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Links of prime ideals and their Rees algebras

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Abstract

In a previous paper we exhibited the somewhat surprising property that most direct links of prime ideals in Gorenstein rings are equimultiple ideals with reduction number 1. This led to the construction of large families of Cohen–Macaulay Rees algebras. The first goal of this paper is to extend this result to arbitrary Cohen–Macaulay rings. The means of the proof are changed since one cannot depend so heavily on linkage theory. We then study the structure of the Rees algebra of these links, more specifically we describe their canonical module in sufficient detail to be able to characterize self–linked prime ideals. In the last section multiplicity estimates for classes of such ideals are established.

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1 Introduction

Let R be a Noetherian local ring and let I be one of its ideals. Two of the most important graded algebras associated to I that have been extensively studied in the past few years are the $Rees\ algebra$ of I

$$R[It] = \bigoplus_{i>0} I^i t^i,$$

and the associated graded ring

$$\operatorname{gr}_I(R) = \bigoplus_{i>0} I^i/I^{i+1}.$$

A recurrent theme is to find out conditions that make them Cohen–Macaulay rings. A tool that has proved its usefulness is the notion of the reduction of an ideal (see [11]). A reduction of I is an ideal $J \subset I$ for which there exists an integer n such that $I^{n+1} = JI^n$; the least such integer is called the reduction number of I with respect to J, in symbols $r_J(I)$. The reduction number of I is the minimum of $r_J(I)$ taken over all the minimal reductions of I. Phrased in other words, J is a reduction of I if and only if the morphism $R[Jt] \hookrightarrow R[It]$ is finite. It seems reasonable to expect to recover some of the properties of R[It] from those of R[Jt]. It seems even more reasonable to expect better results on R[It] when I has an amenable structure, such as being a complete intersection.

In the case in which the residue field of the ring is infinite, the minimal number of generators of a minimal reduction of I does not depend on the reduction; this number, usually denoted with $\ell(I)$, is called the *analytic spread* of I and is equal to the dimension of the fibre ring of I, i.e. $R[It] \otimes R/\mathfrak{m}$. The relation height $(I) \leq \ell(I) \leq \dim(R)$ is a natural source of several deviation measures. One of them was introduced by S. Huckaba and C. Huneke in [8], namely the *analytic deviation* of an ideal

$$ad(I) = \ell(I) - height(I)$$
.

ad(I) measures how far a minimal reduction of an ideal is from being a complete intersection.

The ideals having analytic deviation zero are called *equimultiple ideals*. New families of such ideals were introduced in [3] for Gorenstein rings by the process of linkage. For more details on linkage we refer the reader to [12] while, for the time being, it will be enough to know that two proper ideals I and J of height g in a Cohen–Macaulay ring R are said to be (*directly*) linked, in symbols $I \sim J$, if there exists a regular sequence $\mathbf{z} = (z_1, ..., z_g) \subset I \cap J$ such that $J = \mathbf{z}$: I and $I = \mathbf{z}$: I.

To be specific, if \mathfrak{p} is a prime ideal of height g and $\mathbf{z} = (z_1, \ldots, z_g) \subset \mathfrak{p}$ is a regular sequence, [3] focused on the direct link $I = \mathbf{z} : \mathfrak{p}$, and the following two conditions played significant roles there:

- (L₁) $R_{\mathfrak{p}}$ is not a regular local ring;
- (L₂) $R_{\mathfrak{p}}$ is a regular local ring of dimension at least 2 and two elements in the sequence \mathbf{z} lie in the symbolic square $\mathfrak{p}^{(2)}$.

Using these notations [3, Theorem 2.1, Theorem 3.1, Corollary 3.2]:

Theorem A Let R be a Cohen–Macaulay ring, \mathfrak{p} a prime ideal of codimension g, and let $\mathbf{z} = (z_1, \ldots, z_g) \subset \mathfrak{p}$ be a regular sequence. Set $J = (\mathbf{z})$ and $I = J : \mathfrak{p}$. Suppose that $R_{\mathfrak{p}}$ is a Gorenstein ring. Then I is an equimultiple ideal with reduction number one, more precisely,

$$I^2 = JI$$
,

if either condition L_1 or L_2 holds.

Theorem B Let R be a Gorenstein local ring, \mathfrak{p} a Cohen–Macaulay prime ideal of codimension g, $J = (z_1, ..., z_g)$ a complete intersection, and set $I = J : \mathfrak{p}$. If either condition L_1 or L_2 holds then the associated graded ring $\operatorname{gr}_I(R)$ is Cohen–Macaulay.

Corollary C If in Theorem B we have $\dim(R_{\mathfrak{p}}) \geq 2$ (which is automatic in the second case), then the Rees algebra of I is Cohen–Macaulay.

As a result, the process described above gives us a powerful tool for the production of Cohen–Macaulay Rees algebras.

It is now straightforward to describe our first main result, the extension of Theorem A from Gorenstein rings to general Cohen–Macaulay rings.

Theorem 2.3 Let (R, \mathfrak{m}) be a Cohen–Macaulay local ring and let \mathfrak{p} be a prime ideal of R of height g such that $R_{\mathfrak{p}}$ is not a regular local ring. If $J = (z_1, ..., z_g)$ denotes a regular sequence inside \mathfrak{p} , then the link $I = J : \mathfrak{p}$ is equimultiple with reduction number 1.

This will be proved in Section 2. The approach is fully ideal—theoretic, in contrast to [3], when we could appeal to basic properties of linkage theory. The fact that we are dealing with direct links of prime ideals still permits certain calculations to go through.

To benefit however from this close relationship between the link and its reduction to study the Rees algebra of a link I of a Cohen–Macaulay prime ideal $\mathfrak p$ still requires that R be a Gorenstein ring. We carry out in Section 3 the determination of the canonical module of R[It] in full detail. Self–linked prime ideals will correspond to the case in which the associated graded ring is Gorenstein.

In the final section, we establish two multiplicity formulae. The first says that the multiplicity of R[Jt] and R[It] are the same. Since there are explicit formulae for the multiplicities of some Rees algebras of complete intersection (see [6]), this is useful for

computational purposes. The second is an estimate relating the multiplicity of a Cohen–Macaulay self-linked ideal I with Cohen–Macaulay first Koszul homology (e.g. ideals in the linkage class of complete intersections) to its number of generators.

As a general reference for unexplained notations or basic results we will use three invaluable sources: [2], [10] and [18]—the first one especially for the sections on the canonical module and multiplicity.

Last but not least, it is a pleasure to thank our advisor, Professor Wolmer V. Vasconcelos, for fruitful conversations and the referee for his valuable suggestions regarding this paper.

2 Links of prime ideals in Cohen–Macaulay rings

In this section we describe under which conditions the results of [3] can be extended to the link of a prime ideal in an arbitrary Cohen–Macaulay ring. More precisely, Theorem 2.3 generalizes Theorem A to a Cohen–Macaulay ring; the only assumption is that $R_{\mathfrak{p}}$ cannot be a regular local ring. Actually, this is not a restriction because the case of a regular ring was thoroughly studied in [3].

We first prove the theorem in the case of links of the maximal ideal; we then show how the general case can be reduced to this particular one. For that, one needs to know that if I and J are linked ideals then $\operatorname{Ass}(R/I) \cup \operatorname{Ass}(R/J) = \operatorname{Ass}(R/\mathbf{z})$; in particular every associated prime of I and J has the same height.

An ideal-theoretic fact is needed in order to obtain the first result.

Proposition 2.1 Let A and B be two ideals of a local ring (R, \mathfrak{m}) and let $\mathbf{z} = z_1, ..., z_n$ be a regular sequence contained both in A and in B. If $AB \subset (\mathbf{z})$ but $AB \not\subset \mathfrak{m}(\mathbf{z})$ then A and B are both generated by regular sequences of length n.

Proof. Consider first the case n=1 and write $z=z_1$, for sake of simplicity. By assumption, there exist $a \in A$ and $b \in B$ such that $ab=\alpha z$ with $\alpha \notin \mathfrak{m}$. We may even assume z=ab. Pick now any x in A; since $xb \in AB \subset (z)$, xb=cz for some $c \in R$ and so

$$xz = x(ab) = a(xb) = a(cz) = (ac)z.$$

But z is a regular element, thus x = ac, i.e. $x \in (a)$. Since x is an arbitrary element of A we then conclude that A = (a); similarly, B = (b). Note that if the product of two elements is regular then they are both regular, and this establishes the first case.

Consider now the case n > 1. Since $AB \not\subset \mathfrak{m}(\mathbf{z})$, there exist $a \in A$ and $b \in B$ such that $ab = \alpha_1 z_1 + \cdots + \alpha_n z_n$ and at least one of the α_i does not belong to \mathfrak{m} ; by reordering the z_i 's we may assume $\alpha_n \not\in \mathfrak{m}$. Let "-" denote the homomorphic image modulo $(z_1, ..., z_{n-1})$: we

are then in the case of two ideals \overline{A} and \overline{B} of a local ring $(\overline{R}, \overline{\mathfrak{m}})$ both containing the regular element \overline{z}_n . In addition, $\overline{AB} \subset (\overline{z}_n)$ but $\overline{AB} \not\subset \mathfrak{m}(\overline{z}_n)$. Using the previous argument, we then conclude that $\overline{A} = (\overline{a})$ and $\overline{B} = (\overline{b})$, which implies

$$A = (z_1, ..., z_{n-1}, a)$$
 and $B = (z_1, ..., z_{n-1}, b),$

as claimed. \Box

Theorem 2.2 Let (R, \mathfrak{m}) be a d-dimensional Cohen–Macaulay local ring, which is not a regular local ring. Let $I = J : \mathfrak{m}$ where $J = (z_1, ..., z_d)$ is a system of parameters. Then $I^2 = JI$.

Proof. We divide the proof into two parts, first showing that $\mathfrak{m}I = \mathfrak{m}J$, and then that $I^2 = JI$.

Clearly, $\mathfrak{m}I \supseteq \mathfrak{m}J$; since $I = J : \mathfrak{m}$ we have that $\mathfrak{m}I \subset J$, so if $\mathfrak{m}I \not\subseteq \mathfrak{m}J$, Proposition 2.1 with A = I, $B = \mathfrak{m}$ and $(\mathbf{z}) = J$ would imply that \mathfrak{m} is generated by a regular sequence, contradicting the assumption that R is not a regular local ring.

It will suffice to show that the product of any two elements of I is contained in JI. Pick $a, b \in I$; since $ab \in I^2 \subset \mathfrak{m}I \subset J$, ab can be written in the following way

$$ab = \sum_{i=1}^{d} \alpha_i z_i. \tag{1}$$

The proof will be complete once it is shown that $\alpha_i \in I$, for i = 1, ..., d; to this end it is enough to prove that $\alpha_i x \in J$ for any $x \in \mathfrak{m}$.

Since R is a local ring regular sequences permute, thus we can key on any of the coefficients, say α_d . Note that $xb \in \mathfrak{m}I = \mathfrak{m}J$, hence xb satisfies an equation of the form

$$xb = \sum_{i=1}^{d} \beta_i z_i, \qquad \beta_i \in \mathfrak{m}. \tag{2}$$

Multiply equation (1) by x and equation (2) by a; a quick comparison between these new equations yields the following result

$$\sum_{i=1}^{d} \alpha_i x z_i = \sum_{i=1}^{d} a \beta_i z_i,$$

which can also be written as

$$\sum_{i=1}^{d} (\alpha_i x - a\beta_i) z_i = 0.$$
(3)

Equation (3) modulo the ideal $J_1 = (z_1, ..., z_{d-1})$ gives that $(\alpha_d x - a\beta_d)z_d$ is zero modulo J_1 ; but z_d is regular in R/J_1 hence

$$\alpha_d x - a\beta_d \in J_1 = (z_1, ..., z_{d-1}) \subset J.$$

Since $a\beta_d \in I\mathfrak{m} \subset J$ we then have that $\alpha_d x \in J$ as well, or equivalently $\alpha_d \in I$.

Theorem 2.3 Let (R, \mathfrak{m}) be a Cohen–Macaulay local ring and let \mathfrak{p} be a prime ideal of R of height g such that $R_{\mathfrak{p}}$ is not a regular local ring. If $J = (z_1, ..., z_g)$ denotes a regular sequence inside \mathfrak{p} , then the link $I = J : \mathfrak{p}$ is equimultiple with reduction number 1.

Proof. It is enough to reduce to the case in which (R, \mathfrak{p}) is a local ring and then apply Theorem 2.2.

In order to show the equality $I^2 = JI$ it suffices to establish it at the associated primes of R/JI. From the short exact sequence

$$0 \to J/JI \longrightarrow R/JI \longrightarrow R/J \to 0$$

and the fact that $J/JI = J/J^2 \otimes R/I = (R/I)^g$, it follows that the associated primes of R/JI are contained in those of R/J and of R/I. On the other hand, we claim that $\mathrm{Ass}(R/I) \subset \mathrm{Ass}(R/J)$; indeed, $I = J \colon \mathfrak{p}$ with \mathfrak{p} a prime ideal implies that I and \mathfrak{p} are directly linked. Thus the associated primes of R/JI must have height g, since J is a complete intersection.

If we consider $\mathfrak{q} \in \mathrm{Ass}(R/J)$, then either $\mathfrak{q} = \mathfrak{p}$ or $\mathfrak{q} \not\supset \mathfrak{p}$. The first case is the situation of a link with the maximal ideal; in the second case $I_{\mathfrak{q}} = J_{\mathfrak{q}}$ and so we are done.

The next corollary generalizes Theorem B and Corollary C to the case of Cohen–Macaulay links of prime ideals in arbitrary Cohen–Macaulay rings.

Corollary 2.4 With the same assumptions of Theorem 2.3 if I is a Cohen–Macaulay ideal then the associated graded ring of I is Cohen–Macaulay. If in addition $\dim(R_{\mathfrak{p}}) \geq 2$ then the Rees algebra of I is Cohen–Macaulay as well.

Proof. The proof is exactly the same as the one in [3]. There, the Cohen–Macaulayness of I was a consequence of linkage in Gorenstein rings; here, we have to ask explicitly for it. \square

Remark 2.5 The previous assertions leave out the cases in which $R_{\mathfrak{p}}$ is a regular local ring. When R is Gorenstein this is taken care of by [3] provided that condition L_2 is satisfied. We would like to remark however that the associated graded ring of the link I is Cohen–Macaulay without any assumption if \mathfrak{p} is supposed to be a Gorenstein ideal. This is an easy consequence of Theorems A and B together with the fact that in a Gorenstein setting a strongly Cohen–Macaulay ideal satisfying property \mathcal{F}_1 has Cohen–Macaulay associated graded ring (see [4, Theorem 5.3]).

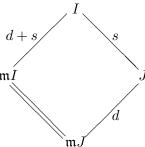
Remark 2.6 In the situation of Theorem 2.2

$$I/J = \operatorname{socle}(R/J) = \operatorname{Hom}_{R/J}(R/\mathfrak{m}, R/J) \simeq (R/\mathfrak{m})^s,$$

where s is the Cohen–Macaulay type of R. This and the fact that $\mathfrak{m}I = \mathfrak{m}J$ imply that the generators of J are among the minimal generators of I and that s extra minimal generators are needed to get the whole I, i.e.

$$I = (J, a_1, ..., a_s).$$

In a diagram this can be seen as follows (note that the numbers represent the length of each quotient):



Remark 2.7 Let (R, \mathfrak{m}) be a Cohen–Macaulay local ring of dimension d and type $s \geq 2$ and let $I = J : \mathfrak{m}$, with J generated by a regular sequence of length d inside \mathfrak{m} . Then the following formula is valid

$$\lambda(\delta(I)) = \binom{s+1}{2},\tag{4}$$

where $\delta(I)$ is the kernel of the natural surjection from $S_2(I)$ to I^2 and $\lambda(\underline{\ })$ is the length function.

Proof. Let us consider the following commutative diagram

where \cdot denotes the product in the symmetric algebra. Since J is a regular sequence inside I we conclude that $J \cdot I \simeq JI$ (see the proof of [17, Theorem 2.4]) or, equivalently, $\star = 0$. Thus, using the additivity of the length on the third column of (5) and the fact that $\delta(I) \simeq \star \star$, one gets (see also [3, Remark 2.4])

$$\lambda(I^2/JI) + \lambda(\delta(I)) = \lambda(S_2(I/J)).$$

Finally, (4) follows from $I/J \simeq (R/\mathfrak{m})^s$ and $I^2 = JI$.

Example 2.8 The assumption of R being Cohen–Macaulay cannot be dropped in Theorem 2.3 as the following example shows; this example, in particular, proves that the previous results cannot even be extended to the case of systems of parameters (which are d-sequences) in Buchsbaum rings.

Let k be any field, set $R = k[X,Y]/(X^2,XY) = k[x,y]$ where x and y denote the images of X and Y modulo (X^2,XY) . It can be easily seen that R is a one dimensional Buchsbaum ring which is not Cohen–Macaulay (e.g. [15, Example 5]). Let $J = (y^3)$ be a system of parameters in R and consider I = J: \mathfrak{m} , where $\mathfrak{m} = (x,y)$; a computation using the computer system Macaulay shows that $I^2 \neq JI$.

3 The canonical module

If (R, \mathfrak{m}) is a Cohen–Macaulay local ring, a faithful maximal Cohen–Macaulay module ω_R of type 1 is called a *canonical module* of R. When it exists this module is unique up to isomorphism. If R is a Gorenstein local ring then ω_R is isomorphic to R; moreover, if $\varphi \colon S \longrightarrow R$ is a local homomorphism of Cohen–Macaulay local rings such that R is a finite S-module and ω_S is the canonical module of S then ω_R exists and is given by $\omega_R \simeq \operatorname{Ext}_S^g(R, \omega_S)$, where $g = \dim(S) - \dim(R)$.

In this section we describe the structure of the canonical module of the Rees algebra of a Cohen–Macaulay, equimultiple ideal I with reduction number 1. This allows us to give a necessary and sufficient condition for the associated graded ring of I to be Gorenstein. As an application, we compute the canonical module of the Rees algebra of the link I of a prime ideal \mathfrak{p} satisfying either condition L_1 or L_2 .

For convenience we recall some additional facts from linkage theory. Let R be a Gorenstein local ring, I an unmixed ideal of grade g and \mathbf{z} a regular sequence of length g inside I. If we set $J = \mathbf{z}$: I then $I \sim J$. In addition, if $I \sim J$ then I is Cohen–Macaulay if and only if J is Cohen–Macaulay. Furthermore, $\omega_{R/I} \simeq J/\mathbf{z}$ and $\omega_{R/J} \simeq I/\mathbf{z}$.

Theorem 3.1 Let R be a d-dimensional Gorenstein local ring and let I be a Cohen–Macaulay ideal of R, of height $g \geq 2$. Let $J = (a_1, ..., a_g) \subset I$ be a reduction of I with $I^2 = JI$. Then the canonical module of R[It] has the form

$$\omega_{R[It]} = (t(1,t)^{g-3} + Lt^{g-1}),$$

where L is given by J:I.

Proof. Set A = R[Jt] and B = R[It] and let L = J: I. Since I is a Cohen–Macaulay ideal, then I = J: L and L is a Cohen–Macaulay ideal (see [12]).

The canonical module of A is the fractional ideal of A given by $\omega_A = (1, t)^{g-2}[-1]$ (see [1, 7]). More explicitly

$$\omega_A = Rt + Rt^2 + \dots + Rt^{g-1} + Jt^g + J^2t^{g+1} + \dots$$

Since J is a reduction of I, the extension $A \hookrightarrow B$ is finite; therefore the canonical module of B is given by

$$\omega_B = \operatorname{Hom}_A(B, \omega_A); \tag{6}$$

as a graded ideal, which we represent as

$$\omega_B = \omega_1 t + \omega_2 t^2 + \cdots.$$

Indeed, localizing ω_B at the multiplicative set given by the powers of x, where $x \in J$ is a regular element, we get that $\omega_{B_x} = \omega_{R_x}[t][-1] = R_x[t][-1]$, since $B_x = R_x[t]$ and R_x is Gorenstein. This means that 1 is the initial degree of ω_{B_x} . However, as x is regular on B (and therefore on ω_B as well) we have the inclusion $\omega_0 \hookrightarrow (\omega_0)_x = 0$, so that $\omega_0 = 0$.

The problem is to recover the ω_i from the formal expression of ω_B as the dual above. Consider the natural exact sequence of R[Jt]-modules,

$$0 \to A \longrightarrow B \longrightarrow C \to 0, \tag{7}$$

where C is $I/J+I^2/J^2+\cdots+I^n/J^n+\cdots$. Note that C is a Cohen–Macaulay R[Jt]-module of dimension d. Indeed, from (7) and the depth lemma (see [2, Lemma 1.2.9]) we have that $depth(C) \geq d$. On the other hand, the dimension of C is less or equal than d since JR[Jt] has height 1 and is contained in the annihilator of C.

Dualizing (7) with ω_A , we have the following exact sequence

$$0 \to \omega_B \longrightarrow \omega_A \longrightarrow \omega_C = \operatorname{Ext}_A^1(C, \omega_A) \to 0.$$
 (8)

It follows that if one knows what $\operatorname{Ext}_A^1(C,\omega_A)$ looks like, ω_B can be recovered. In order to describe $\operatorname{Ext}_A^1(C,\omega_A)$ we have to examine more closely C. Since $I^2=JI$, we have

that $C = I/J \cdot R[Jt][-1]$, i.e. $C_i = (I/J \cdot R[Jt])_{i-1}$; therefore from the presentation $R[x_1, ..., x_g] \longrightarrow R[Jt] \to 0$, we have the following surjection

$$I/J \otimes R[x_1, ..., x_q][-1] = I/J[x_1, ..., x_q][-1] \longrightarrow C = I/J \cdot R[Jt][-1] \to 0.$$
 (9)

We want to show that

$$C \simeq I/J[x_1, ..., x_q][-1].$$

Let K be the kernel of the map in (9)

$$0 \to K \longrightarrow I/J[x_1, ..., x_a][-1] \longrightarrow C \to 0; \tag{10}$$

to show that K=0 it will be enough to show that $K_{\mathfrak{q}}=0$ for all $\mathfrak{q}\in \mathrm{Ass}(I/J[x_1,...,x_g])$. Note that $I/J=(J:L)/J=\omega_{R/L}$ is a Cohen–Macaulay module of dimension d-g. In particular $I/J[x_1,...,x_g]$ is a Cohen-Macaulay module of dimension d. By the depth lemma we also have that $\mathrm{depth}(K)\geq d$, hence K is Cohen-Macaulay as well. Next, we observe that the associated primes of K are included in the associated primes of $I/J[x_1,...,x_g]$; and that $\mathrm{Ass}(I/J)=\mathrm{Ass}(R/L)$. Localize (10) at \mathfrak{q} to get the exact sequence

$$0 \to K_{\mathfrak{q}} \longrightarrow (I/J)_{\mathfrak{q}}[x_1,...,x_g][-1] = \omega_{R_{\mathfrak{q}}/L_{\mathfrak{q}}}[x_1,...,x_g][-1] \longrightarrow C_{\mathfrak{q}} \to 0, \tag{11}$$

of Cohen–Macaulay modules of dimension g. Let $D = R_{\mathfrak{q}}/L_{\mathfrak{q}}$; D is an Artinian ring with maximal ideal $\mathfrak{M} = \mathfrak{q}R_{\mathfrak{q}}/L_{\mathfrak{q}}$. Set $D[x_1,...,x_g] = D[\underline{x}]$ and dualize (11) with $\operatorname{Hom}_{D[\underline{x}]}(\underline{\ },\omega_{D[\underline{x}]})$ to obtain the exact sequence

$$0 \to \operatorname{Hom}_{D[\underline{x}]}(C_{\mathfrak{q}}, \omega_{D[\underline{x}]}) \longrightarrow \operatorname{Hom}_{D[\underline{x}]}(\omega_{D}[\underline{x}][-1], \omega_{D[\underline{x}]}) \longrightarrow \operatorname{Hom}_{D[\underline{x}]}(K_{\mathfrak{q}}, \omega_{D[\underline{x}]}) \to 0$$

or

$$0 \to C_{\mathfrak{q}}^{\vee} \longrightarrow D[\underline{x}][1-g] \longrightarrow K_{\mathfrak{q}}^{\vee} \to 0, \tag{12}$$

of Cohen–Macaulay modules of dimension g. It will be enough to prove that $\operatorname{height}(C_{\mathfrak{q}}^{\vee}) \geq 1$, since this would imply that $\dim(K_{\mathfrak{q}}^{\vee}) \leq g-1$. In order to show $\operatorname{height}(C_{\mathfrak{q}}^{\vee}) \neq 0$ we have to show that $C_{\mathfrak{q}}^{\vee} \not\subseteq \mathfrak{M}[\underline{x}]$, as $\mathfrak{M}[\underline{x}]$ is the unique minimal prime of $D[\underline{x}]$. Assume otherwise; thus $\operatorname{Ann}_D(C_{\mathfrak{q}}^{\vee}) \supseteq \operatorname{Ann}_D(\mathfrak{M}[\underline{x}]) \supseteq \mathfrak{M}^{r-1} \neq 0$, where r is the index of nilpotency of the maximal ideal \mathfrak{M} of the Artinian ring D. This leads to a contradiction since $\operatorname{Ann}_D(C_{\mathfrak{q}}^{\vee}) = 0$. Indeed, $C_{\mathfrak{q}}^{\vee} = \operatorname{Hom}_{D[\underline{x}]}(C_{\mathfrak{q}}, \omega_{D[\underline{x}]})$ and $C_{\mathfrak{q}} \simeq \operatorname{Hom}_{D[\underline{x}]}(C_{\mathfrak{q}}^{\vee}, \omega_{D[\underline{x}]})$ imply that $\operatorname{Ann}_D(C_{\mathfrak{q}}^{\vee}) = \operatorname{Ann}_D(C_{\mathfrak{q}}) \subseteq \operatorname{Ann}_D((I/J)_{\mathfrak{q}}) = 0$. The last part follows from the fact that $C_{\mathfrak{q}} = (I/J)_{\mathfrak{q}} + (I^2/J^2)_{\mathfrak{q}} + \cdots$ and that $(I/J)_{\mathfrak{q}} = \omega_D$ is D-faithful, since $D = \operatorname{End}_D(\omega_D)$. Hence

$$C = \omega_{R/L}[x_1,...,x_g][-1] = \omega_{R/L[x_1,...,x_g]}[g-1].$$

Finally, by duality we have that

$$\omega_C = \operatorname{Ext}_A^1(C, \omega_A) = \operatorname{Ext}_A^1(\omega_{R/L[x_1, ..., x_q]}[g-1], \omega_A) = R/L[x_1, ..., x_q][-g+1].$$

Notice that $R/L[x_1,...,x_g] \simeq \operatorname{gr}_J(R) \otimes R/L$; therefore two of the modules in the short exact sequence (8) are

$$\omega_A = (t(1,t)^{g-2}),$$

$$\omega_C = (\operatorname{gr}_J(R) \otimes R/L)[-g+1].$$

From (8) we conclude that $\omega_i \simeq R$ for all i = 1, ..., g-2 and that ω_{g-1} is given by $R/\omega_{g-1} \simeq R/L$. If we show that $\omega_{g-1} \subseteq L$ we get that $\omega_{g-1} \simeq L$. In order to show such an inclusion, we have to look at the structure of the canonical module in a different way. This alternative approach is based on the following observation

$$\omega_B = \operatorname{Hom}_A(B, \omega_A) \simeq \omega_A :_{R[t]} B.$$

Thus ω_{g-1} is going to satisfy the following set of relations

$$\omega_{g-1}R \subset R, \ \omega_{g-1}I \subset J, \ \omega_{g-1}I^2 \subset J^2, \dots$$

The first inclusion says that $\omega_{g-1} \subseteq R$; since $I^2 = JI$, the only important condition is given by $\omega_{g-1}I \subset J$. From this equation we get the desired inclusion since I = J: L.

Finally, observe that ω_A modulo the submodule of ω_B generated by the elements up to degree g-1 is already ω_C .

Corollary 3.2 Let I be an ideal as in Theorem 3.1, then $gr_I(R)$ is Gorenstein if and only if I = J: I.

Proof. By a result of Herzog–Simis–Vasconcelos (see [5, Corollary 2.5]), $\operatorname{gr}_I(R)$ is Gorenstein if and only if the canonical module of R[It] has the so called *expected form*, i.e. $\omega_{R[It]} \simeq \omega_R t(1,t)^m$ for some $m \geq -1$. By Theorem 3.1,

$$\omega_{R[It]} = (t(1,t)^{g-3} + Lt^{g-1}),$$

hence it assumes the expected form if and only if J: I = L = I.

Let us now apply Theorem 3.1 and Corollary 3.2 to the *leitmotiv* of this paper, i.e. to the case of links of prime ideals.

Corollary 3.3 Let R be a Gorenstein local ring of dimension d, \mathfrak{p} a Cohen-Macaulay prime ideal of codimension $g \geq 2$, $J = (x_1, ..., x_g)$ a complete intersection contained in \mathfrak{p} , and let I = J: \mathfrak{p} . If either L_1 or L_2 holds, then the canonical module of R[It] has the form

$$\omega_{R[It]} = (t(1,t)^{g-3} + \mathfrak{p}t^{g-1}).$$

Proof. From [3], I is an equimultiple ideal with reduction number 1 and Cohen–Macaulay Rees algebra, so that Theorem 3.1 applies.

Remark 3.4 In the case of Corollary 3.3 the isomorphism

$$C \simeq I/J[x_1, ..., x_g][-1]$$

can be established in a more direct manner. Indeed, the only associated prime of I/J is \mathfrak{p} . Localizing (10) at \mathfrak{p} , we get the short exact sequence

$$0 \to K_{\mathfrak{p}} \longrightarrow (I/J)_{\mathfrak{p}}[x_1,...,x_q][-1] \longrightarrow C_{\mathfrak{p}} \to 0,$$

where $(I/J)_{\mathfrak{p}}[x_1,..,x_g]$ is isomorphic to a polynomial ring of dimension g over a field and $K_{\mathfrak{p}}$ to one of its ideals. Thus $\dim(C_{\mathfrak{p}}) \leq g-1$. But this is a contradiction since $C_{\mathfrak{p}}$ is still a Cohen–Macaulay module of dimension g.

4 On the multiplicity of special rings

In this section we study the multiplicity of two families of rings: Rees algebras of links of the maximal ideal of a Gorenstein local ring, and quotients of a Gorenstein local ring modulo a self-linked ideal.

4.1 The Rees algebra of links

The following result is a very useful tool in the computation of the multiplicity of certain Rees algebras.

Theorem 4.1 Let (R, \mathfrak{m}) be a d-dimensional Gorenstein local ring and let $I = J : \mathfrak{m}$ where J is a regular sequence of R of length d. Let $\mathfrak{M} = (\mathfrak{m}, It)$ be the maximal homogeneous ideal of R[It] and $\mathfrak{N} = (\mathfrak{m}, Jt)$ be the maximal homogeneous ideal of R[Jt]. If either L_1 or L_2 holds, then the multiplicities of those two Rees algebras with respect to their maximal ideals are equal,

$$e(\mathfrak{M}, R[It]) = e(\mathfrak{N}, R[Jt]).$$

Proof. R[It] is a finite R[Jt]-module of rank 1, as J is a reduction of I. Applying [2, Corollary 4.6.9] to the ring $(R[Jt], \mathfrak{N})$ and to the module M = R[It] we get that

$$e(\mathfrak{N}R[It], R[It]) = e(\mathfrak{N}, R[Jt]).$$

On the other hand, since $I = J : \mathfrak{m}$, or more specifically $\mathfrak{m}I \subset J$, and $I^2 = JI$ (with J a minimal reduction of I) we have that $\mathfrak{m}I = \mathfrak{m}J$. So, the following calculation goes through

$$(\mathfrak{N}R[It]) \mathfrak{M} = (\mathfrak{m} + Jt + JIt^2 + JI^2t^3 + \cdots)(\mathfrak{m} + It + I^2t^2 + I^3t^3 + \cdots)$$

= $\mathfrak{m}^2 + \mathfrak{m}It + JIt^2 + JI^2t^3 + \cdots$
= \mathfrak{M}^2 .

that is to say $\mathfrak{N}R[It]$ is a reduction of \mathfrak{M} . Therefore the assertion follows from [2, Lemma 4.5.5] when applied to the ring $(R[It], \mathfrak{M})$ and to the module M = R[It].

Example 4.2 Thanks to Theorem 4.1, the problem of computing $e(\mathfrak{M}, R[It])$ is then reduced to the case in which we are dealing with the Rees algebra R[Jt] of a regular sequence. The latter problem has been recently studied in [6] in the case in which R is an homogeneous algebra over a field k and J is an \mathfrak{m} -primary ideal generated by an homogeneous regular sequence.

The next example is taken from [19]. Let (R, \mathfrak{m}) be a 3-dimensional regular local ring containing a field k and assume that the maximal ideal is given by the following regular system of parameters $\mathfrak{m} = (x, y, z)$. Let I and J be the following ideals

$$I = J: \mathfrak{m}$$
 $J = (x, y^2, z^2).$

It turns out that $I=(x,y^2,yz,z^2)$; moreover, since the generators of J are homogeneous of degrees

$$a_1 = \deg(x) = 1$$
, $a_2 = \deg(y^2) = 2$, $a_3 = \deg(z^2) = 2$,

our result combined with [6, Corollary 1.5] gives directly

$$e(\mathfrak{M}, R[It]) = e(\mathfrak{N}, R[Jt]) = (1 + a_1 + a_1 a_2)e(R) = (1 + 1 + 1 \cdot 2)e(R) = 4 \cdot 1 = 4.$$

4.2 Quotients of self-linked primes

Finally, we study the multiplicity of R/I where I is a self-linked ideal. We give a lower bound for this number when, essentially, the first Koszul homology module $H_1(I)$ is Cohen–Macaulay.

Theorem 4.3 Let R be a Gorenstein local ring and I a self-linked ideal of R of height g; assume that I is generically Gorenstein and that it lies in the linkage class of a complete intersection (licci). Let β_1 denote the minimal number of generators of I. Then

$$e(R/I) \ge {\beta_1 - g + 1 \choose 2},$$

where e(R/I) denotes the multiplicity of R/I.

Proof. Write I = J: I, where J is an ideal generated by a regular sequence of length g inside I. Obviously I/J is an R/I-module. For every associated prime \mathfrak{q} of I, $(I/J)_{\mathfrak{q}} \simeq \omega_{(R/I)_{\mathfrak{q}}} \simeq (R/I)_{\mathfrak{q}}$, which shows that I/J is an R/I-module of rank 1, and hence so is $S_2(I/J)$. But the latter module is Cohen–Macaulay since

$$\operatorname{depth}(H_1(I)) = \operatorname{depth}(S_2(I/J)) = \operatorname{depth}(S_2(\omega_{R/I}))$$

(see [16, Theorem 3.1]) and since $H_1(I)$ is Cohen–Macaulay by [9]. Finally, from a general formula due to D. Rees (see [13, Corollary 5.3.3]), we have that

$$e(R/I) \ge \mu(S_2(I/J)) \ge {\beta_1 - g + 1 \choose 2},$$

as claimed.

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