

Ordered Kripke Model, Permissibility, and Convergence of Probabilistic Kripke Model[☆]

Shuige Liu

Faculty of Political Science and Economics, Waseda University, 1-6-1 Nishi-Waseda, Shinjuku-Ku, 169-8050, Tokyo, Japan

Abstract

We define a modification of the standard Kripke model, called the ordered Kripke model, by introducing a linear order on the set of accessible states of each state. We first show this model can be used to describe the lexicographic belief hierarchy in epistemic game theory, and perfect rationalizability can be characterized within this model. Then we show that each ordered Kripke model is the limit of a sequence of standard probabilistic Kripke models with a modified (common) belief operator, in the senses of structure and the (ε -)permissibilities characterized within them.

Keywords: ordered Kripke model, lexicographic belief, probabilistic Kripke model, permissibility

1. Preliminaries

In this section we give surveys on lexicographic belief and permissibility (Section 1.1) and on probabilistic Kripke model for games (Section 1.2). These will be preparation for the introduction of ordered Kripke model in Section 2.

1.1. Lexicographic belief and permissibility

In this subsection we give a survey on lexicographic epistemic model (with complete information) and the definition of permissibility. For a details, see Perea [16], Chapter 5. Consider a finite 2-person strategic form game $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ where $I = 1, 2$. A finite *lexicographic epistemic model* for G is a tuple $M^{lex} = (\Theta_i, \beta_i)_{i \in N}$ where

- (a) Θ_i is a finite set of types, and
- (b) β_i is a mapping that assigns to every $\theta_i \in \Theta_i$ a lexicographic belief over $\Delta(S_j \times \Theta_j)$, i.e., $\beta_i(\theta_i) = (\beta_{i1}, \beta_{i2}, \dots, \beta_{iK})$ where $\beta_{ik} \in \Delta(S_j \times \Theta_j)$ for $k = 1, \dots, K$.

[☆]The author would like to thank Andrés Perea and Zsombor Z. Méder for their valuable comments and encouragements. She gratefully acknowledge the support of Grant-in-Aids for Young Scientists (B) of JSPS No. 17K13707, Grant for Special Research Project No. 2017K-016 of Waseda University.

Email address: shuige_liu@aoni.waseda.jp (Shuige Liu)

Let $\theta_i \in \Theta_i$ with $\beta_i(\theta_i) = (\beta_{i1}, \beta_{i2}, \dots, \beta_{iK})$. Each β_{ik} ($k = 1, \dots, K$) is called θ_i 's *level- k* belief. For $(s_j, \theta_j) \in S_j \times \Theta_j$, we say θ_i *deems* (s_j, θ_j) *possible* iff $\beta_{ik}(s_j, \theta_j) > 0$ for some $k \in \{1, \dots, K\}$. We say θ_i *deems* $\theta_j \in \Theta_j$ *possible* iff θ_i deems (s_j, θ_j) possible for some $s_j \in S_j$. For each $\theta_i \in \Theta_i$, we denote by $\Theta_j(\theta_i)$ the set of all $\theta_j \in \Theta_j$ deemed possible by θ_i . **Definition 1.1 (Caution)** Type $\theta_i \in \Theta_i$ is *cautious* iff for each $\theta_j \in \Theta_j(\theta_i)$ and each $s_j \in S_j$, it deems (s_j, θ_j) possible.

For each $s_i \in S_i$, let $u_i(s_i, \theta_i) = (u_i(s_i, \theta_{i1}), \dots, u_i(s_i, \theta_{iK}))$ where for each $k = 1, \dots, K$, $u_i(s_i, \theta_{ik}) := \sum_{(s_j, \theta_j) \in C_j \times T_j} \beta_{ik}(s_j, \theta_j) u_i(s_i, s_j)$, that is, each $u_i(s_i, \theta_{ik})$ is the expected utility for s_i over θ_{ik} and $u_i(s_i, \theta_i)$ is a vector of expected utilities. For each $s_i, s'_i \in S_i$, we say that θ_i *prefers* s_i *to* s'_i , denoted by $u_i(s_i, \theta_i) > u_i(s'_i, \theta_i)$, iff there is $k \in \{1, \dots, K\}$ such that the following two conditions are satisfied:

- (a) $u_i(s_i, \theta_{i\ell}) = u_i(s'_i, \theta_{i\ell})$ for $\ell = 1, \dots, k$, and
- (b) $u_i(s_i, \theta_{i,k+1}) > u_i(s'_i, \theta_{i,k+1})$.

We say that θ_i is *indifferent between* s_i *and* s'_i , denoted by $u_i(s_i, \theta_i) = u_i(s'_i, \theta_i)$, iff $u_i(s_i, \theta_{ik}) = u_i(s'_i, \theta_{ik})$ for each $k = 1, \dots, K$. It can be seen that the preference relation on S_i under each type θ_i is a linear order. s_i is *rational* (or *optimal*) for θ_i iff θ_i does not prefer any choice to s_i .

Definition 1.2 (Primary belief in the opponent's rationality) Let $\theta_i \in \Theta_i$ with $\beta_i(\theta_i) = (\beta_{i1}, \beta_{i2}, \dots, \beta_{iK})$. θ_i *primarily believes in j 's rationality* iff θ_i 's primary belief θ_{i1} only assigns positive probability to those (s_j, θ_j) where s_j is rational for θ_j .

Definition 1.3 (Common full belief in a property) Let P be an arbitrary property of lexicographic types.

- (a) $\theta_i \in \Theta_i$ *expresses 0-fold full belief in P* iff θ_i satisfies P ;
- (b) For each $n \in \mathbb{N}$, $\theta_i \in \Theta_i$ *expresses $(n+1)$ -fold full belief in P* iff θ_i only deems possible j 's types that express n -fold full belief in P .

θ_i *expresses common full belief in P* iff it expresses n -fold full belief in P for each $n \in \mathbb{N}$.

Definition 1.4 (Permissibility). Given a lexicographic epistemic model $M^{lex} = (\Theta_i, \beta_i)_{i \in N}$ for a game $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$, $s_i \in S_i$ is *permissible* iff it is optimal to some $\theta_i \in \Theta_i$ which expresses common full belief in caution and primary belief in rationality.

Example 1.1 Consider the game G as follows (Myerson [15]):

$u_1 \backslash u_2$	C	D
A	1, 1	0, 0
B	0, 0	0, 0

and $M^{lex} = (\Theta_i, \beta_i)_{i \in N}$ for G where $\Theta_1 = \{\theta_1\}$, $\Theta_2 = \{\theta_2\}$, and

$$\beta_1(\theta_1) = ((C, \theta_2), (D, \theta_2)), \quad \beta_2(\theta_2) = ((A, \theta_1), (B, \theta_1)).$$

It can be seen that A is permissible since it is optimal to t_1 which expresses common full belief in caution and primary belief in rationality.

It is shown by Proposition 5.2 in Asheim and Dufwenberg [2] that a strategy is permissible if and only if it survives an algorithm called *Dekel-Fudenberg procedure* (Dekel and Fudenberg [12]). Given a game G , by Dekel-Fudenberg procedure we mean the process that (1) at first round we eliminate all weakly dominated strategies in G , and (2) then iteratedly eliminate dominated strategies until no strategies can be eliminated.

1.2. Probabilistic Kripke model for games

In this subsection we give a survey of the probabilistic Kripke model for games which is a generalization of the standard Kripke model that is able to capture both pure and mixed strategies. For details, see Bonanno [6], [7]. Let $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ be a 2-person strategic form game. A *probabilistic Kripke model* of G is a tuple $\mathcal{M} = (W, \{R_i\}_{i \in N}, \{p_i\}_{i \in N}, \{\sigma_i\}_{i \in N})$ where

- (1) $W \neq \emptyset$ is the set of *states* (or *possible worlds*), sometimes called the *domain* of \mathcal{M} and is denoted by $\mathcal{D}(\mathcal{M})$;
- (2) For each $i \in N$, $R_i \subseteq W \times W$ is the *accessibility relation* for player i . For each $w \in W$, we use $R_i(w)$ to denote the set of all accessible states from w , i.e., $R_i(w) = \{w' \in W : w R_i w'\}$;
- (3) For each $i \in N$, p_i is a mapping from W to $\Delta(W)$ satisfying (a) for each $w \in W$, $\text{supp } p_i(w) \subseteq R_i(w)$, and (b) for each $w' \in R_i(w)$, $p_i(w') = p_i(w)$;
- (4) For each $i \in N$, σ_i is a mapping from W to S_i such that for each $w' \in R_i(w)$, $\sigma_i(w') = \sigma_i(w)$.

$(W, \{R_i\}_{i \in N}, \{\sigma_i\}_{i \in N})$ is a *standard Kripke model* of G . $\mathcal{M}^o = (W, \{R_i\}_{i \in N})$ is called the *Kripke frame* of \mathcal{M} . Here we follow the literatures and assume that \mathcal{M}^o is a KD45 frame, i.e., each R_i is serial, transitive, and Euclidean. For each $i \in N$, a *semantic belief operator* is a function $\mathbb{B}_i : 2^W \rightarrow 2^W$ such that for each $E \subseteq W$,

$$\mathbb{B}_i(E) = \{w \in W : R_i(w) \subseteq E\}. \quad (1)$$

A *semantic common belief operator* is a function $\mathbb{CB} : 2^W \rightarrow 2^W$ such that for each $E \subseteq W$,

$$\mathbb{CB}(E) = \{w \in W : \cup_{i \in N} R_i(w) \subseteq E\}. \quad (2)$$

It can be seen that \mathbb{B}_i and \mathbb{CB} correspond to Aumann [3]'s standard concept “knowledge” and “common knowledge”.

At $w \in W$ the strategy $s_i \in S_i$ with s_i is *at least as preferred* to s'_i iff $u_i(s_i, \sum_{w' \in R_i(w)} p_i(w)(w') \sigma_j(w')) \geq u_i(s'_i, \sum_{w' \in R_i(w)} p_i(w)(w') \sigma_j(w'))$. s_i is *preferred* to s'_i at w iff the strict inequality holds. s_i is *optimal* at w iff there is no strategy preferred to s_i at w . A state w is *rational* for i iff $\sigma_i(w)$ is optimal at w . We use RAT_i to denote the set of all rational states for player i , and define $RAT = \cap_{i \in N} RAT_i$.

The following statement connects iterated elimination of pure dominated strategies (an algorithm) to rationality (an epistemic concept). Its proof can be found in Bonanno [7], p.452.

Theorem 1.1 (Iterated elimination of dominated strategies and Kripke model). Let $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ and S^{IEDS} be the set of strategy profiles surviving iterated

elimination of dominated strategies. Then

- (1) given an arbitrary probabilistic Kripke model of G , if $w \in \mathbb{CB}(RAT)$, then $\sigma(w) \in S^{IEDS}$;
- (2) for each $s \in S^{IEDS}$, there is a probabilistic Kripke model of G and a state w such that $\sigma(w) = s$ and $w \in \mathbb{CB}(RAT)$.

2. Ordered Kripke Model of Games and Permissibility

In this section we define the ordered Kripke model as a modification of the standard one and show how it can be used to describe the lexicographic reasoning in game theory.

Definition 2.1 (Ordered epistemic model) Let $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ be a 2-person strategic form game. An *ordered Kripke model* of G is a tuple $\overline{\mathcal{M}} = (W, \{R_i\}_{i \in N}, \{\sigma_i\}_{i \in N}, \{\lambda_i\}_{i \in N})$ where

- (1) $(W, \{R_i\}_{i \in N}, \{\sigma_i\}_{i \in N})$ is a standard Kripke model of G , and
- (2) For each $i \in N$, λ_i assigns to each $w \in W$ an injection from a cut $\{1, \dots, K\}$ of natural numbers to the set of probability distributions (with finite supports) over $R_i(w)$, i.e., $\lambda_i(w) : \{1, \dots, K\} \rightarrow \Delta(R_i(w))$. $\lambda_i(w)$ can be interpreted as a linear order on a finite subset of $\Delta(R_i(w))$. We use $\mathcal{D}(\lambda_i(w))$ and $\mathcal{R}(\lambda_i(w))$ to denote the domain and the range of $\lambda_i(w)$, i.e., $\mathcal{D}(\lambda_i(w)) = \{1, \dots, K\}$ and $\mathcal{R}(\lambda_i(w)) = \{\lambda_i(w)(1), \dots, \lambda_i(w)(K)\}$.

Definition 2.2 (Caution). Let $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ be a strategic form game and $\overline{\mathcal{M}} = (W, \{R_i\}_{i \in N}, \{\sigma_i\}_{i \in N}, \{\lambda_i\}_{i \in N})$ an ordered Kripke model for G . R_i is *cautious* at $w \in W$ iff for any $s_j \in S_j$ ($j \neq i$), there exists w' which is assigned a positive probability by some element in $\mathcal{R}(\lambda_i(w))$ such that $\sigma_j(w') = s_j$. We say $\overline{\mathcal{M}}$ is *cautious* iff for each $i \in N$, R_i is cautious at every $w \in W$.

The difference between the ordered Kripke model and the standard one is that the former assigns a linear order $\lambda_i(w)$ on $R_i(w)$ for each state w . This order is used to define the preferences in the model. We have the following definition.

Definition 2.3 (Lexicographic preferences) Let $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ be a strategic form game and $\overline{\mathcal{M}} = (W, \{R_i\}_{i \in N}, \{\sigma_i\}_{i \in N}, \{\lambda_i\}_{i \in N})$ an ordered Kripke model for G . At $w \in W$ the strategy $s_i \in S_i$ is *at least as lexicographically preferred to* s'_i , denoted by $s_i \succeq_w s'_i$, iff $\exists k \in \{0, \dots, |\mathcal{D}(\lambda_i(w))|\}$ such that

- (a) $u_i(s_i, \sigma_j(\sigma_j(\lambda_i(w)(t)))) = u_i(s'_i, \sigma_i(\lambda_i(w)(t)))$ for all $t \leq k$;
- (b) $u_i(s_i, \sigma_j(\lambda_i(w)(k+1))) = u_i(s'_i, \sigma_i(\lambda_i(w)(k+1)))$.

Here by $\sigma_j(\lambda_i(w)(t))$ we mean the mixture of strategies in $\sigma_j(\lambda_i(w)(t))$. Therefore

$$u_i(s_i, \sigma_j(\sigma_j(\lambda_i(w)(t)))) = \sum_{w' \in R_i(w)} \lambda_i(w)(t)(w') u_i(s_i, \sigma_i(w')).$$

It can be seen that when $k = |\mathcal{D}(\lambda_i(w))|$, s_i and s'_i generates the same payoff for player i along $\lambda_i(w)$. This case is denoted by $s_i \simeq_w s'_i$. When $k \neq |\mathcal{D}(\lambda_i(w))|$, we say that s_i is *lexicographically preferred to* s'_i at w , denoted by $s_i \succ_w s'_i$. s_i is *optimal* at w iff there is no $s'_i \in S_i$ such that $s'_i \succ_w s_i$. We say a state w is *lexicographically rational* for i iff the choice $\sigma_i(w)$ is optimal for i . For each $i \in N$, let $LRAT_i$ be the set of rational states for player i

and $LRAT = \cap_{i \in N} LRAT_i$.

Example 2.1. Consider the following game G in Example 1.1:

$u_1 \backslash u_2$	C	D
A	1, 1	0, 0
B	0, 0	0, 0

and an ordered Kripke model $\overline{\mathcal{M}}$ as follows:

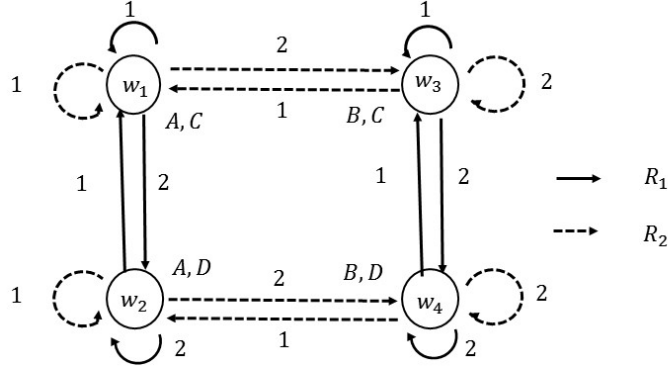


Figure 1: An ordered Kripke model for G

It can be seen that $\overline{\mathcal{M}}$ is cautious. It can be seen that A and C are optimal in each state, w_1 and w_2 are rational for player 1, and w_1 and w_3 are rational for player 2. Therefore, $LRAT_1 = \{w_1, w_2\}$, $LRAT_2 = \{w_1, w_3\}$, and $LRAT = \{w_1\}$. On the other hand, as mentioned in Example 1.1, since both $\sigma_1(w_2) = A$ and $\sigma_2(w_2) = D$ are permissible strategies, lexicographic rationality in the ordered Kripke model here captures the concept of “a strategy is rational under a lexicographic belief” in the first order. Now the problem is how to define belief hierarchy and common belief in this model. It can be seen that we cannot adopt \mathbb{B}_i and \mathbb{CB} in standard approach. Indeed, in Example 2.1 $\mathbb{B}_i(LRAT) = \mathbb{CB}(LRAT) = \emptyset$, which is contradictory to our intention to preserve w_2 . Here we give one approach. For each $i \in N$ and $w \in W$, let $R_i^1(w) = \{w' \in W : \lambda_i(w)(1)(w') > 0\}$ and $R^1 = \cup_{i \in N} R_i^1$. A *semantic level-1 belief operator* for player i is a mapping $\mathbb{B}_i^1 : 2^W \rightarrow 2^W$ such that for each $E \subseteq W$,

$$\mathbb{B}_i^1(E) = \{w \in W : R_i^1(w) \subseteq E\}. \quad (3)$$

Similarly, a *semantic common level-1 belief operator* is a mapping $\mathbb{CB}^1 : 2^W \rightarrow 2^W$ such that for each $E \subseteq W$,

$$\mathbb{CB}^1(E) = \{w \in W : \cup_{i \in N} R_i^1(w) \subseteq E\}. \quad (4)$$

It can be seen that $\mathbb{B}_i^1(LRAT) = \mathbb{CB}^1(LRAT) = \{w_1\}$ in Example 2.1. In general, we have the following result.

Theorem 2.1 (Permissibility and semantic common level-1 belief). Let $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ be a strategic form game and $S^{PER} \subseteq S$ be the set of permissible strategy profiles. Then

- (1) given an arbitrary cautious ordered Kripke model of G , if $w \in \mathbb{CB}^1(LRAT)$, then $\sigma(w) \in S^{PER}$, and
- (2) for each $s \in S^{PER}$, there exists a cautious ordered Kripke model of G such that $\sigma(w) = s$ and $w \in \mathbb{CB}^1(LRAT)$.

To show Theorem 2.1, we need the following lemma.

Lemma 2.1. Let $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ be a strategic form game and $S^{DF} \subseteq S$ be the set of strategy profiles surviving Dekel-Fudenberg procedure. Then given an arbitrary cautious ordered epistemic model of G , if $w \in \mathbb{CB}^1(LRAT)$, then $\sigma(w) \in S^{DF}$.

Proof. For each $n \in \mathbb{N}$, we use S^{DFn} to denote the set of strategy profiles surviving the first n rounds of Dekel-Fudenberg procedure. Let $\bar{M} = (W, \{R_i\}_{i \in N}, \{\sigma_i\}_{i \in N}, \{\lambda_i\}_{i \in N})$ be a cautious ordered epistemic model of G and $w \in W$. We show that if $w \in \mathbb{CB}^1(LRAT)$, then $\sigma(w) \in S^{DFn}$ for each $n \in \mathbb{N}$. First, since \bar{M} is cautious, it can be seen that $\sigma(w) \in S^{DF1}$. Indeed, if there is some $i \in N$ such that $\sigma_i(w)$ is eliminated in the first round of Dekel-Fudenberg procedure, then there is some $r_i \in \Delta(S_i)$. Then it follows from Theorem 5.8.3 in Perea [16] (p.215, 221-226) $\sigma_i(w)$ cannot be optimal to any cautious belief, i.e., it cannot be optimal on $\lambda_i(w)$, which is contradictory since $w \in LRAT_i$.

Now we show that $\sigma(w) \in S^{DF2}$. Suppose for some $i \in N$, $\sigma_i(w)$ is eliminated in the second round of Dekel-Fudenberg procedure, i.e., there exists $r_i \in \Delta(S_i^{DF1})$ such that $u_i(r_i, s_j) > u_i(s_i, s_j)$ for all $s_j \in S_j^{DF1}$. On the other hand, since $w \in \mathbb{CB}^1(LRAT)$, $\sigma_i(w)$ is optimal to $\lambda_i(w)(1)$. This implied that some strategies supporting $\lambda_i(w)(1)$ has been eliminated in the first round. However, since $w \in \mathbb{CB}^1(LRAT)$, it follows from the definition that $\text{supp } \lambda_i(w)(1) \subseteq LRAT$, which, from the argument above, implies that all strategies $w' \in \text{supp } \lambda_i(w)(1)$ should have survived the first round and $\sigma_j(w')$ stay in S_j^{DF1} , a contradiction.

Now suppose that $\sigma(w) \in S^{DF1} \cap \dots \cap S^{DFn}$ but disappeared in S^{DFn+1} . This could happen only if some strategies supporting $\lambda_i(w)(1)$ had been eliminated in the n -th round, which is because some strategies supporting that strategy in $\lambda_i(w)(1)$ in $(n-1)$ -th round, etc. Finally this leads to the second and first rounds, which, by the argument above, is impossible. Therefore $\sigma(w) \in S^{DFn+1}$. //

Proof of Theorem 2.1: (1) Since, by Proposition 5.2 in Asheim and Dufwenberg [2], any strategy surviving Dekel-Fudenberg procedure is permissible and vice versa, i.e., $S^{PER} = S^{DF}$, (1) directly follows from Lemma 2.1.

(2) Let $s \in S^{PER}$, that is, for each $i \in N$, s_i is optimal to some type expressing common full belief in caution and primary belief in rationality in a lexicographic epistemic model $M^{lex} = (T_j, b_j)_{j \in N}$. We construct an ordered Kripke model $\bar{M} = (W, \{R_i\}_{i \in N}, \{\sigma_i\}_{i \in N}, \{\lambda_i\}_{i \in N})$ based on M^{lex} as follows:

- (1) Let $W = T \times S$, here $T = \prod_{i \in N} T_i$;
- (2) for each $w = (t_1, t_2, s_1, s_2)$, $\sigma_i(w) = s_i$;
- (3) Connectiong each state in W according to M^{lex} , i.e., for each $w = (t_1, t_2, s_1, s_2)$, $w' = (t'_1, t'_2, s'_1, s'_2) \in T \times S$, $w' = \lambda_i(w)(k)$ iff $t_i = t'_i$, $s_i = s'_i$, and (s'_j, t'_j) is the k -th entry in $b_i(t_i)$; mixed strategy-type pairs are defined in a similar way.

Without loss of generality, we can assume that each type in M^{lex} is cautious.¹ It can be seen that $\overline{\mathcal{M}}$ is also cautious, and there is $w \in W$ with $\sigma(w) = s$ and $w \in \mathbb{CB}^1(LRAT)$. //

3. Ordered Kripke Model as the Limit of Probabilistic Kripke Models

Though the ordered Kripke model is not the first framework combining standard Kripke model with an order on (a subset of) each $R_i(w)$ (cf. Baltag and Smets [4], [5]), here we are interested in how such a model can be connected to the probabilistic Kripke model for games introduced in Section 1. In this section we will first introduce a probabilistic Kripke model with modified belief operators under which ε -perfect rationalizability can be characterized. Then we will show that an ordered Kripke model can be seen as a “limit” of a sequence of probabilistic Kripke models.

3.1. Probabilistic belief and ε -perfect rationalizability

In this subsection we give a survey on probabilistic epistemic model (with complete information) and the definition of ε -perfect rationalizability. See Perea [16], Chapter 2 for the detail of the former. ε -permissible, which originates from Myerson [15], is defined in a similar way as ε -proper rationalizability as in Schuhmacher [18] and Perea and Roy [17]. Consider a finite 2-person strategic form game $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$. A finite *probabilistic epistemic model* for G is a tuple $M^{pro} = (T_i, b_i)_{i \in N}$ where

- (a) T_i is a finite set of types, and
- (b) b_i is a mapping that assigns to every $t_i \in T_i$ a probability distribution over $\Delta(S_j \times T_j)$.

For each $s_i \in S_i$ and $t_i \in T_i$, we define $u_i(s_i, t_i) = \sum_{(s_j, t_j) \in S_j \times T_j} b_i(t_i)(s_j, t_j) u_i(s_i, s_j)$. s_i is *optimal* (or *rational*) for t_i iff $u_i(s_i, t_i) \geq u_i(s'_i, t_i)$ for any $s'_i \in S_i$. For each $s_i, s'_i \in S_i$ and $t_i \in T_i$, s_i is *preferred to* s'_i under t_i iff $u_i(s_i, t_i) > u_i(s'_i, t_i)$. Given $t_i \in T_i$, for each $(s_j, t_j) \in S_j \times T_j$, we say t_i *deems* (s_j, t_j) *possible* iff $b_i(t_i)(s_j, t_j) > 0$. We say t_i *deems* $t_j \in T_j$ *possible* iff t_i deems (s_j, t_j) possible for some $s_j \in S_j$. For each $t_i \in T_i$, we denote by $T_j(t_i)$ the set of all t_j 's deemed possible by t_i .

Definition 3.1 (Caution) Type $t_i \in T_i$ is *cautious* iff for each $t_j \in T_j(t_i)$ and each $s_j \in S_j$, t_i deems (s_j, t_j) possible.

Definition 3.2 (ε -perfect trembling condition) Type $t_i \in T_i$ satisfies *ε -perfect trembling condition* iff for any $s_j \in S_j$ and $t_j \in T_j(t_i)$ such that t_i deems (s_j, t_j) possible, if s_j is not optimal under t_j then $b_i(t_i)(s_j, t_j) \leq \varepsilon$.

Definition 3.3 (Common full belief in a property) Let P be an arbitrary property of probabilistic types.

- (a) $t_i \in T_i$ *expresses 0-fold full belief in* P iff t_i satisfies P ;
 - (b) For each $n \in \mathbb{N}$, $t_i \in T_i$ *expresses $(n+1)$ -fold full belief in* P iff t_i only deems possible j 's types that express n -fold full belief in P .
- t_i *expresses common full belief in* P iff it expresses n -fold full belief in P for each $n \in \mathbb{N}$.

¹For a state that is not cautious we can extend it into a cautious one. See Liu [14].

Definition 3.4 (ε -Perfect rationalizability). Given a probabilistic epistemic model $M^{pro} = (T_i, b_i)_{i \in N}$ for a game $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$, $s_i \in S_i$ is ε -permissible iff it is optimal to some $t_i \in T_i$ which expresses common full belief in caution and ε -perfect trembling condition.

Example 3.1. Consider the following game G (from Myerson [15]):

$u_1 \backslash u_2$	C	D
A	1, 1	0, 0
B	0, 0	0, 0

and $M^{pro} = (T_i, b_i)_{i \in N}$ for G where $T_1 = \{t_1\}$, $T_2 = \{t_2\}$, and

$$b_1(t_1) = (1 - \varepsilon)(C, t_2) + \varepsilon(D, t_2), \quad b_2(t_2) = (1 - \varepsilon)(A, t_1) + \varepsilon(B, t_1),$$

where $\varepsilon \in (0, 1)$. It can be seen that A is ε -permissible since it is optimal to t_1 which expresses common full belief in caution and ε -perfect trembling condition.

Originally, by the definition in Myerson [15], perfect equilibrium is the limit of ε -perfect equilibrium. Though permissibility is the concepts in epistemic game theory and is defined in a different manner, it still holds tha permissibility is the limit of ε -permissibility. See Schuhmacher [18].

3.2. Characterizing ε -permissibility in probabilistic Kripke model

In this subsection, we show how to use probabilistic Kripke model to describe ε -permissibility. Let $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ be a 2-person strategic form game and $\mathcal{M} = (W, \{R_i\}_{i \in N}, \{p_i\}_{i \in N}, \{\sigma_i\}_{i \in N})$ a probabilistic Kripke model for G . We give the following definitions

Definition 3.5 (Caution). \mathcal{M} is *cautious* at $w \in W$ for $i \in N$ iff for any $s_j \in S_j$ ($j \neq i$), there exists $w' \in R_i(w)$ satisfying $p_i(w)(w') > 0$ and $\sigma_j(w') = s_j$. We say \mathcal{M} is *cautious* iff \mathcal{M} is cautious at every $w \in W$ for each $i \in N$.

Definition 3.6 (ε -perfect trembling condition). \mathcal{M} satisfies ε -perfect trembling condition at $w \in W$ for $i \in N$ iff for each $w' \in R_i(w)$, if $\sigma_i(w')$ (i.e., $\sigma_i(w)$) is not optimal to $\sigma_j(w')$, then $p_i(w)(w') \leq \varepsilon$.

The above two concepts are illustrated in the following example.

Example 3.1. Consider the game G in Example 1.1:

$u_1 \backslash u_2$	C	D
A	1, 1	0, 0
B	0, 0	0, 0

and a probabilistic Kripke model depicted as in Figure 2.

It can be seen that $RAT_1 = \{w_1, w_2\}$, $RAT_2 = \{w_1, w_3\}$, and $RAT = RAT_1 \cap RAT_2 = \{w_1\}$. Since $\sigma(w_1) = (A, C)$ is a pair of ε -perfect rationalizable strategies, RAT can still be used in this framework for the first-order. Now the problem is how to describe higher orders, i.e., interpersonal belief and common full belief in this framework. \mathbb{B}_i and \mathbb{CB} in

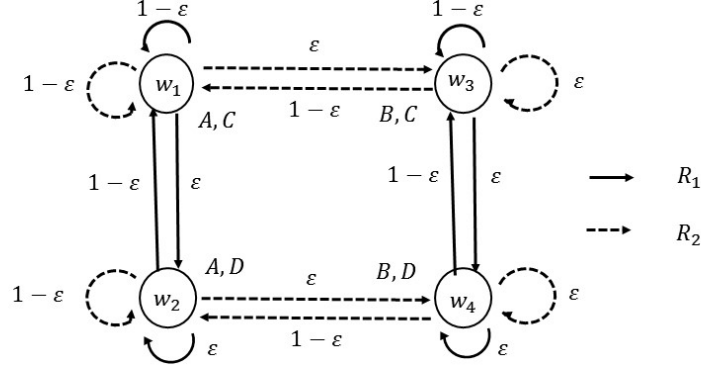


Figure 2: An probabilistic Kripke model for G

standard probabilistic model do not work here since $\mathbb{B}_i(RAT) = \mathbb{CB}(RAT) = \emptyset$, while we want to keep w_1 . Here we provide an approach. Let $\mathcal{M} = (W, \{R_i\}_{i \in N}, \{p_i\}_{i \in N}, \{\sigma_i\}_{i \in N},)$ a probabilistic Kripke model for G satisfying caution and ε -perfect trembling condition. For each $i \in N$, we define $R_i^{>\varepsilon} = \{w' \in R_i(w) : p_i(w)(w') > \varepsilon\}$. An *upper ε semantic belief operator* is a function $\mathbb{B}_i^{>\varepsilon} : 2^W \rightarrow 2^W$ such that for each $E \subseteq W$,

$$\mathbb{B}_i^{>\varepsilon}(E) = \{w \in W : R_i^{>\varepsilon}(w) \subseteq E\}. \quad (5)$$

An *upper ε semantic common belief operator* is a function $\mathbb{CB}^{>\varepsilon} : 2^W \rightarrow 2^W$ such that for each $E \subseteq W$,

$$\mathbb{CB}^{>\varepsilon}(E) = \{w \in W : \cup_{i \in N} R_i^{>\varepsilon}(w) \subseteq E\}. \quad (6)$$

When $\varepsilon < \frac{1}{2}$, it can be seen that in Example 3.1, $\mathbb{CB}^{>\varepsilon}(RAT) = \{w_1\}$.

In general, we have the following statement.

Theorem 3.1 (Characterizing ε -perfect rationalizability). Let $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ be a 2-person strategic form game, $\varepsilon < \frac{1}{2}$,² and $S^{\varepsilon PER} \subseteq S$ be the set of ε -permissible strategy profiles. Then

- (1) given an arbitrary probabilistic Kripke model of G satisfying caution and ε -perfect trembling condition, if $w \in \mathbb{CB}^{>\varepsilon}(RAT)$, then $\sigma(w) \in S^{\varepsilon PER}$, and
- (2) for each $s \in S^{\varepsilon PER}$, there exists a probabilistic Kripke model of G satisfying caution and ε -perfect trembling condition such that $\sigma(w) = s$ and $w \in \mathbb{CB}^{>\varepsilon}(RAT)$.

Proof. (2) can be proved in a similar way as Theorem 2.1. Here we only prove (1). Since there is no algorithm like Dekel-Fudenberg procedure that can screen out ε -permissibility, we show how to construct a type which expresses common full belief in caution and ε -perfect trembling condition. Let $\mathcal{M} = (W, \{R_i\}_{i \in N}, \{p_i\}_{i \in N}, \{\sigma_i\}_{i \in N})$ be a probabilistic Kripke model for G satisfying caution and ε -perfect trembling condition and $w \in \mathbb{CB}^{>\varepsilon}(RAT)$. For $i \in N$, we define a partition $\mathbb{E}_i = \{E_{i1}, \dots, E_{i\ell_i}\}$ of W and satisfies that for each $w', w'' \in W$,

²It should be noted that $\varepsilon < \frac{1}{2}$ makes sure that $R_i^{>\varepsilon}(w) \neq R_i(w)$, though from the viewpoint of convergence/limit this is just a technical requirement.

w' and w'' belong to the same equivalent class E_i if and only if $R_i(w') = R_i(w'')$ and $p_i(w') = p_i(w'')$. For each $E_i \in \mathbb{E}_i$ we assign a symbol $t_i(E_i)$. Without loss of generality, we can assume that for each $s_i \in S_i$ and each E_{ik} , there is some $w' \in E_{ik}$ such that $\sigma_i(w') = s_i$.³ Let $T_i = \{t_i(E_i)\}_{E_i \in \mathbb{E}_i}$, and define $b_i(t_i(E_i))$ with the same probability as $p_i(w')$, where $w' \in E_i$, and the corresponding $t_j(E_j)$. It can be seen that $b_i(t_i(E_i))$ is well-defined since every state in one E_i has identified distributions. It can be seen straightforwardly that $\sigma(w) \in S^{\varepsilon PER}$ since for each $i \in N$, $\sigma_i(w)$ is optimal to the type corresponding to w which expresses common full belief in caution and ε -perfect trembling condition since \mathcal{M} satisfies caution and ε -perfect trembling condition. //

3.3. Probabilistic Kripke models converge to ordered Kripke model

In this subsection we show that any ordered Kripke model is the limit of a sequence of probabilistic Kripke models. This can be intuitively seen by comparing the two Kripke models in Example 2.1 and 3.1: Indeed, Figure 2 can be obtained by replacing $1-\varepsilon$ with 1 and ε with 2 in Figure 1. Also, this can be seen from that perfect rationalizability characterized by the former is the limit of a sequence of ε -perfect rationalizabilities characterized by the later. In this section we show how to formulate this idea.

Let $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ be a 2-person strategic form game and $\overline{\mathcal{M}} = (W, \{R_i\}_{i \in N}, \{\sigma_i\}_{i \in N}, \{\lambda_i\}_{i \in N})$ be an ordered Kripke model of G . Without loss of generality, we assume that $\overline{\mathcal{M}}$ satisfies the following two conditions:

(Disjoint supports) For each $i \in N$, $w \in W$, and $k, k' \in \mathcal{D}(\lambda_i(w))$, $\text{supp } \lambda_i(w)(k) \cap \text{supp } \lambda_i(w)(k') \neq \emptyset$ if and only if $k \neq k'$;⁴

(Surjection) For each $w \in W$ and each $w' \in R_i(w)$, there is some $k \in \mathcal{D}(\lambda_i(w))$ such that $\lambda_i(w)(k)(w') > 0$.⁵

Let $\varepsilon \in (0, 1)$. Consider a probabilistic Kripke model $\mathcal{M}^\varepsilon = (W^\varepsilon, \{R_i^\varepsilon\}_{i \in N}, \{p_i^\varepsilon\}_{i \in N}, \{\sigma_i^\varepsilon\}_{i \in N})$ of G satisfying

- (a) $W^\varepsilon = W$, $R_i^\varepsilon = R_i$ and $\sigma_i^\varepsilon = \sigma_i$ for each $i \in N$;
- (b) for each $i \in N$, $w \in W$, and $w', w'' \in \text{supp } \lambda_i(w)(k)$ for some $k \in \mathcal{D}(\lambda_i(w))$, it is satisfied that $p_i^\varepsilon(w)(w')/p_i^\varepsilon(w)(w'') = \lambda_i(w)(k)(w')/\lambda_i(w)(k)(w'')$;
- (c) for each $i \in N$, $w \in W$, and $w' \in R_i(w)$, $0 < p_i^\varepsilon(w)(w') \leq \varepsilon$ if $\lambda_i(w)(1)(w') = 0$.

³This corresponds to caution for probabilistic epistemic model M^{pro} . It should be noted that even this condition is not satisfied, we can construct “dummies” to make this condition satisfied without hurt the model.

⁴This condition is adopted in some papers such as Blume et al. [8], [9] while is not required in some others such as the standard textbook of Perea [16]. Technically, this condition is not necessary in characterizing rationalizabilities. Here we use it out of simplification.

⁵Surjection is different from caution. Caution requires each strategy of the opponent should appear in the range. When there are multiple states in $R_i(w)$ which are assigned the same strategy, to be cautious only means that at least one of those state should appear in the range, while surjection requires that each of these states should appear. On the other hand, it does not mean that surjection implies caution since surjection has nothing with strategies assigned to each state in $R_i(w)$. In finite models, when surjection is not satisfied, we can faithfully extend each model into one which satisfies surjection without hurting $LRAT$ and $\mathbb{CB}(LRAT)$.

It can be seen that when ε is small enough, such \mathcal{M}^ε exists (not unique). Let $\{\varepsilon_n\}_{n \in \mathbb{N}}$ be a sequence in $(0, 1)$ that converges to 0 such that for each ε_n , there is some probabilistic Kripke model satisfying (a) - (c). We choose an arbitrary $\mathcal{M}^{\varepsilon_n}$ satisfying (a) - (c) for each ε_n . It can be seen that the sequence $\{\mathcal{M}^{\varepsilon_n}\}_{n \in \mathbb{N}}$ “converges” to $\overline{\mathcal{M}}$ in the sense that

(1) for each $i \in N$, $w \in W$, and $w' \in R_i(w)$ such that $p_i^{\varepsilon_n}(w)(w') \rightarrow 0$, w' does not appear in $\lambda_i(w)(1)$;

(2) for each $i \in N$, $w \in W$, and $w' \in R_i(w)$ such that $p_i^{\varepsilon_n}(w)(w') \nrightarrow 0$, $w' \in \text{supp } \lambda_i(w)(1)$ and $p_i^{\varepsilon_n}(w)(w') \rightarrow \lambda_i(w)(1)(w')$;

(3) The convergence is propotional within each level of $\lambda_i(w)$.⁶

More formally, this convergence can be seen from the rationalizabilities they characterize. We have the following statement.

Theorem 3.2 (Probabilistic models converge to ordered model). Let G be a 2-person strategic game, $\overline{\mathcal{M}}$ a cautious ordered Kripke model of G satisfying disjoint supports and surjection, $\{\varepsilon_n\}_{n \in \mathbb{N}}$ a sequence in $(0, 1)$ converging to 0, and $\{\mathcal{M}^{\varepsilon_n}\}_{n \in \mathbb{N}}$ be a sequence of probabilistic model of G satisfying condition (a) - (c) above for each ε_n . Then each w which is commonly believed to be lexicographically rational in $\overline{\mathcal{M}}$ is commonly believed to be ε_n -upper rational in $\mathcal{M}^{\varepsilon_n}$ for each ε_n , i.e., $\mathbb{CB}^1(LRAT) = \bigcup_{M \in \mathbb{N}} \bigcap_{n > M} \mathbb{CB}^{>\varepsilon_n}(RAT_{\varepsilon_n})$.

Proof. (\subseteq) Let $w \in \mathbb{CB}^1(LRAT)$. It follows from Theorem 2.1 that $\sigma(w) \in S^{PER}$, that is, there is a lexicographic model $(\Theta_i, \beta_i)_{i \in N}$ such for each $i \in N$, $\sigma_i(w)$ is optimal to some $\theta_i \in \Theta_i$ which expresses common full belief in caution and primary belief in rationality. Based on each $\mathcal{M}^{\varepsilon_n}$, θ_i can be accordingly translated into a state $t_i^{\varepsilon_n}$ “starting” from w in probabilistic model. Since $\sigma_i(w)$ is optimal to some θ_i , when ε_n is small enough, $\sigma_i(w)$ is optimal to $t_i^{\varepsilon_n}$, and $t_i^{\varepsilon_n}$ expresses common full belief on caution and ε_n -perfect trembling condition. This argument holds for each $i \in N$. Then by Theorem 3.1 $\sigma(w) \in \mathbb{CB}^{>\varepsilon_n}(RAT_{\varepsilon_n})$, and consequently $w \in \bigcup_{M \in \mathbb{N}} \bigcap_{n > M} \mathbb{CB}^{>\varepsilon_n}(RAT_{\varepsilon_n})$.

(\supseteq) Let $w \in \bigcup_{M \in \mathbb{N}} \bigcap_{n > M} \mathbb{CB}^{>\varepsilon_n}(RAT_{\varepsilon_n})$, that is, for some $M \in \mathbb{N}$, $w \in \mathbb{CB}^{>\varepsilon_n}(RAT_{\varepsilon_n})$ for all $n \geq M$. Since $\varepsilon_n \rightarrow 0$, it follows that for each $i \in N$, $\sigma_i(w)$ is optimal on $p_i^{\varepsilon_n}(w)$ for infinitely small ε_n . Since each $\mathcal{M}^{\varepsilon_n}$ keeps the propotion between states within each level of $\lambda_i(w)$, this implies that $\sigma_i(w)$ is optimal to $\lambda_i(w)$. Also, the state in the probabilistic model of G corresponding to $\mathcal{M}^{\varepsilon_n}$ supporting $\sigma_i(w)$ expresses common full belief in caution and ε -perfect trembling condition. Since $\overline{\mathcal{M}}$ is surjective, it follows that the corresponding type in the lexicographic model for $\overline{\mathcal{M}}$ expresses common full belief in caution and primary belief in rationality. Therefore $\sigma_i(w)$ is perfect rationalizable in $\overline{\mathcal{M}}$. Since this argument holds for all $i \in N$, it follows that $w \in \mathbb{CB}^1(LRAT)$. //

⁶To dealing those “irrational” choice which is assigned in probability 0 in any rational belief is one of the motivation for the introduce of lexicographic belief and studies from conditional probability. See Blume et al. [8], [9], Brandenburger et al. [11], Halpern [13].

4. Concluding Remarks

4.1. Convergence and proper rationalizability

In Section 2, we characterized permissibility by ordered Kripke model. Though it is desirable to characterize other rationalizability concepts, e.g., proper rationalizability (Asheim [1]), in the ordered Kripke model, we think it is difficult, if not impossible. The reason is that in this framework, the difference between perfect and proper rationalizabilities is at what kind of order $\lambda_i(w)$ gives on $R_i(w)$, which more relies on the interpretation than on the structure. In other words, by changing the order on accessible states we can characterize proper rationalizability; but this is attributed to the interpretation we give to each state, not to any structural properties of the Kripke frame $(W, \{R_i\}_{i \in N})$ like seriality or transitivity.

On the other hand, using the approach introduced in Section 3.3, proper rationalizability can be discussed as the limit of probabilistic Kripke models. Let G be a 2-person strategic form game and a cautious ordered Kripke model $\overline{\mathcal{M}}$ satisfies a condition parallel to “respecting the opponent’s preferences”. For $\varepsilon > 0$, consider $\mathcal{M}^\varepsilon = (W^\varepsilon, \{R_i^\varepsilon\}_{i \in N}, \{p_i^\varepsilon\}_{i \in N}, \{\sigma_i^\varepsilon\}_{i \in N})$ a probabilistic Kripke model of G satisfying conditions (a), (b) in Section 3.3 and (c’) for each $i \in N$, $w \in W$, and $w', w'' \in R_i(w)$, $0 < p_i^\varepsilon(w)(w') \leq \varepsilon p_i^\varepsilon(w)(w'')$ if $\lambda_i(w)(k')(w') > 0$, $\lambda_i(w)(k'')(w'') > 0$, and $k'' > k'$. It can be seen that (1) each probabilistic Kripke model satisfying (a), (b), and (c’) characterizes some ε -perfect rationalizable strategies; (2) $\overline{\mathcal{M}}$ is the limit of a sequence $\{\mathcal{M}^{\varepsilon_n}\}_{n \in \mathbb{N}}$ satisfying (a), (b), and (c’); and (3) the perfect rationalizable strategies characterized in $\overline{\mathcal{M}}$ is limits of ε_n -perfect rationalizable strategies characterized by $\{\mathcal{M}^{\varepsilon_n}\}_{n \in \mathbb{N}}$.

4.2. Syntactical system

In this paper we have defined the ordered Kripke model to capture the concept of rationality under lexicographic belief hierarchy by a semantical approach. It is wondered that whether there exists a syntactic approach corresponding to that semantic framework, like the one developed in Bonanno [6] for the standard Kripke model for games. A critical property for that syntactic system, if exists, is that the change of the criterion for truth value from the first order to higher orders in the hierarchy, that is, in the first order we need (at most) to check every accessible state, while in the second order \mathbb{B}_i^1 we need only to check the first level states, etc. Works are expected in this direction.

References

References

- [1] Asheim, G.B., (2001). Proper rationalizability in lexicographic beliefs. *International Journal of Game Theory* **30**, 453-478.
- [2] Asheim, G.B., Dufwenberg, M. 2003. Admissibility and common belief. *Games and Economic Behavior* **42**, 208-234.
- [3] Aumann, R., 1976. Agreeing to disagree. *The Annals of Statistics* **4**, 1236-1239.
- [4] Baltag, A., Smets, A., 2006. Conditional doxastic models: a qualitative approach to dynamic belief revision. *Electronic Notes in Theoretical Computer Science* **165**, 5-21.

- [5] Baltag, A., Smets, A., 2007. From conditional probability to the logic of doxastic actions. *Proceedings of TARK XI*, Samet, D. ed, 52-61.
- [6] Bonanno, G., 2008. A syntactic approach to rationality in games with ordinal payoffs, In *Logic and the Foundation of Game and Decision Theory (LOFT 7)*, volumn 3 of *Texts in Logic and Games*, Bonanno, D., van der Hoek, W., Wooldridge, M., eds, Amsterdam University Press, 59-86.
- [7] Bonanno, G., 2015. Epistemic foundation of game theory, Chapter 9 of *Handbook of Epistemic Logic*, van Ditmarsch, H., Halpern, J.Y., van der Hoek, W., Kooi, B., eds, College Publications, 443-487.
- [8] Blume, L., Brandenburger, A., Dekel, E., 1991. Lexicographic probabilities and choice under uncertainty. *Econometrica* 59, 61-79.
- [9] Blume, L., Brandenburger, A., Dekel, E., 1991. Lexicographic probabilities and equilibrium refinements. *Econometrica* 59, 81-98.
- [10] Brandenburger, A. (1992). Lexicographic probabilities and iterated admissibility. In *Economic Analysis of Markets and Games*, ed. P. Dasgupta, et al. MIT Press, 282-290.
- [11] Brandenburger, A., Friedenberg, A., Keisler, H.J., 2007. Notes on the relationship between strong belief and assumption, working paper.
- [12] Dekel, E., Fudenberg, D., 1990. Rational behvaior with payoff uncertainty. *Journal of Economic Theory* 89, 165-185.
- [13] Halpern, J.Y., 2010. Lexicographic probability, conditional probability, and nonstandard probability. *Games and Economic Behavior* 68, 155-179.
- [14] Liu, S. (2018). Characterizing permissibility and proper rationalizability by incomplete information. EPICENTER Working Paper No.14, Maastricht University.
- [15] Myerson, R.B., 1978. Refinements of the Nash Equilibrium Concept. *International Journal of Game Theory* 7, 73-80.
- [16] Perea, A., 2012. *Epistemic Game Theory: Reasoning and Choice*. Cambridge University Press.
- [17] Perea, A., Roy, S., 2017. A new epistemic characterization of ε -proper rationalizability. *Games and Economic Behavior* 104: 309-328.
- [18] Schuhmacher, F., 1999. Proper rationalizability and backward induction. *International Journal of Game Theory* 28, 599-615.