COL 7004 Due: August 21, 2025

Assignment 1

1. Show by mathematical induction that

$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 2.$$

2. Give a bijection between the following sets A and B (and show that the function is a bijection). Note that it is not enough to show (in case A and B are finite) that they are of the same size: you need to explicitly define a bijective function that given an element x of A efficiently outputs the function value $f(x)$:

   (a) X is a finite set $A = \{S \subset X : |S| \text{ is even}\}$ and $B = \{S \subset X : |S| \text{ is odd}\}$.

   (b) Let A be the length-n sequences of digits $[0, ..., 9]$ whose sum of digits is less than $9n/2$ and B be the length-n sequences of digits $[0, ..., 9]$ whose sum of digits is greater than $9n/2$. Here n is a positive integer.

   (c) Let A be the set of all infinite strings where each character belongs to $\{1, 2, 3\}$ and B be the set of all infinite strings where each each character belongs to $\{4, 5\}$.

3. Prove that $\log_7 n$ is either an integer or an irrational number , where n is a positive integer.

4. Prove or disprove the following facts. Here A is an infinite set.

   (a) Let B be a countable set. Then there is a bijection from A to $A \cup B$.

   (b) Let B be an uncountable set. Then there is a bijection from A to $A \cup B$.

   (c) A real number is called quadratic when it is a root of a degree two polynomial with integer coefficients. The number of quadratic reals is countable.

5. Write the following statements and their negation in predicate logic. Further, prove or disprove these statements:

   (a) If m^2 is a multiple of n, then m is also a multiple of n. Here m, n are positive integers. You can use N to denote the set of positive integers and predicate $P(x, y)$ denoting x is a multiple of y.

   (b) Every subset of real numbers which has a lower bound has a least element. You can use R to denote the set of real numbers. A subset S has a lower bound if there is a number which is smaller than all the numbers in S.

   (c) If a positive number n is composite then it has a factor (other than 1) which is at most $\sqrt{n}$. Here you can use N to denotethe set of positive numbers and the predicate $P(x, y)$ indicating that x divides y.