March 27, 2014

MATH 323
Graph Theory

Cr. 3

Lecture 1

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Graph Theory

- Fundamental Concepts
- Trees
- Planar Graphs
- Graph Coloring
- Matching

Leonhard Paul Euler

(1707 – 1783)

Born on April 15, 1707, in Basel, Switzerland, Leonhard Euler was one of math's most pioneering thinkers, establishing a career as an academy scholar and contributing greatly to the fields of geometry, trigonometry and calculus, among many others. He released hundreds of articles and publications during his lifetime, and continued to publish after losing his sight. He died on September 18, 1783.

Graphs

In mathematics and computer science, **graph theory** is the study of *graphs*, which are mathematical structures used to model pairwise relations between objects.

Graph theory has proven useful in the design of integrated circuits for computers and other electronic devices.

Fundamental Concepts

- Examples:

1. Königsberg Bridge (1700)

To walk across a route that crossed each of these bridges exactly once.

In 1736 Leonhard Euler gave a negative proof.

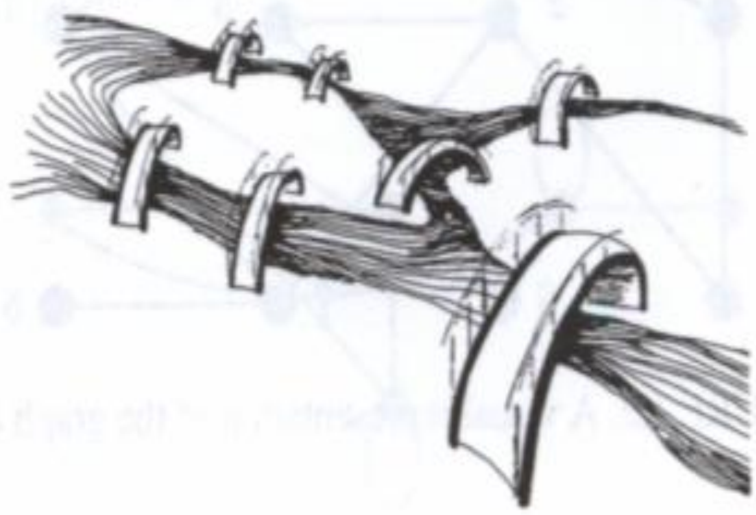


FIGURE 1.1. The bridges in Königsberg.

Examples (Contd.)

1.Four Color Map

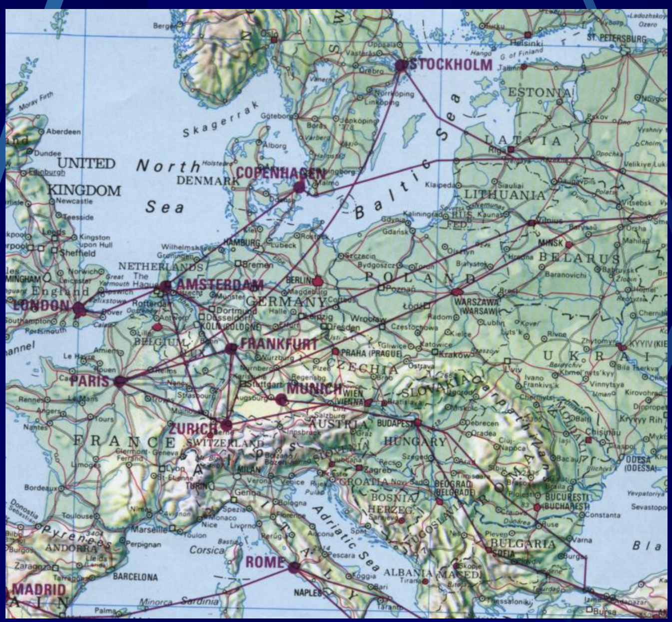
(1852 – DeMorgan)

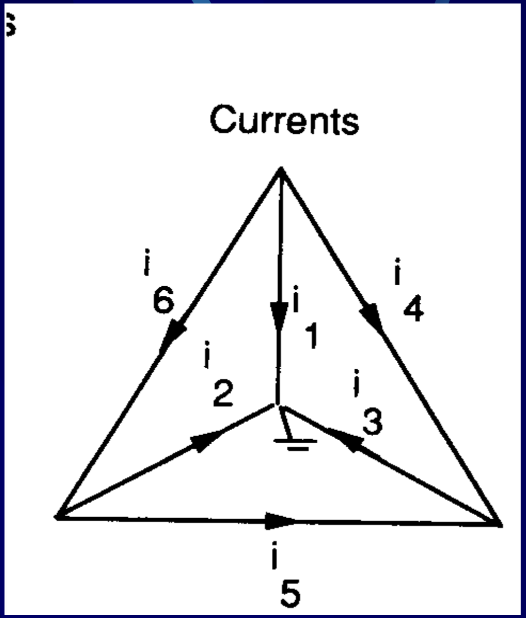
Four is the maximum number of colors required to color any map where bordering regions are colored differently.

In 1976 Appel and Haken proved the conjecture positively.

More Examples

- Air Route
- Electrical Circuit





Examples (Contd.)

Interior angles of a polygon

Polygon	No. of sides	ΣInterior angles
Triangle	3	π
Rectangle	4	2π
Pentagon	5	3π
Hexagon	6	4π
n-sided	*n*	$(n-2)\pi$

Examples (Contd.)

- Euler Formula

Figure	Vertex (V)	Edge (E)	Face (F)
3-pyramid	4	6	4
4-pyramid	5	8	5
Cube	8	12	6

Relation:	$V + F = E + 2$

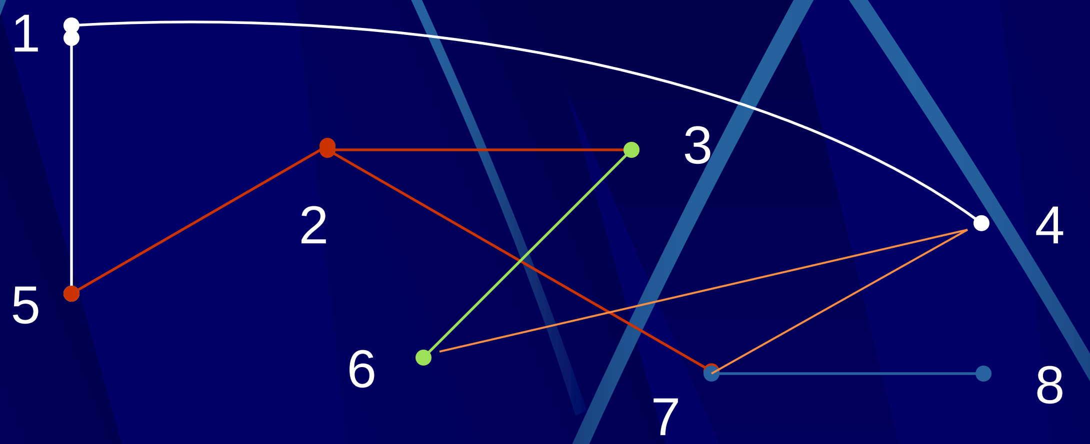
Some Definitions

Graph – A <u>graph G(V,E)</u> with *n vertices* and *m edges* consists of

a vertex set $V(G) = (v_1, v_2, \ldots, v_n)$ and

an edge set $E(G) = (e_1, e_2, \ldots, e_m)$, where each edge consists of two (possibly equal) vertices called endpoints.

Visual Representation



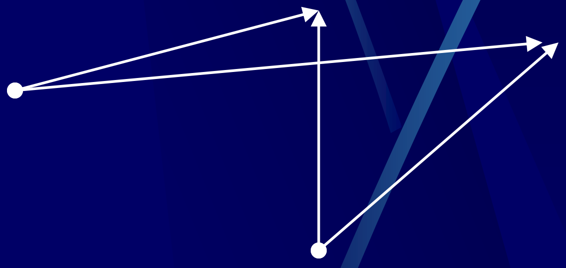
Vertex set, V = {1, 2, 3, 4, 5, 6, 7, 8}

Edge set, E = {{1,4}, {1,5}, {2,3},{2,5},

{2,7}, {3,6}, {4,6}, {4,7}, {7,8}}

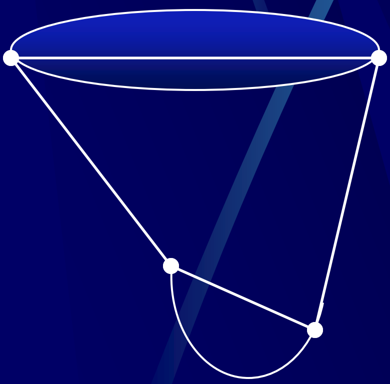
Types of Graphs

1. By replacing set E with a set of *ordered pairs* of vertices, we obtain a *directed graph*, or *digraph*.



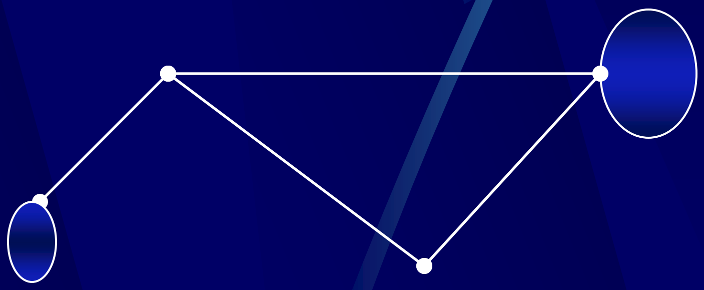
Types of Graph

2. If we allow repeated elements in the edge set, we obtain *multigraph*.



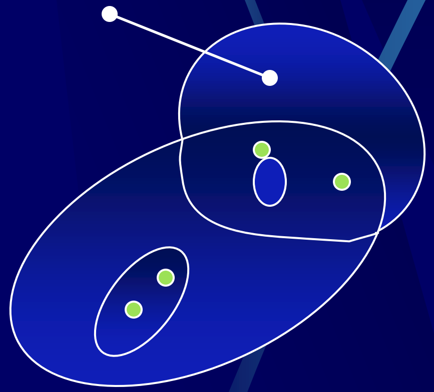
Types of Graph

3. By allowing edges to connect a vertex to itself (*loops*), we obtain a *pseudograph*.



Types of Graph

4. Allowing the edges to be arbitrary subsets of vertices gives *hypergraphs*.



A hypergraph with 6 vertices and 5 edges

Types of Graph

5. By allowing V or E to be infinite set, we obtain infinite graphs.

Some Notations & Definitions

An edge is denoted $\{u, v\}$ or simply as uv.

The *order* of a graph G is the cardinality of its vertex set V.

The *size* of a graph G is the cardinality of its edge set E.

Given two vertices u and v, if $uv \in E$, then u and v are said to be *adjacent*. If $uv \notin E$, u and v are *nonadjacent*.

If an edge e has a vertex v as an endpoint, then u and e are *incident*.

The *neighborhoods* of a vertex v, denoted by $N(v)$, is the set of vertices adjacent to v: $N(v) = \{x \in V \mid vx \in E\}$

More definitions

The *degree* of v, denoted by $\deg(v)$, is the number of edges incident with v. In simple graphs, this is the same as the cardinality of the neighborhood of v.

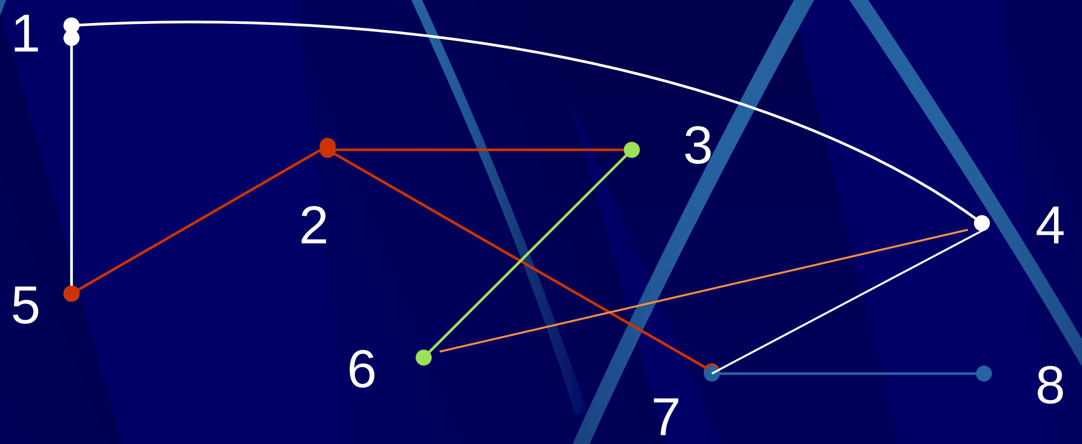
The *maximum degree* of a graph G, denoted by $\Delta(G)$, is defined to be

$$\Delta(G) = \max\{\deg(v) \mid v \in V(G)\}$$

The *minimum degree* of a graph G, denoted by $\delta(G)$, is defined to be

$$\delta(G) = \min\{\deg(v) \mid v \in V(G)\}$$

Example



Vertices 1 and 5 are adjacent, vertices 1 and 2 are nonadjacent.

deg(3) = 2, deg(7) = 3, N(4) = {1, 6, 7}, $\Delta(G) = 3$, $\delta(G) = 1$.

First Theorem of GT

- Theorem 1.

  In a graph G, the sum of the degrees of the vertices is equal to twice the number of edges.

  Alternatively, the number of vertices with odd degree is even.

Proof

Let $S = \sum_{v \in V} \deg(v)$.

Notice that in counting S, we count each edge exactly twice.

Thus S = 2|E| (the sum of the degrees is twice the number of edges).

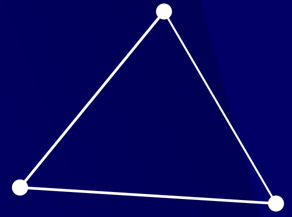
Since S is even, it must be that the number of vertices with odd degree is even.

(Also known as *Handshaking Lemma*.

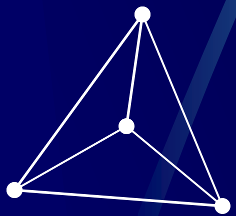
If several people shake hands, then the number of hands shaken is even)

Special Graphs

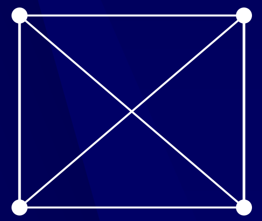
- A graph G is *trivial*, if it has only one vertex, otherwise G is *nontrivial.*
- A graph is *Complete*, if every two vertices are adjacent. The complete graph of order n with n vertices is denoted by K_n.



K_3



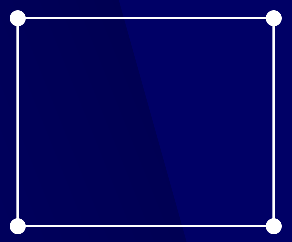
K_4



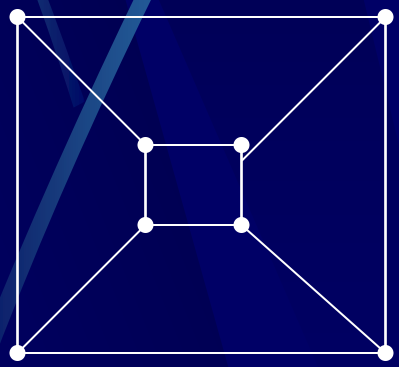
K_4

Special Graphs

- A graph G is said to be *regular* , if every vertex has the same degree. If this degree is equal to r, then G is *r-regular*.



2-regular



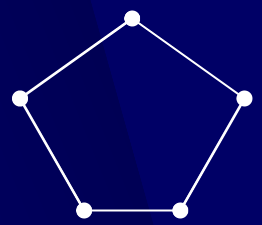
3-regular

Note:

A complete graph K_n is (N-1)-regular.

Special Graphs

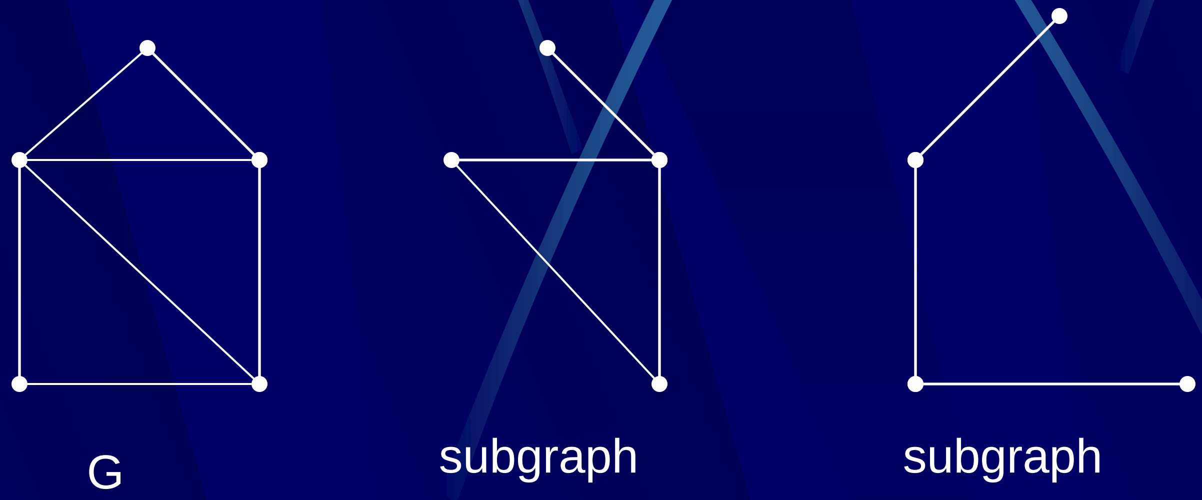
- The *complement* (*clique*) of G, denoted by $\overline{G}$, is the graph whose vertex set is the same as G's and whose edge set consists of all the edges that are not present in G.



The complement of the complete graph is called *discrete Graph*.

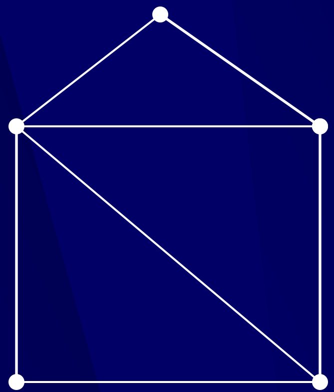
Special Graphs

- A graph H is a *subgraph* of graph G, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

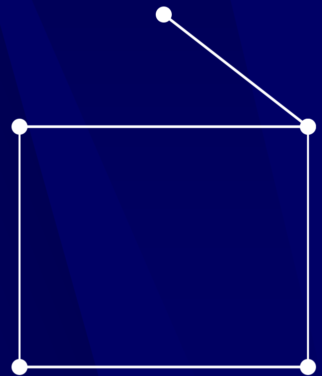


Special Graphs

- If H is a subgraph of G and V(H) = V(G), then H is said to be the *spanning subgraph* of G.



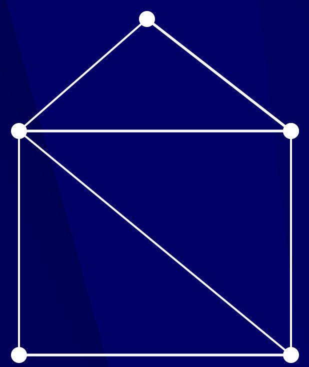
G

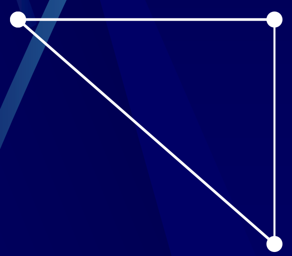


Spanning subgraph

Special Graphs

- A subgraph H of graph G is an *induced subgraph* , if $E(H) = E(G) \cap E(V(H))$.





NEXT

1. Graph (contd.)
2. Trees