} + 1)) \mathbf{wp3}(\mathbf{k} + 1) (\mathbf{qtpthlp}(\mathbf{k}) + \mathbf{qtpthlp}(\mathbf{k} + 1))}{\max(\mathbf{wp2}(\mathbf{k}) + \mathbf{wp2}(\mathbf{k} + 1), 2\epsilon)} \right. \\
&\quad \left. - \frac{(\mathbf{a1m}(\mathbf{k} - 1) + \mathbf{a1m}(\mathbf{k})) \mathbf{wp3}(\mathbf{k}) (\mathbf{qtpthlp}(\mathbf{k} - 1) + \mathbf{qtpthlp}(\mathbf{k}))}{\max(\mathbf{wp2}(\mathbf{k} - 1) + \mathbf{wp2}(\mathbf{k}), 2\epsilon)} \right]
\end{aligned} \tag{63}$$

Separating out the contributions:

$$\begin{aligned}
\mathbf{a}(\mathbf{k}) &= \mathbf{a}(\mathbf{k}) - \frac{\beta}{6\mathbf{dzm}(\mathbf{k})} \frac{(\mathbf{a1m}(\mathbf{k} - 1) + \mathbf{a1m}(\mathbf{k})) \mathbf{wp3}(\mathbf{k})}{\max(\mathbf{wp2}(\mathbf{k} - 1) + \mathbf{wp2}(\mathbf{k}), 2\epsilon)} \\
\mathbf{b}(\mathbf{k}) &= \mathbf{b}(\mathbf{k}) + \frac{\beta}{6\mathbf{dzm}(\mathbf{k})} \left(\frac{(\mathbf{a1m}(\mathbf{k}) + \mathbf{a1m}(\mathbf{k} + 1)) \mathbf{wp3}(\mathbf{k} + 1)}{\max(\mathbf{wp2}(\mathbf{k}) + \mathbf{wp2}(\mathbf{k} + 1), 2\epsilon)} - \frac{(\mathbf{a1m}(\mathbf{k} - 1) + \mathbf{a1m}(\mathbf{k})) \mathbf{wp3}(\mathbf{k})}{\max(\mathbf{wp2}(\mathbf{k} - 1) + \mathbf{wp2}(\mathbf{k}), 2\epsilon)} \right) \\
\mathbf{c}(\mathbf{k}) &= \mathbf{c}(\mathbf{k}) + \frac{\beta}{6\mathbf{dzm}(\mathbf{k})} \frac{(\mathbf{a1m}(\mathbf{k}) + \mathbf{a1m}(\mathbf{k} + 1)) \mathbf{wp3}(\mathbf{k} + 1)}{\max(\mathbf{wp2}(\mathbf{k}) + \mathbf{wp2}(\mathbf{k} + 1), 2\epsilon)}
\end{aligned} \tag{64}$$

4.2.4 Term 4

$$-\nu_2 \nabla_z^2 \overline{q'_t \theta'_l} \Big|_{\mathbf{zm}(\mathbf{k})} = \frac{\nu_2}{\mathbf{dzm}(\mathbf{k})} \left(\frac{\mathbf{qptthlp}(\mathbf{k} + 1) - \mathbf{qptthlp}(\mathbf{k})}{\mathbf{dzt}(\mathbf{k} + 1)} - \frac{\mathbf{qptthlp}(\mathbf{k}) - \mathbf{qptthlp}(\mathbf{k} - 1)}{\mathbf{dzt}(\mathbf{k})} \right) \tag{65}$$

Separating out the contributions:

$$\begin{aligned}
a(k) &= a(k) - \frac{\nu_2}{dzm(k)dz\tau(k)} \\
b(k) &= b(k) + \frac{\nu_2}{dzm(k)} \left(\frac{1}{dz\tau(k+1)} + \frac{1}{dz\tau(k)} \right) \\
c(k) &= c(k) - \frac{\nu_2}{dzm(k)dz\tau(k+1)}
\end{aligned} \tag{66}$$

4.2.5 Term 5

$$\begin{aligned}
& - \left(1 - \frac{1}{3}\beta \right) \frac{\partial}{\partial z} \left(a_2 \frac{\overline{w'^3}}{\overline{w'^2}^2} \overline{w'q'_t} \overline{w'\theta'_t} \right) \Big|_{zm(k)} \\
&= - \frac{1 - \frac{1}{3}\beta}{2dzm(k)} \\
& \times \left[\frac{(a2m(k) + a2m(k+1)) wp3(k+1) (wpqtp(k) + wpqtp(k+1)) (wpthlp(k) + wpthlp(k+1))}{\max(wp2(k) + wp2(k+1), 2\epsilon)^2} \right. \\
& \quad \left. - \frac{(a2m(k-1) + a2m(k)) wp3(k) (wpqtp(k-1) + wpqtp(k)) (wpthlp(k-1) + wpthlp(k))}{\max(wp2(k-1) + wp2(k), 2\epsilon)^2} \right]
\end{aligned} \tag{67}$$

Separating out the contributions:

$$\begin{aligned}
d(k) &= d(k) \\
& - \frac{1 - \frac{1}{3}\beta}{2dzm(k)} \\
& \times \left[\frac{(a2m(k) + a2m(k+1)) wp3(k+1) (wpqtp(k) + wpqtp(k+1)) (wpthlp(k) + wpthlp(k+1))}{\max(wp2(k) + wp2(k+1), 2\epsilon)^2} \right. \\
& \quad \left. - \frac{(a2m(k-1) + a2m(k)) wp3(k) (wpqtp(k-1) + wpqtp(k)) (wpthlp(k-1) + wpthlp(k))}{\max(wp2(k-1) + wp2(k), 2\epsilon)^2} \right]
\end{aligned} \tag{68}$$

4.2.6 Terms 6 and 7

$$\begin{aligned}
& - \overline{w'q'_t} \frac{\partial \bar{\theta}_t}{\partial z} \Big|_{zm(k)} - \overline{w'\theta'_t} \frac{\partial \bar{q}_t}{\partial z} \Big|_{zm(k)} \\
&= -wpqtp(k) \frac{thlm(k+1) - thlm(k)}{dzm(k)} - wpthlp(k) \frac{qtm(k+1) - qtm(k)}{dzm(k)}
\end{aligned} \tag{69}$$

Separating out the contributions:

$$d(k) = d(k) - wpqtp(k) \frac{thlm(k+1) - thlm(k)}{dzm(k)} - wpthlp(k) \frac{qtm(k+1) - qtm(k)}{dzm(k)} \tag{70}$$

5 Implicit solutions for the means and fluxes

\bar{q}_t and $\overline{w'q'_t}$ can be solved simultaneously and implicitly. Start with eqs (3), (9) and substitute expression for the transport term (25):

$$\frac{\partial \bar{q}_t}{\partial t} = -\bar{w} \frac{\partial \bar{q}_t}{\partial z} - \frac{\partial \overline{w'q'_t}}{\partial z} + \frac{\partial \bar{q}_t}{\partial t} \Big|_{\text{ls}} \quad (71)$$

$$\begin{aligned} \frac{\partial \overline{w'q'_t}}{\partial t} = & -\bar{w} \frac{\partial \overline{w'q'_t}}{\partial z} - \frac{\partial}{\partial z} \left(a_1 \frac{\overline{w'^3}}{\overline{w'^2}} \overline{w'q'_t} \right) - \overline{w'^2} \frac{\partial \bar{q}_t}{\partial z} - (1 - C_7) \overline{w'q'_t} \frac{\partial \bar{w}}{\partial z} + (1 - C_7) \frac{g}{\theta_0} \overline{q'_t \theta'_v} \\ & - \frac{C_6}{\tau} \overline{w'q'_t} + \nu_6 \nabla_z^2 \overline{w'q'_t} \end{aligned} \quad (72)$$

After discretizing the time derivative and rearranging terms:

$$\begin{aligned} \frac{\bar{q}_t^{t+\Delta t}}{\Delta t} + \bar{w} \frac{\partial \bar{q}_t^{t+\Delta t}}{\partial z} + \frac{\partial \overline{w'q'_t}^{t+\Delta t}}{\partial z} \\ = \frac{\bar{q}_t^t}{\Delta t} + \frac{\partial \bar{q}_t}{\partial t} \Big|_{\text{ls}} \end{aligned} \quad (73)$$

$$\begin{aligned} \frac{\overline{w'q'_t}^{t+\Delta t}}{\Delta t} + \bar{w} \frac{\partial \overline{w'q'_t}^{t+\Delta t}}{\partial z} + \frac{\partial}{\partial z} \left(a_1 \frac{\overline{w'^3}}{\overline{w'^2}} \overline{w'q'_t}^{t+\Delta t} \right) + \overline{w'^2} \frac{\partial \bar{q}_t^{t+\Delta t}}{\partial z} \\ + (1 - C_7) \overline{w'q'_t}^{t+\Delta t} \frac{\partial \bar{w}}{\partial z} + \frac{C_6}{\tau} \overline{w'q'_t}^{t+\Delta t} - \nu_6 \nabla_z^2 \overline{w'q'_t}^{t+\Delta t} \\ = \frac{\overline{w'q'_t}^t}{\Delta t} + (1 - C_7) \frac{g}{\theta_0} \overline{q'_t \theta'_v}^t \end{aligned} \quad (74)$$

The LHSs of (73)-(74) is linear in \bar{q} and $\overline{w'q'_t}$ and can therefore be rewritten in matrix form:

$$A \begin{pmatrix} \vdots \\ \bar{q}_{k-1}^{t+\Delta t} \\ \overline{w'q'_t}_{k-1}^{t+\Delta t} \\ \bar{q}_k^{t+\Delta t} \\ \overline{w'q'_t}_k^{t+\Delta t} \\ \bar{q}_{k+1}^{t+\Delta t} \\ \overline{w'q'_t}_{k+1}^{t+\Delta t} \\ \vdots \end{pmatrix} = d \quad (75)$$

The matrix A is obtained by vertical discretization of the LHSs, and the vector d by discretization of the RHSs. A is band-diagonal with two rows above and two below the main diagonal. A is stored in compact form in a array with dimensions $(2 \text{ nnzp}, 5)$. d is a vector with dimension (2 nnzp) . A can be

inverted efficiently using a LU decomposition algorithm for band diagonal matrices. The construction of the matrix A and vector d are as follows.

First, we compute the finite difference equivalent to (73):

$$\begin{aligned} & \frac{\text{qtm}^{\text{new}}(\mathbf{k})}{\text{dt}} + \frac{\text{wmt}(\mathbf{k})}{2\text{dzt}(\mathbf{k})}\text{qtm}^{\text{new}}(\mathbf{k} + 1) - \frac{\text{wmt}(\mathbf{k})}{2\text{dzt}(\mathbf{k})}\text{qtm}^{\text{new}}(\mathbf{k} - 1) + \frac{\text{wpqtp}^{\text{new}}(\mathbf{k})}{\text{dzt}(\mathbf{k})} - \frac{\text{wpqtp}^{\text{new}}(\mathbf{k} - 1)}{\text{dzt}(\mathbf{k})} \\ &= \frac{\text{qtm}(\mathbf{k})}{\text{dt}} + \text{qtm_ls}(\mathbf{k}) \end{aligned} \tag{76}$$

Contributions to A from (76) are:

$$A(\mathbf{k_xm}, 1) = A(\mathbf{k_xm}, 1) - \frac{\text{wmt}(\mathbf{k})}{2\text{dzt}(\mathbf{k})} \tag{77}$$

$$A(\mathbf{k_xm}, 2) = A(\mathbf{k_xm}, 2) - \frac{1}{\text{dzt}(\mathbf{k})} \tag{78}$$

$$A(\mathbf{k_xm}, 3) = A(\mathbf{k_xm}, 3) + \frac{1}{\text{dt}} \tag{79}$$

$$A(\mathbf{k_xm}, 4) = A(\mathbf{k_xm}, 4) + \frac{1}{\text{dzt}(\mathbf{k})} \tag{80}$$

$$A(\mathbf{k_xm}, 5) = A(\mathbf{k_xm}, 5) + \frac{\text{wmt}(\mathbf{k})}{2\text{dzt}(\mathbf{k})} \tag{81}$$

Contributions to d from (76) are:

$$d(\mathbf{k_xm}) = d(\mathbf{k_xm}) + \frac{\text{qtm}(\mathbf{k})}{\text{dt}} + \text{qtm_ls}(\mathbf{k}) \tag{82}$$

where $\mathbf{k_xm} = 2\mathbf{k} - 1$.

We now write the finite difference equivalent to (74):

$$\begin{aligned}
& \frac{\text{wpqtp}^{\text{new}}(k)}{dt} + \frac{\text{wmm}(k)}{2dzm(k)} \text{wpqtp}^{\text{new}}(k+1) - \frac{\text{wmm}(k)}{2dzm(k)} \text{wpqtp}^{\text{new}}(k-1) \\
& + \frac{1}{2dzm(k)} \left[- \frac{(\text{a1m}(k-1) + \text{a1m}(k)) \text{wp3}(k)}{\max(\text{wp2}(k-1) + \text{wp2}(k), 2\epsilon)} \text{wpqtp}^{\text{new}}(k-1) \right. \\
& \quad + \left(\frac{(\text{a1m}(k) + \text{a1m}(k+1)) \text{wp3}(k+1)}{\max(\text{wp2}(k) + \text{wp2}(k+1), 2\epsilon)} - \frac{(\text{a1m}(k-1) + \text{a1m}(k)) \text{wp3}(k)}{\max(\text{wp2}(k-1) + \text{wp2}(k), 2\epsilon)} \right) \text{wpqtp}^{\text{new}}(k) \\
& \quad \left. + \frac{(\text{a1m}(k) + \text{a1m}(k+1)) \text{wp3}(k+1)}{\max(\text{wp2}(k) + \text{wp2}(k+1), 2\epsilon)} \text{wpqtp}^{\text{new}}(k+1) \right] \\
& + \text{wp2}(k) \frac{\text{qtm}^{\text{new}}(k+1) - \text{qtm}^{\text{new}}(k)}{dzm(k)} + (1 - C_7) \text{wpqtp}^{\text{new}}(k) \frac{\text{wmt}(k+1) - \text{wmt}(k)}{dzm(k)} \\
& + \frac{C_6}{\text{taum}(k)} \text{wpqtp}^{\text{new}}(k) \\
& - \frac{\nu_6}{dzm(k) dzt(k)} \text{wpqtp}^{\text{new}}(k-1) \\
& + \frac{\nu_6}{dzm(k)} \left(\frac{1}{dzt(k+1)} + \frac{1}{dzt(k)} \right) \text{wpqtp}^{\text{new}}(k) \\
& - \frac{\nu_6}{dzm(k) dzt(k+1)} \text{wpqtp}^{\text{new}}(k+1) \\
& = \frac{\text{wpqtp}(k)}{dt} + (1 - C_7) \frac{g}{\theta_0} \text{qtpthvp}(k)
\end{aligned} \tag{83}$$

Contributions to A from (83) are:

$$\begin{aligned}
& A(k_wpxp, 1) = A(k_wpxp, 1) \\
& - \frac{\text{wmm}(k)}{2dzm(k)} - \frac{1}{2dzm(k)} \frac{(\text{a1m}(k-1) + \text{a1m}(k)) \text{wp3}(k)}{\max(\text{wp2}(k-1) + \text{wp2}(k), 2\epsilon)} - \frac{\nu_6}{dzm(k) dzt(k)}
\end{aligned} \tag{84}$$

$$A(k_wpxp, 2) = A(k_wpxp, 2) - \frac{\text{wp2}(k)}{dzm(k)} \tag{85}$$

$$\begin{aligned}
& A(k_wpxp, 3) = A(k_wpxp, 3) \\
& + \frac{1}{dt} + \frac{1}{2dzm(k)} \left(\frac{(\text{a1m}(k) + \text{a1m}(k+1)) \text{wp3}(k+1)}{\max(\text{wp2}(k) + \text{wp2}(k+1), 2\epsilon)} - \frac{(\text{a1m}(k-1) + \text{a1m}(k)) \text{wp3}(k)}{\max(\text{wp2}(k-1) + \text{wp2}(k), 2\epsilon)} \right) \\
& + (1 - C_7) \frac{\text{wmt}(k+1) - \text{wmt}(k)}{dzm(k)} + \frac{C_6}{\text{taum}(k)} + \frac{\nu_6}{dzm(k)} \left(\frac{1}{dzt(k+1)} + \frac{1}{dzt(k)} \right)
\end{aligned} \tag{86}$$

$$A(k_wpxp, 4) = A(k_wpxp, 4) + \frac{\text{wp2}(k)}{dzm(k)} \tag{87}$$

$$\begin{aligned}
& A(k_wpxp, 5) = A(k_wpxp, 5) \\
& + \frac{\text{wmm}(k)}{2dzm(k)} + \frac{1}{2dzm(k)} \frac{(\text{a1m}(k) + \text{a1m}(k+1)) \text{wp3}(k+1)}{\max(\text{wp2}(k) + \text{wp2}(k+1), 2\epsilon)} - \frac{\nu_6}{dzm(k) dzt(k+1)}
\end{aligned} \tag{88}$$

Contributions to d from (83) are:

$$d(\mathbf{k_wpxp}) = d(\mathbf{k_wpxp}) + \frac{\mathbf{wpqtp}(\mathbf{k})}{dt} + (1 - C_7) \frac{g}{\theta_0} \mathbf{qtpthvp}(\mathbf{k}) \quad (89)$$

where $\mathbf{k_wpxp} = 2\mathbf{k}$.

The procedure for solving implicitly for $\bar{\theta}_l$ and $\overline{w'\theta'_l}$ is identical. It leads to the same matrix A , so A needs to be inverted only once.

6 Implicit solution for the vertical velocity moments

Start with equations (5) and (11):

$$\begin{aligned} \frac{\partial \overline{w'^2}}{\partial t} = & -\bar{w} \frac{\partial \overline{w'^2}}{\partial z} - \frac{\partial \overline{w'^3}}{\partial z} - 2\overline{w'^2} \frac{\partial \bar{w}}{\partial z} + \frac{2g \overline{w'\theta'_v}}{\theta_0} \\ & - \frac{C_4}{\tau} \left(\overline{w'^2} - \frac{2}{3} \bar{e} \right) - C_5 \left(-2\overline{w'^2} \frac{\partial \bar{w}}{\partial z} + \frac{2g \overline{w'\theta'_v}}{\theta_0} \right) + \frac{2}{3} C_5 \left(\frac{g}{\theta_0} \overline{w'\theta'_v} - \overline{u'w'} \frac{\partial \bar{u}}{\partial z} - \overline{v'w'} \frac{\partial \bar{v}}{\partial z} \right) \\ & - \frac{C_1}{\tau} \overline{w'^2} + \nu_1 \nabla_z^2 \overline{w'^2} \end{aligned} \quad (90)$$

$$\begin{aligned} \frac{\partial \overline{w'^3}}{\partial t} = & -\bar{w} \frac{\partial \overline{w'^3}}{\partial z} - \frac{\partial \overline{w'^4}}{\partial z} + 3\overline{w'^2} \frac{\partial \overline{w'^2}}{\partial z} - 3\overline{w'^3} \frac{\partial \bar{w}}{\partial z} + \frac{3g \overline{w'^2 \theta'_v}}{\theta_0} \\ & - \frac{C_8}{\tau} (C_{8b} Skw^4 + 1) \overline{w'^3} - C_{11} \left(-3\overline{w'^3} \frac{\partial \bar{w}}{\partial z} + \frac{3g \overline{w'^2 \theta'_v}}{\theta_0} \right) + (K_w + \nu_8) \nabla_z^2 \overline{w'^3} \end{aligned} \quad (91)$$

Using (23), we can rewrite the transport and production terms in (91):

$$\begin{aligned} & - \frac{\partial \overline{w'^4}}{\partial z} + 3\overline{w'^2} \frac{\partial \overline{w'^2}}{\partial z} \\ = & - \frac{\partial}{\partial z} \left(\overline{w'^4} - \frac{3}{2} \overline{w'^2}^2 \right) \\ = & - \frac{\partial}{\partial z} \left(a_3 \overline{w'^2}^2 \right) - \frac{\partial}{\partial z} \left(a_1 \frac{\overline{w'^3}^2}{\overline{w'^2}} \right) \end{aligned} \quad (92)$$

Rearranging terms and making use of (12):

$$\begin{aligned} & \frac{\partial \overline{w'^2}}{\partial t} + \frac{\partial \overline{w'^3}}{\partial z} + \frac{C_1}{\tau} \overline{w'^2} - \nu_1 \nabla_z^2 \overline{w'^2} \\ = & -\bar{w} \frac{\partial \overline{w'^2}}{\partial z} + (1 - C_5) \frac{2g \overline{w'\theta'_v}}{\theta_0} - 2(1 - C_5) \overline{w'^2} \frac{\partial \bar{w}}{\partial z} + \frac{2}{3} C_5 \left(\frac{g}{\theta_0} \overline{w'\theta'_v} - \overline{u'w'} \frac{\partial \bar{u}}{\partial z} - \overline{v'w'} \frac{\partial \bar{v}}{\partial z} \right) \end{aligned} \quad (93)$$

$$\begin{aligned} & \frac{\partial \overline{w'^3}}{\partial t} - (K_w + \nu_8) \nabla_z^2 \overline{w'^3} + \frac{C_8}{\tau} (C_{8b} Skw^4 + 1) \overline{w'^3} + \frac{\partial}{\partial z} \left(a_3 \overline{w'^2}^2 \right) + \frac{\partial}{\partial z} \left(a_1 \frac{\overline{w'^3}^2}{\overline{w'^2}} \right) \\ = & -\bar{w} \frac{\partial \overline{w'^3}}{\partial z} + (1 - C_{11}) \frac{3g \overline{w'^2 \theta'_v}}{\theta_0} - 3(1 - C_{11}) \overline{w'^3} \frac{\partial \bar{w}}{\partial z} \end{aligned} \quad (94)$$

6.1 $\overline{w'^2}$

Terms on the LHS of (93) are treated fully implicitly, except for the diffusion term which is treated with a Crank-Nicholson time step. Terms on the RHS explicitly:

$$\frac{\overline{w'^2}^{t+\Delta t}}{\Delta t} + \frac{\partial \overline{w'^3}^{t+\Delta t}}{\partial z} + \frac{C_1 \overline{w'^2}^{t+\Delta t}}{\tau} - \frac{1}{2} \nu_1 \nabla_z^2 \overline{w'^2}^{t+\Delta t} = \frac{\overline{w'^2}^t}{\Delta t} + \frac{1}{2} \nu_1 \nabla_z^2 \overline{w'^2}^t + \overline{w'^2} \Big|_{\text{expl}} \quad (95)$$

where

$$\overline{w'^2} \Big|_{\text{expl}} = -\bar{w} \frac{\partial \overline{w'^2}^t}{\partial z} + (1 - C_5) \frac{2g}{\theta_0} \overline{w' \theta'_v}^t - 2(1 - C_5) \overline{w'^2}^t \frac{\partial \bar{w}}{\partial z} + \frac{2}{3} C_5 \left(\frac{g}{\theta_0} \overline{w' \theta'_v}^t - \overline{u' w'}^t \frac{\partial \bar{u}}{\partial z} - \overline{v' w'}^t \frac{\partial \bar{v}}{\partial z} \right)^t \quad (96)$$

The next step consists of writing the finite difference equivalent to (95):

$$\begin{aligned} & \frac{\text{wp2}^{\text{new}}(\mathbf{k})}{\text{dt}} + \frac{\text{wp3}^{\text{new}}(\mathbf{k} + 1) - \text{wp3}^{\text{new}}(\mathbf{k})}{\text{dzm}(\mathbf{k})} + \frac{C_1}{\text{taum}(\mathbf{k})} \text{wp2}^{\text{new}}(\mathbf{k}) \\ & - \frac{\nu_1}{2 \text{dzm}(\mathbf{k}) \text{dzt}(\mathbf{k})} \text{wp2}^{\text{new}}(\mathbf{k} - 1) \\ & + \frac{\nu_1}{2 \text{dzm}(\mathbf{k})} \left(\frac{1}{\text{dzt}(\mathbf{k} + 1)} + \frac{1}{\text{dzt}(\mathbf{k})} \right) \text{wp2}^{\text{new}}(\mathbf{k}) \\ & - \frac{\nu_1}{2 \text{dzm}(\mathbf{k}) \text{dzt}(\mathbf{k} + 1)} \text{wp2}^{\text{new}}(\mathbf{k} + 1) \\ & = \frac{\text{wp2}(\mathbf{k})}{\text{dt}} \\ & + \frac{\nu_1}{2 \text{dzm}(\mathbf{k}) \text{dzt}(\mathbf{k})} \text{wp2}(\mathbf{k} - 1) \\ & - \frac{\nu_1}{2 \text{dzm}(\mathbf{k})} \left(\frac{1}{\text{dzt}(\mathbf{k} + 1)} + \frac{1}{\text{dzt}(\mathbf{k})} \right) \text{wp2}(\mathbf{k}) \\ & + \frac{\nu_1}{2 \text{dzm}(\mathbf{k}) \text{dzt}(\mathbf{k} + 1)} \text{wp2}(\mathbf{k} + 1) \\ & + \text{wp2t}(\mathbf{k}) \end{aligned} \quad (97)$$

where $\text{wp2t}(\mathbf{k})$ is the finite difference equivalent to (96) at level $\text{zm}(\mathbf{k})$.

6.2 $\overline{w'^3}$

The first two terms on the LHS of (94) are treated implicitly, the last three terms on the LHS are treated semi-implicitly (they are linearized and the linearized portion is treated implicitly, the rest explicitly) and terms on the RHS explicitly. Let's focus first on the third term on the LHS, L_3 :

$$L_3 \equiv \frac{C_8}{\tau} (C_{8b} S k w^4 + 1) \overline{w'^3} = \frac{C_8}{\tau} \left(C_{8b} \frac{\overline{w'^3}^5}{\overline{w'^2}^6} + \overline{w'^3} \right) \quad (98)$$

We linearize L_3 with respect to $\overline{w'^3}$:

$$L_3 \left(\overline{w'^3}^{t+\Delta t} \right) \approx L_3 \left(\overline{w'^3}^t \right) + \left. \frac{\partial L_3}{\partial \overline{w'^3}} \right|_t \left(\overline{w'^3}^{t+\Delta t} - \overline{w'^3}^t \right) \quad (99)$$

where

$$\left. \frac{\partial L_3}{\partial \overline{w'^3}} \right|_t = \frac{C_8}{\tau} \left(5 C_{8b} \frac{\overline{w'^3}^4}{\overline{w'^2}^6} + 1 \right) \quad (100)$$

Combining (98), (100) with (99):

$$\begin{aligned} & L_3 \left(\overline{w'^3}^{t+\Delta t} \right) \\ &= \frac{C_8}{\tau} \left(C_{8b} \frac{\overline{w'^3}^{t5}}{\overline{w'^2}^{t6}} + \overline{w'^3}^t \right) + \frac{C_8}{\tau} \left(5 C_{8b} \frac{\overline{w'^3}^{t4}}{\overline{w'^2}^{t6}} + 1 \right) \left(\overline{w'^3}^{t+\Delta t} - \overline{w'^3}^t \right) \\ &= -\frac{C_8}{\tau} \left(4 C_{8b} Skw^{t4} \right) \overline{w'^3}^t + \frac{C_8}{\tau} \left(5 C_{8b} Skw^{t4} + 1 \right) \overline{w'^3}^{t+\Delta t} \end{aligned} \quad (101)$$

In a similar fashion, let's now linearize the fourth term on the LHS of (94):

$$\begin{aligned} & \frac{\partial}{\partial z} \left[a_3 \left(\overline{w'^2}^{t+\Delta t} \right)^2 \right] \\ & \approx \frac{\partial}{\partial z} \left[a_3 \overline{w'^2}^{t2} + 2a_3 \overline{w'^2}^t \left(\overline{w'^2}^{t+\Delta t} - \overline{w'^2}^t \right) \right] \\ &= \frac{\partial}{\partial z} \left(2a_3 \overline{w'^2}^t \overline{w'^2}^{t+\Delta t} \right) - \frac{\partial}{\partial z} \left(a_3 \overline{w'^2}^{t2} \right) \end{aligned} \quad (102)$$

We repeat for the fifth term on the LHS of (94):

$$\begin{aligned} & \frac{\partial}{\partial z} \left(a_1 \frac{\left(\overline{w'^3}^{t+\Delta t} \right)^2}{\overline{w'^2}^t} \right) \\ & \approx \frac{\partial}{\partial z} \left[a_1 \frac{\left(\overline{w'^3}^t \right)^2}{\overline{w'^2}^t} + 2a_1 \frac{\overline{w'^3}^t}{\overline{w'^2}^t} \left(\overline{w'^3}^{t+\Delta t} - \overline{w'^3}^t \right) \right] \\ &= \frac{\partial}{\partial z} \left(2a_1 \frac{\overline{w'^3}^t \overline{w'^3}^{t+\Delta t}}{\overline{w'^2}^t} \right) - \frac{\partial}{\partial z} \left(a_1 \frac{\left(\overline{w'^3}^t \right)^2}{\overline{w'^2}^t} \right) \end{aligned} \quad (103)$$

We can now assemble the time discrete equivalent to (94) using (101), (102) and (103):

$$\begin{aligned}
& \frac{\overline{w'^3}^{t+\Delta t}}{\Delta t} - \frac{1}{2}(K_w + \nu_8) \nabla_z^2 \overline{w'^3}^{t+\Delta t} + \frac{C_8}{\tau} \left(5 C_{8b} S k w^{t4} + 1 \right) \overline{w'^3}^{t+\Delta t} \\
& + \frac{\partial}{\partial z} \left(2a_3 \overline{w'^2}^t \overline{w'^2}^{t+\Delta t} \right) + \frac{\partial}{\partial z} \left(2a_1 \frac{\overline{w'^3}^t \overline{w'^3}^{t+\Delta t}}{\overline{w'^2}^t} \right) \\
& = \frac{\overline{w'^3}^t}{\Delta t} + \frac{1}{2}(K_w + \nu_8) \nabla_z^2 \overline{w'^3}^t + \frac{C_8}{\tau} \left(4 C_{8b} S k w^{t4} \right) \overline{w'^3}^t \\
& + \frac{\partial}{\partial z} \left(a_3 \overline{w'^2}^{t2} \right) + \frac{\partial}{\partial z} \left(a_1 \frac{\left(\overline{w'^3}^t \right)^2}{\overline{w'^2}^t} \right) + \overline{w'^3} \Big|_{\text{expl}}
\end{aligned} \tag{104}$$

where

$$\overline{w'^3} \Big|_{\text{expl}} = -\bar{w} \frac{\partial \overline{w'^3}^t}{\partial z} + (1 - C_{11}) \frac{3g}{\theta_0} \overline{w'^2}^t \theta'_v - 3(1 - C_{11}) \overline{w'^3}^t \frac{\partial \bar{w}}{\partial z} \tag{105}$$

Finally, we derive the finite difference form of (104):

$$\begin{aligned}
& \frac{\text{wp3}^{\text{new}}(\mathbf{k})}{\text{dt}} - \frac{K\text{wt}(\mathbf{k}) + \nu_8}{2\text{dzt}(\mathbf{k})} \left(\frac{\text{wp3}^{\text{new}}(\mathbf{k} + 1) - \text{wp3}^{\text{new}}(\mathbf{k})}{\text{dzm}(\mathbf{k})} - \frac{\text{wp3}^{\text{new}}(\mathbf{k}) - \text{wp3}^{\text{new}}(\mathbf{k} - 1)}{\text{dzm}(\mathbf{k} - 1)} \right) \\
& + \frac{C_8}{\text{taut}(\mathbf{k})} (5 C_{8b} S k \text{wt}(\mathbf{k})^4 + 1) \text{wp3}^{\text{new}}(\mathbf{k}) \\
& + \frac{2}{\text{dzt}(\mathbf{k})} (a_3 \text{m}(\mathbf{k}) \text{wp2}(\mathbf{k}) \text{wp2}^{\text{new}}(\mathbf{k}) - a_3 \text{m}(\mathbf{k} - 1) \text{wp2}(\mathbf{k} - 1) \text{wp2}^{\text{new}}(\mathbf{k} - 1)) \\
& + \frac{1}{2 \text{dzt}(\mathbf{k})} \left(\frac{a_1 \text{m}(\mathbf{k}) (\text{wp3}(\mathbf{k}) + \text{wp3}(\mathbf{k} + 1)) (\text{wp3}^{\text{new}}(\mathbf{k}) + \text{wp3}^{\text{new}}(\mathbf{k} + 1))}{\max(\text{wp2}(\mathbf{k}), \epsilon)} \right. \\
& \quad \left. - \frac{a_1 \text{m}(\mathbf{k} - 1) (\text{wp3}(\mathbf{k} - 1) + \text{wp3}(\mathbf{k})) (\text{wp3}^{\text{new}}(\mathbf{k} - 1) + \text{wp3}^{\text{new}}(\mathbf{k}))}{\max(\text{wp2}(\mathbf{k} - 1), \epsilon)} \right) \\
& = \frac{\text{wp3}(\mathbf{k})}{\text{dt}} + \frac{K\text{wt}(\mathbf{k}) + \nu_8}{2\text{dzt}(\mathbf{k})} \left(\frac{\text{wp3}(\mathbf{k} + 1) - \text{wp3}(\mathbf{k})}{\text{dzm}(\mathbf{k})} - \frac{\text{wp3}(\mathbf{k}) - \text{wp3}(\mathbf{k} - 1)}{\text{dzm}(\mathbf{k} - 1)} \right) \\
& + \frac{C_8}{\text{taut}(\mathbf{k})} (4 C_{8b} S k \text{wt}(\mathbf{k})^4) \text{wp3}(\mathbf{k}) \\
& + \frac{a_3 \text{m}(\mathbf{k}) \text{wp2}(\mathbf{k})^2 - a_3 \text{m}(\mathbf{k} - 1) \text{wp2}(\mathbf{k} - 1)^2}{\text{dzt}(\mathbf{k})} \\
& + \frac{1}{4 \text{dzt}(\mathbf{k})} \left(\frac{a_1 \text{m}(\mathbf{k}) (\text{wp3}(\mathbf{k}) + \text{wp3}(\mathbf{k} + 1))^2}{\max(\text{wp2}(\mathbf{k}), \epsilon)} - \frac{a_1 \text{m}(\mathbf{k} - 1) (\text{wp3}(\mathbf{k} - 1) + \text{wp3}(\mathbf{k}))^2}{\max(\text{wp2}(\mathbf{k} - 1), \epsilon)} \right) \\
& + \text{wp3t}(\mathbf{k})
\end{aligned} \tag{106}$$

where $\text{wp3t}(\mathbf{k})$ is the finite difference equivalent to (105) at level $\text{zt}(\mathbf{k})$.

6.3 Matrix form

The final step is to rewrite (97) and (106) in matrix form:

$$A_{\text{wp23}} \begin{pmatrix} \vdots \\ \text{wp3}^{\text{new}}(\text{k} - 1) \\ \text{wp2}^{\text{new}}(\text{k} - 1) \\ \text{wp3}^{\text{new}}(\text{k}) \\ \text{wp2}^{\text{new}}(\text{k}) \\ \text{wp3}^{\text{new}}(\text{k} + 1) \\ \text{wp2}^{\text{new}}(\text{k} + 1) \\ \vdots \end{pmatrix} = d_{\text{wp23}} \quad (107)$$

A_{wp23} is a band-diagonal matrix with two rows above and two below the main diagonal. A_{wp23} is stored in compact form in a array with dimensions $(2\text{nnzp}, 5)$. d_{wp23} is a vector with dimension (2nnzp) . A_{wp23} can be inverted efficiently using a LU decomposition algorithm for band diagonal matrices.

Contributions to A_{wp23} from (97):

$$A(\text{k_wp2}, 1) = A(\text{k_wp2}, 1) - \frac{\nu_1}{2\text{dzm}(\text{k})\text{dzt}(\text{k})} \quad (108)$$

$$A(\text{k_wp2}, 2) = A(\text{k_wp2}, 2) - \frac{1}{\text{dzm}(\text{k})} \quad (109)$$

$$A(\text{k_wp2}, 3) = A(\text{k_wp2}, 3) + \frac{1}{\text{dt}} + \frac{\text{C}_1}{\text{taum}(\text{k})} + \frac{\nu_1}{2\text{dzm}(\text{k})} \left(\frac{1}{\text{dzt}(\text{k} + 1)} + \frac{1}{\text{dzt}(\text{k})} \right) \quad (110)$$

$$A(\text{k_wp2}, 4) = A(\text{k_wp2}, 4) + \frac{1}{\text{dzm}(\text{k})} \quad (111)$$

$$A(\text{k_wp2}, 5) = A(\text{k_wp2}, 5) - \frac{\nu_1}{2\text{dzm}(\text{k})\text{dzt}(\text{k} + 1)} \quad (112)$$

Contributions to $d_{\text{wp}23}$ from (97):

$$\begin{aligned}
d(k_{\text{wp}2}) &= d(k_{\text{wp}2}) \\
&+ \frac{\text{wp}2(k)}{dt} \\
&+ \frac{\nu_1}{2dzm(k)dz t(k)} \text{wp}2(k-1) \\
&- \frac{\nu_1}{2dzm(k)} \left(\frac{1}{dz t(k+1)} + \frac{1}{dz t(k)} \right) \text{wp}2(k) \\
&+ \frac{\nu_1}{2dzm(k)dz t(k+1)} \text{wp}2(k+1) \\
&+ \text{wp}2t(k)
\end{aligned} \tag{113}$$

where

$$k_{\text{wp}2} = 2k \tag{114}$$

Contributions to $A_{\text{wp}23}$ from (106):

$$\begin{aligned}
A(k_{\text{wp}3}, 1) &= A(k_{\text{wp}3}, 1) \\
&- \frac{Kwt(k) + \nu_8}{2dz t(k)dzm(k-1)} - \frac{1}{2dz t(k)} \frac{a1m(k-1)(\text{wp}3(k-1) + \text{wp}3(k))}{\max(\text{wp}2(k-1), \epsilon)}
\end{aligned} \tag{115}$$

$$A(k_{\text{wp}3}, 2) = A(k_{\text{wp}3}, 2) - \frac{2a3m(k-1)\text{wp}2(k-1)}{dz t(k)} \tag{116}$$

$$\begin{aligned}
A(k_{\text{wp}3}, 3) &= A(k_{\text{wp}3}, 3) \\
&+ \frac{1}{dt} + \frac{C_8}{\text{taut}(k)} (5 C_{8b} S kwt(k)^4 + 1) + \frac{Kwt(k) + \nu_8}{2dz t(k)} \left(\frac{1}{dzm(k-1)} + \frac{1}{dzm(k)} \right) \\
&+ \frac{1}{2dz t(k)} \left(\frac{a1m(k)(\text{wp}3(k) + \text{wp}3(k+1))}{\max(\text{wp}2(k), \epsilon)} - \frac{a1m(k-1)(\text{wp}3(k-1) + \text{wp}3(k))}{\max(\text{wp}2(k-1), \epsilon)} \right)
\end{aligned} \tag{117}$$

$$A(k_{\text{wp}3}, 4) = A(k_{\text{wp}3}, 4) + \frac{2a3m(k)\text{wp}2(k)}{dz t(k)} \tag{118}$$

$$\begin{aligned}
A(k_{\text{wp}3}, 5) &= A(k_{\text{wp}3}, 5) \\
&- \frac{Kwt(k) + \nu_8}{2dz t(k)dzm(k)} + \frac{1}{2dz t(k)} \frac{a1m(k)(\text{wp}3(k) + \text{wp}3(k+1))}{\max(\text{wp}2(k), \epsilon)}
\end{aligned} \tag{119}$$

Contributions to $d_{\text{wp}23}$ from (106):

$$\begin{aligned}
d(k_{\text{wp}3}) &= d(k_{\text{wp}3}) \\
&+ \frac{\text{wp}3(k)}{dt} + \frac{Kwt(k) + \nu_8}{2dz_t(k)} \left(\frac{\text{wp}3(k+1) - \text{wp}3(k)}{dz_m(k)} - \frac{\text{wp}3(k) - \text{wp}3(k-1)}{dz_m(k-1)} \right) \\
&+ \frac{C_8}{\tau_{\text{aut}}(k)} (4 C_{8b} S_{kwt}(k)^4) \text{wp}3(k) \\
&+ \frac{a_{3m}(k)\text{wp}2(k)^2 - a_{3m}(k-1)\text{wp}2(k-1)^2}{dz_t(k)} \\
&+ \frac{1}{4dz_t(k)} \left(\frac{a_{1m}(k)(\text{wp}3(k) + \text{wp}3(k+1))^2}{\max(\text{wp}2(k), \epsilon)} - \frac{a_{1m}(k-1)(\text{wp}3(k-1) + \text{wp}3(k))^2}{\max(\text{wp}2(k-1), \epsilon)} \right) \\
&+ \text{wp}3t(k)
\end{aligned} \tag{120}$$

where

$$k_{\text{wp}3} = 2k - 1 \tag{121}$$

7 Grid configuration

Figure 1 shows the vertical grid configuration for HOC. The grid consists of two types of levels: **zm** and **zt**. Predictive mean variables and third order moments reside on the thermodynamic levels (**zt**). Second and fourth order moments reside on the momentum levels (**zm**).

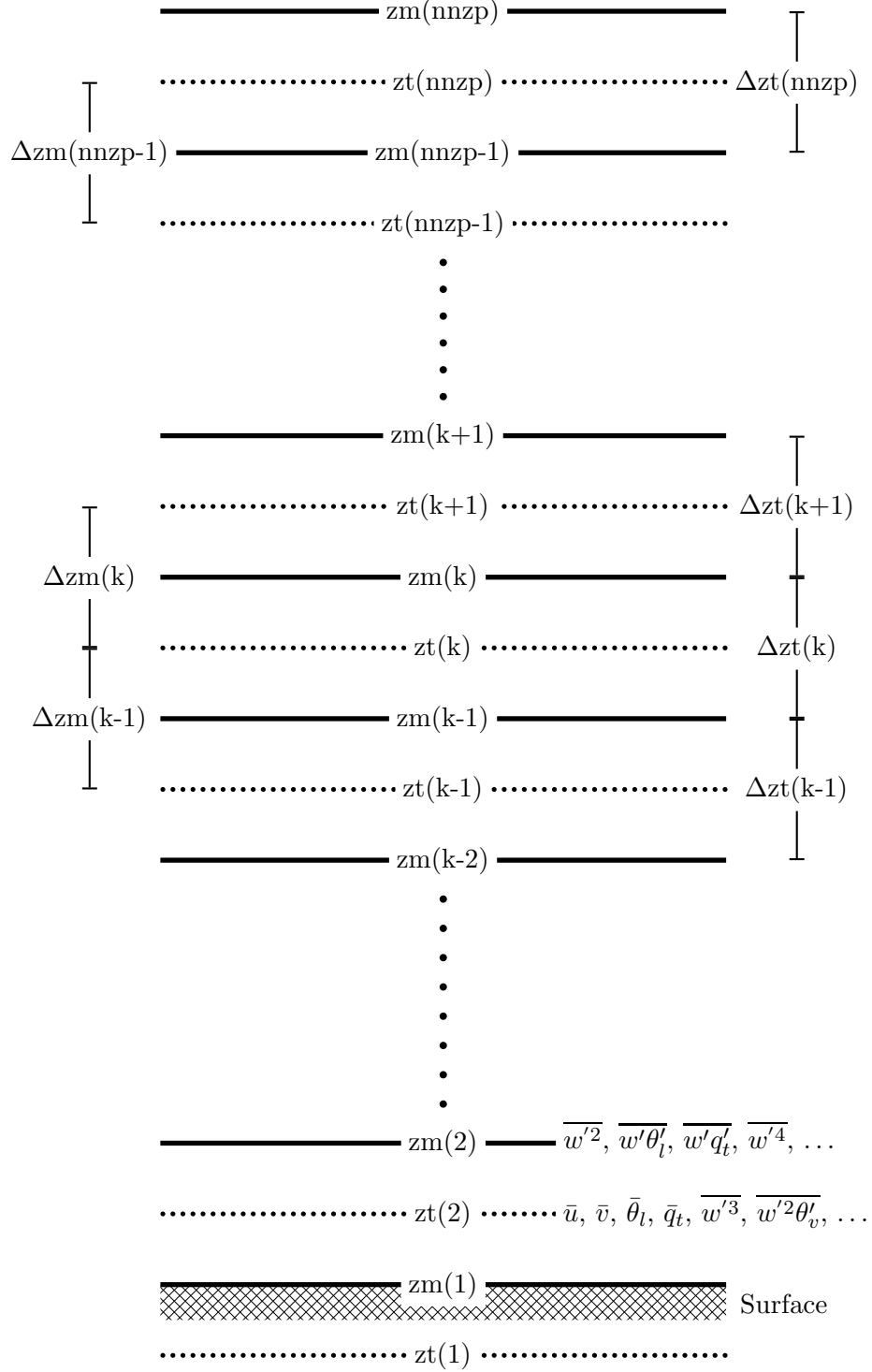


Figure 1: Vertical grid configuration

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