Quiz 1

Name: **Entry Number:**

1. (5 marks) Let $Q(x, y)$ be the proposition that x has appeared on the quiz show y, where the domain of x is the set of all students in your class (call this set A) and the domain of y is the set of all quiz shows on TV (call this set B). Express the following sentence using logical quantifiers. Any quantifiers $\exists$ or $\forall$ must appear at the beginning of the statement. Besides these quantifiers and A, B, Q, you can use: $\in, \notin, \vee, \wedge, \rightarrow, =, \neq, \neg$ (the Boolean negation operator):

   "No two students in your class have appeared in the same quiz show."

   No reasoning is required, just write the expression.

   Solution: : Here one we are saying that for every pair of distinct students and for every pair quiz show, both have not appeared. So one of way of saying this is:

   $$\forall x \in A \forall x' \in A \forall y \in B Q(x, y) \wedge Q(x', y) \rightarrow (x = x').$$

2. (5 marks) Write a recurrence for the number of ternary strings (i.e., strings over the alphabet $\{0, 1, 2\}$) that do not contain 00, i.e., two consecutive 0. Give a short explanation why the recurrence is correct.

   Solution: Let X_n be the number of ternary strings which do not contain 00. Consider such a string s. If the last digit is 0, then the second last digit has to be 1 or 2 – for each such choice of the second last digit the remaining string of length $(n-2)$ satisfies the condition that it does not contain 00. Thus there are $2X_{n-2}$ such strings. Finally, if the last digit is 1 or 2, then the remaining $n-1$ digits can be any ternary string which does not contain 00. There are $2X_{n-1}$ such strings. Thus, we get:

   $$X_n = 2X_{n-1} + 2X_{n-2}.$$

   As base cases, we have $X_0 = 1, X_1 = 3$.

3. (10 marks) You are given $2n + 1$ distinct numbers from the set $\{1, 2, \ldots, 3n\}$. Prove that there must be 3 consecutive numbers in this set (for example, if $n = 3$, and you are given any 7 numbers from $\{1, 2, \ldots, 9\}$, there must be 3 consecutive numbers in this set). [Hint: You may want to use pigeonhole Principle]

   Solution: Let A be the set of $2n+1$ distinct numbers selected from the set $\{1, 2, \ldots, 3n\}$. Let B be a set of size n consisting of triplets $(1, 2, 3), (4, 5, 6), ..., (3n-2, 3n-1, 3n)$. We define a function $f : A \rightarrow B$ as follows. For any $x \in A$, there is a unique triplet $(t_1, t_2, t_3) \in B$ such that $x \in \{t_1, t_2, t_3\}$. We define $f(x) = t$. Now, A is a set of size $2n + 1$, and B is a set of size n. Therefore there must exists a triplet (t_1, t_2, t_3) such that $f^{-1}(t_1, t_2, t_3)$ has size 3. But then $t_1, t_2, t_3 \in A$. Thus, A has three consecutive numbers.

4. (10 marks) Suppose that for every pair of cities in a country, there is a direct one way road connecting them in one direction or the other, i.e., if A, B are two cities, then there is either a road from A to B, or from B to A (but not both). Use mathematical induction to show that there is a city that can be reached from every other city either directly or via exactly one other city.

   Solution: First write the induction hypothesis $S(n)$: in any set of n cities, where we have one way road between every pair of cities, there is a city that can be reached from every other city either directly or via exactly one other city.

   As base case, $S(1)$ is trivially true: there is only one city, and hence, it can be directly reached from itself.

   Now assume $S(n)$ is true, and we need to prove that $S(n+1)$ is true. Let A be a set of $n+1$ cities. Let c be any city in A, and let B be the set $A \setminus \{c\}$, i.e., cities in A other than c. Since B has size n, we can imply induction hypothesis on B – there is a city $c^* \in B$ such that every city in B has a one way road to c^* or has a one way road to c^* via some other city in B. If there is a one way road from c to c^*, then c^* is the desired city and we are done.

   Hence, the trickier case is when there is a one way road from c^* to c. Let B_1 be the set of cities in B which have one way road to c, and B_2 be the remaining cities in B for which there is a one way road from c to these cities. Note that $c^* \in B_1$. Now, two cases arise:

   - There is a city $x \in B_2$ such that there is a road from x to c^*. Then we can go from c to x to c^*. Hence, c^* is the desired city.
   - For every city $x \in B_2$ there is a road from c^* to x. But now, consider a city $x \in B_2$. We know that either there is a direct road to c^* or via some other city y. The first option is ruled out by assumption. For the second case to happen, y must lie in B_1 (otherwise we won't have a road from y to c_1^*). Since $y \in B_1$, we have a road from y to c. Thus every city in B_2 can reach c in two steps. Every city in B_1 can reach c in 1 step. Thus, c is the desired city.