Gaussian Process for Time Series Analysis

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Overview

Multivariate Normal Distribution

$X = (X_1, \cdots, X_d)$ has a **multivariate normal distribution** if every linear combination is normally distributed. In this case it has density of the form

$$p(x|m, K_0) = \frac{1}{\sqrt{(2\pi)^d |K_0|}} \exp\left(-\frac{1}{2}(x-m)^T K_0^{-1}(x-m)\right)$$

where $m \in \mathbb{R}^d$ is the **mean vector** and $K_0 \in M_d(\mathbb{R})$ is the (symmetric, positive definite) **covariance matrix**.

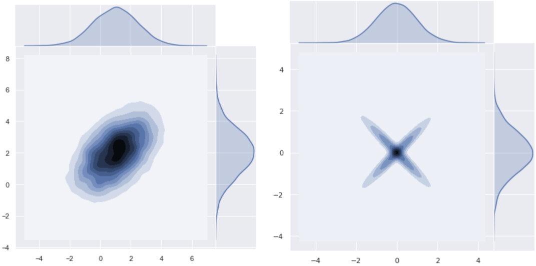


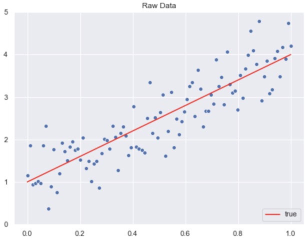
Figure: Left: Multivariate Normal Distribution, Right: Non-Multivariate Normal Distribution

Regularized Bayesian Linear Regression

Let $x_1, \cdots, x_n \in \mathbb{R}^d$ and $y_1, \cdots, y_n$ be a set of observations (data). We want to fit the linear model

$$f(x) = x^T b \quad \text{and} \quad y = f(x) + \varepsilon, \quad \text{with} \quad \varepsilon \sim N(0, \sigma_n^2)$$

where $b \in \mathbb{R}^d$ denotes the parameter vector. Let $X \in M_{d \times n}$ be denote the observation matrix.



We want to compute $p(b|X, y)$ using the Bayes theorem

$$p(b|X, y) = \frac{p(y|X, b)p(b)}{p(y|X)} \propto \text{likelihood} \times \text{prior}$$

Prior Distribution

- Likelihood

$$p(y|X,b) = \prod_{i=1}^{n} p(y_i|x_i,b) = N(X^T b, \sigma_n^2 I)$$

- Prior

$$b \sim N(0, \Sigma_p), \quad \Sigma_p \in M_d(\mathbb{R})$$

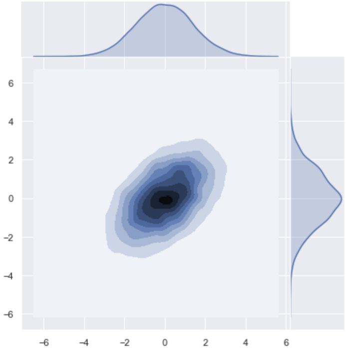


Figure: Prior Distribution

Posterior Distribution

- Posterior

$$p(b|y,X) = N\left(\bar{b} = \frac{1}{\sigma_n^2}A^{-1}Xy, A^{-1}\right)$$

where $A = \sigma_n^{-2}XX^T + \Sigma_p^{-1}$

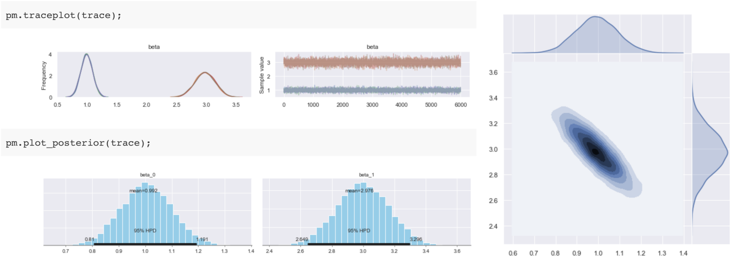


Figure: Posterior Distribution

Predictive Distribution

$$p(f_*|x_*, X, y) = \int p(f_*|x_*, b)p(b|X, y)db$$
$$= N\left(\frac{1}{\sigma_n^2} x_*^T A^{-1} Xy, x_*^T A^{-1} x_*\right)$$

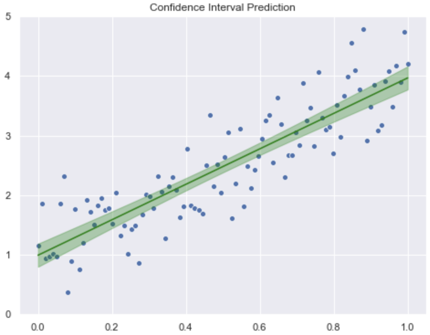


Figure: Left: Join Posterior Distribution, Right: Prediction + Confidence Interval

References

Slides and notebook available at juanitorduz.github.io