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Equations for HOC

1 Predictive equations

$$\frac{\partial \bar{u}}{\partial t} = -\bar{w}\frac{\partial \bar{u}}{\partial z} - f(v_g - \bar{v}) - \frac{\partial}{\partial z}\overline{u'w'} \tag{1}$$

$$\frac{\partial \bar{v}}{\partial t} = -\bar{w}\frac{\partial \bar{v}}{\partial z} + f(u_g - \bar{u}) - \frac{\partial}{\partial z}\overline{v'w'} \tag{2}$$

$$\frac{\partial \bar{q}_t}{\partial t} = -\bar{w}\frac{\partial \bar{q}_t}{\partial z} - \frac{\partial}{\partial z}\overline{w'q_t'} + \left.\frac{\partial \bar{q}_t}{\partial t}\right|_{\mathrm{ls}} \tag{3}$$

$$\frac{\partial \bar{\theta}_l}{\partial t} = -\bar{w}\frac{\partial \bar{\theta}_l}{\partial z} - \frac{\partial}{\partial z}\overline{w'\theta_l'} + \bar{R} + \left.\frac{\partial \bar{\theta}_l}{\partial t}\right|_{\mathrm{ls}} \tag{4}$$

$$\begin{aligned}\frac{\partial \overline{w'^2}}{\partial t} = &- \bar{w}\frac{\partial \overline{w'^2}}{\partial z} - \frac{\partial \overline{w'^3}}{\partial z} - 2\overline{w'^2}\frac{\partial \bar{w}}{\partial z} + \frac{2g}{\theta_0}\overline{w'\theta_v'} \\ &- \frac{C_4}{\tau}\left(\overline{w'^2} - \frac{2}{3}\bar{e}\right) - C_5\left(-2\overline{w'^2}\frac{\partial \bar{w}}{\partial z} + \frac{2g}{\theta_0}\overline{w'\theta_v'}\right) + \frac{2}{3}C_5\left(\frac{g}{\theta_0}\overline{w'\theta_v'} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{v'w'}\frac{\partial \bar{v}}{\partial z}\right) \\ &- \frac{C_1}{\tau}\overline{w'^2} + \nu_1 \nabla_z^2\,\overline{w'^2}\end{aligned} \tag{5}$$

$$\frac{\partial \overline{q_t'^2}}{\partial t} = -\bar{w}\frac{\partial \overline{q_t'^2}}{\partial z} - \frac{\partial \overline{w'q_t'^2}}{\partial z} - 2\overline{w'q_t'}\frac{\partial \bar{q}_t}{\partial z} - \frac{C_2}{\tau}\overline{q_t'^2} + \nu_2\nabla_z^2\overline{q_t'^2} \tag{6}$$

$$\frac{\partial \overline{\theta_l'^2}}{\partial t} = -\bar{w}\frac{\partial \overline{\theta_l'^2}}{\partial z} - \frac{\partial \overline{w'\theta_l'^2}}{\partial z} - 2\overline{w'\theta_l'}\frac{\partial \bar{\theta}_l}{\partial z} - \frac{C_2}{\tau}\overline{\theta_l'^2} + \nu_2\nabla_z^2\overline{\theta_l'^2} \tag{7}$$

$$\frac{\partial \overline{q_t'\theta_l'}}{\partial t} = -\bar{w}\frac{\partial \overline{q_t'\theta_l'}}{\partial z} - \frac{\partial \overline{w'q_t'\theta_l'}}{\partial z} - \overline{w'q_t'}\frac{\partial \bar{\theta}_l}{\partial z} - \overline{w'\theta_l'}\frac{\partial \bar{q}_t}{\partial z} - \frac{C_2}{\tau}\overline{q_t'\theta_l'} + \nu_2\nabla_z^2\overline{q_t'\theta_l'} \tag{8}$$

$$\begin{aligned}\frac{\partial \overline{w'q_t'}}{\partial t} = &- \bar{w}\frac{\partial \overline{w'q_t'}}{\partial z} - \frac{\partial \overline{w'^2q_t'}}{\partial z} - \overline{w'^2}\frac{\partial \bar{q}_t}{\partial z} - \overline{w'q_t'}\frac{\partial \bar{w}}{\partial z} + \frac{g}{\theta_0}\overline{q_t'\theta_v'} \\ &- \frac{C_6}{\tau}\overline{w'q_t'} - C_7\left(-\overline{w'q_t'}\frac{\partial \bar{w}}{\partial z} + \frac{g}{\theta_0}\overline{q_t'\theta_v'}\right) + \nu_6\nabla_z^2\overline{w'q_t'}\end{aligned} \tag{9}$$

$$\begin{aligned}\frac{\partial \overline{w'\theta_l'}}{\partial t} = &- \bar{w}\frac{\partial \overline{w'\theta_l'}}{\partial z} - \frac{\partial \overline{w'^2\theta_l'}}{\partial z} - \overline{w'^2}\frac{\partial \bar{\theta}_l}{\partial z} - \overline{w'\theta_l'}\frac{\partial \bar{w}}{\partial z} + \frac{g}{\theta_0}\overline{\theta_l'\theta_v'} \\ &- \frac{C_6}{\tau}\overline{w'\theta_l'} - C_7\left(-\overline{w'\theta_l'}\frac{\partial \bar{w}}{\partial z} + \frac{g}{\theta_0}\overline{\theta_l'\theta_v'}\right) + \nu_6\nabla_z^2\overline{w'\theta_l'}\end{aligned} \tag{10}$$

$$
\begin{aligned}
\frac{\partial \overline{w'^3}}{\partial t} = & -\bar{w}\frac{\partial \overline{w'^3}}{\partial z} - \frac{\partial \overline{w'^4}}{\partial z} + 3\overline{w'^2}\frac{\partial \overline{w'^2}}{\partial z} - 3\overline{w'^3}\frac{\partial \bar{w}}{\partial z} + \frac{3g}{\theta_0}\overline{w'^2\theta'_v} \\
& - \frac{C_8}{\tau}\left(C_{8b}\, Skw^4 + 1\right)\overline{w'^3} - C_{11}\left(-3\overline{w'^3}\frac{\partial \bar{w}}{\partial z} + \frac{3g}{\theta_0}\overline{w'^2\theta'_v}\right) + \left(K_w + \nu_8\right)\nabla_z^2\overline{w'^3}
\end{aligned}
\tag{11}
$$

$\bar{R}$ is the radiative heating rate, f the Coriolis parameter and u_g, v_g the geostrophic winds. $\left.\frac{\partial \bar{q}_t}{\partial t}\right|_{\mathrm{ls}}$ and $\left.\frac{\partial \bar{\theta}_l}{\partial t}\right|_{\mathrm{ls}}$ are large-scale moisture and temperature forcings. g is the gravity, ρ_0 and θ_0 the reference density and potential temperature.

If the model does not predict any higher-order moments of the horizontal winds, we assume that the turbulence kinetic energy, $\bar{e}$, is proportional to the vertical velocity variance $\overline{w'^2}$:

$$
\bar{e} = \frac{3}{2}\overline{w'^2}. \tag{12}
$$

Alternatively, if higher-order moments of the horizontal winds are computed, then turbulence kinetic energy, $\bar{e}$, is a function of the vertical velocity variance $\overline{w'^2}$, latitudinal wind variance $\overline{v'^2}$, and longitudinal wind variance $\overline{u'^2}$:

$$
\bar{e} = \frac{1}{2}\left(\overline{w'^2} + \overline{u'^2} + \overline{v'^2}\right). \tag{13}
$$

In the second case, the horizontal wind variance terms are determined as in **?**) and given by by the equations:

$$
\begin{aligned}
\frac{\partial \overline{u'^2}}{\partial t} = & -\bar{w}\frac{\partial \overline{u'^2}}{\partial z} - \frac{\partial \overline{w'u'^2}}{\partial z} - \left(1 - C_5\right)2\overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \frac{2}{3}\epsilon + \frac{2}{3}C_5\left(\frac{g}{\theta_0}\overline{w'\theta'_v} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{v'w'}\frac{\partial \bar{v}}{\partial z}\right) \\
& - \frac{C_4}{\tau}\left(\overline{u'^2} - \frac{2}{3}\bar{e}\right) + \nu_9\nabla_z^2\overline{u'^2}
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
\frac{\partial \overline{v'^2}}{\partial t} = & -\bar{w}\frac{\partial \overline{v'^2}}{\partial z} - \frac{\partial \overline{w'v'^2}}{\partial z} - \left(1 - C_5\right)2\overline{v'w'}\frac{\partial \bar{v}}{\partial z} - \frac{2}{3}\epsilon + \frac{2}{3}C_5\left(\frac{g}{\theta_0}\overline{w'\theta'_v} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{v'w'}\frac{\partial \bar{v}}{\partial z}\right) \\
& - \frac{C_4}{\tau}\left(\overline{v'^2} - \frac{2}{3}\bar{e}\right) + \nu_9\nabla_z^2\overline{v'^2}
\end{aligned}
\tag{15}
$$

Where, ϵ, the dissipation of $\bar{e}$, is defined in HOC as:

$$
\epsilon = -C_{14}\frac{\bar{e}}{\tau} \tag{16}
$$

The time scale τ is:

$$\tau = \begin{cases} \dfrac{L}{\sqrt{\overline{e}}}; & L/\sqrt{\overline{e}} \leqslant \tau_{\max} \\ \tau_{\max}; & L/\sqrt{\overline{e}} > \tau_{\max} \end{cases} . \tag{17}$$

Additionally, τ is set to a minimum value $\tau_{\min}$ whenever $\overline{w'^2} \leqslant 0.005$ m^2 s^{-2}.

The eddy diffusivity coefficient K_w is

$$K_w = 0.22\, L\, \overline{e}^{\,1/2}. \tag{18}$$

The momentum fluxes are closed using a down gradient approach:

$$\overline{u'w'} = -K_m \frac{\partial \bar{u}}{\partial z} \tag{19a}$$

$$\overline{v'w'} = -K_m \frac{\partial \bar{v}}{\partial z} \tag{19b}$$

where the turbulent-transfer coefficient K_m is given by:

$$K_m = c_K\, L\, \overline{e}^{\,1/2} \tag{20}$$

with $c_K = c_\mu^{1/4} = 0.548$ as in **?**).

The specific values of the constants C_i and ν_i are as follows: $C_1 = 2.5$; $C_2 = 1.0$; $C_4 = 5.2$; $C_5 = 0.3$; $C_6 = 6.0$; $C_7 = 0.1$; $C_8 = 3.0$; $C_{11} = 0.75$; $C_{14} = 1.0$; $\nu_1 = \nu_8 = \nu_9 = 20$ (m^2/s); and $\nu_2 = \nu_6 = 5$ (m^2/s).

2 PDF closure

Details of the PDF closure can be found in **?**), hereafter referred to as LG. We only briefly summarize key aspects here.

2.1 Transport terms

The transport terms appearing in Eqs (1)-(11) are closed as follows. First, we define $c_{w\theta_l}$ and c_{wq_t} as in Eqs (LG15) and (LG16):

$$c_{w\theta_l} = \frac{\overline{w'\theta_l'}}{\sqrt{\overline{w'^2}}\sqrt{\overline{\theta_l'^2}}} \tag{21}$$

$$c_{wq_t} = \frac{\overline{w'q_t'}}{\sqrt{\overline{w'^2}}\sqrt{\overline{q_t'^2}}} \tag{22}$$

The width of the individual w plumes is given by (LG37):

$$\tilde{\sigma}_w^2 = \gamma \left[1 - \max\left(c_{w\theta_l}^2, c_{wq_t}^2\right)\right] \tag{23}$$

We define the following quantities in order to simplify the notation:

$$a_1 = \frac{1}{(1-\tilde{\sigma}_w^2)} \tag{24}$$

$$a_2 = \frac{1}{(1-\tilde{\sigma}_w^2)^2} \tag{25}$$

$$a_3 = 3\tilde{\sigma}_w^4 + 6(1-\tilde{\sigma}_w^2)\tilde{\sigma}_w^2 + (1-\tilde{\sigma}_w^2)^2 - \frac{3}{2} \tag{26}$$

The turbulence moment $\overline{w'^4}$ is given by (LG40):

$$\overline{w'^4} = \overline{w'^2}^2 \left(a_3 + \frac{3}{2}\right) + a_1 \frac{\overline{w'^3}^2}{\overline{w'^2}} \tag{27}$$

The flux transport terms are given by (LG42):

$$\overline{w'^2\theta_l'} = a_1 \frac{\overline{w'^3}}{\overline{w'^2}} \overline{w'\theta_l'} \tag{28}$$

$$\overline{w'^2 q_t'} = a_1 \frac{\overline{w'^3}}{\overline{w'^2}} \overline{w'q_t'} \tag{29}$$

The variance transport terms follow (LG46):

$$\overline{w'\theta_l'^2} = \frac{1}{3}\beta a_1 \frac{\overline{w'^3}}{\overline{w'^2}} \overline{\theta_l'^2} + \left(1 - \frac{1}{3}\beta\right) a_2 \frac{\overline{w'^3}}{\overline{w'^2}^2} \overline{w'\theta_l'}^2 \tag{30}$$

$$\overline{w'q_t'^2} = \frac{1}{3}\beta a_1 \frac{\overline{w'^3}}{\overline{w'^2}} \overline{q_t'^2} + \left(1 - \frac{1}{3}\beta\right) a_2 \frac{\overline{w'^3}}{\overline{w'^2}^2} \overline{w'q_t'}^2 \tag{31}$$

Finally, the covariance term is obtained substituting (LG56) into (LG48):

$$\overline{w'q_t'\theta_l'} = \frac{1}{3}\beta a_1 \frac{\overline{w'^3}}{\overline{w'^2}} \overline{q_t'\theta_l'} + \left(1 - \frac{1}{3}\beta\right) a_2 \frac{\overline{w'^3}}{\overline{w'^2}^2} \overline{w'q_t'}\,\overline{w'\theta_l'} \tag{32}$$

In the anisotropic case, the horizontal wind variance terms are obtained by:

$$\overline{w'u'^2} = \frac{1}{3}\beta a_1 \frac{\overline{w'^3}}{\overline{w'^2}} \overline{u'^2} + \left(1 - \frac{1}{3}\beta\right) a_2 \frac{\overline{w'^3}}{\overline{w'^2}^2} \overline{w'u'}^2 \tag{33}$$

$$\overline{w'v'^2} = \frac{1}{3}\beta a_1 \frac{\overline{w'^3}}{\overline{w'^2}} \overline{v'^2} + \left(1 - \frac{1}{3}\beta\right) a_2 \frac{\overline{w'^3}}{\overline{w'^2}^2} \overline{w'v'}^2 \tag{34}$$

2.2 Buoyancy terms

There are more unclosed terms involving θ_v. They are $\overline{w'\theta_v'}$, $\overline{q_t'\theta_v'}$, $\overline{\theta_l'\theta_v'}$, and $\overline{w'^2\theta_v'}$ and can be written as:

$$\overline{\chi'\theta_v'} = \overline{\chi'\theta_l'} + \underbrace{\frac{1-\epsilon_0}{\epsilon_0}\theta_0}_{\equiv A\,(\approx 200K)} \overline{\chi'q_t'} + \underbrace{\left(\frac{L_v}{c_p}\left(\frac{p_0}{p}\right)^{R_d/c_p} - \frac{1}{\epsilon_0}\theta_0\right)}_{\equiv B\,(\approx 2000K)} \overline{\chi'q_l'}\ , \tag{35}$$

where χ' represents w', q_t', θ_l' or w'^2. Here $\epsilon_0 = R_d/R_v$, R_d is the gas constant of dry air, R_v is the gas constant of water vapor, L_v is the latent heat of vaporization, c_p is the heat capacity of air, and p_0 is a reference pressure. The correlations involving liquid water ($\overline{\chi'q_l'}$) can be computed for the given family of PDFs (see next section).

3 Cloud properties

The cloud properties, such as cloud fraction, mean liquid water and correlations involving liquid water ($\overline{\chi'q_l'}$) are obtained from the PDF. To do so, a certain number of properties are computed for each Gaussian ($i = 1, 2$):

$$T_{li} = \theta_{li}\left(\frac{p}{p_0}\right)^{R_d/c_p} \tag{36}$$

$$q_{si} = \frac{R_d}{R_v}\,\frac{e_s(T_{li})}{p - [1 - (R_d/R_v)]e_s(T_{li})} \tag{37}$$

$$\beta_i = \frac{R_d}{R_v}\left(\frac{L}{R_d T_{li}}\right)\left(\frac{L}{c_p T_{li}}\right) \tag{38}$$

$$s_i = q_{ti} - q_{si}\frac{1+\beta_i q_{ti}}{1+\beta_i q_{si}} \tag{39}$$

$$c_{q_{ti}} = \frac{1}{1+\beta_i q_{si}} \tag{40}$$

$$c_{\theta_{li}} = \frac{1+\beta_i q_{ti}}{[1+\beta_i q_{si}]^2}\frac{c_p}{L}\beta_i q_{si}\left(\frac{p}{p_0}\right)^{R_d/c_p} \tag{41}$$

$$\sigma_{si}^2 = c_{\theta_{li}}^2\sigma_{\theta_{li}}^2 + c_{q_{ti}}^2\sigma_{q_{ti}}^2 - 2c_{\theta_{li}}\sigma_{\theta_{li}}c_{q_{ti}}\sigma_{q_{ti}}r_{q_t\theta_l} \tag{42}$$

$$C_i = \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{s_i}{\sqrt{2}\sigma_{si}}\right)\right] \tag{43}$$

$$q_{li} = s_i C_i + \frac{\sigma_{si}}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{s_i}{\sigma_{si}}\right)^2\right] \tag{44}$$

where C_i and q_{li} are the cloud fractions and liquid water of each individual Gaussian.

The layer-averaged cloud properties are given by:

$$\overline{C} = aC_1 + (1-a)C_2 \tag{45}$$

$$\overline{q_l} = aq_{l1} + (1-a)q_{l2} \tag{46}$$

$$\overline{w'q_l'} = a(w_1-\bar{w})q_{l1} + (1-a)(w_2-\bar{w})q_{l2} \tag{47}$$

$$\overline{w'^2q_l'} = a\left((w_1-\bar{w})^2+\sigma_{w1}^2\right)q_{l1} + (1-a)\left((w_2-\bar{w})^2+\sigma_{w2}^2\right)q_{l2} - \overline{w'^2}\left(aq_{l1}+(1-a)q_{l2}\right) \tag{48}$$

$$\begin{aligned}\overline{\theta_l'q_l'} =& a\left[(\theta_{l1}-\bar{\theta}_l)q_{l1} - C_1\left(c_{\theta_{l1}}\sigma_{\theta_{l1}}^2 - r_{q_t\theta_l}c_{q_{t1}}\sigma_{q_{t1}}\sigma_{\theta_{l1}}\right)\right] \\ &+ (1-a)\left[(\theta_{l2}-\bar{\theta}_l)q_{l2} - C_2\left(c_{\theta_{l2}}\sigma_{\theta_{l2}}^2 - r_{q_t\theta_l}c_{q_{t2}}\sigma_{q_{t2}}\sigma_{\theta_{l2}}\right)\right]\end{aligned} \tag{49}$$

$$\begin{aligned}\overline{q_t'q_l'} =& a\left[(q_{t1}-\bar{q}_t)q_{l1} + C_1\left(c_{q_{t1}}\sigma_{q_{t1}}^2 - r_{q_t\theta_l}c_{\theta_{l1}}\sigma_{q_{t1}}\sigma_{\theta_{l1}}\right)\right] \\ &+ (1-a)\left[(q_{t2}-\bar{q}_t)q_{l2} + C_2\left(c_{q_{t2}}\sigma_{q_{t2}}^2 - r_{q_t\theta_l}c_{\theta_{l2}}\sigma_{q_{t2}}\sigma_{\theta_{l2}}\right)\right]\end{aligned} \tag{50}$$

4 Steady-state solutions for the variances

4.1 $\overline{q_t'^2}$ and $\overline{\theta_l'^2}$

Start with (6), substitute (31), assume steady-state and rearrange:

$$
\begin{aligned}
&\frac{C_2}{\tau}\overline{q_t'^2} + \bar{w}\frac{\partial \overline{q_t'^2}}{\partial z} + \frac{1}{3}\beta\frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}}{\overline{w'^2}}\,\overline{q_t'^2}\right) - \nu_2\nabla_z^2\overline{q_t'^2} \\
&= -\left(1-\frac{1}{3}\beta\right)\frac{\partial}{\partial z}\left(a_2\frac{\overline{w'^3}}{\overline{w'^2}^2}\,\overline{w'q_t'}^2\right) - 2\,\overline{w'q_t'}\,\frac{\partial \bar{q}_t}{\partial z}
\end{aligned}
\tag{51}
$$

The goal is to recast (51) so that $\overline{q_t'^2}$ can be computed using a tridiagonal solver:

$$
\underbrace{\begin{bmatrix} & (1,2) & \cdots & (1,\mathrm{nnzp}-1) & (1,\mathrm{nnzp}) \\ (2,1) & (2,2) & \cdots & (2,\mathrm{nnzp}-1) & (2,\mathrm{nnzp}) \\ (3,1) & (3,2) & \cdots & (3,\mathrm{nnzp}-1) & \end{bmatrix}}_{\texttt{LHS}\text{(Stored in compact format)}}
\begin{bmatrix} \texttt{qtp2}(1) \\ \texttt{qtp2}(2) \\ \vdots \\ \texttt{qtp2}(\mathrm{nnzp}-1) \\ \texttt{qtp2}(\mathrm{nnzp}) \end{bmatrix}
=
\underbrace{\begin{bmatrix} (1) \\ (2) \\ \vdots \\ (\mathrm{nnzp}-1) \\ (\mathrm{nnzp}) \end{bmatrix}}_{\texttt{RHS}}
$$

$$
\texttt{lhs}(3,\texttt{k})\,\texttt{qtp2}(\texttt{k}-1) + \texttt{lhs}(2,\texttt{k})\,\texttt{qtp2}(\texttt{k}) + \texttt{lhs}(1,\texttt{k})\,\texttt{qtp2}(\texttt{k}+1) = \texttt{rhs}(\texttt{k}) \tag{52}
$$

We now compute the contributions of each term in (51) to $\texttt{lhs}(3,\texttt{k})$, $\texttt{lhs}(2,\texttt{k})$, $\texttt{lhs}(1,\texttt{k})$, and $\texttt{rhs}(\texttt{k})$.

4.1.1 Term 1

$$
\texttt{lhs}(2,\texttt{k}) = \texttt{lhs}(2,\texttt{k}) + \frac{\texttt{C}_2}{\texttt{taum}(\texttt{k})} \tag{53}
$$

4.1.2 Term 2

$$
\begin{aligned}
&\bar{w}\frac{\partial \overline{q_t'^2}}{\partial z}\bigg|_{\texttt{zm(k)}} \\
=&\frac{\texttt{wmm}(\texttt{k})}{\texttt{dzm}(\texttt{k})}\left(\frac{1}{2}\left(\texttt{qtp2}(\texttt{k})+\texttt{qtp2}(\texttt{k}+1)\right) - \frac{1}{2}\left(\texttt{qtp2}(\texttt{k}-1)+\texttt{qtp2}(\texttt{k})\right)\right) \\
=&\frac{\texttt{wmm}(\texttt{k})}{2\texttt{dzm}(\texttt{k})}\texttt{qtp2}(\texttt{k}+1) - \frac{\texttt{wmm}(\texttt{k})}{2\texttt{dzm}(\texttt{k})}\texttt{qtp2}(\texttt{k}-1)
\end{aligned}
\tag{54}
$$

Separating out the contributions:

$$
\begin{aligned}
\mathtt{lhs(3,k)} &= \mathtt{lhs(3,k)} - \frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}} \\
\mathtt{lhs(1,k)} &= \mathtt{lhs(1,k)} + \frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}}
\end{aligned} \tag{55}
$$

4.1.3 Term 3

$$
\begin{aligned}
&\frac{1}{3}\beta\frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}}{\overline{w'^2}}\,\overline{q_t'^2}\right)\Bigg|_{\mathtt{zm(k)}} \\
&= \frac{\beta}{\mathtt{6dzm(k)}}\Bigg[\frac{(\mathtt{a1m(k)}+\mathtt{a1m(k+1)})\,\mathtt{wp3(k+1)}\,(\mathtt{qtp2(k)}+\mathtt{qtp2(k+1)})}{\max(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon)} \\
&\qquad - \frac{(\mathtt{a1m(k-1)}+\mathtt{a1m(k)})\,\mathtt{wp3(k)}\,(\mathtt{qtp2(k-1)}+\mathtt{qtp2(k)})}{\max(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon)}\Bigg]
\end{aligned} \tag{56}
$$

Separating out the contributions:

$$
\begin{aligned}
\mathtt{lhs(3,k)} &= \mathtt{lhs(3,k)} - \frac{\beta}{\mathtt{6dzm(k)}}\frac{(\mathtt{a1m(k-1)}+\mathtt{a1m(k)})\,\mathtt{wp3(k)}}{\max(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon)} \\
\mathtt{lhs(2,k)} &= \mathtt{lhs(2,k)} + \frac{\beta}{\mathtt{6dzm(k)}}\left(\frac{(\mathtt{a1m(k)}+\mathtt{a1m(k+1)})\,\mathtt{wp3(k+1)}}{\max(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon)} - \frac{(\mathtt{a1m(k-1)}+\mathtt{a1m(k)})\,\mathtt{wp3(k)}}{\max(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon)}\right) \\
\mathtt{lhs(1,k)} &= \mathtt{lhs(1,k)} + \frac{\beta}{\mathtt{6dzm(k)}}\frac{(\mathtt{a1m(k)}+\mathtt{a1m(k+1)})\,\mathtt{wp3(k+1)}}{\max(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon)}
\end{aligned} \tag{57}
$$

4.1.4 Term 4

$$
-\nu_2\nabla_z^2\overline{q_t'^2}\Big|_{\mathtt{zm(k)}} = \frac{\nu_2}{\mathtt{dzm(k)}}\left(\frac{\mathtt{qtp2(k+1)}-\mathtt{qtp2(k)}}{\mathtt{dzt(k+1)}} - \frac{\mathtt{qtp2(k)}-\mathtt{qtp2(k-1)}}{\mathtt{dzt(k)}}\right) \tag{58}
$$

Separating out the contributions:

$$
\begin{aligned}
\mathtt{lhs(3,k)} &= \mathtt{lhs(3,k)} - \frac{\nu_2}{\mathtt{dzm(k)dzt(k)}} \\
\mathtt{lhs(2,k)} &= \mathtt{lhs(2,k)} + \frac{\nu_2}{\mathtt{dzm(k)}}\left(\frac{1}{\mathtt{dzt(k+1)}} + \frac{1}{\mathtt{dzt(k)}}\right) \\
\mathtt{lhs(1,k)} &= \mathtt{lhs(1,k)} - \frac{\nu_2}{\mathtt{dzm(k)dzt(k+1)}}
\end{aligned} \tag{59}
$$

4.1.5 Term 5

$$
\begin{aligned}
&-\left(1-\frac{1}{3}\beta\right)\frac{\partial}{\partial z}\left(a_2\frac{\overline{w'^3}}{\overline{w'^2}^2}\,\overline{w'q_t'}^2\right)\Bigg|_{\mathtt{zm(k)}} \\
=&-\frac{1-\frac{1}{3}\beta}{\mathtt{4dzm(k)}}\left[\frac{(\mathtt{a1m(k)}+\mathtt{a1m(k+1)})^2\,\mathtt{wp3(k+1)}\,(\mathtt{wpqtp(k)}+\mathtt{wpqtp(k+1)})^2}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)^2}\right. \\
&\qquad\left.-\frac{(\mathtt{a1m(k-1)}+\mathtt{a1m(k)})^2\,\mathtt{wp3(k)}\,(\mathtt{wpqtp(k-1)}+\mathtt{wpqtp(k)})^2}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)^2}\right]
\end{aligned}
\tag{60}
$$

Separating out the contributions:

$$
\begin{aligned}
&\mathtt{rhs(k)} \\
=&\mathtt{rhs(k)}-\frac{1-\frac{1}{3}\beta}{\mathtt{4dzm(k)}}\left[\frac{(\mathtt{a1m(k)}+\mathtt{a1m(k+1)})^2\,\mathtt{wp3(k+1)}\,(\mathtt{wpqtp(k)}+\mathtt{wpqtp(k+1)})^2}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)^2}\right. \\
&\qquad\left.-\frac{(\mathtt{a1m(k-1)}+\mathtt{a1m(k)})^2\,\mathtt{wp3(k)}\,(\mathtt{wpqtp(k-1)}+\mathtt{wpqtp(k)})^2}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)^2}\right]
\end{aligned}
\tag{61}
$$

4.1.6 Term 6

$$
-2\,\overline{w'q_t'}\,\frac{\partial\bar{q}_t}{\partial z}\Bigg|_{\mathtt{zm(k)}} = -\mathtt{2wpqtp(k)}\frac{\mathtt{qtm(k+1)}-\mathtt{qtm(k)}}{\mathtt{dzm(k)}}
\tag{62}
$$

Separating out the contributions:

$$
\mathtt{rhs(k)} = \mathtt{rhs(k)} - \mathtt{2wpqtp(k)}\frac{\mathtt{qtm(k+1)}-\mathtt{qtm(k)}}{\mathtt{dzm(k)}}
\tag{63}
$$

4.2 $\overline{q_t'\theta_l'}$

Start with (8), substitute (32), assume steady-state and rearrange:

$$
\begin{aligned}
&\frac{C_2}{\tau}\overline{q_t'\theta_l'}+\bar{w}\frac{\partial\overline{q_t'\theta_l'}}{\partial z}+\frac{1}{3}\beta\frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}}{\overline{w'^2}}\,\overline{q_t'\theta_l'}\right)-\nu_2\nabla_z^2\overline{q_t'\theta_l'} \\
=&-\left(1-\frac{1}{3}\beta\right)\frac{\partial}{\partial z}\left(a_2\frac{\overline{w'^3}}{\overline{w'^2}^2}\,\overline{w'q_t'}\,\overline{w'\theta_l'}\right)-\overline{w'q_t'}\frac{\partial\bar{\theta}_l}{\partial z}-\overline{w'\theta_l'}\frac{\partial\bar{q}_t}{\partial z}
\end{aligned}
\tag{64}
$$

As for the variances, the goal is to recast (64) so that $\overline{q_t'\theta_l'}$ can be computed using a tridiagonal solver:

$$
\mathtt{lhs(3,k)}\,\mathtt{qtpthlp(k-1)}+\mathtt{lhs(2,k)}\,\mathtt{qtpthlp(k)}+\mathtt{lhs(1,k)}\,\mathtt{qtpthlp(k+1)}=\mathtt{rhs(k)}
\tag{65}
$$

We now compute the contributions of each term in (64) to $\mathtt{lhs(3,k)}$, $\mathtt{lhs(2,k)}$, $\mathtt{lhs(1,k)}$, and $\mathtt{rhs(k)}$.

4.2.1 Term 1

$$
\texttt{lhs}(2,\texttt{k}) = \texttt{lhs}(2,\texttt{k}) + \frac{\texttt{C}_2}{\texttt{taum}(\texttt{k})} \tag{66}
$$

4.2.2 Term 2

$$
\begin{aligned}
&\bar{w}\frac{\partial\overline{q_t'\theta_l'}}{\partial z}\bigg|_{\texttt{zm}(\texttt{k})} \\
=&\frac{\texttt{wmm}(\texttt{k})}{\texttt{dzm}(\texttt{k})}\left(\frac{1}{2}\left(\texttt{qtpthlp}(\texttt{k})+\texttt{qtpthlp}(\texttt{k}+1)\right)-\frac{1}{2}\left(\texttt{qtpthlp}(\texttt{k}-1)+\texttt{qtpthlp}(\texttt{k})\right)\right) \\
=&\frac{\texttt{wmm}(\texttt{k})}{2\texttt{dzm}(\texttt{k})}\texttt{qtpthlp}(\texttt{k}+1)-\frac{\texttt{wmm}(\texttt{k})}{2\texttt{dzm}(\texttt{k})}\texttt{qtpthlp}(\texttt{k}-1)
\end{aligned} \tag{67}
$$

Separating out the contributions:

$$
\begin{aligned}
\texttt{lhs}(3,\texttt{k}) &= \texttt{lhs}(3,\texttt{k}) - \frac{\texttt{wmm}(\texttt{k})}{2\texttt{dzm}(\texttt{k})} \\
\texttt{lhs}(1,\texttt{k}) &= \texttt{lhs}(1,\texttt{k}) + \frac{\texttt{wmm}(\texttt{k})}{2\texttt{dzm}(\texttt{k})}
\end{aligned} \tag{68}
$$

4.2.3 Term 3

$$
\begin{aligned}
&\frac{1}{3}\beta\frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}}{\overline{w'^2}}\overline{q_t'\theta_l'}\right)\bigg|_{\texttt{zm}(\texttt{k})} \\
&= \frac{\beta}{6\texttt{dzm}(\texttt{k})}\left[\frac{\left(\texttt{a1m}(\texttt{k})+\texttt{a1m}(\texttt{k}+1)\right)\texttt{wp3}(\texttt{k}+1)\left(\texttt{qtpthlp}(\texttt{k})+\texttt{qtpthlp}(\texttt{k}+1)\right)}{\max\left(\texttt{wp2}(\texttt{k})+\texttt{wp2}(\texttt{k}+1),2\epsilon\right)}\right. \\
&\qquad\left.-\frac{\left(\texttt{a1m}(\texttt{k}-1)+\texttt{a1m}(\texttt{k})\right)\texttt{wp3}(\texttt{k})\left(\texttt{qtpthlp}(\texttt{k}-1)+\texttt{qtpthlp}(\texttt{k})\right)}{\max\left(\texttt{wp2}(\texttt{k}-1)+\texttt{wp2}(\texttt{k}),2\epsilon\right)}\right]
\end{aligned} \tag{69}
$$

Separating out the contributions:

$$
\begin{aligned}
\texttt{lhs}(3,\texttt{k}) &= \texttt{lhs}(3,\texttt{k}) - \frac{\beta}{6\texttt{dzm}(\texttt{k})}\frac{\left(\texttt{a1m}(\texttt{k}-1)+\texttt{a1m}(\texttt{k})\right)\texttt{wp3}(\texttt{k})}{\max\left(\texttt{wp2}(\texttt{k}-1)+\texttt{wp2}(\texttt{k}),2\epsilon\right)} \\
\texttt{lhs}(2,\texttt{k}) &= \texttt{lhs}(2,\texttt{k}) + \frac{\beta}{6\texttt{dzm}(\texttt{k})}\left(\frac{\left(\texttt{a1m}(\texttt{k})+\texttt{a1m}(\texttt{k}+1)\right)\texttt{wp3}(\texttt{k}+1)}{\max\left(\texttt{wp2}(\texttt{k})+\texttt{wp2}(\texttt{k}+1),2\epsilon\right)}-\frac{\left(\texttt{a1m}(\texttt{k}-1)+\texttt{a1m}(\texttt{k})\right)\texttt{wp3}(\texttt{k})}{\max\left(\texttt{wp2}(\texttt{k}-1)+\texttt{wp2}(\texttt{k}),2\epsilon\right)}\right) \\
\texttt{lhs}(1,\texttt{k}) &= \texttt{lhs}(1,\texttt{k}) + \frac{\beta}{6\texttt{dzm}(\texttt{k})}\frac{\left(\texttt{a1m}(\texttt{k})+\texttt{a1m}(\texttt{k}+1)\right)\texttt{wp3}(\texttt{k}+1)}{\max\left(\texttt{wp2}(\texttt{k})+\texttt{wp2}(\texttt{k}+1),2\epsilon\right)}
\end{aligned} \tag{70}
$$

4.2.4 Term 4

$$-\nu_2 \nabla_z^2 \overline{q_t' \theta_l'}\Big|_{\mathtt{zm(k)}} = \frac{\nu_2}{\mathtt{dzm(k)}} \left(\frac{\mathtt{qptthlp(k+1)} - \mathtt{qptthlp(k)}}{\mathtt{dzt(k+1)}} - \frac{\mathtt{qptthlp(k)} - \mathtt{qptthlp(k-1)}}{\mathtt{dzt(k)}} \right) \tag{71}$$

Separating out the contributions:

$$\begin{aligned}
\mathtt{lhs(3,k)} &= \mathtt{lhs(3,k)} - \frac{\nu_2}{\mathtt{dzm(k)dzt(k)}} \\
\mathtt{lhs(2,k)} &= \mathtt{lhs(2,k)} + \frac{\nu_2}{\mathtt{dzm(k)}} \left(\frac{1}{\mathtt{dzt(k+1)}} + \frac{1}{\mathtt{dzt(k)}} \right) \\
\mathtt{lhs(1,k)} &= \mathtt{lhs(1,k)} - \frac{\nu_2}{\mathtt{dzm(k)dzt(k+1)}}
\end{aligned} \tag{72}$$

4.2.5 Term 5

$$\begin{aligned}
&- \left(1 - \frac{1}{3}\beta\right) \frac{\partial}{\partial z} \left(a_2 \frac{\overline{w'^3}}{\overline{w'^2}^2} \overline{w' q_t'}\, \overline{w' \theta_l'} \right)\Bigg|_{\mathtt{zm(k)}} \\
=& - \frac{1 - \frac{1}{3}\beta}{\mathtt{4dzm(k)}} \\
&\times \Bigg[\frac{(\mathtt{a1m(k)} + \mathtt{a1m(k+1)})^2\, \mathtt{wp3(k+1)}\, (\mathtt{wpqtp(k)} + \mathtt{wpqtp(k+1)})\, (\mathtt{wpthlp(k)} + \mathtt{wpthlp(k+1)})}{\max\left(\mathtt{wp2(k)} + \mathtt{wp2(k+1)}, 2\epsilon\right)^2} \\
&\quad - \frac{(\mathtt{a1m(k-1)} + \mathtt{a1m(k)})^2\, \mathtt{wp3(k)}\, (\mathtt{wpqtp(k-1)} + \mathtt{wpqtp(k)})\, (\mathtt{wpthlp(k-1)} + \mathtt{wpthlp(k)})}{\max\left(\mathtt{wp2(k-1)} + \mathtt{wp2(k)}, 2\epsilon\right)^2} \Bigg]
\end{aligned} \tag{73}$$

Separating out the contributions:

$$\begin{aligned}
\mathtt{rhs(k)} =& \mathtt{rhs(k)} \\
&- \frac{1 - \frac{1}{3}\beta}{\mathtt{4dzm(k)}} \\
&\times \Bigg[\frac{(\mathtt{a1m(k)} + \mathtt{a1m(k+1)})^2\, \mathtt{wp3(k+1)}\, (\mathtt{wpqtp(k)} + \mathtt{wpqtp(k+1)})\, (\mathtt{wpthlp(k)} + \mathtt{wpthlp(k+1)})}{\max\left(\mathtt{wp2(k)} + \mathtt{wp2(k+1)}, 2\epsilon\right)^2} \\
&\quad - \frac{(\mathtt{a1m(k-1)} + \mathtt{a1m(k)})^2\, \mathtt{wp3(k)}\, (\mathtt{wpqtp(k-1)} + \mathtt{wpqtp(k)})\, (\mathtt{wpthlp(k-1)} + \mathtt{wpthlp(k)})}{\max\left(\mathtt{wp2(k-1)} + \mathtt{wp2(k)}, 2\epsilon\right)^2} \Bigg]
\end{aligned} \tag{74}$$

4.2.6 Terms 6 and 7

$$
\begin{aligned}
& -\overline{w'q_t'}\frac{\partial\bar{\theta}_l}{\partial z}\bigg|_{\mathtt{zm(k)}} - \overline{w'\theta_l'}\frac{\partial\bar{q}_t}{\partial z}\bigg|_{\mathtt{zm(k)}} \\
& = -\mathtt{wpqtp(k)}\frac{\mathtt{thlm(k+1)} - \mathtt{thlm(k)}}{\mathtt{dzm(k)}} - \mathtt{wpthlp(k)}\frac{\mathtt{qtm(k+1)} - \mathtt{qtm(k)}}{\mathtt{dzm(k)}}
\end{aligned}
\tag{75}
$$

Separating out the contributions:

$$
\mathtt{rhs(k)} = \mathtt{rhs(k)} - \mathtt{wpqtp(k)}\frac{\mathtt{thlm(k+1)} - \mathtt{thlm(k)}}{\mathtt{dzm(k)}} - \mathtt{wpthlp(k)}\frac{\mathtt{qtm(k+1)} - \mathtt{qtm(k)}}{\mathtt{dzm(k)}}
\tag{76}
$$

5 Implicit solutions for the means and fluxes

$\bar{q}_t$ and $\overline{w'q_t'}$ can be solved simultaneously and implicitly. Start with eqs (3), (9) and substitute expression for the transport term (29):

$$
\frac{\partial\bar{q}_t}{\partial t} = -\bar{w}\frac{\partial\bar{q}_t}{\partial z} - \frac{\partial}{\partial z}\overline{w'q_t'} + \frac{\partial\bar{q}_t}{\partial t}\bigg|_{\mathrm{ls}}
\tag{77}
$$

$$
\begin{aligned}
\frac{\partial\overline{w'q_t'}}{\partial t} = & -\bar{w}\frac{\partial\overline{w'q_t'}}{\partial z} - \frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}}{\overline{w'^2}}\overline{w'q_t'}\right) - \overline{w'^2}\frac{\partial\bar{q}_t}{\partial z} - (1-C_7)\overline{w'q_t'}\frac{\partial\bar{w}}{\partial z} + (1-C_7)\frac{g}{\theta_0}\overline{q_t'\theta_v'} \\
& - \frac{C_6}{\tau}\overline{w'q_t'} + \nu_6\nabla_z^2\overline{w'q_t'}
\end{aligned}
\tag{78}
$$

After discretizing the time derivative and rearranging terms:

$$
\begin{aligned}
& \frac{\bar{q}_t^{\,t+\Delta t}}{\Delta t} + \bar{w}\frac{\partial\bar{q}_t^{\,t+\Delta t}}{\partial z} + \frac{\partial}{\partial z}\overline{w'q_t'}^{\,t+\Delta t} \\
& = \frac{\bar{q}_t^{\,t}}{\Delta t} + \frac{\partial\bar{q}_t}{\partial t}\bigg|_{\mathrm{ls}}
\end{aligned}
\tag{79}
$$

$$
\begin{aligned}
& \frac{\overline{w'q_t'}^{\,t+\Delta t}}{\Delta t} + \bar{w}\frac{\partial}{\partial z}\overline{w'q_t'}^{\,t+\Delta t} + \frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}}{\overline{w'^2}}\overline{w'q_t'}^{\,t+\Delta t}\right) + \overline{w'^2}\frac{\partial\bar{q}_t^{\,t+\Delta t}}{\partial z} \\
& + (1-C_7)\overline{w'q_t'}^{\,t+\Delta t}\frac{\partial\bar{w}}{\partial z} + \frac{C_6}{\tau}\overline{w'q_t'}^{\,t+\Delta t} - \nu_6\nabla_z^2\overline{w'q_t'}^{\,t+\Delta t} \\
& = \frac{\overline{w'q_t'}^{\,t}}{\Delta t} + (1-C_7)\frac{g}{\theta_0}\overline{q_t'\theta_v'}^{\,t}
\end{aligned}
\tag{80}
$$

The LHSs of (79)-(80) is linear in $\bar{q}$ and $\overline{w'q'_t}$ and can therefore be rewritten in matrix form:

$$
\underbrace{\begin{pmatrix}
\cdots & \bar{q}_{k}^{\,\text{impl.}} & \overline{w'q'_t}_{k}^{\,\text{impl.}} & \bar{q}_{k+1}^{\,\text{impl.}} & \overline{w'q'_t}_{k+1}^{\,\text{impl.}} & \bar{q}_{k+2}^{\,\text{impl.}} & \overline{w'q'_t}_{k+2}^{\,\text{impl.}} & \cdots \\
\cdots & \overline{w'q'_t}_{k-1}^{\,\text{impl.}} & \bar{q}_{k}^{\,\text{impl.}} & \overline{w'q'_t}_{k}^{\,\text{impl.}} & \bar{q}_{k+1}^{\,\text{impl.}} & \overline{w'q'_t}_{k+1}^{\,\text{impl.}} & \bar{q}_{k+2}^{\,\text{impl.}} & \cdots \\
\cdots & \bar{q}_{k-1}^{\,\text{impl.}} & \overline{w'q'_t}_{k-1}^{\,\text{impl.}} & \bar{q}_{k}^{\,\text{impl.}} & \overline{w'q'_t}_{k}^{\,\text{impl.}} & \bar{q}_{k+1}^{\,\text{impl.}} & \overline{w'q'_t}_{k+1}^{\,\text{impl.}} & \cdots \\
\cdots & \overline{w'q'_t}_{k-2}^{\,\text{impl.}} & \bar{q}_{k-1}^{\,\text{impl.}} & \overline{w'q'_t}_{k-1}^{\,\text{impl.}} & \bar{q}_{k}^{\,\text{impl.}} & \overline{w'q'_t}_{k}^{\,\text{impl.}} & \bar{q}_{k+1}^{\,\text{impl.}} & \cdots \\
\cdots & \bar{q}_{k-2}^{\,\text{impl.}} & \overline{w'q'_t}_{k-2}^{\,\text{impl.}} & \bar{q}_{k-1}^{\,\text{impl.}} & \overline{w'q'_t}_{k-1}^{\,\text{impl.}} & \bar{q}_{k}^{\,\text{impl.}} & \overline{w'q'_t}_{k}^{\,\text{impl.}} & \cdots
\end{pmatrix}}_{\texttt{LHS}\text{(Stored in compact format)}}
\begin{pmatrix}
\vdots \\
\bar{q}_{k-1}^{\,t+\Delta t} \\
\overline{w'q'_t}_{k-1}^{\,t+\Delta t} \\
\bar{q}_{k}^{\,t+\Delta t} \\
\overline{w'q'_t}_{k}^{\,t+\Delta t} \\
\bar{q}_{k+1}^{\,t+\Delta t} \\
\overline{w'q'_t}_{k+1}^{\,t+\Delta t} \\
\vdots
\end{pmatrix}
=
\underbrace{\begin{pmatrix}
\vdots \\
\bar{q}_{k-1}^{\,\text{expl.}} \\
\overline{w'q'_t}_{k-1}^{\,\text{expl.}} \\
\bar{q}_{k}^{\,\text{expl.}} \\
\overline{w'q'_t}_{k}^{\,\text{expl.}} \\
\bar{q}_{k+1}^{\,\text{expl.}} \\
\overline{w'q'_t}_{k+1}^{\,\text{expl.}} \\
\vdots
\end{pmatrix}}_{\texttt{RHS}}
\tag{81}
$$

The matrix lhs is obtained by vertical discretization of the LHSs, and the vector rhs by discretization of the RHSs. lhs is band-diagonal with two rows above and two below the main diagonal. lhs is stored in compact form in a array with dimensions $(5, 2\,\texttt{nnzp})$. rhs is a vector with dimension $(2\,\texttt{nnzp})$. lhs can be inverted efficiently using an LU decomposition algorithm for band diagonal matrices. The construction of the matrix lhs and vector rhs are as follows.

First, we compute the finite difference equivalent to (79):

$$
\begin{aligned}
&\frac{\texttt{qtm}^{\texttt{new}}(\texttt{k})}{\texttt{dt}} + \frac{\texttt{wmt}(\texttt{k})}{2\texttt{dzt}(\texttt{k})}\texttt{qtm}^{\texttt{new}}(\texttt{k}+1) - \frac{\texttt{wmt}(\texttt{k})}{2\texttt{dzt}(\texttt{k})}\texttt{qtm}^{\texttt{new}}(\texttt{k}-1) + \frac{\texttt{wpqtp}^{\texttt{new}}(\texttt{k})}{\texttt{dzt}(\texttt{k})} - \frac{\texttt{wpqtp}^{\texttt{new}}(\texttt{k}-1)}{\texttt{dzt}(\texttt{k})} \\
=&\frac{\texttt{qtm}(\texttt{k})}{\texttt{dt}} + \texttt{qtm\_ls}(\texttt{k})
\end{aligned}
\tag{82}
$$

Contributions to lhs from (82) are:

$$\texttt{lhs}(5,\texttt{k\_xm}) = \texttt{lhs}(5,\texttt{k\_xm}) - \frac{\texttt{wmt}(\texttt{k})}{2\texttt{dzt}(\texttt{k})} \tag{83}$$

$$\texttt{lhs}(4,\texttt{k\_xm}) = \texttt{lhs}(4,\texttt{k\_xm}) - \frac{1}{\texttt{dzt}(\texttt{k})} \tag{84}$$

$$\texttt{lhs}(3,\texttt{k\_xm}) = \texttt{lhs}(3,\texttt{k\_xm}) + \frac{1}{\texttt{dt}} \tag{85}$$

$$\texttt{lhs}(2,\texttt{k\_xm}) = \texttt{lhs}(2,\texttt{k\_xm}) + \frac{1}{\texttt{dzt}(\texttt{k})} \tag{86}$$

$$\texttt{lhs}(1,\texttt{k\_xm}) = \texttt{lhs}(1,\texttt{k\_xm}) + \frac{\texttt{wmt}(\texttt{k})}{2\texttt{dzt}(\texttt{k})} \tag{87}$$

Contributions to rhs from (82) are:

$$\texttt{rhs}(\texttt{k\_xm}) = \texttt{rhs}(\texttt{k\_xm}) + \frac{\texttt{qtm}(\texttt{k})}{\texttt{dt}} + \texttt{qtm\_ls}(\texttt{k}) \tag{88}$$

where $\texttt{k\_xm} = 2\texttt{k} - 1$.

We now write the finite difference equivalent to (80):

$$\begin{aligned}
&\frac{\texttt{wpqtp}^{\texttt{new}}(\texttt{k})}{\texttt{dt}} + \frac{\texttt{wmm}(\texttt{k})}{2\texttt{dzm}(\texttt{k})}\texttt{wpqtp}^{\texttt{new}}(\texttt{k}+1) - \frac{\texttt{wmm}(\texttt{k})}{2\texttt{dzm}(\texttt{k})}\texttt{wpqtp}^{\texttt{new}}(\texttt{k}-1) \\
&+\frac{1}{2\texttt{dzm}(\texttt{k})}\Bigg[-\frac{(\texttt{a1m}(\texttt{k}-1)+\texttt{a1m}(\texttt{k}))\,\texttt{wp3}(\texttt{k})}{\max(\texttt{wp2}(\texttt{k}-1)+\texttt{wp2}(\texttt{k}),2\epsilon)}\texttt{wpqtp}^{\texttt{new}}(\texttt{k}-1) \\
&\qquad +\left(\frac{(\texttt{a1m}(\texttt{k})+\texttt{a1m}(\texttt{k}+1))\,\texttt{wp3}(\texttt{k}+1)}{\max(\texttt{wp2}(\texttt{k})+\texttt{wp2}(\texttt{k}+1),2\epsilon)} - \frac{(\texttt{a1m}(\texttt{k}-1)+\texttt{a1m}(\texttt{k}))\,\texttt{wp3}(\texttt{k})}{\max(\texttt{wp2}(\texttt{k}-1)+\texttt{wp2}(\texttt{k}),2\epsilon)}\right)\texttt{wpqtp}^{\texttt{new}}(\texttt{k}) \\
&\qquad +\frac{(\texttt{a1m}(\texttt{k})+\texttt{a1m}(\texttt{k}+1))\,\texttt{wp3}(\texttt{k}+1)}{\max(\texttt{wp2}(\texttt{k})+\texttt{wp2}(\texttt{k}+1),2\epsilon)}\texttt{wpqtp}^{\texttt{new}}(\texttt{k}+1)\Bigg] \\
&+\texttt{wp2}(\texttt{k})\,\frac{\texttt{qtm}^{\texttt{new}}(\texttt{k}+1)-\texttt{qtm}^{\texttt{new}}(\texttt{k})}{\texttt{dzm}(\texttt{k})} + (1-\texttt{C}_7)\,\texttt{wpqtp}^{\texttt{new}}(\texttt{k})\,\frac{\texttt{wmt}(\texttt{k}+1)-\texttt{wmt}(\texttt{k})}{\texttt{dzm}(\texttt{k})} \\
&+\frac{\texttt{C}_6}{\texttt{taum}(\texttt{k})}\,\texttt{wpqtp}^{\texttt{new}}(\texttt{k}) \\
&-\frac{\nu_6}{\texttt{dzm}(\texttt{k})\texttt{dzt}(\texttt{k})}\texttt{wpqtp}^{\texttt{new}}(\texttt{k}-1) \\
&+\frac{\nu_6}{\texttt{dzm}(\texttt{k})}\left(\frac{1}{\texttt{dzt}(\texttt{k}+1)}+\frac{1}{\texttt{dzt}(\texttt{k})}\right)\texttt{wpqtp}^{\texttt{new}}(\texttt{k}) \\
&-\frac{\nu_6}{\texttt{dzm}(\texttt{k})\texttt{dzt}(\texttt{k}+1)}\texttt{wpqtp}^{\texttt{new}}(\texttt{k}+1) \\
=&\frac{\texttt{wpqtp}(\texttt{k})}{\texttt{dt}} + (1-\texttt{C}_7)\frac{\texttt{g}}{\theta_0}\texttt{qtpthvp}(\texttt{k})
\end{aligned} \tag{89}$$

Contributions to lhs from (89) are:

$$\begin{aligned}
&\texttt{lhs}(5,\texttt{k_wpxp}) = \texttt{lhs}(5,\texttt{k_wpxp}) \\
&-\frac{\texttt{wmm}(\texttt{k})}{2\texttt{dzm}(\texttt{k})} - \frac{1}{2\texttt{dzm}(\texttt{k})}\frac{(\texttt{a1m}(\texttt{k}-1)+\texttt{a1m}(\texttt{k}))\,\texttt{wp3}(\texttt{k})}{\max(\texttt{wp2}(\texttt{k}-1)+\texttt{wp2}(\texttt{k}),2\epsilon)} - \frac{\nu_6}{\texttt{dzm}(\texttt{k})\texttt{dzt}(\texttt{k})}
\end{aligned} \tag{90}$$

$$\texttt{lhs}(4,\texttt{k\_wpxp}) = \texttt{lhs}(4,\texttt{k\_wpxp}) - \frac{\texttt{wp2}(\texttt{k})}{\texttt{dzm}(\texttt{k})} \tag{91}$$

$$\begin{aligned}
&\texttt{lhs}(3,\texttt{k_wpxp}) = \texttt{lhs}(3,\texttt{k_wpxp}) \\
&+\frac{1}{\texttt{dt}} + \frac{1}{2\texttt{dzm}(\texttt{k})}\left(\frac{(\texttt{a1m}(\texttt{k})+\texttt{a1m}(\texttt{k}+1))\,\texttt{wp3}(\texttt{k}+1)}{\max(\texttt{wp2}(\texttt{k})+\texttt{wp2}(\texttt{k}+1),2\epsilon)} - \frac{(\texttt{a1m}(\texttt{k}-1)+\texttt{a1m}(\texttt{k}))\,\texttt{wp3}(\texttt{k})}{\max(\texttt{wp2}(\texttt{k}-1)+\texttt{wp2}(\texttt{k}),2\epsilon)}\right) \\
&+(1-\texttt{C}_7)\,\frac{\texttt{wmt}(\texttt{k}+1)-\texttt{wmt}(\texttt{k})}{\texttt{dzm}(\texttt{k})} + \frac{\texttt{C}_6}{\texttt{taum}(\texttt{k})} + \frac{\nu_6}{\texttt{dzm}(\texttt{k})}\left(\frac{1}{\texttt{dzt}(\texttt{k}+1)}+\frac{1}{\texttt{dzt}(\texttt{k})}\right)
\end{aligned} \tag{92}$$

$$\texttt{lhs(2,k\_wpxp)} = \texttt{lhs(2,k\_wpxp)} + \frac{\texttt{wp2(k)}}{\texttt{dzm(k)}} \tag{93}$$

$$\begin{aligned}&\texttt{lhs(1,k\_wpxp)} = \texttt{lhs(1,k\_wpxp)} \\ &+\frac{\texttt{wmm(k)}}{\texttt{2dzm(k)}} + \frac{\texttt{1}}{\texttt{2dzm(k)}} \frac{\left(\texttt{a1m(k)} + \texttt{a1m(k+1)}\right)\texttt{wp3(k+1)}}{\max\left(\texttt{wp2(k)} + \texttt{wp2(k+1)}, 2\epsilon\right)} - \frac{\nu_6}{\texttt{dzm(k)dzt(k+1)}}\end{aligned} \tag{94}$$

Contributions to rhs from (89) are:

$$\texttt{rhs(k\_wpxp)} = \texttt{rhs(k\_wpxp)} + \frac{\texttt{wpqtp(k)}}{\texttt{dt}} + (\texttt{1} - \texttt{C}_7)\frac{\texttt{g}}{\theta_0}\texttt{qtpthvp(k)} \tag{95}$$

where $\texttt{k\_wpxp} = \texttt{2k}$.

The procedure for solving implicitly for $\bar{\theta}_l$ and $\overline{w'\theta_l'}$ is identical. It leads to the same matrix lhs, so lhs needs to be inverted only once.

6 Implicit solution for the vertical velocity moments

Start with equations (5) and (11):

$$\begin{aligned}\frac{\partial\overline{w'^2}}{\partial t} =& -\bar{w}\frac{\partial\overline{w'^2}}{\partial z} - \frac{\partial\overline{w'^3}}{\partial z} - 2\overline{w'^2}\frac{\partial\bar{w}}{\partial z} + \frac{2g}{\theta_0}\overline{w'\theta_v'} \\ & - \frac{C_4}{\tau}\left(\overline{w'^2} - \frac{2}{3}\bar{e}\right) - C_5\left(-2\overline{w'^2}\frac{\partial\bar{w}}{\partial z} + \frac{2g}{\theta_0}\overline{w'\theta_v'}\right) + \frac{2}{3}C_5\left(\frac{g}{\theta_0}\overline{w'\theta_v'} - \overline{u'w'}\frac{\partial\bar{u}}{\partial z} - \overline{v'w'}\frac{\partial\bar{v}}{\partial z}\right) \\ & - \frac{C_1}{\tau}\overline{w'^2} + \nu_1\nabla_z^2\overline{w'^2}\end{aligned} \tag{96}$$

$$\begin{aligned}\frac{\partial\overline{w'^3}}{\partial t} =& -\bar{w}\frac{\partial\overline{w'^3}}{\partial z} - \frac{\partial\overline{w'^4}}{\partial z} + 3\overline{w'^2}\frac{\partial\overline{w'^2}}{\partial z} - 3\overline{w'^3}\frac{\partial\bar{w}}{\partial z} + \frac{3g}{\theta_0}\overline{w'^2\theta_v'} \\ & - \frac{C_8}{\tau}\left(C_{8b}\,Skw^4 + 1\right)\overline{w'^3} - C_{11}\left(-3\overline{w'^3}\frac{\partial\bar{w}}{\partial z} + \frac{3g}{\theta_0}\overline{w'^2\theta_v'}\right) + (K_w + \nu_8)\nabla_z^2\overline{w'^3}\end{aligned} \tag{97}$$

Using (27), we can rewrite the transport and production terms in (97):

$$\begin{aligned}& - \frac{\partial\overline{w'^4}}{\partial z} + 3\overline{w'^2}\frac{\partial\overline{w'^2}}{\partial z} \\ & = -\frac{\partial}{\partial z}\left(\overline{w'^4} - \frac{3}{2}\overline{w'^2}^2\right) \\ & = -\frac{\partial}{\partial z}\left(a_3\overline{w'^2}^2\right) - \frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}^2}{\overline{w'^2}}\right)\end{aligned} \tag{98}$$

Rearranging terms and making use of (12):

$$\frac{\partial \overline{w'^2}}{\partial t} + \frac{\partial \overline{w'^3}}{\partial z} + \frac{C_1}{\tau}\overline{w'^2} - \nu_1 \nabla_z^2 \, \overline{w'^2}$$
$$= -\bar{w}\frac{\partial \overline{w'^2}}{\partial z} + (1-C_5)\frac{2g}{\theta_0}\overline{w'\theta_v'} - 2(1-C_5)\overline{w'^2}\frac{\partial \bar{w}}{\partial z} + \frac{2}{3}C_5\left(\frac{g}{\theta_0}\overline{w'\theta_v'} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{v'w'}\frac{\partial \bar{v}}{\partial z}\right) \tag{99}$$

$$\frac{\partial \overline{w'^3}}{\partial t} - (K_w + \nu_8)\nabla_z^2\overline{w'^3} + \frac{C_8}{\tau}\left(C_{8b}\, Skw^4 + 1\right)\overline{w'^3} + \frac{\partial}{\partial z}\left(a_3\overline{w'^2}^2\right) + \frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}^2}{\overline{w'^2}}\right)$$
$$= -\bar{w}\frac{\partial \overline{w'^3}}{\partial z} + (1-C_{11})\frac{3g}{\theta_0}\overline{w'^2\theta_v'} - 3(1-C_{11})\overline{w'^3}\frac{\partial \bar{w}}{\partial z} \tag{100}$$

6.1 $\overline{w'^2}$

Terms on the LHS of (99) are treated fully implicitly, except for the diffusion term which is treated with a Crank-Nicholson time step. Terms on the RHS explicitly:

$$\frac{\overline{w'^2}^{t+\Delta t}}{\Delta t} + \frac{\partial \overline{w'^3}^{t+\Delta t}}{\partial z} + \frac{C_1}{\tau}\overline{w'^2}^{t+\Delta t} - \frac{1}{2}\nu_1\nabla_z^2\,\overline{w'^2}^{t+\Delta t} = \frac{\overline{w'^2}^{t}}{\Delta t} + \frac{1}{2}\nu_1\nabla_z^2\,\overline{w'^2}^{t} + \overline{w'^2}\Big|_{\mathrm{expl}} \tag{101}$$

where

$$\overline{w'^2}\Big|_{\mathrm{expl}} = -\bar{w}\frac{\partial \overline{w'^2}^{t}}{\partial z} + (1-C_5)\frac{2g}{\theta_0}\overline{w'\theta_v'}^{t} - 2(1-C_5)\overline{w'^2}^{t}\frac{\partial \bar{w}}{\partial z} + \frac{2}{3}C_5\left(\frac{g}{\theta_0}\overline{w'\theta_v'} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{v'w'}\frac{\partial \bar{v}}{\partial z}\right)^{t} \tag{102}$$

The next step consists of writing the finite difference equivalent to (101):

$$\begin{aligned}
&\frac{\mathtt{wp2}^{\mathtt{new}}(\mathtt{k})}{\mathtt{dt}} + \frac{\mathtt{wp3}^{\mathtt{new}}(\mathtt{k+1}) - \mathtt{wp3}^{\mathtt{new}}(\mathtt{k})}{\mathtt{dzm(k)}} + \frac{\mathtt{C_1}}{\mathtt{taum(k)}}\mathtt{wp2}^{\mathtt{new}}(\mathtt{k}) \\
&-\frac{\nu_1}{\mathtt{2dzm(k)dzt(k)}}\mathtt{wp2}^{\mathtt{new}}(\mathtt{k-1}) \\
&+\frac{\nu_1}{\mathtt{2dzm(k)}}\left(\frac{\mathtt{1}}{\mathtt{dzt(k+1)}} + \frac{\mathtt{1}}{\mathtt{dzt(k)}}\right)\mathtt{wp2}^{\mathtt{new}}(\mathtt{k}) \\
&-\frac{\nu_1}{\mathtt{2dzm(k)dzt(k+1)}}\mathtt{wp2}^{\mathtt{new}}(\mathtt{k+1}) \\
=&\frac{\mathtt{wp2(k)}}{\mathtt{dt}} \\
&+\frac{\nu_1}{\mathtt{2dzm(k)dzt(k)}}\mathtt{wp2(k-1)} \\
&-\frac{\nu_1}{\mathtt{2dzm(k)}}\left(\frac{\mathtt{1}}{\mathtt{dzt(k+1)}} + \frac{\mathtt{1}}{\mathtt{dzt(k)}}\right)\mathtt{wp2(k)} \\
&+\frac{\nu_1}{\mathtt{2dzm(k)dzt(k+1)}}\mathtt{wp2(k+1)} \\
&+\mathtt{wp2t(k)}
\end{aligned} \tag{103}$$

where `wp2t(k)` is the finite difference equivalent to (102) at level `zm(k)`.

6.1.1 Using an anisotropic solution for the horizontal wind

As an alternative to assuming $\bar{e} = \frac{3}{2}\overline{w'^2}$, we can calculate $\overline{v'^2}$ and $\overline{u'^2}$ and then compute $\bar{e}$ accordingly. The term with a C_4 coefficient in $\overline{w'^2}$ equation is then non-zero and must be accounted for. Starting with the 5$^{\text{th}}$ term of the original $\overline{w'^2}$ equation:

$$\frac{\partial \overline{w'^2}}{\partial t} = \cdots - \frac{C_4}{\tau}\left(\overline{w'^2} - \frac{2}{3}\bar{e}\right)\cdots \tag{104}$$

From which we obtain the finite difference equivalent:

$$\begin{aligned}
&-\frac{C_4}{\tau}\left(\overline{w'^2} - \frac{2}{3}\bar{e}\right)\Bigg|_{\mathtt{zm(k)}} \\
&= -\frac{\mathtt{C_4}}{\mathtt{taum(k)}}\left(\mathtt{wp2(k)} - \frac{2}{3}\mathtt{em(k)}\right) \\
&= -\frac{\mathtt{C_4}}{\mathtt{taum(k)}}\left(\mathtt{wp2(k)} - \frac{\mathtt{wp2(k)} + \mathtt{up2(k)} + \mathtt{vp2(k)}}{3}\right) \\
&= -\frac{2\,\mathtt{C_4}\,\mathtt{wp2(k)}}{\mathtt{3taum(k)}} + \frac{\mathtt{C_4}\,(\mathtt{up2(k)} + \mathtt{vp2(k)})}{\mathtt{3taum(k)}}
\end{aligned} \tag{105}$$

Separating out the contributions:

$$\begin{aligned}
\mathtt{lhs(3,k_wp2)} &= \mathtt{lhs(3,k_wp2)} + \frac{\mathtt{C_4}\,(\mathtt{up2(k)} + \mathtt{vp2(k)})}{\mathtt{3taum(k)}} \\
\mathtt{rhs(k_wp2)} &= \mathtt{rhs(k_wp2)} + \frac{2\,\mathtt{C_4}\,\mathtt{wp2(k)}}{\mathtt{3taum(k)}}
\end{aligned} \tag{106}$$

6.2 $\overline{w'^3}$

The first two terms on the LHS of (100) are treated implicitly, the last three terms on the LHS are teated semi-implicitly (they are linearized and the linearized portion is treated implicitly, the rest explicitly) and terms on the RHS explicitly. Let's focus first on the third term on the LHS, L_3:

$$L_3 \equiv \frac{C_8}{\tau}\left(C_{8b}\,Skw^4 + 1\right)\overline{w'^3} = \frac{C_8}{\tau}\left(C_{8b}\,\frac{\overline{w'^3}^5}{\overline{w'^2}^6} + \overline{w'^3}\right) \tag{107}$$

We linearize L_3 with respect to $\overline{w'^3}$:

$$L_3\left(\overline{w'^3}^{t+\Delta t}\right) \approx L_3\left(\overline{w'^3}^{t}\right) + \frac{\partial L_3}{\partial \overline{w'^3}}\bigg|_t\left(\overline{w'^3}^{t+\Delta t} - \overline{w'^3}^{t}\right) \tag{108}$$

where

$$\left.\frac{\partial L_3}{\partial \overline{w'^3}}\right|_t = \frac{C_8}{\tau}\left(5\,C_{8b}\,\frac{\overline{w'^3}^4}{\overline{w'^2}^6}+1\right) \tag{109}$$

Combining (107), (109) with (108):

$$\begin{aligned}
&L_3\left(\overline{w'^3}^{t+\Delta t}\right)\\
&=\frac{C_8}{\tau}\left(C_{8b}\,\frac{\overline{w'^3}^{t5}}{\overline{w'^2}^{t6}}+\overline{w'^3}^t\right)+\frac{C_8}{\tau}\left(5\,C_{8b}\,\frac{\overline{w'^3}^{t4}}{\overline{w'^2}^{t6}}+1\right)\left(\overline{w'^3}^{t+\Delta t}-\overline{w'^3}^t\right)\\
&=-\frac{C_8}{\tau}\left(4\,C_{8b}\,Skw^{t4}\right)\overline{w'^3}^t+\frac{C_8}{\tau}\left(5\,C_{8b}\,Skw^{t4}+1\right)\overline{w'^3}^{t+\Delta t}
\end{aligned} \tag{110}$$

In a similar fashion, let's now linearize the fourth term on the LHS of (100):

$$\begin{aligned}
&\frac{\partial}{\partial z}\left[a_3\left(\overline{w'^2}^{t+\Delta t}\right)^2\right]\\
&\approx\frac{\partial}{\partial z}\left[a_3\overline{w'^2}^{t2}+2a_3\overline{w'^2}^t\left(\overline{w'^2}^{t+\Delta t}-\overline{w'^2}^t\right)\right]\\
&=\frac{\partial}{\partial z}\left(2a_3\overline{w'^2}^t\,\overline{w'^2}^{t+\Delta t}\right)-\frac{\partial}{\partial z}\left(a_3\overline{w'^2}^{t2}\right)
\end{aligned} \tag{111}$$

We repeat for the fifth term on the LHS of (100):

$$\begin{aligned}
&\frac{\partial}{\partial z}\left(a_1\frac{\left(\overline{w'^3}^{t+\Delta t}\right)^2}{\overline{w'^2}^t}\right)\\
&\approx\frac{\partial}{\partial z}\left[a_1\frac{\left(\overline{w'^3}^t\right)^2}{\overline{w'^2}^t}+2a_1\frac{\overline{w'^3}^t}{\overline{w'^2}^t}\left(\overline{w'^3}^{t+\Delta t}-\overline{w'^3}^t\right)\right]\\
&=\frac{\partial}{\partial z}\left(2a_1\frac{\overline{w'^3}^t\,\overline{w'^3}^{t+\Delta t}}{\overline{w'^2}^t}\right)-\frac{\partial}{\partial z}\left(a_1\frac{\left(\overline{w'^3}^t\right)^2}{\overline{w'^2}^t}\right)
\end{aligned} \tag{112}$$

We can now assemble the time discrete equivalent to (100) using (110), (111) and (112):

$$\begin{aligned}
&\frac{\overline{w'^3}^{t+\Delta t}}{\Delta t}-\frac{1}{2}(K_w+\nu_8)\nabla_z^2\,\overline{w'^3}^{t+\Delta t}+\frac{C_8}{\tau}\left(5\,C_{8b}\,Skw^{t4}+1\right)\overline{w'^3}^{t+\Delta t}\\
&+\frac{\partial}{\partial z}\left(2a_3\overline{w'^2}^t\,\overline{w'^2}^{t+\Delta t}\right)+\frac{\partial}{\partial z}\left(2a_1\frac{\overline{w'^3}^t\,\overline{w'^3}^{t+\Delta t}}{\overline{w'^2}^t}\right)\\
&=\frac{\overline{w'^3}^t}{\Delta t}+\frac{1}{2}(K_w+\nu_8)\nabla_z^2\,\overline{w'^3}^t+\frac{C_8}{\tau}\left(4\,C_{8b}\,Skw^{t4}\right)\overline{w'^3}^t\\
&\quad+\frac{\partial}{\partial z}\left(a_3\overline{w'^2}^{t2}\right)+\frac{\partial}{\partial z}\left(a_1\frac{\left(\overline{w'^3}^t\right)^2}{\overline{w'^2}^t}\right)+\left.\overline{w'^3}\right|_{\mathrm{expl}}
\end{aligned} \tag{113}$$

where

$$\left.\overline{w'^3}\right|_{\text{expl}} = -\bar{w}\frac{\partial \overline{w'^3}^t}{\partial z} + (1 - C_{11})\frac{3g}{\theta_0}\overline{w'^2\theta'_v}^t - 3(1 - C_{11})\overline{w'^3}^t\frac{\partial \bar{w}}{\partial z} \tag{114}$$

Finally, we derive the finite difference form of (113):

$$\begin{aligned}
&\frac{\mathtt{wp3}^{\mathtt{new}}(\mathtt{k})}{\mathtt{dt}} - \frac{\mathtt{Kwt}(\mathtt{k}) + \nu_8}{2\mathtt{dzt}(\mathtt{k})}\left(\frac{\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}+1) - \mathtt{wp3}^{\mathtt{new}}(\mathtt{k})}{\mathtt{dzm}(\mathtt{k})} - \frac{\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}) - \mathtt{wp3}^{\mathtt{new}}(\mathtt{k}-1)}{\mathtt{dzm}(\mathtt{k}-1)}\right) \\
&+\frac{\mathtt{C_8}}{\mathtt{taut}(\mathtt{k})}\left(5\,\mathtt{C_{8b}}\,\mathtt{Skwt}(\mathtt{k})^4 + 1\right)\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}) \\
&+\frac{2}{\mathtt{dzt}(\mathtt{k})}\left(\mathtt{a3m}(\mathtt{k})\mathtt{wp2}(\mathtt{k})\mathtt{wp2}^{\mathtt{new}}(\mathtt{k}) - \mathtt{a3m}(\mathtt{k}-1)\mathtt{wp2}(\mathtt{k}-1)\mathtt{wp2}^{\mathtt{new}}(\mathtt{k}-1)\right) \\
&+\frac{1}{2\,\mathtt{dzt}(\mathtt{k})}\left(\frac{\mathtt{a1m}(\mathtt{k})\left(\mathtt{wp3}(\mathtt{k}) + \mathtt{wp3}(\mathtt{k}+1)\right)\left(\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}) + \mathtt{wp3}^{\mathtt{new}}(\mathtt{k}+1)\right)}{\max\left(\mathtt{wp2}(\mathtt{k}), \epsilon\right)}\right. \\
&\qquad\qquad\left. - \frac{\mathtt{a1m}(\mathtt{k}-1)\left(\mathtt{wp3}(\mathtt{k}-1) + \mathtt{wp3}(\mathtt{k})\right)\left(\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}-1) + \mathtt{wp3}^{\mathtt{new}}(\mathtt{k})\right)}{\max\left(\mathtt{wp2}(\mathtt{k}-1), \epsilon\right)}\right) \\
=&\frac{\mathtt{wp3}(\mathtt{k})}{\mathtt{dt}} + \frac{\mathtt{Kwt}(\mathtt{k}) + \nu_8}{2\mathtt{dzt}(\mathtt{k})}\left(\frac{\mathtt{wp3}(\mathtt{k}+1) - \mathtt{wp3}(\mathtt{k})}{\mathtt{dzm}(\mathtt{k})} - \frac{\mathtt{wp3}(\mathtt{k}) - \mathtt{wp3}(\mathtt{k}-1)}{\mathtt{dzm}(\mathtt{k}-1)}\right) \\
&+\frac{\mathtt{C_8}}{\mathtt{taut}(\mathtt{k})}\left(4\,\mathtt{C_{8b}}\,\mathtt{Skwt}(\mathtt{k})^4\right)\mathtt{wp3}(\mathtt{k}) \\
&+\frac{\mathtt{a3m}(\mathtt{k})\mathtt{wp2}(\mathtt{k})^2 - \mathtt{a3m}(\mathtt{k}-1)\mathtt{wp2}(\mathtt{k}-1)^2}{\mathtt{dzt}(\mathtt{k})} \\
&+\frac{1}{4\,\mathtt{dzt}(\mathtt{k})}\left(\frac{\mathtt{a1m}(\mathtt{k})\left(\mathtt{wp3}(\mathtt{k}) + \mathtt{wp3}(\mathtt{k}+1)\right)^2}{\max\left(\mathtt{wp2}(\mathtt{k}), \epsilon\right)} - \frac{\mathtt{a1m}(\mathtt{k}-1)\left(\mathtt{wp3}(\mathtt{k}-1) + \mathtt{wp3}(\mathtt{k})\right)^2}{\max\left(\mathtt{wp2}(\mathtt{k}-1), \epsilon\right)}\right) \\
&+\mathtt{wp3t}(\mathtt{k})
\end{aligned} \tag{115}$$

where $\mathtt{wp3t}(\mathtt{k})$ is the finite difference equivalent to (114) at level $\mathtt{zt}(\mathtt{k})$.

6.3 Matrix form

The final step is to rewrite (103) and (115) in matrix form:

$$
\underbrace{\begin{pmatrix}
\cdots & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}{+}1) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}{+}1) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}{+}2) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}{+}2) & \cdots \\
\cdots & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}{-}1) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}{+}1) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}{+}1) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}{+}2) & \cdots \\
\cdots & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}{-}1) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}{-}1) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}{+}1) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}{+}1) & \cdots \\
\cdots & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}{-}2) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}{-}1) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}{-}1) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}{+}1) & \cdots \\
\cdots & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}{-}2) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}{-}2) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}{-}1) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}{-}1) & \mathtt{wp3}^{\mathtt{impl}}(\mathtt{k}) & \mathtt{wp2}^{\mathtt{impl}}(\mathtt{k}) & \cdots
\end{pmatrix}}_{\mathtt{LHS}_{\mathtt{wp23}}\text{(Stored in compact format)}}
\begin{pmatrix}
\vdots \\
\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}-1) \\
\mathtt{wp2}^{\mathtt{new}}(\mathtt{k}-1) \\
\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}) \\
\mathtt{wp2}^{\mathtt{new}}(\mathtt{k}) \\
\mathtt{wp3}^{\mathtt{new}}(\mathtt{k}+1) \\
\mathtt{wp2}^{\mathtt{new}}(\mathtt{k}+1) \\
\vdots
\end{pmatrix}
$$

$$
= \underbrace{\begin{pmatrix}
\vdots \\
\mathtt{wp3}^{\mathtt{expl}}(\mathtt{k}{-}1) \\
\mathtt{wp2}^{\mathtt{expl}}(\mathtt{k}{-}1) \\
\mathtt{wp3}^{\mathtt{expl}}(\mathtt{k}) \\
\mathtt{wp2}^{\mathtt{expl}}(\mathtt{k}) \\
\mathtt{wp3}^{\mathtt{expl}}(\mathtt{k}{+}1) \\
\mathtt{wp2}^{\mathtt{expl}}(\mathtt{k}{+}1) \\
\vdots
\end{pmatrix}}_{\mathtt{RHS}_{\mathtt{wp23}}}
\tag{116}
$$

$lhs_{\mathtt{wp23}}$ is a band-diagonal matrix with two rows above and two below the main diagonal. $lhs_{\mathtt{wp23}}$ is stored in compact form in a array with dimensions $(5, 2\,\mathtt{nnzp})$. $rhs_{\mathtt{wp23}}$ is a vector with dimension $(2\,\mathtt{nnzp})$. $lhs_{\mathtt{wp23}}$ can be inverted efficiently using a LU decomposition algorithm for band diagonal matrices.

Contributions to $lhs_{\mathtt{wp23}}$ from (103):

$$\mathtt{lhs}(\mathtt{k\_wp2}, 5) = \mathtt{lhs}(\mathtt{k\_wp2}, 5) - \frac{\nu_1}{2\mathtt{dzm}(\mathtt{k})\mathtt{dzt}(\mathtt{k})} \tag{117}$$

$$\mathtt{lhs}(\mathtt{k\_wp2}, 4) = \mathtt{lhs}(\mathtt{k\_wp2}, 4) - \frac{1}{\mathtt{dzm}(\mathtt{k})} \tag{118}$$

$$\mathtt{lhs}(\mathtt{k\_wp2}, 3) = \mathtt{lhs}(\mathtt{k\_wp2}, 3) + \frac{1}{\mathtt{dt}} + \frac{\mathtt{C}_1}{\mathtt{taum}(\mathtt{k})} + \frac{\nu_1}{2\mathtt{dzm}(\mathtt{k})} \left(\frac{1}{\mathtt{dzt}(\mathtt{k}+1)} + \frac{1}{\mathtt{dzt}(\mathtt{k})} \right) \tag{119}$$

$$\mathtt{lhs}(\mathtt{k\_wp2}, 2) = \mathtt{lhs}(\mathtt{k\_wp2}, 2) + \frac{1}{\mathtt{dzm}(\mathtt{k})} \tag{120}$$

$$\texttt{lhs(k\_wp2,1)} = \texttt{lhs(k\_wp2,1)} - \frac{\nu_1}{\texttt{2dzm(k)dzt(k+1)}} \tag{121}$$

Contributions to $rhs_{\texttt{wp23}}$ from (103):

$$\begin{aligned}
&\texttt{rhs(k_wp2)} = \texttt{rhs(k_wp2)} \\
&+\frac{\texttt{wp2(k)}}{\texttt{dt}} \\
&+\frac{\nu_1}{\texttt{2dzm(k)dzt(k)}}\texttt{wp2(k-1)} \\
&-\frac{\nu_1}{\texttt{2dzm(k)}}\left(\frac{1}{\texttt{dzt(k+1)}} + \frac{1}{\texttt{dzt(k)}}\right)\texttt{wp2(k)} \\
&+\frac{\nu_1}{\texttt{2dzm(k)dzt(k+1)}}\texttt{wp2(k+1)} \\
&+\texttt{wp2t(k)}
\end{aligned} \tag{122}$$

where

$$\texttt{k\_wp2} = \texttt{2k} \tag{123}$$

Contributions to $lhs_{\texttt{wp23}}$ from (115):

$$\begin{aligned}
&\texttt{lhs(k_wp3,5)} = \texttt{lhs(k_wp3,5)} \\
&-\frac{\texttt{Kwt(k)} + \nu_8}{\texttt{2dzt(k)dzm(k-1)}} - \frac{1}{\texttt{2 dzt(k)}}\frac{\texttt{a1m(k-1)}\left(\texttt{wp3(k-1)} + \texttt{wp3(k)}\right)}{\max\left(\texttt{wp2(k-1)}, \epsilon\right)}
\end{aligned} \tag{124}$$

$$\texttt{lhs(k\_wp3,4)} = \texttt{lhs(k\_wp3,4)} - \frac{\texttt{2a3m(k-1)wp2(k-1)}}{\texttt{dzt(k)}} \tag{125}$$

$$\begin{aligned}
&\texttt{lhs(k_wp3,3)} = \texttt{lhs(k_wp3,3)} \\
&+\frac{1}{\texttt{dt}} + \frac{\texttt{C}_8}{\texttt{taut(k)}}\left(5\,\texttt{C}_{\texttt{8b}}\,\texttt{Skwt(k)}^4 + 1\right) + \frac{\texttt{Kwt(k)} + \nu_8}{\texttt{2dzt(k)}}\left(\frac{1}{\texttt{dzm(k-1)}} + \frac{1}{\texttt{dzm(k)}}\right) \\
&+\frac{1}{\texttt{2 dzt(k)}}\left(\frac{\texttt{a1m(k)}\left(\texttt{wp3(k)} + \texttt{wp3(k+1)}\right)}{\max\left(\texttt{wp2(k)}, \epsilon\right)} - \frac{\texttt{a1m(k-1)}\left(\texttt{wp3(k-1)} + \texttt{wp3(k)}\right)}{\max\left(\texttt{wp2(k-1)}, \epsilon\right)}\right)
\end{aligned} \tag{126}$$

$$\texttt{lhs(k\_wp3,2)} = \texttt{lhs(k\_wp3,2)} + \frac{\texttt{2a3m(k)wp2(k)}}{\texttt{dzt(k)}} \tag{127}$$

$$\begin{aligned}
&\texttt{lhs(k_wp3,1)} = \texttt{lhs(k_wp3,1)} \\
&-\frac{\texttt{Kwt(k)} + \nu_8}{\texttt{2dzt(k)dzm(k)}} + \frac{1}{\texttt{2 dzt(k)}}\frac{\texttt{a1m(k)}\left(\texttt{wp3(k)} + \texttt{wp3(k+1)}\right)}{\max\left(\texttt{wp2(k)}, \epsilon\right)}
\end{aligned} \tag{128}$$

Contributions to $rhs_{\mathtt{wp23}}$ from (115):

$$
\begin{aligned}
&\mathtt{rhs(k\_wp3)} = \mathtt{rhs(k\_wp3)} \\
&+\frac{\mathtt{wp3(k)}}{\mathtt{dt}} + \frac{\mathtt{Kwt(k)} + \nu_8}{\mathtt{2dzt(k)}} \left(\frac{\mathtt{wp3(k+1)} - \mathtt{wp3(k)}}{\mathtt{dzm(k)}} - \frac{\mathtt{wp3(k)} - \mathtt{wp3(k-1)}}{\mathtt{dzm(k-1)}} \right) \\
&+\frac{\mathtt{C_8}}{\mathtt{taut(k)}} \left(4\, \mathtt{C_{8b}}\, \mathtt{Skwt(k)}^4 \right) \mathtt{wp3(k)} \\
&+\frac{\mathtt{a3m(k)wp2(k)}^2 - \mathtt{a3m(k-1)wp2(k-1)}^2}{\mathtt{dzt(k)}} \\
&+\frac{1}{4\,\mathtt{dzt(k)}} \left(\frac{\mathtt{a1m(k)}\left(\mathtt{wp3(k)} + \mathtt{wp3(k+1)}\right)^2}{\max\left(\mathtt{wp2(k)}, \epsilon\right)} - \frac{\mathtt{a1m(k-1)}\left(\mathtt{wp3(k-1)} + \mathtt{wp3(k)}\right)^2}{\max\left(\mathtt{wp2(k-1)}, \epsilon\right)} \right) \\
&+\mathtt{wp3t(k)}
\end{aligned}
\tag{129}
$$

where

$$
\mathtt{k\_wp3} = \mathtt{2k-1} \tag{130}
$$

7 High-order Solution to the Horizontal Wind

As an alternative to assuming $\bar{e} = \frac{3}{2}\overline{w'^2}$, we can obtain an anisotropic solution using a semi-implicit discretization for $\overline{u'^2}$ and $\overline{v'^2}$. Similarly to (6), start with equations (14) and (15), substitute (33) and (34) respectively.

7.1 $\overline{u'^2}$

Assume a steady-state and rearrange $\overline{u'^2}$ for a semi-implicit solution to obtain:

$$
\begin{aligned}
&\underbrace{\frac{C_4}{\tau}\left(\overline{u'^2}-\frac{2}{3}\bar{e}\right)}_{dp1}\underbrace{+\frac{2}{3}\left(C_{14}\frac{\bar{e}}{\tau}\right)}_{pr1}\underbrace{+\bar{w}\frac{\partial\overline{u'^2}}{\partial z}}_{ma}\underbrace{+\frac{1}{3}\beta\frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}}{\overline{\overline{w'^2}}}\,\overline{u'^2}\right)}_{ta}\underbrace{-\nu_9\nabla_z^2\overline{u'^2}}_{dp2}\\
=&\underbrace{-\left(1-\frac{1}{3}\beta\right)\frac{\partial}{\partial z}\left(a_2\frac{\overline{w'^3}}{\overline{w'^2}^2}\,\overline{w'u'}^2\right)}_{ta}\underbrace{-2(1-C_5)\,\overline{w'u'}\,\frac{\partial\bar{u}}{\partial z}}_{tp}\\
&\underbrace{+\frac{2}{3}C_5\left(\frac{g}{\theta_0}\overline{w'\theta_v'}-\overline{u'w'}\frac{\partial\bar{u}}{\partial z}-\overline{v'w'}\frac{\partial\bar{v}}{\partial z}\right)}_{pr2}
\end{aligned}
\tag{131}
$$

As in the case of $\overline{q_t'^2}$ and $\overline{\theta_l'^2}$, the horizontal wind variance terms are solved using a tridiagonal matrix.

7.1.1 Terms 1 and 2

$$
\begin{aligned}
&\frac{C_4}{\tau}\left(\overline{u'^2}-\frac{2}{3}\bar{e}\right)+\frac{2}{3}C_{14}\frac{\bar{e}}{\tau}\bigg|_{\mathtt{zm(k)}}\\
=&\frac{\mathtt{C_4}}{\mathtt{taum(k)}}\mathtt{up2(k)}-\frac{\mathtt{C_4}}{\mathtt{taum(k)}}\frac{2}{3}\mathtt{em(k)}+\frac{2}{3}\mathtt{C_{14}}\frac{\mathtt{em(k)}}{\mathtt{taum(k)}}\\
=&\frac{\mathtt{C_4}}{\mathtt{taum(k)}}\mathtt{up2(k)}-\frac{2}{3}\mathtt{em(k)}\left(\frac{\mathtt{C_4}}{\mathtt{taum(k)}}-\frac{\mathtt{C_{14}}}{\mathtt{taum(k)}}\right)\\
=&\frac{\mathtt{C_4}}{\mathtt{taum(k)}}\mathtt{up2(k)}-\frac{2}{3}\left[\frac{\mathtt{up2(k)}+\mathtt{vp2(k)}+\mathtt{wp2(k)}}{2}\right]\left(\frac{\mathtt{C_4}}{\mathtt{taum(k)}}-\frac{\mathtt{C_{14}}}{\mathtt{taum(k)}}\right)\\
=&\mathtt{up2(k)}\frac{1}{3}\left(\frac{\mathtt{2C_4}+\mathtt{C_{14}}}{\mathtt{taum(k)}}\right)-\left(\frac{1}{3}\left(\mathtt{C_4}-\mathtt{C_{14}}\right)\left(\frac{\mathtt{vp2(k)}+\mathtt{wp2(k)}}{\mathtt{taum(k)}}\right)\right)
\end{aligned}
\tag{132}
$$

Separating out the contributions:

$$
\begin{aligned}
&\mathtt{lhs(2,k)}=\mathtt{lhs(2,k)}+\frac{\mathtt{2C_4}+\mathtt{C_{14}}}{\mathtt{3taum(k)}}\\
&\mathtt{rhs(k)}=\mathtt{rhs(k)}+\frac{1}{3}\left(\mathtt{C_4}-\mathtt{C_{14}}\right)\left(\frac{\mathtt{vp2(k)}+\mathtt{wp2(k)}}{\mathtt{taum(k)}}\right)
\end{aligned}
\tag{133}
$$

7.1.2 Term 3

$$
\begin{aligned}
&\bar{w}\frac{\partial\overline{u'^2}}{\partial z}\bigg|_{\mathtt{zm(k)}}\\
=&\frac{\mathtt{wmm(k)}}{\mathtt{dzm(k)}}\left(\frac{1}{2}\left(\mathtt{up2(k)}+\mathtt{up2(k+1)}\right)-\frac{1}{2}\left(\mathtt{up2(k-1)}+\mathtt{up2(k)}\right)\right)\\
=&\frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}}\mathtt{up2(k+1)}-\frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}}\mathtt{up2(k-1)}
\end{aligned}
\tag{134}
$$

Separating out the contributions:

$$
\begin{aligned}
\mathtt{lhs(1,k)} &= \mathtt{lhs(1,k)} + \frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}} \\
\mathtt{lhs(3,k)} &= \mathtt{lhs(3,k)} - \frac{\mathtt{wmm(k)}}{\mathtt{2dzm(k)}}
\end{aligned} \tag{135}
$$

7.1.3 Term 4

$$
\begin{aligned}
&\frac{1}{3}\beta\frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}}{\overline{w'^2}}\overline{u'^2}\right)\Bigg|_{\mathtt{zm(k)}} \\
&= \frac{\beta}{\mathtt{6dzm(k)}}\Bigg[\frac{\left(\mathtt{a1m(k)}+\mathtt{a1m(k+1)}\right)\mathtt{wp3(k+1)}\left(\mathtt{up2(k)}+\mathtt{up2(k+1)}\right)}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)} \\
&\qquad\qquad - \frac{\left(\mathtt{a1m(k-1)}+\mathtt{a1m(k)}\right)\mathtt{wp3(k)}\left(\mathtt{up2(k-1)}+\mathtt{up2(k)}\right)}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)}\Bigg]
\end{aligned} \tag{136}
$$

Separating out the contributions:

$$
\begin{aligned}
\mathtt{lhs(3,k)} &= \mathtt{lhs(3,k)} - \frac{\beta}{\mathtt{6dzm(k)}}\frac{\left(\mathtt{a1m(k-1)}+\mathtt{a1m(k)}\right)\mathtt{wp3(k)}}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)} \\
\mathtt{lhs(2,k)} &= \mathtt{lhs(2,k)} + \frac{\beta}{\mathtt{6dzm(k)}}\left(\frac{\left(\mathtt{a1m(k)}+\mathtt{a1m(k+1)}\right)\mathtt{wp3(k+1)}}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)} - \frac{\left(\mathtt{a1m(k-1)}+\mathtt{a1m(k)}\right)\mathtt{wp3(k)}}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)}\right) \\
\mathtt{lhs(1,k)} &= \mathtt{lhs(1,k)} + \frac{\beta}{\mathtt{6dzm(k)}}\frac{\left(\mathtt{a1m(k)}+\mathtt{a1m(k+1)}\right)\mathtt{wp3(k+1)}}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)}
\end{aligned} \tag{137}
$$

7.1.4 Term 5

$$
-\nu_9\nabla_z^2\overline{u'^2}\Big|_{\mathtt{zm(k)}} = \frac{\nu_9}{\mathtt{dzm(k)}}\left(\frac{\mathtt{up2(k+1)}-\mathtt{up2(k)}}{\mathtt{dzt(k+1)}} - \frac{\mathtt{up2(k)}-\mathtt{up2(k-1)}}{\mathtt{dzt(k)}}\right) \tag{138}
$$

Separating out the contributions:

$$
\begin{aligned}
\mathtt{lhs(3,k)} &= \mathtt{lhs(3,k)} - \frac{\nu_9}{\mathtt{dzm(k)dzt(k)}} \\
\mathtt{lhs(2,k)} &= \mathtt{lhs(2,k)} + \frac{\nu_9}{\mathtt{dzm(k)}}\left(\frac{1}{\mathtt{dzt(k+1)}}+\frac{1}{\mathtt{dzt(k)}}\right) \\
\mathtt{lhs(1,k)} &= \mathtt{lhs(1,k)} - \frac{\nu_9}{\mathtt{dzm(k)dzt(k+1)}}
\end{aligned} \tag{139}
$$

7.1.5 Term 6

$$
\begin{aligned}
& -\left(1-\frac{1}{3}\beta\right)\frac{\partial}{\partial z}\left(a_2\frac{\overline{w'^3}}{\overline{w'^2}^2}\,\overline{w'u'}^2\right)\Bigg|_{\mathtt{zm(k)}} \\
=& -\frac{1-\frac{1}{3}\beta}{4\mathtt{dzm(k)}}\left[\frac{\left(\mathtt{a1m(k)}+\mathtt{a1m(k+1)}\right)^2\mathtt{wp3(k+1)}\left(\mathtt{upwp(k)}+\mathtt{upwp(k+1)}\right)^2}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)^2}\right. \\
& \left.-\frac{\left(\mathtt{a1m(k-1)}+\mathtt{a1m(k)}\right)^2\mathtt{wp3(k)}\left(\mathtt{upwp(k-1)}+\mathtt{upwp(k)}\right)^2}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)^2}\right]
\end{aligned}
\tag{140}
$$

Separating out the contributions:

$$
\begin{aligned}
& \mathtt{rhs(k)} \\
=& \mathtt{rhs(k)}-\frac{1-\frac{1}{3}\beta}{4\mathtt{dzm(k)}}\left[\frac{\left(\mathtt{a1m(k)}+\mathtt{a1m(k+1)}\right)^2\mathtt{wp3(k+1)}\left(\mathtt{upwp(k)}+\mathtt{upwp(k+1)}\right)^2}{\max\left(\mathtt{wp2(k)}+\mathtt{wp2(k+1)},2\epsilon\right)^2}\right. \\
& \left.-\frac{\left(\mathtt{a1m(k-1)}+\mathtt{a1m(k)}\right)^2\mathtt{wp3(k)}\left(\mathtt{upwp(k-1)}+\mathtt{upwp(k)}\right)^2}{\max\left(\mathtt{wp2(k-1)}+\mathtt{wp2(k)},2\epsilon\right)^2}\right]
\end{aligned}
\tag{141}
$$

7.1.6 Term 7

$$
-2\,(1-C_5)\,\overline{w'u'}\,\frac{\partial\bar{u}}{\partial z}\bigg|_{\mathtt{zm(k)}} = -2\left(1-\mathtt{C}_5\right)\mathtt{upwp(k)}\frac{\mathtt{um(k+1)}-\mathtt{um(k)}}{\mathtt{dzm(k)}}
\tag{142}
$$

Separating out the contributions:

$$
\mathtt{rhs(k)} = \mathtt{rhs(k)} - 2\left(1-\mathtt{C}_5\right)\mathtt{upwp(k)}\frac{\mathtt{um(k+1)}-\mathtt{um(k)}}{\mathtt{dzm(k)}}
\tag{143}
$$

7.1.7 Term 8

$$
\begin{aligned}
& \frac{2}{3}C_5\left(\frac{g}{\theta_0}\overline{w'\theta'_v}-\overline{u'w'}\frac{\partial\bar{u}}{\partial z}-\overline{v'w'}\frac{\partial\bar{v}}{\partial z}\right)\big|_{\mathtt{zm(k)}} \\
=& \frac{2}{3}\mathtt{C}_5\left(\frac{\mathtt{grav}}{\mathtt{T0}}\mathtt{wpthvp(k)}-\mathtt{upwp(k)}\frac{\mathtt{um(k+1)}-\mathtt{um(k)}}{\mathtt{dzm(k)}}-\mathtt{vpwp(k)}\frac{\mathtt{vm(k+1)}-\mathtt{vm(k)}}{\mathtt{dzm(k)}}\right)
\end{aligned}
\tag{144}
$$

Separating out the contributions:

$$
\mathtt{rhs(k)} = \mathtt{rhs(k)} + \frac{2}{3}\mathtt{C}_5\left(\frac{\mathtt{grav}}{\mathtt{T0}}\mathtt{wpthvp(k)}-\mathtt{upwp(k)}\frac{\mathtt{um(k+1)}-\mathtt{um(k)}}{\mathtt{dzm(k)}}-\mathtt{vpwp(k)}\frac{\mathtt{vm(k+1)}-\mathtt{vm(k)}}{\mathtt{dzm(k)}}\right)
\tag{145}
$$

7.2 $\overline{v'^2}$

As in $u^{'2}$ assume a steady-state and rearrange $\overline{u'^2}$ for a semi-implicit solution to obtain:

$$
\begin{aligned}
&\frac{C_4}{\tau}\left(\overline{v'^2}-\frac{2}{3}\bar{e}\right)+\frac{2}{3}\left(C_{14}\frac{\bar{e}}{\tau}\right)+\bar{w}\frac{\partial\overline{v'^2}}{\partial z}+\frac{1}{3}\beta\frac{\partial}{\partial z}\left(a_1\frac{\overline{w'^3}}{\overline{w'^2}}\,\overline{v'^2}\right)-\nu_9\nabla_z^2\overline{v'^2}\\
&=-\left(1-\frac{1}{3}\beta\right)\frac{\partial}{\partial z}\left(a_2\frac{\overline{w'^3}}{\overline{w'^2}^2}\,\overline{w'v'}^2\right)-2(1-C_5)\,\overline{w'v'}\,\frac{\partial\bar{v}}{\partial z}\\
&\quad+\frac{2}{3}C_5\left(\frac{g}{\theta_0}\overline{w'\theta_v'}-\overline{u'w'}\frac{\partial\bar{u}}{\partial z}-\overline{v'w'}\frac{\partial\bar{v}}{\partial z}\right)
\end{aligned}
\tag{146}
$$

The discretization for $v^{'2}$ follows in the same way as $u^{'2}$.

8 Grid configuration

Figure 1 shows the vertical grid configuration for HOC. The grid consists of two types of levels: `zm` and `zt`. Predictive mean variables and third order moments reside on the thermodynamic levels (`zt`). Second and fourth order moments reside on the momentum levels (`zm`).

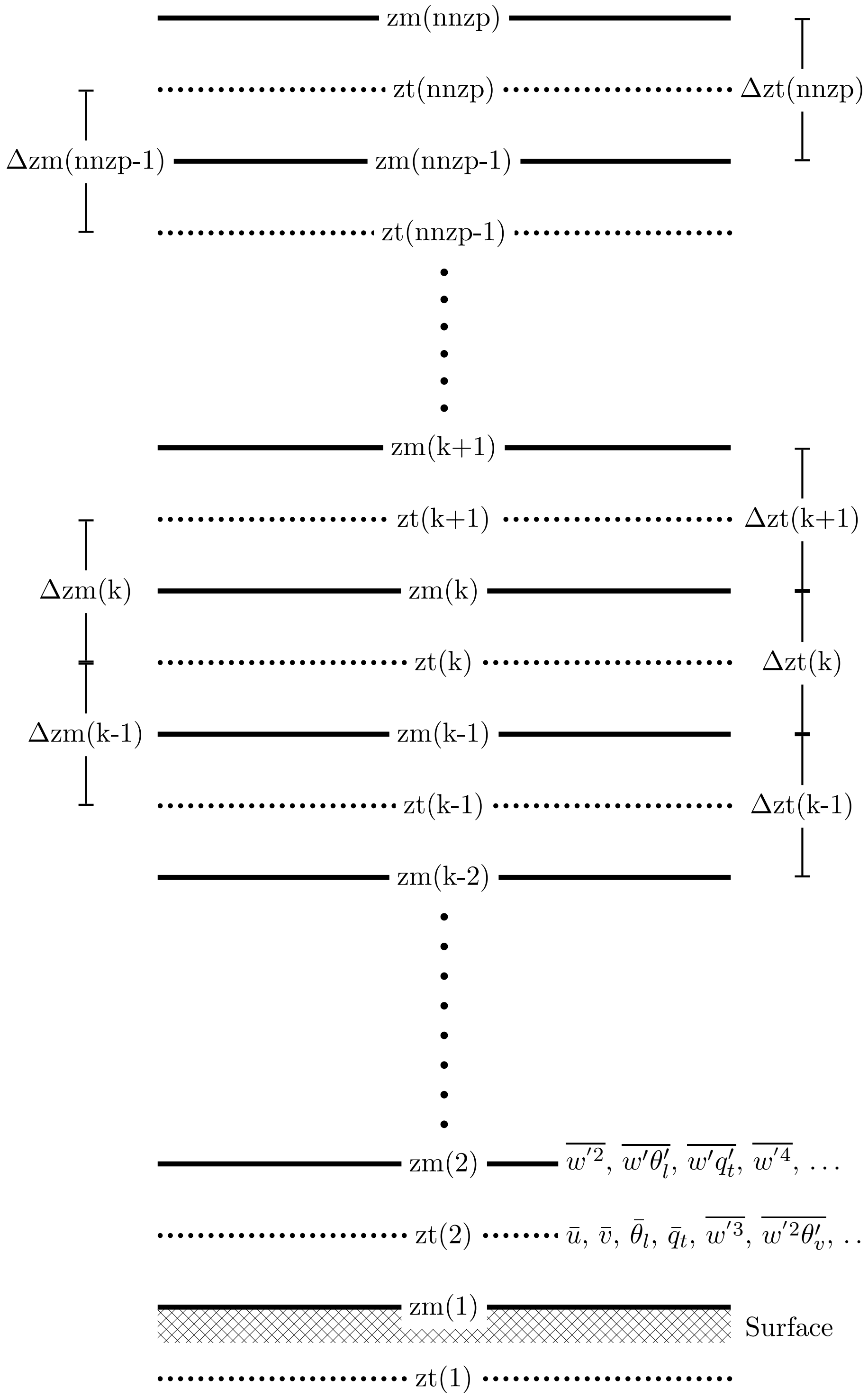

Figure 1: Vertical grid configuration

9 Predictive equations in Conjunction with Host Model

The HOC parameterization can be used in conjunction with a larger host model. Some of the main predictive equations (Eqns. 1, 2, 3, and 4) have to be divided in the following manner:

$$\frac{\partial \bar{u}}{\partial t} = -\bar{w}\frac{\partial \bar{u}}{\partial z} - f(v_g - \bar{v}) - \frac{\partial}{\partial z}\overline{u'w'} \tag{147}$$

$$\frac{\partial \bar{v}}{\partial t} = -\bar{w}\frac{\partial \bar{v}}{\partial z} + f(u_g - \bar{u}) - \frac{\partial}{\partial z}\overline{v'w'} \tag{148}$$

$$\frac{\partial \bar{q}_t}{\partial t} = -\bar{w}\frac{\partial \bar{q}_t}{\partial z} - \frac{\partial}{\partial z}\overline{w'q_t'} + \left.\frac{\partial \bar{q}_t}{\partial t}\right|_{\mathrm{ls}} \tag{149}$$

$$\frac{\partial \bar{\theta}_l}{\partial t} = -\bar{w}\frac{\partial \bar{\theta}_l}{\partial z} - \frac{\partial}{\partial z}\overline{w'\theta_l'} + \bar{R} + \left.\frac{\partial \bar{\theta}_l}{\partial t}\right|_{\mathrm{ls}} \tag{150}$$

The variables in red, $\bar{u}$, $\bar{v}$, $\bar{q}_t$, and $\bar{\theta}_l$, are supplied from the host model to the HOC parameterization. The variables in blue, $\overline{u'w'}$, $\overline{v'w'}$, $\overline{w'q_t'}$, and $\overline{w'\theta_l'}$, are computed in the HOC parameterization and then sent back to the host model. The vertical derivatives of these variables are then used to effect the time tendencies of their related variables. The terms listed in magenta, which include the vertical mean advection terms, the coriolis terms, the radiative heating term, and the large-scale moisture and temperature forcings, are terms that are usually calculated in HOC when HOC does not run with any host model. However, in cases where a host model is involved, the host model should calculate all of these terms. As before, $\bar{R}$ is the radiative heating rate, f the Coriolis parameter and u_g, v_g the geostrophic winds. $\left.\frac{\partial \bar{q}_t}{\partial t}\right|_{\mathrm{ls}}$ and $\left.\frac{\partial \bar{\theta}_l}{\partial t}\right|_{\mathrm{ls}}$ are large-scale moisture and temperature forcings.

The $\frac{\partial \overline{w'^2}}{\partial t}$ equation (eq. 5), the $\frac{\partial \overline{q_t'^2}}{\partial t}$ equation (eq. 6), the $\frac{\partial \overline{\theta_l'^2}}{\partial t}$ equation (eq. 7), the $\frac{\partial \overline{q_t'\theta_l'}}{\partial t}$ equation (eq. 8), the $\frac{\partial \overline{w'q_t'}}{\partial t}$ equation (eq. 9), the $\frac{\partial \overline{w'\theta_t'}}{\partial t}$ equation (eq. 10), and the $\frac{\partial \overline{w'^3}}{\partial t}$ equation (eq. 11) all remain unchanged. All of these variables are computed and used completely within the structure of the HOC parameterization.

Within the structure of a computer code, the HOC parameterization requires that the values of certain variables be saved at all grid points for use during the next timestep. Since the HOC parameterization is a one-dimensional parameterization (in the vertical), or a single-column parameterization, a three-dimensional host model must call the HOC parameterization once for every grid column that it has.

Therefore, the values of all these variables must be saved from timestep to timestep at every grid point in the three dimensions.

The variables that need to be saved as such are the following:

On the momentum (or full) levels		
Description	Variable	Variable name in HOC code
Turbulent Flux of θ_l	$\overline{w'\theta_l'}$	wpthlp
Turbulent Flux of q_t	$\overline{w'q_t'}$	wprtp
Variance of w	$\overline{w'^2}$	wp2
Variance of q_t	$\overline{q_t'^2}$	rtp2
Variance of θ_l	$\overline{\theta_l'^2}$	thlp2
Covariance of q_t and θ_l	$\overline{q_t'\theta_l'}$	rtpthlp
Covariance of u and w	$\overline{u'w'}$	upwp
Covariance of v and w	$\overline{v'w'}$	vpwp
Time scale	τ	taum
Width of the individual w plumes	$\tilde{\sigma}_w^2$	Scm
On the thermodynamic (or half) levels		
Description	Variable	Variable name in HOC code
Third-order Moment of w	$\overline{w'^3}$	wp3
Rain water mixing ratio	q_r	rrm
Rain drop concentration	N_r	Nrm
Cloud drop concentration	N_c	Ncm
Cloud water mixing ratio	q_c	rcm
Cloud fraction	cf	cf

It is only necessary to save cloud water mixing ratio from timestep to timestep if the host model would need information on the subgrid value of that variable. It is also necessary to provide HOC with information on cloud water mixing ratio for the initial timestep of the run. Cloud fraction is usually saved for output purposes only.