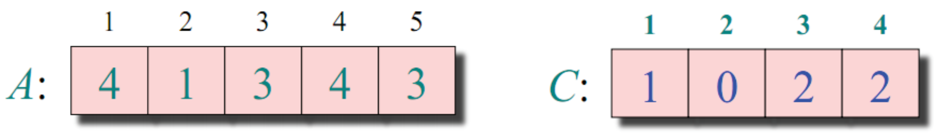
Counting Sort

Counting sort is an algorithm that takes an array A of n elements in the range $\{1, 2, ..., k\}$ and sorts the array in $O(n + k)$ time. Counting sort uses no comparisons and uses the fact that the n elements are in a limited range to beat the $O(n \log n)$ limit of comparison sorts.

Algorithm: Counting sort keeps an auxiliary array C with k elements, all initialized to 0. We make one pass through the input array A and for each element i in A that we see, we increment $C[i]$ by 1. After we iterate through the n elements of A and update C, the value at index j of C corresponds to how many times j appeared in A. This step takes $O(n)$ time to iterate through A.

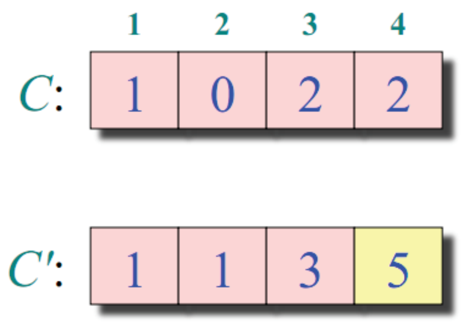


Once we have C, we can construct the sorted version of A by iterating through C and inserting each element j a total of $C[j]$ times into a new list (or A itself). Iterating through C takes $O(k)$ time.

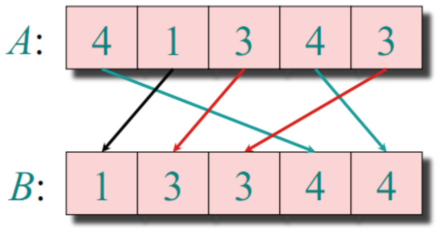
The end result is a sorted A and in total it took $O(n + k)$ time to do so.

Note that this does not permute the elements in A into a sorted list. If A had two 3s for example, there's no distinction which 3 mapped to which 3 in the sorted result. We just counted two 3s and arbitrarily stuck two 3s in the sorted list. This is perfectly fine in many cases, but you'll see later on in radix sort why in some cases it is preferable to be able to provide a permutation that transforms A into a sorted version of itself.

To do this, we continue from the point where C is an array where $C[j]$ refers to how many times j appears in A. We transform C to an array where $C[j]$ refers to how many elements are $\leq j$. We do this by iterating through C and adding the value at the previous index to the value at the current index, since the number of elements $\leq j$ is equal to the number of elements $\leq j - 1$ (i.e. the value at the previous index) plus the number of elements = j (i.e. the value at the current index). The final result is a matrix C where the value of $C[j]$ is the number of elements $\leq j$ in A.



Now we iterate through A backwards starting from the last element of A. For each element i we see, we check $C[i]$ to find out how many elements are there $\leq i$. From this information, we know exactly where we can put i in the sorted array. Once we insert i into the sorted array, we decrement $C[i]$ so that if we see a duplicate element, we know that we have to insert it right before the previous i. Once we finish iterating through A, we will get a sorted list as before. This time, we provided a mapping from each element A to the sorted list. Note that since we iterated through A backwards and decrement $C[i]$ every time we see i. we preserve the order of duplicates in A. That is, if there are two 3s in A, we map the first 3 to an index before the second 3. A sort that has this property is called a **stable sort**. We will need the stableness of counting sort when we use radix sort.

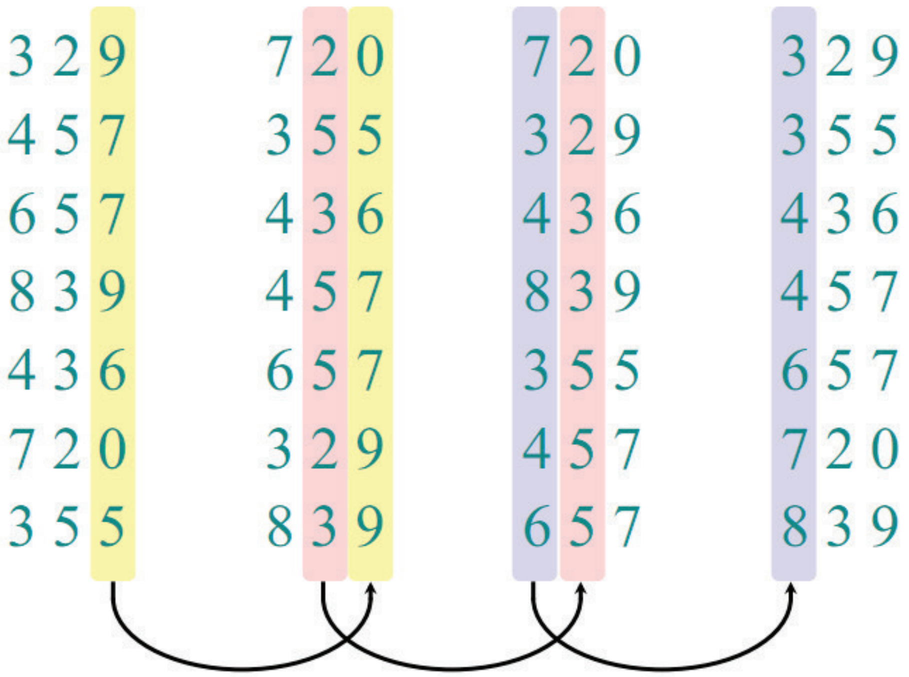


Iterating through C to change $C[j]$ from being the number of times j is found in A to being the number of times an element $\leq j$ is found in A takes $O(k)$ time. Iterating through A to map the elements of A to the sorted list takes $O(n)$ time. Since filling up C to begin with also took $O(n)$ time, the total runtime of this stable version of counting sort is $O(n+k+n) = O(2n+k) = O(n+k)$.

Radix Sort

The downfall of counting sort is that it may not be too practical if the range of elements is too large. For example, if the range of the n elements we need to sort was from 1 to n^3, then simply creating the auxiliary array C will take $O(n^3)$ time and counting sort will asymptotically do worse than insertion sort. This also takes $O(n^3)$ space which is significant larger than any of space used by any other sorting algorithm we've learned so far.

Radix sort helps solve this problem by sorting the elements digit by digit. The idea is that we can sort integers by first sorting them by their least significant digit (i.e. the ones digit), then sorting the result of that sort by their next significant digit (i.e. the tens digit), and so on until we sort the integers by their most significant digit.



How do we sort these elements by digit? Counting sort! Note that the k factor in counting sorting by digit is restricted to the range of each digit instead of the range of the elements. We have to use the stable variant of counting sort in radix sort. If we used the first version of counting sort, we wouldn't have a mapping from element to element. In the example above, when we sort the ones digit, we would be able to sort them in order but we would have no indication of whether the first 7 corresponds to 457 or 657. By using the stable variant, we get a mapping from element to element and can map the 457 to the first 7 since 457 appeared earlier in the list than 657. This stable property also ensures that as we sort more and more significant digits, duplicate digits stay in the right order according to the less significant digits, thus preserving the correctness of radix sort.

If there are n integers to sort in radix sort and the integers are represented in base k (i.e. there are k possible values for each digit) and the maximum length of the integers is d, then radix sort will execute a $O(n + k)$ counting sort for each of the d digits, resulting in a $O(d(n + k))$ runtime for radix sort.

Note that what base we choose to represent the integers in affects the value of k and d. Large bases have the advantage of shorter integers digit-wise, but the counting sort on each digit will take longer and vice versa for small bases.

Knowing n, the ideal base for radix sort is $k = n$. Say that the integers range from 0 to $u - 1$. The number of digits in the integers is at most $\log_k u$ for base k. We can substitute this into the runtime for radix sort to get $O((n+k)\log_k u)$. Using some fancy math, we can find that the best k to choose to minimize this runtime is $k = n$. In this case, our runtime for radix sort is $O(n \log_n u)$. When $u = n^{O(1)}$, the $\log_n u$ term becomes a constant and our runtime for radix sort turns out to be $O(n)$, giving us a linear time sorting algorithm if the range of the integers we're sorting is polynomial in the number of integers we're sorting.