Topics learned.

* Indicator Random Variable.

↳ A variable that maps every Random variable to either 1 or 0

$$I_R \begin{cases} 1 \\ 0 \end{cases}$$

↳ Independence

↳ R1 and R2 are Independent iff

$\forall x_1, x_2 \in \Omega$

$[R_1 = x_1]$ and $[R_2 = x_2]$ are independent

(#) Two events are independent if their I_R are independent.

If 2 coins are tossed

↳ Let R = outcome of first toss is (H/T)

↳ Let S = outcome of second toss

Define

$$F(R) \begin{cases} 1 & R = H \\ 0 & R = T \end{cases} \qquad F(S) \begin{cases} 1 & R = H \\ 0 & R = T \end{cases}$$

then

F(R) and F(S) are also independent event.

* Numbers game

↳ pick an x that you think is b/w Low(L) and High(H) card

↳ when you pick a card and it is less than x it might be L you switch otherwise keep

$\boxed{\text{we win} \geq 1/2 + 1/2n}$

for n=100

$P(\text{win}) \geq \frac{1}{2} + \frac{1}{200}$

≥ 0.50005

* General Binomial distribution

$$\boxed{F_{n,p}(k) = {}^nC_k\, p^k q^{n-k}}$$

↳ ${}^nC_k$ → Sequence with exactly k heads and n−k tails

p → prob of heads

q → Prob of tails

☆ Great Expectations.

↳ average value

↳ If R is non-negative real value random variable defined on a sample space S then

$$\boxed{Ex[R] = \sum_{\omega \in S} R(\omega) \cdot Pr[\omega]} \rightarrow \text{eq } (1.1)$$

(#) $Ex[1/R] \neq 1/Ex[R]$

(#) If I_A is an Indicator RV for an event A then

$Ex[I_A] = Pr[A]$. ——— eq (1.2)

↳ from eqⁿ 1.1

$Ex[I_A] = \sum_{\omega \in S} R(\omega) \cdot Pr[\omega]$ here

$= 1 \cdot Pr[I_A = 1] + 0 \cdot Pr[I_A = 0]$

$= Pr[A]$.

* Conditional Expectations

$$\boxed{Ex[R \mid A] = \sum_{r \in \text{range}(R)} r \cdot Pr[R = r \mid A]} \rightarrow \text{eq } (1.3)$$

For example, we can compute the expected value of a roll of a fair die, given that the number rolled is at least 4. We do this by letting R be the outcome of a roll of the die. Then by equation (19.4),

$$\text{Ex}[R \mid R \geq 4] = \sum_{i=1}^{6} i \cdot \Pr[R = i \mid R \geq 4] = 1 \cdot 0 + 2 \cdot 0 + 3 \cdot 0 + 4 \cdot \tfrac{1}{3} + 5 \cdot \tfrac{1}{3} + 6 \cdot \tfrac{1}{3} = 5.$$

Law of Total Expectations

$$\boxed{Ex[R] = \sum_i Ex[R \mid A_i] \cdot Pr[A_i]} \text{ ——— eq } (1.4)$$

(#) A random variable R has a geometric distribution with parameter p.

↳ $Pr(R=i) = q^{i-1} \cdot p$

↳ $Ex[R] = 1/p$.

↳ $Pr[H]=0.2$ $Pr[T]=0.8$

↳ R is the no. of tosses required to get heads

$Ex[R] = 1/0.2 = 5$.

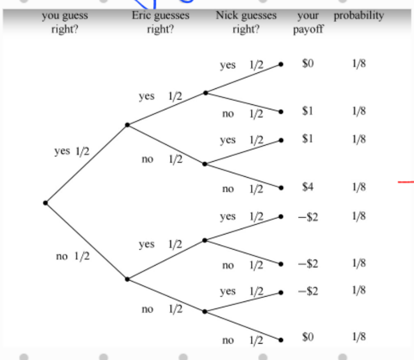
* Gambling games

↳ win w with prob p

↳ loss x with prob (1-q)

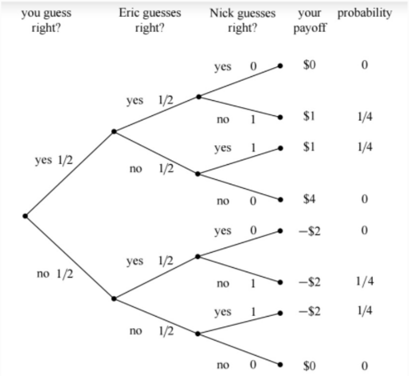
net = pw - qx

* pot splitting game



Ex[Payoff] = 0

→ this changes if Nick and Eric were made the same guess.



☆ Linearity of Expectations.

↳ if $X = X_1+X_2+X_3+ \cdots X_n$

$Ex = Ex_1 + Ex_2 + \cdots Ex_n$

↳ regardless of their independence

Corollary 19.5.3 (Linearity of Expectation). *For any random variables $R_1, \ldots, R_k$ and constants $a_1, \ldots, a_k \in \mathbb{R}$,*

$$\text{Ex}\left[\sum_{i=1}^{k} a_i R_i\right] = \sum_{i=1}^{k} a_i \,\text{Ex}[R_i].$$

↳ Expected value of dice

$$Ex = \sum_{i=1}^{6} x_i \cdot Pr[x=i] = 7/2.$$

↳ Expected value of 2 dice

$E[X_A+X_B] = Ex_A + Ex_B = 7$.

* If n men gets their hats back after randomizing what is the expected number of people who get their hat back.

↳ Prob [getting hat back] = 1/n

let $X = X_1+X_2+ \cdots X_3$

$X_i \begin{cases} 1 & \text{if } i^{th} \text{ man gets their hat} \\ 0 & \text{otherwise.} \end{cases}$

$Ex_i = Pr[G_i=1] = 1/n$.

i.e

$Ex = 1/n + 1/n + 1/n \cdots$ n times

$\boxed{Ex = 1}$

Mean Time to Failure

If a system independently fails at each time step with probability p, then the expected number of steps up to the first failure is $1/p$.

↳ If we get a new coloured car out of n coloured cars uniformly, what is the expected no. of cars we need to get in order to have all cars.

let $X = X_1+X_2+ \cdots X_n$.

at X_K we have K coloured cars.

i.e each new time we might get a new car colour with prob $\frac{n-K}{n}$.

i.e $Pr[X_K] = \frac{n-K}{n}$.

by mean time to failure

$Ex_K = \frac{n}{n-K}$.

i.e $E_0 = 1$ $E_1 = \frac{n}{n-1}$ ---

$Ex = n(1+\frac{1}{2}+\frac{1}{3}+ \cdots \frac{1}{n})$

$\boxed{Ex = nH_n}$