MSRI UP 2023 - HW 1 - DAY 1

Due: Wednesday 6/14 by 7:00am. Submit your assignment to your personal dropbox.

Important: You don't have to complete the entire assignment, but do as much of it as possible. Make sure to present your (partial) solutions in a neat and organized manner. Discuss the problems with at least two other people, but write your solutions on your own.

(1) Let (X, d_X) be a metric space. Show that the following statements are equivalent:
  (a) (X, d_X) is compact.
  (b) Every sequence in X has a convergent subsequence. That is, if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X, then there exists an strictly increasing function $\eta : \mathbb{N} \to \mathbb{N}$ so that the sequence $\{x_{\eta(k)}\}_{k\in\mathbb{N}}$ converges.
  (c) X is complete and totally bounded. That is, every Cauchy sequence in X converges, and for any $\epsilon > 0$ there exist $x_1, \ldots, x_N \in X$ so that $X \subset B_\epsilon(x_1) \cup \cdots \cup B_\epsilon(x_N)$.

(2) Show that the following metric spaces are **not** complete:
  (a) $([0, 1), d_1)$
  (b) $(\mathbb{R}^\infty, d_2)$
  (c) $(\mathcal{D}_0, d_B)$

(3) A collection K of finite nonempty sets is said to be a *simplicial complex* if the following is true: For any $\sigma \in K$, if $\emptyset \neq \tau \subset \sigma$, then $\tau \in K$. Let (X, d_X) be a metric space and for $\epsilon > 0$ let

$$R_\epsilon(X, d_X) = \left\{\sigma \subset X \ : \ 0 < \#(\sigma) < \infty \ \text{ and } \ \mathsf{diam}(\sigma) < \epsilon\right\}$$

where $\mathsf{diam}(\sigma) = \max\{d_X(x, x') \ : \ x, x' \in \sigma\}$ is the diameter of σ.
  (a) Show that $R_\epsilon(X, d_X)$ is a simplicial complex.
  (b) Show that if $0 < \epsilon \leq \epsilon'$, then $R_\epsilon(X, d_X) \subset R_{\epsilon'}(X, d_X)$.