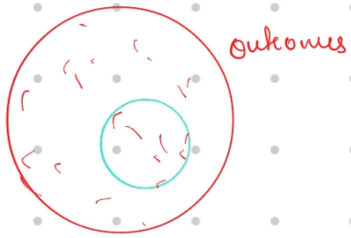
4^{th} Sept

☆ Recap

$\Pr[X=a] \equiv \Pr[\text{outcomes } \omega : X(\omega)=a]$



$$\boxed{\mathbb{E}X = \sum_a a \Pr[X=a]}$$

↳ To simplify these eqn we break them into simpler random variable.

i.e $X = x_1 + x_2 + \dots x_3$

$$\boxed{\mathbb{E}X = \mathbb{E}X_1 + \mathbb{E}X_2 + \dots \mathbb{E}X_3}$$

↳ linearity of Expectation.

* Proof of linearity of Expectation

↳ prove for smaller expression

i.e if $X = Y+Z$

$\mathbb{E}X = \mathbb{E}Y + \mathbb{E}Z$

(#) we can use this to prove for multiple variables.

i.e $X = Y+Z+W$

$X = Y + X'$

$= \mathbb{E}X = \mathbb{E}Y + \mathbb{E}X'$

$= \mathbb{E}X = \mathbb{E}Y + \mathbb{E}Z + \mathbb{E}W$.

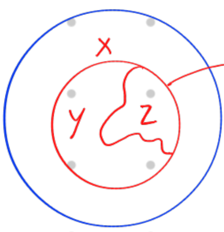
Proof

$LHS = \mathbb{E}X$

$\sum_a a \Pr[X=a] \Rightarrow \sum_a a[\Pr[Y+Z=a]]$

↳ for $y+z$ to be equal to a only if $y=b$, then $Z=a-b$

$[Y+Z=a] = \bigcup_b [Y=b, Z=a-b]$.



Y and Z are disjoint events.
↳ (a) why?

by (a) we know that probability of union of disjoint events is sum of all events

i.e $\sum_a a\left\{\sum_{\substack{b,c\\ b+c=a}} \Pr[Y=b,Z=c]\right\}$.

$= \sum_a \left\{\sum (b+c) \Pr[Y=b,Z=c]\right\}$ ∵ $a=b+c$

(#) our eqn is of form

$$\sum_a \sum_{\substack{b,c\\ b+c=a}} f(b,c)$$

which is equivalent to

$\sum_{b,c} f(b,c)$ ∵ we are basically find and summing all possible pairs of b,c which equal to some a.

$$= \sum_{\substack{b,c\\ b+c=a}} \{\Pr[Y=b,Z=c].(b+c)\}$$

$$= \underbrace{\sum_{b,c} b\{\Pr[Y=b,Z=c]\}}_{(b)} + \underbrace{\sum_{b,c} c\{\Pr[Y=b,Z=c]\}}_{(c)}$$

(#) we can say that

$[Y=b] = \bigcup_c [Y=b,Z=c]$.

$\Pr[Y=b] = \sum_c \Pr[Y=b,Z=c]. \longrightarrow (d)$

↳ Saying as we toss a coin and ask what is the probability of first coin toss is heads.

↳ Pr[first heads] = Σ Pr[first head, others head or tails].

↳ $\Pr[Y=H] = \Pr[Y=H,Z=H] + \Pr[Y=H,Z=T]$

by using (b)

$\sum_b b\left\{\sum_c \Pr[Y=b,Z=c]\right\}$

$\sum_b b.\Pr[Y=b]$ —— by (d)

Similarly for (c) we get $\sum_c c.[\Pr[Z=c]]$

hence

$\mathbb{E}X = \sum_b b.\Pr[Y=b] + \sum_c c.\Pr[Z=c]$

$\boxed{\mathbb{E}X = \mathbb{E}Y + \mathbb{E}Z}$ hence proved.

Proove: $\mathbb{E}X = \mathbb{E}Y \cdot \mathbb{E}Z$

↳ $\sum_a a \sum_{\substack{b,c \\ b \cdot c = a}} Pr[Y=b, Z=a]$ — using the similar steps as before just we get $b \cdot c = a$

↳ but now we cannot decompose this but suppose we could, then.

$\Rightarrow Pr[Y=b] \cdot Pr[Z=c]$ ——— ⓐ

$\Rightarrow \sum_{b,c} Pr[Y=b] Pr[Z=c] (b \cdot c)$

$\Rightarrow \sum_{b,c} b\, Pr[Y=b] \cdot c\, Pr[Z=c]$

↓ how is it possible

assume $\sum x_i y_j$

$= (x_1 y_1) + (x_2 y_2) + (x_1 y_2) + (x_2 y_1)$

$\Rightarrow (x_1 + x_2)(y_1 + y_2)$

$\Rightarrow \sum_b b\, Pr[Y=b] \cdot \sum_c Pr[Z=c] \cdot c$

$\Rightarrow \mathbb{E}Y \cdot \mathbb{E}Z.$

For this to happen step ⓐ must be possible but in general it is not possible.

Example $X = \begin{cases} 1 & \text{if } H \\ 0 & \text{if } T \end{cases}$

$Y = X, Z = X$ is $X = YZ$?

↳ yes for this binary ex.

by $\mathbb{E}X = 1/2$.

So, $\mathbb{E}X = \mathbb{E}Y \cdot \mathbb{E}Z$

$= 1/2 = 1/2 \cdot 1/2$

$\Rightarrow 1/2 = 1/4$ which is not possible

hence this only works if Y and Z are independent

i.e

$$\boxed{\text{if } Y \text{ and } Z \text{ are independent. } Pr[Y=b, Z=c] = Pr[Y=b] \cdot Pr[Z=c].}$$ ☆

For Sum, this is true regardless.

Example

① Toss 2 coins

$X \begin{cases} 1 & \text{if 1st coin is } H \\ 0 & \text{otherwise} \end{cases}$ $Y \begin{cases} 1 & \text{if 2nd is } H \\ 0 & \text{otherwise} \end{cases}$

$Pr[X=1, Y=1] = 1/4 = Pr[X=H] \cdot Pr[Y=H].$

∴ X and Y are independent.

② when throwing a dice

$X \begin{cases} 1 & \text{if number is even} \\ 0 & \text{otherwise.} \end{cases}$

$Y = \begin{cases} 1 & \text{number is greater than 3} \\ 0 & \text{otherwise} \end{cases}$

X and Y are not independent.

i.e

$P[X=1, Y=1] \neq Pr[X=\text{even}] \cdot Pr[Y = >3]$

③ Toss n coins

$X = \#$ heads.

$X = X_1 + X_2 + \cdots X_n$

$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ coin toss is } H \\ 0 & \text{otherwise} \end{cases}$

$\mathbb{E}X = \mathbb{E}X_1 + \mathbb{E}X_2 + \cdots \mathbb{E}X_n$

$\mathbb{E}X = 1/2 + 1/2 + \cdots 1/2 = n/2.$

④ n letters in n envelopes

X : # number of letter that go in correct envelope.

$X = X_1 + X_2 + \cdots X_n$

$X_i \begin{cases} 1 & \text{if } i^{th} \text{ letter is in correct envelope} \\ 0 & \text{otherwise.} \end{cases}$

$Pr[X_i = 1] = 1/n$ [equally likely to go in any envelope]

i.e $\mathbb{E}X_i = 1/n.$

$\mathbb{E}X = \mathbb{E}X_1 + \mathbb{E}X_2 \cdots \mathbb{E}X_n$

$= n \times 1/n$

$\boxed{\mathbb{E}X = 1}$

(5) we are given an array of length n with distinct element and we want to find the 'max'

i.e
```
Max = A[0]
for i=1 to n-1
    if [A[i] > max]
        max = A[i]   <-------- (a)
```

↳ In worst case we will execute (a) $n-1$ times.

↳ Start with a random permutation of $\{1 --- n\}$

X = # of times (a) is executed.

what is $\mathbb{E}X$.

$X = X_1 + X_2 + ---- X_{n-1}$ → not independent because if max is at j then every $i \le j$ there is a iteration.

$$X_i = \begin{cases} 1 & \text{if (a) is executed in } i^{th} \text{ iteration} \\ 0 & \text{otherwise.} \end{cases}$$

$\mathbb{E}X = \mathbb{E}X_1 + \mathbb{E}X_2 + --- \mathbb{E}X_{n-1}$

$\mathbb{E}X_i = P_r[X_i = 1] \cdot 1$

↳ we will execute (a) at i^{th} iteration iff $A[i] > A[0]$ to $A[i-1]$

i.e $P_r[X_i = 1] = 1/i$

$\mathbb{E}X = 1 + 1/2 + 1/3 + --- 1/(n-1) \approx \log_e n.$

$$\boxed{\mathbb{E}X \approx \log_e n}$$

(6) Random Walk.

X = Position after n Step.

$X = Y_1 + Y_2 + --- Y_n.$ → independent

$$Y_i = \begin{cases} 1 & \text{if i move right on } i^{th} \text{ Step.} \\ -1 & \text{if i move left on } i^{th} \text{ Step.} \end{cases}$$

$\mathbb{E}X_n = \mathbb{E}Y_1 + \mathbb{E}Y_2 ---- \mathbb{E}Y_n.$

$\mathbb{E}Y_i = 0 \quad \because 1 \cdot 1/2 + (-1) \cdot 1/2$ —— (a)

$\mathbb{E}X = 0.$

now, if i want to calculate the distance travell we will need to ignore the sign

i.e $\mathbb{E}(X_n^2)$

Let $X_n = Y_1 + Y_2 + --- Y_n$.

$X_n^2 = (Y_1 + Y_2 + --- Y_n)^2$

$= X_n^2 = Y_1^2 + Y_2^2 + -- Y_n^2 + 2 \sum_{i \ne j} Y_i Y_j$

$\mathbb{E}[X_n^2] = \mathbb{E}(Y_1)^2 + \mathbb{E}(Y_2)^2 + --- 2 \sum_{\substack{i,j \\ i \ne j}} \mathbb{E}(Y_i \cdot Y_j)$

$\mathbb{E}(Y_i)^2 = 1.$

$\mathbb{E}(Y_i \cdot Y_j) = \mathbb{E}Y_i \cdot \mathbb{E}Y_j \quad \because$ they are independent and $\mathbb{E}Y_i, \mathbb{E}Y_j = 0$

↳ by (a)

i.e

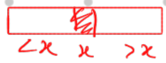
$\mathbb{E}(X_n)^2 = 1 + 1 + -- n \text{ times} + 0$

$$\boxed{\mathbb{E}(X_n)^2 = n.}$$

(7) Randomize Quicksort.

(1) pick a random element

(2) divide the array in 2 parts.



(3) Sort both pieces.

(#) Every pair is going to be compared at most once i.e running time is at most n^2.

↓ remaining.