COL 7004

Sample Solutions to Tutorial Problems

1.2 We prove this statement by contradiction. Suppose $A \cap C \neq \emptyset$ and $B \cap D \neq \emptyset$. We will prove that $(A \times B) \cap (C \times D) \neq \emptyset$. To prove this, let $x \in A \cap C$ and $y \in B \cap D$ (elements x and y exist because these two sets have been assumed to be non-empty). If $x \in A \cap C$, then $x \in A$. If $y \in B \cap D$, then $y \in B$. Hence, $(x, y) \in A \times B$. Similarly, $(x, y) \in C \times D$. This implies that $(x, y) \in (A \times B) \cap (C \times D)$, and therefore, $(A \times B) \cap (C \times D) \neq \emptyset$.

2.1 To show that number of equivalence class is infinite, we need to show that there are an infinite number of binary strings $x_1, x_2, \ldots$, such that no two of them are related to each other. Since each of them will lie in a distinct equivalence class, this would imply that there are infinite number of equivalence classes.

Now define string s_i as follows: it is 1 at coordinates $i, 2i, 3i, ...$; and 0 at all other coordinates. Now consider two strings s_{i_1} and s_{i_2} where $i_1 < i_2$. For any positive integer k and coordinate $j \in (ki_2, (k+1)i_2)$, $s_2(j) = 0$. But there are exactlt $i_2 - 1 \geq i_1$ coordinates in the interval $(ki_2, (k+1)i_2)$, and hence, one of these coordinates j will be a multiple of i_1. Then $s_1(j) = 1$. Thus, s_1 and s_2 differ in at least one coordinate in the interval $(ki_2, (k+1)i_2)$. Since k can take infinite values, s_1 and s_2 also differ in infinite coordinates.

2.9 Let $x < y$ be two rational numbers. Consider the number $z = \frac{\sqrt{2}x+y}{\sqrt{2}+1}$. Observe that $z > x$ because this inequality is equivalent to showing

$$\sqrt{x}x + y > \sqrt{x} + x,$$

which is true because $x < y$. Similarly $z < y$. Now we claim that z is irrational. Suppose not. Then z can be written as p/q, where p and q are integers. But then

$$\frac{p}{q} = \frac{\sqrt{2}x + y}{\sqrt{2} + 1}.$$

Simplifying the above expression, we get

$$\sqrt{2} = \frac{qy - p}{p - qx}.$$

In the right hand side, all the four quantities x, y, p, q are rationals, and hence the right hand side is a rational number. But this contradicts the fact that $\sqrt{2}$ is irrational. Therefore, z must be irrational.

3.3 We prove this by the pigeonhole principle. Let A denote the set of 52 positive integers. We define a set B of size 51 and mapping $f : A \to B$. The set B has pairs of the form $(i, 100-i)$ for $i = 1, \ldots, 49$, and it also has two more elements: 0 and 50. Now, we define the mapping f. For a positive integer x, let r be the remainder when it is divided by 100. If $r = 0$, we set $f(x) = 0$. Similarly if $r = 50$, we set $f(x) = 50$. Otherwise there is unique i such that $r \in \{i, 100-i\}$. We set $f(x) = (i, 100-i)$.

Now pigeonhole principle implies that there exist two distinct elements x and y such that $f(x) = f(y)$. If $f(x) = 0$ or $f(x) = 50$, both x and y have the same remainder when divided by 100. So $x - y$ is multiple of 100. Suppose $f(x) = (i, 100-i)$. Two cases arise: both x and y leave remainder i when divided by 100. In this case, $x - y$ is a multiple of 100. Second case is that x leaves remainder i, but y leaves remainder $100 - i$ (or the opposite case). In this case, $x + y$ is a multiple of 100.

3.8 We shall prove that the following invariant property is maintained throughout the process. Property $P(t)$: After t steps, the sum of the remaining numbers is even. We prove this by induction on t.

For base case, i.e. $P(0)$, we show that the initially the sum of the numbers is even. This is true because $1 + 2 + \ldots + 11 = 11 \times 12/2 = 66$, which is even. Suppose $P(t)$ is true. Let $x_1, \ldots, x_k$ be the numbers after t steps. We need to show that $P(t+1)$ is also true. Suppose in step $(t+1)$, we select numbers x_i and x_j, and assume $x_i < x_j$. Then we replace x_i and x_j by $x_j - x_i$. The change in the sum of the numbers is equal to $(x_j - x_i) - (x_i + x_j) = -2x_i$, which is an even number. Therefore, if $x_1 + \ldots + x_k$ is even, then the new sum, i.e., $x_1 + \ldots + x_k - 2x_i$ is also even. This proves that $P(t+1)$ is also true. At the last step, we have just one number. The invariant property shows that this must be an even number.