MSRI UP 2023 - HW 1 - DAY 2

Due: Thursday 6/15 by 7:00am. Submit your assignment to your personal dropbox.

Important: You don't have to complete the entire assignment, but do as much of it as possible. Make sure to present your (partial) solutions in a neat and organized manner. Discuss the problems with at least two other people, but write your solutions on your own.

(1) Let $(\mathbb{M}, d_{\mathbb{M}})$ be a metric space and let $X, Y \subset \mathbb{M}$ be so that $d_H^{\mathbb{M}}(X, Y) < \delta < \infty$.

(a) Show that there are functions $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow X$ so that

$$d_{\mathbb{M}}(x, \varphi(x)) < \delta \quad \text{and} \quad d_{\mathbb{M}}(y, \psi(y)) < \delta$$

for all $x \in X$ and all $y \in Y$.

(b) Let d_X be the restriction of $d_{\mathbb{M}}$ to $X \times X$, and similarly, let d_Y be the restriction of $= d_{\mathbb{M}}$ to $Y \times Y$. Show that if $\sigma \in R_\alpha(X, d_X)$, then $\varphi(\sigma) \in R_{\alpha+2\delta}(Y, d_Y)$. Let $\varphi_\alpha : R_\alpha(X, d_X) \longrightarrow R_{\alpha+2\delta}(Y, d_Y)$ be the resulting function, and similarly, let $\psi_\alpha : R_\alpha(Y, d_Y) \longrightarrow R_{\alpha+2\delta}(X, d_X)$ denote the one induced by ψ.

(c) Show that if $\sigma \in R_\alpha(X, d_X)$, then $\big(\sigma \cup \psi_{\alpha+2\delta} \circ \varphi_\alpha(\sigma)\big) \in R_{\alpha+4\delta}(X, d_X)$.

(2) Let K and L be simplicial complexes. A function $f : K \longrightarrow L$ is said to be a *simplicial map* if $f\left(K^{(0)}\right) \subset L^{(0)}$ and $f(\{x_0, \ldots, x_n\}) = \{f(x_0), \ldots, f(x_n)\}$ for every $\{x_0, \ldots, x_n\} \in K$. That is, if f takes each vertex of K to a vertex of L, and the value of f on an n-simplex is given by the value on its vertices. Let f be a simplicial map and let $\mathbb{F}$ be a field.

(a) Show that there is a unique linear transformation $f_{n\#} : C_n(K; \mathbb{F}) \longrightarrow C_n(L; \mathbb{F})$ satisfying

$$f_{n\#}(\sigma) = \begin{cases} f(\sigma) & \text{if } f(\sigma) \in L^{(n)} \\ 0 & \text{else} \end{cases}$$

for each $\sigma \in K^{(n)}$.

(b) Let ∂_n^K and ∂_n^L denote the n-boundary maps for K and L, respectively. Show that $f_{n-1\#} \circ \partial_n^K = \partial_n^L \circ f_{n\#}$ for every $n \geq 1$, and use this to conclude that $f_{n\#}\big(Z_n(K; \mathbb{F})\big) \subset Z_n(L; \mathbb{F})$ and $f_{n\#}\big(B_n(K; \mathbb{F})\big) \subset B_n(L; \mathbb{F})$.

(c) Show that the function $f_{n_*} : H_n(K; \mathbb{F}) \longrightarrow H_n(L; \mathbb{F})$ given by $f_{n_*}([\zeta]) = [f_{n\#}(\zeta)]$ for each $\zeta \in Z_n(K; \mathbb{F})$, is a well-defined linear transformation.