Equations for CLUBB

1 Predictive equations

$$\frac{\partial \bar{u}}{\partial t} = \underbrace{-\bar{w}\frac{\partial \bar{u}}{\partial z}}_{ma} \underbrace{-f(v_g - \bar{v})}_{cf/gf} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{u'w'}}{\partial z}}_{ta} + \left.\frac{\partial \bar{u}}{\partial t}\right|_{\mathrm{ls}} + \left.\frac{\partial \bar{u}}{\partial t}\right|_{\mathrm{ndg}} + \left.\frac{\partial \bar{u}}{\partial t}\right|_{\mathrm{sdmp}} \tag{1}$$

$$\frac{\partial \bar{v}}{\partial t} = \underbrace{-\bar{w}\frac{\partial \bar{v}}{\partial z}}_{ma} \underbrace{+f(u_g - \bar{u})}_{cf/gf} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{v'w'}}{\partial z}}_{ta} + \left.\frac{\partial \bar{v}}{\partial t}\right|_{\mathrm{ls}} + \left.\frac{\partial \bar{v}}{\partial t}\right|_{\mathrm{ndg}} + \left.\frac{\partial \bar{v}}{\partial t}\right|_{\mathrm{sdmp}} \tag{2}$$

$$\frac{\partial \bar{r}_t}{\partial t} = \underbrace{-\bar{w}\frac{\partial \bar{r}_t}{\partial z}}_{ma} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{w'r_t'}}{\partial z}}_{ta} + \left.\frac{\partial \bar{r}_t}{\partial t}\right|_{\mathrm{ls}} + \left.\frac{\partial \bar{r}_t}{\partial t}\right|_{\mathrm{cl}} + \left.\frac{\partial \bar{r}_t}{\partial t}\right|_{\mathrm{mfl}} + \left.\frac{\partial \bar{r}_t}{\partial t}\right|_{\mathrm{tacl}} + \left.\frac{\partial \bar{r}_t}{\partial t}\right|_{\mathrm{sdmp}} \tag{3}$$

$$\frac{\partial \bar{\theta}_l}{\partial t} = \underbrace{-\bar{w}\frac{\partial \bar{\theta}_l}{\partial z}}_{ma} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{w'\theta_l'}}{\partial z}}_{ta} + \bar{R} + \left.\frac{\partial \bar{\theta}_l}{\partial t}\right|_{\mathrm{ls}} + \left.\frac{\partial \bar{\theta}_l}{\partial t}\right|_{\mathrm{cl}} + \left.\frac{\partial \bar{\theta}_l}{\partial t}\right|_{\mathrm{mfl}} + \left.\frac{\partial \bar{\theta}_l}{\partial t}\right|_{\mathrm{tacl}} + \left.\frac{\partial \bar{\theta}_l}{\partial t}\right|_{\mathrm{sdmp}} \tag{4}$$

$$\begin{aligned}\frac{\partial \overline{w'^2}}{\partial t} = & \underbrace{-\bar{w}\frac{\partial \overline{w'^2}}{\partial z}}_{ma} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{w'^3}}{\partial z}}_{ta} \underbrace{-2\overline{w'^2}\frac{\partial \bar{w}}{\partial z}}_{ac} \underbrace{+\frac{2g}{\theta_{vs}}\overline{w'\theta_v'}}_{bp} \underbrace{-\frac{C_4}{\tau}\left(\overline{w'^2} - \frac{2}{3}\bar{e}\right)}_{pr1} \\ & \underbrace{-C_5\left(-2\overline{w'^2}\frac{\partial \bar{w}}{\partial z} + \frac{2g}{\theta_{vs}}\overline{w'\theta_v'}\right)}_{pr2} \underbrace{+\frac{2}{3}C_5\left(\frac{g}{\theta_{vs}}\overline{w'\theta_v'} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{v'w'}\frac{\partial \bar{v}}{\partial z}\right)}_{pr3} \\ & \underbrace{-\frac{C_1}{\tau}\left(\overline{w'^2} - w|_{\mathrm{tol}}^2\right)}_{dp1} \underbrace{+\frac{\partial}{\partial z}\left[\left(K_{w1} + \nu_1\right)\frac{\partial}{\partial z}\overline{w'^2}\right]}_{dp2} + \left.\frac{\partial \overline{w'^2}}{\partial t}\right|_{\mathrm{pd}} + \left.\frac{\partial \overline{w'^2}}{\partial t}\right|_{\mathrm{cl}}\end{aligned} \tag{5}$$

$$\begin{aligned}\frac{\partial \overline{r_t'^2}}{\partial t} = & \underbrace{-\bar{w}\frac{\partial \overline{r_t'^2}}{\partial z}}_{ma} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{w'r_t'^2}}{\partial z}}_{ta} \underbrace{-2\overline{w'r_t'}\frac{\partial \bar{r}_t}{\partial z}}_{tp} \underbrace{-\frac{C_2}{\tau}\left(\overline{r_t'^2} - r_t|_{\mathrm{tol}}^2\right)}_{dp1} \\ & \underbrace{+\frac{\partial}{\partial z}\left[\left(K_{w2} + \nu_2\right)\frac{\partial}{\partial z}\overline{r_t'^2}\right]}_{dp2} + \left.\frac{\partial \overline{r_t'^2}}{\partial t}\right|_{\mathrm{pd}} + \left.\frac{\partial \overline{r_t'^2}}{\partial t}\right|_{\mathrm{cl}}\end{aligned} \tag{6}$$

$$
\begin{aligned}
\frac{\partial \overline{\theta_l'^2}}{\partial t} = & \underbrace{-\bar{w}\frac{\partial \overline{\theta_l'^2}}{\partial z}}_{ma} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{w'\theta_l'^2}}{\partial z}}_{ta} \underbrace{-2\overline{w'\theta_l'}\frac{\partial \bar{\theta}_l}{\partial z}}_{tp} \underbrace{-\frac{C_2}{\tau}\left(\overline{\theta_l'^2} - \theta_l|_{\mathrm{tol}}^2\right)}_{dp1} \\
& \underbrace{+\frac{\partial}{\partial z}\left[(K_{w2}+\nu_2)\frac{\partial}{\partial z}\overline{\theta_l'^2}\right]}_{dp2} + \left.\frac{\partial \overline{\theta_l'^2}}{\partial t}\right|_{\mathrm{pd}} + \left.\frac{\partial \overline{\theta_l'^2}}{\partial t}\right|_{\mathrm{cl}}
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\frac{\partial \overline{r_t'\theta_l'}}{\partial t} = & \underbrace{-\bar{w}\frac{\partial \overline{r_t'\theta_l'}}{\partial z}}_{ma} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{w'r_t'\theta_l'}}{\partial z}}_{ta} \underbrace{-\overline{w'r_t'}\frac{\partial \bar{\theta}_l}{\partial z}}_{tp1} \underbrace{-\overline{w'\theta_l'}\frac{\partial \bar{r}_t}{\partial z}}_{tp2} \underbrace{-\frac{C_2}{\tau}\overline{r_t'\theta_l'}}_{dp1} \\
& \underbrace{+\frac{\partial}{\partial z}\left[(K_{w2}+\nu_2)\frac{\partial}{\partial z}\overline{r_t'\theta_l'}\right]}_{dp2} + \left.\frac{\partial \overline{r_t'\theta_l'}}{\partial t}\right|_{\mathrm{cl}}
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
\frac{\partial \overline{w'r_t'}}{\partial t} = & \underbrace{-\bar{w}\frac{\partial \overline{w'r_t'}}{\partial z}}_{ma} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{w'^2 r_t'}}{\partial z}}_{ta} \underbrace{-\overline{w'^2}\frac{\partial \bar{r}_t}{\partial z}}_{tp} \underbrace{-\overline{w'r_t'}\frac{\partial \bar{w}}{\partial z}}_{ac} \underbrace{+\frac{g}{\theta_{vs}}\overline{r_t'\theta_v'}}_{bp} \underbrace{-\frac{C_6}{\tau}\overline{w'r_t'}}_{pr1} \\
& \underbrace{+C_7\overline{w'r_t'}\frac{\partial \bar{w}}{\partial z}}_{pr2} \underbrace{-C_7\frac{g}{\theta_{vs}}\overline{r_t'\theta_v'}}_{pr3} \underbrace{+\frac{\partial}{\partial z}\left[(K_{w6}+\nu_6)\frac{\partial}{\partial z}\overline{w'r_t'}\right]}_{dp1} + \left.\frac{\partial \overline{w'r_t'}}{\partial t}\right|_{\mathrm{sicl}} + \left.\frac{\partial \overline{w'r_t'}}{\partial t}\right|_{\mathrm{cl}} + \left.\frac{\partial \overline{w'r_t'}}{\partial t}\right|_{\mathrm{mfl}}
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
\frac{\partial \overline{w'\theta_l'}}{\partial t} = & \underbrace{-\bar{w}\frac{\partial \overline{w'\theta_l'}}{\partial z}}_{ma} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{w'^2 \theta_l'}}{\partial z}}_{ta} \underbrace{-\overline{w'^2}\frac{\partial \bar{\theta}_l}{\partial z}}_{tp} \underbrace{-\overline{w'\theta_l'}\frac{\partial \bar{w}}{\partial z}}_{ac} \underbrace{+\frac{g}{\theta_{vs}}\overline{\theta_l'\theta_v'}}_{bp} \underbrace{-\frac{C_6}{\tau}\overline{w'\theta_l'}}_{pr1} \\
& \underbrace{+C_7\overline{w'\theta_l'}\frac{\partial \bar{w}}{\partial z}}_{pr2} \underbrace{-C_7\frac{g}{\theta_{vs}}\overline{\theta_l'\theta_v'}}_{pr3} \underbrace{+\frac{\partial}{\partial z}\left[(K_{w6}+\nu_6)\frac{\partial}{\partial z}\overline{w'\theta_l'}\right]}_{dp1} + \left.\frac{\partial \overline{w'\theta_l'}}{\partial t}\right|_{\mathrm{sicl}} + \left.\frac{\partial \overline{w'\theta_l'}}{\partial t}\right|_{\mathrm{cl}} + \left.\frac{\partial \overline{w'\theta_l'}}{\partial t}\right|_{\mathrm{mfl}}
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
\frac{\partial \overline{w'^3}}{\partial t} = & \underbrace{-\bar{w}\frac{\partial \overline{w'^3}}{\partial z}}_{ma} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{w'^4}}{\partial z}}_{ta} \underbrace{+3\frac{\overline{w'^2}}{\rho_s}\frac{\partial \rho_s \overline{w'^2}}{\partial z}}_{tp} \underbrace{-3\overline{w'^3}\frac{\partial \bar{w}}{\partial z}}_{ac} \underbrace{+\frac{3g}{\theta_{vs}}\overline{w'^2\theta_v'}}_{bp1} \\
& \underbrace{-C_{15}K_m\left(\frac{g}{\theta_{vs}}\frac{\partial \overline{w'\theta_v'}}{\partial z} - \left(\frac{\partial \overline{u'w'}}{\partial z}\frac{\partial^2 \bar{u}}{\partial z^2} - \frac{\partial \overline{v'w'}}{\partial z}\frac{\partial^2 \bar{v}}{\partial z^2}\right)\right)}_{bp2} \\
& \underbrace{-\frac{C_8}{\tau}\left(C_{8b}\,Skw^4+1\right)\overline{w'^3}}_{pr1} \underbrace{-C_{11}\left(-3\overline{w'^3}\frac{\partial \bar{w}}{\partial z} + \frac{3g}{\theta_{vs}}\overline{w'^2\theta_v'}\right)}_{pr2} \underbrace{+\frac{\partial}{\partial z}\left[(K_{w8}+\nu_8)\frac{\partial}{\partial z}\overline{w'^3}\right]}_{dp1} + \left.\frac{\partial \overline{w'^3}}{\partial t}\right|_{\mathrm{cl}}
\end{aligned} \tag{11}
$$

where $\bar{R}$ is the radiative heating rate, f the Coriolis parameter, and u_g and v_g the geostrophic winds. Furthermore, $\left.\frac{\partial \bar{r}_t}{\partial t}\right|_{\mathrm{ls}}$ and $\left.\frac{\partial \bar{\theta}_l}{\partial t}\right|_{\mathrm{ls}}$ are large-scale moisture and temperature forcings, respectively, and g

is acceleration due to gravity. The set of equations is an anelastic set of equations, where ρ_s is the dry, static, base-state density, which only changes with respect to altitude; and where θ_{vs} is the dry, base-state θ_v, which also only changes with respect to altitude. Threshold values of the variances are established, such that $w|_{\mathrm{tol}}^2$ is the minimum threshold value for $\overline{w'^2}$; $r_t|_{\mathrm{tol}}^2$ is the minimum threshold value for $\overline{r_t'^2}$; and $\theta_l|_{\mathrm{tol}}^2$ is the minimum threshold value for $\overline{\theta_l'^2}$. The subscript $|_{\mathrm{pd}}$ stands for the rate of change due to the positive-definite hole-filling scheme, the subscript $|_{\mathrm{sicl}}$ stands for the rate of change due to the semi-implicit clipping scheme, the subscript $|_{\mathrm{cl}}$ stands for the rate of change due to completely explicit clipping, the subscript $|_{\mathrm{mfl}}$ denotes adjustments from the monotonic flux limiter, the subscript the $|_{\mathrm{tacl}}$ denotes turbulent advection clipping, and finally the subscript $|_{\mathrm{sdmp}}$ denotes sponge layer damping.

If the model does not predict any higher-order moments of the horizontal winds, we assume that the turbulence kinetic energy, $\bar{e}$, is proportional to the vertical velocity variance $\overline{w'^2}$:

$$\bar{e} = \frac{3}{2}\overline{w'^2}. \tag{12}$$

Alternatively, if higher-order moments of the horizontal winds are computed, then turbulence kinetic energy, $\bar{e}$, is a function of the vertical velocity variance $\overline{w'^2}$, latitudinal wind variance $\overline{v'^2}$, and longitudinal wind variance $\overline{u'^2}$:

$$\bar{e} = \frac{1}{2}\left(\overline{w'^2} + \overline{u'^2} + \overline{v'^2}\right). \tag{13}$$

In the second case, the horizontal wind variance terms are determined as in Bougeault (1981a) and given by the equations:

$$\begin{aligned}
\frac{\partial \overline{u'^2}}{\partial t} = & \underbrace{-\bar{w}\frac{\partial \overline{u'^2}}{\partial z}}_{ma} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{w'u'^2}}{\partial z}}_{ta} \underbrace{-(1-C_5)\, 2\overline{u'w'}\frac{\partial \bar{u}}{\partial z}}_{tp} \underbrace{-\frac{2}{3}C_{14}\frac{\bar{e}}{\tau}}_{pr1} \\
& \underbrace{+\frac{2}{3}C_5\left(\frac{g}{\theta_{vs}}\overline{w'\theta_v'} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{v'w'}\frac{\partial \bar{v}}{\partial z}\right)}_{pr2} \underbrace{-\frac{C_4}{\tau}\left(\overline{u'^2} - \frac{2}{3}\bar{e}\right)}_{dp1} \\
& + \underbrace{\frac{\partial}{\partial z}\left[(K_{w9} + \nu_9)\frac{\partial}{\partial z}\overline{u'^2}\right]}_{dp2} + \left.\frac{\partial \overline{u'^2}}{\partial t}\right|_{\mathrm{pd}} + \left.\frac{\partial \overline{u'^2}}{\partial t}\right|_{\mathrm{cl}}
\end{aligned} \tag{14}$$

$$
\begin{aligned}
\frac{\partial\overline{v'^2}}{\partial t} = & \underbrace{-\bar{w}\frac{\partial\overline{v'^2}}{\partial z}}_{ma} \underbrace{-\frac{1}{\rho_s}\frac{\partial\rho_s\overline{w'v'^2}}{\partial z}}_{ta} \underbrace{-(1-C_5)\,2\overline{v'w'}\frac{\partial\bar{v}}{\partial z}}_{tp} \underbrace{-\frac{2}{3}C_{14}\frac{\bar{e}}{\tau}}_{pr1} \\
& \underbrace{+\frac{2}{3}C_5\left(\frac{g}{\theta_{vs}}\overline{w'\theta_v'} - \overline{u'w'}\frac{\partial\bar{u}}{\partial z} - \overline{v'w'}\frac{\partial\bar{v}}{\partial z}\right)}_{pr2} \underbrace{-\frac{C_4}{\tau}\left(\overline{v'^2} - \frac{2}{3}\bar{e}\right)}_{dp1} \\
& + \underbrace{\frac{\partial}{\partial z}\left[(K_{w9}+\nu_9)\frac{\partial}{\partial z}\overline{v'^2}\right]}_{dp2} + \left.\frac{\partial\overline{v'^2}}{\partial t}\right|_{\mathrm{pd}} + \left.\frac{\partial\overline{v'^2}}{\partial t}\right|_{\mathrm{cl}}
\end{aligned}
\tag{15}
$$

Where, ϵ in Bougeault (1981b), the dissipation of $\bar{e}$, has been defined in CLUBB by the equation:

$$
\epsilon = C_{14}\frac{\bar{e}}{\tau} \tag{16}
$$

The time scale τ is:

$$
\tau = \begin{cases} \dfrac{L}{\sqrt{\bar{e}}}; & L/\sqrt{\bar{e}} \leqslant \tau_{\max} \\ \tau_{\max}; & L/\sqrt{\bar{e}} > \tau_{\max} \end{cases} . \tag{17}
$$

The momentum fluxes are closed using a down gradient approach:

$$
\overline{u'w'} = -K_m\frac{\partial\bar{u}}{\partial z} \tag{18a}
$$

$$
\overline{v'w'} = -K_m\frac{\partial\bar{v}}{\partial z} \tag{18b}
$$

The momentum fluxes, $\overline{u'w'}$ and $\overline{v'w'}$, are also subject to completely explicit clipping. The turbulent-transfer coefficient K_m is given by:

$$
K_m = c_K\,L\,\overline{e}^{1/2}. \tag{19}
$$

$c_K = c_\mu^{1/4} = 0.548$ in Duynkerke and Driedonks (1987), but CLUBB reduces the value to better fit LES output.

The eddy diffusivity coefficients in Equation (5) through Equation (11) and in Equations (14) and (15) are as follows:

$$K_{w1} = c_{K1} K_m + c_{Ksqd} \overline{w'^2}^2 \Big|_{3 \text{ pnt avg}}$$

$$K_{w2} = c_{K2} K_m + c_{Ksqd} \lambda|_{3 \text{ pnt avg}}$$

$$K_{w6} = c_{K6} K_m + c_{Ksqd} \phi|_{3 \text{ pnt avg}}$$

$$K_{w8} = c_{K8} K_m + c_{Ksqd} \overline{w'^3}^2 \Big|_{3 \text{ pnt avg}}$$

$$K_{w9} = c_{K9} K_m + c_{Ksqd} \eta|_{3 \text{ pnt avg}}$$

where λ is $10^{12} \overline{r_t'^2}^2$ for Equation (6), $\overline{\theta_l'^2}^2$ for Equation (7), and $10^6 \overline{r_t'\theta_l'}^2$ for Equation (8); ϕ is $10^6 \overline{w'r_t'}^2$ for Equation (9) and $\overline{w'\theta_l'}^2$ for Equation (10); and η is $\overline{u'^2}^2$ for Equation (14) and $\overline{v'^2}^2$ for Equation (15).

2 PDF closure

Details of the PDF closure can be found in Larson and Golaz (2005), hereafter referred to as LG. We only briefly summarize key aspects here.

2.1 Transport terms

The transport terms appearing in Eqs (1)-(11) are closed as follows. First, we define $c_{w\theta_l}$ and c_{wr_t} as in Eqs (LG15) and (LG16):

$$c_{w\theta_l} = \frac{\overline{w'\theta_l'}}{\sqrt{\overline{w'^2}}\sqrt{\overline{\theta_l'^2}}} \tag{20}$$

$$c_{wr_t} = \frac{\overline{w'r_t'}}{\sqrt{\overline{w'^2}}\sqrt{\overline{r_t'^2}}} \tag{21}$$

The width of the individual w plumes is given by (LG37):

$$\tilde{\sigma}_w^2 = \gamma \left[1 - \max\left(c_{w\theta_l}^2, c_{wr_t}^2\right)\right] \tag{22}$$

We define the following quantities in order to simplify the notation:

$$a_1 = \frac{1}{(1 - \tilde{\sigma}_w^2)} \tag{23}$$

$$a_2 = \frac{1}{(1-\tilde{\sigma}_w^2)^2} \tag{24}$$

$$a_3 = 3\tilde{\sigma}_w^4 + 6(1-\tilde{\sigma}_w^2)\tilde{\sigma}_w^2 + (1-\tilde{\sigma}_w^2)^2 - 3 \tag{25}$$

The turbulence moment $\overline{w'^4}$ is given by (LG40):

$$\overline{w'^4} = \overline{w'^2}^2\,(a_3+3) + a_1\frac{\overline{w'^3}^2}{\overline{w'^2}} \tag{26}$$

The flux transport terms are given by (LG42):

$$\overline{w'^2\theta_l'} = a_1\frac{\overline{w'^3}}{\overline{w'^2}}\,\overline{w'\theta_l'} \tag{27}$$

$$\overline{w'^2 r_t'} = a_1\frac{\overline{w'^3}}{\overline{w'^2}}\,\overline{w'r_t'} \tag{28}$$

The variance transport terms follow (LG46):

$$\overline{w'\theta_l'^2} = \frac{1}{3}\beta a_1\frac{\overline{w'^3}}{\overline{w'^2}}\overline{\theta_l'^2} + \left(1-\frac{1}{3}\beta\right)a_2\frac{\overline{w'^3}}{\overline{w'^2}^2}\overline{w'\theta_l'}^2 \tag{29}$$

$$\overline{w'r_t'^2} = \frac{1}{3}\beta a_1\frac{\overline{w'^3}}{\overline{w'^2}}\overline{r_t'^2} + \left(1-\frac{1}{3}\beta\right)a_2\frac{\overline{w'^3}}{\overline{w'^2}^2}\overline{w'r_t'}^2 \tag{30}$$

Finally, the covariance term is obtained substituting (LG56) into (LG48):

$$\overline{w'r_t'\theta_l'} = \frac{1}{3}\beta a_1\frac{\overline{w'^3}}{\overline{w'^2}}\,\overline{r_t'\theta_l'} + \left(1-\frac{1}{3}\beta\right)a_2\frac{\overline{w'^3}}{\overline{w'^2}^2}\overline{w'r_t'}\,\overline{w'\theta_l'} \tag{31}$$

In the anisotropic case, the horizontal wind variance terms are obtained by:

$$\overline{w'u'^2} = \frac{1}{3}\beta a_1\frac{\overline{w'^3}}{\overline{w'^2}}\overline{u'^2} + \left(1-\frac{1}{3}\beta\right)a_2\frac{\overline{w'^3}}{\overline{w'^2}^2}\overline{w'u'}^2 \tag{32}$$

$$\overline{w'v'^2} = \frac{1}{3}\beta a_1\frac{\overline{w'^3}}{\overline{w'^2}}\overline{v'^2} + \left(1-\frac{1}{3}\beta\right)a_2\frac{\overline{w'^3}}{\overline{w'^2}^2}\overline{w'v'}^2 \tag{33}$$

2.2 Buoyancy terms

There are more unclosed terms involving θ_v. They are $\overline{w'\theta_v'}$, $\overline{r_t'\theta_v'}$, $\overline{\theta_l'\theta_v'}$, and $\overline{w'^2\theta_v'}$ and can be written as:

$$\overline{\chi'\theta_v'} = \overline{\chi'\theta_l'} + \underbrace{\frac{1-\epsilon_0}{\epsilon_0}\theta_0}_{\equiv A\,(\approx 200K)}\,\overline{\chi'r_t'} + \underbrace{\left(\frac{L_v}{c_p}\left(\frac{p_0}{p}\right)^{R_d/c_p} - \frac{1}{\epsilon_0}\theta_0\right)}_{\equiv B\,(\approx 2000K)}\overline{\chi'r_l'}\ , \tag{34}$$

where χ' represents w', r_t', θ_l', w'^2, or a passive scalar. Here $\epsilon_0 = R_d/R_v$, R_d is the gas constant of dry air, R_v is the gas constant of water vapor, L_v is the latent heat of vaporization, c_p is the heat capacity of air, and p_0 is a reference pressure. The correlations involving liquid water $(\overline{\chi' r_l'})$ can be computed for the given family of PDFs (see next section).

3 Cloud properties

The cloud properties, such as cloud fraction, mean liquid water and correlations involving liquid water $(\overline{\chi' r_l'})$ are obtained from the PDF. To do so, a certain number of properties are computed for each Gaussian $(i = 1, 2)$:

$$T_{li} = \theta_{li} \left(\frac{p}{p_0} \right)^{R_d/c_p} \tag{35}$$

$$r_{si} = \frac{R_d}{R_v} \frac{e_s(T_{li})}{p - [1 - (R_d/R_v)] e_s(T_{li})} \tag{36}$$

$$\beta_i = \frac{R_d}{R_v} \left(\frac{L}{R_d T_{li}} \right) \left(\frac{L}{c_p T_{li}} \right) \tag{37}$$

$$s_i = r_{ti} - r_{si} \frac{1 + \beta_i r_{ti}}{1 + \beta_i r_{si}} \tag{38}$$

$$c_{r_{ti}} = \frac{1}{1 + \beta_i r_{si}} \tag{39}$$

$$c_{\theta_{li}} = \frac{1 + \beta_i r_{ti}}{[1 + \beta_i r_{si}]^2} \frac{c_p}{L} \beta_i r_{si} \left(\frac{p}{p_0} \right)^{R_d/c_p} \tag{40}$$

$$\sigma_{si}^2 = c_{\theta_{li}}^2 \sigma_{\theta_{li}}^2 + c_{r_{ti}}^2 \sigma_{r_{ti}}^2 - 2 c_{\theta_{li}} \sigma_{\theta_{li}} c_{r_{ti}} \sigma_{r_{ti}} r_{r_t \theta_l} \tag{41}$$

$$C_i = \frac{1}{2} \left[1 + \mathrm{erf} \left(\frac{s_i}{\sqrt{2} \sigma_{si}} \right) \right] \tag{42}$$

$$r_{li} = s_i C_i + \frac{\sigma_{si}}{\sqrt{2\pi}} \exp \left[-\frac{1}{2} \left(\frac{s_i}{\sigma_{si}} \right)^2 \right] \tag{43}$$

where C_i and r_{li} are the cloud fractions and liquid water of each individual Gaussian.

The layer-averaged cloud properties are given by:

$$\overline{C} = a C_1 + (1 - a) C_2 \tag{44}$$

$$\overline{r_l} = a r_{l1} + (1 - a) r_{l2} \tag{45}$$

$$\overline{w'r'_l} = a(w_1 - \bar{w})r_{l1} + (1-a)(w_2 - \bar{w})r_{l2} \tag{46}$$

$$\overline{w'^2 r'_l} = a\left((w_1 - \bar{w})^2 + \sigma_{w1}^2\right) r_{l1} + (1-a)\left((w_2 - \bar{w})^2 + \sigma_{w2}^2\right) r_{l2} - \overline{w'^2}\,(a r_{l1} + (1-a) r_{l2}) \tag{47}$$

$$\begin{aligned}\overline{\theta'_l r'_l} =& a\left[(\theta_{l1} - \bar{\theta}_l) r_{l1} - C_1\left(c_{\theta_{l1}}\sigma_{\theta_{l1}}^2 - r_{r_t\theta_l} c_{r_{t1}}\sigma_{r_{t1}}\sigma_{\theta_{l1}}\right)\right] \\ &+ (1-a)\left[(\theta_{l2} - \bar{\theta}_l) r_{l2} - C_2\left(c_{\theta_{l2}}\sigma_{\theta_{l2}}^2 - r_{r_t\theta_l} c_{r_{t2}}\sigma_{r_{t2}}\sigma_{\theta_{l2}}\right)\right]\end{aligned} \tag{48}$$

$$\begin{aligned}\overline{r'_t r'_l} =& a\left[(r_{t1} - \bar{r}_t) r_{l1} + C_1\left(c_{r_{t1}}\sigma_{r_{t1}}^2 - r_{r_t\theta_l} c_{\theta_{l1}}\sigma_{r_{t1}}\sigma_{\theta_{l1}}\right)\right] \\ &+ (1-a)\left[(r_{t2} - \bar{r}_t) r_{l2} + C_2\left(c_{r_{t2}}\sigma_{r_{t2}}^2 - r_{r_t\theta_l} c_{\theta_{l2}}\sigma_{r_{t2}}\sigma_{\theta_{l2}}\right)\right]\end{aligned} \tag{49}$$

4 Steady-state solutions for the variances

4.1 $\overline{r_t'^2}$ and $\overline{\theta_l'^2}$

Start with (6) (for simplicity, neglect the $|_{\mathrm{pd}}$ and $|_{\mathrm{cl}}$ terms), substitute (30), assume steady-state and rearrange:

$$\begin{aligned}&\frac{C_2}{\tau}\overline{r_t'^2} + \bar{w}\frac{\partial \overline{r_t'^2}}{\partial z} + \frac{1}{3}\beta\frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}}{\overline{w'^2}}\,\overline{r_t'^2}\right) - \frac{\partial}{\partial z}\left[(K_{w2} + \nu_2)\,\frac{\partial}{\partial z}\overline{r_t'^2}\right] \\ &= -\left(1 - \frac{1}{3}\beta\right)\frac{\partial}{\partial z}\left(a_2\frac{\overline{w'^3}}{\overline{w'^2}^2}\,\overline{w'r_t'}^2\right) - 2\,\overline{w'r'_t}\,\frac{\partial \bar{r}_t}{\partial z} + \frac{C_2}{\tau}\,r_t|_{\mathrm{tol}}^2\end{aligned} \tag{50}$$

The goal is to recast (50) so that $\overline{r_t'^2}$ can be computed using a tridiagonal solver:

$$\underbrace{\begin{bmatrix} & (1,2) & \cdots & (1,\mathrm{nzmax}-1) & (1,\mathrm{nzmax}) \\ (2,1) & (2,2) & \cdots & (2,\mathrm{nzmax}-1) & (2,\mathrm{nzmax}) \\ (3,1) & (3,2) & \cdots & (3,\mathrm{nzmax}-1) & \end{bmatrix}}_{\texttt{LHS}\text{(Stored in compact format)}} \begin{bmatrix} \texttt{rtp2}(1) \\ \texttt{rtp2}(2) \\ \vdots \\ \texttt{rtp2}(\mathrm{nzmax}-1) \\ \texttt{rtp2}(\mathrm{nzmax}) \end{bmatrix} = \underbrace{\begin{bmatrix} (1) \\ (2) \\ \vdots \\ (\mathrm{nzmax}-1) \\ (\mathrm{nzmax}) \end{bmatrix}}_{\texttt{RHS}}$$

$$\texttt{lhs}(3,\texttt{k})\,\texttt{rtp2}(\texttt{k}-1) + \texttt{lhs}(2,\texttt{k})\,\texttt{rtp2}(\texttt{k}) + \texttt{lhs}(1,\texttt{k})\,\texttt{rtp2}(\texttt{k}+1) = \texttt{rhs}(\texttt{k}) \tag{51}$$

We now compute the contributions of each term in (50) to $\texttt{lhs}(3,\texttt{k})$, $\texttt{lhs}(2,\texttt{k})$, $\texttt{lhs}(1,\texttt{k})$, and $\texttt{rhs}(\texttt{k})$.

4.1.1 Term 1: dp1, implicit component

$$\mathtt{lhs(2,k)} = \mathtt{lhs(2,k)} + \frac{\mathtt{C_2}}{\mathtt{taum(k)}} \tag{52}$$

4.1.2 Term 2: ma

$$\begin{aligned}
&\bar{w}\frac{\partial \overline{q_t'^2}}{\partial z}\bigg|_{\mathtt{zm(k)}} \\
=&\frac{\mathtt{wmm(k)}}{\mathtt{dzm(k)}}\left(\frac{1}{2}\left(\mathtt{rtp2(k)}+\mathtt{rtp2(k+1)}\right)-\frac{1}{2}\left(\mathtt{rtp2(k-1)}+\mathtt{rtp2(k)}\right)\right) \\
=&\frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}}\mathtt{rtp2(k+1)}-\frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}}\mathtt{rtp2(k-1)}
\end{aligned} \tag{53}$$

Separating out the contributions:

$$\begin{aligned}
\mathtt{lhs(3,k)} &= \mathtt{lhs(3,k)} - \frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}} \\
\mathtt{lhs(1,k)} &= \mathtt{lhs(1,k)} + \frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}}
\end{aligned} \tag{54}$$

4.1.3 Term 3: ta, implicit component

$$\begin{aligned}
&\frac{1}{3}\beta\frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}}{\overline{w'^2}}\overline{r_t'^2}\right)\bigg|_{\mathtt{zm(k)}} \\
=&\frac{\beta}{\mathtt{6dzm(k)}}\Bigg[\frac{\left(\mathtt{a1m(k)}+\mathtt{a1m(k+1)}\right)\mathtt{wp3(k+1)}\left(\mathtt{rtp2(k)}+\mathtt{rtp2(k+1)}\right)}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)} \\
&\qquad -\frac{\left(\mathtt{a1m(k-1)}+\mathtt{a1m(k)}\right)\mathtt{wp3(k)}\left(\mathtt{rtp2(k-1)}+\mathtt{rtp2(k)}\right)}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)}\Bigg]
\end{aligned} \tag{55}$$

Separating out the contributions:

$$\begin{aligned}
\mathtt{lhs(3,k)} &= \mathtt{lhs(3,k)} - \frac{\beta}{\mathtt{6dzm(k)}}\frac{\left(\mathtt{a1m(k-1)}+\mathtt{a1m(k)}\right)\mathtt{wp3(k)}}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)} \\
\mathtt{lhs(2,k)} &= \mathtt{lhs(2,k)} + \frac{\beta}{\mathtt{6dzm(k)}}\left(\frac{\left(\mathtt{a1m(k)}+\mathtt{a1m(k+1)}\right)\mathtt{wp3(k+1)}}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)}-\frac{\left(\mathtt{a1m(k-1)}+\mathtt{a1m(k)}\right)\mathtt{wp3(k)}}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)}\right) \\
\mathtt{lhs(1,k)} &= \mathtt{lhs(1,k)} + \frac{\beta}{\mathtt{6dzm(k)}}\frac{\left(\mathtt{a1m(k)}+\mathtt{a1m(k+1)}\right)\mathtt{wp3(k+1)}}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)}
\end{aligned} \tag{56}$$

In order to increase numerical stability in the model, a_1 has been brought outside of the derivative.

This is not mathematically correct, but it does help to increase stability. Brian Griffin. Feb. 21, 2008.

$$\begin{aligned}
&a_1 \frac{1}{3}\beta\frac{\partial}{\partial z}\left(\frac{\overline{w'^3}}{\overline{w'^2}}\overline{r_t'^2}\right)\Bigg|_{\mathtt{zm(k)}} \\
&= \mathtt{a1m(k)}\frac{\beta}{\mathtt{3dzm(k)}}\left[\frac{\mathtt{wp3(k+1)}\left(\mathtt{rtp2(k)}+\mathtt{rtp2(k+1)}\right)}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)}-\frac{\mathtt{wp3(k)}\left(\mathtt{rtp2(k-1)}+\mathtt{rtp2(k)}\right)}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)}\right]
\end{aligned} \tag{57}$$

Separating out the contributions:

$$\begin{aligned}
\mathtt{lhs(3,k)} &= \mathtt{lhs(3,k)} - \mathtt{a1m(k)}\frac{\beta}{\mathtt{3dzm(k)}}\frac{\mathtt{wp3(k)}}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)} \\
\mathtt{lhs(2,k)} &= \mathtt{lhs(2,k)} + \mathtt{a1m(k)}\frac{\beta}{\mathtt{3dzm(k)}}\left(\frac{\mathtt{wp3(k+1)}}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)}-\frac{\mathtt{wp3(k)}}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)}\right) \\
\mathtt{lhs(1,k)} &= \mathtt{lhs(1,k)} + \mathtt{a1m(k)}\frac{\beta}{\mathtt{3dzm(k)}}\frac{\mathtt{wp3(k+1)}}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)}
\end{aligned} \tag{58}$$

4.1.4 Term 4: dp2

$$\begin{aligned}
&-\frac{\partial}{\partial z}\left[\left(K_{w2}+\nu_2\right)\frac{\partial}{\partial z}\overline{q_t'^2}\right] \\
&= -\frac{1}{\mathtt{dzm(k)}}\left(\frac{\left(\mathtt{Kw2(k+1)}+\nu_2\right)\left(\mathtt{rtp2(k+1)}-\mathtt{rtp2(k)}\right)}{\mathtt{dzt(k+1)}}\right. \\
&\qquad\left.-\frac{\left(\mathtt{Kw2(k)}+\nu_2\right)\left(\mathtt{rtp2(k)}-\mathtt{rtp2(k-1)}\right)}{\mathtt{dzt(k)}}\right)
\end{aligned} \tag{59}$$

Separating out the contributions:

$$\begin{aligned}
\mathtt{lhs(3,k)} &= \mathtt{lhs(3,k)} - \frac{\mathtt{Kw2(k)}+\nu_2}{\mathtt{dzm(k)dzt(k)}} \\
\mathtt{lhs(2,k)} &= \mathtt{lhs(2,k)} + \frac{1}{\mathtt{dzm(k)}}\left(\frac{\mathtt{Kw2(k+1)}+\nu_2}{\mathtt{dzt(k+1)}}+\frac{\mathtt{Kw2(k)}+\nu_2}{\mathtt{dzt(k)}}\right) \\
\mathtt{lhs(1,k)} &= \mathtt{lhs(1,k)} - \frac{\mathtt{Kw2(k+1)}+\nu_2}{\mathtt{dzm(k)dzt(k+1)}}
\end{aligned} \tag{60}$$

4.1.5 Term 5: ta, explicit component

$$\begin{aligned}
&-\left(1-\frac{1}{3}\beta\right)\frac{\partial}{\partial z}\left(a_2\frac{\overline{w'^3}}{\overline{w'^2}^2}\overline{w'q_t'}^2\right)\Bigg|_{\mathtt{zm(k)}} \\
&= -\frac{1-\frac{1}{3}\beta}{\mathtt{4dzm(k)}}\left[\frac{\left(\mathtt{a1m(k)}+\mathtt{a1m(k+1)}\right)^2\mathtt{wp3(k+1)}\left(\mathtt{wprtp(k)}+\mathtt{wprtp(k+1)}\right)^2}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)^2}\right. \\
&\qquad\left.-\frac{\left(\mathtt{a1m(k-1)}+\mathtt{a1m(k)}\right)^2\mathtt{wp3(k)}\left(\mathtt{wprtp(k-1)}+\mathtt{wprtp(k)}\right)^2}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)^2}\right]
\end{aligned} \tag{61}$$

Separating out the contributions:

$$
\begin{aligned}
&\mathtt{rhs(k)} \\
&= \mathtt{rhs(k)} - \frac{1-\frac{1}{3}\beta}{4\mathtt{dzm(k)}}\left[\frac{\left(\mathtt{a1m(k)}+\mathtt{a1m(k+1)}\right)^2 \mathtt{wp3(k+1)}\left(\mathtt{wprtp(k)}+\mathtt{wprtp(k+1)}\right)^2}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)}, 2\epsilon\right)^2}\right. \\
&\qquad\qquad\left. - \frac{\left(\mathtt{a1m(k-1)}+\mathtt{a1m(k)}\right)^2 \mathtt{wp3(k)}\left(\mathtt{wprtp(k-1)}+\mathtt{wprtp(k)}\right)^2}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)}, 2\epsilon\right)^2}\right]
\end{aligned}
\tag{62}
$$

In order to increase numerical stability in the model, a_1 has been brought outside of the derivative. This is not mathematically correct, but it does help to increase stability. Brian Griffin. Feb. 21, 2008.

$$
\begin{aligned}
&- a_2\left(1-\frac{1}{3}\beta\right)\frac{\partial}{\partial z}\left(\frac{\overline{w'^3}}{\overline{w'^2}^2}\,\overline{w'r_t'}^2\right)\Bigg|_{\mathtt{zm(k)}} \\
&= -\mathtt{a1m(k)}^2\frac{1-\frac{1}{3}\beta}{\mathtt{dzm(k)}}\left[\frac{\mathtt{wp3(k+1)}\left(\mathtt{wprtp(k)}+\mathtt{wprtp(k+1)}\right)^2}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)}, 2\epsilon\right)^2}\right. \\
&\qquad\qquad\left. - \frac{\mathtt{wp3(k)}\left(\mathtt{wprtp(k-1)}+\mathtt{wprtp(k)}\right)^2}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)}, 2\epsilon\right)^2}\right]
\end{aligned}
\tag{63}
$$

Separating out the contributions:

$$
\begin{aligned}
&\mathtt{rhs(k)} \\
&= \mathtt{rhs(k)} - \mathtt{a1m(k)}^2\frac{1-\frac{1}{3}\beta}{\mathtt{dzm(k)}}\left[\frac{\mathtt{wp3(k+1)}\left(\mathtt{wprtp(k)}+\mathtt{wprtp(k+1)}\right)^2}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)}, 2\epsilon\right)^2}\right. \\
&\qquad\qquad\left. - \frac{\mathtt{wp3(k)}\left(\mathtt{wprtp(k-1)}+\mathtt{wprtp(k)}\right)^2}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)}, 2\epsilon\right)^2}\right]
\end{aligned}
\tag{64}
$$

4.1.6 Term 6: tp

$$
-2\,\overline{w'r_t'}\,\frac{\partial \bar{r}_t}{\partial z}\bigg|_{\mathtt{zm(k)}} = -2\mathtt{wprtp(k)}\frac{\mathtt{rtm(k+1)}-\mathtt{rtm(k)}}{\mathtt{dzm(k)}}
\tag{65}
$$

Separating out the contributions:

$$
\mathtt{rhs(k)} = \mathtt{rhs(k)} - 2\mathtt{wprtp(k)}\frac{\mathtt{rtm(k+1)}-\mathtt{rtm(k)}}{\mathtt{dzm(k)}}
\tag{66}
$$

4.1.7 Term 7: dp1, explicit component

$$
\mathtt{rhs(k)} = \mathtt{rhs(k)} + \frac{\mathtt{C_2}}{\mathtt{taum(k)}}\mathtt{rttol}^2
\tag{67}
$$

4.2 $\overline{q_t'\theta_l'}$

Start with (8) (for simplicity, neglect the $|_{\mathrm{cl}}$ term), substitute (31), assume steady-state and rearrange:

$$\begin{aligned}&\frac{C_2}{\tau}\overline{r_t'\theta_l'} + \bar{w}\frac{\partial\overline{r_t'\theta_l'}}{\partial z} + \frac{1}{3}\beta\frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}}{\overline{w'^2}}\overline{r_t'\theta_l'}\right) - \frac{\partial}{\partial z}\left[(K_{w2}+\nu_2)\frac{\partial}{\partial z}\overline{r_t'\theta_l'}\right] \\ &= -\left(1-\frac{1}{3}\beta\right)\frac{\partial}{\partial z}\left(a_2\frac{\overline{w'^3}}{\overline{w'^2}^2}\overline{w'r_t'}\,\overline{w'\theta_l'}\right) - \overline{w'r_t'}\frac{\partial\bar{\theta}_l}{\partial z} - \overline{w'\theta_l'}\frac{\partial\bar{r}_t}{\partial z}\end{aligned} \tag{68}$$

As for the variances, the goal is to recast (68) so that $\overline{r_t'\theta_l'}$ can be computed using a tridiagonal solver:

$$\mathtt{lhs(3,k)}\,\mathtt{rtpthlp(k-1)} + \mathtt{lhs(2,k)}\,\mathtt{rtpthlp(k)} + \mathtt{lhs(1,k)}\,\mathtt{rtpthlp(k+1)} = \mathtt{rhs(k)} \tag{69}$$

We now compute the contributions of each term in (68) to `lhs(3,k)`, `lhs(2,k)`, `lhs(1,k)`, and `rhs(k)`.

4.2.1 Term 1: dp1

$$\mathtt{lhs(2,k)} = \mathtt{lhs(2,k)} + \frac{\mathtt{C_2}}{\mathtt{taum(k)}} \tag{70}$$

4.2.2 Term 2: ma

$$\begin{aligned}&\bar{w}\frac{\partial\overline{r_t'\theta_l'}}{\partial z}\bigg|_{\mathtt{zm(k)}} \\ &= \frac{\mathtt{wmm(k)}}{\mathtt{dzm(k)}}\left(\frac{1}{2}\left(\mathtt{rtpthlp(k)} + \mathtt{rtpthlp(k+1)}\right) - \frac{1}{2}\left(\mathtt{rtpthlp(k-1)} + \mathtt{rtpthlp(k)}\right)\right) \\ &= \frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}}\mathtt{rtpthlp(k+1)} - \frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}}\mathtt{rtpthlp(k-1)}\end{aligned} \tag{71}$$

Separating out the contributions:

$$\begin{aligned}\mathtt{lhs(3,k)} &= \mathtt{lhs(3,k)} - \frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}} \\ \mathtt{lhs(1,k)} &= \mathtt{lhs(1,k)} + \frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}}\end{aligned} \tag{72}$$

4.2.3 Term 3: ta, implicit component

$$
\begin{aligned}
&\frac{1}{3}\beta\frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}}{\overline{w'^2}}\,\overline{r_t'\theta_l'}\right)\Bigg|_{\mathtt{zm(k)}} \\
&= \frac{\beta}{\mathtt{6dzm(k)}}\Bigg[\frac{(\mathtt{a1m(k)}+\mathtt{a1m(k+1)})\,\mathtt{wp3(k+1)}\,(\mathtt{rtpthlp(k)}+\mathtt{rtpthlp(k+1)})}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)} \\
&\qquad -\frac{(\mathtt{a1m(k-1)}+\mathtt{a1m(k)})\,\mathtt{wp3(k)}\,(\mathtt{rtpthlp(k-1)}+\mathtt{rtpthlp(k)})}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)}\Bigg]
\end{aligned}
\tag{73}
$$

Separating out the contributions:

$$
\begin{aligned}
\mathtt{lhs(3,k)} &= \mathtt{lhs(3,k)} - \frac{\beta}{\mathtt{6dzm(k)}}\frac{(\mathtt{a1m(k-1)}+\mathtt{a1m(k)})\,\mathtt{wp3(k)}}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)} \\
\mathtt{lhs(2,k)} &= \mathtt{lhs(2,k)} + \frac{\beta}{\mathtt{6dzm(k)}}\left(\frac{(\mathtt{a1m(k)}+\mathtt{a1m(k+1)})\,\mathtt{wp3(k+1)}}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)} - \frac{(\mathtt{a1m(k-1)}+\mathtt{a1m(k)})\,\mathtt{wp3(k)}}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)}\right) \\
\mathtt{lhs(1,k)} &= \mathtt{lhs(1,k)} + \frac{\beta}{\mathtt{6dzm(k)}}\frac{(\mathtt{a1m(k)}+\mathtt{a1m(k+1)})\,\mathtt{wp3(k+1)}}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)}
\end{aligned}
\tag{74}
$$

In order to increase numerical stability in the model, a_1 has been brought outside of the derivative. This is not mathematically correct, but it does help to increase stability. Brian Griffin. Feb. 21, 2008.

$$
\begin{aligned}
&a_1\frac{1}{3}\beta\frac{\partial}{\partial z}\left(\frac{\overline{w'^3}}{\overline{w'^2}}\,\overline{r_t'\theta_l'}\right)\Bigg|_{\mathtt{zm(k)}} \\
&= \mathtt{a1m(k)}\frac{\beta}{\mathtt{3dzm(k)}}\Bigg[\frac{\mathtt{wp3(k+1)}\,(\mathtt{rtpthlp(k)}+\mathtt{rtpthlp(k+1)})}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)} \\
&\qquad -\frac{\mathtt{wp3(k)}\,(\mathtt{rtpthlp(k-1)}+\mathtt{rtpthlp(k)})}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)}\Bigg]
\end{aligned}
\tag{75}
$$

Separating out the contributions:

$$
\begin{aligned}
\mathtt{lhs(3,k)} &= \mathtt{lhs(3,k)} - \mathtt{a1m(k)}\frac{\beta}{\mathtt{3dzm(k)}}\frac{\mathtt{wp3(k)}}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)} \\
\mathtt{lhs(2,k)} &= \mathtt{lhs(2,k)} + \mathtt{a1m(k)}\frac{\beta}{\mathtt{3dzm(k)}}\left(\frac{\mathtt{wp3(k+1)}}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)} - \frac{\mathtt{wp3(k)}}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)}\right) \\
\mathtt{lhs(1,k)} &= \mathtt{lhs(1,k)} + \mathtt{a1m(k)}\frac{\beta}{\mathtt{3dzm(k)}}\frac{\mathtt{wp3(k+1)}}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)}
\end{aligned}
\tag{76}
$$

4.2.4 Term 4: dp2

$$
\begin{aligned}
&-\frac{\partial}{\partial z}\left[\left(K_{w2}+\nu_2\right)\frac{\partial}{\partial z}\overline{r_t'\theta_l'}\right] \\
&=-\frac{1}{\mathtt{dzm(k)}}\left(\frac{\left(\mathtt{Kw2(k+1)}+\nu_2\right)\left(\mathtt{rtpthlp(k+1)}-\mathtt{rtpthlp(k)}\right)}{\mathtt{dzt(k+1)}}\right. \\
&\qquad\qquad\left.-\frac{\left(\mathtt{Kw2(k)}+\nu_2\right)\left(\mathtt{rtpthlp(k)}-\mathtt{rtpthlp(k-1)}\right)}{\mathtt{dzt(k)}}\right)
\end{aligned} \tag{77}
$$

Separating out the contributions:

$$
\begin{aligned}
\mathtt{lhs(3,k)} &= \mathtt{lhs(3,k)} - \frac{\mathtt{Kw2(k)}+\nu_2}{\mathtt{dzm(k)dzt(k)}} \\
\mathtt{lhs(2,k)} &= \mathtt{lhs(2,k)} + \frac{1}{\mathtt{dzm(k)}}\left(\frac{\mathtt{Kw2(k+1)}+\nu_2}{\mathtt{dzt(k+1)}}+\frac{\mathtt{Kw2(k)}+\nu_2}{\mathtt{dzt(k)}}\right) \\
\mathtt{lhs(1,k)} &= \mathtt{lhs(1,k)} - \frac{\mathtt{Kw2(k+1)}+\nu_2}{\mathtt{dzm(k)dzt(k+1)}}
\end{aligned} \tag{78}
$$

4.2.5 Term 5: ta, explicit component

$$
\begin{aligned}
&-\left(1-\frac{1}{3}\beta\right)\frac{\partial}{\partial z}\left(a_2\frac{\overline{w'^3}}{\overline{w'^2}^2}\,\overline{w'r_t'}\;\overline{w'\theta_l'}\right)\Bigg|_{\mathtt{zm(k)}} \\
&=-\frac{1-\frac{1}{3}\beta}{\mathtt{4dzm(k)}} \\
&\quad\times\left[\frac{\left(\mathtt{a1m(k)}+\mathtt{a1m(k+1)}\right)^2\mathtt{wp3(k+1)}\left(\mathtt{wprtp(k)}+\mathtt{wprtp(k+1)}\right)\left(\mathtt{wpthlp(k)}+\mathtt{wpthlp(k+1)}\right)}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)^2}\right. \\
&\qquad\left.-\frac{\left(\mathtt{a1m(k-1)}+\mathtt{a1m(k)}\right)^2\mathtt{wp3(k)}\left(\mathtt{wprtp(k-1)}+\mathtt{wprtp(k)}\right)\left(\mathtt{wpthlp(k-1)}+\mathtt{wpthlp(k)}\right)}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)^2}\right]
\end{aligned} \tag{79}
$$

Separating out the contributions:

$$
\begin{aligned}
\mathtt{rhs(k)} =&\,\mathtt{rhs(k)} \\
&-\frac{1-\frac{1}{3}\beta}{\mathtt{4dzm(k)}} \\
&\times\left[\frac{\left(\mathtt{a1m(k)}+\mathtt{a1m(k+1)}\right)^2\mathtt{wp3(k+1)}\left(\mathtt{wprtp(k)}+\mathtt{wprtp(k+1)}\right)\left(\mathtt{wpthlp(k)}+\mathtt{wpthlp(k+1)}\right)}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)^2}\right. \\
&\qquad\left.-\frac{\left(\mathtt{a1m(k-1)}+\mathtt{a1m(k)}\right)^2\mathtt{wp3(k)}\left(\mathtt{wprtp(k-1)}+\mathtt{wprtp(k)}\right)\left(\mathtt{wpthlp(k-1)}+\mathtt{wpthlp(k)}\right)}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)^2}\right]
\end{aligned} \tag{80}
$$

In order to increase numerical stability in the model, a_1 has been brought outside of the derivative. This is not mathematically correct, but it does help to increase stability. Brian Griffin. Feb. 21, 2008.

$$
\begin{aligned}
&- a_2 \left(1 - \frac{1}{3}\beta\right) \frac{\partial}{\partial z} \left(\frac{\overline{w'^3}}{\overline{w'^2}^2} \, \overline{w' r_t'} \, \overline{w' \theta_l'} \right) \Bigg|_{\mathtt{zm(k)}} \\
&= - \mathtt{a1m(k)}^2 \frac{1 - \frac{1}{3}\beta}{\mathtt{dzm(k)}} \\
&\quad \times \left[\frac{\mathtt{wp3(k+1)} \left(\mathtt{wprtp(k)} + \mathtt{wprtp(k+1)} \right) \left(\mathtt{wpthlp(k)} + \mathtt{wpthlp(k+1)} \right)}{\max \left(\mathtt{wp2(k)} + \mathtt{wp2(k+1)}, 2\epsilon \right)^2} \right. \\
&\qquad \left. - \frac{\mathtt{wp3(k)} \left(\mathtt{wprtp(k-1)} + \mathtt{wprtp(k)} \right) \left(\mathtt{wpthlp(k-1)} + \mathtt{wpthlp(k)} \right)}{\max \left(\mathtt{wp2(k-1)} + \mathtt{wp2(k)}, 2\epsilon \right)^2} \right]
\end{aligned}
\tag{81}
$$

Separating out the contributions:

$$
\begin{aligned}
\mathtt{rhs(k)} = &\mathtt{rhs(k)} \\
&- \mathtt{a1m(k)}^2 \frac{1 - \frac{1}{3}\beta}{\mathtt{dzm(k)}} \\
&\times \left[\frac{\mathtt{wp3(k+1)} \left(\mathtt{wprtp(k)} + \mathtt{wprtp(k+1)} \right) \left(\mathtt{wpthlp(k)} + \mathtt{wpthlp(k+1)} \right)}{\max \left(\mathtt{wp2(k)} + \mathtt{wp2(k+1)}, 2\epsilon \right)^2} \right. \\
&\quad \left. - \frac{\mathtt{wp3(k)} \left(\mathtt{wprtp(k-1)} + \mathtt{wprtp(k)} \right) \left(\mathtt{wpthlp(k-1)} + \mathtt{wpthlp(k)} \right)}{\max \left(\mathtt{wp2(k-1)} + \mathtt{wp2(k)}, 2\epsilon \right)^2} \right]
\end{aligned}
\tag{82}
$$

4.2.6 Terms 6 and 7: tp1 and tp2, respectively

$$
\begin{aligned}
&- \overline{w' q_t'} \frac{\partial \bar{\theta}_l}{\partial z} \Bigg|_{\mathtt{zm(k)}} - \overline{w' \theta_l'} \frac{\partial \bar{q}_t}{\partial z} \Bigg|_{\mathtt{zm(k)}} \\
&= - \mathtt{wprtp(k)} \frac{\mathtt{thlm(k+1)} - \mathtt{thlm(k)}}{\mathtt{dzm(k)}} - \mathtt{wpthlp(k)} \frac{\mathtt{rtm(k+1)} - \mathtt{rtm(k)}}{\mathtt{dzm(k)}}
\end{aligned}
\tag{83}
$$

Separating out the contributions:

$$
\mathtt{rhs(k)} = \mathtt{rhs(k)} - \mathtt{wprtp(k)} \frac{\mathtt{thlm(k+1)} - \mathtt{thlm(k)}}{\mathtt{dzm(k)}} - \mathtt{wpthlp(k)} \frac{\mathtt{rtm(k+1)} - \mathtt{rtm(k)}}{\mathtt{dzm(k)}}
\tag{84}
$$

5 Implicit solutions for the means and fluxes

$\bar{r}_t$ and $\overline{w' r_t'}$ can be solved simultaneously and implicitly. Start with eqs (3), (9) (for simplicity, neglect the $|_{\mathrm{sicl}}$ and $|_{\mathrm{cl}}$ terms), and substitute expression for the transport term (28):

$$
\frac{\partial \bar{r}_t}{\partial t} = - \bar{w} \frac{\partial \bar{r}_t}{\partial z} - \frac{\partial}{\partial z} \overline{w' r_t'} + \frac{\partial \bar{r}_t}{\partial t} \bigg|_{\mathrm{ls}}
\tag{85}
$$

$$
\begin{aligned}
\frac{\partial \overline{w'r_t'}}{\partial t} =& - \bar{w}\frac{\partial \overline{w'r_t'}}{\partial z} - \frac{\partial}{\partial z}\left(a_1 \frac{\overline{w'^3}}{\overline{w'^2}} \overline{w'r_t'} \right) - \overline{w'^2}\frac{\partial \bar{r}_t}{\partial z} - (1-C_7)\overline{w'r_t'}\frac{\partial \bar{w}}{\partial z} + (1-C_7)\frac{g}{\theta_0}\overline{r_t'\theta_v'} \\
& - \frac{C_6}{\tau}\overline{w'r_t'} + \frac{\partial}{\partial z}\left[(K_{w6}+\nu_6)\frac{\partial}{\partial z}\overline{w'r_t'} \right]
\end{aligned} \tag{86}
$$

After discretizing the time derivative and rearranging terms:

$$
\begin{aligned}
&\frac{\bar{r}_t^{t+\Delta t}}{\Delta t} + \bar{w}\frac{\partial \bar{r}_t^{t+\Delta t}}{\partial z} + \frac{\partial}{\partial z}\overline{w'r_t'}^{t+\Delta t} \\
&= \frac{\bar{r}_t^{t}}{\Delta t} + \left.\frac{\partial \bar{r}_t}{\partial t}\right|_{\mathrm{ls}}
\end{aligned} \tag{87}
$$

$$
\begin{aligned}
&\frac{\overline{w'r_t'}^{t+\Delta t}}{\Delta t} + \bar{w}\frac{\partial}{\partial z}\overline{w'r_t'}^{t+\Delta t} + \frac{\partial}{\partial z}\left(a_1 \frac{\overline{w'^3}}{\overline{w'^2}} \overline{w'r_t'}^{t+\Delta t} \right) + \overline{w'^2}\frac{\partial \bar{r}_t^{t+\Delta t}}{\partial z} \\
&+ (1-C_7)\overline{w'r_t'}^{t+\Delta t}\frac{\partial \bar{w}}{\partial z} + \frac{C_6}{\tau}\overline{w'r_t'}^{t+\Delta t} - \frac{\partial}{\partial z}\left[(K_{w6}+\nu_6)\frac{\partial}{\partial z}\overline{w'r_t'}^{t+\Delta t} \right] \\
&= \frac{\overline{w'r_t'}^{t}}{\Delta t} + (1-C_7)\frac{g}{\theta_0}\overline{r_t'\theta_v'}^{t}
\end{aligned} \tag{88}
$$

The LHSs of (87)-(88) are linear in $\bar{r}_t$ and $\overline{w'r_t'}$ and can therefore be rewritten in matrix form:

$$
\underbrace{\begin{pmatrix}
\cdots & \bar{r}_{t\,k}^{\,\mathrm{impl.}} & \overline{w'r_t'}_{k}^{\,\mathrm{impl.}} & \bar{r}_{t\,k+1}^{\,\mathrm{impl.}} & \overline{w'r_t'}_{k+1}^{\,\mathrm{impl.}} & \bar{r}_{t\,k+2}^{\,\mathrm{impl.}} & \overline{w'r_t'}_{k+2}^{\,\mathrm{impl.}} & \cdots \\
\cdots & \overline{w'r_t'}_{k-1}^{\,\mathrm{impl.}} & \bar{r}_{t\,k}^{\,\mathrm{impl.}} & \overline{w'r_t'}_{k}^{\,\mathrm{impl.}} & \bar{r}_{t\,k+1}^{\,\mathrm{impl.}} & \overline{w'r_t'}_{k+1}^{\,\mathrm{impl.}} & \bar{r}_{t\,k+2}^{\,\mathrm{impl.}} & \cdots \\
\cdots & \bar{r}_{t\,k-1}^{\,\mathrm{impl.}} & \overline{w'r_t'}_{k-1}^{\,\mathrm{impl.}} & \bar{r}_{t\,k}^{\,\mathrm{impl.}} & \overline{w'r_t'}_{k}^{\,\mathrm{impl.}} & \bar{r}_{t\,k+1}^{\,\mathrm{impl.}} & \overline{w'r_t'}_{k+1}^{\,\mathrm{impl.}} & \cdots \\
\cdots & \overline{w'r_t'}_{k-2}^{\,\mathrm{impl.}} & \bar{r}_{t\,k-1}^{\,\mathrm{impl.}} & \overline{w'r_t'}_{k-1}^{\,\mathrm{impl.}} & \bar{r}_{t\,k}^{\,\mathrm{impl.}} & \overline{w'r_t'}_{k}^{\,\mathrm{impl.}} & \bar{r}_{t\,k+1}^{\,\mathrm{impl.}} & \cdots \\
\cdots & \bar{r}_{t\,k-2}^{\,\mathrm{impl.}} & \overline{w'r_t'}_{k-2}^{\,\mathrm{impl.}} & \bar{r}_{t\,k-1}^{\,\mathrm{impl.}} & \overline{w'r_t'}_{k-1}^{\,\mathrm{impl.}} & \bar{r}_{t\,k}^{\,\mathrm{impl.}} & \overline{w'r_t'}_{k}^{\,\mathrm{impl.}} & \cdots
\end{pmatrix}}_{\texttt{LHS}\text{(Stored in compact format)}}
\begin{pmatrix}
\vdots \\
\bar{r}_{t\,k-1}^{\,t+\Delta t} \\
\overline{w'r_t'}_{k-1}^{\,t+\Delta t} \\
\bar{r}_{t\,k}^{\,t+\Delta t} \\
\overline{w'r_t'}_{k}^{\,t+\Delta t} \\
\bar{r}_{t\,k+1}^{\,t+\Delta t} \\
\overline{w'r_t'}_{k+1}^{\,t+\Delta t} \\
\vdots
\end{pmatrix}
=
\underbrace{\begin{pmatrix}
\vdots \\
\bar{r}_{t\,k-1}^{\,\mathrm{expl.}} \\
\overline{w'r_t'}_{k-1}^{\,\mathrm{expl.}} \\
\bar{r}_{t\,k}^{\,\mathrm{expl.}} \\
\overline{w'r_t'}_{k}^{\,\mathrm{expl.}} \\
\bar{r}_{t\,k+1}^{\,\mathrm{expl.}} \\
\overline{w'r_t'}_{k+1}^{\,\mathrm{expl.}} \\
\vdots
\end{pmatrix}}_{\texttt{RHS}} \tag{89}
$$

The matrix lhs is obtained by vertical discretization of the LHSs, and the vector rhs by discretization of the RHSs. lhs is band-diagonal with two rows above and two below the main diagonal. lhs is stored in compact form in an array with dimensions $(\texttt{5}, \texttt{2}\,\texttt{nzmax})$. rhs is a vector with dimension $(\texttt{2}\,\texttt{nzmax})$. lhs can be inverted efficiently using an LU decomposition algorithm for band diagonal matrices. The construction of the matrix lhs and vector rhs are as follows.

First, we compute the finite difference equivalent to (87):

$$
\begin{aligned}
&\frac{\texttt{rtm}^{\texttt{new}}(\texttt{k})}{\texttt{dt}} + \frac{\texttt{wmt}(\texttt{k})}{\texttt{2dzt}(\texttt{k})}\texttt{rtm}^{\texttt{new}}(\texttt{k}+\texttt{1}) - \frac{\texttt{wmt}(\texttt{k})}{\texttt{2dzt}(\texttt{k})}\texttt{rtm}^{\texttt{new}}(\texttt{k}-\texttt{1}) + \frac{\texttt{wprtp}^{\texttt{new}}(\texttt{k})}{\texttt{dzt}(\texttt{k})} - \frac{\texttt{wprtp}^{\texttt{new}}(\texttt{k}-\texttt{1})}{\texttt{dzt}(\texttt{k})} \\
=&\frac{\texttt{rtm}(\texttt{k})}{\texttt{dt}} + \texttt{rtm\_ls}(\texttt{k})
\end{aligned}
$$

$$(90)$$

Contributions to lhs from (90) are:

$$\mathtt{lhs(5,k\_xm)} = \mathtt{lhs(5,k\_xm)} - \frac{\mathtt{wmt(k)}}{\mathtt{2dzt(k)}} \tag{91}$$

$$\mathtt{lhs(4,k\_xm)} = \mathtt{lhs(4,k\_xm)} - \frac{\mathtt{1}}{\mathtt{dzt(k)}} \tag{92}$$

$$\mathtt{lhs(3,k\_xm)} = \mathtt{lhs(3,k\_xm)} + \frac{\mathtt{1}}{\mathtt{dt}} \tag{93}$$

$$\mathtt{lhs(2,k\_xm)} = \mathtt{lhs(2,k\_xm)} + \frac{\mathtt{1}}{\mathtt{dzt(k)}} \tag{94}$$

$$\mathtt{lhs(1,k\_xm)} = \mathtt{lhs(1,k\_xm)} + \frac{\mathtt{wmt(k)}}{\mathtt{2dzt(k)}} \tag{95}$$

Contributions to rhs from (90) are:

$$\mathtt{rhs(k\_xm)} = \mathtt{rhs(k\_xm)} + \frac{\mathtt{rtm(k)}}{\mathtt{dt}} + \mathtt{rtm\_ls(k)} \tag{96}$$

where $\mathtt{k\_xm} = \mathtt{2k} - \mathtt{1}$.

We now write the finite difference equivalent to (88):

$$\begin{aligned}
&\frac{\mathtt{wprtp^{new}(k)}}{\mathtt{dt}} + \frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}}\mathtt{wprtp^{new}(k+1)} - \frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}}\mathtt{wprtp^{new}(k-1)} \\
&+\frac{\mathtt{1}}{\mathtt{2dzm(k)}}\Bigg[-\frac{\left(\mathtt{a1m(k-1)}+\mathtt{a1m(k)}\right)\mathtt{wp3(k)}}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)}\mathtt{wprtp^{new}(k-1)} \\
&\qquad+\left(\frac{\left(\mathtt{a1m(k)}+\mathtt{a1m(k+1)}\right)\mathtt{wp3(k+1)}}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)} - \frac{\left(\mathtt{a1m(k-1)}+\mathtt{a1m(k)}\right)\mathtt{wp3(k)}}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)}\right)\mathtt{wprtp^{new}(k)} \\
&\qquad+\frac{\left(\mathtt{a1m(k)}+\mathtt{a1m(k+1)}\right)\mathtt{wp3(k+1)}}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)}\mathtt{wprtp^{new}(k+1)}\Bigg] \\
&+\mathtt{wp2(k)}\,\frac{\mathtt{rtm^{new}(k+1)}-\mathtt{rtm^{new}(k)}}{\mathtt{dzm(k)}} + \left(1-\mathtt{C_7}\right)\mathtt{wprtp^{new}(k)}\,\frac{\mathtt{wmt(k+1)}-\mathtt{wmt(k)}}{\mathtt{dzm(k)}} \\
&+\frac{\mathtt{C_6}}{\mathtt{taum(k)}}\,\mathtt{wprtp^{new}(k)} \\
&-\frac{\mathtt{Kw6(k)}+\nu_6}{\mathtt{dzm(k)dzt(k)}}\mathtt{wprtp^{new}(k-1)} \\
&+\frac{\mathtt{1}}{\mathtt{dzm(k)}}\left(\frac{\mathtt{Kw6(k+1)}+\nu_6}{\mathtt{dzt(k+1)}} + \frac{\mathtt{Kw6(k)}+\nu_6}{\mathtt{dzt(k)}}\right)\mathtt{wprtp^{new}(k)} \\
&-\frac{\mathtt{Kw6(k+1)}+\nu_6}{\mathtt{dzm(k)dzt(k+1)}}\mathtt{wprtp^{new}(k+1)} \\
=&\frac{\mathtt{wprtp(k)}}{\mathtt{dt}} + \left(1-\mathtt{C_7}\right)\frac{\mathtt{g}}{\theta_0}\mathtt{rtpthvp(k)}
\end{aligned}$$

$$(97)$$

Contributions to *lhs* from (97) are:

$$\begin{aligned}&\texttt{lhs}(5,\texttt{k\_wpxp}) = \texttt{lhs}(5,\texttt{k\_wpxp})\\ &-\frac{\texttt{wmm(k)}}{2\texttt{dzm(k)}} - \frac{1}{2\texttt{dzm(k)}}\frac{(\texttt{a1m(k}-1) + \texttt{a1m(k)})\,\texttt{wp3(k)}}{\max(\texttt{wp2(k}-1) + \texttt{wp2(k)}, 2\epsilon)} - \frac{\texttt{Kw6(k)} + \nu_6}{\texttt{dzm(k)dzt(k)}}\end{aligned} \tag{98}$$

$$\texttt{lhs}(4,\texttt{k\_wpxp}) = \texttt{lhs}(4,\texttt{k\_wpxp}) - \frac{\texttt{wp2(k)}}{\texttt{dzm(k)}} \tag{99}$$

$$\begin{aligned}&\texttt{lhs}(3,\texttt{k\_wpxp}) = \texttt{lhs}(3,\texttt{k\_wpxp})\\ &+\frac{1}{\texttt{dt}} + \frac{1}{2\texttt{dzm(k)}}\left(\frac{(\texttt{a1m(k)} + \texttt{a1m(k}+1))\,\texttt{wp3(k}+1)}{\max(\texttt{wp2(k)} + \texttt{wp2(k}+1), 2\epsilon)} - \frac{(\texttt{a1m(k}-1) + \texttt{a1m(k)})\,\texttt{wp3(k)}}{\max(\texttt{wp2(k}-1) + \texttt{wp2(k)}, 2\epsilon)}\right)\\ &+(1-\texttt{C}_7)\frac{\texttt{wmt(k}+1) - \texttt{wmt(k)}}{\texttt{dzm(k)}} + \frac{\texttt{C}_6}{\texttt{taum(k)}} + \frac{1}{\texttt{dzm(k)}}\left(\frac{\texttt{Kw6(k}+1) + \nu_6}{\texttt{dzt(k}+1)} + \frac{\texttt{Kw6(k)} + \nu_6}{\texttt{dzt(k)}}\right)\end{aligned} \tag{100}$$

$$\texttt{lhs}(2,\texttt{k\_wpxp}) = \texttt{lhs}(2,\texttt{k\_wpxp}) + \frac{\texttt{wp2(k)}}{\texttt{dzm(k)}} \tag{101}$$

$$\begin{aligned}&\texttt{lhs}(1,\texttt{k\_wpxp}) = \texttt{lhs}(1,\texttt{k\_wpxp})\\ &+\frac{\texttt{wmm(k)}}{2\texttt{dzm(k)}} + \frac{1}{2\texttt{dzm(k)}}\frac{(\texttt{a1m(k)} + \texttt{a1m(k}+1))\,\texttt{wp3(k}+1)}{\max(\texttt{wp2(k)} + \texttt{wp2(k}+1), 2\epsilon)} - \frac{\texttt{Kw6(k}+1) + \nu_6}{\texttt{dzm(k)dzt(k}+1)}\end{aligned} \tag{102}$$

Contributions to *rhs* from (97) are:

$$\texttt{rhs}(\texttt{k\_wpxp}) = \texttt{rhs}(\texttt{k\_wpxp}) + \frac{\texttt{wprtp(k)}}{\texttt{dt}} + (1-\texttt{C}_7)\frac{\texttt{g}}{\theta_0}\texttt{rtpthvp(k)} \tag{103}$$

where $\texttt{k\_wpxp} = 2\texttt{k}$.

The procedure for solving implicitly for $\bar{\theta}_l$ and $\overline{w'\theta_l'}$ is identical. It leads to the same matrix *lhs*, so *lhs* needs to be inverted only once.

6 Implicit solution for the vertical velocity moments

Start with equations (5) and (11) (for simplicity, neglect the $|_{\rm pd}$ and $|_{\rm cl}$ terms):

$$\begin{aligned}\frac{\partial\overline{w'^2}}{\partial t} =& -\bar{w}\frac{\partial\overline{w'^2}}{\partial z} - \frac{\partial\overline{w'^3}}{\partial z} - 2\overline{w'^2}\frac{\partial\bar{w}}{\partial z} + \frac{2g}{\theta_0}\overline{w'\theta_v'}\\ &-\frac{C_4}{\tau}\left(\overline{w'^2} - \frac{2}{3}\bar{e}\right) - C_5\left(-2\overline{w'^2}\frac{\partial\bar{w}}{\partial z} + \frac{2g}{\theta_0}\overline{w'\theta_v'}\right) + \frac{2}{3}C_5\left(\frac{g}{\theta_0}\overline{w'\theta_v'} - \overline{u'w'}\frac{\partial\bar{u}}{\partial z} - \overline{v'w'}\frac{\partial\bar{v}}{\partial z}\right)\\ &-\frac{C_1}{\tau}\left(\overline{w'^2} - w|_{\rm tol}^2\right) + \frac{\partial}{\partial z}\left[(K_{w1} + \nu_1)\frac{\partial}{\partial z}\overline{w'^2}\right]\end{aligned}$$

$$\tag{104}$$

$$
\begin{aligned}
\frac{\partial \overline{w'^3}}{\partial t} =& -\bar{w}\frac{\partial \overline{w'^3}}{\partial z} - \frac{\partial \overline{w'^4}}{\partial z} + 3\overline{w'^2}\frac{\partial \overline{w'^2}}{\partial z} - 3\overline{w'^3}\frac{\partial \bar{w}}{\partial z} + \frac{3g}{\theta_0}\overline{w'^2\theta'_v} \\
& - \frac{C_8}{\tau}\left(C_{8b}\, Skw^4 + 1\right)\overline{w'^3} - C_{11}\left(-3\overline{w'^3}\frac{\partial \bar{w}}{\partial z} + \frac{3g}{\theta_0}\overline{w'^2\theta'_v}\right) + \frac{\partial}{\partial z}\left[\left(K_{w8} + \nu_8\right)\frac{\partial}{\partial z}\overline{w'^3}\right]
\end{aligned}
\tag{105}
$$

Using (26), we can rewrite the transport and production terms in (105):

$$
\begin{aligned}
& -\frac{\partial \overline{w'^4}}{\partial z} + 3\overline{w'^2}\frac{\partial \overline{w'^2}}{\partial z} \\
&= -\frac{\partial}{\partial z}\left(\overline{w'^4} - \frac{3}{2}\overline{w'^2}^2\right) \\
&= -\frac{\partial}{\partial z}\left(\tilde{a}_3\overline{w'^2}^2\right) - \frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}^2}{\overline{w'^2}}\right)
\end{aligned}
\tag{106}
$$

where $\tilde{a}_3 = a_3 + 3/2$. Rearranging terms and making use of (12):

$$
\begin{aligned}
& \frac{\partial \overline{w'^2}}{\partial t} + \bar{w}\frac{\partial \overline{w'^2}}{\partial z} + \frac{\partial \overline{w'^3}}{\partial z} + \frac{C_1}{\tau}\overline{w'^2} - \frac{\partial}{\partial z}\left[\left(K_{w1} + \nu_1\right)\frac{\partial}{\partial z}\overline{w'^2}\right] \\
&= +(1 - C_5)\frac{2g}{\theta_0}\overline{w'\theta'_v} - 2(1 - C_5)\overline{w'^2}\frac{\partial \bar{w}}{\partial z} + \frac{C_1}{\tau}\left. w\right|^2_{\text{tol}} + \frac{2}{3}C_5\left(\frac{g}{\theta_0}\overline{w'\theta'_v} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{v'w'}\frac{\partial \bar{v}}{\partial z}\right)
\end{aligned}
\tag{107}
$$

$$
\begin{aligned}
& \frac{\partial \overline{w'^3}}{\partial t} + \bar{w}\frac{\partial \overline{w'^3}}{\partial z} - \frac{\partial}{\partial z}\left[\left(K_{w8} + \nu_8\right)\frac{\partial}{\partial z}\overline{w'^3}\right] + \frac{C_8}{\tau}\left(C_{8b}\, Skw^4 + 1\right)\overline{w'^3} \\
& \qquad + \frac{\partial}{\partial z}\left(\tilde{a}_3\overline{w'^2}^2\right) + \frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}^2}{\overline{w'^2}}\right) \\
&= +(1 - C_{11})\frac{3g}{\theta_0}\overline{w'^2\theta'_v} - 3(1 - C_{11})\overline{w'^3}\frac{\partial \bar{w}}{\partial z}
\end{aligned}
\tag{108}
$$

6.1 $\overline{w'^2}$

Terms on the LHS of (107) are treated fully implicitly, except for the diffusion term which is treated with a Crank-Nicholson time step. Terms on the RHS explicitly:

$$
\begin{aligned}
& \frac{\overline{w'^2}^{t+\Delta t}}{\Delta t} + \bar{w}\frac{\partial \overline{w'^2}^{t+\Delta t}}{\partial z} + \frac{\partial \overline{w'^3}^{t+\Delta t}}{\partial z} + \frac{C_1}{\tau}\overline{w'^2}^{t+\Delta t} - \frac{1}{2}\frac{\partial}{\partial z}\left[\left(K_{w1} + \nu_1\right)\frac{\partial}{\partial z}\overline{w'^2}^{t+\Delta t}\right] \\
&= \frac{\overline{w'^2}^t}{\Delta t} + \frac{1}{2}\frac{\partial}{\partial z}\left[\left(K_{w1} + \nu_1\right)\frac{\partial}{\partial z}\overline{w'^2}^t\right] + \left.\overline{w'^2}\right|_{\text{expl}}
\end{aligned}
\tag{109}
$$

where

$$
\begin{aligned}
\left.\overline{w'^2}\right|_{\text{expl}} =& + (1 - C_5)\frac{2g}{\theta_0}\overline{w'\theta'_v}^t - 2(1 - C_5)\overline{w'^2}^t\frac{\partial \bar{w}}{\partial z} + \frac{C_1}{\tau}\left. w\right|^2_{\text{tol}} \\
& + \frac{2}{3}C_5\left(\frac{g}{\theta_0}\overline{w'\theta'_v} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{v'w'}\frac{\partial \bar{v}}{\partial z}\right)^t
\end{aligned}
\tag{110}
$$

The next step consists of writing the finite difference equivalent to (109):

$$
\begin{aligned}
&\frac{\mathtt{wp2}^{\mathtt{new}}(\mathtt{k})}{\mathtt{dt}} + \mathtt{wmm}(\mathtt{k}) \frac{\mathtt{wp2}^{\mathtt{new}}(\mathtt{k}+1) - \mathtt{wp2}^{\mathtt{new}}(\mathtt{k}-1)}{2\ \mathtt{dzm}(\mathtt{k})} \\
&+\frac{\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}+1) - \mathtt{wp3}^{\mathtt{new}}(\mathtt{k})}{\mathtt{dzm}(\mathtt{k})} + \frac{\mathtt{C_1}}{\mathtt{taum}(\mathtt{k})} \mathtt{wp2}^{\mathtt{new}}(\mathtt{k}) \\
&-\frac{\mathtt{Kw1}(\mathtt{k}) + \nu_1}{\mathtt{2dzm}(\mathtt{k})\mathtt{dzt}(\mathtt{k})} \mathtt{wp2}^{\mathtt{new}}(\mathtt{k}-1) \\
&+\frac{1}{\mathtt{2dzm}(\mathtt{k})} \left(\frac{\mathtt{Kw1}(\mathtt{k}+1) + \nu_1}{\mathtt{dzt}(\mathtt{k}+1)} + \frac{\mathtt{Kw1}(\mathtt{k}) + \nu_1}{\mathtt{dzt}(\mathtt{k})} \right) \mathtt{wp2}^{\mathtt{new}}(\mathtt{k}) \\
&-\frac{\mathtt{Kw1}(\mathtt{k}+1) + \nu_1}{\mathtt{2dzm}(\mathtt{k})\mathtt{dzt}(\mathtt{k}+1)} \mathtt{wp2}^{\mathtt{new}}(\mathtt{k}+1) \\
=&\frac{\mathtt{wp2}(\mathtt{k})}{\mathtt{dt}} \\
&+\frac{\mathtt{Kw1}(\mathtt{k}) + \nu_1}{\mathtt{2dzm}(\mathtt{k})\mathtt{dzt}(\mathtt{k})} \mathtt{wp2}(\mathtt{k}-1) \\
&-\frac{1}{\mathtt{2dzm}(\mathtt{k})} \left(\frac{\mathtt{Kw1}(\mathtt{k}+1) + \nu_1}{\mathtt{dzt}(\mathtt{k}+1)} + \frac{\mathtt{Kw1}(\mathtt{k}) + \nu_1}{\mathtt{dzt}(\mathtt{k})} \right) \mathtt{wp2}(\mathtt{k}) \\
&+\frac{\mathtt{Kw1}(\mathtt{k}+1) + \nu_1}{\mathtt{2dzm}(\mathtt{k})\mathtt{dzt}(\mathtt{k}+1)} \mathtt{wp2}(\mathtt{k}+1) \\
&+\mathtt{wp2t}(\mathtt{k})
\end{aligned}
\tag{111}
$$

where $\mathtt{wp2t}(\mathtt{k})$ is the finite difference equivalent to (110) at level $\mathtt{zm}(\mathtt{k})$.

6.1.1 Using an anisotropic solution for the horizontal wind

As an alternative to assuming $\bar{e} = \frac{3}{2}\overline{w'^2}$, we can calculate $\overline{v'^2}$ and $\overline{u'^2}$ and then compute $\bar{e}$ accordingly. The term with a C_4 coefficient in $\overline{w'^2}$ equation is then non-zero and must be accounted for. Starting with the 5^{th} term of the original $\overline{w'^2}$ equation:

$$
\frac{\partial \overline{w'^2}}{\partial t} = \cdots - \frac{C_4}{\tau} \left(\overline{w'^2} - \frac{2}{3}\bar{e} \right) \cdots \tag{112}
$$

From which we obtain the finite difference equivalent:

$$
\begin{aligned}
&-\frac{C_4}{\tau}\left(\overline{w'^2}-\frac{2}{3}\bar{e}\right)\Big|_{\mathtt{zm(k)}} \\
&=-\frac{\mathtt{C_4}}{\mathtt{taum(k)}}\left(\mathtt{wp2(k)}-\frac{2}{3}\mathtt{em(k)}\right) \\
&=-\frac{\mathtt{C_4}}{\mathtt{taum(k)}}\left(\mathtt{wp2(k)}-\frac{\mathtt{wp2(k)}+\mathtt{up2(k)}+\mathtt{vp2(k)}}{3}\right) \\
&=-\frac{2\,\mathtt{C_4}\,\mathtt{wp2(k)}}{3\mathtt{taum(k)}}+\frac{\mathtt{C_4}\,(\mathtt{up2(k)}+\mathtt{vp2(k)})}{3\mathtt{taum(k)}}
\end{aligned}
\tag{113}
$$

Separating out the contributions:

$$
\begin{aligned}
\mathtt{lhs(3,k\_wp2)} &= \mathtt{lhs(3,k\_wp2)}+\frac{\mathtt{C_4}\,(\mathtt{up2(k)}+\mathtt{vp2(k)})}{3\mathtt{taum(k)}} \\
\mathtt{rhs(k\_wp2)} &= \mathtt{rhs(k\_wp2)}+\frac{2\,\mathtt{C_4}\,\mathtt{wp2(k)}}{3\mathtt{taum(k)}}
\end{aligned}
\tag{114}
$$

6.2 $\overline{w'^3}$

The first two terms on the LHS of (108) (i.e. mean advection dissipation) are treated implicitly, and the last three terms on the LHS are teated semi-implicitly (they are linearized and the linearized portion is treated implicitly, the rest explicitly). The terms on the RHS are treated fully explicit discretization. Let's focus first on the third term on the LHS, L_3:

$$
L_3 \equiv \frac{C_8}{\tau}\left(C_{8b}\,Skw^4+1\right)\overline{w'^3}=\frac{C_8}{\tau}\left(C_{8b}\,\frac{\overline{w'^3}^5}{\overline{w'^2}^6}+\overline{w'^3}\right)
\tag{115}
$$

We linearize L_3 with respect to $\overline{w'^3}$:

$$
L_3\left(\overline{w'^3}^{t+\Delta t}\right)\approx L_3\left(\overline{w'^3}^{t}\right)+\frac{\partial L_3}{\partial\overline{w'^3}}\Big|_t\left(\overline{w'^3}^{t+\Delta t}-\overline{w'^3}^{t}\right)
\tag{116}
$$

where

$$
\frac{\partial L_3}{\partial\overline{w'^3}}\Big|_t=\frac{C_8}{\tau}\left(5\,C_{8b}\,\frac{\overline{w'^3}^4}{\overline{w'^2}^6}+1\right)
\tag{117}
$$

Combining (115), (117) with (116):

$$
\begin{aligned}
&L_3\left(\overline{w'^3}^{t+\Delta t}\right) \\
&=\frac{C_8}{\tau}\left(C_{8b}\,\frac{\overline{w'^3}^{t\,5}}{\overline{w'^2}^{t\,6}}+\overline{w'^3}^{t}\right)+\frac{C_8}{\tau}\left(5\,C_{8b}\,\frac{\overline{w'^3}^{t\,4}}{\overline{w'^2}^{t\,6}}+1\right)\left(\overline{w'^3}^{t+\Delta t}-\overline{w'^3}^{t}\right) \\
&=-\frac{C_8}{\tau}\left(4\,C_{8b}\,Skw^{t\,4}\right)\overline{w'^3}^{t}+\frac{C_8}{\tau}\left(5\,C_{8b}\,Skw^{t\,4}+1\right)\overline{w'^3}^{t+\Delta t}
\end{aligned}
\tag{118}
$$

For reasons of numerical stability we now linearize the fourth term on the LHS in a formulation that is fully implicit (108):

$$\begin{aligned}
&\frac{\partial}{\partial z}\left[\tilde{a}_3\left(\overline{w'^2}^{t+\Delta t}\right)^2\right] \\
&\approx \frac{\partial}{\partial z}\left[\tilde{a}_3\overline{w'^2}^{t2} + 2\tilde{a}_3\overline{w'^2}^{t}\left(\overline{w'^2}^{t+\Delta t} - \overline{w'^2}^{t}\right)\right] \\
&= \frac{\partial}{\partial z}\left(2\tilde{a}_3\overline{w'^2}^{t}\,\overline{w'^2}^{t+\Delta t}\right) - \frac{\partial}{\partial z}\left(\tilde{a}_3\overline{w'^2}^{t2}\right)
\end{aligned} \tag{119}$$

We repeat for the fifth term on the LHS of (108):

$$\begin{aligned}
&\frac{\partial}{\partial z}\left(a_1\frac{\left(\overline{w'^3}^{t+\Delta t}\right)^2}{\overline{w'^2}^{t}}\right) \\
&\approx \frac{\partial}{\partial z}\left[a_1\frac{\left(\overline{w'^3}^{t}\right)^2}{\overline{w'^2}^{t}} + 2a_1\frac{\overline{w'^3}^{t}}{\overline{w'^2}^{t}}\left(\overline{w'^3}^{t+\Delta t} - \overline{w'^3}^{t}\right)\right] \\
&= \frac{\partial}{\partial z}\left(2a_1\frac{\overline{w'^3}^{t}\,\overline{w'^3}^{t+\Delta t}}{\overline{w'^2}^{t}}\right) - \frac{\partial}{\partial z}\left(a_1\frac{\left(\overline{w'^3}^{t}\right)^2}{\overline{w'^2}^{t}}\right)
\end{aligned} \tag{120}$$

We can now assemble the time discrete equivalent to (108) using (118), (119) and (120):

$$\begin{aligned}
&\frac{\overline{w'^3}^{t+\Delta t}}{\Delta t} + \bar{w}\frac{\partial\overline{w'^3}^{t+\Delta t}}{\partial z} - \frac{1}{2}\frac{\partial}{\partial z}\left[(K_{w8}+\nu_8)\frac{\partial}{\partial z}\overline{w'^3}^{t+\Delta t}\right] + \frac{C_8}{\tau}\left(5\,C_{8b}\,Skw^{t4} + 1\right)\overline{w'^3}^{t+\Delta t} \\
&+ \frac{\partial}{\partial z}\left(\tilde{a}_3\overline{w'^2}^{t}\,\overline{w'^2}^{t+\Delta t}\right) + \frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}^{t}\,\overline{w'^3}^{t+\Delta t}}{\overline{w'^2}^{t}}\right) \\
&= \frac{\overline{w'^3}^{t}}{\Delta t} + \frac{1}{2}\frac{\partial}{\partial z}\left[(K_{w8}+\nu_8)\frac{\partial}{\partial z}\overline{w'^3}^{t}\right] + \frac{C_8}{\tau}\left(4\,C_{8b}\,Skw^{t4}\right)\overline{w'^3}^{t} + \left.\overline{w'^3}\right|_{\mathrm{expl}}
\end{aligned} \tag{121}$$

where

$$\left.\overline{w'^3}\right|_{\mathrm{expl}} = +(1-C_{11})\frac{3g}{\theta_0}\overline{w'^2\theta'_v}^{t} - 3(1-C_{11})\overline{w'^3}^{t}\frac{\partial\bar{w}}{\partial z} \tag{122}$$

Finally, we derive the finite difference form of (121):

$$
\begin{aligned}
&\frac{\mathtt{wp3}^{\mathtt{new}}(\mathtt{k})}{\mathtt{dt}} + \mathtt{wmt}(\mathtt{k})\frac{\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}+1) - \mathtt{wp3}^{\mathtt{new}}(\mathtt{k}-1)}{2\ \mathtt{dzt}(\mathtt{k})} \\
&-\frac{1}{2\mathtt{dzt}(\mathtt{k})}\Bigg((\mathtt{Kw8}(\mathtt{k}) + \nu_8)\ \frac{\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}+1) - \mathtt{wp3}^{\mathtt{new}}(\mathtt{k})}{\mathtt{dzm}(\mathtt{k})} \\
&\qquad\qquad - (\mathtt{Kw8}(\mathtt{k}-1) + \nu_8)\ \frac{\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}) - \mathtt{wp3}^{\mathtt{new}}(\mathtt{k}-1)}{\mathtt{dzm}(\mathtt{k}-1)}\Bigg) \\
&+\frac{\mathtt{C}_8}{\mathtt{taut}(\mathtt{k})}\left(5\,\mathtt{C}_{8\mathtt{b}}\,\mathtt{Skwt}(\mathtt{k})^4 + 1\right)\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}) \\
&+\frac{1}{\mathtt{dzt}(\mathtt{k})}\left(\mathtt{a3m}(\mathtt{k})\mathtt{wp2}(\mathtt{k})\mathtt{wp2}^{\mathtt{new}}(\mathtt{k}) - \mathtt{a3m}(\mathtt{k}-1)\mathtt{wp2}(\mathtt{k}-1)\mathtt{wp2}^{\mathtt{new}}(\mathtt{k}-1)\right) \\
&+\frac{1}{2\,\mathtt{dzt}(\mathtt{k})}\Bigg(\frac{\mathtt{a1m}(\mathtt{k})\left(\mathtt{wp3}(\mathtt{k}) + \mathtt{wp3}(\mathtt{k}+1)\right)\left(\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}) + \mathtt{wp3}^{\mathtt{new}}(\mathtt{k}+1)\right)}{\max\left(\mathtt{wp2}(\mathtt{k}), \epsilon\right)} \\
&\qquad\qquad -\frac{\mathtt{a1m}(\mathtt{k}-1)\left(\mathtt{wp3}(\mathtt{k}-1) + \mathtt{wp3}(\mathtt{k})\right)\left(\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}-1) + \mathtt{wp3}^{\mathtt{new}}(\mathtt{k})\right)}{\max\left(\mathtt{wp2}(\mathtt{k}-1), \epsilon\right)}\Bigg) \\
=&\frac{\mathtt{wp3}(\mathtt{k})}{\mathtt{dt}} + \frac{1}{2\mathtt{dzt}(\mathtt{k})}\Bigg((\mathtt{Kw8}(\mathtt{k}) + \nu_8)\ \frac{\mathtt{wp3}(\mathtt{k}+1) - \mathtt{wp3}(\mathtt{k})}{\mathtt{dzm}(\mathtt{k})} \\
&\qquad\qquad - (\mathtt{Kw8}(\mathtt{k}-1) + \nu_8)\ \frac{\mathtt{wp3}(\mathtt{k}) - \mathtt{wp3}(\mathtt{k}-1)}{\mathtt{dzm}(\mathtt{k}-1)}\Bigg) \\
&+\frac{\mathtt{C}_8}{\mathtt{taut}(\mathtt{k})}\left(4\,\mathtt{C}_{8\mathtt{b}}\,\mathtt{Skwt}(\mathtt{k})^4\right)\mathtt{wp3}(\mathtt{k}) + \mathtt{wp3t}(\mathtt{k})
\end{aligned}
\tag{123}
$$

where $\mathtt{wp3t}(\mathtt{k})$ is the finite difference equivalent to (122) at level $\mathtt{zt}(\mathtt{k})$.

In order to increase numerical stability in the model, a_1 has been brought outside of the derivative. Besides a_1, a_3 has been previously brought outside of the derivative for the same purpose. This is not mathematically correct, but it does help to increase stability. Brian Griffin. Feb. 21, 2008.

$$
\begin{aligned}
&\frac{\mathtt{wp3}^{\mathtt{new}}(\mathtt{k})}{\mathtt{dt}} + \mathtt{wmt}(\mathtt{k})\frac{\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}+1) - \mathtt{wp3}^{\mathtt{new}}(\mathtt{k}-1)}{2\;\mathtt{dzt}(\mathtt{k})} \\
&-\frac{1}{2\mathtt{dzt}(\mathtt{k})}\Bigg((\mathtt{Kw8}(\mathtt{k}) + \nu_8)\,\frac{\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}+1) - \mathtt{wp3}^{\mathtt{new}}(\mathtt{k})}{\mathtt{dzm}(\mathtt{k})} \\
&\qquad\qquad - (\mathtt{Kw8}(\mathtt{k}-1) + \nu_8)\,\frac{\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}) - \mathtt{wp3}^{\mathtt{new}}(\mathtt{k}-1)}{\mathtt{dzm}(\mathtt{k}-1)}\Bigg) \\
&+\frac{\mathtt{C_8}}{\mathtt{taut}(\mathtt{k})}\left(5\,\mathtt{C_{8b}}\,\mathtt{Skwt}(\mathtt{k})^4 + 1\right)\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}) \\
&+\left(\frac{\mathtt{a3m}(\mathtt{k}) + \mathtt{a3m}(\mathtt{k}-1)}{2}\right)\frac{2}{\mathtt{dzt}(\mathtt{k})}\left(\mathtt{wp2}(\mathtt{k})\mathtt{wp2}^{\mathtt{new}}(\mathtt{k}) - \mathtt{wp2}(\mathtt{k}-1)\mathtt{wp2}^{\mathtt{new}}(\mathtt{k}-1)\right) \\
&+\left(\frac{\mathtt{a1m}(\mathtt{k}) + \mathtt{a1m}(\mathtt{k}-1)}{2}\right)\frac{1}{2\,\mathtt{dzt}(\mathtt{k})}\Bigg(\frac{(\mathtt{wp3}(\mathtt{k}) + \mathtt{wp3}(\mathtt{k}+1))\,(\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}) + \mathtt{wp3}^{\mathtt{new}}(\mathtt{k}+1))}{\max\left(\mathtt{wp2}(\mathtt{k}), \epsilon\right)} \\
&\qquad\qquad - \frac{(\mathtt{wp3}(\mathtt{k}-1) + \mathtt{wp3}(\mathtt{k}))\,(\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}-1) + \mathtt{wp3}^{\mathtt{new}}(\mathtt{k}))}{\max\left(\mathtt{wp2}(\mathtt{k}-1), \epsilon\right)}\Bigg) \\
=&\frac{\mathtt{wp3}(\mathtt{k})}{\mathtt{dt}} + \frac{1}{2\mathtt{dzt}(\mathtt{k})}\Bigg((\mathtt{Kw8}(\mathtt{k}) + \nu_8)\,\frac{\mathtt{wp3}(\mathtt{k}+1) - \mathtt{wp3}(\mathtt{k})}{\mathtt{dzm}(\mathtt{k})} \\
&\qquad\qquad - (\mathtt{Kw8}(\mathtt{k}-1) + \nu_8)\,\frac{\mathtt{wp3}(\mathtt{k}) - \mathtt{wp3}(\mathtt{k}-1)}{\mathtt{dzm}(\mathtt{k}-1)}\Bigg) \\
&+\frac{\mathtt{C_8}}{\mathtt{taut}(\mathtt{k})}\left(4\,\mathtt{C_{8b}}\,\mathtt{Skwt}(\mathtt{k})^4\right)\mathtt{wp3}(\mathtt{k}) \\
&+\left(\frac{\mathtt{a3m}(\mathtt{k}) + \mathtt{a3m}(\mathtt{k}-1)}{2}\right)\frac{\mathtt{wp2}(\mathtt{k})^2 - \mathtt{wp2}(\mathtt{k}-1)^2}{\mathtt{dzt}(\mathtt{k})} \\
&+\left(\frac{\mathtt{a1m}(\mathtt{k}) + \mathtt{a1m}(\mathtt{k}-1)}{2}\right)\frac{1}{4\,\mathtt{dzt}(\mathtt{k})}\left(\frac{(\mathtt{wp3}(\mathtt{k}) + \mathtt{wp3}(\mathtt{k}+1))^2}{\max\left(\mathtt{wp2}(\mathtt{k}), \epsilon\right)} - \frac{(\mathtt{wp3}(\mathtt{k}-1) + \mathtt{wp3}(\mathtt{k}))^2}{\max\left(\mathtt{wp2}(\mathtt{k}-1), \epsilon\right)}\right) \\
&+\mathtt{wp3t}(\mathtt{k})
\end{aligned}
\tag{124}
$$

6.3 Matrix form

The final step is to rewrite (111) and (123) in matrix form:

$$\underbrace{\begin{pmatrix} \cdots & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}{+}1) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}{+}1) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}{+}2) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}{+}2) & \cdots \\ \cdots & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}{-}1) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}{+}1) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}{+}1) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}{+}2) & \cdots \\ \cdots & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}{-}1) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}{-}1) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}{+}1) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}{+}1) & \cdots \\ \cdots & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}{-}2) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}{-}1) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}{-}1) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}{+}1) & \cdots \\ \cdots & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}{-}2) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}{-}2) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}{-}1) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}{-}1) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}) & \cdots \end{pmatrix}}_{\mathtt{LHS}_{\mathtt{wp23}}\text{(Stored in compact format)}} \begin{pmatrix} \vdots \\ \mathtt{wp3}^{\mathtt{new}}(\mathtt{k}-1) \\ \mathtt{wp2}^{\mathtt{new}}(\mathtt{k}-1) \\ \mathtt{wp3}^{\mathtt{new}}(\mathtt{k}) \\ \mathtt{wp2}^{\mathtt{new}}(\mathtt{k}) \\ \mathtt{wp3}^{\mathtt{new}}(\mathtt{k}+1) \\ \mathtt{wp2}^{\mathtt{new}}(\mathtt{k}+1) \\ \vdots \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \vdots \\ \mathtt{wp3}^{\mathtt{expl}}(\mathtt{k}{-}1) \\ \mathtt{wp2}^{\mathtt{expl}}(\mathtt{k}{-}1) \\ \mathtt{wp3}^{\mathtt{expl}}(\mathtt{k}) \\ \mathtt{wp2}^{\mathtt{expl}}(\mathtt{k}) \\ \mathtt{wp3}^{\mathtt{expl}}(\mathtt{k}{+}1) \\ \mathtt{wp2}^{\mathtt{expl}}(\mathtt{k}{+}1) \\ \vdots \end{pmatrix}}_{\mathtt{RHS}_{\mathtt{wp23}}} \tag{125}$$

$lhs_{\mathtt{wp23}}$ is a band-diagonal matrix with two rows above and two below the main diagonal. $lhs_{\mathtt{wp23}}$ is stored in compact form in an array with dimensions $(5, 2\,\mathtt{nzmax})$. $rhs_{\mathtt{wp23}}$ is a vector with dimension $(2\,\mathtt{nzmax})$. $lhs_{\mathtt{wp23}}$ can be inverted efficiently using a LU decomposition algorithm for band diagonal matrices.

Contributions to $lhs_{\mathtt{wp23}}$ from (111):

$$\mathtt{lhs}(\mathtt{k\_wp2},5) = \mathtt{lhs}(\mathtt{k\_wp2},5) - \frac{\mathtt{Kw1}(\mathtt{k}) + \nu_1}{\mathtt{2dzm}(\mathtt{k})\mathtt{dzt}(\mathtt{k})} - \frac{\mathtt{wmm}(\mathtt{k})}{2\ \mathtt{dzm}(\mathtt{k})} \tag{126}$$

$$\mathtt{lhs}(\mathtt{k\_wp2},4) = \mathtt{lhs}(\mathtt{k\_wp2},4) - \frac{1}{\mathtt{dzm}(\mathtt{k})} \tag{127}$$

$$\mathtt{lhs}(\mathtt{k\_wp2},3) = \mathtt{lhs}(\mathtt{k\_wp2},3) + \frac{1}{\mathtt{dt}} + \frac{\mathtt{C}_1}{\mathtt{taum}(\mathtt{k})} + \frac{1}{\mathtt{2dzm}(\mathtt{k})}\left(\frac{\mathtt{Kw1}(\mathtt{k}+1) + \nu_1}{\mathtt{dzt}(\mathtt{k}+1)} + \frac{\mathtt{Kw1}(\mathtt{k}) + \nu_1}{\mathtt{dzt}(\mathtt{k})}\right) \tag{128}$$

$$\mathtt{lhs}(\mathtt{k\_wp2},2) = \mathtt{lhs}(\mathtt{k\_wp2},2) + \frac{1}{\mathtt{dzm}(\mathtt{k})} \tag{129}$$

$$\mathtt{lhs(k\_wp2,1)} = \mathtt{lhs(k\_wp2,1)} - \frac{\mathtt{Kw1(k+1)}+\nu_1}{\mathtt{2dzm(k)dzt(k+1)}} + \frac{\mathtt{wmm(k)}}{\mathtt{2\ dzm(k)}} \tag{130}$$

Contributions to $rhs_{\mathtt{wp23}}$ from (111):

$$\begin{aligned}
&\mathtt{rhs(k_wp2)} = \mathtt{rhs(k_wp2)} \\
&+\frac{\mathtt{wp2(k)}}{\mathtt{dt}} \\
&+\frac{\mathtt{Kw1(k)}+\nu_1}{\mathtt{2dzm(k)dzt(k)}}\mathtt{wp2(k-1)} \\
&-\frac{1}{\mathtt{2dzm(k)}}\left(\frac{\mathtt{Kw1(k+1)}+\nu_1}{\mathtt{dzt(k+1)}} + \frac{\mathtt{Kw1(k)}+\nu_1}{\mathtt{dzt(k)}}\right)\mathtt{wp2(k)} \\
&+\frac{\mathtt{Kw1(k+1)}+\nu_1}{\mathtt{2dzm(k)dzt(k+1)}}\mathtt{wp2(k+1)} \\
&+\mathtt{wp2t(k)}
\end{aligned} \tag{131}$$

where

$$\mathtt{k\_wp2} = \mathtt{2k} \tag{132}$$

Contributions to $lhs_{\mathtt{wp23}}$ from (123):

$$\begin{aligned}
&\mathtt{lhs(k_wp3,5)} = \mathtt{lhs(k_wp3,5)} \\
&-\frac{\mathtt{Kw8(k-1)}+\nu_8}{\mathtt{2dzt(k)dzm(k-1)}} - \frac{\mathtt{wmt(k)}}{\mathtt{2\ dzt(k)}} - \frac{1}{\mathtt{2\,dzt(k)}}\frac{\mathtt{a1m(k-1)}\left(\mathtt{wp3(k-1)}+\mathtt{wp3(k)}\right)}{\max\left(\mathtt{wp2(k-1)},\epsilon\right)}
\end{aligned} \tag{133}$$

$$\mathtt{lhs(k\_wp3,4)} = \mathtt{lhs(k\_wp3,4)} - \frac{\mathtt{2a3m(k-1)wp2(k-1)}}{\mathtt{dzt(k)}} \tag{134}$$

$$\begin{aligned}
&\mathtt{lhs(k_wp3,3)} = \mathtt{lhs(k_wp3,3)} \\
&+\frac{1}{\mathtt{dt}} + \frac{\mathtt{C_8}}{\mathtt{taut(k)}}\left(5\,\mathtt{C_{8b}}\,\mathtt{Skwt(k)}^4+1\right) + \frac{1}{\mathtt{2dzt(k)}}\left(\frac{\mathtt{Kw8(k)}+\nu_8}{\mathtt{dzm(k)}} + \frac{\mathtt{Kw8(k-1)}+\nu_8}{\mathtt{dzm(k-1)}}\right) \\
&+\frac{1}{\mathtt{2\,dzt(k)}}\left(\frac{\mathtt{a1m(k)}\left(\mathtt{wp3(k)}+\mathtt{wp3(k+1)}\right)}{\max\left(\mathtt{wp2(k)},\epsilon\right)} - \frac{\mathtt{a1m(k-1)}\left(\mathtt{wp3(k-1)}+\mathtt{wp3(k)}\right)}{\max\left(\mathtt{wp2(k-1)},\epsilon\right)}\right)
\end{aligned} \tag{135}$$

$$\mathtt{lhs(k\_wp3,2)} = \mathtt{lhs(k\_wp3,2)} + \frac{\mathtt{2a3m(k)wp2(k)}}{\mathtt{dzt(k)}} \tag{136}$$

$$\begin{aligned}
&\mathtt{lhs(k_wp3,1)} = \mathtt{lhs(k_wp3,1)} \\
&-\frac{\mathtt{Kw8(k)}+\nu_8}{\mathtt{2dzt(k)dzm(k)}} + \frac{\mathtt{wmt(k)}}{\mathtt{2\ dzt(k)}} + \frac{1}{\mathtt{2\,dzt(k)}}\frac{\mathtt{a1m(k)}\left(\mathtt{wp3(k)}+\mathtt{wp3(k+1)}\right)}{\max\left(\mathtt{wp2(k)},\epsilon\right)}
\end{aligned} \tag{137}$$

Contributions to $rhs_{\mathtt{wp23}}$ from (123):

$$\begin{aligned}
&\mathtt{rhs(k_wp3)} = \mathtt{rhs(k_wp3)} \\
&+\frac{\mathtt{wp3(k)}}{\mathtt{dt}} + \frac{1}{\mathtt{2dzt(k)}}\Bigg((\mathtt{Kw8(k)} + \nu_8)\,\frac{\mathtt{wp3(k+1)} - \mathtt{wp3(k)}}{\mathtt{dzm(k)}} \\
&\qquad\qquad - (\mathtt{Kw8(k-1)} + \nu_8)\,\frac{\mathtt{wp3(k)} - \mathtt{wp3(k-1)}}{\mathtt{dzm(k-1)}}\Bigg) \\
&+\frac{\mathtt{C_8}}{\mathtt{taut(k)}}\left(4\,\mathtt{C_{8b}}\,\mathtt{Skwt(k)}^4\right)\mathtt{wp3(k)} \\
&+\frac{\mathtt{a3m(k)wp2(k)}^2 - \mathtt{a3m(k-1)wp2(k-1)}^2}{\mathtt{dzt(k)}} \\
&+\frac{1}{4\,\mathtt{dzt(k)}}\left(\frac{\mathtt{a1m(k)}\,(\mathtt{wp3(k)} + \mathtt{wp3(k+1)})^2}{\max(\mathtt{wp2(k)}, \epsilon)} - \frac{\mathtt{a1m(k-1)}\,(\mathtt{wp3(k-1)} + \mathtt{wp3(k)})^2}{\max(\mathtt{wp2(k-1)}, \epsilon)}\right) \\
&+\mathtt{wp3t(k)}
\end{aligned} \tag{138}$$

where

$$\mathtt{k\_wp3} = 2\mathtt{k} - 1 \tag{139}$$

In order to increase numerical stability in the model, a_1 has been brought outside of the derivative. Besides a_1, a_3 has been previously brought outside of the derivative for the same purpose. This is not mathematically correct, but it does help to increase stability. Brian Griffin. Feb. 21, 2008. Contributions to $lhs_{\mathtt{wp23}}$ from (123):

$$\begin{aligned}
&\mathtt{lhs(k_wp3,5)} = \mathtt{lhs(k_wp3,5)} \\
&-\frac{\mathtt{Kw8(k-1)} + \nu_8}{\mathtt{2dzt(k)dzm(k-1)}} - \frac{\mathtt{wmt(k)}}{2\ \mathtt{dzt(k)}} - \left(\frac{\mathtt{a1m(k)} + \mathtt{a1m(k-1)}}{2}\right)\frac{1}{2\,\mathtt{dzt(k)}}\frac{(\mathtt{wp3(k-1)} + \mathtt{wp3(k)})}{\max(\mathtt{wp2(k-1)}, \epsilon)}
\end{aligned} \tag{140}$$

$$\mathtt{lhs(k\_wp3,4)} = \mathtt{lhs(k\_wp3,4)} - \left(\frac{\mathtt{a3m(k)} + \mathtt{a3m(k-1)}}{2}\right)\frac{\mathtt{2wp2(k-1)}}{\mathtt{dzt(k)}} \tag{141}$$

$$\begin{aligned}
&\mathtt{lhs(k_wp3,3)} = \mathtt{lhs(k_wp3,3)} \\
&+\frac{1}{\mathtt{dt}} + \frac{\mathtt{C_8}}{\mathtt{taut(k)}}\left(5\,\mathtt{C_{8b}}\,\mathtt{Skwt(k)}^4 + 1\right) + \frac{1}{\mathtt{2dzt(k)}}\left(\frac{\mathtt{Kw8(k)} + \nu_8}{\mathtt{dzm(k)}} + \frac{\mathtt{Kw8(k-1)} + \nu_8}{\mathtt{dzm(k-1)}}\right) \\
&+\left(\frac{\mathtt{a1m(k)} + \mathtt{a1m(k-1)}}{2}\right)\frac{1}{2\,\mathtt{dzt(k)}}\left(\frac{(\mathtt{wp3(k)} + \mathtt{wp3(k+1)})}{\max(\mathtt{wp2(k)}, \epsilon)} - \frac{(\mathtt{wp3(k-1)} + \mathtt{wp3(k)})}{\max(\mathtt{wp2(k-1)}, \epsilon)}\right)
\end{aligned} \tag{142}$$

$$\mathtt{lhs(k\_wp3,2)} = \mathtt{lhs(k\_wp3,2)} + \left(\frac{\mathtt{a3m(k)} + \mathtt{a3m(k-1)}}{2}\right)\frac{\mathtt{2wp2(k)}}{\mathtt{dzt(k)}} \tag{143}$$

$$\begin{aligned}
&\mathtt{lhs(k_wp3,1)} = \mathtt{lhs(k_wp3,1)} \\
&-\frac{\mathtt{Kw8(k)} + \nu_8}{\mathtt{2dzt(k)dzm(k)}} + \frac{\mathtt{wmt(k)}}{2\ \mathtt{dzt(k)}} + \left(\frac{\mathtt{a1m(k)} + \mathtt{a1m(k-1)}}{2}\right)\frac{1}{2\,\mathtt{dzt(k)}}\frac{(\mathtt{wp3(k)} + \mathtt{wp3(k+1)})}{\max(\mathtt{wp2(k)}, \epsilon)}
\end{aligned} \tag{144}$$

Contributions to $rhs_{\mathtt{wp23}}$ from (123):

$$
\begin{aligned}
&\mathtt{rhs(k\_wp3)} = \mathtt{rhs(k\_wp3)} \\
&+\frac{\mathtt{wp3(k)}}{\mathtt{dt}} + \frac{1}{\mathtt{2dzt(k)}}\Bigg((\mathtt{Kw8(k)} + \nu_8)\, \frac{\mathtt{wp3(k+1)} - \mathtt{wp3(k)}}{\mathtt{dzm(k)}} \\
&\qquad\qquad\qquad\qquad - (\mathtt{Kw8(k-1)} + \nu_8)\, \frac{\mathtt{wp3(k)} - \mathtt{wp3(k-1)}}{\mathtt{dzm(k-1)}} \Bigg) \\
&+\frac{\mathtt{C_8}}{\mathtt{taut(k)}} \left(4\, \mathtt{C_{8b}}\, \mathtt{Skwt(k)}^4\right) \mathtt{wp3(k)} \\
&+\left(\frac{\mathtt{a3m(k)} + \mathtt{a3m(k-1)}}{2}\right) \frac{\mathtt{wp2(k)}^2 - \mathtt{wp2(k-1)}^2}{\mathtt{dzt(k)}} \\
&+\left(\frac{\mathtt{a1m(k)} + \mathtt{a1m(k-1)}}{2}\right) \frac{1}{4\,\mathtt{dzt(k)}} \left(\frac{(\mathtt{wp3(k)} + \mathtt{wp3(k+1)})^2}{\max\left(\mathtt{wp2(k)}, \epsilon\right)} - \frac{(\mathtt{wp3(k-1)} + \mathtt{wp3(k)})^2}{\max\left(\mathtt{wp2(k-1)}, \epsilon\right)} \right) \\
&+\mathtt{wp3t(k)}
\end{aligned}
\tag{145}
$$

7 High-order Solution to the Horizontal Wind

As an alternative to assuming $\bar{e} = \frac{3}{2}\overline{w'^2}$, we can obtain an anisotropic solution using a semi-implicit discretization for $\overline{u'^2}$ and $\overline{v'^2}$. Similarly to (6), start with equations (14) and (15) (for simplicity, neglect the $|_{\mathrm{pd}}$ and $|_{\mathrm{cl}}$ terms), substitute (32) and (33) respectively.

7.1 $\overline{u'^2}$

Assume a steady-state and rearrange $\overline{u'^2}$ for a semi-implicit solution to obtain:

$$
\begin{aligned}
&\underbrace{\frac{C_4}{\tau}\left(\overline{u'^2} - \frac{2}{3}\bar{e}\right)}_{dp1} \underbrace{+\frac{2}{3}\left(C_{14}\frac{\bar{e}}{\tau}\right)}_{pr1} \underbrace{+\bar{w}\frac{\partial \overline{u'^2}}{\partial z}}_{ma} \underbrace{+\frac{1}{3}\beta\frac{\partial}{\partial z}\left(a_1 \frac{\overline{w'^3}}{\overline{w'^2}}\,\overline{u'^2}\right)}_{ta} \underbrace{-\frac{\partial}{\partial z}\left[\left(K_{w9} + \nu_9\right)\frac{\partial}{\partial z}\overline{u'^2}\right]}_{dp2} \\
&= \underbrace{-\left(1 - \frac{1}{3}\beta\right)\frac{\partial}{\partial z}\left(a_2 \frac{\overline{w'^3}}{\overline{w'^2}^2}\,\overline{w'u'}^2\right)}_{ta} \underbrace{-2(1 - C_5)\,\overline{w'u'}\,\frac{\partial \bar{u}}{\partial z}}_{tp} \\
&\quad \underbrace{+\frac{2}{3}C_5\left(\frac{g}{\theta_0}\overline{w'\theta'_v} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{v'w'}\frac{\partial \bar{v}}{\partial z}\right)}_{pr2}
\end{aligned}
\tag{146}
$$

As in the case of $\overline{r_t'^2}$ and $\overline{\theta_l'^2}$, the horizontal wind variance terms are solved using a tridiagonal matrix.

7.1.1 Terms 1 and 2: dp1 and pr1, respectively

$$
\begin{aligned}
&\frac{C_4}{\tau}\left(\overline{u'^2}-\frac{2}{3}\bar{e}\right)+\frac{2}{3}C_{14}\frac{\bar{e}}{\tau}\bigg|_{\mathtt{zm(k)}}\\
=&\frac{\mathtt{C_4}}{\mathtt{taum(k)}}\mathtt{up2(k)}-\frac{\mathtt{C_4}}{\mathtt{taum(k)}}\frac{2}{3}\mathtt{em(k)}+\frac{2}{3}\mathtt{C_{14}}\frac{\mathtt{em(k)}}{\mathtt{taum(k)}}\\
=&\frac{\mathtt{C_4}}{\mathtt{taum(k)}}\mathtt{up2(k)}-\frac{2}{3}\mathtt{em(k)}\left(\frac{\mathtt{C_4}}{\mathtt{taum(k)}}-\frac{\mathtt{C_{14}}}{\mathtt{taum(k)}}\right)\\
=&\frac{\mathtt{C_4}}{\mathtt{taum(k)}}\mathtt{up2(k)}-\frac{2}{3}\left[\frac{\mathtt{up2(k)}+\mathtt{vp2(k)}+\mathtt{wp2(k)}}{2}\right]\left(\frac{\mathtt{C_4}}{\mathtt{taum(k)}}-\frac{\mathtt{C_{14}}}{\mathtt{taum(k)}}\right)\\
=&\mathtt{up2(k)}\frac{1}{3}\left(\frac{2\mathtt{C_4}+\mathtt{C_{14}}}{\mathtt{taum(k)}}\right)-\left(\frac{1}{3}\left(\mathtt{C_4}-\mathtt{C_{14}}\right)\left(\frac{\mathtt{vp2(k)}+\mathtt{wp2(k)}}{\mathtt{taum(k)}}\right)\right)
\end{aligned}
\tag{147}
$$

Separating out the contributions:

$$
\begin{aligned}
&\mathtt{lhs(2,k)}=\mathtt{lhs(2,k)}+\frac{2\mathtt{C_4}+\mathtt{C_{14}}}{3\mathtt{taum(k)}}\\
&\mathtt{rhs(k)}=\mathtt{rhs(k)}+\frac{1}{3}\left(\mathtt{C_4}-\mathtt{C_{14}}\right)\left(\frac{\mathtt{vp2(k)}+\mathtt{wp2(k)}}{\mathtt{taum(k)}}\right)
\end{aligned}
\tag{148}
$$

7.1.2 Term 3: ma

$$
\begin{aligned}
&\bar{w}\frac{\partial\overline{u'^2}}{\partial z}\bigg|_{\mathtt{zm(k)}}\\
=&\frac{\mathtt{wmm(k)}}{\mathtt{dzm(k)}}\left(\frac{1}{2}\left(\mathtt{up2(k)}+\mathtt{up2(k+1)}\right)-\frac{1}{2}\left(\mathtt{up2(k-1)}+\mathtt{up2(k)}\right)\right)\\
=&\frac{\mathtt{wmm(k)}}{2\mathtt{dzm(k)}}\mathtt{up2(k+1)}-\frac{\mathtt{wmm(k)}}{2\mathtt{dzm(k)}}\mathtt{up2(k-1)}
\end{aligned}
\tag{149}
$$

Separating out the contributions:

$$
\begin{aligned}
&\mathtt{lhs(1,k)}=\mathtt{lhs(1,k)}+\frac{\mathtt{wmm(k)}}{2\mathtt{dzm(k)}}\\
&\mathtt{lhs(3,k)}=\mathtt{lhs(3,k)}-\frac{\mathtt{wmm(k)}}{2\mathtt{dzm(k)}}
\end{aligned}
\tag{150}
$$

7.1.3 Term 4: ta, implicit component

$$
\begin{aligned}
&\frac{1}{3}\beta\frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}}{\overline{w'^2}}\,\overline{u'^2}\right)\Bigg|_{\texttt{zm(k)}}\\
&=\frac{\beta}{\texttt{6dzm(k)}}\Bigg[\frac{(\texttt{a1m(k)}+\texttt{a1m(k+1)})\,\texttt{wp3(k+1)}\,(\texttt{up2(k)}+\texttt{up2(k+1)})}{\max\left(\texttt{wp2(k)}+\texttt{wp2(k+1)},2\epsilon\right)}\\
&\qquad-\frac{(\texttt{a1m(k-1)}+\texttt{a1m(k)})\,\texttt{wp3(k)}\,(\texttt{up2(k-1)}+\texttt{up2(k)})}{\max\left(\texttt{wp2(k-1)}+\texttt{wp2(k)},2\epsilon\right)}\Bigg]
\end{aligned}
\tag{151}
$$

Separating out the contributions:

$$
\begin{aligned}
\texttt{lhs(3,k)}&=\texttt{lhs(3,k)}-\frac{\beta}{\texttt{6dzm(k)}}\frac{(\texttt{a1m(k-1)}+\texttt{a1m(k)})\,\texttt{wp3(k)}}{\max\left(\texttt{wp2(k-1)}+\texttt{wp2(k)},2\epsilon\right)}\\
\texttt{lhs(2,k)}&=\texttt{lhs(2,k)}+\frac{\beta}{\texttt{6dzm(k)}}\left(\frac{(\texttt{a1m(k)}+\texttt{a1m(k+1)})\,\texttt{wp3(k+1)}}{\max\left(\texttt{wp2(k)}+\texttt{wp2(k+1)},2\epsilon\right)}-\frac{(\texttt{a1m(k-1)}+\texttt{a1m(k)})\,\texttt{wp3(k)}}{\max\left(\texttt{wp2(k-1)}+\texttt{wp2(k)},2\epsilon\right)}\right)\\
\texttt{lhs(1,k)}&=\texttt{lhs(1,k)}+\frac{\beta}{\texttt{6dzm(k)}}\frac{(\texttt{a1m(k)}+\texttt{a1m(k+1)})\,\texttt{wp3(k+1)}}{\max\left(\texttt{wp2(k)}+\texttt{wp2(k+1)},2\epsilon\right)}
\end{aligned}
\tag{152}
$$

In order to increase numerical stability in the model, a_1 has been brought outside of the derivative. This is not mathematically correct, but it does help to increase stability. Brian Griffin. Feb. 21, 2008.

$$
\begin{aligned}
&a_1\frac{1}{3}\beta\frac{\partial}{\partial z}\left(\frac{\overline{w'^3}}{\overline{w'^2}}\,\overline{u'^2}\right)\Bigg|_{\texttt{zm(k)}}\\
&=\texttt{a1m(k)}\frac{\beta}{\texttt{3dzm(k)}}\left[\frac{\texttt{wp3(k+1)}\,(\texttt{up2(k)}+\texttt{up2(k+1)})}{\max\left(\texttt{wp2(k)}+\texttt{wp2(k+1)},2\epsilon\right)}-\frac{\texttt{wp3(k)}\,(\texttt{up2(k-1)}+\texttt{up2(k)})}{\max\left(\texttt{wp2(k-1)}+\texttt{wp2(k)},2\epsilon\right)}\right]
\end{aligned}
\tag{153}
$$

Separating out the contributions:

$$
\begin{aligned}
\texttt{lhs(3,k)}&=\texttt{lhs(3,k)}-\texttt{a1m(k)}\frac{\beta}{\texttt{3dzm(k)}}\frac{\texttt{wp3(k)}}{\max\left(\texttt{wp2(k-1)}+\texttt{wp2(k)},2\epsilon\right)}\\
\texttt{lhs(2,k)}&=\texttt{lhs(2,k)}+\texttt{a1m(k)}\frac{\beta}{\texttt{3dzm(k)}}\left(\frac{\texttt{wp3(k+1)}}{\max\left(\texttt{wp2(k)}+\texttt{wp2(k+1)},2\epsilon\right)}-\frac{\texttt{wp3(k)}}{\max\left(\texttt{wp2(k-1)}+\texttt{wp2(k)},2\epsilon\right)}\right)\\
\texttt{lhs(1,k)}&=\texttt{lhs(1,k)}+\texttt{a1m(k)}\frac{\beta}{\texttt{3dzm(k)}}\frac{\texttt{wp3(k+1)}}{\max\left(\texttt{wp2(k)}+\texttt{wp2(k+1)},2\epsilon\right)}
\end{aligned}
\tag{154}
$$

7.1.4 Term 5: dp2

$$
\begin{aligned}
&-\frac{\partial}{\partial z}\left[\left(K_{w9}+\nu_9\right)\frac{\partial}{\partial z}\overline{u'^2}\right]\Bigg|_{\texttt{zm(k)}}\\
&=-\frac{1}{\texttt{dzm(k)}}\left(\frac{(\texttt{Kw9(k+1)}+\nu_9)\,(\texttt{up2(k+1)}-\texttt{up2(k)})}{\texttt{dzt(k+1)}}-\frac{(\texttt{Kw9(k)}+\nu_9)\,(\texttt{up2(k)}-\texttt{up2(k-1)})}{\texttt{dzt(k)}}\right)
\end{aligned}
$$

$$(155)$$

Separating out the contributions:

$$\begin{aligned}
\mathtt{lhs(3,k)} &= \mathtt{lhs(3,k)} - \frac{\mathtt{Kw9(k)} + \nu_9}{\mathtt{dzm(k)dzt(k)}} \\
\mathtt{lhs(2,k)} &= \mathtt{lhs(2,k)} + \frac{1}{\mathtt{dzm(k)}}\left(\frac{\mathtt{Kw9(k+1)} + \nu_9}{\mathtt{dzt(k+1)}} + \frac{\mathtt{Kw9(k)} + \nu_9}{\mathtt{dzt(k)}}\right) \\
\mathtt{lhs(1,k)} &= \mathtt{lhs(1,k)} - \frac{\mathtt{Kw9(k+1)} + \nu_9}{\mathtt{dzm(k)dzt(k+1)}}
\end{aligned} \tag{156}$$

7.1.5 Term 6: ta, explicit component

$$\begin{aligned}
&-\left(1-\frac{1}{3}\beta\right)\frac{\partial}{\partial z}\left(a_2\frac{\overline{w'^3}}{\overline{w'^2}^2}\,\overline{w'u'}^2\right)\Bigg|_{\mathtt{zm(k)}} \\
=&-\frac{1-\frac{1}{3}\beta}{\mathtt{4dzm(k)}}\Bigg[\frac{\left(\mathtt{a1m(k)}+\mathtt{a1m(k+1)}\right)^2\mathtt{wp3(k+1)}\left(\mathtt{upwp(k)}+\mathtt{upwp(k+1)}\right)^2}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)^2} \\
&\qquad -\frac{\left(\mathtt{a1m(k-1)}+\mathtt{a1m(k)}\right)^2\mathtt{wp3(k)}\left(\mathtt{upwp(k-1)}+\mathtt{upwp(k)}\right)^2}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)^2}\Bigg]
\end{aligned} \tag{157}$$

Separating out the contributions:

$$\begin{aligned}
&\mathtt{rhs(k)} \\
=&\mathtt{rhs(k)} - \frac{1-\frac{1}{3}\beta}{\mathtt{4dzm(k)}}\Bigg[\frac{\left(\mathtt{a1m(k)}+\mathtt{a1m(k+1)}\right)^2\mathtt{wp3(k+1)}\left(\mathtt{upwp(k)}+\mathtt{upwp(k+1)}\right)^2}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)^2} \\
&\qquad -\frac{\left(\mathtt{a1m(k-1)}+\mathtt{a1m(k)}\right)^2\mathtt{wp3(k)}\left(\mathtt{upwp(k-1)}+\mathtt{upwp(k)}\right)^2}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)^2}\Bigg]
\end{aligned} \tag{158}$$

In order to increase numerical stability in the model, a_1 has been brought outside of the derivative. This is not mathematically correct, but it does help to increase stability. Brian Griffin. Feb. 21, 2008.

$$\begin{aligned}
&-a_2\left(1-\frac{1}{3}\beta\right)\frac{\partial}{\partial z}\left(\frac{\overline{w'^3}}{\overline{w'^2}^2}\,\overline{w'u'}^2\right)\Bigg|_{\mathtt{zm(k)}} \\
=&-\mathtt{a1m(k)}^2\frac{1-\frac{1}{3}\beta}{\mathtt{dzm(k)}}\Bigg[\frac{\mathtt{wp3(k+1)}\left(\mathtt{upwp(k)}+\mathtt{upwp(k+1)}\right)^2}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)^2} \\
&\qquad -\frac{\mathtt{wp3(k)}\left(\mathtt{upwp(k-1)}+\mathtt{upwp(k)}\right)^2}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)^2}\Bigg]
\end{aligned} \tag{159}$$

Separating out the contributions:

$$\begin{aligned}
&\mathtt{rhs(k)} \\
=&\mathtt{rhs(k)} - \mathtt{a1m(k)}^2\frac{1-\frac{1}{3}\beta}{\mathtt{dzm(k)}}\Bigg[\frac{\mathtt{wp3(k+1)}\left(\mathtt{upwp(k)}+\mathtt{upwp(k+1)}\right)^2}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)^2} \\
&\qquad -\frac{\mathtt{wp3(k)}\left(\mathtt{upwp(k-1)}+\mathtt{upwp(k)}\right)^2}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)^2}\Bigg]
\end{aligned} \tag{160}$$

7.1.6 Term 7: tp

$$-2\,(1-C_5)\,\overline{w'u'}\,\frac{\partial \bar{u}}{\partial z}\bigg|_{\mathtt{zm(k)}} = -2\,(1-\mathtt{C_5})\,\mathtt{upwp(k)}\frac{\mathtt{um(k+1)}-\mathtt{um(k)}}{\mathtt{dzm(k)}} \tag{161}$$

Separating out the contributions:

$$\mathtt{rhs(k)} = \mathtt{rhs(k)} - 2\,(1-\mathtt{C_5})\,\mathtt{upwp(k)}\frac{\mathtt{um(k+1)}-\mathtt{um(k)}}{\mathtt{dzm(k)}} \tag{162}$$

7.1.7 Term 8: pr2

$$\begin{aligned}&\frac{2}{3}C_5\left(\frac{g}{\theta_0}\overline{w'\theta_v'} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{v'w'}\frac{\partial \bar{v}}{\partial z}\right)|_{\mathtt{zm(k)}}\\ =&\frac{2}{3}\mathtt{C_5}\left(\frac{\mathtt{grav}}{\mathtt{T0}}\mathtt{wpthvp(k)} - \mathtt{upwp(k)}\frac{\mathtt{um(k+1)}-\mathtt{um(k)}}{\mathtt{dzm(k)}} - \mathtt{vpwp(k)}\frac{\mathtt{vm(k+1)}-\mathtt{vm(k)}}{\mathtt{dzm(k)}}\right)\end{aligned} \tag{163}$$

Separating out the contributions:

$$\mathtt{rhs(k)} = \mathtt{rhs(k)} + \frac{2}{3}\mathtt{C_5}\left(\frac{\mathtt{grav}}{\mathtt{T0}}\mathtt{wpthvp(k)} - \mathtt{upwp(k)}\frac{\mathtt{um(k+1)}-\mathtt{um(k)}}{\mathtt{dzm(k)}} - \mathtt{vpwp(k)}\frac{\mathtt{vm(k+1)}-\mathtt{vm(k)}}{\mathtt{dzm(k)}}\right) \tag{164}$$

7.2 $\overline{v'^2}$

As in $\overline{u'^2}$ assume a steady-state and rearrange $\overline{u'^2}$ for a semi-implicit solution to obtain:

$$\begin{aligned}&\frac{C_4}{\tau}\left(\overline{v'^2} - \frac{2}{3}\bar{e}\right) + \frac{2}{3}\left(C_{14}\frac{\bar{e}}{\tau}\right) + \bar{w}\frac{\partial \overline{v'^2}}{\partial z} + \frac{1}{3}\beta\frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}}{\overline{w'^2}}\,\overline{v'^2}\right) - \frac{\partial}{\partial z}\left[\left(K_{w9}+\nu_9\right)\frac{\partial}{\partial z}\overline{v'^2}\right]\\ =&-\left(1-\frac{1}{3}\beta\right)\frac{\partial}{\partial z}\left(a_2\frac{\overline{w'^3}}{\overline{w'^2}^2}\,\overline{w'v'}^2\right) - 2(1-C_5)\,\overline{w'v'}\,\frac{\partial \bar{v}}{\partial z}\\ &+\frac{2}{3}C_5\left(\frac{g}{\theta_0}\overline{w'\theta_v'} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{v'w'}\frac{\partial \bar{v}}{\partial z}\right)\end{aligned} \tag{165}$$

The discretization for $\overline{v'^2}$ follows in the same way as $\overline{u'^2}$.

8 Grid Configuration

Figure 1 shows the vertical grid configuration for CLUBB. The grid consists of two types of levels: `zm` and `zt`. Predictive mean variables and third order moments reside on the thermodynamic levels (`zt`). Second and fourth order moments reside on the momentum levels (`zm`).

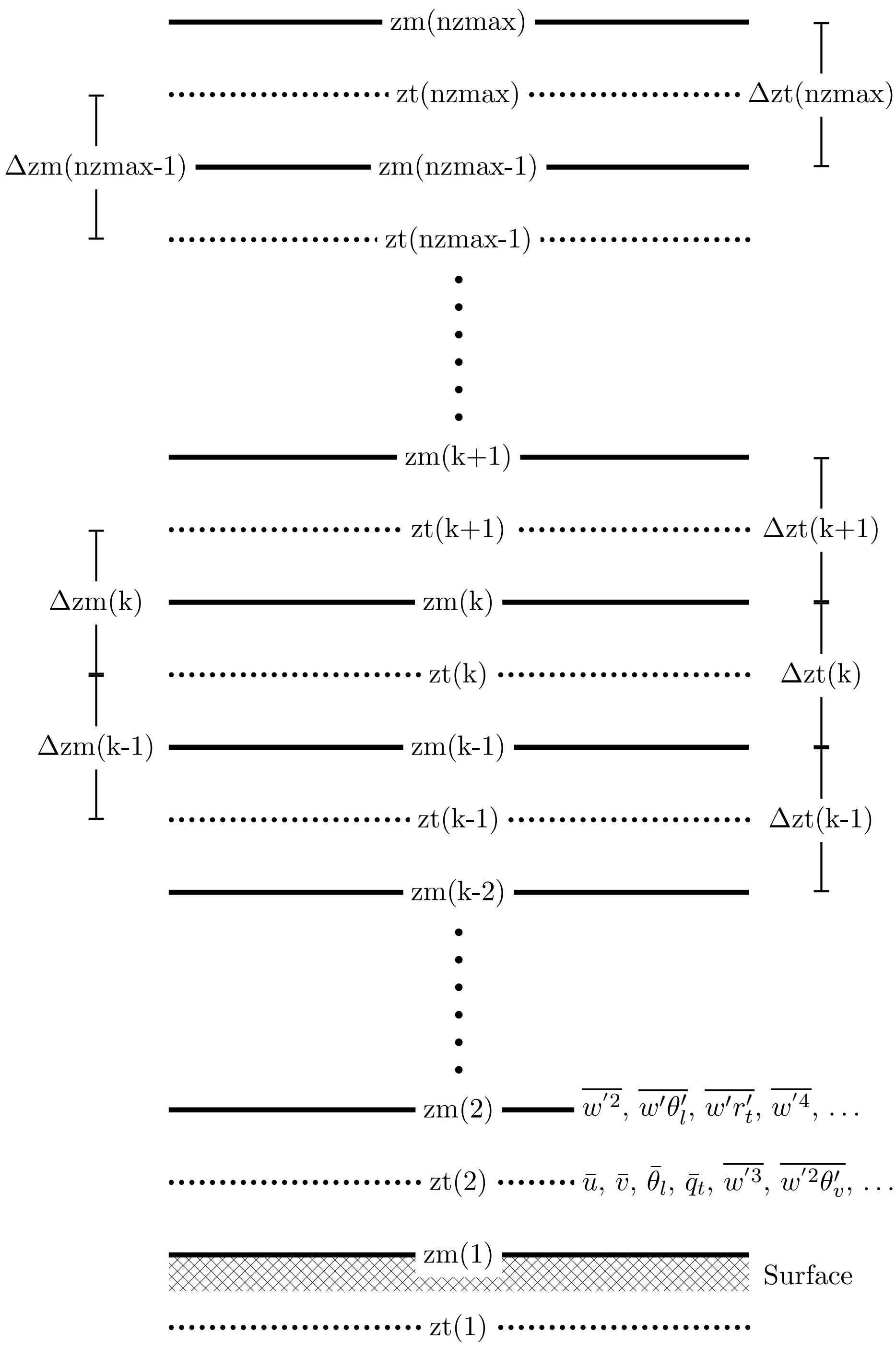
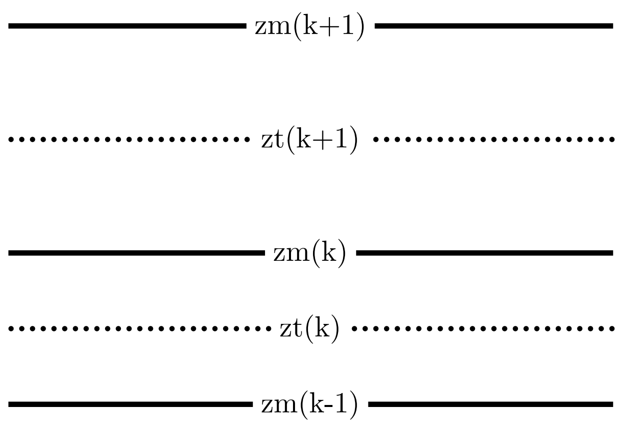


Figure 1: Vertical grid configuration

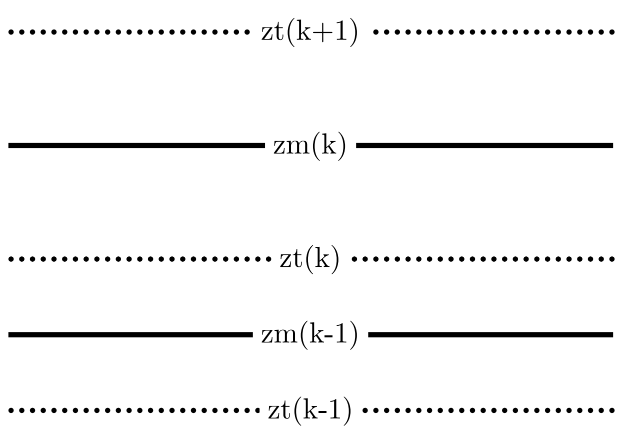
8.1 Stretched (unevenly-spaced) grid configuration

The grid setup is compatible with a stretched (unevenly-spaced) grid configuration. Thus, the distance between successive vertical grid levels may not always be constant.

The following diagram is an example of a stretched grid that is defined on momentum levels. The thermodynamic levels are placed exactly halfway between the momentum levels. However, the momentum levels do not fall halfway between the thermodynamic levels.



The following diagram is an example of a stretched grid that is defined on thermodynamic levels. The momentum levels are placed exactly halfway between the thermodynamic levels. However, the thermodynamic levels do not fall halfway between the momentum levels.



8.2 Generalized grid functions

Each variable in the CLUBB parameterization resides at certain discrete points in the vertical, whether the points be on momentum levels or on thermodynamic levels. The values of each variable are considered to behave in a linear fashion in the sections between the levels where each variable resides. Thus, linear interpolation is used to find the values of a variable at the levels where it does not reside. Since a variable is considered to behave linearly between two successive levels where it resides, the linear derivative of that variable at any point in the section between two successive levels where it resides is always the same.

Any future computer code written for use in the CLUBB parameterization should use interpolation formulas consistent with a stretched grid. The simplest way to do so is to use the appropriate equation listed below in equations 166 through 173. Interpolations should not be handled in the form of: $(\mathtt{var_m(k)}+\mathtt{var_m(k-1)})/2$; nor in the form of: $0.5\times(\mathtt{var_t(k+1)}+\mathtt{var_t(k)})$. Rather, all explicit interpolations should use the appropriate equation from equations 166, 167, 170, or 171; while interpolations for a variable being solved for implicitly in the model code should use the appropriate equation set from equations 168 and 169, or 172 and 173. The formula for a linear derivative is the same whether an evenly-spaced grid or stretched grid is in use.

8.2.1 Momentum grid levels to thermodynamic grid levels

A standard linear interpolation formula, equation 166, is used to interpolate a variable that resides on momentum levels, $\mathtt{var_m}$, to thermodynamic levels (as $\mathtt{var_t}$) on any type of grid configuration.

$$\mathtt{var_t(k)} = \left(\frac{\mathtt{var_m(k)} - \mathtt{var_m(k-1)}}{\mathtt{zm(k)} - \mathtt{zm(k-1)}}\right)\Big(\mathtt{zt(k)} - \mathtt{zm(k-1)}\Big) + \mathtt{var_m(k-1)} \tag{166}$$

When converting a variable from momentum levels to thermodynamic levels, there is one instance on the grid, at the lowermost level ($\mathtt{k} = \mathtt{1}$), where a linear extension, equation 167, is required.

$$\mathtt{var_t(1)} = \left(\frac{\mathtt{var_m(2)} - \mathtt{var_m(1)}}{\mathtt{zm(2)} - \mathtt{zm(1)}}\right)\Big(\mathtt{zt(1)} - \mathtt{zm(1)}\Big) + \mathtt{var_m(1)} \tag{167}$$

When a variable that is being solved for implicitly in an equation is also being interpolated from momentum levels to intermediate thermodynamic levels, the interpolation formulas listed above will not work. Rather, interpolation weights are necessary. The weight of the upper momentum level (index $\mathtt{k}$) on the intermediate thermodynamic level (index $\mathtt{k}$) is found according to equation 168.

$$\mathtt{weights\_zm2zt(m\_above,k)} = \left(\frac{\mathtt{zt(k)} - \mathtt{zm(k-1)}}{\mathtt{zm(k)} - \mathtt{zm(k-1)}} \right) \tag{168}$$

The weight of the lower momentum level (index $\mathtt{k-1}$) on the intermediate thermodynamic level (index $\mathtt{k}$) is found according to equation 169.

$$\mathtt{weights\_zm2zt(m\_below,k)} = 1 - \left(\frac{\mathtt{zt(k)} - \mathtt{zm(k-1)}}{\mathtt{zm(k)} - \mathtt{zm(k-1)}} \right) \tag{169}$$

8.2.2 Thermodynamic grid levels to momentum grid levels

A standard linear interpolation formula, equation 170, is used to interpolate a variable that resides on thermodynamic levels, $\mathtt{var_t}$, to momentum levels (as $\mathtt{var_m}$) on any type of grid configuration.

$$\mathtt{var_m(k)} = \left(\frac{\mathtt{var_t(k+1)} - \mathtt{var_t(k)}}{\mathtt{zt(k+1)} - \mathtt{zt(k)}} \right) \Big(\mathtt{zm(k)} - \mathtt{zt(k)} \Big) + \mathtt{var_t(k)} \tag{170}$$

When converting a variable from thermodynamic levels to momentum levels, there is one instance on the grid, at the uppermost level ($\mathtt{k} = \mathtt{nzmax}$), where a linear extension, equation 171, is required.

$$\mathtt{var_m(nzmax)} = \left(\frac{\mathtt{var_t(nzmax)} - \mathtt{var_t(nzmax-1)}}{\mathtt{zt(nzmax)} - \mathtt{zt(nzmax-1)}} \right) \Big(\mathtt{zm(nzmax)} - \mathtt{zt(nzmax)} \Big) + \mathtt{var_t(nzmax)} \tag{171}$$

When a variable that is being solved for implicitly in an equation is also being interpolated from thermodynamic levels to intermediate momentum levels, the interpolation formulas listed above will not work. Rather, interpolation weights are necessary. The weight of the upper thermodynamic level (index $\mathtt{k+1}$) on the intermediate momentum level (index $\mathtt{k}$) is found according to equation 172.

$$\mathtt{weights\_zt2zm(t\_above,k)} = \left(\frac{\mathtt{zm(k)} - \mathtt{zt(k)}}{\mathtt{zt(k+1)} - \mathtt{zt(k)}} \right) \tag{172}$$

The weight of the lower thermodynamic level (index $\mathtt{k}$) on the intermediate momentum level (index $\mathtt{k}$) is found according to equation 173.

$$\mathtt{weights\_zt2zm(t\_below,k)} = 1 - \left(\frac{\mathtt{zm(k)} - \mathtt{zt(k)}}{\mathtt{zt(k+1)} - \mathtt{zt(k)}} \right) \tag{173}$$

9 Predictive equations in Conjunction with Host Model

The CLUBB parameterization can be used in conjunction with a larger host model. Some of the main predictive equations (Eqns. 1, 2, 3, and 4) have to be divided in the following manner:

$$\frac{\partial \bar{u}}{\partial t} = -\bar{w}\frac{\partial \bar{u}}{\partial z} - f(v_g - \bar{v}) - \frac{\partial}{\partial z}\overline{u'w'} \tag{174}$$

$$\frac{\partial \bar{v}}{\partial t} = -\bar{w}\frac{\partial \bar{v}}{\partial z} + f(u_g - \bar{u}) - \frac{\partial}{\partial z}\overline{v'w'} \tag{175}$$

$$\frac{\partial \bar{q}_t}{\partial t} = -\bar{w}\frac{\partial \bar{q}_t}{\partial z} - \frac{\partial}{\partial z}\overline{w'q_t'} + \left.\frac{\partial \bar{q}_t}{\partial t}\right|_{\mathrm{ls}} \tag{176}$$

$$\frac{\partial \bar{\theta}_l}{\partial t} = -\bar{w}\frac{\partial \bar{\theta}_l}{\partial z} - \frac{\partial}{\partial z}\overline{w'\theta_l'} + \bar{R} + \left.\frac{\partial \bar{\theta}_l}{\partial t}\right|_{\mathrm{ls}} \tag{177}$$

The variables in red, $\bar{u}$, $\bar{v}$, $\bar{q}_t$, and $\bar{\theta}_l$, are supplied from the host model to the CLUBB parameterization. The variables in blue, $\overline{u'w'}$, $\overline{v'w'}$, $\overline{w'q_t'}$, and $\overline{w'\theta_l'}$, are computed in the CLUBB parameterization and then sent back to the host model. The vertical derivatives of these variables are then used to effect the time tendencies of their related variables. The terms listed in magenta, which include the vertical mean advection terms, the coriolis terms, the radiative heating term, and the large-scale moisture and temperature forcings, are terms that are usually calculated in CLUBB when CLUBB does not run with any host model. However, in cases where a host model is involved, the host model should calculate all of these terms. As before, $\bar{R}$ is the radiative heating rate, f the Coriolis parameter and u_g, v_g the geostrophic winds. $\left.\frac{\partial \bar{q}_t}{\partial t}\right|_{\mathrm{ls}}$ and $\left.\frac{\partial \bar{\theta}_l}{\partial t}\right|_{\mathrm{ls}}$ are large-scale moisture and temperature forcings.

The $\frac{\partial \overline{w'^2}}{\partial t}$ equation (eq. 5), the $\frac{\partial \overline{r_t'^2}}{\partial t}$ equation (eq. 6), the $\frac{\partial \overline{\theta_l'^2}}{\partial t}$ equation (eq. 7), the $\frac{\partial \overline{q_t'\theta_l'}}{\partial t}$ equation (eq. 8), the $\frac{\partial \overline{w'q_t'}}{\partial t}$ equation (eq. 9), the $\frac{\partial \overline{w'\theta_t'}}{\partial t}$ equation (eq. 10), and the $\frac{\partial \overline{w'^3}}{\partial t}$ equation (eq. 11) all remain unchanged. All of these variables are computed and used completely within the structure of the CLUBB parameterization.

Within the structure of a computer code, the CLUBB parameterization requires that the values of certain variables be saved at all grid points for use during the next timestep. Since the CLUBB parameterization is a one-dimensional parameterization (in the vertical), or a single-column parameterization, a three-dimensional host model must call the CLUBB parameterization once for every grid column that it has. Therefore, the values of all these variables must be saved from timestep to timestep at every grid point in the three dimensions.

The variables that need to be saved as such are the following:

On the momentum (or full) levels		
Description	Variable	Variable name in CLUBB code
Turbulent Flux of θ_l	$\overline{w'\theta_l'}$	wpthlp
Turbulent Flux of r_t	$\overline{w'r_t'}$	wprtp
Variance of w	$\overline{w'^2}$	wp2
Variance of u	$\overline{u'^2}$	up2
Variance of v	$\overline{v'^2}$	vp2
Variance of r_t	$\overline{r_t'^2}$	rtp2
Variance of θ_l	$\overline{\theta_l'^2}$	thlp2
Covariance of r_t and θ_l	$\overline{r_t'\theta_l'}$	rtpthlp
Covariance of u and w	$\overline{u'w'}$	upwp
Covariance of v and w	$\overline{v'w'}$	vpwp
Time scale	τ	taum
Width of the individual w plumes	$\tilde{\sigma}_w^2$	sigma_sqd_w
On the thermodynamic (or half) levels		
Description	Variable	Variable name in CLUBB code
Third-order Moment of w	$\overline{w'^3}$	wp3
Eddy-diffusivity	K_w	Kh_zt
Cloud water mixing ratio	$\overline{r_c}$	rcm
Cloud fraction	cf	cloud_frac

It is only necessary to save cloud water mixing ratio from timestep to timestep if the host model would need information on the subgrid value of that variable. It is also necessary to provide CLUBB with information on cloud water mixing ratio for the initial timestep of the run. Cloud fraction is usually saved for output purposes only.

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