September 25, 2018

Equations for CLUBB

1 Predictive equations

$$\frac{\partial \bar{u}}{\partial t} = \underbrace{-\bar{w}\frac{\partial \bar{u}}{\partial z}}_{ma} \underbrace{-f(v_g - \bar{v})}_{cf/gf} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{u'w'}}{\partial z}}_{ta} + \left.\frac{\partial \bar{u}}{\partial t}\right|_{\mathrm{ls}} + \left.\frac{\partial \bar{u}}{\partial t}\right|_{\mathrm{ndg}} + \left.\frac{\partial \bar{u}}{\partial t}\right|_{\mathrm{sdmp}} \tag{1}$$

$$\frac{\partial \bar{v}}{\partial t} = \underbrace{-\bar{w}\frac{\partial \bar{v}}{\partial z}}_{ma} \underbrace{+f(u_g - \bar{u})}_{cf/gf} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{v'w'}}{\partial z}}_{ta} + \left.\frac{\partial \bar{v}}{\partial t}\right|_{\mathrm{ls}} + \left.\frac{\partial \bar{v}}{\partial t}\right|_{\mathrm{ndg}} + \left.\frac{\partial \bar{v}}{\partial t}\right|_{\mathrm{sdmp}} \tag{2}$$

$$\frac{\partial \bar{r}_t}{\partial t} = \underbrace{-\bar{w}\frac{\partial \bar{r}_t}{\partial z}}_{ma} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{w'r_t'}}{\partial z}}_{ta} + \left.\frac{\partial \bar{r}_t}{\partial t}\right|_{\mathrm{ls}} + \left.\frac{\partial \bar{r}_t}{\partial t}\right|_{\mathrm{cl}} + \left.\frac{\partial \bar{r}_t}{\partial t}\right|_{\mathrm{mfl}} + \left.\frac{\partial \bar{r}_t}{\partial t}\right|_{\mathrm{tacl}} + \left.\frac{\partial \bar{r}_t}{\partial t}\right|_{\mathrm{sdmp}} \tag{3}$$

$$\frac{\partial \bar{\theta}_l}{\partial t} = \underbrace{-\bar{w}\frac{\partial \bar{\theta}_l}{\partial z}}_{ma} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{w'\theta_l'}}{\partial z}}_{ta} + \bar{R} + \left.\frac{\partial \bar{\theta}_l}{\partial t}\right|_{\mathrm{ls}} + \left.\frac{\partial \bar{\theta}_l}{\partial t}\right|_{\mathrm{cl}} + \left.\frac{\partial \bar{\theta}_l}{\partial t}\right|_{\mathrm{mfl}} + \left.\frac{\partial \bar{\theta}_l}{\partial t}\right|_{\mathrm{tacl}} + \left.\frac{\partial \bar{\theta}_l}{\partial t}\right|_{\mathrm{sdmp}} \tag{4}$$

$$\begin{aligned}\frac{\partial \overline{w'^2}}{\partial t} = & \underbrace{-\bar{w}\frac{\partial \overline{w'^2}}{\partial z}}_{ma} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{w'^3}}{\partial z}}_{ta} \underbrace{-2\overline{w'^2}\frac{\partial \bar{w}}{\partial z}}_{ac} \underbrace{+\frac{2g}{\theta_{vs}}\overline{w'\theta_v'}}_{bp} \underbrace{-\frac{C_4}{\tau}\left(\overline{w'^2} - \frac{2}{3}\bar{e}\right)}_{pr1} \\ & \underbrace{-C_5\left(-2\overline{w'^2}\frac{\partial \bar{w}}{\partial z} + \frac{2g}{\theta_{vs}}\overline{w'\theta_v'}\right)}_{pr2} \underbrace{+\frac{2}{3}C_5\left(\frac{g}{\theta_{vs}}\overline{w'\theta_v'} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{v'w'}\frac{\partial \bar{v}}{\partial z}\right)}_{pr3} \\ & \underbrace{-\frac{C_1}{\tau}\left(\overline{w'^2} - w|_{\mathrm{tol}}^2\right)}_{dp1} \underbrace{+\frac{\partial}{\partial z}\left[\left(K_{w1} + \nu_1\right)\frac{\partial}{\partial z}\overline{w'^2}\right]}_{dp2} + \left.\frac{\partial \overline{w'^2}}{\partial t}\right|_{\mathrm{pd}} + \left.\frac{\partial \overline{w'^2}}{\partial t}\right|_{\mathrm{cl}}\end{aligned} \tag{5}$$

$$\begin{aligned}\frac{\partial \overline{r_t'^2}}{\partial t} = & \underbrace{-\bar{w}\frac{\partial \overline{r_t'^2}}{\partial z}}_{ma} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{w'r_t'^2}}{\partial z}}_{ta} \underbrace{-2\overline{w'r_t'}\frac{\partial \bar{r}_t}{\partial z}}_{tp} \underbrace{-\frac{C_2}{\tau}\left(\overline{r_t'^2} - r_t|_{\mathrm{tol}}^2\right)}_{dp1} \\ & \underbrace{+\frac{\partial}{\partial z}\left[\left(K_{w2} + \nu_2\right)\frac{\partial}{\partial z}\overline{r_t'^2}\right]}_{dp2} + \left.\frac{\partial \overline{r_t'^2}}{\partial t}\right|_{\mathrm{pd}} + \left.\frac{\partial \overline{r_t'^2}}{\partial t}\right|_{\mathrm{cl}}\end{aligned} \tag{6}$$

$$
\begin{aligned}
\frac{\partial \overline{\theta_l'^2}}{\partial t} = & \underbrace{-\bar{w}\frac{\partial \overline{\theta_l'^2}}{\partial z}}_{ma} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{w'\theta_l'^2}}{\partial z}}_{ta} \underbrace{-2\overline{w'\theta_l'}\frac{\partial \bar{\theta}_l}{\partial z}}_{tp} \underbrace{-\frac{C_2}{\tau}\left(\overline{\theta_l'^2} - \theta_l|^2_{\mathrm{tol}}\right)}_{dp1} \\
& \underbrace{+\frac{\partial}{\partial z}\left[(K_{w2}+\nu_2)\frac{\partial}{\partial z}\overline{\theta_l'^2}\right]}_{dp2} + \left.\frac{\partial \overline{\theta_l'^2}}{\partial t}\right|_{\mathrm{pd}} + \left.\frac{\partial \overline{\theta_l'^2}}{\partial t}\right|_{\mathrm{cl}}
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\frac{\partial \overline{r_t'\theta_l'}}{\partial t} = & \underbrace{-\bar{w}\frac{\partial \overline{r_t'\theta_l'}}{\partial z}}_{ma} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{w'r_t'\theta_l'}}{\partial z}}_{ta} \underbrace{-\overline{w'r_t'}\frac{\partial \bar{\theta}_l}{\partial z}}_{tp1} \underbrace{-\overline{w'\theta_l'}\frac{\partial \bar{r}_t}{\partial z}}_{tp2} \underbrace{-\frac{C_2}{\tau}\overline{r_t'\theta_l'}}_{dp1} \\
& \underbrace{+\frac{\partial}{\partial z}\left[(K_{w2}+\nu_2)\frac{\partial}{\partial z}\overline{r_t'\theta_l'}\right]}_{dp2} + \left.\frac{\partial \overline{r_t'\theta_l'}}{\partial t}\right|_{\mathrm{cl}}
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
\frac{\partial \overline{w'r_t'}}{\partial t} = & \underbrace{-\bar{w}\frac{\partial \overline{w'r_t'}}{\partial z}}_{ma} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{w'^2r_t'}}{\partial z}}_{ta} \underbrace{-\overline{w'^2}\frac{\partial \bar{r}_t}{\partial z}}_{tp} \underbrace{-\overline{w'r_t'}\frac{\partial \bar{w}}{\partial z}}_{ac} \underbrace{+\frac{g}{\theta_{vs}}\overline{r_t'\theta_v'}}_{bp} \underbrace{-\frac{C_6}{\tau}\overline{w'r_t'}}_{pr1} \\
& \underbrace{+C_7\overline{w'r_t'}\frac{\partial \bar{w}}{\partial z}}_{pr2} \underbrace{-C_7\frac{g}{\theta_{vs}}\overline{r_t'\theta_v'}}_{pr3} \underbrace{+\frac{\partial}{\partial z}\left[(K_{w6}+\nu_6)\frac{\partial}{\partial z}\overline{w'r_t'}\right]}_{dp1} + \left.\frac{\partial \overline{w'r_t'}}{\partial t}\right|_{\mathrm{sicl}} + \left.\frac{\partial \overline{w'r_t'}}{\partial t}\right|_{\mathrm{cl}} + \left.\frac{\partial \overline{w'r_t'}}{\partial t}\right|_{\mathrm{mfl}}
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
\frac{\partial \overline{w'\theta_l'}}{\partial t} = & \underbrace{-\bar{w}\frac{\partial \overline{w'\theta_l'}}{\partial z}}_{ma} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{w'^2\theta_l'}}{\partial z}}_{ta} \underbrace{-\overline{w'^2}\frac{\partial \bar{\theta}_l}{\partial z}}_{tp} \underbrace{-\overline{w'\theta_l'}\frac{\partial \bar{w}}{\partial z}}_{ac} \underbrace{+\frac{g}{\theta_{vs}}\overline{\theta_l'\theta_v'}}_{bp} \underbrace{-\frac{C_6}{\tau}\overline{w'\theta_l'}}_{pr1} \\
& \underbrace{+C_7\overline{w'\theta_l'}\frac{\partial \bar{w}}{\partial z}}_{pr2} \underbrace{-C_7\frac{g}{\theta_{vs}}\overline{\theta_l'\theta_v'}}_{pr3} \underbrace{+\frac{\partial}{\partial z}\left[(K_{w6}+\nu_6)\frac{\partial}{\partial z}\overline{w'\theta_l'}\right]}_{dp1} + \left.\frac{\partial \overline{w'\theta_l'}}{\partial t}\right|_{\mathrm{sicl}} + \left.\frac{\partial \overline{w'\theta_l'}}{\partial t}\right|_{\mathrm{cl}} + \left.\frac{\partial \overline{w'\theta_l'}}{\partial t}\right|_{\mathrm{mfl}}
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
\frac{\partial \overline{w'^3}}{\partial t} = & \underbrace{-\bar{w}\frac{\partial \overline{w'^3}}{\partial z}}_{ma} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{w'^4}}{\partial z}}_{ta} \underbrace{+3\frac{\overline{w'^2}}{\rho_s}\frac{\partial \rho_s \overline{w'^2}}{\partial z}}_{tp} \underbrace{-3\overline{w'^3}\frac{\partial \bar{w}}{\partial z}}_{ac} \underbrace{+\frac{3g}{\theta_{vs}}\overline{w'^2\theta_v'}}_{bp1} \\
& \underbrace{-C_{15}K_m\left(\frac{g}{\theta_{vs}}\frac{\partial \overline{w'\theta_v'}}{\partial z} - \left(\frac{\partial\left(\overline{u'w'}\frac{\partial \bar{u}}{\partial z}\right)}{\partial z} + \frac{\partial\left(\overline{v'w'}\frac{\partial \bar{v}}{\partial z}\right)}{\partial z}\right)\right)}_{bp2} \\
& \underbrace{-\frac{C_8}{\tau}\left(C_{8b}\,Skw^4+1\right)\overline{w'^3}}_{pr1} \underbrace{-C_{11}\left(-3\overline{w'^3}\frac{\partial \bar{w}}{\partial z} + \frac{3g}{\theta_{vs}}\overline{w'^2\theta_v'}\right)}_{pr2} \underbrace{+\frac{\partial}{\partial z}\left[(K_{w8}+\nu_8)\frac{\partial}{\partial z}\overline{w'^3}\right]}_{dp1} + \left.\frac{\partial \overline{w'^3}}{\partial t}\right|_{\mathrm{cl}}
\end{aligned} \tag{11}
$$

where $\bar{R}$ is the radiative heating rate, f the Coriolis parameter, and u_g and v_g the geostrophic winds. Furthermore, $\left.\frac{\partial \bar{r}_t}{\partial t}\right|_{\mathrm{ls}}$ and $\left.\frac{\partial \bar{\theta}_l}{\partial t}\right|_{\mathrm{ls}}$ are large-scale moisture and temperature forcings, respectively, and g

is acceleration due to gravity. The set of equations is an anelastic set of equations, where ρ_s is the dry, static, base-state density, which only changes with respect to altitude; and where θ_{vs} is the dry, base-state θ_v, which also only changes with respect to altitude. Threshold values of the variances are established, such that $w|_{\mathrm{tol}}^2$ is the minimum threshold value for $\overline{w'^2}$; $r_t|_{\mathrm{tol}}^2$ is the minimum threshold value for $\overline{r_t'^2}$; and $\theta_l|_{\mathrm{tol}}^2$ is the minimum threshold value for $\overline{\theta_l'^2}$. The subscript $|_{\mathrm{pd}}$ stands for the rate of change due to the positive-definite hole-filling scheme, the subscript $|_{\mathrm{sicl}}$ stands for the rate of change due to the semi-implicit clipping scheme, the subscript $|_{\mathrm{cl}}$ stands for the rate of change due to completely explicit clipping, the subscript $|_{\mathrm{mfl}}$ denotes adjustments from the monotonic flux limiter, the subscript the $|_{\mathrm{tacl}}$ denotes turbulent advection clipping, and finally the subscript $|_{\mathrm{sdmp}}$ denotes sponge layer damping.

If the model does not predict any higher-order moments of the horizontal winds, we assume that the turbulence kinetic energy, $\bar{e}$, is proportional to the vertical velocity variance $\overline{w'^2}$:

$$\bar{e} = \frac{3}{2}\overline{w'^2}. \tag{12}$$

Alternatively, if higher-order moments of the horizontal winds are computed, then turbulence kinetic energy, $\bar{e}$, is a function of the vertical velocity variance $\overline{w'^2}$, latitudinal wind variance $\overline{v'^2}$, and longitudinal wind variance $\overline{u'^2}$:

$$\bar{e} = \frac{1}{2}\left(\overline{w'^2} + \overline{u'^2} + \overline{v'^2}\right). \tag{13}$$

In the second case, the horizontal wind variance terms are determined as in Bougeault (1981a) and given by the equations:

$$\begin{aligned}
\frac{\partial \overline{u'^2}}{\partial t} = & \underbrace{-\bar{w}\frac{\partial \overline{u'^2}}{\partial z}}_{ma} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{w'u'^2}}{\partial z}}_{ta} \underbrace{-\left(1-C_5\right)2\overline{u'w'}\frac{\partial \bar{u}}{\partial z}}_{tp} \underbrace{-\frac{2}{3}C_{14}\frac{\bar{e}}{\tau}}_{pr1} \\
& \underbrace{+\frac{2}{3}C_5\left(\frac{g}{\theta_{vs}}\overline{w'\theta_v'} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{v'w'}\frac{\partial \bar{v}}{\partial z}\right)}_{pr2} \underbrace{-\frac{C_4}{\tau}\left(\overline{u'^2} - \frac{2}{3}\bar{e}\right)}_{dp1} \\
& + \underbrace{\frac{\partial}{\partial z}\left[\left(K_{w9}+\nu_9\right)\frac{\partial}{\partial z}\overline{u'^2}\right]}_{dp2} + \left.\frac{\partial \overline{u'^2}}{\partial t}\right|_{\mathrm{pd}} + \left.\frac{\partial \overline{u'^2}}{\partial t}\right|_{\mathrm{cl}}
\end{aligned} \tag{14}$$

$$
\begin{aligned}
\frac{\partial \overline{v'^2}}{\partial t} = & \underbrace{-\bar{w}\frac{\partial \overline{v'^2}}{\partial z}}_{ma} \underbrace{-\frac{1}{\rho_s}\frac{\partial \rho_s \overline{w'v'^2}}{\partial z}}_{ta} \underbrace{-\left(1-C_5\right) 2\overline{v'w'}\frac{\partial \bar{v}}{\partial z}}_{tp} \underbrace{-\frac{2}{3}C_{14}\frac{\bar{e}}{\tau}}_{pr1} \\
& \underbrace{+\frac{2}{3}C_5\left(\frac{g}{\theta_{vs}}\overline{w'\theta_v'} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{v'w'}\frac{\partial \bar{v}}{\partial z}\right)}_{pr2} \underbrace{-\frac{C_4}{\tau}\left(\overline{v'^2} - \frac{2}{3}\bar{e}\right)}_{dp1} \\
& + \underbrace{\frac{\partial}{\partial z}\left[\left(K_{w9} + \nu_9\right)\frac{\partial}{\partial z}\overline{v'^2}\right]}_{dp2} + \left.\frac{\partial \overline{v'^2}}{\partial t}\right|_{\mathrm{pd}} + \left.\frac{\partial \overline{v'^2}}{\partial t}\right|_{\mathrm{cl}}
\end{aligned} \tag{15}
$$

Where, ϵ in Bougeault (1981b), the dissipation of $\bar{e}$, has been defined in CLUBB by the equation:

$$
\epsilon = C_{14}\frac{\bar{e}}{\tau} \tag{16}
$$

The time scale τ is:

$$
\tau = \begin{cases} \dfrac{L}{\sqrt{\bar{e}}}; & L/\sqrt{\bar{e}} \leqslant \tau_{\max} \\ \tau_{\max}; & L/\sqrt{\bar{e}} > \tau_{\max} \end{cases} . \tag{17}
$$

The momentum fluxes are closed using a down gradient approach:

$$
\overline{u'w'} = -K_m\frac{\partial \bar{u}}{\partial z} \tag{18a}
$$

$$
\overline{v'w'} = -K_m\frac{\partial \bar{v}}{\partial z} \tag{18b}
$$

The momentum fluxes, $\overline{u'w'}$ and $\overline{v'w'}$, are also subject to completely explicit clipping. The turbulent-transfer coefficient K_m is given by:

$$
K_m = c_K\, L\,\bar{e}^{1/2}. \tag{19}
$$

$c_K = c_\mu^{1/4} = 0.548$ in Duynkerke and Driedonks (1987), but CLUBB reduces the value to better fit LES output.

The eddy diffusivity coefficients in Equation (5) through Equation (11) and in Equations (14) and (15) are as follows:

$$K_{w1} = c_{K1}K_m + c_{Ksqd}\,\overline{w'^2}^2\Big|_{3\text{ pnt avg}}$$

$$K_{w2} = c_{K2}K_m + c_{Ksqd}\,\lambda|_{3\text{ pnt avg}}$$

$$K_{w6} = c_{K6}K_m + c_{Ksqd}\,\phi|_{3\text{ pnt avg}}$$

$$K_{w8} = c_{K8}K_m + c_{Ksqd}\,\overline{w'^3}^2\Big|_{3\text{ pnt avg}}$$

$$K_{w9} = c_{K9}K_m + c_{Ksqd}\,\eta|_{3\text{ pnt avg}}$$

where λ is $10^{12}\,\overline{r_t'^2}^2$ for Equation (6), $\overline{\theta_l'^2}^2$ for Equation (7), and $10^6\,\overline{r_t'\theta_l'}^2$ for Equation (8); ϕ is $10^6\,\overline{w'r_t'}^2$ for Equation (9) and $\overline{w'\theta_l'}^2$ for Equation (10); and η is $\overline{u'^2}^2$ for Equation (14) and $\overline{v'^2}^2$ for Equation (15).

2 PDF closure

Details of the PDF closure can be found in Larson and Golaz (2005), hereafter referred to as LG. We only briefly summarize key aspects here.

2.1 Transport terms

The transport terms appearing in Eqs (1)-(11) are closed as follows. First, we define $c_{w\theta_l}$ and c_{wr_t} as in Eqs (LG15) and (LG16):

$$c_{w\theta_l} = \frac{\overline{w'\theta_l'}}{\sqrt{\overline{w'^2}}\sqrt{\overline{\theta_l'^2}}} \tag{20}$$

$$c_{wr_t} = \frac{\overline{w'r_t'}}{\sqrt{\overline{w'^2}}\sqrt{\overline{r_t'^2}}} \tag{21}$$

The width of the individual w plumes is given by (LG37):

$$\tilde{\sigma}_w^2 = \gamma\left[1 - \max\left(c_{w\theta_l}^2, c_{wr_t}^2\right)\right] \tag{22}$$

We define the following quantities in order to simplify the notation:

$$a_1 = \frac{1}{(1-\tilde{\sigma}_w^2)} \tag{23}$$

$$a_2 = \frac{1}{(1-\tilde{\sigma}_w^2)^2} \tag{24}$$

$$a_3 = 3\tilde{\sigma}_w^4 + 6(1-\tilde{\sigma}_w^2)\tilde{\sigma}_w^2 + (1-\tilde{\sigma}_w^2)^2 - 3 \tag{25}$$

The turbulence moment $\overline{w'^4}$ is given by (LG40):

$$\overline{w'^4} = \overline{w'^2}^2\,(a_3+3) + a_1 \frac{\overline{w'^3}^2}{\overline{w'^2}} \tag{26}$$

The flux transport terms are given by (LG42):

$$\overline{w'^2\theta_l'} = a_1 \frac{\overline{w'^3}}{\overline{w'^2}}\,\overline{w'\theta_l'} \tag{27}$$

$$\overline{w'^2 r_t'} = a_1 \frac{\overline{w'^3}}{\overline{w'^2}}\,\overline{w' r_t'} \tag{28}$$

The variance transport terms follow (LG46):

$$\overline{w'\theta_l'^2} = \frac{1}{3}\beta a_1 \frac{\overline{w'^3}}{\overline{w'^2}}\overline{\theta_l'^2} + \left(1-\frac{1}{3}\beta\right) a_2 \frac{\overline{w'^3}}{\overline{w'^2}^2}\overline{w'\theta_l'}^2 \tag{29}$$

$$\overline{w' r_t'^2} = \frac{1}{3}\beta a_1 \frac{\overline{w'^3}}{\overline{w'^2}}\overline{r_t'^2} + \left(1-\frac{1}{3}\beta\right) a_2 \frac{\overline{w'^3}}{\overline{w'^2}^2}\overline{w' r_t'}^2 \tag{30}$$

Finally, the covariance term is obtained substituting (LG56) into (LG48):

$$\overline{w' r_t'\theta_l'} = \frac{1}{3}\beta a_1 \frac{\overline{w'^3}}{\overline{w'^2}}\,\overline{r_t'\theta_l'} + \left(1-\frac{1}{3}\beta\right) a_2 \frac{\overline{w'^3}}{\overline{w'^2}^2}\overline{w' r_t'}\,\overline{w'\theta_l'} \tag{31}$$

In the anisotropic case, the horizontal wind variance terms are obtained by:

$$\overline{w'u'^2} = \frac{1}{3}\beta a_1 \frac{\overline{w'^3}}{\overline{w'^2}}\overline{u'^2} + \left(1-\frac{1}{3}\beta\right) a_2 \frac{\overline{w'^3}}{\overline{w'^2}^2}\overline{w'u'}^2 \tag{32}$$

$$\overline{w'v'^2} = \frac{1}{3}\beta a_1 \frac{\overline{w'^3}}{\overline{w'^2}}\overline{v'^2} + \left(1-\frac{1}{3}\beta\right) a_2 \frac{\overline{w'^3}}{\overline{w'^2}^2}\overline{w'v'}^2 \tag{33}$$

2.2 Buoyancy terms

There are more unclosed terms involving θ_v. They are $\overline{w'\theta_v'}$, $\overline{r_t'\theta_v'}$, $\overline{\theta_l'\theta_v'}$, and $\overline{w'^2\theta_v'}$ and can be written as:

$$\overline{\chi'\theta_v'} = \overline{\chi'\theta_l'} + \underbrace{\frac{1-\epsilon_0}{\epsilon_0}\theta_0}_{\equiv A\,(\approx 200K)}\,\overline{\chi' r_t'} + \underbrace{\left(\frac{L_v}{c_p}\left(\frac{p_0}{p}\right)^{R_d/c_p} - \frac{1}{\epsilon_0}\theta_0\right)}_{\equiv B\,(\approx 2000K)}\overline{\chi' r_l'}\,, \tag{34}$$

where χ' represents w', r_t', θ_l', w'^2, or a passive scalar. Here $\epsilon_0 = R_d/R_v$, R_d is the gas constant of dry air, R_v is the gas constant of water vapor, L_v is the latent heat of vaporization, c_p is the heat capacity of air, and p_0 is a reference pressure. The correlations involving liquid water ($\overline{\chi' r_l'}$) can be computed for the given family of PDFs (see next section).

3 Cloud properties

The cloud properties, such as cloud fraction, mean liquid water and correlations involving liquid water ($\overline{\chi' r_l'}$) are obtained from the PDF. To do so, a certain number of properties are computed for each Gaussian ($i = 1, 2$):

$$T_{li} = \theta_{li} \left(\frac{p}{p_0} \right)^{R_d/c_p} \tag{35}$$

$$r_{si} = \frac{R_d}{R_v} \frac{e_s(T_{li})}{p - [1 - (R_d/R_v)] e_s(T_{li})} \tag{36}$$

$$\beta_i = \frac{R_d}{R_v} \left(\frac{L}{R_d T_{li}} \right) \left(\frac{L}{c_p T_{li}} \right) \tag{37}$$

$$s_i = r_{ti} - r_{si} \frac{1 + \beta_i r_{ti}}{1 + \beta_i r_{si}} \tag{38}$$

$$c_{r_{ti}} = \frac{1}{1 + \beta_i r_{si}} \tag{39}$$

$$c_{\theta_{li}} = \frac{1 + \beta_i r_{ti}}{[1 + \beta_i r_{si}]^2} \frac{c_p}{L} \beta_i r_{si} \left(\frac{p}{p_0} \right)^{R_d/c_p} \tag{40}$$

$$\sigma_{si}^2 = c_{\theta_{li}}^2 \sigma_{\theta_{li}}^2 + c_{r_{ti}}^2 \sigma_{r_{ti}}^2 - 2 c_{\theta_{li}} \sigma_{\theta_{li}} c_{r_{ti}} \sigma_{r_{ti}} r_{r_t \theta_l} \tag{41}$$

$$C_i = \frac{1}{2} \left[1 + \mathrm{erf} \left(\frac{s_i}{\sqrt{2} \sigma_{si}} \right) \right] \tag{42}$$

$$r_{li} = s_i C_i + \frac{\sigma_{si}}{\sqrt{2\pi}} \exp \left[-\frac{1}{2} \left(\frac{s_i}{\sigma_{si}} \right)^2 \right] \tag{43}$$

where C_i and r_{li} are the cloud fractions and liquid water of each individual Gaussian.

The layer-averaged cloud properties are given by:

$$\overline{C} = a C_1 + (1 - a) C_2 \tag{44}$$

$$\overline{r_l} = a r_{l1} + (1 - a) r_{l2} \tag{45}$$

$$\overline{w'r'_l} = a(w_1 - \bar{w})r_{l1} + (1-a)(w_2 - \bar{w})r_{l2} \tag{46}$$

$$\overline{w'^2r'_l} = a\left((w_1 - \bar{w})^2 + \sigma_{w1}^2\right) r_{l1} + (1-a)\left((w_2 - \bar{w})^2 + \sigma_{w2}^2\right) r_{l2} - \overline{w'^2}\,(ar_{l1} + (1-a)r_{l2}) \tag{47}$$

$$\begin{aligned}\overline{\theta'_l r'_l} =& a\left[(\theta_{l1} - \bar{\theta}_l)r_{l1} - C_1\left(c_{\theta_{l1}}\sigma_{\theta_{l1}}^2 - r_{r_t\theta_l}c_{r_{t1}}\sigma_{r_{t1}}\sigma_{\theta_{l1}}\right)\right] \\ &+ (1-a)\left[(\theta_{l2} - \bar{\theta}_l)r_{l2} - C_2\left(c_{\theta_{l2}}\sigma_{\theta_{l2}}^2 - r_{r_t\theta_l}c_{r_{t2}}\sigma_{r_{t2}}\sigma_{\theta_{l2}}\right)\right]\end{aligned} \tag{48}$$

$$\begin{aligned}\overline{r'_t r'_l} =& a\left[(r_{t1} - \bar{r}_t)r_{l1} + C_1\left(c_{r_{t1}}\sigma_{r_{t1}}^2 - r_{r_t\theta_l}c_{\theta_{l1}}\sigma_{r_{t1}}\sigma_{\theta_{l1}}\right)\right] \\ &+ (1-a)\left[(r_{t2} - \bar{r}_t)r_{l2} + C_2\left(c_{r_{t2}}\sigma_{r_{t2}}^2 - r_{r_t\theta_l}c_{\theta_{l2}}\sigma_{r_{t2}}\sigma_{\theta_{l2}}\right)\right]\end{aligned} \tag{49}$$

4 Steady-state solutions for the variances

4.1 $\overline{r_t'^2}$ and $\overline{\theta_l'^2}$

Start with (6) (for simplicity, neglect the $|_{\text{pd}}$ and $|_{\text{cl}}$ terms), substitute (30), assume steady-state and rearrange:

$$\begin{aligned}&\frac{C_2}{\tau}\overline{r_t'^2} + \bar{w}\frac{\partial\overline{r_t'^2}}{\partial z} + \frac{1}{3}\beta\frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}}{\overline{w'^2}}\,\overline{r_t'^2}\right) - \frac{\partial}{\partial z}\left[(K_{w2} + \nu_2)\frac{\partial}{\partial z}\overline{r_t'^2}\right] \\ =& -\left(1 - \frac{1}{3}\beta\right)\frac{\partial}{\partial z}\left(a_2\frac{\overline{w'^3}}{\overline{w'^2}^2}\,\overline{w'r_t'}^2\right) - 2\,\overline{w'r'_t}\,\frac{\partial\bar{r}_t}{\partial z} + \frac{C_2}{\tau}\,r_t|^2_{\text{tol}}\end{aligned} \tag{50}$$

The goal is to recast (50) so that $\overline{r_t'^2}$ can be computed using a tridiagonal solver:

$$\underbrace{\begin{bmatrix} & (1,2) & \cdots & (1,\text{nzmax}-1) & (1,\text{nzmax}) \\ (2,1) & (2,2) & \cdots & (2,\text{nzmax}-1) & (2,\text{nzmax}) \\ (3,1) & (3,2) & \cdots & (3,\text{nzmax}-1) & \end{bmatrix}}_{\texttt{LHS}\text{(Stored in compact format)}} \begin{bmatrix} \texttt{rtp2}(1) \\ \texttt{rtp2}(2) \\ \vdots \\ \texttt{rtp2}(\text{nzmax}-1) \\ \texttt{rtp2}(\text{nzmax}) \end{bmatrix} = \underbrace{\begin{bmatrix} (1) \\ (2) \\ \vdots \\ (\text{nzmax}-1) \\ (\text{nzmax}) \end{bmatrix}}_{\texttt{RHS}}$$

$$\texttt{lhs(3,k)}\,\texttt{rtp2(k-1)} + \texttt{lhs(2,k)}\,\texttt{rtp2(k)} + \texttt{lhs(1,k)}\,\texttt{rtp2(k+1)} = \texttt{rhs(k)} \tag{51}$$

We now compute the contributions of each term in (50) to `lhs(3,k)`, `lhs(2,k)`, `lhs(1,k)`, and `rhs(k)`.

4.1.1 Term 1: dp1, implicit component

$$\mathtt{lhs(2,k)} = \mathtt{lhs(2,k)} + \frac{\mathtt{C_2}}{\mathtt{taum(k)}} \tag{52}$$

4.1.2 Term 2: ma

$$\begin{aligned}
&\bar{w}\frac{\partial \overline{q_t'^2}}{\partial z}\bigg|_{\mathtt{zm(k)}} \\
=&\frac{\mathtt{wmm(k)}}{\mathtt{dzm(k)}}\left(\frac{1}{2}\left(\mathtt{rtp2(k)} + \mathtt{rtp2(k+1)}\right) - \frac{1}{2}\left(\mathtt{rtp2(k-1)} + \mathtt{rtp2(k)}\right)\right) \\
=&\frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}}\mathtt{rtp2(k+1)} - \frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}}\mathtt{rtp2(k-1)}
\end{aligned} \tag{53}$$

Separating out the contributions:

$$\begin{aligned}
\mathtt{lhs(3,k)} &= \mathtt{lhs(3,k)} - \frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}} \\
\mathtt{lhs(1,k)} &= \mathtt{lhs(1,k)} + \frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}}
\end{aligned} \tag{54}$$

4.1.3 Term 3: ta, implicit component

$$\begin{aligned}
&\frac{1}{3}\beta\frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}}{\overline{w'^2}}\overline{r_t'^2}\right)\bigg|_{\mathtt{zm(k)}} \\
=&\frac{\beta}{\mathtt{6dzm(k)}}\Bigg[\frac{\left(\mathtt{a1m(k)} + \mathtt{a1m(k+1)}\right)\mathtt{wp3(k+1)}\left(\mathtt{rtp2(k)} + \mathtt{rtp2(k+1)}\right)}{\max\left(\mathtt{wp2(k)} + \mathtt{wp2(k+1)}, 2\epsilon\right)} \\
&- \frac{\left(\mathtt{a1m(k-1)} + \mathtt{a1m(k)}\right)\mathtt{wp3(k)}\left(\mathtt{rtp2(k-1)} + \mathtt{rtp2(k)}\right)}{\max\left(\mathtt{wp2(k-1)} + \mathtt{wp2(k)}, 2\epsilon\right)}\Bigg]
\end{aligned} \tag{55}$$

Separating out the contributions:

$$\begin{aligned}
\mathtt{lhs(3,k)} &= \mathtt{lhs(3,k)} - \frac{\beta}{\mathtt{6dzm(k)}}\frac{\left(\mathtt{a1m(k-1)} + \mathtt{a1m(k)}\right)\mathtt{wp3(k)}}{\max\left(\mathtt{wp2(k-1)} + \mathtt{wp2(k)}, 2\epsilon\right)} \\
\mathtt{lhs(2,k)} &= \mathtt{lhs(2,k)} + \frac{\beta}{\mathtt{6dzm(k)}}\left(\frac{\left(\mathtt{a1m(k)} + \mathtt{a1m(k+1)}\right)\mathtt{wp3(k+1)}}{\max\left(\mathtt{wp2(k)} + \mathtt{wp2(k+1)}, 2\epsilon\right)} - \frac{\left(\mathtt{a1m(k-1)} + \mathtt{a1m(k)}\right)\mathtt{wp3(k)}}{\max\left(\mathtt{wp2(k-1)} + \mathtt{wp2(k)}, 2\epsilon\right)}\right) \\
\mathtt{lhs(1,k)} &= \mathtt{lhs(1,k)} + \frac{\beta}{\mathtt{6dzm(k)}}\frac{\left(\mathtt{a1m(k)} + \mathtt{a1m(k+1)}\right)\mathtt{wp3(k+1)}}{\max\left(\mathtt{wp2(k)} + \mathtt{wp2(k+1)}, 2\epsilon\right)}
\end{aligned} \tag{56}$$

In order to increase numerical stability in the model, a_1 has been brought outside of the derivative.

This is not mathematically correct, but it does help to increase stability. Brian Griffin. Feb. 21, 2008.

$$
\begin{aligned}
& a_1 \frac{1}{3} \beta \frac{\partial}{\partial z} \left(\frac{\overline{w'^3}}{\overline{w'^2}} \overline{r_t'^2} \right) \Bigg|_{\mathtt{zm(k)}} \\
& = \mathtt{a1m(k)} \frac{\beta}{\mathtt{3dzm(k)}} \left[\frac{\mathtt{wp3(k+1)}\left(\mathtt{rtp2(k)}+\mathtt{rtp2(k+1)}\right)}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)}, 2\epsilon\right)} - \frac{\mathtt{wp3(k)}\left(\mathtt{rtp2(k-1)}+\mathtt{rtp2(k)}\right)}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)}, 2\epsilon\right)} \right]
\end{aligned}
\tag{57}
$$

Separating out the contributions:

$$
\begin{aligned}
\mathtt{lhs(3,k)} &= \mathtt{lhs(3,k)} - \mathtt{a1m(k)} \frac{\beta}{\mathtt{3dzm(k)}} \frac{\mathtt{wp3(k)}}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)}, 2\epsilon\right)} \\
\mathtt{lhs(2,k)} &= \mathtt{lhs(2,k)} + \mathtt{a1m(k)} \frac{\beta}{\mathtt{3dzm(k)}} \left(\frac{\mathtt{wp3(k+1)}}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)}, 2\epsilon\right)} - \frac{\mathtt{wp3(k)}}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)}, 2\epsilon\right)} \right) \\
\mathtt{lhs(1,k)} &= \mathtt{lhs(1,k)} + \mathtt{a1m(k)} \frac{\beta}{\mathtt{3dzm(k)}} \frac{\mathtt{wp3(k+1)}}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)}, 2\epsilon\right)}
\end{aligned}
\tag{58}
$$

4.1.4 Term 4: dp2

$$
\begin{aligned}
& - \frac{\partial}{\partial z} \left[\left(K_{w2} + \nu_2\right) \frac{\partial}{\partial z} \overline{q_t'^2} \right] \\
& = - \frac{1}{\mathtt{dzm(k)}} \Bigg(\frac{\left(\mathtt{Kw2(k+1)} + \nu_2\right)\left(\mathtt{rtp2(k+1)} - \mathtt{rtp2(k)}\right)}{\mathtt{dzt(k+1)}} \\
& \qquad - \frac{\left(\mathtt{Kw2(k)} + \nu_2\right)\left(\mathtt{rtp2(k)} - \mathtt{rtp2(k-1)}\right)}{\mathtt{dzt(k)}} \Bigg)
\end{aligned}
\tag{59}
$$

Separating out the contributions:

$$
\begin{aligned}
\mathtt{lhs(3,k)} &= \mathtt{lhs(3,k)} - \frac{\mathtt{Kw2(k)} + \nu_2}{\mathtt{dzm(k)dzt(k)}} \\
\mathtt{lhs(2,k)} &= \mathtt{lhs(2,k)} + \frac{1}{\mathtt{dzm(k)}} \left(\frac{\mathtt{Kw2(k+1)} + \nu_2}{\mathtt{dzt(k+1)}} + \frac{\mathtt{Kw2(k)} + \nu_2}{\mathtt{dzt(k)}} \right) \\
\mathtt{lhs(1,k)} &= \mathtt{lhs(1,k)} - \frac{\mathtt{Kw2(k+1)} + \nu_2}{\mathtt{dzm(k)dzt(k+1)}}
\end{aligned}
\tag{60}
$$

4.1.5 Term 5: ta, explicit component

$$
\begin{aligned}
& - \left(1 - \frac{1}{3}\beta\right) \frac{\partial}{\partial z} \left(a_2 \frac{\overline{w'^3}}{\overline{w'^2}^2} \overline{w' q_t'}^2 \right) \Bigg|_{\mathtt{zm(k)}} \\
=& - \frac{1 - \frac{1}{3}\beta}{\mathtt{4dzm(k)}} \Bigg[\frac{\left(\mathtt{a1m(k)} + \mathtt{a1m(k+1)}\right)^2 \mathtt{wp3(k+1)} \left(\mathtt{wprtp(k)} + \mathtt{wprtp(k+1)}\right)^2}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)}, 2\epsilon\right)^2} \\
& \qquad - \frac{\left(\mathtt{a1m(k-1)} + \mathtt{a1m(k)}\right)^2 \mathtt{wp3(k)} \left(\mathtt{wprtp(k-1)} + \mathtt{wprtp(k)}\right)^2}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)}, 2\epsilon\right)^2} \Bigg]
\end{aligned}
\tag{61}
$$

Separating out the contributions:

$$
\begin{aligned}
&\mathtt{rhs(k)} \\
=&\mathtt{rhs(k)} - \frac{1-\frac{1}{3}\beta}{\mathtt{4dzm(k)}} \Bigg[\frac{(\mathtt{a1m(k)}+\mathtt{a1m(k+1)})^2\, \mathtt{wp3(k+1)}\, (\mathtt{wprtp(k)}+\mathtt{wprtp(k+1)})^2}{\max(\mathtt{wp2(k)}+\mathtt{wp2(k+1)}, 2\epsilon)^2} \\
&\qquad - \frac{(\mathtt{a1m(k-1)}+\mathtt{a1m(k)})^2\, \mathtt{wp3(k)}\, (\mathtt{wprtp(k-1)}+\mathtt{wprtp(k)})^2}{\max(\mathtt{wp2(k-1)}+\mathtt{wp2(k)}, 2\epsilon)^2} \Bigg]
\end{aligned} \tag{62}
$$

In order to increase numerical stability in the model, a_1 has been brought outside of the derivative. This is not mathematically correct, but it does help to increase stability. Brian Griffin. Feb. 21, 2008.

$$
\begin{aligned}
&- a_2 \left(1 - \frac{1}{3}\beta\right) \frac{\partial}{\partial z} \left(\frac{\overline{w'^3}}{\overline{w'^2}^2} \overline{w'r_t'}^2 \right) \Bigg|_{\mathtt{zm(k)}} \\
=&-\mathtt{a1m(k)}^2 \frac{1-\frac{1}{3}\beta}{\mathtt{dzm(k)}} \Bigg[\frac{\mathtt{wp3(k+1)}\, (\mathtt{wprtp(k)}+\mathtt{wprtp(k+1)})^2}{\max(\mathtt{wp2(k)}+\mathtt{wp2(k+1)}, 2\epsilon)^2} \\
&\qquad - \frac{\mathtt{wp3(k)}\, (\mathtt{wprtp(k-1)}+\mathtt{wprtp(k)})^2}{\max(\mathtt{wp2(k-1)}+\mathtt{wp2(k)}, 2\epsilon)^2} \Bigg]
\end{aligned} \tag{63}
$$

Separating out the contributions:

$$
\begin{aligned}
&\mathtt{rhs(k)} \\
=&\mathtt{rhs(k)} - \mathtt{a1m(k)}^2 \frac{1-\frac{1}{3}\beta}{\mathtt{dzm(k)}} \Bigg[\frac{\mathtt{wp3(k+1)}\, (\mathtt{wprtp(k)}+\mathtt{wprtp(k+1)})^2}{\max(\mathtt{wp2(k)}+\mathtt{wp2(k+1)}, 2\epsilon)^2} \\
&\qquad - \frac{\mathtt{wp3(k)}\, (\mathtt{wprtp(k-1)}+\mathtt{wprtp(k)})^2}{\max(\mathtt{wp2(k-1)}+\mathtt{wp2(k)}, 2\epsilon)^2} \Bigg]
\end{aligned} \tag{64}
$$

4.1.6 Term 6: tp

$$
-2\, \overline{w'r_t'} \frac{\partial \bar{r}_t}{\partial z} \Bigg|_{\mathtt{zm(k)}} = -2\mathtt{wprtp(k)} \frac{\mathtt{rtm(k+1)} - \mathtt{rtm(k)}}{\mathtt{dzm(k)}} \tag{65}
$$

Separating out the contributions:

$$
\mathtt{rhs(k)} = \mathtt{rhs(k)} - 2\mathtt{wprtp(k)} \frac{\mathtt{rtm(k+1)} - \mathtt{rtm(k)}}{\mathtt{dzm(k)}} \tag{66}
$$

4.1.7 Term 7: dp1, explicit component

$$
\mathtt{rhs(k)} = \mathtt{rhs(k)} + \frac{\mathtt{C_2}}{\mathtt{taum(k)}} \mathtt{rttol}^2 \tag{67}
$$

4.2 $\overline{q_t'\theta_l'}$

Start with (8) (for simplicity, neglect the $|_{\mathrm{cl}}$ term), substitute (31), assume steady-state and rearrange:

$$
\begin{aligned}
&\frac{C_2}{\tau}\overline{r_t'\theta_l'} + \bar{w}\frac{\partial \overline{r_t'\theta_l'}}{\partial z} + \frac{1}{3}\beta\frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}}{\overline{w'^2}}\,\overline{r_t'\theta_l'}\right) - \frac{\partial}{\partial z}\left[(K_{w2}+\nu_2)\,\frac{\partial}{\partial z}\overline{r_t'\theta_l'}\right] \\
&= -\left(1-\frac{1}{3}\beta\right)\frac{\partial}{\partial z}\left(a_2\frac{\overline{w'^3}}{\overline{w'^2}^2}\,\overline{w'r_t'}\,\overline{w'\theta_l'}\right) - \overline{w'r_t'}\frac{\partial\bar{\theta}_l}{\partial z} - \overline{w'\theta_l'}\frac{\partial\bar{r}_t}{\partial z}
\end{aligned}
\tag{68}
$$

As for the variances, the goal is to recast (68) so that $\overline{r_t'\theta_l'}$ can be computed using a tridiagonal solver:

$$
\mathtt{lhs(3,k)}\,\mathtt{rtpthlp(k-1)} + \mathtt{lhs(2,k)}\,\mathtt{rtpthlp(k)} + \mathtt{lhs(1,k)}\,\mathtt{rtpthlp(k+1)} = \mathtt{rhs(k)} \tag{69}
$$

We now compute the contributions of each term in (68) to $\mathtt{lhs(3,k)}$, $\mathtt{lhs(2,k)}$, $\mathtt{lhs(1,k)}$, and $\mathtt{rhs(k)}$.

4.2.1 Term 1: dp1

$$
\mathtt{lhs(2,k)} = \mathtt{lhs(2,k)} + \frac{\mathtt{C_2}}{\mathtt{taum(k)}} \tag{70}
$$

4.2.2 Term 2: ma

$$
\begin{aligned}
&\bar{w}\frac{\partial \overline{r_t'\theta_l'}}{\partial z}\bigg|_{\mathtt{zm(k)}} \\
&= \frac{\mathtt{wmm(k)}}{\mathtt{dzm(k)}}\left(\frac{1}{2}\left(\mathtt{rtpthlp(k)}+\mathtt{rtpthlp(k+1)}\right) - \frac{1}{2}\left(\mathtt{rtpthlp(k-1)}+\mathtt{rtpthlp(k)}\right)\right) \\
&= \frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}}\mathtt{rtpthlp(k+1)} - \frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}}\mathtt{rtpthlp(k-1)}
\end{aligned}
\tag{71}
$$

Separating out the contributions:

$$
\begin{aligned}
\mathtt{lhs(3,k)} &= \mathtt{lhs(3,k)} - \frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}} \\
\mathtt{lhs(1,k)} &= \mathtt{lhs(1,k)} + \frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}}
\end{aligned}
\tag{72}
$$

4.2.3 Term 3: ta, implicit component

$$
\begin{aligned}
&\frac{1}{3}\beta\frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}}{\overline{w'^2}}\overline{r_t'\theta_l'}\right)\Bigg|_{\texttt{zm(k)}} \\
&= \frac{\beta}{\texttt{6dzm(k)}}\left[\frac{(\texttt{a1m(k)}+\texttt{a1m(k+1)})\,\texttt{wp3(k+1)}\,(\texttt{rtpthlp(k)}+\texttt{rtpthlp(k+1)})}{\max(\texttt{wp2(k)}+\texttt{wp2(k+1)},2\epsilon)}\right. \\
&\qquad\left.-\frac{(\texttt{a1m(k-1)}+\texttt{a1m(k)})\,\texttt{wp3(k)}\,(\texttt{rtpthlp(k-1)}+\texttt{rtpthlp(k)})}{\max(\texttt{wp2(k-1)}+\texttt{wp2(k)},2\epsilon)}\right]
\end{aligned}
\tag{73}
$$

Separating out the contributions:

$$
\begin{aligned}
\texttt{lhs(3,k)} &= \texttt{lhs(3,k)} - \frac{\beta}{\texttt{6dzm(k)}}\frac{(\texttt{a1m(k-1)}+\texttt{a1m(k)})\,\texttt{wp3(k)}}{\max(\texttt{wp2(k-1)}+\texttt{wp2(k)},2\epsilon)} \\
\texttt{lhs(2,k)} &= \texttt{lhs(2,k)} + \frac{\beta}{\texttt{6dzm(k)}}\left(\frac{(\texttt{a1m(k)}+\texttt{a1m(k+1)})\,\texttt{wp3(k+1)}}{\max(\texttt{wp2(k)}+\texttt{wp2(k+1)},2\epsilon)} - \frac{(\texttt{a1m(k-1)}+\texttt{a1m(k)})\,\texttt{wp3(k)}}{\max(\texttt{wp2(k-1)}+\texttt{wp2(k)},2\epsilon)}\right) \\
\texttt{lhs(1,k)} &= \texttt{lhs(1,k)} + \frac{\beta}{\texttt{6dzm(k)}}\frac{(\texttt{a1m(k)}+\texttt{a1m(k+1)})\,\texttt{wp3(k+1)}}{\max(\texttt{wp2(k)}+\texttt{wp2(k+1)},2\epsilon)}
\end{aligned}
\tag{74}
$$

In order to increase numerical stability in the model, a_1 has been brought outside of the derivative. This is not mathematically correct, but it does help to increase stability. Brian Griffin. Feb. 21, 2008.

$$
\begin{aligned}
&a_1\frac{1}{3}\beta\frac{\partial}{\partial z}\left(\frac{\overline{w'^3}}{\overline{w'^2}}\overline{r_t'\theta_l'}\right)\Bigg|_{\texttt{zm(k)}} \\
&= \texttt{a1m(k)}\frac{\beta}{\texttt{3dzm(k)}}\left[\frac{\texttt{wp3(k+1)}\,(\texttt{rtpthlp(k)}+\texttt{rtpthlp(k+1)})}{\max(\texttt{wp2(k)}+\texttt{wp2(k+1)},2\epsilon)}\right. \\
&\qquad\left.-\frac{\texttt{wp3(k)}\,(\texttt{rtpthlp(k-1)}+\texttt{rtpthlp(k)})}{\max(\texttt{wp2(k-1)}+\texttt{wp2(k)},2\epsilon)}\right]
\end{aligned}
\tag{75}
$$

Separating out the contributions:

$$
\begin{aligned}
\texttt{lhs(3,k)} &= \texttt{lhs(3,k)} - \texttt{a1m(k)}\frac{\beta}{\texttt{3dzm(k)}}\frac{\texttt{wp3(k)}}{\max(\texttt{wp2(k-1)}+\texttt{wp2(k)},2\epsilon)} \\
\texttt{lhs(2,k)} &= \texttt{lhs(2,k)} + \texttt{a1m(k)}\frac{\beta}{\texttt{3dzm(k)}}\left(\frac{\texttt{wp3(k+1)}}{\max(\texttt{wp2(k)}+\texttt{wp2(k+1)},2\epsilon)} - \frac{\texttt{wp3(k)}}{\max(\texttt{wp2(k-1)}+\texttt{wp2(k)},2\epsilon)}\right) \\
\texttt{lhs(1,k)} &= \texttt{lhs(1,k)} + \texttt{a1m(k)}\frac{\beta}{\texttt{3dzm(k)}}\frac{\texttt{wp3(k+1)}}{\max(\texttt{wp2(k)}+\texttt{wp2(k+1)},2\epsilon)}
\end{aligned}
\tag{76}
$$

4.2.4 Term 4: dp2

$$
\begin{aligned}
& -\frac{\partial}{\partial z}\left[\left(K_{w2}+\nu_2\right)\frac{\partial}{\partial z}\overline{r_t'\theta_l'}\right] \\
& = -\frac{1}{\mathtt{dzm(k)}}\left(\frac{\left(\mathtt{Kw2(k+1)}+\nu_2\right)\left(\mathtt{rtpthlp(k+1)}-\mathtt{rtpthlp(k)}\right)}{\mathtt{dzt(k+1)}}\right. \\
& \qquad \left. -\frac{\left(\mathtt{Kw2(k)}+\nu_2\right)\left(\mathtt{rtpthlp(k)}-\mathtt{rtpthlp(k-1)}\right)}{\mathtt{dzt(k)}}\right)
\end{aligned}
\tag{77}
$$

Separating out the contributions:

$$
\begin{aligned}
\mathtt{lhs(3,k)} &= \mathtt{lhs(3,k)} - \frac{\mathtt{Kw2(k)}+\nu_2}{\mathtt{dzm(k)dzt(k)}} \\
\mathtt{lhs(2,k)} &= \mathtt{lhs(2,k)} + \frac{1}{\mathtt{dzm(k)}}\left(\frac{\mathtt{Kw2(k+1)}+\nu_2}{\mathtt{dzt(k+1)}}+\frac{\mathtt{Kw2(k)}+\nu_2}{\mathtt{dzt(k)}}\right) \\
\mathtt{lhs(1,k)} &= \mathtt{lhs(1,k)} - \frac{\mathtt{Kw2(k+1)}+\nu_2}{\mathtt{dzm(k)dzt(k+1)}}
\end{aligned}
\tag{78}
$$

4.2.5 Term 5: ta, explicit component

$$
\begin{aligned}
& -\left(1-\frac{1}{3}\beta\right)\frac{\partial}{\partial z}\left(a_2\frac{\overline{w'^3}}{\overline{w'^2}^2}\,\overline{w'r_t'}\,\overline{w'\theta_l'}\right)\Bigg|_{\mathtt{zm(k)}} \\
= & -\frac{1-\frac{1}{3}\beta}{\mathtt{4dzm(k)}} \\
& \times\left[\frac{\left(\mathtt{a1m(k)}+\mathtt{a1m(k+1)}\right)^2\mathtt{wp3(k+1)}\left(\mathtt{wprtp(k)}+\mathtt{wprtp(k+1)}\right)\left(\mathtt{wpthlp(k)}+\mathtt{wpthlp(k+1)}\right)}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)^2}\right. \\
& \quad\left.-\frac{\left(\mathtt{a1m(k-1)}+\mathtt{a1m(k)}\right)^2\mathtt{wp3(k)}\left(\mathtt{wprtp(k-1)}+\mathtt{wprtp(k)}\right)\left(\mathtt{wpthlp(k-1)}+\mathtt{wpthlp(k)}\right)}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)^2}\right]
\end{aligned}
\tag{79}
$$

Separating out the contributions:

$$
\begin{aligned}
\mathtt{rhs(k)} = & \mathtt{rhs(k)} \\
& -\frac{1-\frac{1}{3}\beta}{\mathtt{4dzm(k)}} \\
& \times\left[\frac{\left(\mathtt{a1m(k)}+\mathtt{a1m(k+1)}\right)^2\mathtt{wp3(k+1)}\left(\mathtt{wprtp(k)}+\mathtt{wprtp(k+1)}\right)\left(\mathtt{wpthlp(k)}+\mathtt{wpthlp(k+1)}\right)}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)^2}\right. \\
& \quad\left.-\frac{\left(\mathtt{a1m(k-1)}+\mathtt{a1m(k)}\right)^2\mathtt{wp3(k)}\left(\mathtt{wprtp(k-1)}+\mathtt{wprtp(k)}\right)\left(\mathtt{wpthlp(k-1)}+\mathtt{wpthlp(k)}\right)}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)^2}\right]
\end{aligned}
\tag{80}
$$

In order to increase numerical stability in the model, a_1 has been brought outside of the derivative. This is not mathematically correct, but it does help to increase stability. Brian Griffin. Feb. 21, 2008.

$$
\begin{aligned}
& - a_2 \left(1 - \frac{1}{3}\beta\right) \frac{\partial}{\partial z} \left(\frac{\overline{w'^3}}{\overline{w'^2}^2} \overline{w'r_t'}\, \overline{w'\theta_l'} \right) \Bigg|_{\mathtt{zm(k)}} \\
=& -\mathtt{a1m(k)}^2 \frac{1 - \frac{1}{3}\beta}{\mathtt{dzm(k)}} \\
& \times \left[\frac{\mathtt{wp3(k+1)}\,(\mathtt{wprtp(k)} + \mathtt{wprtp(k+1)})\,(\mathtt{wpthlp(k)} + \mathtt{wpthlp(k+1)})}{\max\left(\mathtt{wp2(k)} + \mathtt{wp2(k+1)}, 2\epsilon\right)^2} \right. \\
& \qquad \left. - \frac{\mathtt{wp3(k)}\,(\mathtt{wprtp(k-1)} + \mathtt{wprtp(k)})\,(\mathtt{wpthlp(k-1)} + \mathtt{wpthlp(k)})}{\max\left(\mathtt{wp2(k-1)} + \mathtt{wp2(k)}, 2\epsilon\right)^2} \right]
\end{aligned} \tag{81}
$$

Separating out the contributions:

$$
\begin{aligned}
\mathtt{rhs(k)} =& \mathtt{rhs(k)} \\
& -\mathtt{a1m(k)}^2 \frac{1 - \frac{1}{3}\beta}{\mathtt{dzm(k)}} \\
& \times \left[\frac{\mathtt{wp3(k+1)}\,(\mathtt{wprtp(k)} + \mathtt{wprtp(k+1)})\,(\mathtt{wpthlp(k)} + \mathtt{wpthlp(k+1)})}{\max\left(\mathtt{wp2(k)} + \mathtt{wp2(k+1)}, 2\epsilon\right)^2} \right. \\
& \qquad \left. - \frac{\mathtt{wp3(k)}\,(\mathtt{wprtp(k-1)} + \mathtt{wprtp(k)})\,(\mathtt{wpthlp(k-1)} + \mathtt{wpthlp(k)})}{\max\left(\mathtt{wp2(k-1)} + \mathtt{wp2(k)}, 2\epsilon\right)^2} \right]
\end{aligned} \tag{82}
$$

4.2.6 Terms 6 and 7: tp1 and tp2, respectively

$$
\begin{aligned}
& - \overline{w'q_t'} \frac{\partial \bar{\theta}_l}{\partial z} \bigg|_{\mathtt{zm(k)}} - \overline{w'\theta_l'} \frac{\partial \bar{q}_t}{\partial z} \bigg|_{\mathtt{zm(k)}} \\
& = -\mathtt{wprtp(k)} \frac{\mathtt{thlm(k+1)} - \mathtt{thlm(k)}}{\mathtt{dzm(k)}} - \mathtt{wpthlp(k)} \frac{\mathtt{rtm(k+1)} - \mathtt{rtm(k)}}{\mathtt{dzm(k)}}
\end{aligned} \tag{83}
$$

Separating out the contributions:

$$
\mathtt{rhs(k)} = \mathtt{rhs(k)} - \mathtt{wprtp(k)} \frac{\mathtt{thlm(k+1)} - \mathtt{thlm(k)}}{\mathtt{dzm(k)}} - \mathtt{wpthlp(k)} \frac{\mathtt{rtm(k+1)} - \mathtt{rtm(k)}}{\mathtt{dzm(k)}} \tag{84}
$$

5 Implicit solutions for the means and fluxes

$\bar{r}_t$ and $\overline{w'r_t'}$ can be solved simultaneously and implicitly. Start with eqs (3), (9) (for simplicity, neglect the $|_{\mathrm{sicl}}$ and $|_{\mathrm{cl}}$ terms), and substitute expression for the transport term (28):

$$
\frac{\partial \bar{r}_t}{\partial t} = -\bar{w} \frac{\partial \bar{r}_t}{\partial z} - \frac{\partial}{\partial z} \overline{w'r_t'} + \frac{\partial \bar{r}_t}{\partial t} \bigg|_{\mathrm{ls}} \tag{85}
$$

$$
\begin{aligned}
\frac{\partial \overline{w'r_t'}}{\partial t} =& -\bar{w}\frac{\partial \overline{w'r_t'}}{\partial z} - \frac{\partial}{\partial z}\left(a_1 \frac{\overline{w'^3}}{\overline{w'^2}}\overline{w'r_t'}\right) - \overline{w'^2}\frac{\partial \bar{r}_t}{\partial z} - (1-C_7)\overline{w'r_t'}\frac{\partial \bar{w}}{\partial z} + (1-C_7)\frac{g}{\theta_0}\overline{r_t'\theta_v'} \\
& - \frac{C_6}{\tau}\overline{w'r_t'} + \frac{\partial}{\partial z}\left[(K_{w6}+\nu_6)\frac{\partial}{\partial z}\overline{w'r_t'}\right]
\end{aligned}
\tag{86}
$$

After discretizing the time derivative and rearranging terms:

$$
\begin{aligned}
&\frac{\bar{r}_t^{t+\Delta t}}{\Delta t} + \bar{w}\frac{\partial \bar{r}_t^{t+\Delta t}}{\partial z} + \frac{\partial}{\partial z}\overline{w'r_t'}^{t+\Delta t} \\
&= \frac{\bar{r}_t^{t}}{\Delta t} + \left.\frac{\partial \bar{r}_t}{\partial t}\right|_{\mathrm{ls}}
\end{aligned}
\tag{87}
$$

$$
\begin{aligned}
&\frac{\overline{w'r_t'}^{t+\Delta t}}{\Delta t} + \bar{w}\frac{\partial}{\partial z}\overline{w'r_t'}^{t+\Delta t} + \frac{\partial}{\partial z}\left(a_1 \frac{\overline{w'^3}}{\overline{w'^2}}\overline{w'r_t'}^{t+\Delta t}\right) + \overline{w'^2}\frac{\partial \bar{r}_t^{t+\Delta t}}{\partial z} \\
&+ (1-C_7)\overline{w'r_t'}^{t+\Delta t}\frac{\partial \bar{w}}{\partial z} + \frac{C_6}{\tau}\overline{w'r_t'}^{t+\Delta t} - \frac{\partial}{\partial z}\left[(K_{w6}+\nu_6)\frac{\partial}{\partial z}\overline{w'r_t'}^{t+\Delta t}\right] \\
&= \frac{\overline{w'r_t'}^{t}}{\Delta t} + (1-C_7)\frac{g}{\theta_0}\overline{r_t'\theta_v'}^{t}
\end{aligned}
\tag{88}
$$

The LHSs of (87)-(88) are linear in $\bar{r}_t$ and $\overline{w'r_t'}$ and can therefore be rewritten in matrix form:

$$
\underbrace{\begin{pmatrix}
\cdots & \bar{r}_{t\,k}^{\text{impl.}} & \overline{w'r_t'}_{k}^{\text{impl.}} & \bar{r}_{t\,k+1}^{\text{impl.}} & \overline{w'r_t'}_{k+1}^{\text{impl.}} & \bar{r}_{t\,k+2}^{\text{impl.}} & \overline{w'r_t'}_{k+2}^{\text{impl.}} & \cdots \\
\cdots & \overline{w'r_t'}_{k-1}^{\text{impl.}} & \bar{r}_{t\,k}^{\text{impl.}} & \overline{w'r_t'}_{k}^{\text{impl.}} & \bar{r}_{t\,k+1}^{\text{impl.}} & \overline{w'r_t'}_{k+1}^{\text{impl.}} & \bar{r}_{t\,k+2}^{\text{impl.}} & \cdots \\
\cdots & \bar{r}_{t\,k-1}^{\text{impl.}} & \overline{w'r_t'}_{k-1}^{\text{impl.}} & \bar{r}_{t\,k}^{\text{impl.}} & \overline{w'r_t'}_{k}^{\text{impl.}} & \bar{r}_{t\,k+1}^{\text{impl.}} & \overline{w'r_t'}_{k+1}^{\text{impl.}} & \cdots \\
\cdots & \overline{w'r_t'}_{k-2}^{\text{impl.}} & \bar{r}_{t\,k-1}^{\text{impl.}} & \overline{w'r_t'}_{k-1}^{\text{impl.}} & \bar{r}_{t\,k}^{\text{impl.}} & \overline{w'r_t'}_{k}^{\text{impl.}} & \bar{r}_{t\,k+1}^{\text{impl.}} & \cdots \\
\cdots & \bar{r}_{t\,k-2}^{\text{impl.}} & \overline{w'r_t'}_{k-2}^{\text{impl.}} & \bar{r}_{t\,k-1}^{\text{impl.}} & \overline{w'r_t'}_{k-1}^{\text{impl.}} & \bar{r}_{t\,k}^{\text{impl.}} & \overline{w'r_t'}_{k}^{\text{impl.}} & \cdots
\end{pmatrix}}_{\texttt{LHS}\text{(Stored in compact format)}}
\begin{pmatrix}
\vdots \\
\bar{r}_{t\,k-1}^{t+\Delta t} \\
\overline{w'r_t'}_{k-1}^{t+\Delta t} \\
\bar{r}_{t\,k}^{t+\Delta t} \\
\overline{w'r_t'}_{k}^{t+\Delta t} \\
\bar{r}_{t\,k+1}^{t+\Delta t} \\
\overline{w'r_t'}_{k+1}^{t+\Delta t} \\
\vdots
\end{pmatrix}
=
\underbrace{\begin{pmatrix}
\vdots \\
\bar{r}_{t\,k-1}^{\text{expl.}} \\
\overline{w'r_t'}_{k-1}^{\text{expl.}} \\
\bar{r}_{t\,k}^{\text{expl.}} \\
\overline{w'r_t'}_{k}^{\text{expl.}} \\
\bar{r}_{t\,k+1}^{\text{expl.}} \\
\overline{w'r_t'}_{k+1}^{\text{expl.}} \\
\vdots
\end{pmatrix}}_{\texttt{RHS}}
\tag{89}
$$

The matrix lhs is obtained by vertical discretization of the LHSs, and the vector rhs by discretization of the RHSs. lhs is band-diagonal with two rows above and two below the main diagonal. lhs is stored in compact form in an array with dimensions $(5, 2\,\texttt{nzmax})$. rhs is a vector with dimension $(2\,\texttt{nzmax})$. lhs can be inverted efficiently using an LU decomposition algorithm for band diagonal matrices. The construction of the matrix lhs and vector rhs are as follows.

First, we compute the finite difference equivalent to (87):

$$
\begin{aligned}
&\frac{\texttt{rtm}^{\texttt{new}}(\texttt{k})}{\texttt{dt}} + \frac{\texttt{wmt(k)}}{\texttt{2dzt(k)}}\texttt{rtm}^{\texttt{new}}(\texttt{k}+1) - \frac{\texttt{wmt(k)}}{\texttt{2dzt(k)}}\texttt{rtm}^{\texttt{new}}(\texttt{k}-1) + \frac{\texttt{wprtp}^{\texttt{new}}(\texttt{k})}{\texttt{dzt(k)}} - \frac{\texttt{wprtp}^{\texttt{new}}(\texttt{k}-1)}{\texttt{dzt(k)}} \\
=&\frac{\texttt{rtm(k)}}{\texttt{dt}} + \texttt{rtm\_ls(k)}
\end{aligned}
$$

$$(90)$$

Contributions to *lhs* from (90) are:

$$\mathtt{lhs}(5,\mathtt{k\_xm}) = \mathtt{lhs}(5,\mathtt{k\_xm}) - \frac{\mathtt{wmt}(\mathtt{k})}{2\mathtt{dzt}(\mathtt{k})} \tag{91}$$

$$\mathtt{lhs}(4,\mathtt{k\_xm}) = \mathtt{lhs}(4,\mathtt{k\_xm}) - \frac{1}{\mathtt{dzt}(\mathtt{k})} \tag{92}$$

$$\mathtt{lhs}(3,\mathtt{k\_xm}) = \mathtt{lhs}(3,\mathtt{k\_xm}) + \frac{1}{\mathtt{dt}} \tag{93}$$

$$\mathtt{lhs}(2,\mathtt{k\_xm}) = \mathtt{lhs}(2,\mathtt{k\_xm}) + \frac{1}{\mathtt{dzt}(\mathtt{k})} \tag{94}$$

$$\mathtt{lhs}(1,\mathtt{k\_xm}) = \mathtt{lhs}(1,\mathtt{k\_xm}) + \frac{\mathtt{wmt}(\mathtt{k})}{2\mathtt{dzt}(\mathtt{k})} \tag{95}$$

Contributions to *rhs* from (90) are:

$$\mathtt{rhs}(\mathtt{k\_xm}) = \mathtt{rhs}(\mathtt{k\_xm}) + \frac{\mathtt{rtm}(\mathtt{k})}{\mathtt{dt}} + \mathtt{rtm\_ls}(\mathtt{k}) \tag{96}$$

where $\mathtt{k\_xm} = 2\mathtt{k} - 1$.

We now write the finite difference equivalent to (88):

$$\begin{aligned}
&\frac{\mathtt{wprtp}^{\mathtt{new}}(\mathtt{k})}{\mathtt{dt}} + \frac{\mathtt{wmm}(\mathtt{k})}{2\mathtt{dzm}(\mathtt{k})}\mathtt{wprtp}^{\mathtt{new}}(\mathtt{k}+1) - \frac{\mathtt{wmm}(\mathtt{k})}{2\mathtt{dzm}(\mathtt{k})}\mathtt{wprtp}^{\mathtt{new}}(\mathtt{k}-1) \\
&+\frac{1}{2\mathtt{dzm}(\mathtt{k})}\Bigg[-\frac{(\mathtt{a1m}(\mathtt{k}-1)+\mathtt{a1m}(\mathtt{k}))\,\mathtt{wp3}(\mathtt{k})}{\max\left(\mathtt{wp2}(\mathtt{k}-1)+\mathtt{wp2}(\mathtt{k}),2\epsilon\right)}\mathtt{wprtp}^{\mathtt{new}}(\mathtt{k}-1) \\
&\qquad+\left(\frac{(\mathtt{a1m}(\mathtt{k})+\mathtt{a1m}(\mathtt{k}+1))\,\mathtt{wp3}(\mathtt{k}+1)}{\max\left(\mathtt{wp2}(\mathtt{k})+\mathtt{wp2}(\mathtt{k}+1),2\epsilon\right)} - \frac{(\mathtt{a1m}(\mathtt{k}-1)+\mathtt{a1m}(\mathtt{k}))\,\mathtt{wp3}(\mathtt{k})}{\max\left(\mathtt{wp2}(\mathtt{k}-1)+\mathtt{wp2}(\mathtt{k}),2\epsilon\right)}\right)\mathtt{wprtp}^{\mathtt{new}}(\mathtt{k}) \\
&\qquad+\frac{(\mathtt{a1m}(\mathtt{k})+\mathtt{a1m}(\mathtt{k}+1))\,\mathtt{wp3}(\mathtt{k}+1)}{\max\left(\mathtt{wp2}(\mathtt{k})+\mathtt{wp2}(\mathtt{k}+1),2\epsilon\right)}\mathtt{wprtp}^{\mathtt{new}}(\mathtt{k}+1)\Bigg] \\
&+\mathtt{wp2}(\mathtt{k})\,\frac{\mathtt{rtm}^{\mathtt{new}}(\mathtt{k}+1)-\mathtt{rtm}^{\mathtt{new}}(\mathtt{k})}{\mathtt{dzm}(\mathtt{k})} + (1-\mathtt{C}_7)\,\mathtt{wprtp}^{\mathtt{new}}(\mathtt{k})\,\frac{\mathtt{wmt}(\mathtt{k}+1)-\mathtt{wmt}(\mathtt{k})}{\mathtt{dzm}(\mathtt{k})} \\
&+\frac{\mathtt{C}_6}{\mathtt{taum}(\mathtt{k})}\,\mathtt{wprtp}^{\mathtt{new}}(\mathtt{k}) \\
&-\frac{\mathtt{Kw6}(\mathtt{k})+\nu_6}{\mathtt{dzm}(\mathtt{k})\mathtt{dzt}(\mathtt{k})}\mathtt{wprtp}^{\mathtt{new}}(\mathtt{k}-1) \\
&+\frac{1}{\mathtt{dzm}(\mathtt{k})}\left(\frac{\mathtt{Kw6}(\mathtt{k}+1)+\nu_6}{\mathtt{dzt}(\mathtt{k}+1)} + \frac{\mathtt{Kw6}(\mathtt{k})+\nu_6}{\mathtt{dzt}(\mathtt{k})}\right)\mathtt{wprtp}^{\mathtt{new}}(\mathtt{k}) \\
&-\frac{\mathtt{Kw6}(\mathtt{k}+1)+\nu_6}{\mathtt{dzm}(\mathtt{k})\mathtt{dzt}(\mathtt{k}+1)}\mathtt{wprtp}^{\mathtt{new}}(\mathtt{k}+1) \\
=&\frac{\mathtt{wprtp}(\mathtt{k})}{\mathtt{dt}} + (1-\mathtt{C}_7)\frac{\mathtt{g}}{\theta_0}\mathtt{rtpthvp}(\mathtt{k})
\end{aligned}$$

$$ \tag{97} $$

Contributions to lhs from (97) are:

$$
\begin{aligned}
&\mathtt{lhs(5,k\_wpxp)} = \mathtt{lhs(5,k\_wpxp)} \\
&-\frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}} - \frac{1}{\mathtt{2dzm(k)}} \frac{(\mathtt{a1m(k-1)} + \mathtt{a1m(k)})\,\mathtt{wp3(k)}}{\max(\mathtt{wp2(k-1)} + \mathtt{wp2(k)}, 2\epsilon)} - \frac{\mathtt{Kw6(k)} + \nu_6}{\mathtt{dzm(k)dzt(k)}}
\end{aligned}
\tag{98}
$$

$$
\mathtt{lhs(4,k\_wpxp)} = \mathtt{lhs(4,k\_wpxp)} - \frac{\mathtt{wp2(k)}}{\mathtt{dzm(k)}}
\tag{99}
$$

$$
\begin{aligned}
&\mathtt{lhs(3,k\_wpxp)} = \mathtt{lhs(3,k\_wpxp)} \\
&+\frac{1}{\mathtt{dt}} + \frac{1}{\mathtt{2dzm(k)}} \left(\frac{(\mathtt{a1m(k)} + \mathtt{a1m(k+1)})\,\mathtt{wp3(k+1)}}{\max(\mathtt{wp2(k)} + \mathtt{wp2(k+1)}, 2\epsilon)} - \frac{(\mathtt{a1m(k-1)} + \mathtt{a1m(k)})\,\mathtt{wp3(k)}}{\max(\mathtt{wp2(k-1)} + \mathtt{wp2(k)}, 2\epsilon)} \right) \\
&+(1 - \mathtt{C_7}) \frac{\mathtt{wmt(k+1)} - \mathtt{wmt(k)}}{\mathtt{dzm(k)}} + \frac{\mathtt{C_6}}{\mathtt{taum(k)}} + \frac{1}{\mathtt{dzm(k)}} \left(\frac{\mathtt{Kw6(k+1)} + \nu_6}{\mathtt{dzt(k+1)}} + \frac{\mathtt{Kw6(k)} + \nu_6}{\mathtt{dzt(k)}} \right)
\end{aligned}
\tag{100}
$$

$$
\mathtt{lhs(2,k\_wpxp)} = \mathtt{lhs(2,k\_wpxp)} + \frac{\mathtt{wp2(k)}}{\mathtt{dzm(k)}}
\tag{101}
$$

$$
\begin{aligned}
&\mathtt{lhs(1,k\_wpxp)} = \mathtt{lhs(1,k\_wpxp)} \\
&+\frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}} + \frac{1}{\mathtt{2dzm(k)}} \frac{(\mathtt{a1m(k)} + \mathtt{a1m(k+1)})\,\mathtt{wp3(k+1)}}{\max(\mathtt{wp2(k)} + \mathtt{wp2(k+1)}, 2\epsilon)} - \frac{\mathtt{Kw6(k+1)} + \nu_6}{\mathtt{dzm(k)dzt(k+1)}}
\end{aligned}
\tag{102}
$$

Contributions to rhs from (97) are:

$$
\mathtt{rhs(k\_wpxp)} = \mathtt{rhs(k\_wpxp)} + \frac{\mathtt{wprtp(k)}}{\mathtt{dt}} + (1 - \mathtt{C_7}) \frac{\mathtt{g}}{\theta_0} \mathtt{rtpthvp(k)}
\tag{103}
$$

where $\mathtt{k\_wpxp} = 2\mathtt{k}$.

The procedure for solving implicitly for $\bar{\theta}_l$ and $\overline{w'\theta_l'}$ is identical. It leads to the same matrix lhs, so lhs needs to be inverted only once.

6 Implicit solution for the vertical velocity moments

Start with equations (5) and (11) (for simplicity, neglect the $|_{\mathrm{pd}}$ and $|_{\mathrm{cl}}$ terms):

$$
\begin{aligned}
\frac{\partial \overline{w'^2}}{\partial t} = &- \bar{w} \frac{\partial \overline{w'^2}}{\partial z} - \frac{\partial \overline{w'^3}}{\partial z} - 2\overline{w'^2} \frac{\partial \bar{w}}{\partial z} + \frac{2g}{\theta_0} \overline{w'\theta_v'} \\
&- \frac{C_4}{\tau} \left(\overline{w'^2} - \frac{2}{3} \bar{e} \right) - C_5 \left(-2\overline{w'^2} \frac{\partial \bar{w}}{\partial z} + \frac{2g}{\theta_0} \overline{w'\theta_v'} \right) + \frac{2}{3} C_5 \left(\frac{g}{\theta_0} \overline{w'\theta_v'} - \overline{u'w'} \frac{\partial \bar{u}}{\partial z} - \overline{v'w'} \frac{\partial \bar{v}}{\partial z} \right) \\
&- \frac{C_1}{\tau} \left(\overline{w'^2} - w|_{\mathrm{tol}}^2 \right) + \frac{\partial}{\partial z} \left[(K_{w1} + \nu_1) \frac{\partial}{\partial z} \overline{w'^2} \right]
\end{aligned}
$$

$$ \tag{104} $$

$$
\begin{aligned}
\frac{\partial \overline{w'^3}}{\partial t} =& -\bar{w}\frac{\partial \overline{w'^3}}{\partial z} - \frac{\partial \overline{w'^4}}{\partial z} + 3\overline{w'^2}\frac{\partial \overline{w'^2}}{\partial z} - 3\overline{w'^3}\frac{\partial \bar{w}}{\partial z} + \frac{3g}{\theta_0}\overline{w'^2\theta'_v} \\
&- \frac{C_8}{\tau}\left(C_{8b}\, Skw^4 + 1\right)\overline{w'^3} - C_{11}\left(-3\overline{w'^3}\frac{\partial \bar{w}}{\partial z} + \frac{3g}{\theta_0}\overline{w'^2\theta'_v}\right) + \frac{\partial}{\partial z}\left[\left(K_{w8} + \nu_8\right)\frac{\partial}{\partial z}\overline{w'^3}\right]
\end{aligned}
\tag{105}
$$

Using (26), we can rewrite the transport and production terms in (105):

$$
\begin{aligned}
&- \frac{\partial \overline{w'^4}}{\partial z} + 3\overline{w'^2}\frac{\partial \overline{w'^2}}{\partial z} \\
&= -\frac{\partial}{\partial z}\left(\overline{w'^4} - \frac{3}{2}\overline{w'^2}^2\right) \\
&= -\frac{\partial}{\partial z}\left(\tilde{a}_3\overline{w'^2}^2\right) - \frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}^2}{\overline{w'^2}}\right)
\end{aligned}
\tag{106}
$$

where $\tilde{a}_3 = a_3 + 3/2$. Rearranging terms and making use of (12):

$$
\begin{aligned}
&\frac{\partial \overline{w'^2}}{\partial t} + \bar{w}\frac{\partial \overline{w'^2}}{\partial z} + \frac{\partial \overline{w'^3}}{\partial z} + \frac{C_1}{\tau}\overline{w'^2} - \frac{\partial}{\partial z}\left[\left(K_{w1} + \nu_1\right)\frac{\partial}{\partial z}\overline{w'^2}\right] \\
&= +(1 - C_5)\frac{2g}{\theta_0}\overline{w'\theta'_v} - 2(1 - C_5)\overline{w'^2}\frac{\partial \bar{w}}{\partial z} + \frac{C_1}{\tau}\, w|^2_{\mathrm{tol}} + \frac{2}{3}C_5\left(\frac{g}{\theta_0}\overline{w'\theta'_v} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{v'w'}\frac{\partial \bar{v}}{\partial z}\right)
\end{aligned}
\tag{107}
$$

$$
\begin{aligned}
&\frac{\partial \overline{w'^3}}{\partial t} + \bar{w}\frac{\partial \overline{w'^3}}{\partial z} - \frac{\partial}{\partial z}\left[\left(K_{w8} + \nu_8\right)\frac{\partial}{\partial z}\overline{w'^3}\right] + \frac{C_8}{\tau}\left(C_{8b}\, Skw^4 + 1\right)\overline{w'^3} \\
&\qquad + \frac{\partial}{\partial z}\left(\tilde{a}_3\overline{w'^2}^2\right) + \frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}^2}{\overline{w'^2}}\right) \\
&= +(1 - C_{11})\frac{3g}{\theta_0}\overline{w'^2\theta'_v} - 3(1 - C_{11})\overline{w'^3}\frac{\partial \bar{w}}{\partial z}
\end{aligned}
\tag{108}
$$

6.1 $\overline{w'^2}$

Terms on the LHS of (107) are treated fully implicitly, except for the diffusion term which is treated with a Crank-Nicholson time step. Terms on the RHS explicitly:

$$
\begin{aligned}
&\frac{\overline{w'^2}^{t+\Delta t}}{\Delta t} + \bar{w}\frac{\partial \overline{w'^2}^{t+\Delta t}}{\partial z} + \frac{\partial \overline{w'^3}^{t+\Delta t}}{\partial z} + \frac{C_1}{\tau}\overline{w'^2}^{t+\Delta t} - \frac{1}{2}\frac{\partial}{\partial z}\left[\left(K_{w1} + \nu_1\right)\frac{\partial}{\partial z}\overline{w'^2}^{t+\Delta t}\right] \\
&= \frac{\overline{w'^2}^t}{\Delta t} + \frac{1}{2}\frac{\partial}{\partial z}\left[\left(K_{w1} + \nu_1\right)\frac{\partial}{\partial z}\overline{w'^2}^t\right] + \left.\overline{w'^2}\right|_{\mathrm{expl}}
\end{aligned}
\tag{109}
$$

where

$$
\begin{aligned}
\left.\overline{w'^2}\right|_{\mathrm{expl}} =& + (1 - C_5)\frac{2g}{\theta_0}\overline{w'\theta'_v}^t - 2(1 - C_5)\overline{w'^2}^t\frac{\partial \bar{w}}{\partial z} + \frac{C_1}{\tau}\, w|^2_{\mathrm{tol}} \\
&+ \frac{2}{3}C_5\left(\frac{g}{\theta_0}\overline{w'\theta'_v} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{v'w'}\frac{\partial \bar{v}}{\partial z}\right)^t
\end{aligned}
\tag{110}
$$

The next step consists of writing the finite difference equivalent to (109):

$$
\begin{aligned}
&\frac{\mathtt{wp2^{new}(k)}}{\mathtt{dt}} + \mathtt{wmm(k)}\frac{\mathtt{wp2^{new}(k+1)} - \mathtt{wp2^{new}(k-1)}}{\mathtt{2\ dzm(k)}} \\
&+\frac{\mathtt{wp3^{new}(k+1)} - \mathtt{wp3^{new}(k)}}{\mathtt{dzm(k)}} + \frac{\mathtt{C_1}}{\mathtt{taum(k)}}\mathtt{wp2^{new}(k)} \\
&-\frac{\mathtt{Kw1(k)} + \nu_1}{\mathtt{2dzm(k)dzt(k)}}\mathtt{wp2^{new}(k-1)} \\
&+\frac{1}{\mathtt{2dzm(k)}}\left(\frac{\mathtt{Kw1(k+1)} + \nu_1}{\mathtt{dzt(k+1)}} + \frac{\mathtt{Kw1(k)} + \nu_1}{\mathtt{dzt(k)}}\right)\mathtt{wp2^{new}(k)} \\
&-\frac{\mathtt{Kw1(k+1)} + \nu_1}{\mathtt{2dzm(k)dzt(k+1)}}\mathtt{wp2^{new}(k+1)} \\
=&\frac{\mathtt{wp2(k)}}{\mathtt{dt}} \\
&+\frac{\mathtt{Kw1(k)} + \nu_1}{\mathtt{2dzm(k)dzt(k)}}\mathtt{wp2(k-1)} \\
&-\frac{1}{\mathtt{2dzm(k)}}\left(\frac{\mathtt{Kw1(k+1)} + \nu_1}{\mathtt{dzt(k+1)}} + \frac{\mathtt{Kw1(k)} + \nu_1}{\mathtt{dzt(k)}}\right)\mathtt{wp2(k)} \\
&+\frac{\mathtt{Kw1(k+1)} + \nu_1}{\mathtt{2dzm(k)dzt(k+1)}}\mathtt{wp2(k+1)} \\
&+\mathtt{wp2t(k)}
\end{aligned}
\tag{111}
$$

where `wp2t(k)` is the finite difference equivalent to (110) at level `zm(k)`.

6.1.1 Using an anisotropic solution for the horizontal wind

As an alternative to assuming $\bar{e} = \frac{3}{2}\overline{w'^2}$, we can calculate $\overline{v'^2}$ and $\overline{u'^2}$ and then compute $\bar{e}$ accordingly. The term with a C_4 coefficient in $\overline{w'^2}$ equation is then non-zero and must be accounted for. Starting with the 5$^{\mathrm{th}}$ term of the original $\overline{w'^2}$ equation:

$$
\frac{\partial\overline{w'^2}}{\partial t} = \cdots - \frac{C_4}{\tau}\left(\overline{w'^2} - \frac{2}{3}\bar{e}\right)\cdots
\tag{112}
$$

From which we obtain the finite difference equivalent:

$$
\begin{aligned}
& -\frac{C_4}{\tau}\left(\overline{w'^2} - \frac{2}{3}\bar{e}\right)\bigg|_{\mathtt{zm(k)}} \\
&= -\frac{\mathtt{C_4}}{\mathtt{taum(k)}}\left(\mathtt{wp2(k)} - \frac{2}{3}\mathtt{em(k)}\right) \\
&= -\frac{\mathtt{C_4}}{\mathtt{taum(k)}}\left(\mathtt{wp2(k)} - \frac{\mathtt{wp2(k)} + \mathtt{up2(k)} + \mathtt{vp2(k)}}{3}\right) \\
&= -\frac{2\,\mathtt{C_4}\,\mathtt{wp2(k)}}{3\mathtt{taum(k)}} + \frac{\mathtt{C_4}\left(\mathtt{up2(k)} + \mathtt{vp2(k)}\right)}{3\mathtt{taum(k)}}
\end{aligned}
\tag{113}
$$

Separating out the contributions:

$$
\begin{aligned}
\mathtt{lhs(3,k\_wp2)} &= \mathtt{lhs(3,k\_wp2)} + \frac{\mathtt{C_4}\left(\mathtt{up2(k)} + \mathtt{vp2(k)}\right)}{3\mathtt{taum(k)}} \\
\mathtt{rhs(k\_wp2)} &= \mathtt{rhs(k\_wp2)} + \frac{2\,\mathtt{C_4}\,\mathtt{wp2(k)}}{3\mathtt{taum(k)}}
\end{aligned}
\tag{114}
$$

6.2 $\overline{w'^3}$

The first two terms on the LHS of (108) (i.e. mean advection dissipation) are treated implicitly, and the last three terms on the LHS are teated semi-implicitly (they are linearized and the linearized portion is treated implicitly, the rest explicitly). The terms on the RHS are treated fully explicit discretization. Let's focus first on the third term on the LHS, L_3:

$$
L_3 \equiv \frac{C_8}{\tau}\left(C_{8b}\,Skw^4 + 1\right)\overline{w'^3} = \frac{C_8}{\tau}\left(C_{8b}\,\frac{\overline{w'^3}^5}{\overline{w'^2}^6} + \overline{w'^3}\right) \tag{115}
$$

We linearize L_3 with respect to $\overline{w'^3}$:

$$
L_3\left(\overline{w'^3}^{t+\Delta t}\right) \approx L_3\left(\overline{w'^3}^t\right) + \frac{\partial L_3}{\partial \overline{w'^3}}\bigg|_t \left(\overline{w'^3}^{t+\Delta t} - \overline{w'^3}^t\right) \tag{116}
$$

where

$$
\frac{\partial L_3}{\partial \overline{w'^3}}\bigg|_t = \frac{C_8}{\tau}\left(5\,C_{8b}\,\frac{\overline{w'^3}^4}{\overline{w'^2}^6} + 1\right) \tag{117}
$$

Combining (115), (117) with (116):

$$
\begin{aligned}
& L_3\left(\overline{w'^3}^{t+\Delta t}\right) \\
&= \frac{C_8}{\tau}\left(C_{8b}\,\frac{\overline{w'^3}^{t5}}{\overline{w'^2}^{t6}} + \overline{w'^3}^t\right) + \frac{C_8}{\tau}\left(5\,C_{8b}\,\frac{\overline{w'^3}^{t4}}{\overline{w'^2}^{t6}} + 1\right)\left(\overline{w'^3}^{t+\Delta t} - \overline{w'^3}^t\right) \\
&= -\frac{C_8}{\tau}\left(4\,C_{8b}\,Skw^{t4}\right)\overline{w'^3}^t + \frac{C_8}{\tau}\left(5\,C_{8b}\,Skw^{t4} + 1\right)\overline{w'^3}^{t+\Delta t}
\end{aligned}
\tag{118}
$$

For reasons of numerical stability we now linearize the fourth term on the LHS in a formulation that is fully implicit (108):

$$
\begin{aligned}
&\frac{\partial}{\partial z}\left[\tilde{a}_3\left(\overline{w'^2}^{t+\Delta t}\right)^2\right] \\
&\approx \frac{\partial}{\partial z}\left[\tilde{a}_3\overline{w'^2}^{t2} + 2\tilde{a}_3\overline{w'^2}^{t}\left(\overline{w'^2}^{t+\Delta t} - \overline{w'^2}^{t}\right)\right] \\
&= \frac{\partial}{\partial z}\left(2\tilde{a}_3\overline{w'^2}^{t}\,\overline{w'^2}^{t+\Delta t}\right) - \frac{\partial}{\partial z}\left(\tilde{a}_3\overline{w'^2}^{t2}\right)
\end{aligned}
\tag{119}
$$

We repeat for the fifth term on the LHS of (108):

$$
\begin{aligned}
&\frac{\partial}{\partial z}\left(a_1\frac{\left(\overline{w'^3}^{t+\Delta t}\right)^2}{\overline{w'^2}^{t}}\right) \\
&\approx \frac{\partial}{\partial z}\left[a_1\frac{\left(\overline{w'^3}^{t}\right)^2}{\overline{w'^2}^{t}} + 2a_1\frac{\overline{w'^3}^{t}}{\overline{w'^2}^{t}}\left(\overline{w'^3}^{t+\Delta t} - \overline{w'^3}^{t}\right)\right] \\
&= \frac{\partial}{\partial z}\left(2a_1\frac{\overline{w'^3}^{t}\,\overline{w'^3}^{t+\Delta t}}{\overline{w'^2}^{t}}\right) - \frac{\partial}{\partial z}\left(a_1\frac{\left(\overline{w'^3}^{t}\right)^2}{\overline{w'^2}^{t}}\right)
\end{aligned}
\tag{120}
$$

We can now assemble the time discrete equivalent to (108) using (118), (119) and (120):

$$
\begin{aligned}
&\frac{\overline{w'^3}^{t+\Delta t}}{\Delta t} + \bar{w}\frac{\partial\overline{w'^3}^{t+\Delta t}}{\partial z} - \frac{1}{2}\frac{\partial}{\partial z}\left[(K_{w8}+\nu_8)\frac{\partial}{\partial z}\overline{w'^3}^{t+\Delta t}\right] + \frac{C_8}{\tau}\left(5\,C_{8b}\,Skw^{t4}+1\right)\overline{w'^3}^{t+\Delta t} \\
&+ \frac{\partial}{\partial z}\left(\tilde{a}_3\overline{w'^2}^{t}\,\overline{w'^2}^{t+\Delta t}\right) + \frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}^{t}\,\overline{w'^3}^{t+\Delta t}}{\overline{w'^2}^{t}}\right) \\
&= \frac{\overline{w'^3}^{t}}{\Delta t} + \frac{1}{2}\frac{\partial}{\partial z}\left[(K_{w8}+\nu_8)\frac{\partial}{\partial z}\overline{w'^3}^{t}\right] + \frac{C_8}{\tau}\left(4\,C_{8b}\,Skw^{t4}\right)\overline{w'^3}^{t} + \left.\overline{w'^3}\right|_{\mathrm{expl}}
\end{aligned}
\tag{121}
$$

where

$$
\left.\overline{w'^3}\right|_{\mathrm{expl}} = +(1-C_{11})\frac{3g}{\theta_0}\overline{w'^2\theta'_v}^{t} - 3(1-C_{11})\overline{w'^3}^{t}\frac{\partial\bar{w}}{\partial z}
\tag{122}
$$

Finally, we derive the finite difference form of (121):

$$
\begin{aligned}
&\frac{\mathtt{wp3}^{\mathtt{new}}(\mathtt{k})}{\mathtt{dt}} + \mathtt{wmt}(\mathtt{k}) \frac{\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}+1) - \mathtt{wp3}^{\mathtt{new}}(\mathtt{k}-1)}{2\,\mathtt{dzt}(\mathtt{k})} \\
&- \frac{1}{2\mathtt{dzt}(\mathtt{k})} \Bigg((\mathtt{Kw8}(\mathtt{k}) + \nu_8) \frac{\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}+1) - \mathtt{wp3}^{\mathtt{new}}(\mathtt{k})}{\mathtt{dzm}(\mathtt{k})} \\
&\qquad\qquad - (\mathtt{Kw8}(\mathtt{k}-1) + \nu_8) \frac{\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}) - \mathtt{wp3}^{\mathtt{new}}(\mathtt{k}-1)}{\mathtt{dzm}(\mathtt{k}-1)} \Bigg) \\
&+ \frac{\mathtt{C}_8}{\mathtt{taut}(\mathtt{k})} \left(5\,\mathtt{C}_{8\mathtt{b}}\,\mathtt{Skwt}(\mathtt{k})^4 + 1\right) \mathtt{wp3}^{\mathtt{new}}(\mathtt{k}) \\
&+ \frac{1}{\mathtt{dzt}(\mathtt{k})} \left(\mathtt{a3m}(\mathtt{k})\mathtt{wp2}(\mathtt{k})\mathtt{wp2}^{\mathtt{new}}(\mathtt{k}) - \mathtt{a3m}(\mathtt{k}-1)\mathtt{wp2}(\mathtt{k}-1)\mathtt{wp2}^{\mathtt{new}}(\mathtt{k}-1)\right) \\
&+ \frac{1}{2\,\mathtt{dzt}(\mathtt{k})} \Bigg(\frac{\mathtt{a1m}(\mathtt{k})\left(\mathtt{wp3}(\mathtt{k}) + \mathtt{wp3}(\mathtt{k}+1)\right)\left(\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}) + \mathtt{wp3}^{\mathtt{new}}(\mathtt{k}+1)\right)}{\max\left(\mathtt{wp2}(\mathtt{k}), \epsilon\right)} \\
&\qquad\qquad - \frac{\mathtt{a1m}(\mathtt{k}-1)\left(\mathtt{wp3}(\mathtt{k}-1) + \mathtt{wp3}(\mathtt{k})\right)\left(\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}-1) + \mathtt{wp3}^{\mathtt{new}}(\mathtt{k})\right)}{\max\left(\mathtt{wp2}(\mathtt{k}-1), \epsilon\right)} \Bigg) \\
=& \frac{\mathtt{wp3}(\mathtt{k})}{\mathtt{dt}} + \frac{1}{2\mathtt{dzt}(\mathtt{k})} \Bigg((\mathtt{Kw8}(\mathtt{k}) + \nu_8) \frac{\mathtt{wp3}(\mathtt{k}+1) - \mathtt{wp3}(\mathtt{k})}{\mathtt{dzm}(\mathtt{k})} \\
&\qquad\qquad - (\mathtt{Kw8}(\mathtt{k}-1) + \nu_8) \frac{\mathtt{wp3}(\mathtt{k}) - \mathtt{wp3}(\mathtt{k}-1)}{\mathtt{dzm}(\mathtt{k}-1)} \Bigg) \\
&+ \frac{\mathtt{C}_8}{\mathtt{taut}(\mathtt{k})} \left(4\,\mathtt{C}_{8\mathtt{b}}\,\mathtt{Skwt}(\mathtt{k})^4\right) \mathtt{wp3}(\mathtt{k}) + \mathtt{wp3t}(\mathtt{k})
\end{aligned}
\tag{123}
$$

where $\mathtt{wp3t}(\mathtt{k})$ is the finite difference equivalent to (122) at level $\mathtt{zt}(\mathtt{k})$.

In order to increase numerical stability in the model, a_1 has been brought outside of the derivative. Besides a_1, a_3 has been previously brought outside of the derivative for the same purpose. This is not mathematically correct, but it does help to increase stability. Brian Griffin. Feb. 21, 2008.

$$
\begin{aligned}
&\frac{\mathtt{wp3}^{\mathtt{new}}(\mathtt{k})}{\mathtt{dt}} + \mathtt{wmt}(\mathtt{k})\frac{\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}+1) - \mathtt{wp3}^{\mathtt{new}}(\mathtt{k}-1)}{2\,\mathtt{dzt}(\mathtt{k})} \\
&-\frac{1}{2\mathtt{dzt}(\mathtt{k})}\Bigg((\mathtt{Kw8}(\mathtt{k}) + \nu_8)\,\frac{\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}+1) - \mathtt{wp3}^{\mathtt{new}}(\mathtt{k})}{\mathtt{dzm}(\mathtt{k})} \\
&\qquad\qquad - (\mathtt{Kw8}(\mathtt{k}-1) + \nu_8)\,\frac{\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}) - \mathtt{wp3}^{\mathtt{new}}(\mathtt{k}-1)}{\mathtt{dzm}(\mathtt{k}-1)}\Bigg) \\
&+\frac{\mathtt{C_8}}{\mathtt{taut}(\mathtt{k})}\left(5\,\mathtt{C_{8b}}\,\mathtt{Skwt}(\mathtt{k})^4 + 1\right)\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}) \\
&+\left(\frac{\mathtt{a3m}(\mathtt{k}) + \mathtt{a3m}(\mathtt{k}-1)}{2}\right)\frac{2}{\mathtt{dzt}(\mathtt{k})}\left(\mathtt{wp2}(\mathtt{k})\mathtt{wp2}^{\mathtt{new}}(\mathtt{k}) - \mathtt{wp2}(\mathtt{k}-1)\mathtt{wp2}^{\mathtt{new}}(\mathtt{k}-1)\right) \\
&+\left(\frac{\mathtt{a1m}(\mathtt{k}) + \mathtt{a1m}(\mathtt{k}-1)}{2}\right)\frac{1}{2\,\mathtt{dzt}(\mathtt{k})}\Bigg(\frac{(\mathtt{wp3}(\mathtt{k}) + \mathtt{wp3}(\mathtt{k}+1))\,(\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}) + \mathtt{wp3}^{\mathtt{new}}(\mathtt{k}+1))}{\max(\mathtt{wp2}(\mathtt{k}),\epsilon)} \\
&\qquad\qquad - \frac{(\mathtt{wp3}(\mathtt{k}-1) + \mathtt{wp3}(\mathtt{k}))\,(\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}-1) + \mathtt{wp3}^{\mathtt{new}}(\mathtt{k}))}{\max(\mathtt{wp2}(\mathtt{k}-1),\epsilon)}\Bigg) \qquad (124) \\
=&\frac{\mathtt{wp3}(\mathtt{k})}{\mathtt{dt}} + \frac{1}{2\mathtt{dzt}(\mathtt{k})}\Bigg((\mathtt{Kw8}(\mathtt{k}) + \nu_8)\,\frac{\mathtt{wp3}(\mathtt{k}+1) - \mathtt{wp3}(\mathtt{k})}{\mathtt{dzm}(\mathtt{k})} \\
&\qquad\qquad - (\mathtt{Kw8}(\mathtt{k}-1) + \nu_8)\,\frac{\mathtt{wp3}(\mathtt{k}) - \mathtt{wp3}(\mathtt{k}-1)}{\mathtt{dzm}(\mathtt{k}-1)}\Bigg) \\
&+\frac{\mathtt{C_8}}{\mathtt{taut}(\mathtt{k})}\left(4\,\mathtt{C_{8b}}\,\mathtt{Skwt}(\mathtt{k})^4\right)\mathtt{wp3}(\mathtt{k}) \\
&+\left(\frac{\mathtt{a3m}(\mathtt{k}) + \mathtt{a3m}(\mathtt{k}-1)}{2}\right)\frac{\mathtt{wp2}(\mathtt{k})^2 - \mathtt{wp2}(\mathtt{k}-1)^2}{\mathtt{dzt}(\mathtt{k})} \\
&+\left(\frac{\mathtt{a1m}(\mathtt{k}) + \mathtt{a1m}(\mathtt{k}-1)}{2}\right)\frac{1}{4\,\mathtt{dzt}(\mathtt{k})}\left(\frac{(\mathtt{wp3}(\mathtt{k}) + \mathtt{wp3}(\mathtt{k}+1))^2}{\max(\mathtt{wp2}(\mathtt{k}),\epsilon)} - \frac{(\mathtt{wp3}(\mathtt{k}-1) + \mathtt{wp3}(\mathtt{k}))^2}{\max(\mathtt{wp2}(\mathtt{k}-1),\epsilon)}\right) \\
&+\mathtt{wp3t}(\mathtt{k})
\end{aligned}
$$

6.3 Matrix form

The final step is to rewrite (111) and (123) in matrix form:

$$
\underbrace{\begin{pmatrix}
\cdots & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k{+}1}) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k{+}1}) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k{+}2}) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k{+}2}) & \cdots \\
\cdots & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k{-}1}) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k{+}1}) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k{+}1}) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k{+}2}) & \cdots \\
\cdots & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k{-}1}) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k{-}1}) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k{+}1}) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k{+}1}) & \cdots \\
\cdots & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k{-}2}) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k{-}1}) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k{-}1}) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k{+}1}) & \cdots \\
\cdots & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k{-}2}) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k{-}2}) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k{-}1}) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k{-}1}) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}) & \cdots
\end{pmatrix}}_{\mathtt{LHS}_{\mathtt{wp23}}\text{(Stored in compact format)}}
\begin{pmatrix}
\vdots \\
\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}-1) \\
\mathtt{wp2}^{\mathtt{new}}(\mathtt{k}-1) \\
\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}) \\
\mathtt{wp2}^{\mathtt{new}}(\mathtt{k}) \\
\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}+1) \\
\mathtt{wp2}^{\mathtt{new}}(\mathtt{k}+1) \\
\vdots
\end{pmatrix}
= \underbrace{\begin{pmatrix}
\vdots \\
\mathtt{wp3}^{\mathtt{expl}}(\mathtt{k{-}1}) \\
\mathtt{wp2}^{\mathtt{expl}}(\mathtt{k{-}1}) \\
\mathtt{wp3}^{\mathtt{expl}}(\mathtt{k}) \\
\mathtt{wp2}^{\mathtt{expl}}(\mathtt{k}) \\
\mathtt{wp3}^{\mathtt{expl}}(\mathtt{k{+}1}) \\
\mathtt{wp2}^{\mathtt{expl}}(\mathtt{k{+}1}) \\
\vdots
\end{pmatrix}}_{\mathtt{RHS}_{\mathtt{wp23}}}
\tag{125}
$$

$lhs_{\mathtt{wp23}}$ is a band-diagonal matrix with two rows above and two below the main diagonal. $lhs_{\mathtt{wp23}}$ is stored in compact form in an array with dimensions $(\mathtt{5}, 2\,\mathtt{nzmax})$. $rhs_{\mathtt{wp23}}$ is a vector with dimension $(2\,\mathtt{nzmax})$. $lhs_{\mathtt{wp23}}$ can be inverted efficiently using a LU decomposition algorithm for band diagonal matrices.

Contributions to $lhs_{\mathtt{wp23}}$ from (111):

$$\mathtt{lhs}(\mathtt{k\_wp2},5) = \mathtt{lhs}(\mathtt{k\_wp2},5) - \frac{\mathtt{Kw1}(\mathtt{k}) + \nu_1}{2\mathtt{dzm}(\mathtt{k})\mathtt{dzt}(\mathtt{k})} - \frac{\mathtt{wmm}(\mathtt{k})}{2\ \mathtt{dzm}(\mathtt{k})} \tag{126}$$

$$\mathtt{lhs}(\mathtt{k\_wp2},4) = \mathtt{lhs}(\mathtt{k\_wp2},4) - \frac{1}{\mathtt{dzm}(\mathtt{k})} \tag{127}$$

$$\mathtt{lhs}(\mathtt{k\_wp2},3) = \mathtt{lhs}(\mathtt{k\_wp2},3) + \frac{1}{\mathtt{dt}} + \frac{\mathtt{C}_1}{\mathtt{taum}(\mathtt{k})} + \frac{1}{2\mathtt{dzm}(\mathtt{k})}\left(\frac{\mathtt{Kw1}(\mathtt{k}+1) + \nu_1}{\mathtt{dzt}(\mathtt{k}+1)} + \frac{\mathtt{Kw1}(\mathtt{k}) + \nu_1}{\mathtt{dzt}(\mathtt{k})}\right) \tag{128}$$

$$\mathtt{lhs}(\mathtt{k\_wp2},2) = \mathtt{lhs}(\mathtt{k\_wp2},2) + \frac{1}{\mathtt{dzm}(\mathtt{k})} \tag{129}$$

$$\mathtt{lhs(k\_wp2,1)} = \mathtt{lhs(k\_wp2,1)} - \frac{\mathtt{Kw1(k+1)} + \nu_1}{\mathtt{2dzm(k)dzt(k+1)}} + \frac{\mathtt{wmm(k)}}{\mathtt{2\ dzm(k)}} \tag{130}$$

Contributions to $rhs_{\mathtt{wp23}}$ from (111):

$$\begin{aligned}
&\mathtt{rhs(k_wp2)} = \mathtt{rhs(k_wp2)}\\
&+\frac{\mathtt{wp2(k)}}{\mathtt{dt}}\\
&+\frac{\mathtt{Kw1(k)}+\nu_1}{\mathtt{2dzm(k)dzt(k)}}\mathtt{wp2(k-1)}\\
&-\frac{1}{\mathtt{2dzm(k)}}\left(\frac{\mathtt{Kw1(k+1)}+\nu_1}{\mathtt{dzt(k+1)}}+\frac{\mathtt{Kw1(k)}+\nu_1}{\mathtt{dzt(k)}}\right)\mathtt{wp2(k)}\\
&+\frac{\mathtt{Kw1(k+1)}+\nu_1}{\mathtt{2dzm(k)dzt(k+1)}}\mathtt{wp2(k+1)}\\
&+\mathtt{wp2t(k)}
\end{aligned} \tag{131}$$

where

$$\mathtt{k\_wp2} = \mathtt{2k} \tag{132}$$

Contributions to $lhs_{\mathtt{wp23}}$ from (123):

$$\begin{aligned}
&\mathtt{lhs(k_wp3,5)} = \mathtt{lhs(k_wp3,5)}\\
&-\frac{\mathtt{Kw8(k-1)}+\nu_8}{\mathtt{2dzt(k)dzm(k-1)}} - \frac{\mathtt{wmt(k)}}{\mathtt{2\ dzt(k)}} - \frac{1}{\mathtt{2\,dzt(k)}}\frac{\mathtt{a1m(k-1)}\left(\mathtt{wp3(k-1)}+\mathtt{wp3(k)}\right)}{\max\left(\mathtt{wp2(k-1)},\epsilon\right)}
\end{aligned} \tag{133}$$

$$\mathtt{lhs(k\_wp3,4)} = \mathtt{lhs(k\_wp3,4)} - \frac{\mathtt{2a3m(k-1)wp2(k-1)}}{\mathtt{dzt(k)}} \tag{134}$$

$$\begin{aligned}
&\mathtt{lhs(k_wp3,3)} = \mathtt{lhs(k_wp3,3)}\\
&+\frac{1}{\mathtt{dt}} + \frac{\mathtt{C_8}}{\mathtt{taut(k)}}\left(5\,\mathtt{C_{8b}}\,\mathtt{Skwt(k)}^4+1\right) + \frac{1}{\mathtt{2dzt(k)}}\left(\frac{\mathtt{Kw8(k)}+\nu_8}{\mathtt{dzm(k)}}+\frac{\mathtt{Kw8(k-1)}+\nu_8}{\mathtt{dzm(k-1)}}\right)\\
&+\frac{1}{\mathtt{2\,dzt(k)}}\left(\frac{\mathtt{a1m(k)}\left(\mathtt{wp3(k)}+\mathtt{wp3(k+1)}\right)}{\max\left(\mathtt{wp2(k)},\epsilon\right)} - \frac{\mathtt{a1m(k-1)}\left(\mathtt{wp3(k-1)}+\mathtt{wp3(k)}\right)}{\max\left(\mathtt{wp2(k-1)},\epsilon\right)}\right)
\end{aligned} \tag{135}$$

$$\mathtt{lhs(k\_wp3,2)} = \mathtt{lhs(k\_wp3,2)} + \frac{\mathtt{2a3m(k)wp2(k)}}{\mathtt{dzt(k)}} \tag{136}$$

$$\begin{aligned}
&\mathtt{lhs(k_wp3,1)} = \mathtt{lhs(k_wp3,1)}\\
&-\frac{\mathtt{Kw8(k)}+\nu_8}{\mathtt{2dzt(k)dzm(k)}} + \frac{\mathtt{wmt(k)}}{\mathtt{2\ dzt(k)}} + \frac{1}{\mathtt{2\,dzt(k)}}\frac{\mathtt{a1m(k)}\left(\mathtt{wp3(k)}+\mathtt{wp3(k+1)}\right)}{\max\left(\mathtt{wp2(k)},\epsilon\right)}
\end{aligned} \tag{137}$$

Contributions to $rhs_{\mathtt{wp23}}$ from (123):

$$
\begin{aligned}
&\mathtt{rhs(k\_wp3)} = \mathtt{rhs(k\_wp3)} \\
&+\frac{\mathtt{wp3(k)}}{\mathtt{dt}} + \frac{1}{\mathtt{2dzt(k)}}\Bigg((\mathtt{Kw8(k)}+\nu_8)\,\frac{\mathtt{wp3(k+1)}-\mathtt{wp3(k)}}{\mathtt{dzm(k)}} \\
&\qquad\qquad - (\mathtt{Kw8(k-1)}+\nu_8)\,\frac{\mathtt{wp3(k)}-\mathtt{wp3(k-1)}}{\mathtt{dzm(k-1)}}\Bigg) \\
&+\frac{\mathtt{C_8}}{\mathtt{taut(k)}}\left(4\,\mathtt{C_{8b}}\,\mathtt{Skwt(k)}^4\right)\mathtt{wp3(k)} \\
&+\frac{\mathtt{a3m(k)wp2(k)}^2-\mathtt{a3m(k-1)wp2(k-1)}^2}{\mathtt{dzt(k)}} \\
&+\frac{1}{4\,\mathtt{dzt(k)}}\left(\frac{\mathtt{a1m(k)}\left(\mathtt{wp3(k)}+\mathtt{wp3(k+1)}\right)^2}{\max\left(\mathtt{wp2(k)},\epsilon\right)} - \frac{\mathtt{a1m(k-1)}\left(\mathtt{wp3(k-1)}+\mathtt{wp3(k)}\right)^2}{\max\left(\mathtt{wp2(k-1)},\epsilon\right)}\right) \\
&+\mathtt{wp3t(k)}
\end{aligned}
\tag{138}
$$

where

$$
\mathtt{k\_wp3} = \mathtt{2k-1} \tag{139}
$$

In order to increase numerical stability in the model, a_1 has been brought outside of the derivative. Besides a_1, a_3 has been previously brought outside of the derivative for the same purpose. This is not mathematically correct, but it does help to increase stability. Brian Griffin. Feb. 21, 2008. Contributions to $lhs_{\mathtt{wp23}}$ from (123):

$$
\begin{aligned}
&\mathtt{lhs(k\_wp3,5)} = \mathtt{lhs(k\_wp3,5)} \\
&-\frac{\mathtt{Kw8(k-1)}+\nu_8}{\mathtt{2dzt(k)dzm(k-1)}} - \frac{\mathtt{wmt(k)}}{\mathtt{2\,dzt(k)}} - \left(\frac{\mathtt{a1m(k)}+\mathtt{a1m(k-1)}}{2}\right)\frac{1}{2\,\mathtt{dzt(k)}}\frac{\left(\mathtt{wp3(k-1)}+\mathtt{wp3(k)}\right)}{\max\left(\mathtt{wp2(k-1)},\epsilon\right)}
\end{aligned}
\tag{140}
$$

$$
\mathtt{lhs(k\_wp3,4)} = \mathtt{lhs(k\_wp3,4)} - \left(\frac{\mathtt{a3m(k)}+\mathtt{a3m(k-1)}}{2}\right)\frac{\mathtt{2wp2(k-1)}}{\mathtt{dzt(k)}} \tag{141}
$$

$$
\begin{aligned}
&\mathtt{lhs(k\_wp3,3)} = \mathtt{lhs(k\_wp3,3)} \\
&+\frac{1}{\mathtt{dt}} + \frac{\mathtt{C_8}}{\mathtt{taut(k)}}\left(5\,\mathtt{C_{8b}}\,\mathtt{Skwt(k)}^4+1\right) + \frac{1}{\mathtt{2dzt(k)}}\left(\frac{\mathtt{Kw8(k)}+\nu_8}{\mathtt{dzm(k)}} + \frac{\mathtt{Kw8(k-1)}+\nu_8}{\mathtt{dzm(k-1)}}\right) \\
&+\left(\frac{\mathtt{a1m(k)}+\mathtt{a1m(k-1)}}{2}\right)\frac{1}{2\,\mathtt{dzt(k)}}\left(\frac{\left(\mathtt{wp3(k)}+\mathtt{wp3(k+1)}\right)}{\max\left(\mathtt{wp2(k)},\epsilon\right)} - \frac{\left(\mathtt{wp3(k-1)}+\mathtt{wp3(k)}\right)}{\max\left(\mathtt{wp2(k-1)},\epsilon\right)}\right)
\end{aligned}
\tag{142}
$$

$$
\mathtt{lhs(k\_wp3,2)} = \mathtt{lhs(k\_wp3,2)} + \left(\frac{\mathtt{a3m(k)}+\mathtt{a3m(k-1)}}{2}\right)\frac{\mathtt{2wp2(k)}}{\mathtt{dzt(k)}} \tag{143}
$$

$$
\begin{aligned}
&\mathtt{lhs(k\_wp3,1)} = \mathtt{lhs(k\_wp3,1)} \\
&-\frac{\mathtt{Kw8(k)}+\nu_8}{\mathtt{2dzt(k)dzm(k)}} + \frac{\mathtt{wmt(k)}}{\mathtt{2\,dzt(k)}} + \left(\frac{\mathtt{a1m(k)}+\mathtt{a1m(k-1)}}{2}\right)\frac{1}{2\,\mathtt{dzt(k)}}\frac{\left(\mathtt{wp3(k)}+\mathtt{wp3(k+1)}\right)}{\max\left(\mathtt{wp2(k)},\epsilon\right)}
\end{aligned}
\tag{144}
$$

Contributions to $rhs_{\mathtt{wp23}}$ from (123):

$$
\begin{aligned}
&\mathtt{rhs(k\_wp3)} = \mathtt{rhs(k\_wp3)} \\
&+\frac{\mathtt{wp3(k)}}{\mathtt{dt}} + \frac{1}{\mathtt{2dzt(k)}}\Bigg((\mathtt{Kw8(k)} + \nu_8)\,\frac{\mathtt{wp3(k+1)} - \mathtt{wp3(k)}}{\mathtt{dzm(k)}} \\
&\qquad\qquad - (\mathtt{Kw8(k-1)} + \nu_8)\,\frac{\mathtt{wp3(k)} - \mathtt{wp3(k-1)}}{\mathtt{dzm(k-1)}}\Bigg) \\
&+\frac{\mathtt{C_8}}{\mathtt{taut(k)}}\left(4\,\mathtt{C_{8b}}\,\mathtt{Skwt(k)}^4\right)\mathtt{wp3(k)} \\
&+\left(\frac{\mathtt{a3m(k)} + \mathtt{a3m(k-1)}}{2}\right)\frac{\mathtt{wp2(k)}^2 - \mathtt{wp2(k-1)}^2}{\mathtt{dzt(k)}} \\
&+\left(\frac{\mathtt{a1m(k)} + \mathtt{a1m(k-1)}}{2}\right)\frac{1}{4\,\mathtt{dzt(k)}}\left(\frac{(\mathtt{wp3(k)} + \mathtt{wp3(k+1)})^2}{\max(\mathtt{wp2(k)}, \epsilon)} - \frac{(\mathtt{wp3(k-1)} + \mathtt{wp3(k)})^2}{\max(\mathtt{wp2(k-1)}, \epsilon)}\right) \\
&+\mathtt{wp3t(k)}
\end{aligned}
\tag{145}
$$

7 High-order Solution to the Horizontal Wind

As an alternative to assuming $\bar{e} = \frac{3}{2}\overline{w'^2}$, we can obtain an anisotropic solution using a semi-implicit discretization for $\overline{u'^2}$ and $\overline{v'^2}$. Similarly to (6), start with equations (14) and (15) (for simplicity, neglect the $|_{\mathrm{pd}}$ and $|_{\mathrm{cl}}$ terms), substitute (32) and (33) respectively.

7.1 $\overline{u'^2}$

Assume a steady-state and rearrange $\overline{u'^2}$ for a semi-implicit solution to obtain:

$$
\begin{aligned}
&\underbrace{\frac{C_4}{\tau}\left(\overline{u'^2} - \frac{2}{3}\bar{e}\right)}_{dp1} \underbrace{+\frac{2}{3}\left(C_{14}\frac{\bar{e}}{\tau}\right)}_{pr1} \underbrace{+\bar{w}\frac{\partial\overline{u'^2}}{\partial z}}_{ma} \underbrace{+\frac{1}{3}\beta\frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}}{\overline{w'^2}}\,\overline{u'^2}\right)}_{ta} \underbrace{-\frac{\partial}{\partial z}\left[(K_{w9} + \nu_9)\,\frac{\partial}{\partial z}\overline{u'^2}\right]}_{dp2} \\
&= \underbrace{-\left(1 - \frac{1}{3}\beta\right)\frac{\partial}{\partial z}\left(a_2\frac{\overline{w'^3}}{\overline{w'^2}^2}\,\overline{w'u'}^2\right)}_{ta} \underbrace{-2(1 - C_5)\,\overline{w'u'}\,\frac{\partial\bar{u}}{\partial z}}_{tp} \\
&\quad \underbrace{+\frac{2}{3}C_5\left(\frac{g}{\theta_0}\overline{w'\theta'_v} - \overline{u'w'}\frac{\partial\bar{u}}{\partial z} - \overline{v'w'}\frac{\partial\bar{v}}{\partial z}\right)}_{pr2}
\end{aligned}
\tag{146}
$$

As in the case of $\overline{r_t'^2}$ and $\overline{\theta_l'^2}$, the horizontal wind variance terms are solved using a tridiagonal matrix.

7.1.1 Terms 1 and 2: dp1 and pr1, respectively

$$
\begin{aligned}
&\frac{C_4}{\tau}\left(\overline{u'^2}-\frac{2}{3}\bar{e}\right)+\frac{2}{3}C_{14}\frac{\bar{e}}{\tau}\bigg|_{\texttt{zm(k)}}\\
=&\frac{\mathtt{C_4}}{\mathtt{taum(k)}}\mathtt{up2(k)}-\frac{\mathtt{C_4}}{\mathtt{taum(k)}}\frac{2}{3}\mathtt{em(k)}+\frac{2}{3}\mathtt{C_{14}}\frac{\mathtt{em(k)}}{\mathtt{taum(k)}}\\
=&\frac{\mathtt{C_4}}{\mathtt{taum(k)}}\mathtt{up2(k)}-\frac{2}{3}\mathtt{em(k)}\left(\frac{\mathtt{C_4}}{\mathtt{taum(k)}}-\frac{\mathtt{C_{14}}}{\mathtt{taum(k)}}\right)\\
=&\frac{\mathtt{C_4}}{\mathtt{taum(k)}}\mathtt{up2(k)}-\frac{2}{3}\left[\frac{\mathtt{up2(k)}+\mathtt{vp2(k)}+\mathtt{wp2(k)}}{2}\right]\left(\frac{\mathtt{C_4}}{\mathtt{taum(k)}}-\frac{\mathtt{C_{14}}}{\mathtt{taum(k)}}\right)\\
=&\mathtt{up2(k)}\frac{1}{3}\left(\frac{\mathtt{2C_4}+\mathtt{C_{14}}}{\mathtt{taum(k)}}\right)-\left(\frac{1}{3}\left(\mathtt{C_4}-\mathtt{C_{14}}\right)\left(\frac{\mathtt{vp2(k)}+\mathtt{wp2(k)}}{\mathtt{taum(k)}}\right)\right)
\end{aligned}
\tag{147}
$$

Separating out the contributions:

$$
\begin{aligned}
\mathtt{lhs(2,k)}&=\mathtt{lhs(2,k)}+\frac{\mathtt{2C_4}+\mathtt{C_{14}}}{\mathtt{3taum(k)}}\\
\mathtt{rhs(k)}&=\mathtt{rhs(k)}+\frac{1}{3}\left(\mathtt{C_4}-\mathtt{C_{14}}\right)\left(\frac{\mathtt{vp2(k)}+\mathtt{wp2(k)}}{\mathtt{taum(k)}}\right)
\end{aligned}
\tag{148}
$$

7.1.2 Term 3: ma

$$
\begin{aligned}
&\bar{w}\frac{\partial\overline{u'^2}}{\partial z}\bigg|_{\texttt{zm(k)}}\\
=&\frac{\mathtt{wmm(k)}}{\mathtt{dzm(k)}}\left(\frac{1}{2}\left(\mathtt{up2(k)}+\mathtt{up2(k+1)}\right)-\frac{1}{2}\left(\mathtt{up2(k-1)}+\mathtt{up2(k)}\right)\right)\\
=&\frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}}\mathtt{up2(k+1)}-\frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}}\mathtt{up2(k-1)}
\end{aligned}
\tag{149}
$$

Separating out the contributions:

$$
\begin{aligned}
\mathtt{lhs(1,k)}&=\mathtt{lhs(1,k)}+\frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}}\\
\mathtt{lhs(3,k)}&=\mathtt{lhs(3,k)}-\frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}}
\end{aligned}
\tag{150}
$$

7.1.3 Term 4: ta, implicit component

$$
\begin{aligned}
&\frac{1}{3}\beta\frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}}{\overline{w'^2}}\overline{u'^2}\right)\Bigg|_{\mathtt{zm(k)}} \\
&= \frac{\beta}{\mathtt{6dzm(k)}}\Bigg[\frac{(\mathtt{a1m(k)}+\mathtt{a1m(k+1)})\,\mathtt{wp3(k+1)}\,(\mathtt{up2(k)}+\mathtt{up2(k+1)})}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)} \\
&\qquad -\frac{(\mathtt{a1m(k-1)}+\mathtt{a1m(k)})\,\mathtt{wp3(k)}\,(\mathtt{up2(k-1)}+\mathtt{up2(k)})}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)}\Bigg]
\end{aligned}
\tag{151}
$$

Separating out the contributions:

$$
\begin{aligned}
\mathtt{lhs(3,k)} &= \mathtt{lhs(3,k)} - \frac{\beta}{\mathtt{6dzm(k)}}\frac{(\mathtt{a1m(k-1)}+\mathtt{a1m(k)})\,\mathtt{wp3(k)}}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)} \\
\mathtt{lhs(2,k)} &= \mathtt{lhs(2,k)} + \frac{\beta}{\mathtt{6dzm(k)}}\left(\frac{(\mathtt{a1m(k)}+\mathtt{a1m(k+1)})\,\mathtt{wp3(k+1)}}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)} - \frac{(\mathtt{a1m(k-1)}+\mathtt{a1m(k)})\,\mathtt{wp3(k)}}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)}\right) \\
\mathtt{lhs(1,k)} &= \mathtt{lhs(1,k)} + \frac{\beta}{\mathtt{6dzm(k)}}\frac{(\mathtt{a1m(k)}+\mathtt{a1m(k+1)})\,\mathtt{wp3(k+1)}}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)}
\end{aligned}
\tag{152}
$$

In order to increase numerical stability in the model, a_1 has been brought outside of the derivative. This is not mathematically correct, but it does help to increase stability. Brian Griffin. Feb. 21, 2008.

$$
\begin{aligned}
&a_1\frac{1}{3}\beta\frac{\partial}{\partial z}\left(\frac{\overline{w'^3}}{\overline{w'^2}}\overline{u'^2}\right)\Bigg|_{\mathtt{zm(k)}} \\
&= \mathtt{a1m(k)}\frac{\beta}{\mathtt{3dzm(k)}}\left[\frac{\mathtt{wp3(k+1)}\,(\mathtt{up2(k)}+\mathtt{up2(k+1)})}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)} - \frac{\mathtt{wp3(k)}\,(\mathtt{up2(k-1)}+\mathtt{up2(k)})}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)}\right]
\end{aligned}
\tag{153}
$$

Separating out the contributions:

$$
\begin{aligned}
\mathtt{lhs(3,k)} &= \mathtt{lhs(3,k)} - \mathtt{a1m(k)}\frac{\beta}{\mathtt{3dzm(k)}}\frac{\mathtt{wp3(k)}}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)} \\
\mathtt{lhs(2,k)} &= \mathtt{lhs(2,k)} + \mathtt{a1m(k)}\frac{\beta}{\mathtt{3dzm(k)}}\left(\frac{\mathtt{wp3(k+1)}}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)} - \frac{\mathtt{wp3(k)}}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)}\right) \\
\mathtt{lhs(1,k)} &= \mathtt{lhs(1,k)} + \mathtt{a1m(k)}\frac{\beta}{\mathtt{3dzm(k)}}\frac{\mathtt{wp3(k+1)}}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)}
\end{aligned}
\tag{154}
$$

7.1.4 Term 5: dp2

$$
\begin{aligned}
&-\frac{\partial}{\partial z}\left[\left(K_{w9}+\nu_9\right)\frac{\partial}{\partial z}\overline{u'^2}\right]\Bigg|_{\mathtt{zm(k)}} \\
&= -\frac{1}{\mathtt{dzm(k)}}\left(\frac{(\mathtt{Kw9(k+1)}+\nu_9)\,(\mathtt{up2(k+1)}-\mathtt{up2(k)})}{\mathtt{dzt(k+1)}} - \frac{(\mathtt{Kw9(k)}+\nu_9)\,(\mathtt{up2(k)}-\mathtt{up2(k-1)})}{\mathtt{dzt(k)}}\right)
\end{aligned}
$$

$$(155)$$

Separating out the contributions:

$$
\begin{aligned}
\texttt{lhs}(3,\texttt{k}) &= \texttt{lhs}(3,\texttt{k}) - \frac{\texttt{Kw9}(\texttt{k}) + \nu_9}{\texttt{dzm}(\texttt{k})\texttt{dzt}(\texttt{k})} \\
\texttt{lhs}(2,\texttt{k}) &= \texttt{lhs}(2,\texttt{k}) + \frac{1}{\texttt{dzm}(\texttt{k})}\left(\frac{\texttt{Kw9}(\texttt{k}+1) + \nu_9}{\texttt{dzt}(\texttt{k}+1)} + \frac{\texttt{Kw9}(\texttt{k}) + \nu_9}{\texttt{dzt}(\texttt{k})}\right) \\
\texttt{lhs}(1,\texttt{k}) &= \texttt{lhs}(1,\texttt{k}) - \frac{\texttt{Kw9}(\texttt{k}+1) + \nu_9}{\texttt{dzm}(\texttt{k})\texttt{dzt}(\texttt{k}+1)}
\end{aligned}
\tag{156}
$$

7.1.5 Term 6: ta, explicit component

$$
\begin{aligned}
&-\left(1-\frac{1}{3}\beta\right)\frac{\partial}{\partial z}\left(a_2\frac{\overline{w'^3}}{\overline{w'^2}^2}\overline{w'u'}^2\right)\Bigg|_{\texttt{zm(k)}} \\
=&-\frac{1-\frac{1}{3}\beta}{4\texttt{dzm}(\texttt{k})}\left[\frac{(\texttt{a1m}(\texttt{k})+\texttt{a1m}(\texttt{k}+1))^2\,\texttt{wp3}(\texttt{k}+1)\,(\texttt{upwp}(\texttt{k})+\texttt{upwp}(\texttt{k}+1))^2}{\max\left(\texttt{wp2}(\texttt{k})+\texttt{wp2}(\texttt{k}+1),2\epsilon\right)^2}\right. \\
&\left.-\frac{(\texttt{a1m}(\texttt{k}-1)+\texttt{a1m}(\texttt{k}))^2\,\texttt{wp3}(\texttt{k})\,(\texttt{upwp}(\texttt{k}-1)+\texttt{upwp}(\texttt{k}))^2}{\max\left(\texttt{wp2}(\texttt{k}-1)+\texttt{wp2}(\texttt{k}),2\epsilon\right)^2}\right]
\end{aligned}
\tag{157}
$$

Separating out the contributions:

$$
\begin{aligned}
&\texttt{rhs}(\texttt{k}) \\
=&\texttt{rhs}(\texttt{k})-\frac{1-\frac{1}{3}\beta}{4\texttt{dzm}(\texttt{k})}\left[\frac{(\texttt{a1m}(\texttt{k})+\texttt{a1m}(\texttt{k}+1))^2\,\texttt{wp3}(\texttt{k}+1)\,(\texttt{upwp}(\texttt{k})+\texttt{upwp}(\texttt{k}+1))^2}{\max\left(\texttt{wp2}(\texttt{k})+\texttt{wp2}(\texttt{k}+1),2\epsilon\right)^2}\right. \\
&\left.-\frac{(\texttt{a1m}(\texttt{k}-1)+\texttt{a1m}(\texttt{k}))^2\,\texttt{wp3}(\texttt{k})\,(\texttt{upwp}(\texttt{k}-1)+\texttt{upwp}(\texttt{k}))^2}{\max\left(\texttt{wp2}(\texttt{k}-1)+\texttt{wp2}(\texttt{k}),2\epsilon\right)^2}\right]
\end{aligned}
\tag{158}
$$

In order to increase numerical stability in the model, a_1 has been brought outside of the derivative. This is not mathematically correct, but it does help to increase stability. Brian Griffin. Feb. 21, 2008.

$$
\begin{aligned}
&-a_2\left(1-\frac{1}{3}\beta\right)\frac{\partial}{\partial z}\left(\frac{\overline{w'^3}}{\overline{w'^2}^2}\overline{w'u'}^2\right)\Bigg|_{\texttt{zm(k)}} \\
=&-\texttt{a1m}(\texttt{k})^2\frac{1-\frac{1}{3}\beta}{\texttt{dzm}(\texttt{k})}\left[\frac{\texttt{wp3}(\texttt{k}+1)\,(\texttt{upwp}(\texttt{k})+\texttt{upwp}(\texttt{k}+1))^2}{\max\left(\texttt{wp2}(\texttt{k})+\texttt{wp2}(\texttt{k}+1),2\epsilon\right)^2}\right. \\
&\left.-\frac{\texttt{wp3}(\texttt{k})\,(\texttt{upwp}(\texttt{k}-1)+\texttt{upwp}(\texttt{k}))^2}{\max\left(\texttt{wp2}(\texttt{k}-1)+\texttt{wp2}(\texttt{k}),2\epsilon\right)^2}\right]
\end{aligned}
\tag{159}
$$

Separating out the contributions:

$$
\begin{aligned}
&\texttt{rhs}(\texttt{k}) \\
=&\texttt{rhs}(\texttt{k})-\texttt{a1m}(\texttt{k})^2\frac{1-\frac{1}{3}\beta}{\texttt{dzm}(\texttt{k})}\left[\frac{\texttt{wp3}(\texttt{k}+1)\,(\texttt{upwp}(\texttt{k})+\texttt{upwp}(\texttt{k}+1))^2}{\max\left(\texttt{wp2}(\texttt{k})+\texttt{wp2}(\texttt{k}+1),2\epsilon\right)^2}\right. \\
&\left.-\frac{\texttt{wp3}(\texttt{k})\,(\texttt{upwp}(\texttt{k}-1)+\texttt{upwp}(\texttt{k}))^2}{\max\left(\texttt{wp2}(\texttt{k}-1)+\texttt{wp2}(\texttt{k}),2\epsilon\right)^2}\right]
\end{aligned}
\tag{160}
$$

7.1.6 Term 7: tp

$$-2\,(1-C_5)\,\overline{w'u'}\,\frac{\partial \bar{u}}{\partial z}\bigg|_{\texttt{zm(k)}} = -2\,(1-\texttt{C}_5)\,\texttt{upwp(k)}\frac{\texttt{um(k+1)}-\texttt{um(k)}}{\texttt{dzm(k)}} \tag{161}$$

Separating out the contributions:

$$\texttt{rhs(k)} = \texttt{rhs(k)} - 2\,(1-\texttt{C}_5)\,\texttt{upwp(k)}\frac{\texttt{um(k+1)}-\texttt{um(k)}}{\texttt{dzm(k)}} \tag{162}$$

7.1.7 Term 8: pr2

$$\begin{aligned}&\frac{2}{3}C_5\left(\frac{g}{\theta_0}\overline{w'\theta_v'} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{v'w'}\frac{\partial \bar{v}}{\partial z}\right)|_{\texttt{zm(k)}}\\ =&\frac{2}{3}\texttt{C}_5\left(\frac{\texttt{grav}}{\texttt{T0}}\texttt{wpthvp(k)} - \texttt{upwp(k)}\frac{\texttt{um(k+1)}-\texttt{um(k)}}{\texttt{dzm(k)}} - \texttt{vpwp(k)}\frac{\texttt{vm(k+1)}-\texttt{vm(k)}}{\texttt{dzm(k)}}\right)\end{aligned} \tag{163}$$

Separating out the contributions:

$$\texttt{rhs(k)} = \texttt{rhs(k)} + \frac{2}{3}\texttt{C}_5\left(\frac{\texttt{grav}}{\texttt{T0}}\texttt{wpthvp(k)} - \texttt{upwp(k)}\frac{\texttt{um(k+1)}-\texttt{um(k)}}{\texttt{dzm(k)}} - \texttt{vpwp(k)}\frac{\texttt{vm(k+1)}-\texttt{vm(k)}}{\texttt{dzm(k)}}\right) \tag{164}$$

7.2 $\overline{v'^2}$

As in $\overline{u'^2}$ assume a steady-state and rearrange $\overline{u'^2}$ for a semi-implicit solution to obtain:

$$\begin{aligned}&\frac{C_4}{\tau}\left(\overline{v'^2} - \frac{2}{3}\bar{e}\right) + \frac{2}{3}\left(C_{14}\frac{\bar{e}}{\tau}\right) + \bar{w}\frac{\partial\overline{v'^2}}{\partial z} + \frac{1}{3}\beta\frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}}{\overline{w'^2}}\,\overline{v'^2}\right) - \frac{\partial}{\partial z}\left[(K_{w9}+\nu_9)\,\frac{\partial}{\partial z}\overline{v'^2}\right]\\ =&-\left(1-\frac{1}{3}\beta\right)\frac{\partial}{\partial z}\left(a_2\frac{\overline{w'^3}}{\overline{w'^2}^2}\,\overline{w'v'}^2\right) - 2(1-C_5)\,\overline{w'v'}\,\frac{\partial \bar{v}}{\partial z}\\ &+\frac{2}{3}C_5\left(\frac{g}{\theta_0}\overline{w'\theta_v'} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{v'w'}\frac{\partial \bar{v}}{\partial z}\right)\end{aligned} \tag{165}$$

The discretization for $\overline{v'^2}$ follows in the same way as $\overline{u'^2}$.

8 Grid Configuration

Figure 1 shows the vertical grid configuration for CLUBB. The grid consists of two types of levels: `zm` and `zt`. Predictive mean variables and third order moments reside on the thermodynamic levels (`zt`). Second and fourth order moments reside on the momentum levels (`zm`).

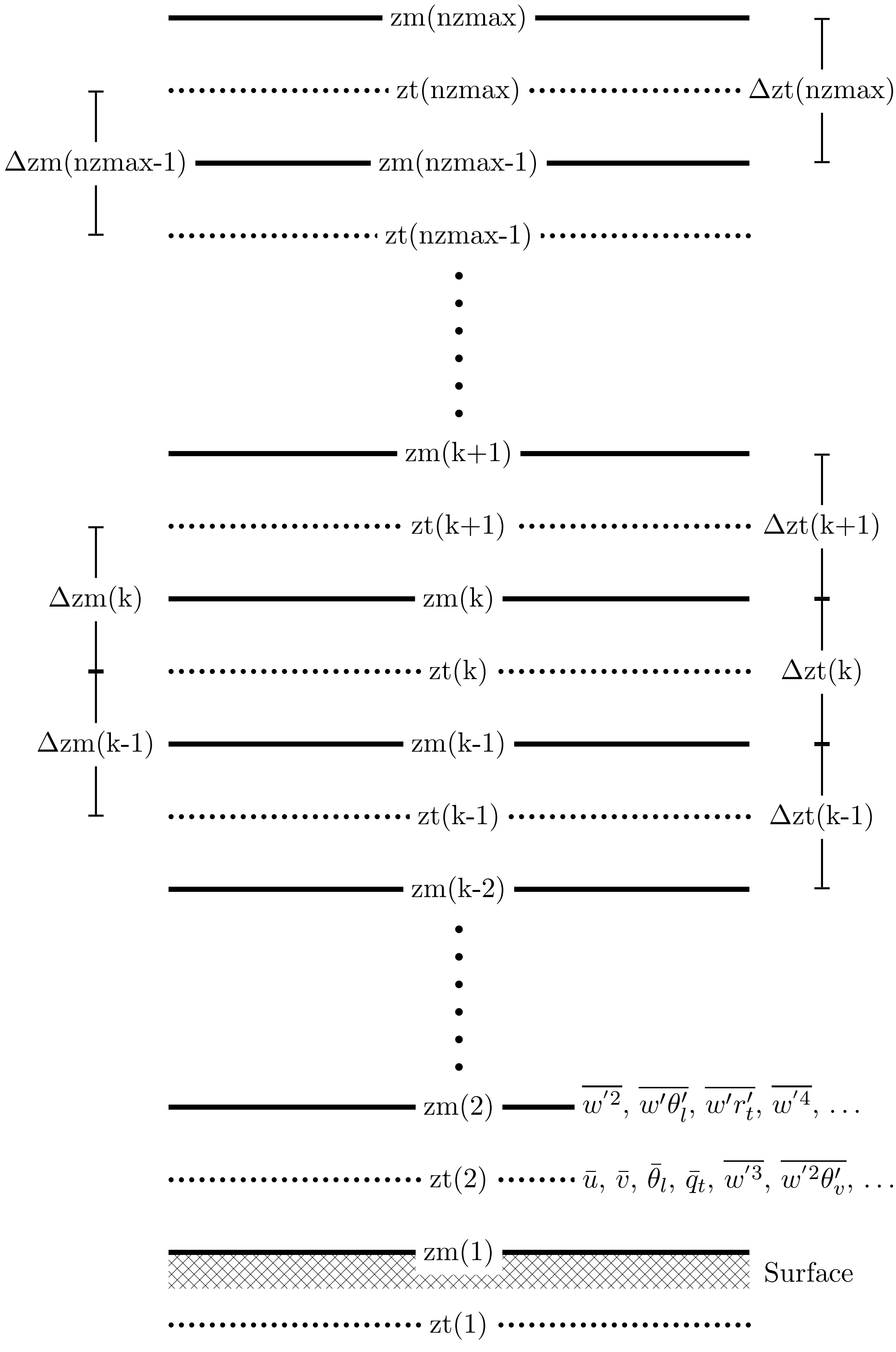
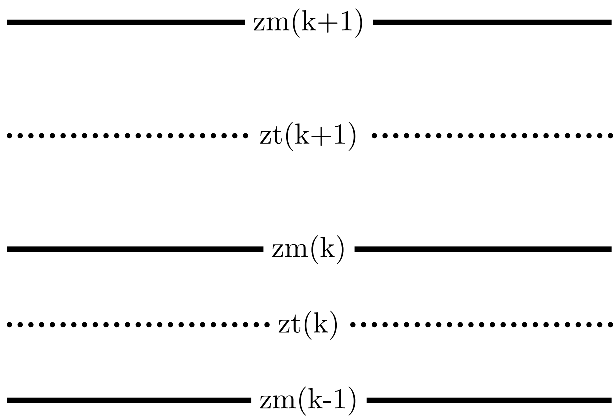


Figure 1: Vertical grid configuration

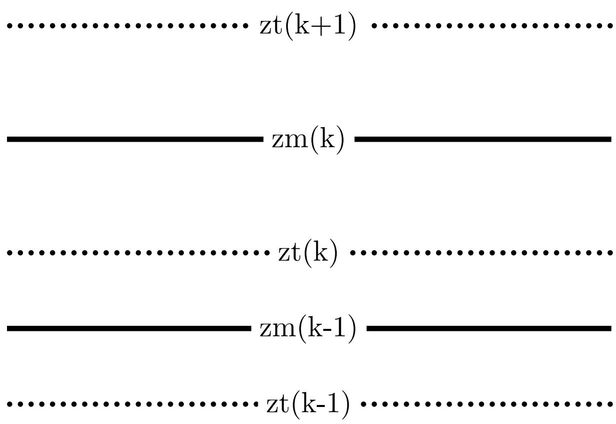
8.1 Stretched (unevenly-spaced) grid configuration

The grid setup is compatible with a stretched (unevenly-spaced) grid configuration. Thus, the distance between successive vertical grid levels may not always be constant.

The following diagram is an example of a stretched grid that is defined on momentum levels. The thermodynamic levels are placed exactly halfway between the momentum levels. However, the momentum levels do not fall halfway between the thermodynamic levels.



The following diagram is an example of a stretched grid that is defined on thermodynamic levels. The momentum levels are placed exactly halfway between the thermodynamic levels. However, the thermodynamic levels do not fall halfway between the momentum levels.



8.2 Generalized grid functions

Each variable in the CLUBB parameterization resides at certain discrete points in the vertical, whether the points be on momentum levels or on thermodynamic levels. The values of each variable are considered to behave in a linear fashion in the sections between the levels where each variable resides. Thus, linear interpolation is used to find the values of a variable at the levels where it does not reside. Since a variable is considered to behave linearly between two successive levels where it resides, the linear derivative of that variable at any point in the section between two successive levels where it resides is always the same.

Any future computer code written for use in the CLUBB parameterization should use interpolation formulas consistent with a stretched grid. The simplest way to do so is to use the appropriate equation listed below in equations 166 through 173. Interpolations should not be handled in the form of: $(\mathtt{var_m}(\mathtt{k})+\mathtt{var_m}(\mathtt{k}-\mathtt{1}))/2$; nor in the form of: $0.5\times(\mathtt{var_t}(\mathtt{k}+\mathtt{1})+\mathtt{var_t}(\mathtt{k}))$. Rather, all explicit interpolations should use the appropriate equation from equations 166, 167, 170, or 171; while interpolations for a variable being solved for implicitly in the model code should use the appropriate equation set from equations 168 and 169, or 172 and 173. The formula for a linear derivative is the same whether an evenly-spaced grid or stretched grid is in use.

8.2.1 Momentum grid levels to thermodynamic grid levels

A standard linear interpolation formula, equation 166, is used to interpolate a variable that resides on momentum levels, $\mathtt{var_m}$, to thermodynamic levels (as $\mathtt{var_t}$) on any type of grid configuration.

$$\mathtt{var_t}(\mathtt{k}) = \left(\frac{\mathtt{var_m}(\mathtt{k}) - \mathtt{var_m}(\mathtt{k}-\mathtt{1})}{\mathtt{zm}(\mathtt{k}) - \mathtt{zm}(\mathtt{k}-\mathtt{1})}\right)\Big(\mathtt{zt}(\mathtt{k}) - \mathtt{zm}(\mathtt{k}-\mathtt{1})\Big) + \mathtt{var_m}(\mathtt{k}-\mathtt{1}) \tag{166}$$

When converting a variable from momentum levels to thermodynamic levels, there is one instance on the grid, at the lowermost level ($\mathtt{k}=\mathtt{1}$), where a linear extension, equation 167, is required.

$$\mathtt{var_t}(\mathtt{1}) = \left(\frac{\mathtt{var_m}(\mathtt{2}) - \mathtt{var_m}(\mathtt{1})}{\mathtt{zm}(\mathtt{2}) - \mathtt{zm}(\mathtt{1})}\right)\Big(\mathtt{zt}(\mathtt{1}) - \mathtt{zm}(\mathtt{1})\Big) + \mathtt{var_m}(\mathtt{1}) \tag{167}$$

When a variable that is being solved for implicitly in an equation is also being interpolated from momentum levels to intermediate thermodynamic levels, the interpolation formulas listed above will not work. Rather, interpolation weights are necessary. The weight of the upper momentum level (index $\mathtt{k}$) on the intermediate thermodynamic level (index $\mathtt{k}$) is found according to equation 168.

$$\mathtt{weights\_zm2zt(m\_above, k)} = \left(\frac{\mathtt{zt(k)} - \mathtt{zm(k-1)}}{\mathtt{zm(k)} - \mathtt{zm(k-1)}} \right) \tag{168}$$

The weight of the lower momentum level (index $\mathtt{k-1}$) on the intermediate thermodynamic level (index $\mathtt{k}$) is found according to equation 169.

$$\mathtt{weights\_zm2zt(m\_below, k)} = 1 - \left(\frac{\mathtt{zt(k)} - \mathtt{zm(k-1)}}{\mathtt{zm(k)} - \mathtt{zm(k-1)}} \right) \tag{169}$$

8.2.2 Thermodynamic grid levels to momentum grid levels

A standard linear interpolation formula, equation 170, is used to interpolate a variable that resides on thermodynamic levels, $\mathtt{var_t}$, to momentum levels (as $\mathtt{var_m}$) on any type of grid configuration.

$$\mathtt{var_m(k)} = \left(\frac{\mathtt{var_t(k+1)} - \mathtt{var_t(k)}}{\mathtt{zt(k+1)} - \mathtt{zt(k)}} \right) \Big(\mathtt{zm(k)} - \mathtt{zt(k)} \Big) + \mathtt{var_t(k)} \tag{170}$$

When converting a variable from thermodynamic levels to momentum levels, there is one instance on the grid, at the uppermost level ($\mathtt{k} = \mathtt{nzmax}$), where a linear extension, equation 171, is required.

$$\mathtt{var_m(nzmax)} = \left(\frac{\mathtt{var_t(nzmax)} - \mathtt{var_t(nzmax-1)}}{\mathtt{zt(nzmax)} - \mathtt{zt(nzmax-1)}} \right) \Big(\mathtt{zm(nzmax)} - \mathtt{zt(nzmax)} \Big) + \mathtt{var_t(nzmax)} \tag{171}$$

When a variable that is being solved for implicitly in an equation is also being interpolated from thermodynamic levels to intermediate momentum levels, the interpolation formulas listed above will not work. Rather, interpolation weights are necessary. The weight of the upper thermodynamic level (index $\mathtt{k+1}$) on the intermediate momentum level (index $\mathtt{k}$) is found according to equation 172.

$$\mathtt{weights\_zt2zm(t\_above, k)} = \left(\frac{\mathtt{zm(k)} - \mathtt{zt(k)}}{\mathtt{zt(k+1)} - \mathtt{zt(k)}} \right) \tag{172}$$

The weight of the lower thermodynamic level (index $\mathtt{k}$) on the intermediate momentum level (index $\mathtt{k}$) is found according to equation 173.

$$\mathtt{weights\_zt2zm(t\_below, k)} = 1 - \left(\frac{\mathtt{zm(k)} - \mathtt{zt(k)}}{\mathtt{zt(k+1)} - \mathtt{zt(k)}} \right) \tag{173}$$

9 Predictive equations in Conjunction with Host Model

The CLUBB parameterization can be used in conjunction with a larger host model. Some of the main predictive equations (Eqns. 1, 2, 3, and 4) have to be divided in the following manner:

$$\frac{\partial \bar{u}}{\partial t} = -\bar{w}\frac{\partial \bar{u}}{\partial z} - f(v_g - \bar{v}) - \frac{\partial}{\partial z}\overline{u'w'} \tag{174}$$

$$\frac{\partial \bar{v}}{\partial t} = -\bar{w}\frac{\partial \bar{v}}{\partial z} + f(u_g - \bar{u}) - \frac{\partial}{\partial z}\overline{v'w'} \tag{175}$$

$$\frac{\partial \bar{q}_t}{\partial t} = -\bar{w}\frac{\partial \bar{q}_t}{\partial z} - \frac{\partial}{\partial z}\overline{w'q_t'} + \left.\frac{\partial \bar{q}_t}{\partial t}\right|_{\mathrm{ls}} \tag{176}$$

$$\frac{\partial \bar{\theta}_l}{\partial t} = -\bar{w}\frac{\partial \bar{\theta}_l}{\partial z} - \frac{\partial}{\partial z}\overline{w'\theta_l'} + \bar{R} + \left.\frac{\partial \bar{\theta}_l}{\partial t}\right|_{\mathrm{ls}} \tag{177}$$

The variables in red, $\bar{u}$, $\bar{v}$, $\bar{q}_t$, and $\bar{\theta}_l$, are supplied from the host model to the CLUBB parameterization. The variables in blue, $\overline{u'w'}$, $\overline{v'w'}$, $\overline{w'q_t'}$, and $\overline{w'\theta_l'}$, are computed in the CLUBB parameterization and then sent back to the host model. The vertical derivatives of these variables are then used to effect the time tendencies of their related variables. The terms listed in magenta, which include the vertical mean advection terms, the coriolis terms, the radiative heating term, and the large-scale moisture and temperature forcings, are terms that are usually calculated in CLUBB when CLUBB does not run with any host model. However, in cases where a host model is involved, the host model should calculate all of these terms. As before, $\bar{R}$ is the radiative heating rate, f the Coriolis parameter and u_g, v_g the geostrophic winds. $\left.\frac{\partial \bar{q}_t}{\partial t}\right|_{\mathrm{ls}}$ and $\left.\frac{\partial \bar{\theta}_l}{\partial t}\right|_{\mathrm{ls}}$ are large-scale moisture and temperature forcings.

The $\frac{\partial \overline{w'^2}}{\partial t}$ equation (eq. 5), the $\frac{\partial \overline{r_t'^2}}{\partial t}$ equation (eq. 6), the $\frac{\partial \overline{\theta_l'^2}}{\partial t}$ equation (eq. 7), the $\frac{\partial \overline{q_t'\theta_l'}}{\partial t}$ equation (eq. 8), the $\frac{\partial \overline{w'q_t'}}{\partial t}$ equation (eq. 9), the $\frac{\partial \overline{w'\theta_l'}}{\partial t}$ equation (eq. 10), and the $\frac{\partial \overline{w'^3}}{\partial t}$ equation (eq. 11) all remain unchanged. All of these variables are computed and used completely within the structure of the CLUBB parameterization.

Within the structure of a computer code, the CLUBB parameterization requires that the values of certain variables be saved at all grid points for use during the next timestep. Since the CLUBB parameterization is a one-dimensional parameterization (in the vertical), or a single-column parameterization, a three-dimensional host model must call the CLUBB parameterization once for every grid column that it has. Therefore, the values of all these variables must be saved from timestep to timestep at every grid point in the three dimensions.

The variables that need to be saved as such are the following:

On the momentum (or full) levels		
Description	Variable	Variable name in CLUBB code
Turbulent Flux of θ_l	$\overline{w'\theta_l'}$	wpthlp
Turbulent Flux of r_t	$\overline{w'r_t'}$	wprtp
Variance of w	$\overline{w'^2}$	wp2
Variance of u	$\overline{u'^2}$	up2
Variance of v	$\overline{v'^2}$	vp2
Variance of r_t	$\overline{r_t'^2}$	rtp2
Variance of θ_l	$\overline{\theta_l'^2}$	thlp2
Covariance of r_t and θ_l	$\overline{r_t'\theta_l'}$	rtpthlp
Covariance of u and w	$\overline{u'w'}$	upwp
Covariance of v and w	$\overline{v'w'}$	vpwp
Time scale	τ	taum
Width of the individual w plumes	$\tilde{\sigma}_w^2$	sigma_sqd_w
On the thermodynamic (or half) levels		
Description	Variable	Variable name in CLUBB code
Third-order Moment of w	$\overline{w'^3}$	wp3
Eddy-diffusivity	K_w	Kh_zt
Cloud water mixing ratio	$\overline{r_c}$	rcm
Cloud fraction	cf	cloud_frac

It is only necessary to save cloud water mixing ratio from timestep to timestep if the host model would need information on the subgrid value of that variable. It is also necessary to provide CLUBB with information on cloud water mixing ratio for the initial timestep of the run. Cloud fraction is usually saved for output purposes only.

10 SILHS

SILHS picks sample points from a joint probability distribution. In CLUBB, SILHS is used to account for subgrid variability in physical processes such as microphysics.

Mathematically, the goal of SILHS is to numerically approximate integrals of the following form:

$$\int h(\mathbf{x})P(\mathbf{x})d\mathbf{x} \tag{178}$$

where $P(\mathbf{x})$ is the PDF that describes CLUBB's joint probability distribution of variables, and $h(\mathbf{x})$ is a function of some or all of these variables. In other words, we want to compute the average value of $h(\mathbf{x})$ over the entire domain.

Briefly, SILHS performs the following tasks to numerically approximate (178):

1. Choose a set of uniform sample points at one vertical level. Currently, the vertical level chosen is where cloud water is maximized. The uniform sample is picked using the Latin hypercube algorithm in order to reduce variance.

2. Vertically correlate these uniform sample points to the other vertical levels in CLUBB. Vertical correlation is based on the spacing between vertical levels, CLUBB's length scale, and an emperical constant.

3. Transform the uniform sample points to CLUBB's PDF. At this point, SILHS has produced a set of subcolumns drawn from the PDF $P(\mathbf{x})$ in (178) (or, in the case of importance sampling, a related PDF, as described below).

4. Evaluate the function $h(\mathbf{x})$ using each of the SILHS sample points. Compute an average of these values to get an estimate of the average value of $h(\mathbf{x})$ over the distribution.

The following sections discuss these tasks in depth.

10.1 CLUBB's PDF

In CLUBB, SILHS draws sample points from CLUBB's PDF, which has the following form.

$$
\begin{aligned}
P(\mathbf{x}) = \sum_{m=1}^{N_{\mathrm{comp}}} \xi_{(m)} \ [\ & f_{p(m)} P_{(m)}(\chi, \eta, w, N_{cn}, \mathbf{hm}) + \\
& (1 - f_{p(m)})\ \delta(\mathbf{hm})\ P_{(m)}(\chi, \eta, w, N_{cn})\ ]
\end{aligned}
\tag{179}
$$

The PDF has N_{comp} components; currently, in CLUBB, $N_{\mathrm{comp}} = 2$. Each component m has a weight $\xi_{(m)}$, where $\sum_{m=1}^{N_{\mathrm{comp}}} \xi_{(m)} = 1$. In each component, a fraction $f_{p(m)}$ of the component is allowed to contain precipitation, where $0 \leq f_{p(m)} \leq 1$. The vector $\mathbf{hm}$ contains hydrometeor species (e.g., rain, snow). The exact type and number of hydrometeors depends on the microphysics scheme used.

In the portions of the PDF that contain precipitation, $P_{(m)}(\chi, \eta, w, N_{cn}, \mathbf{hm})$ is a joint normal-lognormal distribution, where χ, η, and w are normally distributed, and N_{cn} and all the variables in $\mathbf{hm}$ are lognormally distributed. In the parts of the PDF that don't contain precipitation, $P_{(m)}(\chi, \eta, w, N_{cn})$ is a joint normal-lognormal distribution, just like before, but all the hydrometeors are zero, rather than lognormally distributed. ($\delta(\mathbf{hm})$ is short for $\delta(\mathrm{hm}_1)\delta(\mathrm{hm}_2)\cdots\delta(\mathrm{hm}_n)$).

Let d be the total number of variates in the PDF (the four fixed ones plus the number of hydrometeors). The key parameters associated with (179), which must be supplied to SILHS, are:

1. The weight of each PDF component, $\xi_{(m)}$, and the precipitation fraction in each component, $f_{p(m)}$.
2. $\boldsymbol{\mu}_{(m)}$: a vector of d means corresponding to each variate in the PDF, for a given PDF component. These means are "normalized", which means that for the lognormal variates, the corresponding mean that appears in $\boldsymbol{\mu}_{(m)}$ is the mean of the natural logarithm of the variate. Also, these are in-precipitation means. This means that in the portion of the PDF that contains precipitation, $P_{(m)}(\chi, \eta, w, N_{cn}, \mathbf{hm})$ is distributed such that each (normalized) variate has a mean given by

$\boldsymbol{\mu}_{(m)}$. Outside of precipitation, the four variates of $P_{(m)}(\chi, \eta, w, N_{cn})$ have means corresponding to $\boldsymbol{\mu}_{(m)}$, but the hydrometeors are zero, and so have a mean of zero.

3. $\boldsymbol{\sigma}_{(m)}$: a vector of d in-precipitation normalized standard deviations for a given PDF component.

4. $\boldsymbol{\Sigma}_{(m)}$: a $d \times d$ correlation matrix, where $\boldsymbol{\Sigma}_{(m)i,j}$ is the in-precipitation, normalized correlation between the i^{th} and j^{th} variates (in PDF component m).

It is easy to draw sample points directly from a joint normal-lognormal distribution. A simple way to draw sample points from (179) is to draw samples from one of the joint normal-lognormal PDFs indicated in the equation (one of the $P_{(m)}$ functions). How often to sample from each distribution is decided by the weights $\xi_{(m)}$ and $f_{p(m)}$. To do this, (179) can be written in terms of two new variates: u_{d+1} and u_{d+2}. The two variates are uniformly distributed, with $0 \leq u_{d+1}, u_{d+2} \leq 1$. The variate u_{d+1} is used to determine the PDF component to sample from, and u_{d+2} determines whether to sample from the portion of the PDF with precipitation. The PDF in (179) can then be written as follows:

$$P(\chi, \eta, w, N_{cn}, \mathbf{hm}, u_{d+1}, u_{d+2}) = \begin{cases} P_{(m(u_{d+1}))}(\chi, \eta, w, N_{cn}, \mathbf{hm}) & u_{d+2} < f_{p(m(u_{d+1}))} \\ \delta(\mathbf{hm}) P_{(m(u_{d+1}))}(\chi, \eta, w, N_{cn}) & u_{d+2} \geq f_{p(m(u_{d+1}))} \end{cases} \tag{180}$$

where $m(u_{d+1})$ is a function that associates u_{d+1} with a PDF component. In CLUBB's two-component case, the function can simply be defined as:

$$m(u_{d+1}) = \begin{cases} 1 & u_{d+1} < \xi_1 \\ 2 & u_{d+1} \geq \xi_1 \end{cases} \tag{181}$$

10.2 Generation of Points in Uniform Space

As a first step, sample points are picked from a uniform distribution. One uniform variate is picked for each variate in the PDF.

As a variance reduction technique, SILHS employs Latin hypercube sampling. The basic idea of this algorithm is to stratify each distribution variate into several equally-sized regions, and ensure that each region is sampled. The algorithm, which is described in Larson et al. (2005), is implemented as follows.

Let N_s be the number of SILHS sample points used. Each of the $d+2$ uniform variates is split into N_s sections: $\left(0, \frac{1}{N_s}\right), \left(\frac{1}{N_s}, \frac{2}{N_s}\right), \ldots, \left(\frac{N_s-1}{N_s}, 1\right)$. Next, for each variate, an independent permutation of the integers $(0, 1, \ldots, N_s - 1)$ is chosen, corresponding to the N_s sections of the variate. These $d+2$ permutations form a $N_s \times (d+2)$ matrix, $\mathbf{\Pi}$, where each column of the matrix, $\mathbf{\Pi}_{(1 \ldots N_s),j}$ is the permutation corresponding to the j^{th} variate. Finally, we form another $N_s \times (d+2)$ matrix, $\mathbf{U}$, each element of which is a random uniform number between 0 and 1. The purpose of the matrix $\mathbf{U}$ is to choose uniformly a value within each section. Our sample matrix, $\mathbf{V}$, is then given by:

$$\mathbf{V} = \frac{1}{N_s} \left(\mathbf{\Pi} + \mathbf{U} \right) \tag{182}$$

Each of the N_s rows of $\mathbf{V}$ is a SILHS sample. In a given row of $\mathbf{V}$, the value in each of the $(d+2)$ columns is the sample value of the corresponding variate in the PDF. Note that using this method, the association of sections between the variates is scrambled. For example, one sample might contain a sample of one variate in the 2^{nd} section and a sample of another variate in the 5^{th} section.

SILHS also has an option that splits a SILHS sample over multiple timesteps. To use this feature, the user specifies an integer, N_t, which is an integer multiple of N_s. Then, the Latin hypercube algorithm, as described above, generates N_t points instead of N_s points. N_s of these points are used (without replacement) each timestep, until $\frac{N_t}{N_s}$ timesteps have elapsed, at which point the samples are regenerated.

10.3 Importance Sampling

Importance sampling is another technique used in SILHS for variance reduction. The basic idea is to sample some parts of the PDF (the "important" regions) more often than they would normally be sampled.

First, the PDF is split into a set of disjoint categories, C_j. These categories, which span the entire

PDF, are currently defined as follows:

1. In cloud, in precipitation, in mixture component 1

   $(\chi > 0, \quad u_{d+2} < f_{p(1)}, \quad u_{d+1} < \xi_1)$

2. In cloud, in precipitation, in mixture component 2

   $(\chi > 0, \quad u_{d+2} < f_{p(2)}, \quad u_{d+1} \geq \xi_1)$

3. Out of cloud, in precipitation, in mixture component 1

   $(\chi \leq 0, \quad u_{d+2} < f_{p(1)}, \quad u_{d+1} < \xi_1)$

4. Out of cloud, in precipitation, in mixture component 2

   $(\chi \leq 0, \quad u_{d+2} < f_{p(2)}, \quad u_{d+1} \geq \xi_1)$

5. In cloud, out of precipitation, in mixture component 1

   $(\chi > 0, \quad u_{d+2} \geq f_{p(1)}, \quad u_{d+1} < \xi_1)$

6. In cloud, out of precipitation, in mixture component 2

   $(\chi > 0, \quad u_{d+2} \geq f_{p(2)}, \quad u_{d+1} \geq \xi_1)$

7. Out of cloud, out of precipitation, in mixture component 1

   $(\chi \leq 0, \quad u_{d+2} \geq f_{p(1)}, \quad u_{d+1} < \xi_1)$

8. Out of cloud, out of precipitation, in mixture component 2

   $(\chi \leq 0, \quad u_{d+2} \geq f_{p(2)}, \quad u_{d+1} \geq \xi_1)$

Each category C_j is associated with a certain amount of PDF mass, called the category's "PDF probability" and denoted as:

$$p_j = \int \mathbf{1}_j(\mathbf{x}) P(\mathbf{x}) d\mathbf{x} \tag{183}$$

where $\mathbf{1}_j(\mathbf{x})$ is the indicator function of C_j:

$$\mathbf{1}_j(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \in C_j \\ \\ 0 & \mathbf{x} \notin C_j \end{cases} \tag{184}$$

Since the categories C_j span the entire PDF, we have:

$$\sum_{j=1}^{N_c} p_j = 1 \tag{185}$$

where N_c is the number of categories (currently eight in CLUBB).

It can be assumed, without loss of generality, that $p_j > 0$ for all categories, because in any category where $p_j = 0$, the corresponding potion of the integral in (178) is zero, so the region of the PDF belonging to that category can simply be left out of the integral.

It is worth noting that in general, the PDF masses p_j might not always be easy to compute. In the case of SILHS, because of the simple definitions of the categories (given above) and the independence of cloud and precipitation, these masses can be computed using information already provided to SILHS. For example, the mass of category 1, where there is cloud and precipitation in mixture component 1, is given simply by

$$p_1 = f_{c(1)} f_{p(1)} \zeta_1. \tag{186}$$

Next, we prescribe for each category another probability, S_j, called the category's "prescribed probability". The probabilities must be prescribed such that

$$\sum_{j=1}^{N_c} S_j = 1 \tag{187}$$

The prescribed probability S_j of a given category is the probability that any sample will fall in that category. In other words, it is the expected fraction of sample points in the category. Therefore, intuitively, it is advantageous to prescribe the probabilities such that the categories that are "important" for a desired process are sampled more often than the unimportant categories.

We want to generate sample points such that the expected fraction of sample points in the category C_j is S_j rather than p_j. In order to do this, we define a set of new functions:

$$L_j(\mathbf{x}) = \begin{cases} \frac{p_j}{S_j} & \mathbf{x} \in C_j \\ 0 & \text{otherwise} \end{cases}$$

We then rewrite the integral given in (178) as:

$$\int h(\mathbf{x}) P(\mathbf{x}) d\mathbf{x} = \sum_{j=1}^{N_c} \int h(\mathbf{x}) L_j(\mathbf{x}) Q_j(\mathbf{x}) d\mathbf{x} \tag{188}$$

where

$$Q_j(\mathbf{x}) = \begin{cases} \frac{P(\mathbf{x})}{L_j(\mathbf{x})} & \mathbf{x} \in C_j \\ \\ 0 & \text{otherwise} \end{cases}$$

Here, $Q_j(\mathbf{x})$ is a set of new "quasi-PDFs" that has been resized to fit our prescribed probabililties. Each $Q_j(\mathbf{x})$ has area S_j for $j = 1, ..., N_c$. We can easily verify this:

$$\begin{aligned} \int Q_j(\mathbf{x}) d\mathbf{x} &= \int_{C_j} \frac{P(\mathbf{x})}{L_j(\mathbf{x})} d\mathbf{x} && (189) \\ &= \frac{S_j}{p_j} \int_{C_j} P(\mathbf{x}) d\mathbf{x} && (190) \\ &= \frac{S_j}{p_j} p_j && (191) \\ &= S_j && (192) \end{aligned}$$

Now, instead of drawing points from the $P(\mathbf{x})$ distribution and evaluate the function $h(\mathbf{x})$, we draw points from the $Q(\mathbf{x})$ distribution, where

$$Q(\mathbf{x}) = \sum_{j=1}^{N_c} Q_j(\mathbf{x}) \tag{193}$$

and evaluate the function $h(\mathbf{x}) L_j(\mathbf{x})$.

Based on the integral form given in (188), each sample $h(\mathbf{x}_i)$ needs to be multiplied by $L_j(\mathbf{x}_i)$. This factor is referred to as the sample point "weight". Each sample point has a weight, denoted as:

$$\omega_i = \sum_{j=1}^{N_c} \omega_{ij} \tag{194}$$

where

$$\omega_{ij} = L_j(\mathbf{x}_i) = \frac{p_j}{S_j} \; \mathbf{1}_j(\mathbf{x}_i) \tag{195}$$

for $i = 1, ..., N_s$ and $j = 1, ..., N_c$.

So if $\mathbf{x}_i \in C_{j'}$ for some category j', then

$$\omega_i = \frac{p_{j'}}{S_{j'}} \tag{196}$$

Thus each sample point $\mathbf{x}_i$ gets associated with the weight ω_i that corresponds to the category that $\mathbf{x}_i$ belongs to.

10.4 Vertical Correlation of Samples

At this point, a collection of sample points have been generated for a single vertical level. Each vertical level in CLUBB has its own PDF, each of the form given in (179). In principle, we could simply repeat the above process for every vertical level. We would generate a collection of single-level sample points for each vertical level, and arbitrarily match the samples among the vertical levels in order to form subcolumn samples. However, it is found emperically that this "random overlap" assumption does not match what happens in nature.

Each subcolumn physically represents a vertical column in space. It is found that there is some degree of vertical overlap in space. That is, in many cases, the values of sample points are similar between adjacent vertical levels.

In SILHS, this is mimiked by influencing the vertical correlation between uniform sample points. The process is as follows. First, a vertical level, k_s, is chosen to begin sampling (this variable is known as `k_lh_start` in the code. Next, the vertical correlation ρ_k (`vert_corr` in code) is defined for each vertical level k as:

$$\rho_k = \exp\left(\frac{-\alpha\ \Delta z_k}{L}\right) \tag{197}$$

where L is the CLUBB's length scale, Δz_k is the vertical spacing between grid levels at level k, and α is a parameter, known in the code as `vert_decorr_coef`, that controls how correlated fields are in the vertical. Intuitively, it can be seen that if $\alpha = 0$, then $\rho_k = 1$, and this corresponds to maximum vertical correlation (or maximal overlap). As $\alpha \to \infty$, $\rho_k \to 0$, and this corresponds to zero vertical correlation (or random overlap).

The next step is to correlate each of the $d + 2$ uniform sample points in the vertical. For $1 \leq k < k_s$, we set

$$u_k^{'} = u_{k+1} + u^* \left(1 - \rho_k\right) \tag{198}$$

where u_{k+1} is the uniform sample at the height level immediately above k, and u^* is a uniform random number in the range $(-1, 1)$. For $k_s < k \leq \texttt{nzmax}$,

$$u_k^{'} = u_{k-1} + u^* \left(1 - \rho_k\right) \tag{199}$$

Depending on the value of u^*, there is a possibility that $u_k^{'}$ is not in the range $(0, 1)$, and so the value might need to be folded back into the correct range. The actual uniform variate, u_k, is set to be a corrected version of $u_k^{'}$. Specifically,

$$u_k = \begin{cases} 2 - u_k^{'} & u_k^{'} > 1 \\ |u_k^{'}| & u_k^{'} < 0 \\ u_k^{'} & 0 \leq u_k^{'} \leq 1 \end{cases} \tag{200}$$

10.5 Transformation to Desired Distribution

Each sample must now be transformed to be a sample from the PDF in (179). First, the sample values of u_{d+1} and u_{d+2} are used to determine a joint normal-lognormal distribution to sample from, as in (180). Next, the uniform values of the sample, excluding the u_{d+1} and u_{d+2} variates, are transformed to an uncorrelated standard normal sample using the inverse cumulative distribution function of the standard normal distribution:

$$\mathbf{Z} = \Phi^{-1}(\mathbf{u}) = \begin{pmatrix} \Phi^{-1}(u_1) \\ \Phi^{-1}(u_2) \\ \vdots \\ \Phi^{-1}(u_d) \end{pmatrix} \tag{201}$$

A well known result is that given a vector of uncorrelated standard normal values, a sample following the desired joint normal distribution (say, $\mathbf{x}_{\text{norm}}$) can be obtained using the following formula:

$$\mathbf{x}_{\text{norm}} = \mathbf{LZ} + \boldsymbol{\mu} \tag{202}$$

where $\mathbf{L}$ is a matrix that satisfies

$$\mathbf{\Sigma} = \mathbf{L}\mathbf{L}^{\mathbf{T}} \tag{203}$$

and $\mathbf{\Sigma}$ is the covariance matrix of the joint distribution. One way to calculate such a matrix $\mathbf{L}$ is to use an algorithm called the Cholesky decomposition.

In SILHS, instead of the Cholesky decomposition of the covariance matrix, we have the decomposition of the correlation matrix. The decomposition of the covariance matrix is found by multiplying each element of each row of correlation decomposition by the standard deviation of the associated variate. Mathematically,

$$\mathbf{L} = (\mathbf{I}_d \boldsymbol{\sigma})\, \mathbf{L}_{\mathrm{corr}} \tag{204}$$

where $\mathbf{I}_d$ is the $d \times d$ identity matrix and $\boldsymbol{\sigma}$ is the vector of standard deviations.

The last step is to transform from a joint normal distribution to a joint normal-lognormal distribution. This can be done by applying the exponential function to each log-normally distributed variate in the sample (while the normally distributed variates remain the same).

$$\mathbf{x} = \begin{matrix} x_{\mathrm{norm},1} & \chi \\ x_{\mathrm{norm},2} & \eta \\ x_{\mathrm{norm},3} & w \\ \exp(x_{\mathrm{norm},4}) & N_{cn} \\ \exp(x_{\mathrm{norm},5}) & \mathrm{hm}_1 \\ \vdots & \\ \exp(x_{\mathrm{norm},d}) & \mathrm{hm}_n \end{matrix} \tag{205}$$

10.6 Integrating over a PDF using SILHS

When importance sampling is used, the sample points are picked according to the distribution given by the PDF $Q(\mathbf{x})$, as defined in (193). The weight of each sample point, denoted as ω_i, is given in (195). When importance sampling is not used, the sample points are not weighted, or

$$\omega_i = 1 \tag{206}$$

Given SILHS sample points picked according to the distribution given by the PDF $Q(\mathbf{x})$, an integral in the form of (178) can be approximated using the following formulas. To integrate over the entire PDF, use:

$$\int h(\mathbf{x})P(\mathbf{x})d\mathbf{x} \approx \frac{1}{N_s}\sum_{i=1}^{N_s}\omega_i h(\mathbf{x}_i) \tag{207}$$

where $\mathbf{x}_i$ is the i^{th} sample point and N_s is the total number of sample points.

The estimator in (207) gives the mean of $h(\mathbf{x})$ over the entire PDF. To determine the mean over some subset A of the PDF, the following formula should be used:

$$\int h(\mathbf{x})\mathbf{1}_A(\mathbf{x})P(\mathbf{x})d\mathbf{x} \approx \frac{1}{N_s}\sum_{i=1}^{N_s}\omega_i h(\mathbf{x}_i)\mathbf{1}_A(\mathbf{x}_i) \tag{208}$$

However, note that this equation will implicitly be scaled by p_A. To compute a normalized version of the integral, the following could be used instead:

$$\frac{1}{p_A}\int h(\mathbf{x})\mathbf{1}_A(\mathbf{x})P(\mathbf{x})d\mathbf{x} \approx \frac{1}{p_A}\frac{1}{N_s}\sum_{i=1}^{N_s}\omega_i h(\mathbf{x}_i)\mathbf{1}_A(\mathbf{x}_i) \tag{209}$$

10.7 Optimal Allocation of Sample Points

The hope of using the importance sampling method is that our new function will have less variance than the old one. The variance of the new function is given by:

$$v = \int \left[h(\mathbf{x})L(\mathbf{x})\right]^2 Q(\mathbf{x})d\mathbf{x} - \mu^2 \tag{210}$$

We can split the integral up over our importance categories.

$$\begin{aligned}
v &= \sum_{j=1}^{N_c}\left\{\int \left[h(\mathbf{x})L(\mathbf{x})\right]^2 \mathbf{1}_j(\mathbf{x})Q(\mathbf{x})d\mathbf{x}\right\} - \mu^2 && (211)\\
&= \sum_{j=1}^{N_c}\left\{\left(\frac{p_j}{S_j}\right)^2 \int \left[h(\mathbf{x})\right]^2 \mathbf{1}_j(\mathbf{x})\left(\frac{S_j}{p_j}\right) P(\mathbf{x})\ d\mathbf{x}\right\} - \mu^2 && (212)\\
&= \sum_{j=1}^{N_c}\left\{\left(\frac{p_j}{S_j}\right) \int \left[h(\mathbf{x})\right]^2 \mathbf{1}_j(\mathbf{x})P(\mathbf{x})\ d\mathbf{x}\right\} - \mu^2 && (213)
\end{aligned}$$

It is interesting to ask how we might choose our sample probabilities S_j such that (213) is minimized. For convenience, let $u_j = \int [h(\mathbf{x})]^2 \, \mathbf{1}_j(\mathbf{x}) P(\mathbf{x}) \; d\mathbf{x}$. Then we write (213) as

$$v = \left(\sum_{j=1}^{N_c} \frac{p_j u_j}{S_j} \right) - \mu^2 \tag{214}$$

The probabilities S_j must add to one, so we can write this as:

$$v = \left(\sum_{j=1}^{N_c - 1} \frac{p_j u_j}{S_j} \right) + \frac{p_{N_c} u_{N_c}}{1 - \sum_{j=1}^{N_c-1} S_j} - \mu^2 \tag{215}$$

To optimize this, we take the $N_c - 1$ partial derivitives of v with respect to S_j and set them all equal to zero.

$$\frac{\partial v}{\partial S_j} = -\frac{p_j u_j}{S_j^2} + \frac{p_{N_c} u_{N_c}}{\left(1 - \sum_{j=1}^{N_c-1} S_j\right)^2} = 0 \tag{216}$$

$$\frac{p_j u_j}{S_j^2} = \frac{p_{N_c} u_{N_c}}{\left(1 - \sum_{j=1}^{N_c-1} S_j\right)^2} \tag{217}$$

$$\frac{p_j u_j}{S_j^2} = \frac{p_{N_c} u_{N_c}}{S_{N_c}^2} \tag{218}$$

$$\frac{\sqrt{p_j u_j}}{S_j} = \frac{\sqrt{p_{N_c} u_{N_c}}}{S_{N_c}} \tag{219}$$

So, in each category, the sample probability S_j should be proportional to $\sqrt{p_j u_j}$. If we let

$$\alpha = \frac{S_j}{\sqrt{p_j u_j}} \tag{220}$$

and impose the constraint that the sample probabilities add to one, we have

$$\sum_{j=1}^{N_c} \alpha \sqrt{p_j u_j} = 1 \tag{221}$$

$$\alpha = \frac{1}{\sum_{i=j}^{N_c} \sqrt{p_j u_j}} \tag{222}$$

Substituting (222) into (220), we find that

$$S_j = \frac{\sqrt{p_j u_j}}{\sum_{j^*=1}^{N_c} \sqrt{p_{j^*} u_{j^*}}} \tag{223}$$

References

Bougeault, P., 1981a: Modeling the trade-wind cumulus boundary layer. Part I: Testing the ensemble cloud relations against numerical data. *J. Atmos. Sci.*, **38**, 2414–2428.

Bougeault, P., 1981b: Modeling the trade-wind cumulus boundary layer. Part II: A high-order one-dimensional model. *J. Atmos. Sci.*, **38**, 2429–2439.

Duynkerke, P. G. and A. G. M. Driedonks, 1987: A model for the turbulent structure of the stratocumulus-topped atmospheric boundary layer. *J. Atmos. Sci.*, **44**, 43–64.

Larson, V. E. and J.-C. Golaz, 2005: Using probability density functions to derive consistent closure relationships among higher-order moments. *Mon. Wea. Rev.*, **133**, 1023–1042.

Larson, V. E., J.-C. Golaz, H. Jiang, and W. R. Cotton, 2005: Supplying local microphysics parameterizations with information about subgrid variability: Latin hypercube sampling. *Journal of the atmospheric sciences*, **62**, 4010–4026.