Jean Léonard Marie Poiseuille (1797–1869). From a photographic portrait that appeared with the article by Brillouin (1930); oil-painted enhancement by SPS.

THE HISTORY OF POISEUILLE'S LAW

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1. BIOGRAPHICAL HIGHLIGHTS AND MYSTERIES

Jean Léonard Marie Poiseuille entered the Ecole Polytechnique at the age
of 18 in the fall of 1815. His residence there ended April 13, 1816 when
the entire Ecole was disbanded for political reasons. He did not go back
when it reopened but switched to the study of medicine instead. During
his months at Ecole Polytechnique Poiseuille took courses from Cauchy,
Ampère, Hachette, Arago, Petit, and Thénard. Brillouin (1930) attributes
Poiseuille's extraordinary sense of experimental precision to the influence
of his physics professor, the brilliant but short-lived (1791–1820) Alexis
Petit, who along with P. L. Dulong discovered in 1819 that the molar
specific heat of all solids tends to a constant at high temperature (Dulong-
Petit rule). During his doctoral research on *The force of the aortic heart*
(Poiseuille 1828), Poiseuille invented the U-tube mercury manometer
(called the hemodynamometer) and used it to measure pressures in the
arteries of horses and dogs. A recording version of the manometer, named
the Poiseuille-Ludwig hemodynamometer, was used in medical schools
until the 1960s and to this day blood pressures are reported in mm Hg due
to Poiseuille's invention.

    Between 1828 and 1868 Poiseuille published 15 articles ranging from

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brief communications to the French Academy of Sciences to extensive monographs. A complete list of Poiseuille's publications is given under the Literature Cited section (from Pappenheimer 1978). It is remarkable that these few experimental papers have made the name of Poiseuille familiar in a variety of fields including engineering, physics, medicine, and biology. Following completion of his doctoral dissertation on the heart and pulse waves, Poiseuille turned his attention to hemodynamics in micro-circulation. His observations of the mesenteric microcirculation of the frog (Poiseuille 1835) revealed that blood flow in the arterioles and venules features a plasma layer at the vessel wall in which there are few red cells, that "plasma-skimming" occurs at vessel bifurcations, and that white cells tend to adhere to the vessel wall. The realization that uncontrolled in vivo studies would not permit a clear formulation of the laws governing blood flow in microcirculation led him to undertake his careful and extensive studies of the flow of liquids in small diameter glass capillaries.

These studies presumably began sometime in the 1830s since in 1838 he gave a preliminary oral report on the effects of pressure and of tube length to the Société Philomatique (Poiseuille 1838). Then, in 1839, Poiseuille deposited with the French Academy of Sciences a sealed packet containing the results of his studies on the flow of water through glass tubes and the effects of pressure drop, tube length, tube diameter, and temperature. The purpose of this procedure was to establish priority. During the academic year 1840–1841 he made three oral communications (*Mémoires lus*) to the Academy of Sciences. Excerpts of these were subsequently published in the Academy's *Comptes Rendus* (Poiseuille 1840a,b; 1841). In January 1841 Poiseuille deposited another sealed packet of experimental results dealing with the flow of a variety of liquids through glass capillaries. Some of these results were communicated to the Academy in 1843 (Poiseuille 1843).

The results and conclusions presented by Poiseuille in 1840–1841 were considered sufficiently important that the Academy appointed an elite Special Commission to investigate their validity. This Commission, con-sisting of members Arago, Babinet, Piobert, and Régnault, met in 1842 and with Poiseuille repeated some of his experiments using his apparatus. In the course of this review, the Commission prevailed upon Poiseuille to do some new preliminary experiments using mercury and ethyl ether. The Commission reported back to the Academy on December 26, 1842 recommending that Poiseuille's work be approved and included in its entirety in *Mémoires des Savants Etrangers*, a publication of the Academy of Sciences. It actually appeared in the *Mémoires Presentés par Divers Savants à l'Académie Royale des Sciences de l'Institut de France* in 1846, seven full years after he delivered his first sealed packet to the Academy.

A complete English translation of this paper is available (in Bingham 1940). The Commission's report was published in the *Annales de Chimie et Physique* (Régnault et al 1843). Poiseuille's final contribution to the subject of liquid flow in narrow tubes appeared in September 1847. That paper presented measurements for (*i*) dilute aqueous salt solutions, (*ii*) aqueous solutions of bases, (*iii*) aqueous solutions of acids, (*iv*) mineral waters, (*v*) teas, (*vi*) wines and spirits, (*vii*) extracts of plants and roots, (*viii*) bovine serum and acidic solutions thereof, and (*ix*) a mixed group of ethers, alcohols, and solutions of ammonia. In each group flow times were compared to that of distilled water under the same conditions. It appears that these studies were motivated by Poiseuille's interest in the possible facilitation of capillary blood flow through medication.

There is no record of where Poiseuille did his work or how it was supported financially. His apparatus was elaborate and certainly required the services of an expert glassblower. The experiments were time-consuming (the calibration of a single capillary tube took as long as twelve hours) so he probably had technical assistance. Brillouin (1930) suggests the possibility that the well established physiologist Magendie provided space and necessary resources at La Salpêtrière Hospital in Paris. Pappenheimer (1978) suggests that a wealthy father-in-law may have made it possible for Poiseuille to dedicate himself to research. Apparently Poiseuille practiced medicine for a while because he was listed in a Paris directory of physicians dated 1845, but other evidence indicates that he did not practice medicine after 1844.

Original biographical information on Poiseuille's life is scarce. Brillouin (1930), Joly (1968), and Pappenheimer (1978) summarize most of the known information. Joly's biographical note, which was delivered prior to the presentation of the first Poiseuille medal to Robin Fåhraeus in 1966, is an especially eloquent testimony to the many facets of this scientist and his accomplishments. Joly points out that during his lifetime Poiseuille was only modestly recognized. In 1835, the Academy of Sciences awarded him half of the prize for experimental physiology (value unmentioned); in 1845, he won the prize for medicine and surgery (worth 700 francs), and in 1860, he received an honorable mention, again from the Academy of Sciences. Although Poiseuille was an elected member of the Paris Academy of Medicine, his numerous attempts to win election to the Academy of Sciences in the 1840s, 1850s, and 1860s were never successful.

Another mysterious aspect of Poiseuille's life concerns his circumstances and employment in later life. In 1858, he filed an application for a position in the Paris public school system. In 1860, Dr. Poiseuille went to work as Inspector of School Sanitation in the Seine district. Poiseuille, born on April 22, 1799, died in Paris, the city of his birth on December 26, 1869.

2. POISEUILLE'S EXPERIMENTS

Poiseuille set out to find a functional relationship among four variables: the volumetric efflux rate of distilled water from a tube Q, the driving pressure differential P, the tube length L, and the tube diameter D. The diameters of his glass tubes ranged from 0.015 to 0.6 mm, encompassing vessel sizes found in most microcirculatory systems but not quite that of human capillaries (~ 5 to 10 microns). Initially he planned to maintain a constant temperature of 10°C; subsequently he examined the influence of temperature from 0 to 45°C, still using distilled water as the test liquid. He later extended his studies to a great variety of other liquids (Poiseuille 1847).

Figure 1 is a drawing of the frontal view of Poiseuille's experimental apparatus. It is estimated that the apparatus stood between $2\frac{1}{2}$ and 3 meters tall. The heart of the system is the small capillary viscometer labeled c–e–d near the center of the figure just below the spindle-shaped bulb M. Because it is immersed in water inside a glass cylinder the viscometer is shown in dotted outline. A hand-operated pump (h, surrounded by a water jacket X–Y) was used to charge the vertical reservoir on the left with air and simultaneously to raise either a water column in the tall manometer i–i or a mercury column in the short manometer i'–i'. During pressurization the valve R leading to the viscometer was closed. Once the desired pressure, as indicated by one of the manometers, was reached, the pump discharge valve R' was closed, valve R was opened, and flow was driven through the test capillary d.

A close-up enlargement of the viscometer assembly is shown in Figure 2. The pointed bottom of the bulb M served to trap dust particles—which tended to settle out either from the air or the liquid used to clean the glassware—preventing them from falling into the capillary branch (b''–c–e–d). Poiseuille found it necessary to filter his distilled water repeatedly, sometimes as many as 20 times, to be rid of foreign particles. The entire test capillary was situated under water in a glass cylinder (C–D–F–E) which was surrounded by a water bath (G–H–I–K).

The underwater efflux of the capillary tubes was necessitated when Poiseuille discovered that he could not achieve reproducible results when the minuscule liquid flows (some as low as 0.10 cc in several hours) exited in air against the erratic resistance of surface tension. This problem was eliminated by underwater efflux, but required that flow be measured upstream by timing the passage of a liquid meniscus between two lines, C and E (Figure 3), which delimited a known volume of the spherical bulb A–B. The second smaller bulb G in Figure 3 provided the entrance to the horizontal test capillary D which was fused to G so as to provide an abrupt entrance. This was crucial to the accurate definition of tube length.
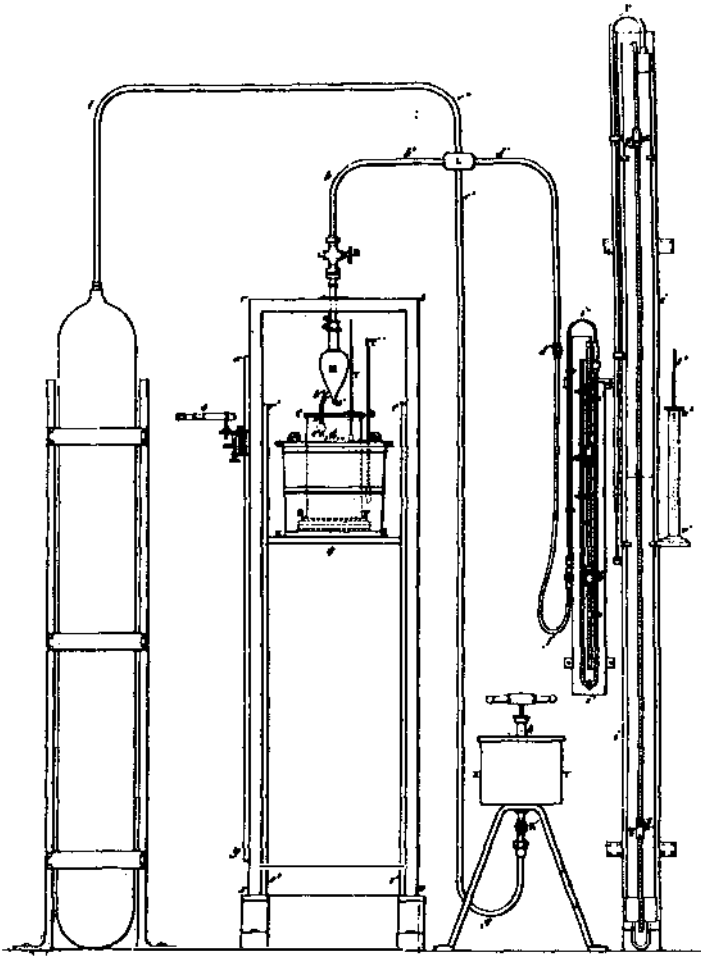
Figure 1   Frontal elevation view of Poiseuille's apparatus. Photocopy of one segment of a ten part fold-out plate published with Poiseuille's summary paper (1846).

Poiseuille extended his study of the influence of pressure up to about eight atmospheres. (At a pressure of 10 atmospheres, one of the bulbs M exploded.) For pressures above one atmosphere the spherical bulbs were replaced by a cylindrical vessel shown in Figure 4. These cylinders and the attached test capillaries (labeled $K–D$ in Figure 4) were tested in air. In these cases, the effluxes were large enough so that surface tension was not a problem.
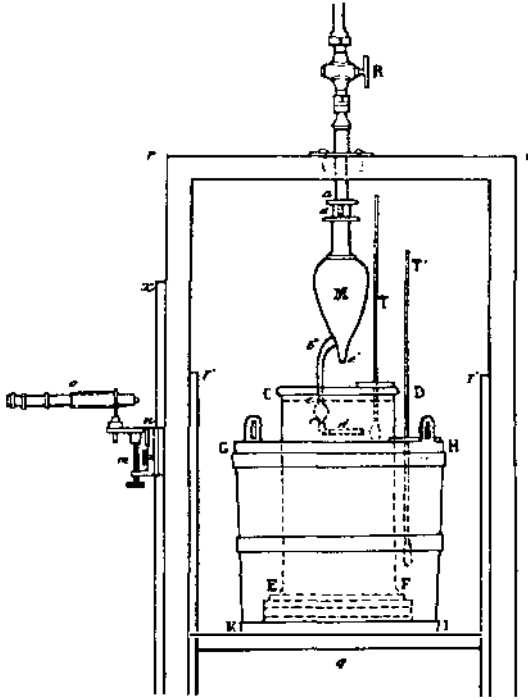
Figure 2   Enlargement of the viscometer assembly, from Figure 1.

The pressure differential was the primary independent variable in Poi-
seuille's experimental design. However, the head declined during outflow
because of changing liquid levels in the manometer, the viscometer bulb,
and the receiver vessel. Following contemporary understanding among
hydraulic engineers, Poiseuille used the arithmetic average of the initial
and final heads for P in his data analysis. He even performed an auxiliary
experiment (one of several) to test the accuracy of this assumption.
Bingham pointed out in his critique (1940) that the arithmetic mean is not
rigorously the correct average to use but that, given the dimensions of his
viscometer bulbs and the total heads applied, Poiseuille probably avoided
any appreciable errors from this approximation. Poiseuille was meticulous
in making second-order corrections for (*a*) the difference in the atmo-
spheric pressures acting on the water in the open manometer leg S' and
the free surface of the receiver vessel, (*b*) the weights of unequal air columns
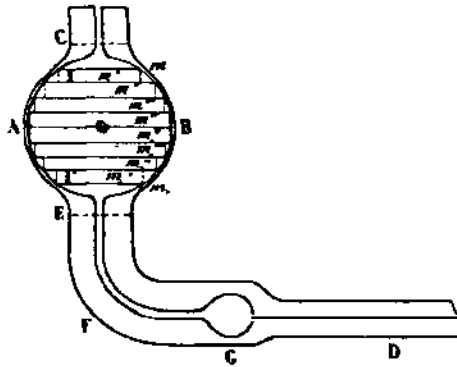confined within the pressurized legs of the apparatus, and (*c*) capillarity in

Figure 3 Detailed drawing of the sphcrical viscometer bulb (O) with attached test capillary (D) (from Poiseuille 1846). The horizontal lines $m, m', \ldots, m_1$ constructed inside the bulb were used by Poiseuille to argue that the elevation of the midplane (AOB) could be used to determine the average pressure under which the bulb volume was discharged.
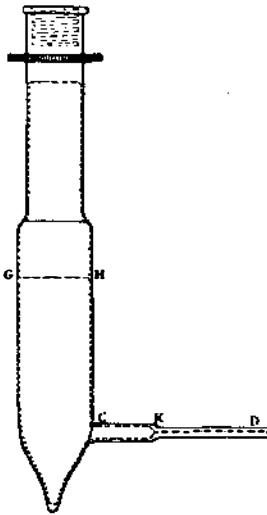

Figure 4 Cylindrical-conical viscometer bulb employed in high pressure experiments. KD is the capillary. From Poiseuille (1846).

the viscometer bulb. In one sample calculation Poiseuille showed that the correction due to different air column weights amounted to about 0.15%.

Correction for the capillarity in the spherical bulbs was more problematic because the area of the air-water interface varied continuously during flow. Poiseuille solved this by running auxiliary trials on graduated cylindrical bulbs wherein the capillarity was constant. First a capillary tube connected to a spherical bulb was tested. Then this tube was detached from the spherical bulb and reattached to a graduated cylindrical bulb. By carefully timing the outflow of a volume equal to that of the spherical bulb through the cylindrical bulb at the same average pressure and temperature, Poiseuille was able to figure the net capillarity correction of the original spherical bulb. This process was repeated "a great number of times" for each and every bulb used in the experiments. Again, by numerical example, Poiseuille also showed the capillarity correction to be of the order of three parts in 2000.

From a great number of glass tubes which he examined, Poiseuille selected a few which appeared to be fairly cylindrical along their length. This first screening was done by measuring the length of a thread of mercury a few centimeters long at different positions along the length of the tube. The cross section of a tube was then examined by cutting a small perpendicular section 2 to 3 mm long from one end, and grinding and polishing its faces until its thickness was reduced to about 0.1 mm. This thin annular disk was then placed between two plates of glass along with some Canada balsam and heated. The heating caused the balsam to flow into the small bore. This sandwich was then examined under the objective of a horizontal *Amici* microscope. Owing to the thinness of the annular disk, problems due to reflection, refraction, and diffraction were eliminated and the image of its bore was distinct and clear. By means of an illuminated chamber, a *camera lucida*, fitted to the microscope, an image of the bore was projected at a known magnification on the horizontal table of the microscope and its maximum and minimum diameters were measured with dividers and a millimeter scale. By this technique Poiseuille specified his tube diameters in millimeters, nominally from 0.015 to 0.6, to four and sometimes five places, i.e. to tenths or hundredths of a micron! One can question the significance of the fourth and fifth digits in these measured diameters, however, given that the original measurements of the magnified projected images were made by dividers and a millimeter scale and could be read perhaps to within $\frac{1}{2}$ part in 10 to 300 mm.

The lengths of the glass tubes were measured, after both ends were ground smooth, by means of a beam-compass equipped with a vernier scale. This tool (which was borrowed from the physical laboratory of the Collège de France courtesy of Monsieur Savart) could be read to within

1/20 to 1/40 mm. The series of seven tubes used in the "length study" ranged from 6.77 to 100.5 mm long.

Poiseuille recorded efflux times to the nearest quarter of a second but did not identify the particular timepiece (chronometer) used.

In most of the experiments dealing with the influence of pressure, tube diameter, and length, Poiseuille maintained the temperature of the bath surrounding the receiver vessel at 10°C. The temperature was indicated by the thermometer T (Figure 2) situated in the receiver with its bulb at the same level as the test capillary. This thermometer had divisions of fifths of a degree Celsius. Poiseuille's papers say nothing about how temperature was controlled. In two subsets of his experiments on the effect of pressure in which the driving heads were high, Poiseuille used tubes that were too long to fit in the receiving vessel. Hence these tests were performed in air at ambient temperatures varying from about 20°C all the way down to 7°C! (Apparently the laboratory was unheated.)

Poiseuille first studied the effect of pressure on flow. He began with a tube referred to as A, which was 100.5 mm long, determined its maximum and minimum internal diameters at each end (in this instance, exit end: 0.1395 mm, 0.1415 mm; entrance end: 0.1405 mm, 0.1430 mm), and fused it to the bulb G (Figure 3). Pressures of 385.870, 739.114, and 773.443 mm Hg at 10°C were established in succession and the corresponding flow times of 13.34085 cm^3 (the bulb volume at 10°C) of distilled water measured. These were 3505.75, 1830.75, and 1750.00 s, respectively. Next, successive portions of the end of the tube were cut off to provide test lengths of 51.1, 25.55, 15.75, 9.55, 6.775, and about 1 mm. The same procedure was followed with tubes B, C, D, E, F, G, H, I, and K with nominal internal diameters of 0.11, 0.085, 0.045, 0.03, 0.65, 0.63, 0.01, 0.09, and 0.13 mm, respectively. The lowest pressure applied was 74.29 mm of water and the highest was over 6000 mm Hg (about 8 atmospheres).

Poiseuille summarized his findings at this stage by the equation $Q = KP$, where the coefficient K was a function, to be determined, of tube length, diameter, and temperature. To investigate the influence of tube length Poiseuille took from his previous experiments on the A series of tubes all the data from those runs where the pressure was close to 775 mm Hg. Then, using his "law of pressures" he adjusted the measured flow times to correspond to a standard P of exactly 775 mm Hg. He was then able to show that the flow time was proportional to tube length (the "law of lengths") in a majority of his experiments. At this point Poiseuille could state that $K = K'/L$ and, therefore, $Q = K'P/L$, where K' was a function of tube diameter and temperature.

To determine the effect of tube diameter on flow Poiseuille (1847) stated that "we have measured the volumes of liquid flowing through tubes of

different diameters under the same pressure, at the same temperature, in the same time, the tubes having the same length; and we have compared the efflux, taking the diameters of the tubes into account." In fact, Poiseuille used the data he already had in hand, interpolating as necessary and applying the "laws of pressure and length", to arrive at a set of volume-diameter data standardized to $P = 775$ mm Hg, $L = 25$ mm, $\delta t = 500$ s and $T = 10°C$. The volume of the bulb used in each experiment was accurately determined by weighing the mercury contained between the lines C and E (Figure 3) to the nearest 0.5 mg. Since these weighings were carried out at ambient temperature, the calculated bulb volumes were corrected for the thermal expansion of glass to find the
the standard temperature of 10°C.

To assign a diameter to one of his noncircular, noncylindrical tubes, Poiseuille first calculated a geometrical average diameter for each end. This was defined as the diameter of the circle having the same area as an ellipse with the maximum and minimum diameters of the tube section. The arithmetic average of the geometrical means at the two ends was taken as the average diameter of the tube.

Following the above scheme, Poiseuille analyzed the data of seven of his previous experiments from which he was able to discern that the efflux volumes (in 500 s) varied directly as the fourth power of the average diameter. He would now claim that

$$Q = K''PD^4/L, \qquad (1)$$

K'' being simply a function of temperature and the type of liquid flowing. For 10°C his data yielded an average value of $K'' = 2495.224$ for distilled water expressed in mixed units of (mg/s)/(mm Hg) mm^3.

In his final series of experiments, Poiseuille explored the influence of temperature from a few tenths of a degree C to 45°C. He used four of his original tubes (before truncating them): A, C, D', and E. In each case he corrected both tube diameter and bulb volume for thermal expansion or contraction relative to the reference state of 10°C. Recognizing that the dependence of K'' on the temperature T was nonlinear, he elected to seek a polynomial fit of the form $K'' = K_1(1 + AT + A'T^2 + A''T^3 + \ldots)$ and found for distilled water:

$$K'' = 1836.7\,(1 + 0.033679T + 0.00022099\,T^2), \qquad (2)$$

where T is in °C.

Poiseuille recognized what are now called entrance effects, but did not come to precise conclusions. In his first series of experiments on the pressure effect beginning with tube A, Poiseuille found that the results obtained from shorter tubes deviated from the proportionality $Q = KP$.

He relegated these experiments to a "Second Series of Experiments" and excluded their data from his subsequent analyses. Poiseuille concluded that the "pressure law" would hold only if tube length exceeded a certain limit and that this limit depended on the tube diameter. He saw that the smaller the diameter, the smaller the limiting or minimum length. Beyond this observation Poiseuille had no explanation for the "Second Series." In one case, referring to the tube that was about 1 mm long, he opined that the "movement of fluid molecules" through the tube was not rectilinear. He recalled his observation of blood flow in a small diameter (0.15 mm), lateral branch from the mesenteric artery of a living frog. The "blood globules" could be seen to move along linear trajectories only if the artery was longer than about 2 mm.

The aberrant experiments (Second Series) encompassed a fairly wide range of Reynolds numbers, from close to 1 to 2600, but Poiseuille did not consider the relative roles of inertial and viscous forces in the development of tube flow. However, he expressed the belief that the pressure-flow proportionality would hold in capillary blood vessels longer than about 300 microns.

3.   DERIVATION OF POISEUILLE'S LAW

Strictly speaking, Poiseuille's law as written by Poiseuille is Equation (1) above. The equation which is more usually referred to as Poiseuille's law was not derived by Poiseuille. The more usual form is:

$$Q = \pi D^4 P / 128 \mu L. \tag{3}$$

The difference between Equation (3) and Poiseuille's Equation (1) is simply that in Equation (3) Poiseuille's constant K'' is replaced by $\pi/128\mu$ where μ is the viscosity of the fluid. Although viscosity had been defined by Navier (1823) no mention of viscosity per se was made by Poiseuille. However, he clearly recognized that K'' was a function of temperature and the flowing liquid. Poiseuille's determinations of K'' for water were so accurate that the viscosity derived from K'' agrees with accepted values within 0.1% (Bingham 1922).

The first derivation of Equation (3) from the Navier-Stokes equations is usually attributed to Eduard Hagenbach (1833–1910), a physicist of Basel. Hagenbach's 1860 paper is reprinted in a book edited by L. Schiller (1933) who states in an appendix that at about the same time that Hagenbach's paper appeared, another derivation of Poiseuille's law was published by H. Jacobson (1860) based on lectures of Franz Neumann, a physicist of Königsberg. Neumann's own treatise did not appear until some years later (Neumann 1883). Bingham (1922) points out that derivations of

Poiseuille's law were also published by H. Helmholtz (1860), J. Stephan (1862), and E. Mathieu (1863).

Sir George Gabriel Stokes (1813–1903) of Cambridge University apparently solved the problem of Poiseuille flow as an application of the Navier-Stokes equations which he derived in the same paper in 1845. However, he did not publish the result because he was unsure of the boundary condition of zero velocity at the tube wall. He writes: "But having calculated, according to the conditions which I have mentioned, the discharge of long straight circular pipes and rectangular canals, and compared the resulting formulae with some of the experiments of Bossut and Du Buat, I found that the formulae did not at all agree with experiment." [Charles Bossut (1730–1814), Pierre Louis Georges Du Buat (1734–1809)]. Stokes was apparently unaware of Poiseuille's work at this time. Later in the same article (Stokes 1845), he discusses the flow in canals and points out the similarity to pipe flow under gravity at constant pressure. For the case of a circular pipe he writes: "In this case the solution is extremely easy" and gives the solution:

$$w = \frac{g\rho \sin\alpha}{4\mu}(a^2 - r^2) + U. \tag{4}$$

Here w is the axial velocity, a and α are the radius and inclination of the pipe and U is the velocity of the fluid at the wall, which Stokes still leaves open. By 1851, Stokes felt quite sure of the no-slip condition for a viscous fluid at a rigid wall as he explicitly discusses it and uses it in his famous paper in which he derives Stokes law of drag on a sphere at low Reynolds number (Stokes 1851). But he did not remark further on pipe flow.

The naming of Equation (3) as Poiseuille's law is due to Hagenbach (1860) who, after giving the derivation, generously suggested calling it Poiseuille's law: "wir werden daher die obige Formel die POISEUILLE'sche Formel nennen." Jacobson (1860) also calls Equation (3) Poiseuille's law.

Hagenbach (1860) indicates a footnote that explains that Navier (1823) had arrived at a different equation, namely $Q = CPD^3/L$ where C is a constant [Claude Louis Marie Henri Navier (1785–1836)]. It is interesting to note that Thomas Young (1773–1829) tried to summarize existing pressure-drop formulae for flow of liquids in tubes in his Croonian Lecture of 1809 which was aimed at studying various aspects of blood flow, including wave propagation, in living organisms. He also quotes data of Bossut and Du Buat. His equations also give a dependence of Q approximately proportional to D^3. This was apparently a widespread opinion and explains why Bingham (1940) remarks on Poiseuille's work: "It was not a simple thing to go exactly counter to all of the established data and proposed

formulas of the hydraulicians. It made it necessary to use the utmost possible precision."

An aspect of Poiseuille's law that is not explicitly covered in Poiseuille's work is the effect of gravity if the capillary is inclined. For this case Poiseuille's law may be written

$$Q = \frac{\pi D^4}{128\mu}\left(\frac{P}{L} + \rho X\right), \tag{5}$$

where ρ is the density of the fluid and X is the component of body force per unit mass in the direction of flow. All of Poiseuille's tests were carried out on horizontal tubes.

Another poorly documented aspect of Poiseuille flow history concerns who first solved and named the unidirectional flow between parallel plates commonly called two-dimensional Poiseuille flow. The form of Poiseuille's law that is the counterpart to Equation (5) in this case is:

$$q = \frac{H^4}{12\mu}\left(\frac{P}{L} + \rho X\right), \tag{6}$$

where q is defined as the discharge rate in a width H of the flow and H is the spacing of the plates. Poiseuille never mentioned flow between parallel plates, but such flows were well known to Stokes (1898) and were probably derived earlier.

4. HAGEN'S EXPERIMENTS

In 1839, a German hydraulic engineer, Gotthilf Heinrich Ludwig Hagen (1797–1884) of Berlin, published a paper on the flow of water in cylindrical tubes. His results were similar to those of Poiseuille, but less extensive and less accurate. However, they included some entrance effects and observations of the differences between laminar and turbulent flows. In the notation used above, Hagen's expression for the driving pressure difference was assumed to be of the form

$$P = \frac{1}{D^4}(ALQ + BQ^2) \tag{7}$$

where A and B are constants. Hagen found A to be dependent on temperature and expressed it in the form

$$A = a - bT + cT^2 \tag{8}$$

where a, b, and c are experimental constants. Hagen appreciated that the

Q^2 term in (7) was associated with generating the kinetic energy of the fluid and the term linear in Q was a fluid friction resistance. It is readily seen that for sufficiently small values of Q, the Q^2 term in Equation (7) should be negligible. Then solving Equation (7) for Q gives the same form as proposed by Poiseuille. Prandtl & Tietjens (1934) have converted Hagen's measurements of the coefficient A in Equations (7) and (8) to derive a plot of a friction factor vs the Reynolds number, $R_N = DV/v$, where V is the mean velocity and v is the kinematic viscosity. Hagen's data fall very close to the theoretical line $f = 64/R_N$ (where f is the usual pipe friction factor) for a range of Reynolds numbers from about 70 to 1000. The coefficient of viscosity of water extracted from Hagen's data is also shown to agree closely with accepted values. In view of the fact that Hagen's results were quite accurate and preceded publication of Poiseuille's main papers in 1840 and 1841, Prandtl & Tietjens suggest that the laminar flow law should be called the Hagen-Poiseuille law as advocated by Ostwald (1925). It seems, however, that the majority opinion, as expressed by common usage, has settled on calling it Poiseuille's law. There are some points of rationale that can be raised in favor of this decision. It appears that Poiseuille and Hagen worked quite independently and were doing their experiments at about the same time. Their papers do not cross-reference each other's work, but Hagen in 1869 published an article pointing out that his 1839 paper preceded Poiseuille's work (Hagen 1869). Poiseuille's first paper is dated 1838, although his main results were not published until 1840 and 1841.

Hagen's tests were on three brass tubes of diameters 0.255, 0.401, and 0.591 cm and lengths of 47.4, 109, and 105 cm, respectively. In seeking the dependence of the pressure drop on tube diameter, he used a least squares fit to determine the appropriate exponent of the diameter and reported a value of -4.12—but suggested that since the possible errors in the measurements were not exactly known, a value of -4.0 be adopted. In Poiseuille's work, several more different diameters were used and the exponent -4.0 was more definitively established. Bingham (1940) concludes,

> It does not appear that entire historical justice can be done in a name and the coupling of several names together is cumbersome and unnecessary. The greatest importance must be attached to the fact that Poiseuille's paper brought conviction, whereas without it the rheological writings of all the others might have long remained unknown or never have been written.

5.   EXTENSIONS AND USES OF POISEUILLE'S LAW

Historically, one of the interesting uses of Poiseuille's careful experiments was to provide evidence as to the correct boundary condition for a viscous

flow at a solid boundary. Lamb (1932) remarks on the occurrence of the factor D^4 in the formula for discharge rate through a tube: "This last result is of importance as furnishing a conclusive proof that there is in these experiments no appreciable slippage of the fluid in contact with the wall." Lamb goes on to show that if there were slippage at the wall, there would be a correction to Poiseuille's law, which, in fact, Poiseuille's experiments show to be zero within measurable accuracy.

Deeley & Parr (1913) proposed naming the C.G.S. unit of viscosity the "Poise" in honor of Poiseuille. We quote from their paper on the viscosity of glacier ice:

> It would be a distinct advantage to have a name for the unit of viscosity expressed in C.G.S. units, and we would suggest that the word Poise be used for this; for it is to Poiseuille that we owe the experimental demonstration that when a liquid flows through a capillary tube of considerable length, at constant temperature, the viscosity is constant at all rates of shear, provided that the flow is not turbulent. In the case of a soft solid (plastic substance) the so-called viscosity is not the same for all rates of shear: whereas the viscosity of a liquid is a physical constant and should be named.

This usage is then found in the standards literature as early as 1918 (Perry 1955) and in later literature on weights, measures, and units (CGPM 1948, Mechtly 1964).

As a practical matter, the capillary viscometer is a simplified version of Poiseuille's test equipment, and its use is based on Poiseuille's law with interpretation based on Hagenbach's derivation, Equation (3).

It is interesting to take stock of progress made toward Poiseuille's original goal of understanding the laws of pressure distribution in a living circulation of blood. A great deal has been learned about the properties of blood cells and the flow of blood in the 20th century. When all is said and done, Poiseuille's law is a good approximation for blood flow provided the appropriate value for the apparent viscosity is used. Therein lies the rub. Red blood cells are very flexible and at low shear rates they aggregate into stacks called rouleaux. At higher shear stresses, disaggregation and deformation of the cells leads to decreasing viscosity (see Chien et al 1984, for example). Nevertheless, when bulk viscosity measurements are made (in large tubes) and compared to the apparent viscosity derived from tube flows in smaller tubes at the same hematocrit (cell concentration) and mean shear rate, the results agree quite closely for diameters down to about 29 μm (Barbee & Cokelet (1971).

Blood cells tend to move away from blood vessel walls [as observed by Poiseuille (1835) and many others] in small diameter vessels. This leads to a reduction in apparent viscosity as the diameter decreases—known as the Fåhraeus-Lindqvist (1931) effect. However, it has been shown that this result is largely due to the reduction in hematocrit which results from

the centralization of the blood cells (Cokelet 1987). At sufficiently small diameters ($D < 8$ mm) there is a reverse Fåhraeus-Lindqvist effect, namely, the apparent viscosity increases with decreasing capillary diameter because the blood cells fill most of the lumen (Skalak 1990); so Poiseuille's law no longer holds.

As a direct proof of the extent of applicability of Poiseuille's law to the in vivo flow of blood, a summary of measurements due to Lipowsky et al (1978) is shown in Figure 5. A good correlation of the resistance per unit length of vessel is obtained with the exponent of vessel radius close to 4.0. Surely Poiseuille would have been glad to see this!

Poiseuille's law is one of the few equations derived from applied mechanics that is well known in the present medical community. It has been used to model other biological flows, besides blood flow. Pappenheimer (1978) explains how he used it to discuss flow through the endothelial layer
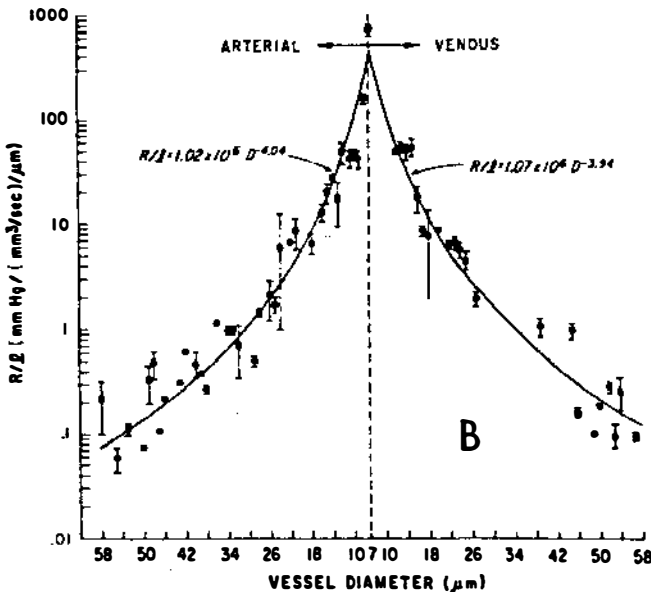

Figure 5   Resistance per unit length of vessel (R/L) where $R = \Delta P/Q$. The resistance R to blood flow is computed from simultaneous measurements of flow Q and pressure drop ΔP, in single unbranched vessels of mesentery. The solid curves are power law regressions of the form $R/L = aD^m$. (From Lipowsky et al 1978, by permission).

of blood vessels to approximate the size of pores or other channels that must exist to account for measured fluid transport.

Similarly, flow through porous media (Batchelor 1967, p. 233) and filters (Skalak et al 1987) have been modeled by defining equivalent Poiscuille flows. Like Stokes drag for sedimenting particles, Poiseuille's law allows an approximate length scale to be defined characterizing the geometry of laminar flows where the geometry is, in fact, much more complex.

Extensions of Poiseuille's law are legion, depending on which definition of an extension is adopted. A broad definition might be classes of flows which, in some limit of the range of the parameters involved, reduce again to Poiseuille's law. Thus, Poiseuille's law is one example of exact solutions of the Navier-Stokes equations (see Wang 1991 for a comprehensive discussion).

A case of interest to blood flow is the exact solution of the sinusoidally oscillatory rectilinear flow of a Newtonian fluid in a circular tube. This solution has also been published independently several times (McDonald 1974), but is known in the blood flow literature through the work of Womersley (1955) and McDonald (1974). At sufficiently low dimensionless frequency [$\alpha = a(\omega/v)^{1/2}$ where a is the tube radius and ω is the frequency], the oscillatory flow velocity profile and pressure gradient approach Poiseuille flow.

Another interesting extension is the so-called Hele-Shaw flows between parallel plates (Hele-Shaw 1898). The theory of these flows was given by Stokes (1898) who showed that the parabolic Poiseuille velocity profile was obtained in each component of velocity parallel to the bounding plates. He also pointed out the analogy of Hele-Shaw flows to two-dimensional inviscid flow.

Extensions and uses of Poiseuille flow listed above are just a few cases which readily come to mind. The authors apologize for neglect of the many additional cases and categories not covered in which the name of Poiseuille is involved. However, one of the most recent such references indicates how far afield Poiseuille's influence has extended (Chamkha 1991). Surely, Poiseuille would be surprised to see his name in the title: "Series solution for unsteady hydromagnetic Poiseuille two-phase flow." It shows how long and far the influence of Poiseuille has been manifest.

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