Chapter 2
Arrays

2.2 The Array as an Abstract Data Type

Array:

- **A set of pairs: <index, value> (correspondence or mapping)**
- **Two operations: retrieve, store**

Now we will use the C++ class to define an ADT.

ADT2.1 GeneralArray

class GeneralArray **{**

// a set of pairs <index, value> where for each value of

// index in IndexSet there is a value of type **float**. IndexSet is

// a finite ordered set of one or more dimensions.

public:

GeneralArray(**int** j, RangeList list, **float** initValue = defaultValue)**;**

// The constructor GeneralArray creates a j

// dimensional array of floats; the range of the kth

// dimension is given by the kth element of list.

// For all $i \in$ IndexSet, insert <i, initValue> into the array.

```
float  Retrieve(index i);
// if (i∈IndexSet) return the float associated with i in the
//  array;else throw an exception.

void Store(index i, float x);
// if (i∈IndexSet) replace the old value associated with i
// by x;  else throw an exception.
}; //end of GeneralArray
```

Note:

- **Not necessarily implemented using consecutive memory**
- **Index can be coded any way**
- **GeneralArray is more general than C++ array as it is more flexible about the composition of the index set**
- **To be simple, we will hereafter use the C++ array**

2.3 The Polynomial Abstract Data Type

Array can be used to implement other abstract data types. The simplest one might be:

Ordered or linear list.

Example:

(Sun, Mon, Tue, Wed, Thu, Fri, Sat)

(2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A)

( )   // empty list

More generally, An ordered list is either empty or $(a_0, a_1, \ldots, a_{n-1})$. // index important

Main operations:

(1) Find the length, n, of the list.

(2) Read the list from left to right (or right to left)

(3) Retrieve the ith element, $0 \le i < n$.

(4) Store a new value into the ith position, $0 \le i < n$.

(5) Insert a new element at position i, $0 \le i < n$, causing elements numbered i, i+1,…n-1 to become numbered i+1, i+2,…n.

(6) Delete the element at position i, $0 \le i < n$, causing elements numbered i+1, i+2,…n-1 to become numbered i, i+1,…n-2.

How to represent ordered list efficiently?

- Use array: $a_i \leftarrow\rightarrow$ index i
- Sequential mapping, because using conventional array representation, we are storing a_i and a_{i+1} into consecutive location i and i+1.
- Random access any element in O(1).
- Operations (5) and (6) need data movement.

Now let us look at a problem requiring ordered list.

Problem: build an ADT for the representation and manipulation of symbolic polynomials.

$A(x)=3x^2+ 2x+4$

$B(x)=x^4+ 10x^3+ 3x^2+1$

Degree: the largest exponent

ADT 2.3 Polynomial

class Polynomial **{**

// $p(x)=a_0x^{e0}+,\ldots,+a_nx^{en}$; a set of ordered pairs of <e_i, a_i>,

// where a_i is a nonzero **float** coefficient and e_i is a

// non-negative exponent

public:

Polynomial ( )**;**

// Construct the polynomial p(x)=0

```
    void AddTerm (Exponent e, Coefficient c);
    // add the term <e,c> to *this, so that it can be initialized

    Polynomial Add (Polynomial poly);
    // return the sum of the polynomials *this and poly

    Polynomial Mult (Polynomial poly);
    // return the product of the polynomials *this and poly

    float Eval ( float f);
    // evaluate polynomial *this at f and return the result
}
```

2.3.1 Polynomial Representation

Representation 1

private:

int degree**;** // degree ≤ MaxDegree

float coef[MaxDegree+1]**;**

Let a be $A(x)=a_nx^n+ a_{n-1}x^{n-1}+,\ldots,+ a_1x+ a_0$

a.degree=n;

a.coef[i] $= a_{n-i}$**,** $0 \leq i \leq n$

Simple algorithms for many operations.

Representation 2

When a.degree << MaxDegree, representation 1 is very poor in memory use. To improve, define variable sized data member as:

```
private:
    int degree;
    float *coef;

Polynomial::Polynomial(int d)
 {
    int degree=d;
    coef= new float[degree+1];
 }
```

Representation 3

Representation 2 is still not desirable. For instance, $x^{1000}+1$ makes 999 entries of the coef be zero.

So, we store only the none zero terms:

$$A(x) = b_m x^{e_m} + b_{m-1} x^{e_{m-1}} + , \ldots , + b_0 x^{e_0}$$

Where $b_i \neq 0$, $\quad e_m > e_{m-1} > , \ldots , e_0 \geq 0$

```
class Polynomial; // forward declaration
class Term {
friend Polynomial;
private:
    float coef; // coefficient
    int exp;    // exponent
};

class Polynomial {
public:
   ……
private:
   Term *termArray;
   int capacity; // size of termArray
   int terms; // number of nonzero terms
}
```

For $A(x) = 2x^{1000} + 1$

A.termArray looks like:

coef	2	1		
exp	1000	0		

Many zero **--- good**

Few zero **--- not very good, may use twice as much space as in presentation 2.**

2.3.2 Polynomial Addition

Use presentation 3 to obtain C = A +B.

Idea:

Because the exponents are in descending order, we can adds A(x) and B(x) term by term to produce C(x).

The terms of C are entered into its termArray by calling function NewTerm.

If the space in termArray is not enough, its capacity is doubled.

```
Polynomial  Polynomial::Add (Polynomial b)
{   // return the sum of the polynomials *this and b.
  Polynomial c;
  int aPos=0, bPos=0;
  while (( aPos < terms) && (bPos < b.terms))
     if (termArray[aPos].exp==b.termArray[bPos].exp) {
        float t = termArray[aPos].coef + termArray[bPos].coef
        if ( t ) c.NewTerm (t, termArray[aPos].exp);
        aPos++;  bPos++;
     }
     else if  (termArray[aPos].exp < b.termArray[bPos].exp) {
  c.NewTerm (b.termArray[bPos].coef, b.termArray[bPos].exp);
        bPos++;
     }
```

```
else {
    c.NewTerm (termArray[aPos].coef, termArray[aPos].exp);
    aPos++;
}
// add in the remaining terms of *this
for ( ; aPos < terms; aPos++ )
    c.NewTerm(termArray[aPos].coef, termArray[aPos].exp );
// add in the remaining terms of b
for ( ; bPos < b.terms; bPos++ )
c.NewTerm(b.termArray[bPos].coef, b.termArray[bPos].exp);
return c;
}
```

```
void Polynomial::NewTerm(const float theCoeff,
                         const int theExp)
{ // add a new term to the end of termArray.
  if (terms == capacity)
  { // double capacity of termArray
      capacity *= 2;
      term *temp = new term[capacity]; // new array
      copy(termArray, termAarry + terms, temp);
      delete [ ] termArray;  // deallocate old memory
      termArray = temp;
  }
  termArray[terms].coef = theCoeff;
  termArray[terms++].exp = theExp;
}
```

Analysis of Add:

Let m, n be the number of nonzero terms in a and b respectively.

- **line 3 and 4---O(1)**
- **in each iteration of the while loop, aPos or bPos or both increase by 1, the number of iterations of this loop $\leq$ m+n-1**
- **if ignore the time for doubling the capacity, each iteration takes O(1)**
- **line 20--- O(m), line 23--- O(n)**

Asymptotic computing time for Add: O(m+n)

Analysis of doubling capacity:

- the time for doubling is linear in the size of new array
- initially, c.capacity is 1
- suppose when Add terminates, c.capacity is 2^k
- the total time spent over all array doubling is

$$O\left(\sum_{i=1}^{k} 2^i \right) = O(2^{k+1}) = O(2^k)$$

- since c.terms > 2^{k-1} and m + n $\geq$ c.terms, the total time for array doubling is

$$O(c.terms) = O(m + n)$$

- so, even consider array doubling, the total run time of Add is $O(m + n)$.
- experiments show that array doubling is responsible for very small fraction of the total run time of Add.

Exercises: P93-2,6, P94-9

2.4 Sparse Matrices

2.4.1 Introduction

A general matrix consists of m rows and n columns (m × n)of numbers, as:

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left[\begin{array}{ccc} -27 & 3 & 4 \\ 6 & 82 & -2 \\ 109 & -64 & 11 \\ 12 & 8 & 9 \\ 48 & 27 & 47 \end{array}\right] \end{array}$$

Fig.2.2(a) 5×3

$$
\begin{array}{c c}
 & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\begin{bmatrix}
15 & 0 & 0 & 22 & 0 & -15 \\
0 & 11 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & -6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
91 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 28 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

Fig. 2.2(b) 6×6

A matrix of $m \times m$ is called a **square**.

A matrix with many zero entries is called **sparse**.

Representation:

- A natural way --- a[m][n], access element by a[i][j], easy operations. But for sparse matrix, wasteful of both memory and time.
- Alternative way --- store nonzero elements explicitly. 0 as default.

ADT 2.4 SparseMatrix

```
class SparseMatrix
{ // a set of <row, column, value>, where row, column are
  // non-negative integers and form a unique combination;
  //  value is also an integer.
public:
    SparseMatrix ( int r, int c, int t);
    // creates a r×c SparseMatrix with a capacity of t nonzero
    // terms
    SparseMatrix Transpose ( );
    // return the SparseMatrix obtained by transposing *this
    SparseMatrix  Add ( SparseMatrix b);
    SparseMatrix  Multiply ( SparseMatrix b);
};
```

2.4.2 Sparse Matrix Representation

Use triple <row, col, value>, sorted in ascending order by <row, col>.

We need also the number of rows and the number of columns and the number of nonzero elements. Hence,

```
class  SparseMatrix;
class  MatrixTerm {
friend class SparseMatrix;
private:
   int row, col, value;
};
```

And in class SparseMatrix:

private:

Int rows, cols, terms, capacity**;**

MatrixTerm *smArray**;**

Now we can store the matrix of Fig.2.2 (b) as Fig.2.3 (a).

	row	col	value
smArray[0]	0	0	15
[1]	0	3	22
[2]	0	5	-15
[3]	1	1	11
[4]	1	2	3
[5]	2	3	-6
[6]	4	0	91
[7]	5	2	28

Fig.2.3 (a)

2.4.3 Transposing a Matrix

Transpose:

If an element is at position [i][j] in the original matrix, then it is at position [j][i] in the transposed matrix.

Fig.2.3(b) shows the transpose of Fig2.3(a).

	row	col	value
smArray[0]	0	0	15
[1]	0	4	91
[2]	1	1	11
[3]	2	1	3
[4]	2	5	28
[5]	3	0	22
[6]	3	2	-6
[7]	5	0	-15

Fig.2.3 (b)

First try:

For (each row i)

take element (i, j, value) and

store it in (j, i, value) of the transpose;

Difficulty: not knowing where to put (j, i, value) until all other elements preceding it have been processed.

Improvement:

For (all elements in column j)

store (i, j, value) of the original matrix as

(j, i, value) of the transpose;

Since the rows are in order, we will locate elements in the correct column order.

```
SparseMatrix SparseMatrix::Transpose ( )
{ // return the transpose of *this
    SparseMatrix b(cols, rows, terms);
    if (terms > 0)
    {   //nonzero matrix
      int currentB = 0;
```

```
      for ( int c=0; c<cols; c++ ) // transpose by columns
        for ( int i=0; i<terms; i++ )
        // find and move terms in column c
          if ( smArray[i].col == c )
          {
            b.smArray[CurrentB].row = c;
            b.smArray[CurrentB].col = smArray[i].row;
            b.smArray[CurrentB++].value= smArray[i].value;
          }
    } // end of if (terms > 0)
  return b;
}
```

Time complexity of Transpose:

- **line 7-15 loop--- cols times**
- **line 10 loop--- terms times**
- **other line--- O(1)**

Total time: O(cols* terms)

Additional space: O(1)

Think:

O(cols* terms) is not good. If terms = O(cols* rows) then it becomes O($cols^2$* rows)---too bad! Since with 2-dimensional representation,

we can get an easy O(cols* rows) algorithm as:

```
for (int j=0;j < columns;j++)
    for (int i=0; i < rows; i++)  B[j][i] = A[i][j];
```

Further improvement:

If we use some more space to store some knowledge about the matrix, we can do much better: doing it in O(cols + terms).

- get the number of elements in each column of *this = the number of elements in each row of b;
- obtain the starting point in b of each of its rows;
- move the elements of *this one by one into their right position in b.

Now the algorithm FastTranspose.

```
SparseMatrix SparseMatrix::FastTranspose ( )
{ // return the transpose of *this in O(terms+cols) time.
   SparseMatrix b(cols, rows, terms);
   if (terms > 0)
   { // nonzero matrix
     int *rowSize = new int[cols];
     int *rowStart = new int[cols];
     // compute rowSize[i] = number of terms in row i of b
     fill(rowSize, rowSize + cols, 0);  // initialze
     for (i=0; i<terms; i++ )  rowSize[smArray[i].col]++;
```

```
// rowStart[i] = starting position of row i in b
rowStart[0] = 0;
for (i=1;i<cols;i++) rowStart[i]=rowStart[i-1]+rowSize[i-1];
for (i=0; i<terms; i++)
{    // copy from *this to b
     int j = rowStart[smArray[i].col];
     b.smArray[j].row = smArray[i].col;
     b.smArray[j].col = smArray[i].row;
     b.smArray[j].value = smArray[i].value;
     rowStart[smArray[i].col]++;
 }   // end of for
```

```
22      delete [ ] rowSize;
23      delete [ ] rowStart;
24   }  // end of if
25   return b;
26 }
```

For Fig.2.3(a), after line 13, we get :

	[0]	[1]	[2]	[3]	[4]	[5]
RowSize=	**2**	**1**	**2**	**2**	**0**	**1**
RowStart=	**0**	**2**	**3**	**5**	**7**	**7**

Note the error in P101 of the text book!

Analysis:

3 loops:

- **line 10--- O(terms)**
- **line 13--- O(cols)**
- **line 14 – 21--- O(terms)**

and line 9--- O(cols), other lines--- O(1)

Total: O(cols+terms)

This is a typical example for trading space for time.

Exercises: P107-1, 2, 4

2.6 The String Abstract data Type

A string $S = s_0, s_1, \ldots, s_{n-1}$,

where $s_i \in$ char, $0 \le i < n$, n is the length.

ADT 2.5 String

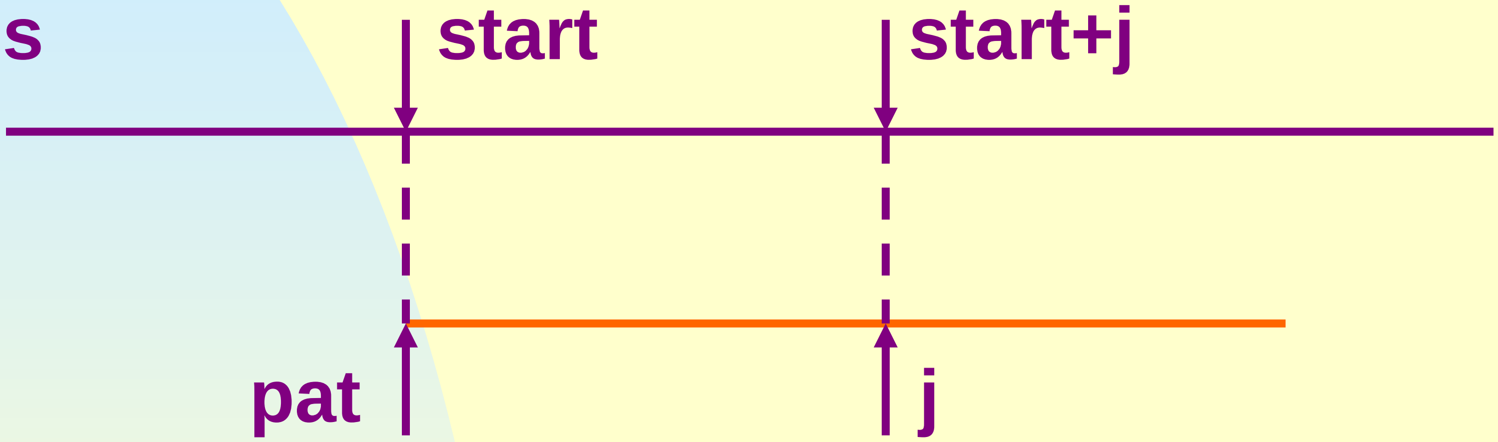
```
class String
{   // a finite set of zero or more characters;
public:
     String (char *init, int m );
     // initialize *this to string init of length m
```

```
    bool operator == (String t );
    // if *this equals t, return true else false.
    bool operator ! (  );
    // if *this is empty return true else false.
    int Length ( );
     // return the number of chars in *this
     String Concat (String t);
     String Substr (int  i, int  j);
     int Find (String pat);
    // return i such that pat matches the substring of *this that
    // begins at position i. Return –1 if pat is either empty or not
    // a substring of *this.
};
```

Assume the String class is represented by:

```
private:
    char* str;
```

2.6.1 String Pattern Matching: A Simple Algorithm



The idea is showed in the function Find .

```
int String::Find ( String pat )
{ // Return –1 if pat does not occur in *this; otherwise
  // return the first position in *this, where  pat begins.
     if (pat.Length( ) == 0) return -1; // pat is empty
     for (int start=0; start<=Length( ) - pat.Length( ); start++)
     { //  check  for match beginning at str[start]
         for (int j=0; j<pat.Length( )&&str[start+j]==pat.str[j];j++)
         if (j== pat.Length( )) return start; // match found
         // no match at position start
      }
      return -1;     // pat does not occur in s
}
```

The complexity of it is O(LengthP * LengthS).
Problem: rescanning.

Even if we check the last character of pat first, the time complexity can't be improved!

2.6.2 String Pattern Matching: The Knuth-Morris-Pratt Algorithm

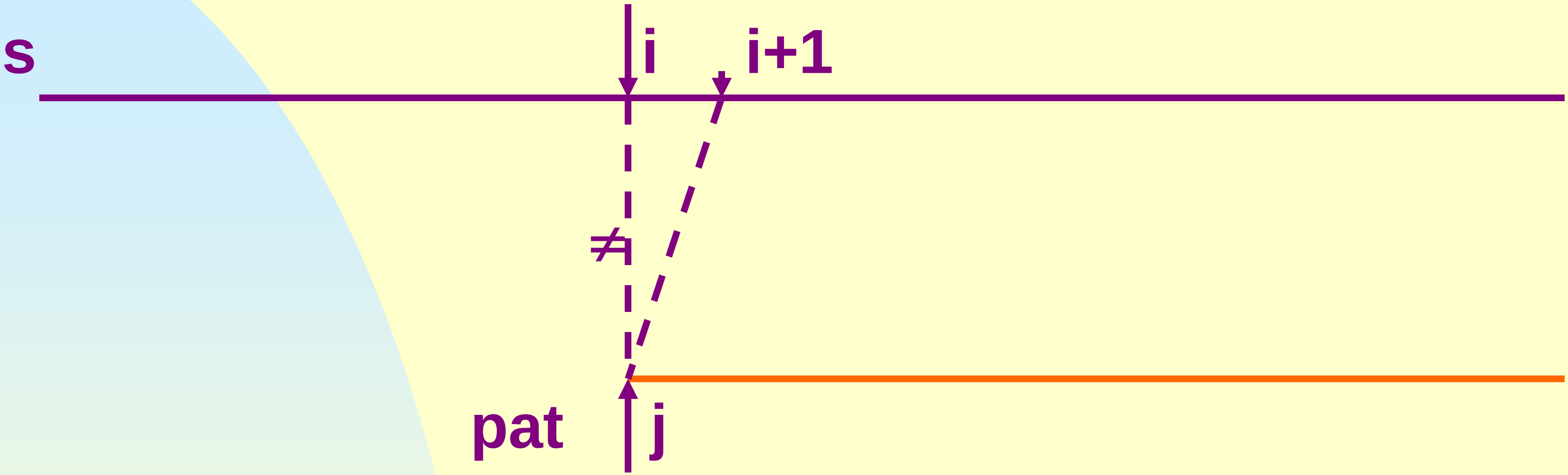
Can we get an algorithm which avoid rescanning the strings and works in O(LengthP + LengthS)?

This is optimal for this problem, as in the worst it is necessary to look at characters in the pattern and string at least once.

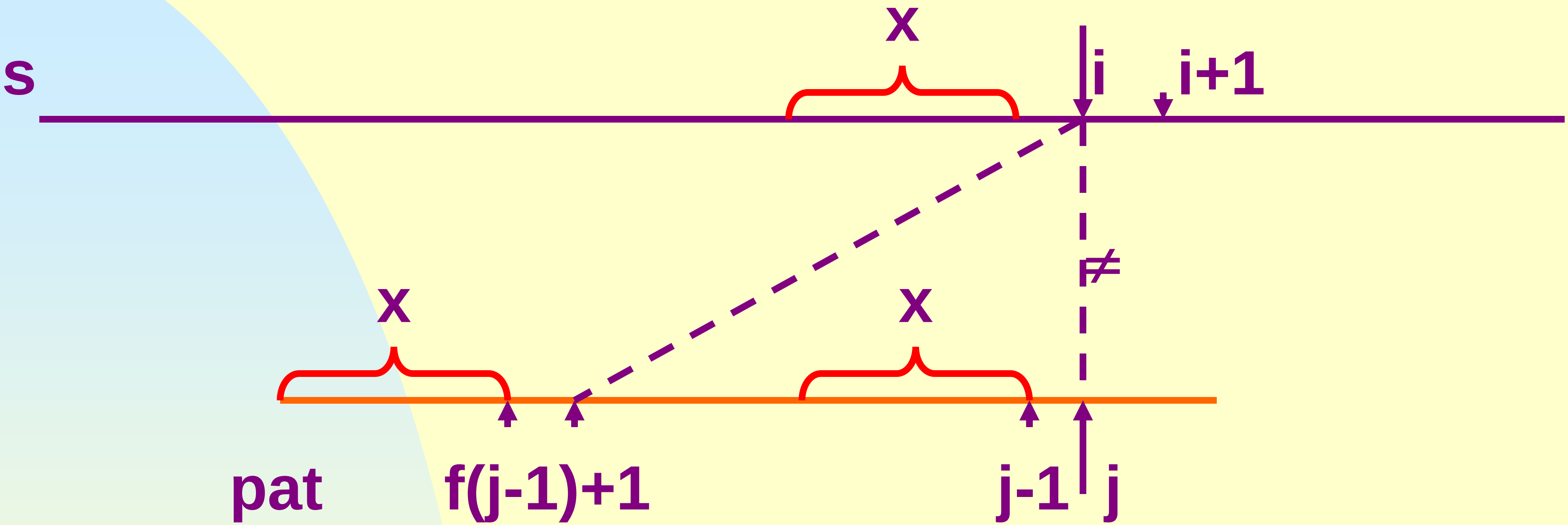
Basic Ideas:

- Rescanning to avoid missing the target --- too conservative. If we can go without rescanning, it is likely to do the job in O(LengthP + LengthS).
- Preprocess the pattern, to get some knowledge of the characters in it and the position in it, so that if a mismatch occurs we can determine where to continue the search and avoid moving backwards in the string.

Now we show details about the idea.



case: j = 0



case: $j \neq 0$

An concrete example:

i

↓

s = . . . a b d **a b** d . . . .

$\neq$

pat = **a b** d **a b** c a c a b

↑

j=5

To formalize the above idea:

Definition: if $p = p_0 p_1 \ldots p_{n-1}$ is a pattern, then its failure function f, is defined as:

$$
f(j) = \begin{cases} \text{largest } k < j, \text{ such that } p_0 p_1 \ldots p_k = p_{j-k} p_{j-k+1} \ldots p_j & \text{if such } k \geq 0 \text{ exists} \\ -1 & \text{otherwise} \end{cases}
$$

For example, pat = a b c a b c a c a b, we have

j	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
pat	**a**	**b**	**c**	**a**	**b**	**c**	**a**	**c**	**a**	**b**
f	**-1**	**-1**	**-1**	**0**	**1**	**2**	**3**	**-1**	**0**	**1**

Note:

- **largest : no match be missed**
- **k < j : avoid dead loop**

From the definition of f, we have the following rule for pattern matching:

If a partial match is found such that $s_{i-j} \ldots s_{i-1} = p_0 p_1 \ldots p_{j-1}$ and $s_i \neq p_j$ then matching may be resumed by comparing s_i and $p_{f(j-1)+1}$ if $j \neq 0$. If $j=0$, then we may continue by comparing s_{i+1} and p_0.

The failure function is represented by an array of integers f, which is a private data member of String.

Now the algorithm FastFind.

```
int String::FastFind (String pat)
{ // Determine if pat is a substring of s
   int PosP = 0,  PosS = 0;
   int LengthP= pat.Length( ), LengthS= Length( );
   while ((PosP  < LengthP) && (PosS  < LengthS))
       if ( pat.str[PosP] == str[PosS] ) { // characters match
       PosP ++; PosS ++;
       }
       else
         if ( PosP==0)
            PosS++;
         else PosP= pat.f [PosP-1] + 1;
   if ((PosP < LengthP) || LengthP==0)) return –1;
   else return PosS  - LengthP ;
}
```

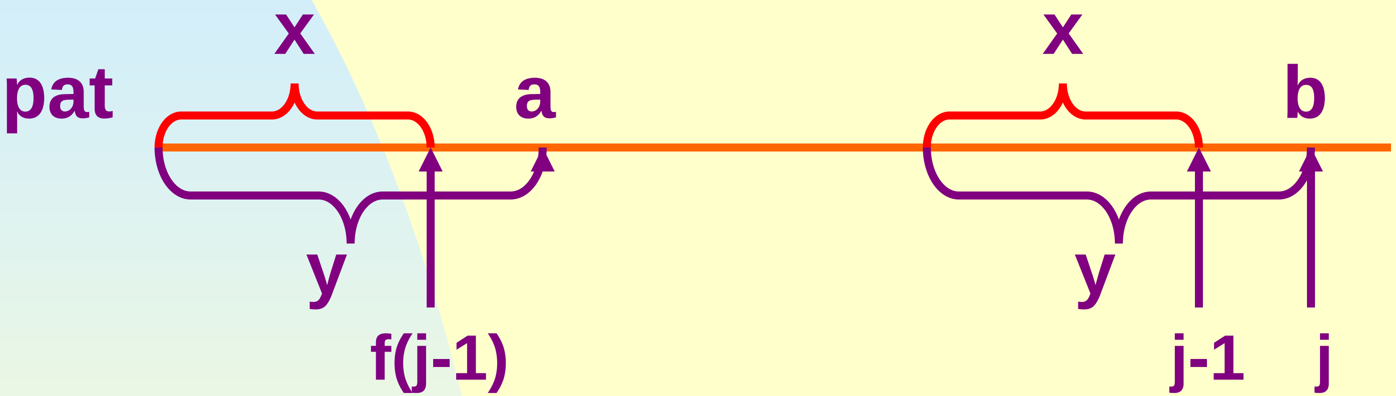
Analysis of FastFind:

- Line 7 and 11 --- at most LengthS times, since PosS is increased but never decreased. So PosP can move right on pat at most LengthS times (line 7).
- Line 12 moves PosP left, it can be done at most LengthS times. Note that f(j-1)+1< j.

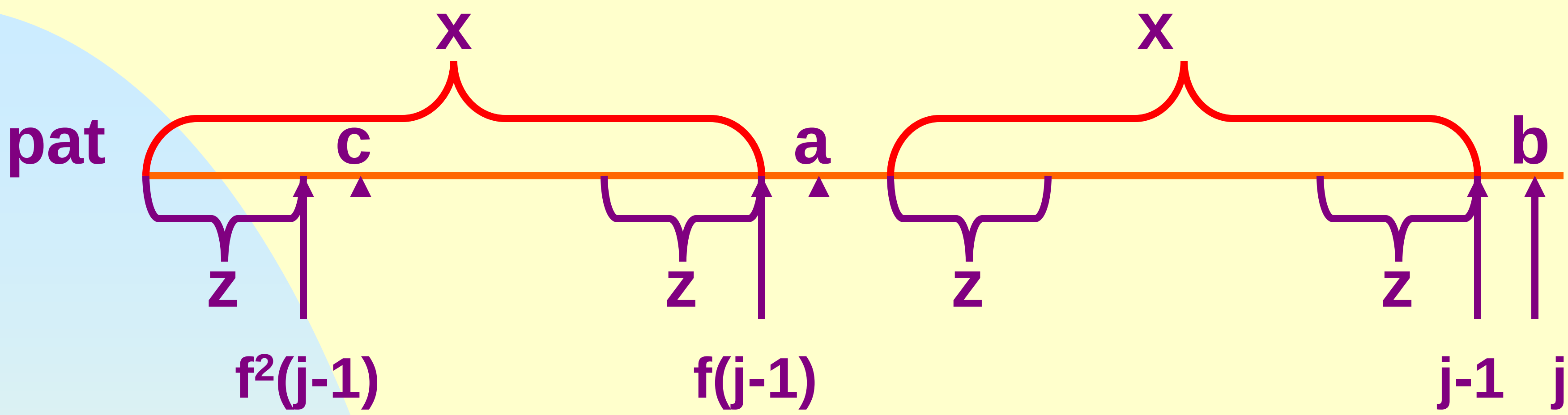
Consequently, the computing time is O(LengthS).

How about the computing of the f for the pattern? By similar idea, we can do it in O(LengthP).

f(0)=-1, now if we have f(j-1), we can compute f(j) from it by the following observation:



If a=b, then f(j)=f(j-1)+1 else



If c=b, $f(j)=f(f(j-1))+1=f^2(j-1)+1$ else ……

In general, we have the following restatement of the failure function:

$$f(j) = \begin{cases} -1 & \text{if } j=0 \\ f^m(j-1)+1 & \text{where } m \text{ is the least } k \text{ for which } p_{f^k(j-1)+1} = p_j \\ -1 & \text{if there is no } k \text{ satisfying the above} \end{cases}$$

Now we get the algorithm to compute f.

```
void String::Failurefunction( )
{ // compute the failure function of the pattern *this.
   int LengthP= Length( );
   f [0]= -1;
   for (int j=1; j< LengthP; j++) // compute  f[j]
      {
        int i=f [j-1];
        while ((*(str+j)!=*(str+i+1)) && (i>=0)) i=f[i]; // try for m
         if ( *(str+j)==*(str+i+1))
             f[j]=i+1;
        else f[j]= -1;
    }
}
```

Analysis of fail:

- In each iteration of the while i decreases (line 8, and f(j)<j)
- i is reset (line 7) to –1 (when the previous iteration went through line 11), or to a value 1 greater than its value on the previous iteration (when through line 10).
- There are only LengthP –1 executions of line 7, the value of i has a total increment of at most LengthP –1.
- i cannot be decremented more than LengthP –1 times, the while is iterated at most LengthP –1 times over the whole algorithm.

Consequently, the computing time is O(LengthP).

Now we can see, when the failure function is not known in advance, pattern matching can be carried out in O(LengthP + LengthS) by first computing the failure function and then using the FastFind.

Exercises: P118-1, P119-7, 9

Experiment 1: P123-8