28th Aug.

↳ Recurrences.

① # of binary string that do not have 2 consecutive zeros (0s)

↳ $\boxed{x^n S_n = x^n S_{n-1} + x^n S_{n-2} \quad n \geq 2 \quad S_0 = 1,\ S_1 = 2}$

generating functions.

$x \sum_{n\geq 2} S_{n-1} x^{n-1} \Rightarrow x(F(x) - S_0)$

↳ $\underbrace{\sum_{n\geq 2} S_n x^n}_{F(x) - S_0 - S_1 x} = \sum_{n\geq 2} S_{n-1} x^n + \underbrace{\sum_{n\geq 2} S_{n-2} x^n}_{x^2(F(x))} \quad n \geq 2$

$\Rightarrow F(x) - 1 - 2x = x(F(x) - 1) + x^2 F(x)$

$\Rightarrow F(x) = \dfrac{1+x}{1+x-x^2}$

$\Rightarrow \dfrac{1+x}{(1-\alpha x)(1-\beta x)}$

$1+x-x^2 = (1-\alpha x)(1-\beta x)$

$= 1 + \beta x^2 - (\alpha + \beta) x$

$\alpha + \beta = -1, \ \alpha\beta = -1, \ 1/\beta + \beta = -1$

$\alpha, \beta = \dfrac{-1 \pm \sqrt{5}}{2}$

$\Rightarrow \dfrac{1+x}{(1-\alpha x)(1-\beta x)} = \dfrac{a}{1-\alpha x} + \dfrac{b}{1-\beta x}$

i.e $1 + x = a - a\beta x + b - b\alpha x$

i.e $a + b = 1$

$(a\beta + b\alpha) = -1$

$\boxed{S_n = a_1 \alpha^n + b_1 \beta^n}$

↳ for fibonacci

$\boxed{\text{(#) Idea is to generate closed form of 'F(x)'}}$

② # of ways of Multiplying $\overbrace{x_0 \cdots x_n}^{c_n}$

we got

↳ $C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots C_{n-1} C_0$

$C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}$

↳ multiply both side by x^n — ① always first thing to do

$\Rightarrow \underbrace{\sum_{n\geq 1} C_n x^n}_{F(x) - C_0} = \sum_{n\geq 1} \sum_{i=0}^{n-1} C_i C_{n-i-1} x^n$

first term is $C_0 = 1$

↳ now if we have $F(x) \cdot F(x)$

$\Rightarrow (C_0 + C_1 x + C_2 x^2 + \cdots)(C_0 + C_1 x + C_2 x^2 + \cdots)$

↳ what is the coefficient of x^n

↳ $\boxed{C_0 C_n + C_1 C_{n-1} + C_2 C_{n-2} + \cdots C_n C_0}$

↳ write x^n as $x^1 \cdot x^{n-1}$ in RHS

i.e

$\Rightarrow x \boxed{\sum_{n\geq 1} \sum_{i=0}^{n-1} C_i C_{n-i-1} x^{n-1}}$

this is of the form $F(x)^2$.

$\Rightarrow F(x) - C_0 = x F(x)^2$

$\Rightarrow F(x) - 1 = x F(x)^2$

$F(x) = \dfrac{1 \pm \sqrt{1-4x}}{2x}$.

we will choose $\dfrac{1 - \sqrt{1-4x}}{2x}$ why?

↳ $F(x) = \dfrac{1 - \sqrt{1-4x}}{2x}$

$(1 + \alpha x)^{1/2} = \sum_{n\geq 0} (\alpha x)^n \binom{1/2}{n}$?!

★ Probability And Random Variable

"Sample Space"
where we are observing something and we get a set of outcomes (Ω)

for each outcome $\omega \in \Omega$ we assign it probability $P(\omega)$.

↳ Probability is a Totally ordered function $P: \Omega \to \mathbb{R}$.

↳ Conditions Satisfied
① $P(\omega) \geq 0 \quad \forall \omega \in \Omega$
② $\sum_{\omega} P(\omega) = 1$

↳ for quicksort : outcomes are a trace of path/partition made by algo.

(#) Infinite set of outcomes → selecting a number b/w $(0, N)$
↳ Prob of getting 0.3 from $(0,1)$ $0 \because \frac{1}{\infty}$

* Event (E) → as E is set we can have general set operations such as $E_1 \cup E_2$, $E_1 \cap E_2$
↳ Subset of Ω
↳ $E \subseteq \Omega$.

$$P(E) = \sum_{\omega \in E} P(\omega).$$

* Independent Event.
↳ for two Events E_1 and E_2
↳ these two events are independent if

$$\boxed{P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)}$$

Examples

① Throw 2 die
E_1 = Sum of the value = 6
E_2 = the first die is odd

$P(E_1) = 5/36 \qquad P(E_2) = 1/3$

$P(E_1 \cap E_2) = 1/13$

i.e $P(E_1) \cdot P(E_2) = 5/36 \times 1/3$

↳ hence they are not independent.

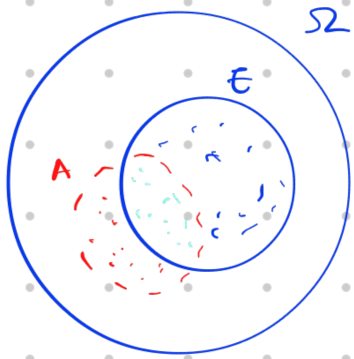
* Conditional prob.
↳ Put some condition on a event.
↳ Suppose E are a set of outcomes that have happened.

↳ now this set E becomes our Ω
↳ $\forall \omega \in E$ $P(\omega)$ changes as other outcomes from Ω are not part of our system.

↳ we will have to scale them, that we will do by.

$$\forall \omega \in E \quad P'(\omega) = \frac{P(\omega)}{\sum_{\omega \in E} P(\omega)} \to P(E).$$

$$\Rightarrow \quad P(\omega/E) = \begin{cases} \dfrac{P(\omega)}{P(E)} & \text{if } \omega \in E \\ 0 & \text{if } \omega \notin E \end{cases}$$



i.e the prob of happening $A \cap E$, $\because$ $A - E$ are all 0

↳ $P(A|E)$ → prob of A given E has happend.

↳ $\sum_{\substack{\omega \in A \\ \omega \notin B}} 0 + \sum_{\substack{\omega \in A \\ \omega \in B}} \frac{P(\omega)}{P(E)}$

↳ $\boxed{\dfrac{P(A \cap E)}{P(E)} = P(A|E)}$

(#) If A and E are independent

$\boxed{P(A|E) = P(A)}$

* Basic facts

1. $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P_1(A_1 \cap A_2)$

Special Case — for die example $A_1 \to (2,4)$, $A_2 = (3,5)$

* $A_1 \cap A_2 = \emptyset$
  ↳ $P(A_1 \cup A_2) = P(A_1) + P(A_2)$
* Union bound → really imp later on
  ↳ $P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$

Examples

1. 4 envelope — name on it
   4 letter — name on it.
   ↳ we put any letter in any envelope.

↳ Possible outcomes $= 4!$

P(Exactly 2 letters are in right envelope)

↳ $\frac{{}^4C_2}{4!}$ $\boxed{A \mid B \mid C \mid D}$

P(at least 3)

↳ P(none correct) + P(1 correct) + P(2 correct) + P(3 correct)

or

1 − P(4 correct).

* Random Variable.

↳ A random Variable X is a mapping from
$\Omega \to \mathbb{R}$

example

1. Throw 10 coins, X = no. of heads.
   $|\Omega| = 2^{10}$, $X = \{0,1,2,3,4,5,6,7,8,9,10\}$
   $|X| = 11$

2. we have n bins and n balls

   □ □ □ - - - - □ n bins

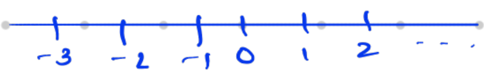
   n balls.

   ↳ we can fill bins with any no. of balls

   X = no. of bins that are empty

   $X = \{0,1,2 - - - n-1\}$

   $|X| = n$.

3. 

you can move (x−1) or (x+1) — why?

↳ Set of outcomes : infinite (countable)

1^{st} Sept

Bayes Rule

↳ $P(A|B)$ — If B has happened what is the probability of A to happen.

hypothesis → observation

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

$$\boxed{P(A|B) = \frac{P(B|A).P(A)}{P(B)}}$$

↳ It is Expressing $P(A|B)$ in terms of $P(B|A)$

Example

(1) There are 2 boxes with coloured balls

5B 5R	9B 1R

↳ choose a random bag and draw a ball

↳ The ball happens to be red. what is the probability that it chose bag 1

↳ A (hypothesis)

→ B (observation)

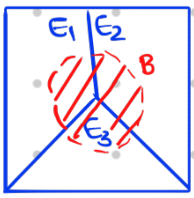
↳ $P(A|B) = \frac{P(B|A).P(A)}{P(B)}$

$$= \frac{\frac{1}{2}\times\frac{1}{2}}{6/20} = 5/6.$$

(#) A set of Events E_1, E_2 is said to be Partition the set of outcomes

$E_1 \cap E_2 = \emptyset$

$\bigcup_i E_i = U$



$$\boxed{Pr[B] = P[B\cap E_1] + P(B\cap E_2) \cdots}$$

Or

$$\boxed{P[B] = P[B|E_1]P(E_1) + P(B|E_2)P(E_2) \cdots}$$

↳ In our example E_1 = bag 1 is chosen
E_2 = bag 2 is chosen

$$P[B] = P[B|E_1]P(E_1) + Pr[B|E_2]P[E_2]$$

$$= 1/4 + 1/20 = 3/10$$

★ Random Variable. $[X : \Omega \to \mathbb{R}]$

Examples (contd.)

(1) Suppose we have a program which tosses x coins.

↳ and to get an output it takes some y_i amount of time $\forall x \in \Omega$.

then $X : \Omega \to \mathbb{R}$
$|\Omega| = 2^{10}$
X: # heads.

as per real life X will be close to $\frac{1}{2}(50)$

Q. how do we capture the expected value of x.

↳ Perform experiment

↳ $\boxed{\mathbb{E}X := \sum i.\, Pr[X=i]}$ ★

(#) $Pr[X=i] \to Pr\left(\begin{array}{l}\text{outcomes } \omega \in \Omega \\ \text{where } X(\omega) = i\end{array}\right)$

↳ Suppose we conduct this experiment N times

Suppose $X = x_j$ in the j^{th} experiment

Output = $\frac{x_1+x_2+\cdots x_n}{N}$ | let $P_i = Pr[X=i]$

(a) Suppose in an "ideal world", we see i heads exactly $P_i N$ times

↳ think with an ex: like $N=100$, $i=10$

$$\frac{x_1+x_2+\cdots x_n}{N} = \frac{\sum i \cdot P_i \cdot \cancel{N}}{\cancel{N}}$$

$$\boxed{= \sum i P_i}$$

↳ mathematical proof of (a)

Law of Large Numbers

↳ Let X be a random Variable

Suppose we Conduct this experiment N times, and x_i denotes the value of X in the i^{th} experiment

Then, $\frac{x_1 + x_2 + \cdots x_N}{N} \longrightarrow \mathbb{E}X$ as $N \to \infty$

* Examples

① Toss n coins X # heads

$$\mathbb{E}X = \sum_{i=0}^{n} i \, P_r[X=i] = \sum_{i=0}^{n} i \cdot \binom{n}{i} \left(\tfrac{1}{2}\right)^n$$
$$= n/2$$

② n envelopes, n letter.

↳ randomly insert.

X = # letters in Correct envelope.

$$\mathbb{E}X = \sum_{i=0}^{n} i \left[\frac{{}^nC_i \, [\text{derangement}]}{n!} \right]$$

→ no. of letters correctly placed.

no. of letters not " "

(#) Derangements : how many permutation of $\{1, 2 \cdots k\}$ are there where $\sigma i \neq i$ for each i

↳ 2143

2341

⋮

$$\mathbb{E}X = \sum_{i=0}^{n} i \left[\frac{{}^nC_i \, [\text{derangement}]}{n!} \right] = 1.$$

③

3.1 : not biased Coin $P_r[H] = P_r[T] = 1/2$

keep tossing 1 coin till you see a head

↳ $H, TH, TTH, TTTH \cdots\cdots T^{(x-1)}H$

↳ X are no of coin tosses.

$$\boxed{\mathbb{E}X = \sum_{i=1}^{\infty} i \cdot \frac{1}{2^i} = 2.}$$

3.2 $P_r[H] = P$ [Biased coin]

$$\boxed{\mathbb{E}X = \sum_i i \cdot \underbrace{(1-P)^{i-1}}_{\text{not getting heads}} P = 1/P}$$

④ Random walk example.

--- --- -3 -2 -1 0 1 2 3 --- ---

↳ equally likely to move left [L] or Right [R] starting from 0

X = distance from origin.

$\boxed{\mathbb{E}X \approx \sqrt{n}}$ DIY : But can?

↳ This means differce of L and R will be $\sqrt{n}$

* Linearity of Expectation.

↳ Let X be a random variable.

– Suppose there are other random variables $X_1, X_2, X_3 \cdots X_k$ such that

$X = X_1 + X_2 + \cdots X_k$.

$$\boxed{\mathbb{E}X = \mathbb{E}X_1 + \mathbb{E}X_2 + \mathbb{E}X_3 + \cdots \mathbb{E}X_k}$$

Examples.

① n coin tosses

X : # heads

Define $X_i \begin{cases} 0 \text{ if } i^{th} \text{ coin toss is tails} \\ 1 \text{ if } i^{th} \text{ coin toss is heads} \end{cases}$

$X = X_1 + X_2 + \cdots X_n$

$\mathbb{E}X = \mathbb{E}X_1 + \mathbb{E}X_2 + \cdots\cdots \mathbb{E}X_n$

↳ $\mathbb{E}X_i = 1 \cdot P[X_i = 1] + P[X_i = 0] \cdot 0$

$\mathbb{E}X_i = 1 \cdot 1/2$

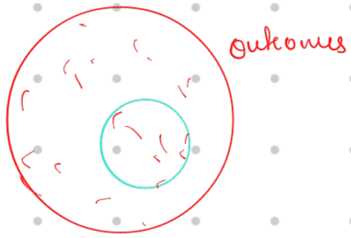
$\mathbb{E}X_i = 1/2$.

$$\boxed{\mathbb{E}X = n/2}$$

4^{th} Sept

☆ Recap

$Pr[X=a] \equiv Pr[\text{outcomes } \omega : X(\omega)=a]$



$$\boxed{\mathbb{E}X = \sum_a a \, Pr[X=a]}$$

↳ To simplify these eqⁿ we break them into simpler random variable.

i.e $X = x_1 + x_2 + \dots x_3$

$$\boxed{\mathbb{E}X = \mathbb{E}X_1 + \mathbb{E}X_2 + \dots \mathbb{E}X_3}$$

↳ linearity of Expectation.

☆ Proof of linearity of Expectation

↳ prove for smaller expression

i.e if $X = Y + Z$

$\mathbb{E}X = \mathbb{E}Y + \mathbb{E}Z$

(#) we can use this to prove for multiple variables.

i.e $X = Y + Z + W$

$X = Y + X'$

$= \mathbb{E}X = \mathbb{E}Y + \mathbb{E}X'$

$= \mathbb{E}X = \mathbb{E}Y + \mathbb{E}Z + \mathbb{E}W$.

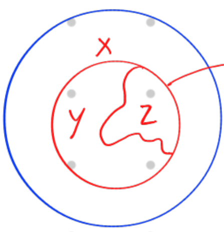
Proof

$LHS = \mathbb{E}X$

$\sum_a a \, Pr[X=a] \Rightarrow \sum_a a[Pr[Y+Z=a]]$

↳ for $y+z$ to be equal to a only if $y=b$, then $Z=a-b$

$[Y+Z=a] = \bigcup_b [Y=b, Z=a-b]$.



y and z are disjoint events.

↳ (a) why?

by (a) we know that probability of union of disjoint events is sum of all events

i.e $\sum_a a \left\{ \sum_{\substack{b,c \\ b+c=a}} Pr[Y=b, Z=c] \right\}$

$= \sum_a \left\{ \sum (b+c) \, Pr[Y=b, Z=c] \right\} \quad \because a = b+c$

(#) our eqⁿ is of form

$$\sum_a \sum_{\substack{b,c \\ b+c=a}} f(b,c)$$

which is equivalent to

$\sum_{b,c} f(b,c)$ ∵ we are basically find and summing all possible pairs of b,c which equal to some a.

$= \sum_{\substack{b,c \\ b+c=a}} \{Pr[Y=b, Z=c] \cdot (b+c)\}$

$$= \underbrace{\sum_{b,c} b \{Pr[Y=b, Z=c]\}}_{(b)} + \underbrace{\sum_{b,c} c \{Pr[Y=b, Z=c]\}}_{(c)}$$

(#) we can say that

$[Y=b] = \bigcup_c [Y=b, Z=c]$.

$Pr[Y=b] = \sum_c Pr[Y=b, Z=c]. \longrightarrow (d)$

↳ Saying as we toss a coin and ask what is the probability of first coin toss is heads.

↳ $Pr[\text{first heads}] = \sum Pr[\text{first head, others head or tail}]$.

↳ $Pr[Y=H] = Pr[Y=H, Z=H] + Pr[Y=H, Z=T]$

by using (b)

$\sum_b b \left\{ \sum_c Pr[Y=b, Z=c] \right\}$

$\sum_b b \cdot Pr[Y=b] \xrightarrow{\quad} \text{by } (d)$

Similarly for (c) we get $\sum_c c \cdot [Pr[Z=c]]$

hence

$\mathbb{E}X = \sum_b b \cdot Pr[Y=b] + \sum_c c \cdot Pr[Z=c]$

$\boxed{\mathbb{E}X = \mathbb{E}Y + \mathbb{E}Z}$ hence proved.

Proove: $\mathbb{E}X = \mathbb{E}Y \cdot \mathbb{E}Z$

↳ $\sum_a a \sum_{\substack{b,c \\ b\cdot c = a}} Pr[Y=b, Z=a]$ — using the similar steps as before just we get $b \cdot c = a$

↳ but now we cannot decompose this but suppose we could. then.

$\Rightarrow Pr[Y=b] \cdot Pr[Z=c]$ —— (a)

$\Rightarrow \sum_{b,c} Pr[Y=b] Pr[Z=c] (b \cdot c)$

$\Rightarrow \sum_{b,c} b\, Pr[Y=b) \cdot c\, Pr[Z=c]$

↓ how is it possible

$\Rightarrow \sum_b b\, Pr[Y=b) \cdot \sum_c Pr[Z=c] \cdot c$

assume $\sum x_i y_j$
$= (x_1 y_1) + (x_2 y_2) + (x_1 y_2) + (x_2 y_1)$
$\Rightarrow (x_1 + x_2)(y_1 + y_2)$

$\Rightarrow \mathbb{E}Y \cdot \mathbb{E}Z.$

for this to happen step (a) must be possible but in general it is not possible.

Example $X = \begin{cases} 1 & \text{if } H \\ 0 & \text{if } T \end{cases}$

$Y = X,\ Z = X$ is $X = YZ$?
↳ yes for this binary. ex.

by $\mathbb{E}X = 1/2.$

So, $\mathbb{E}X = \mathbb{E}Y \cdot \mathbb{E}Z$
$= 1/2 = 1/2 \cdot 1/2$
$\Rightarrow 1/2 = 1/4$ which is not possible

hence this only works if Y and Z are independent
i.e

$\boxed{\text{if } Y \text{ and } Z \text{ are independent. } Pr[Y=b, Z=c] = Pr[Y=b] \cdot Pr[Z=c].}$ ☆

for sum, this is true regardless.

Example

(1) Toss 2 coins

$X \begin{cases} 1 & \text{if 1st coin is H} \\ 0 & \text{otherwise} \end{cases}$ $\quad Y \begin{cases} 1 & \text{if 2nd is H} \\ 0 & \text{otherwise} \end{cases}$

$Pr[X=1, Y=1] = 1/4 = Pr[X=H] \cdot Pr[Y=H].$

$\therefore$ X and Y are independent.

(2) when throwing a dice

$X \begin{cases} 1 & \text{if number is even} \\ 0 & \text{otherwise.} \end{cases}$

$Y = \begin{cases} 1 & \text{number is greater than 3} \\ 0 & \text{otherwise} \end{cases}$

X and Y are not independent.
i.e

$P[X=1, Y=1] \neq Pr[X=\text{even}] \cdot Pr[Y=>3]$

(3) Toss n coins
$X = \#$ heads.

$X = X_1 + X_2 + \dots X_n$

$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ coin toss is H} \\ 0 & \text{otherwise} \end{cases}$

$\mathbb{E}X = \mathbb{E}X_1 + \mathbb{E}X_2 + \dots \mathbb{E}X_n$
$\mathbb{E}X = 1/2 + 1/2 + \dots 1/2. = n/2.$

(4) n letters in n envelopes

X : # number of letter that go in correct envelope.

$X = X_1 + X_2 + \dots X_n$

$X_i \begin{cases} 1 & \text{if } i^{th} \text{ letter is in correct envelope} \\ 0 & \text{otherwise.} \end{cases}$

$Pr[X_i = 1] = 1/n$ [equally likely to go in any envelope]

i.e $\mathbb{E}X_i = 1/n.$

$\mathbb{E}X = \mathbb{E}X_1 + \mathbb{E}X_2 \dots \mathbb{E}X_n$
$= n \times 1/n$

$\boxed{\mathbb{E}X = 1}$

(5) We are given an array of length n with distinct element and we want to find the 'max'

i.e

```
Max = A[0]
for i = 1 to n-1
    if [A[i] > max]
        max = A[i]   <---- (a)
```

↳ In worst case we will execute (a) $n-1$ times.

↳ start with a random permutation of $\{1 --- n\}$

X = # of times (a) is executed.

What is $\mathbb{E}X$.

$X = X_1 + X_2 + ---- X_{n-1}$ → not independent because if max is at j then every $i \le j$ there is a iteration.

$$X_i \begin{cases} 1 & \text{if (a) is executed in } i^{th} \text{ iteration} \\ 0 & \text{otherwise.} \end{cases}$$

$\mathbb{E}X = \mathbb{E}X_1 + \mathbb{E}X_2 + --- \mathbb{E}X_{n-1}$

$\mathbb{E}X_i = P_r[X_i = 1] . 1$

↳ we will execute (a) at i^{th} iteration iff

$A[i] > A[0]$ to $A[i-1]$

i.e $P_r[X_i = 1] = 1/i$

$\mathbb{E}X = 1 + 1/2 + 1/3 + ---- 1/_{n-1} \approx \log_e n.$

$\boxed{\mathbb{E}X \approx \log_e n}$

(6) Random Walk.

X = Position after n Step.

$X = Y_1 + Y_2 + --- Y_n.$ → independent.

$$Y_i \begin{cases} 1 & \text{if i move right on } i^{th} \text{ Step.} \\ -1 & \text{if i move left on } i^{th} \text{ Step.} \end{cases}$$

$\mathbb{E}Y_n = \mathbb{E}Y_1 + \mathbb{E}Y_2 ---- \mathbb{E}Y_n.$

$\mathbb{E}Y_i = 0 \quad \because 1 * 1/2 + (-1) * 1/2$ ---- (a)

$\mathbb{E}X = 0.$

Now, if i want to calculate the distance travell we will need to ignore the sign

i.e $\mathbb{E}(X_n^2)$

let $X_n = Y_1 + Y_2 + --- Y_n.$

$X_n^2 = (Y_1 + Y_2 + --- Y_n)^2$

$= X_n^2 = Y_1^2 + Y_2^2 + -- Y_n^2 + 2 \sum_{i \ne j} Y_i Y_j$

$\mathbb{E}(X_n^2) = \mathbb{E}(Y_1)^2 + \mathbb{E}(Y_2)^2 + ---- 2 \sum_{\substack{i,j \\ i \ne j}} \mathbb{E}(Y_i . Y_j)$

$\mathbb{E}(Y_i)^2 = 1.$

$\mathbb{E}(Y_i . Y_j) = \mathbb{E}Y_i . \mathbb{E}Y_j \quad \because$ they are independent and $\mathbb{E}Y_i, \mathbb{E}Y_j = 0$ ↳ by (a)

i.e

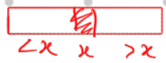
$\mathbb{E}(X_n)^2 = 1 + 1 + -- n \text{ times} + 0$

$\boxed{\mathbb{E}(X_n)^2 = n.}$

(7) Randomize Quicksort.

(1) pick a random element

(2) divide the array in 2 parts.



(3) Sort both pieces.

(#) Every pair is going to be compared at most once i.e running time is at most n^2.

↓ remaining.

8th Sept

* Linearity of Expectations

① R. QuickSort

$$\underbrace{\quad\quad}_{<A_n} \; A_n \; \underbrace{\quad\quad}_{>A_n}$$

Let X be the number of comparisions.

↳ 2 pairs are compared at most once.

Let us say that sorted order is
$a_1 < a_2 < a_3 \cdots a_n$

$$X_{ij} \begin{cases} 1 & \text{if } a_i \text{ and } a_j \text{ are compared} \\ 0 & \text{otherwise.} \end{cases}$$

i.e $X = \sum_{\text{Pair}(i,j)} X_{ij}$

↳ $\mathbb{E}X = \mathbb{E} \sum_{\text{Pair}(i,j)} X_{ij}$ ⊕ we can take expectation inside

↳ $\mathbb{E}X = \sum_{\text{Pair}(i,j)} \mathbb{E}X_{ij}$ → $\Pr[X_{ij}=1]$
⇒ $\Pr$[we compare a_i and a_j]

↳ we don't compare element for the bigger array when we split the array.

↳ we compare when either a_i or a_j becomes Pivot before any other becomes the pivot.

$\Pr[X=1] = \frac{2}{j-i+1}$ } probability of selecting i or j as first pivote.

⇒ $\boxed{\mathbb{E}X = \sum_{\text{Pair}(i,j)} \frac{2}{j-i+1}}$ → expected number of comp required in R. Quicksort.

↳ $\mathbb{E}X = 2 + \frac{3}{3} + \frac{2}{4} + \cdots - \frac{2}{n}$.

$\mathbb{E}X = 2\left(1 + \frac{1}{3} + \frac{1}{4} \cdots \frac{1}{n}\right)$

$\boxed{\mathbb{E}X = 2\log_e n.}$

Q. Can we say that it is unlikely that random variable X is far from $\mathbb{E}X$?

↳ MARKOV'S INEQUALITY

↳ Let $X \geq 0$ be a Random variable

the $\Pr[X > t] \leq \frac{\mathbb{E}X}{t}$ where $t > 0$

↳ lets say in a class of 100 average marks is 20 and we don't know what is most marks

then we can say. → not possible with negative marks.

$\Pr[\text{Scoring more than } 60] \leq \frac{20}{60}$

$\Pr[X > 60] \leq \frac{1}{3}$

* Proof.

$\mathbb{E}X = \sum_{i \geq 0} i . \Pr[X=i]$.

$= \sum_{\substack{i \geq 0 \\ i < t}} i . \Pr[X=i] + \sum_{\substack{i \geq 0 \\ i \geq t}} i . \Pr[X=i]$.

$\geq \sum_{\substack{i \geq 0 \\ i \geq t}} i . \Pr[X=i] \geq t \underbrace{\sum_{i \geq t} \Pr[X=i]}_{\Pr[X \geq t]}$

↳ every i is at least t so we can make it constant.

$\mathbb{E}X \geq t \Pr[X \geq t]$

$\boxed{\Pr[X \geq t] \leq \frac{\mathbb{E}X}{t}}$

Example

① Toss a coin 100 times.

Pr[getting more than 900 Heads]

$\Pr[X \geq 900] \leq$?

⊕ Expectation doest potray the whole story.

Prob ↓	A runs.	B	
1/5	100	20	Match 1.
4/5	0		Match 2.
$\mathbb{E}X$	20	20	

↳ B is more consistence while A has a lot more variance

② We are given MCQ with +1 and +0 Marking scheme

Strategy ①: Pick random from ABCD and then all answers are B.

Strategy ②: for each Question pick ABCD randomly

Let X = no. of correct answers

$$X_i \begin{cases} 1 & \text{if } i^{th} \text{ Question is right} \\ 0 & \text{otherwise.} \end{cases}$$

$P[i^{th}$ question is correct$] = 1/4$

$\boxed{\mathbb{E}X = n/4.}$

(#) But what if the exam setter selected all optⁿ to be same then

Strategy① gets full marks with $1/4$ Prob and 0 marks with $3/4$ prob.

while strategy ② gets the answer correctly consistently.

Q. How do we define Variance

↳ Intuitively it tells us how far away R.V goes away from the mean

Example

↳ Scores of a batsman.

5, 10, 15, 20, 15, 35, 40, 45, 0, 1 - - - -

$\mathbb{E}X = 20$

$[X_i - \mathbb{E}X]$

↳ $-15, -10, -5, 0 - - - - - - -$

↳ we cannot add as +ve and -ve cancel each other out hence we square and then add

i.e

$Var(X) = \mathbb{E}[X_i - \mathbb{E}X]^2$ or $\sum P_r[X=i].(i-\mathbb{E}X)^2$

Or

$\mathbb{E}(X^2 + u^2 - 2Xu)$

$= \mathbb{E}X^2 + u^2 - 2u\mathbb{E}X$

$\Rightarrow \mathbb{E}X^2 + u^2 - 2u^2 \quad \because \mathbb{E}X = u$

$\boxed{Var(X) = \mathbb{E}X^2 - u^2}$

Examples

① batsman

A

Prob	Scores
$1/5$	100
$4/5$	0

Mean = 20

$Var = 1/5(100-20)^2 + 4/5(0-20)^2$

B

Prob	Score
1	20
0	0

$Var = 1(20-20)^2 + 0(0-20)^2$

$\boxed{Var = 0}$

② for question paper example from before examiner sets A as solution for all questions.

	Prob	marks	$(marks)^2$
Strategy ①	$1/4$	n	n^2
	$3/4$	0	0

$\mathbb{E}X = n/4$ $\qquad \mathbb{E}X^2 = n^2/4$

$Var = \mathbb{E}X^2 - n/4.$

$= n^2/4 - n/4 = \frac{n}{4}(n-1).$

✱ Observations.

① $\mathbb{E}KX = K\mathbb{E}X$.

② $Var(KX) = K^2 Var(X).$

③ if $X_1, X_2 - - - X_n$ all are independent

$\Rightarrow Var(X_1) + Var(X_2) + - - - Var(X_n) = \mathbb{E}(X-u)^2.$
$= Var(X)$

↳ Proof

$u_i = \mathbb{E}X_i$

$u = u_1 + u_2 + u_3 + - - u_n$ (Linearity)

$\mathbb{E}(X-u)^2 = \mathbb{E}[(X_1+X_2+X_3 - - X_n) - (u_1+u_2+ - - u_n)]^2$

$= \mathbb{E}[(X_1-u_1)^2 + (X_2-u_2)^2 + - - - (X_n-u_n)^2]$

$\Rightarrow \mathbb{E}(X_1-u_1)^2 + \mathbb{E}(X_2-u_2)^2 + - - - - \mathbb{E}(X_n-u_n)^2 + 2\sum_{i \neq j} \mathbb{E}((X_i-u_i)(X_j-u_j))]$

$\mathbb{E}(x_1-\mu_1)^2 + \mathbb{E}(x_2-\mu_2)^2 + \cdots \mathbb{E}(x_n-\mu_n)^2$
$+ 2\sum_{i\neq j} \underbrace{\mathbb{E}[(x_i-\mu_i)(x_j-\mu_j)]}_{0}$

(4) Let X be a random Variable

$Y = \frac{X_1+X_2+\cdots X_n}{n}$ where x_i is an independent copy of x.

$Var(Y) = \frac{1}{n^2}[Var(X_1+\cdots X_n)]$

$= \frac{1}{n^2}[Var(X_1) + Var(X_2) + \cdots Var(X_n)]$

$\Rightarrow \frac{n\,Var(X)}{n^2} \Rightarrow \boxed{\frac{Var(x)}{n}}$

↳ taking average of multiple experiments reduces the Variance.

Examples

(1) n times tossing of coins
$X \to$ no. of heads.

$\mathbb{E}x = n/2$ $\quad X = X_1+X_2+\cdots X_n$ → independent

$X_i \begin{cases} 1 & \text{if } i^{th} \text{ is head} \\ 0 & \text{otherwise.} \end{cases}$

$Var(X) = Var(X_1) + \cdots Var(X_n)$

$Var(X_i) = \mathbb{E}x_i^2 - (\mathbb{E}x_i)^2$

$Var(X_i) = 1 * 1/2 - (1/2)^2$

$= 1/4.$

$\boxed{Var(x) = n * 1/4}$

(#) If a random Variable has small Variance then Chances of deviating away from $\mathbb{E}x$ are small.

↳ (1) $Pr[X \geq t] \leq \frac{\mathbb{E}x}{t}.$

(2) $Pr[\,|X-\mu| \geq t] \leq \frac{Var(x)}{t^2}.$ [Chebychev]

↳ $Pr[\underbrace{(x-\mu)^2}_{y.} \geq t^2] \leq \frac{\mathbb{E}y}{t^2} = \frac{Var(x)}{t^2}$

↳ $Pr[X \geq 3n/4]$ where x is no. of heads in n tosses

(1) $Pr(X \geq 3n/4) \leq \frac{n/2}{3n/4} = 2/3$

(2) $Pr(\,|X - n/2| \geq n/4) \leq \frac{n/4}{(n/4)^2} = 4/n$

Due to square we get sharper bound.