COL 7004 August 4, 6, 7

TUTORIAL SHEET 2

1. For infinite binary strings x and y, write xRy if x and y differ in only a finite number of corresponding positions. Show that R is an equivalence relation. Show that the number of equivalence classes is infinite.

2. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.

   - $\forall x(A \to P(x)) \equiv A \to \forall x P(x)$.
   - $\exists x(A \to P(x)) \equiv A \to \exists P(x)$.

3. Let $P(x)$ be the statement "x can speak Russian" and $Q(x)$ be the statement "x knows the computer language C++". Express each of these sentences in terms of $P(x), Q(x)$, quantifiers and logical connectives. The domain for quantifiers consists of all students in the class.

   - There is a student who can speak Russian and knows C++.
   - There is a students who speaks Russian but does not know C++.
   - Every student either knows C++ or speaks Russian.
   - Noone knows C++ or speaks Russian.
   - There are only two students who know C++ and speak Russian.

4. Find a common domain for the variables x, y, and z for which the statement $\forall x \forall y((x \neq y) \to \forall z((z = x) \vee (z = y))$ is true and another domain for which it is false.

5. Assuming all quantifiers have the same nonempty domain show that

   - $\forall x P(x) \wedge \exists x Q(x) \equiv \forall x \exists y(P(x) \wedge Q(y)$.
   - $\forall x P(x) \vee \exists x Q(x) \equiv \forall x \exists y(P(x) \vee Q(y))$.

6. Let $P(x), Q(x)$, and $R(x)$ be the statements "x is a clear explanation", "x is satisfactory", and "x is an excuse", respectively. Suppose that the domain for x consists of all English text. Express each of these statements using quantifiers, logical connectives, and $P(x), Q(x)$ and $R(x)$.

   (a) All clear explanations are satisfactory.

   (b) Some excuses are unsatisfactory.

   (c) Some excuses are not clear explanations.

   (d) Does (c) follow from (a) and (b)?

7. Use rules of inference to show that if $\forall x(P(x) \vee Q(x)), \forall x(\neg Q(x) \vee S(x)), \forall x(R(x) \rightarrow \neg S(x))$ and $\exists x \neg P(x)$ are true, then $\exists x \neg R(x)$ is true.

8. Prove or disprove that there is a rational number x and an irrational number y such that xy is irrational.

9. Prove that between every two rational numbers there is an irrational number.

10. Prove by contradiction that there is no rational number r such that $r^3 + r + 1 = 0$.

11. Prove that there is no solution in positive integers x and y to the equation $x^4 + y^4 = 625$.

12. Prove or disprove: There is a rational number x and an irrational number y such that x^y is irrational.