COL 7004 September 8, 10, 11

TUTORIAL SHEET 7

1. In class, we showed that if we toss a coin with probability of Heads equal to p, and X is the number of coin tosses before we see a Heads, then $E[X] = p$. Show this fact by the following alternate method:
   - Let E denote $E[X]$. Show that E satisfies the equation: $E = p + (1 - p)(1 + E)$.
   - You this method to compute the expected number of coin tosses till we see two consecuted Heads (you may want to define another random variable Y which is the number of coin tosses till you see HH provided the previous coin toss was H).

2. There are n balls in a bag, each with a different colour. You repeat the following experiment till you have drawn a ball of each colour: randomly select a ball from the bag, see its colour and put it back in the bag. Let X denote the number of trials in this experiment. What is $E[X]$ (Hint: define random variables X_i denoting the number of trials to see the i^{th} new colour).

3. There are n students studying two different courses: course A and course B. At the end, the professor in each course ranks the students as follows: the professor takes a random ordering of the students and uses this ordering to declare the rank of each student. What is the expected number of students that have a higher rank in course A than in course B?

4. A box initially contains n balls, all colored black. A ball is drawn from the box at random. If the drawn ball is black, then a biased coin with probability, $p > 0$, of coming up heads is flipped. If the coin comes up heads, a white ball is put into the box; otherwise the black ball is returned to the box. This process is repeated until the box contains n white balls. Let D be the number of balls drawn until the process ends with the box full of white balls. What is $E[D]$?

5. A man has a set of n keys, only one of which will fit the lock on the door to his apartment. He tries the keys until he finds the right one. Give the expectation and variance of the number of keys he has to try, when (i) he tries the keys at random (possibly repeating a key tried earlier), (ii) he chooses keys randomly among the ones that he has not yet tried.

6. In the n letter and envelope problem, we showed that the expected number of letters which are in the correct envelope is 1. Use Chebychev inequality to bound the probability that more than 10 letters are in the correct envelope.

7. You have two coins: one has probability p of turning Heads, and the other has $p + \varepsilon$, for some small constant $\varepsilon > 0$. You are given one of these two coins and allowed to

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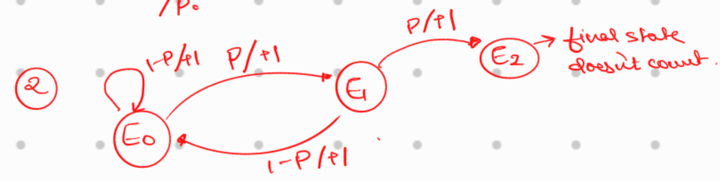
① $E = P + (1-P)(1+E)$

$E(\cancel{1} - \cancel{1} + P) = \cancel{P} + 1 - \cancel{P}$

$E = 1/P.$

②



i.e. $E_0 = P * E_1 + (1-P) * E_0 + 1$

$E_1 = 1 + P*(0) + (1-P)*E_0.$

$E_0 = P[1 + (1-P)E_0] + (1-P)E_0 + 1$

$E_{\cancel{0}} = P + \cancel{E_0 P} - P^2 E_0 + \cancel{E_0} - \cancel{E_0 P} + 1$

$$\boxed{E_0 = \frac{P+1}{P^2}}$$

2. There are n balls in a bag, each with a different colour. You repeat the following experiment till you have drawn a ball of each colour: randomly select a ball from the bag, see its colour and put it back in the bag. Let X denote the number of trials in this experiment. What is $E[X]$ (Hint: define random variables X_i denoting the number of trials to see the i^{th} new colour).

Let $X = X_1 + X_2 + X_3 + \cdots X_n.$

Let $X_i \begin{cases} 1 & \text{getting a new } i^{th} \text{ colour ball.} \\ 0 & \text{otherwise.} \end{cases}$

Probability of seeing a new i^{th} coloured balled given we have seen $i-1$ colours before

$= \frac{n-i+1}{n}.$

by failure principle

$E X_i = \frac{n}{n-i+1} \Rightarrow E X_0 = 1$

$E X_1 = \frac{n}{n-1}$

$\vdots$

$E X_n = n.$

$E X = n\left(\frac{1}{n} + \frac{1}{n-1} + \cdots 1\right)$

$nH_n.$

$$\boxed{E X = n \log n.}$$

3. There are n students studying two different courses: course A and course B. At the end, the professor in each course ranks the students as follows: the professor takes a random ordering of the students and uses this ordering to declare the rank of each student. What is the expected number of students that have a higher rank in course A than in course B?

↳ If there are n students i^{th} rank.

A:

B:

↳ also known as K subset problems

$A[i] > B[i].$

↳ total possible permutations $= n^2.$

Let $X = X_{12} + X_{13} + \cdots X_{nn}$

where X_{ij}, $i \neq j$ $\begin{cases} 1 & \text{rank}(i) > \text{rank}(j) \\ 0 & \text{otherwise.} \end{cases}$

$E X_{ij} = Pr[X_{ij}] = \frac{i(j-1)}{n}.$

$\therefore$

$E X_{12} = 1/n \quad E X_{13} = 2/n \cdots E X_{n-1\,n} = n-1$

i.e. $E X = 1/n + 2/n + \cdots \frac{n-1}{n}.$

$E X = 1/n\,(1 + 2 + \cdots n-1)$

$= \frac{1}{\cancel{n}} \frac{(n-1)(\cancel{n})}{2}$

$= \frac{n-1}{2}.$

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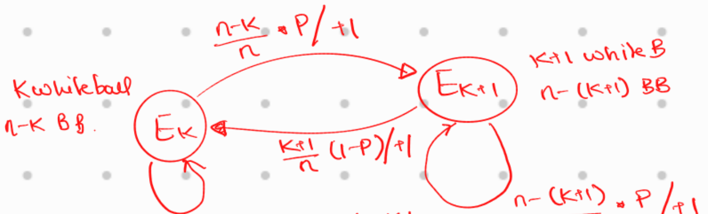


$\frac{K}{n} + \frac{n-K}{n}(1-P) = 1 - \left(\frac{n-K}{n}\right)P$

$E_K = 1 + \left(\frac{n-K}{n}\right)P\, E_{K+1} + \left[1 - \left(\frac{n-K}{n}\right)P\right] E_K$

$E_K = \frac{n}{(n-K)P} + E_{K+1}$

$E_n = 0 = E_{K+1}$

$E_{n-1} = \frac{n}{(n-n+1)P} = n/P.$

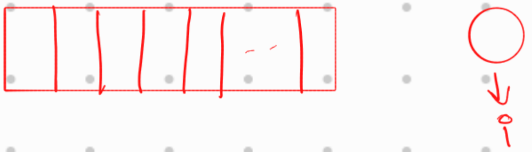
$E_{n-2} = \frac{n}{2P} \qquad E_0 = 1$

$E_{n-3} = \frac{n}{3P}.$

$\Rightarrow \frac{n}{P}\left(1 + 1/2 + 1/3 + \cdots 1/n\right)$

$n/P \log n.$

5. A man has a set of n keys, only one of which will fit the lock on the door to his apartment. He tries the keys until he finds the right one. Give the expectation and variance of the number of keys he has to try, when (i) he tries the keys at random (possibly repeating a key tried earlier), (ii) he chooses keys randomly among the ones that he has not yet tried.



1) $1/n, 1/n, 1/n \to$ G.P

If our probability stays constant over all number of trials then

$\mathbb{E}X = 1/p$

i.e $\mathbb{E}X = P$

$\boxed{\mathbb{E}X = n.}$

$Var(X) =$

?

2) $\frac{1}{n}, \frac{n-1}{n}, \frac{n-2}{n} ----1$ A.P.

$X = X_1 + X_2 + --- X_n$

$x_i \begin{cases} 1 & \text{if he gets the key correct in ith try.} \\ 0 & \text{otherwise.} \end{cases}$

$Pr[X=i] = \frac{n-i+1}{n}$

i.e

$Pr(X=1) = 1$

$Pr(X=2) \ \frac{n-1}{n}$

$EX = \left(\frac{n}{n} + \frac{n-1}{n} + --- \frac{1}{n}\right)$

$= \frac{n(n+1)}{2n}$

$\mathbb{E}X = \frac{n+1}{2}$

6. In the n letter and envelope problem, we showed that the expected number of letters which are in the correct envelope is 1. Use Chebychev inequality to bound the probability that more than 10 letters are in the correct envelope. → 0.001

→ $\mathbb{E}X = \mu = 1$

$\mathbb{E}[|x-\mu| \geq 10] = \frac{Var(x)}{(10)^2}$

$x_i \begin{cases} 1 & \text{if ith person gets his hat} \\ 0 & \text{otherwise.} \end{cases}$

$Pr[X_i = 1] = 1/n.$

$\mathbb{E}X = \frac{1}{n} + 1/n + --- n\text{times}$

$\mathbb{E}X = 1$

$Var(X) = E[(X-\mu)^2]$

$= E[(X-1)^2]$

$= \mathbb{E}[X^2 - 2X + 1]$

$= \mathbb{E}X^2 - 2\mathbb{E}X + 1$

$\mathbb{E}X^2 = \mathbb{E}[x(x-1) + x]$

$= \mathbb{E}[x(x-1)] + \mathbb{E}x$

$\mathbb{E}(X(X-1)) = \left(\sum_i I_i \sum_j I_j\right) - \sum I_j$

$\Rightarrow \sum_{i,j} I_i I_j - \sum_i I_i$

$\Rightarrow \sum_i I_i^2 + \sum_{\substack{i,j \\ i \neq j}} I_i I_j - \sum_i I_i$

$\Rightarrow \sum_i I_i - \sum_i I_i + \sum_{i,j} I_i I_j$

$\mathbb{E}[x(x-1)] \Rightarrow \sum_{\substack{i,j \\ i \neq j}} I_i I_j$

$\Rightarrow \sum_{i \neq j} P(I_i = 1 \text{ and } I_j = 1)$

$P(I_i = 1 \text{ and } I_j = 1) = \frac{(n-2)!}{n!}$

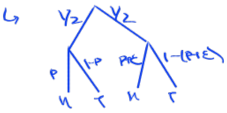
$= \frac{1}{n(n-1)}$

$Var(x) = 1.$

$E[|x-\mu| \geq 10] = \frac{1}{(10)^2} = 0.01.$

7. You have two coins: one has probability p of turning Heads, and the other has $p + \varepsilon$, for some small constant $\varepsilon > 0$. You are given one of these two coins and allowed to toss it n times. What should be the value of n such that with probability 0.9 you can correctly tell which coin was given?

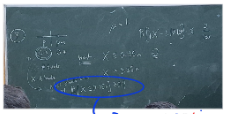
↳ confidence.



n = # trials
X = # heads.

want $X \le \frac{(p + p + \varepsilon)}{2}$

$x > \left(\frac{2p+\varepsilon}{2}\right)n$



$Pr[X > 0.35n] \le 0.1$

$X = x_1 + x_2 + \dots x_n$

$Ex = 0.3n$

$Var X = Var(x_1) + \dots Var(x_n)$

$Ex_i^2 = Ex_i \quad x_i \begin{cases} 1 & p \\ 0 & 1-p \end{cases}$

$Var x = p - p^2$

$0.3 \cdot 0.7 = 0.21$

$Var x = 0.21n$

$Pr[|x - 0.3n| > 0.05n] \le \frac{0.21}{(0.05)^2 n} = 0.1$

$n = \frac{0.21}{(0.1)(0.05)}$

↳ $Pr(|x - \mu| > \text{something}) \le \frac{Var(x)}{(\text{something})^2}$

$E_{c_1} X = n/p$

$E_{c_2} X = n/p+\varepsilon$

for coin 1 let

$X = X_1 + X_2 + \dots X_n.$

$Pr[X_i = 1] = p.$

$E_{c_1} x_i = 1/p.$

$E_{c_1} X = n/p.$

$Var(x) = Var(x_1) + Var(x_2) + \dots$
$\dots Var(x_3)$

$Var(x) = Ex^2 - (Ex)^2.$

now $Ex_i^2 = Ex$

$\because x_i \begin{cases} 1 \\ 0 \end{cases}$ i.e $x_i^2 \begin{cases} 1 \\ 0 \end{cases}$

↳ $Var(x) = Ex - (Ex)^2$

$= n/p - n^2/p \quad = n/p (1 - n).$

by using chebyshev

$Pr(|x - 1/p) \ge \left(\frac{p + p + \varepsilon}{2}\right)) < \frac{Var(x) * 4}{(2p + \varepsilon)^2}.$

ATQ.

$\frac{Var(x) * 4}{(2p + \varepsilon)^2} \ge 0.9$

$\frac{n(1-n)}{p(1-n)} \ge \frac{0.9}{4} (2p + \varepsilon)^2$

$\frac{n - n^2}{p - pn} \ge \frac{0.9}{4} (4p^2 + \varepsilon^2 + 4p\varepsilon)$

$n - n^2 = \frac{(0.9p^2 + 0.225\varepsilon^2 + 0.9p\varepsilon)}{(p - pn)}.$

$S(1) = 2$

$$\boxed{\frac{1}{K}(1) + \frac{K-1}{K}\left(1 + E_{K-1}\right) = E_K}$$

$$E_K = \frac{1}{K}(1) + \frac{K-1}{K}\left(1 + E_{K-1}\right)$$

$$1 + 2\left({}^{n}C_{2}\right)$$

Q.3

A 1 2 3 - - - n

B

X = # of student who get better rank in Case B

$$X_{ij},\ i \neq j \begin{cases} 1 & \text{if } \mathrm{rank}(i) > \mathrm{rank}(j) \\ 0. & \\ \frac{i(i-1)}{n} & \end{cases}$$

$$\frac{0}{n} + \frac{1}{n} + \frac{2}{n} + \cdots \frac{n-1}{n}$$

$$\frac{1}{n}\left(\frac{n(n-1)}{2}\right) = \frac{n-1}{2}.$$