COL 7004 August 11, 12, 14

TUTORIAL SHEET 3

1. Determine whether each of the following sets is countable or uncountable:
   - all positive rational numbers that cannot be written with denominators less than 4.
   - real numbers not containing 0 in their decimal representation
   - real numbers containing only finite number of 1's in their decimal representation
2. Let x be an irrational number. Given a positive integer n, show that there is a positive integer j not exceeding n, such that the absolute value of the difference between jx and the nearest integer to jx is less than $1/n$.
3. Prove that given any set of distinct 52 positive integers, there exist two integers m, n in this set such that either $m + n$ or $m - n$ is divisible by 100.
4. Show that a $6 \times n$ board ($n \geq 2$) can be tiled with L-shaped tiles, without gap and overlapping. Each L-shaped tiles cover three squares.
5. Which of the powers of 9 ($9^0, 9^1, 9^2, \ldots$) have 9 as a unit digit ? Prove by induction.
6. Show that it is possible to arrange the numbers $1, 2, \ldots, n$ in a row such that the average of any two of these numbers never appears between them (Hint: show that it suffices to prove this when n is a power of 2, and use induction in this case).
7. Use the well-ordering principle to show that there is no solution in positive integers to the equation
   $$a^2 + b^2 = 3(s^2 + t^2).$$
   Hint: consider all such tuples (a, b, c, d), and choose the one for which a is smallest. Now argue that a and b both must be multiples of 3, and continue.
8. The numbers $1, 2, \ldots, 11$ are written on a board. We repeat the following process till only one number remains: pick any two numbers, and replace them by the absolute value of their difference. Prove that we cannot end up with 5.