COL 7004 August 25, 26, 28

TUTORIAL SHEET 5

1. You are given a 10×10 grid consisting of 100 cells. Two cells are neighbors if they share a side. Initially 9 cells are arbitrarily chosen and colored, and all other cells are left uncolored. A move colors a cell provided it has two colored neighboring cells. Prove that no matter how the initial cells are chosen, no sequence of moves can result in all cells being colored. [Hint: what happens to the number of edges (an edge is a boundary of a cell) which have only one side colored?]

2. In the class, we had considered the following game: there are three bags A, B and C, each containing a certain number of pebbles. In each step, we select two bags, take one pebble out of each and transfer these two pebbles to the third bag. Our goal is to end up with a configuration where two of the bags are empty. Let a, b, c be the initial number of pebbles in these three bags respectively.We had shown that a necessary condition for the game to end is that at least two of the three numbers, a, b and c, must leave the same remainder mod 3. Now show that this condition is sufficient as follows: suppose $a \equiv b (\text{mod } 3)$ and $a \leq b$. Give a sequence of moves that preserve the following invariant: let a', b', c' be the number of pebbles in the three bags at some point of time. Then, each move decreases b' and maintains $a' \equiv b' (\text{mod } 3)$. Why is this invariant enough to ensure that the game will end?

3. Show using induction that starting from the permutation $(1, 2, ..., n)$, you can arrive at any given permutation σ of $\{1, 2, \ldots, n\}$ by a sequence of transpositions: a transposition involves taking two adjacent symbols and interchanging them. For example, $(1, 2, 3, 4) \to (1, 3, 2, 4)$ is a trasposition.

4. Consider an odd number of cells, i.e., $2n + 1$ for some n, arranged in a row where the center cell is empty, and each cell to the left of the empty cell has a black pebble and each cell to the right has a white pebble. Thus, the initial configuration is $b^n \circ w^n$, where $\circ$ denotes an empty cell. There are two possible moves to rearrange the pebbles: (1) (shift) move a pebble to a neighboring cell if it is empty (i.e., you can go from $p\circ$ to $\circ p$, or from $\circ p$ to $p\circ$, and (2) (jump) jump a pebble over an adjacent pebble to land in an empty cell, i.e., from $pq\circ$ to $q \circ p$, or from $\circ pq$ to $q \circ p$. Show that all the black and white pebbles can be interchanged using only these moves, i.e., you can end up with $w^n \circ b^n$. Use the following sequence of steps:

   - Show that for any string S consisting of w, b and $\circ$, we can go from the configuration $\circ S$ to $S\circ$.
   - Now use the question #3 to show that you can end up with *any* final configuration.

5. $A[0..N-1]$ is an array of distinct integers where N is even and $N > 0$. Find the largest and smallest numbers and store them in variables x and y, respectively, using the following algorithm. First, compare every adjacent pair of numbers, $A[2i]$ with $A[2i+1]$, for all $i, i \geq 0$, storing the smaller one in $A[2i]$ and the larger one in $A[2i+1]$. Next, find the smallest number over all even-indexed elements and store it in x, and find the largest one over all odd-indexed elements and store it in y. Prove the correctness of this algorithm.

6. Let $A[0..N-1, 0..N-1], N \geq 1$, be a matrix consisting of binary numbers where the numbers in each column are non-decreasing. The following program claims to find the first row which has all 1s. Prove its correctness.

```
i = 0, j = N-1
while i < N AND j >= 0:
    if A[i, j] = 1
          then j := j-1
    else i := i+1

if i < N then
     print(row i is the desired row)
else print(there is no such row)
```