Matroids

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Abstract

Matroids were introduced in 1935 by Hassler Whitney. Matroid theory is an active area of research and incorporates concepts from algebra, geometry, combinatorics, and graph theory. This paper includes definitions and examples of matroids, an important theorem of matroid theory, Whitney's 2-isomorphism theorem, an application of using matroids for combinatorial optimization, and a version of the Greedy Algorithm written in the Sage programming language.

1 Introduction

Matroids were first introduced by Hassler Whitney in 1935 [6] as a way of formalizing the similarities between independence and rank in graph theory and linear independence and dimension in vector spaces [7]. In his paper [6], Whitney introduced the concept of a matroid using the columns of a matrix. He also mentioned extending matroids to include vectors, polynomials, and linear graphs. For over twenty years, little work was done with matroids. Then in 1958, Tutte characterized graphic matroids [7]. Since Whitney's original definition [6], the definition of a matroid has evolved from being defined on the columns of a matrix to a more general definition. Matroid theory incorporates concepts from algebra, geometry, combinatorics, and graph theory and has many applications. Matroids are an active area of research [2].

To understand matroids, one must first become familiar with the definitions and terminology associated with matroids. Some of the definitions are unique to matroid theory while others are borrowed from other fields, mainly graph theory and linear algebra. Oxley's book [4] begins with 57 pages of examples and definitions. Only some of those definitions are included here. In addition to the definitions and examples, this paper also includes a very interesting theorem of matroid theory, Whitney's 2-isomorphism theorem, as well as an application of matroids using the Greedy algorithm.

Whitney's 2-isomorphism theorem states that if G and H are loopless graphs with no isolated verticies, then their cycle matroids, $M(G)$ and $M(H)$, are isomorphic if and only if G and H are 2-isomorphic. The Greedy algorithm is an algorithm for solving a combinatorial optimization problem. Given a finite set E, a collection of subsets $\mathcal{I}$ of E, and a weight function w, the Greedy algorithm solves the optimization problem $(\mathcal{I}, w)$ if and only if $(E, \mathcal{I})$ is a matroid. Appendix A includes a version of the Greedy algorithm written in the Sage programming language.

2 Definitions and examples

Matroids can be defined in different ways. The first definition defines matroids based on the ground set and the independent sets of a matroid.

Definition 2.1 (Matroid)**.** Let E be a finite set and let $\mathcal{I}$ be a family of subsets of E. Then the family $\mathcal{I}$ forms the independent sets of a matroid $M = (E, \mathcal{I})$ if

(1) $\emptyset \in \mathcal{I}$,

(2) $I \in \mathcal{I}$ and $I' \subseteq I$ imply that $I' \in \mathcal{I}$, and

(3) I_1, $I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$ implies there is some element $x \in I_2 - I_1$ such that $I_1 \cup \{x\} \in \mathcal{I}$.

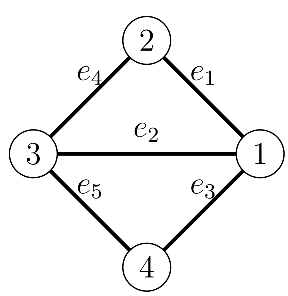
The set E is the *ground set* of matroid M. The members of $\mathcal{I}$ are the *independent sets* of matroid M. The *rank* of matroid M is the size of the largest independent set.

Two of the most basic examples of matroids are vector matroids, $M[A]$, in which the independent set includes all of the sets of linearly independent columns of a matrix as shown in Example 2.2 and $M(G)$ the cycle matroid of a graph as shown in Example 2.3.

Example 2.2. Let E be the columns of a matrix A with entries in $\mathbb{F}$ and let $\mathcal{I}$ be the collection of all subsets of E that are linearly independent. Then $M[A] = (E, \mathcal{I})$ is a matroid, called the vector matroid.

Example 2.3. Let E be the set of edges of graph G and let $\mathcal{I}$ be the collection of all subsets of edges that are acyclic. Then $M(G) = (E, \mathcal{I})$ is a matroid, called the cycle matroid.

Consider the following graph $G = (V, E)$:



Here $E = \{e_1, e_2, e_3, e_4, e_5\}$ and $\mathcal{I} = \{S \subseteq E : S \text{ is not a cycle}\}$. Then $M(G) = (E, \mathcal{I})$ is a matroid.

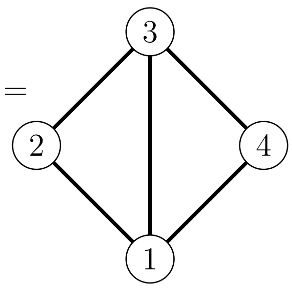
Example 2.4. Let $E = \{e_1, e_2, e_3, e_4, e_5\}$ and let $\mathcal{I}_1$ be the power set of $\{e_1, e_2, e_3\}$:

$$\mathcal{I}_1 = \{\emptyset, \{e_1\}, \{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}, \{e_1, e_2, e_3\}\} = 2^{\{e_1, e_2, e_3\}}.$$

Then $M_1 = (E, \mathcal{I}_1)$ is a matroid.

Example 2.5. Graph and it's corresponding incidence and adjacency matrices:

$G =$



Incidence Matrix, $B_G =$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Adjacency Matrix, $A_G =$

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Matroids can also be defined by their circuits and by their bases.

Definition 2.6 (Circuit of M)**.** A minimal dependent set, a dependent set all of whose proper subsets are independent, of a matroid M is called a circuit of M or $\mathcal{C}(M)$.

Example 2.7. In Example 2.3,

$$\mathcal{C}(M(G)) = \{\{e_1, e_2, e_3, e_4\}, \{e_1, e_2, e_5\}, \{e_3, e_4, e_5\}\}.$$

Definition 2.8 (Basis)**.** If M is a matroid with independent sets $\mathcal{I}$, then B is a basis of matroid M if B is a maximal independent set.

Example 2.9. In Example 2.2, the bases of $M[A]$, are all of the independent subsets of the ground set E of size 3.
In Example 2.3, the bases of $M(G)$ are:

$$B(M(G)) = \{\{e_1, e_2, e_3\}, \{e_1, e_2, e_5\}, \{e_3, e_4, e_5\}\}$$

In Example 2.4, the bases of M_1 are:

$$B(M_1) = \{\{e_1, e_2, e_3\}\}$$

The following are some properties of matroids.

Definition 2.10 (Dual)**.** Let M be a matroid on the ground set E. Then the dual matroid M^* is a matroid on the same ground set E such that

$$\mathcal{B}(M^*) = \{E - B : B \in \mathcal{B}(M)\}.$$

Example 2.11. Recall Example 2.4, we have

$$B(M_1^*) = E - \{e_1, e_2, e_3\} = \{\{e_4, e_5\}\}$$

and

$$\mathcal{I}(M_1^*) = \{\emptyset, \{e_4\}, \{e_5\}, \{e_4, e_5\}\}.$$

Definition 2.12 (Complementary Cocircuit)**.** A cocircuit of M is circuit of M^*.

Some of the operations that can be performed on matroids are deletion, contraction, and direct sum.

Definition 2.13 (Deletion)**.** Let M be a matroid on $(E, \mathcal{I})$ and suppose $X \subseteq E$. Let $\mathcal{I}|X$ be $\{I \subseteq X : I \in \mathcal{I}\}$. Then the matroid $(X, \mathcal{I}|X)$ is the restriction of M to X or the deletion of $E - X$ from M. Denoted $M|X$ or $M \backslash (E - X)$.

Example 2.14. Recall 2.4:
Let

$$X = E - \{e_1\} = \{e_2, e_3, e_4, e_5\}.$$

Then

$$X|\mathcal{I} = \{\emptyset, \{e_2\}, \{e_3\}, \{e_2, e_3\}\}$$

and $(X, \mathcal{I}|X)$ is the restriction of M_1 to X.

Definition 2.15 (Rank)**.** The rank $r(X)$ of X is the cardinality of a basis B of $M|X$.

Definition 2.16 (Contraction)**.** Let M be a matroid on E and T be a subset of E. Let M/T be the contraction of T from M given by $M/T = (M^*\backslash T)^*$.

Example 2.17. Continuing Example 2.4, let $T = \{e_1\}$. The groundset of $M_1^*\backslash T$ is $\{e_2, e_3, e_4, e_5\}$ and $\mathcal{I}(M_1^*\backslash T) = \{\emptyset, \{e_4\}, \{e_5\}, \{e_4, e_5\}\}$. Then $\mathcal{I}(M_1/T) = \mathcal{I}(M_1^*\backslash T)^* = \{\emptyset, \{e_2\}, \{e_3\}, \{e_2, e_3\}\}$.

Definition 2.18 (Closure)**.** Let cl be the function from 2^E to 2^E defined by $\text{cl}(X) = \{x \in E : r(X \cup \{x\}) = r(X)\}$.

Example 2.19. Let M be the matroid from Example 2.4 and set

$$X = E - \{e_1\} = \{e_2, e_3, e_4, e_5\}.$$

Then

$$X|\mathcal{I} = \{\emptyset, \{e_2\}, \{e_3\}, \{e_2, e_3\}\},$$

$$r(X) = 2$$

$$\text{cl}(X) = \{e_2, e_3, e_4, e_5\}.$$

Definition 2.20 (Flat of M)**.** A subset X of $E(M)$ for which $\text{cl}(X) = X$ is a flat or closed set of M. In the above example, X is a flat of M_1.

Definition 2.21 (Hyperplane)**.** A hyperplane of M is a flat of rank $r(M) - 1$.

X above is a hyperplane of M_1 since $r(M_1) = 3$.

Definition 2.22 (Direct Sum)**.** Let M_1 and M_2 be matroids on disjoint ground sets E_1 and E_2 respectively. Define the direct sum $M_1 \oplus M_2$ to be the matroid on the ground set $E = E_1 \cup E_2$ with independent sets $I_1 \cup I_2$, where $I_1 \subseteq E_1$ is independent in M_1 and $I_2 \subseteq E_2$ is independent in M_2.

Example 2.23. Let

$$E_1 = \{1,2\}, \mathcal{I}_1 = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}, M_1 = (E_1, \mathcal{I}_1),$$

$$E_2 = \{3,4\}, \mathcal{I}_2 = \{\emptyset, \{3\}, \{4\}, \{3,4\}\}, M_2 = (E_2, \mathcal{I}_2),$$

. Then

$$M_1 \oplus M_2 = (E, \mathcal{I}), E = \{1,2,3,4\},$$

$$\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\},$$

$$\{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$$

Definition 2.24 (Minor)**.** A minor of a matroid M is a matroid formed by any sequence of deletions and contractions, $M \backslash X/Y$, from M where X and Y are disjoint.

Example 2.25. Let $M(E) = \{e_1, e_2, e_3, e_4\}$ and

$$\mathcal{B}(M) = \{\{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}, \{e_1, e_3, e_4\}, \{e_2, e_3, e_4\}\}.$$

Set $X = \{e_1\}$ and $Y = \{e_2\}$ Then $M \backslash X/Y = M_1/Y = M_2$ where

$$\begin{aligned} M_1(E) &= E - \{e_1\} & \qquad \mathcal{B}(M_1) &= \{e_2, e_3, e_4\}), \\ M_2(E) &= \{e_3, e_4\} & \qquad \mathcal{B}(M_2) &= \{e_3, e_4\}. \end{aligned}$$

Matroids can be classified in different ways. Many properties are unique to certain categories of matroids.

Definition 2.26 (Representable)**.** If M is isomorphic to a vector matroid D over a field $\mathbb{F}$, then M is representable over $\mathbb{F}$.

Each of the matroids in Examples 2.2, 2.3, 2.4 are representable.

Definition 2.27 (Graphic)**.** A matroid that is isomorphic to the cycle matroid of a graph is called graphic.

Each of the matroids in Examples 2.2, 2.3, 2.4 are graphic.

Definition 2.28 (Regular)**.** Matroids that are representable over all fields are called regular or unimodular. Regular matroids can be represented by a totally unimodular matrix, a matrix in which every square submatrix has determinant in $\{0, 1, -1\}$.

The matroid $M[A]$ in Example 2.2 is regular.

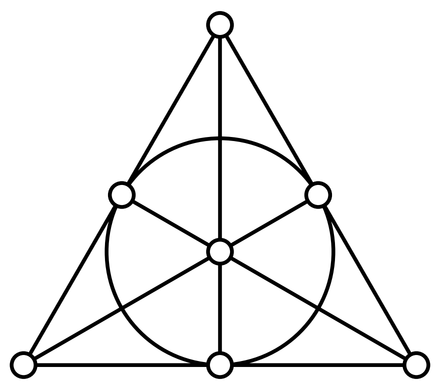
Definition 2.29 (Binary)**.** Binary matroids are matroids representable over $\mathbb{F}_2$.

The matroids in Examples 2.2, 2.3, 2.4 are all binary.

Question 2.30. *Are all binary matroids regular?*

No. There are some matroids that are only representable over fields of characteristic 2, such as the Fano plane.

Example 2.31. Consider the *Fano plane* F_7, shown below.

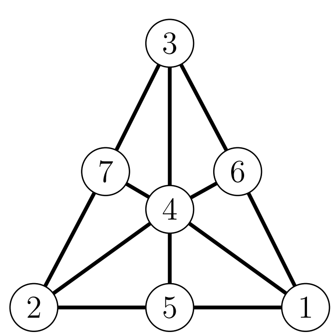


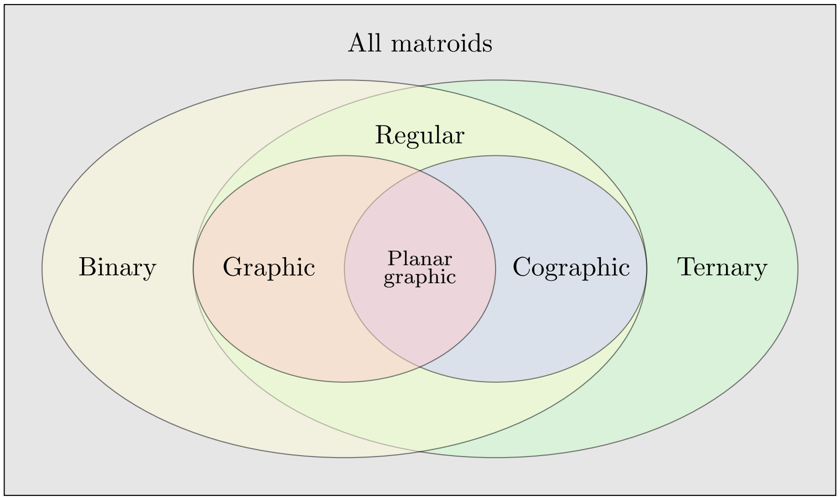
The associated vector matroid for F_7 over $\mathbb{F}_2$ is

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Definition 2.32 (Ternary)**.** Ternary matroids are matroids representable over $\mathbb{F}_3$.

Example 2.33. The non-Fano configuration F_7^- is ternary.





Definition 2.34 (Uniform Matroid)**.** Let $m, n \in \mathbb{Z}, n, m \geq 0, m \leq n$. Let E be an n element set and $\mathcal{B}$ be the collection of m-element subsets of E. Then $\mathcal{B}$ is the set of bases of a uniform matroid of rank m on E, denoted $U_{m,n}$.

Example 2.35. Let $E = \{1, 2, 3, 4\}$ and

$$\mathcal{B} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}, \{1, 3, 4\}\}.$$

Then $(E, \mathcal{B}) = U_{4,3}$ is a uniform matroid.

Definition 2.36 (k-Connected)**.** A graph is k-connected if it remains connected whenever any $k-1$ vertices are removed from the graph.

Graph G in Example 2.3 is 2-connected but not 3-connected.

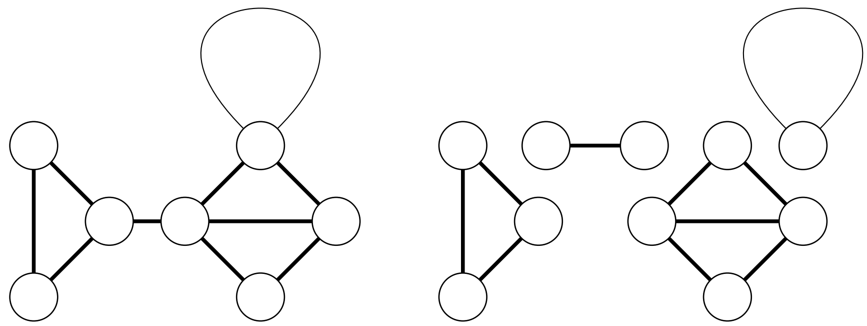
Definition 2.37 (Connected Matroid)**.** A matroid M is connected if M cannot be written as a direct sum of smaller matroids. That is for all $x, y \in E$, there is a circuit C containing both x and y. If $M = M_1 \oplus N$, where M_1 is connected, then M_1 is a connected component of M.

Question 2.38. *Are k-connected and connected matroids related?*

Yes. The matroid $M(G)$ is a connected matroid if and only if G is a 2-connected graph. See Proposition 3.9.

Definition 2.39 (Block)**.** A block is a connected graph whose cycle matroid is connected. A block of a graph is a subgraph that is a block and is maximal with this property.

Example 2.40 (Blocks of a Graph)**.**



Definition 2.41 (Cographic)**.** A matroid M is cographic if and only if M^* is graphic.

Definition 2.42 (Planar)**.** Planar graphs are graphs that can be drawn without the edges intersecting.

Definition 2.43 (Planar Graphic)**.** Planar graphic matroids are isomorphic to the cycle matroids of planar graphs.

The claim was made at the beginning of this section that the object in Example 2.4 is a matroid. The proofs that this object is not only a matroid but also a graphic matroid follow.

Proposition 2.44. *$M_1 = (E, \mathcal{I}_1)$ from Example 2.4 is a matroid.*

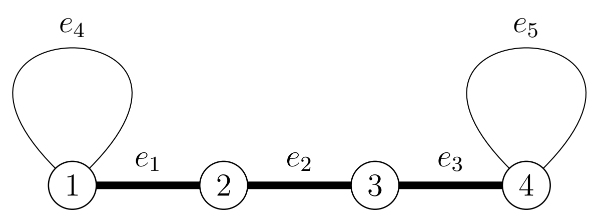
Proof. Condition (1) is satisfied since $\emptyset \in \mathcal{I}_1$. To see that condition (2) is satisfied note that $\mathcal{I}_1 = \mathcal{P}(\{e_1, e_2, e_3\})$. Then for all $I \in \mathcal{I}_1$, $I' \subseteq I$ implies that $I' \in \mathcal{I}_1$. Condition (3) requires that for all I_1, $I_2 \in \mathcal{I}_1$ with $\mid I_1 \mid < \mid I_2 \mid$, there exists an element $x \in I_2 - I_1$ such that $I_1 \cup \{x\} \in \mathcal{I}_1$. Since $\mathcal{I}_1 = \mathcal{P}(\{e_1, e_2, e_3\})$, condition (3) is also satisfied. Therefore M_1 satisfies the three conditions of a matriod. By definition, M_1 is a matroid. ■

Proposition 2.45. *M_1 is graphic.*

Proof. The only excluded minor for graphic matroids with edge sets of order less than or equal to 5 is $U_{2,4}$. Since $|E(M_1)| = 5$, the only possible excluded minor for M_1 is $U_{2,4}$. Suppose for a contradiction that M_1 has a minor M' isomorphic to $U_{2,4}$. Then $|E(M')| = 4$ which implies that either $e_4 \in E(M')$ or $e_5 \in E(M')$. Without loss of generality suppose that $e_4 \in E(M')$. The independent set of M', $\mathcal{I}(M')$, contains every subset of $E(M')$ of order 2. Then $\{e_i, e_4\} \in \mathcal{I}((M')$ where $e_i \in E(M')\backslash e_4$ and $\{e_4\} \in \mathcal{I}((M')$ but $e_4 \notin I$ for any $I \in \mathcal{I}(M_1)$. The matroid M' is obtained from M_1 by contractions and deletions which implies that $e_4 \notin I'$ for any $I' \in \mathcal{I}(M')$. The number of elements in $I(M')$ is less than the number of elements in $I(U_{2,4})$ and any mapping from $I(U_{2,4})$ to $I(M')$ is not one-to-one. Therefore M' is not isomorphic to $U_{2,4}$. Contradiction. The matroid M_1 contains no excluded minors for graphic matroids which implies that M_1 is graphic. ■

$$M[A] \cong M_1 \text{ where } A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

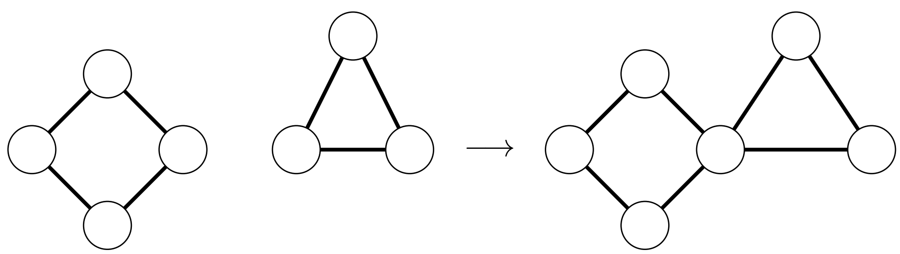
$M(G) \cong M_1$ where $G =$



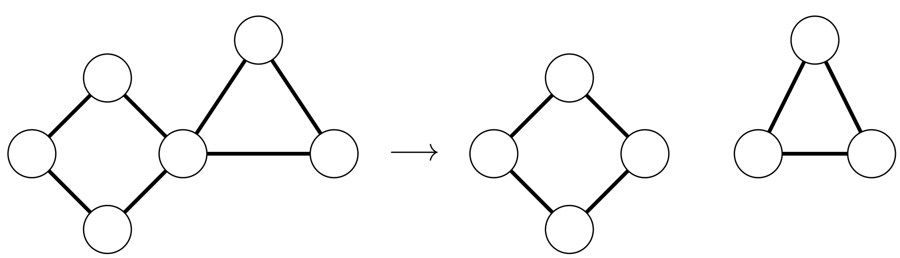
3 Whitney's 2-Isomorphism Theorem

It is possible for two non-isomorphic graphs to have cycle matroids that are isomorphic. Whitney's 2-isomorphism theorem states that for loopless graphs with no isolated verticies this only occurs when the graphs are 2-isomorphic. The three basic operations of vertex identification, vertex cleaving, and twisting define 2-isomorphisms.

Definition 3.1 (Vertex Identification)**.** Let v and v' be verticies of distinct components of a graph G. Modify G by identifying v and v' as a new vertex $\overline{v}$.

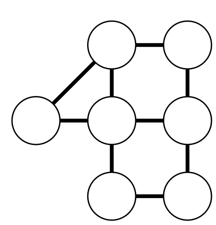


Definition 3.2 (Vertex Cleaving)**.** Vertex cleaving is the reverse operation of vertex identification. A graph can only be cleft at a cut-vertex or at a vertex with a loop.



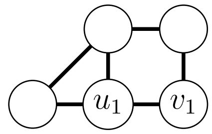
Definition 3.3 (Twisting)**.** Suppose graph G is obtained from disjoint graphs G_1 and G_2. Identify verticies u_1 of G_1 and u_2 of G_2 as vertex u of G and v_1 of G_1 and v_2 of G_2 as vertex v of G. In a twisting of G about $\{u, v\}$ identify u_1 with v_2 and u_2 with v_1.

Example 3.4. Let G be the graph

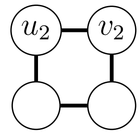


and set

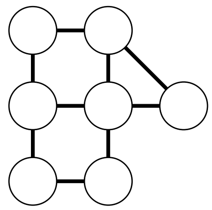
G_1 :



G_2 :



Then the twisting of G is the graph G'



Definition 3.5 (2-Isomorphic)**.** A graph G is 2-isomorphic to graph H if H can be transformed into a graph isomorphic to G by a sequence of vertex identifications, vertex cleavings, and twistings.

The terms generalized cycle, edge cut, and vertex bond are important in the propositions and proofs of Whitney's 2-isomorphism theorem.

Definition 3.6 (Generalized cycle)**.** For $k \geq 2$, a connected graph is a generalized cycle with parts $G_1, G_2, \ldots, G_k$ if the following conditions hold:

(1) Each G_i is a connected subgraph of G having a non-empty edge set, and, if $k = 2$, both G_1 and G_2 have at least three verticies.

(2) The edge sets of $G_1, G_2, \ldots, G_k$ partition the edge set of G, and each G_i shares exactly two verticies, its contact verticies, with $\bigcup_{j \neq i} G_j$.

(3) If each G_i is replaced by an edge joining its contact verticies, the resulting graph is a cycle.

Definition 3.7 (Edge Cut)**.** Let X be a set of edges of a graph G. $G \backslash X$ is the subgraph of G obtained by deleting the edges of X. If $G \backslash X$ has more connected components than G, then X is an edge cut of G.

Definition 3.8 (Vertex Bond)**.** If v is a non-isolated vertex of G and X is the set of edges meeting v, then X is an edge cut. If X is a minimal edge cut, then X is a vertex bond of G.

Proposition 3.9 ([4])**.** *Let G be a loopless graph without isolated verticies and suppose that $|V(G)| \geq 3$. Then $M(G)$ is a connected matroid if and only if G is a 2-connected graph.*

Proof. ($\Rightarrow$) Assume $M(G)$ is connected. Let $e_1 = uv, e_2 = xy \in E(M(G)), e_1$ and e_2 distinct. By definition, there exists $C \in \mathcal{C}(M(G))$ such that e_1 and $e_2 \in C$. Let $P_1 = C \backslash e_1$. Then P_1 is a path connecting u, v, x, y. Since e_1 and e_2 were arbitrary elements of $E(M(G))$, there exists a path P_i connecting any two verticies of $G \backslash P_i$. Therefore G is 2-connected.

($\Leftarrow$) Assume G is 2-connected. Let $e_1 = uv, e_2 = xy \in E(M(G))$, e_1 and e_2 distinct. The graph G is 2-connected which implies that $G \backslash e_1$ is connected. Let P_1 be a path in $G \backslash \{e_1\}$ containing verticies u, v, x, y and edge e_2. Then there exists a cycle $C \subseteq P_1 \cup \{e_1\}$ such that $e_1, e_2 \in C$. The edges e_1 and e_2 were arbitrary. Therefore for any two elements $e_1, e_2 \in E(M(G))$ there exists a circuit C in $M(G))$ such that $e_1, e_2 \in C$. By definition the matroid $M(G)$ is connected. ∎

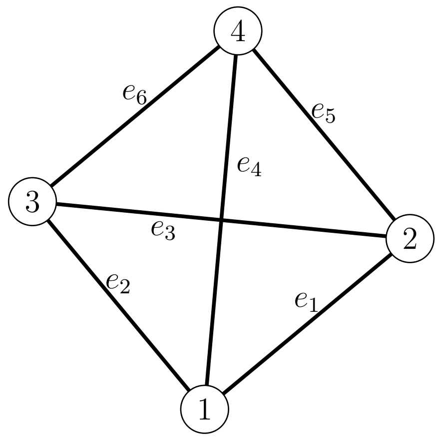
Lemma 3.10. *[[4]] Let G and H be loopless graphs without isolated verticies. Suppose that $\psi\colon E(G) \longrightarrow E(H)$ is an isomorphism from $M(G)$ to $M(H)$. If G is 3-connected, then ψ induces an isomorphism between the graphs G and H.*

Proof. Since G is a loopless graph without isolated verticies, G is a loopless block. A hyperplane of $M(G)$ is connected only if the complementary cocircuit is a vertex bond. The graph G is 3-connected, so for every vertex v of G, $G - v$ is 2-connected. The subgraph $G - v$ is 2-connected which implies that $M(G - v)$ is connected, so $M(G - v)$ is a connected hyperplane and $M(G)$ has exactly $|V(G)|$ connected hyperplanes. The matroid $M(G)$ has exactly $|V(G)|$ connected hyperplanes which implies that, up to a relabeling, $M(G)$ determines the mod-2 incidence matrix of J.

The matroids $M(G)$ and $M(H)$ are isomorphic, and $M(G)$ is connected and loopless, so H is also connected and loopless and H is also a loopless block. Therefore $|V(G)| = r(M(G)) + 1 = r(M(H)) + 1 = |V(H)|$. The matroid $M(H)$ has exactly $|V(H)|$ connected hyperplanes.

To see that G and H are isomorphic note that $|E(M(G))| = |E(M(H))|$ which implies that $|E(G)| = |E(H)|$, $|V(G)| = |V(H)|$, and up to a relabeling, G and H have the same mod-2 incidence matrix. Therefore G is isomorphic to H. ∎

Example 3.11. Let G be the graph



Then $M(G)$ is a regular matroid of rank 3 on 6 elements with 16 bases. G is 3-connected.

The hyperplanes of G are:

$$\{e_1, e_2, e_6\}, \{e_1, e_3\}, \{e_1, e_4, e_5\}, \{e_2, e_3, e_5\}, \{e_2, e_4\}, \{e_3, e_4, e_6\}, \{e_5, e_6\}.$$

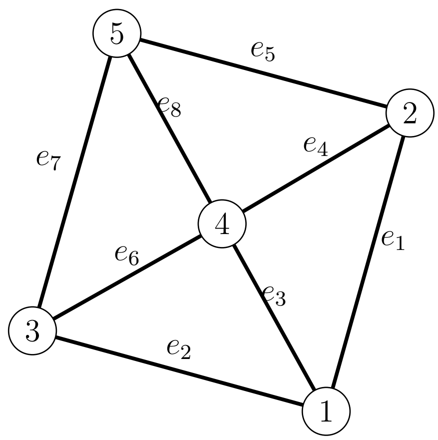
The connected hyperplanes of $M(G)$ are:

$$\{e_1, e_2, e_3\}, \{e_2, e_4, e_6\}, \{e_3, e_5, e_6\}, \{e_1, e_4, e_5\}.$$

The complimentary cocircuit of $\{e_1, e_2, e_3\}$ is $\{e_4, e_5, e_6\}$ which is a vertex bond.
$M(G)$ has $4 = |V(G)|$ connected hyperplanes.
Mod-2 vertex-edge incidence matrix of G:

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Example 3.12. Let G_2 be the graph



The matroid $M(G_2)$ is a regular matroid of rank 4 on 8 elements with 45 bases. The graph G_2 is 3-connected.

The hyperplanes of $M(G_2)$ are:

$$\{e_1,e_2,e_3,e_4,e_6\},\{e_1,e_2,e_5,e_7\},\{e_1,e_2,e_8\},\{e_1,e_3,e_4,e_5,e_8\},\{e_1,e_3,e_4,e_7\},$$
$$\{e_1,e_5,e_6\},\{e_1,e_6,e_7,e_8\},\{e_2,e_3,e_5,e_6\},\{e_2,e_3,e_6,e_7,e_8\},\{e_2,e_4,e_5,e_8\},$$
$$\{e_2,e_4,e_7\},\{e_3,e_5,e_7\},\{e_4,e_5,e_6,e_7,e_8\}$$

The connected hyperplanes of $M(G_2)$ are:

$$\{e_1,e_2,e_3,e_4,e_6\},\{e_1,e_2,e_5,e_7\},\{e_1,e_3,e_4,e_5,e_8\},\{e_2,e_3,e_6,e_7,e_8\},\{e_4,e_5,e_6,e_7,e_8\}$$

The matroid $M(G_2)$ has $5 = |V(G_2)|$ connected hyperplanes. The complimentary cocircuit of $\{e_1,e_2,e_3,e_4,e_6\}$ is $\{e_5,e_7,e_8\}$ which is a vertex bond.

Mod-2 vertex-edge incidence matrix of G_2:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Lemma 3.13 ([4])**.** *Let G be a block having at least four verticies and suppose that G is not 3-connected. Then G has a representation as a generalized cycle, each part* *of which is a block.*

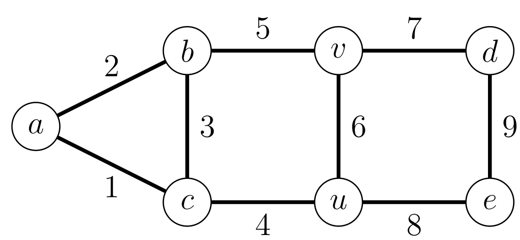
Proof. The graph G is not 3-connected which implies that G has a vertex cut. Since G is a block, the vertex cut is not a cut vertex. Suppose that $\{u, v\}$ is a vertex cut of G. Then $G - \{u, v\}$ has 2 components, call them H_1 and H_2.

Let G_1 be the subgraph of G induced by $V(H_1) \cup \{u, v\}$ and G_2 be the subgraph of G induced by $V(H_2) \cup \{u, v\}$. Let G_2' be the graph obtained from G_2 by deleting the edges joining u and v. If G_1 is a block, then G is a generalized cycle with parts G_1 and G_2'.

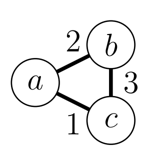
If G_1 is not a block, it is the union of two connected subgraphs that each have at least 2 verticies. Call these subgraphs $G_{1,1}$ and $G_{1,2}$. Since G_1 is not a block, $G_{1,1}$ and $G_{1,2}$ have only one common vertex, call this vertex x. Since G is a block, $x \notin \{u, v\}$, either u or v is in $G_{1,1}$ and the other vertex is in $G_{1,2}$. If $G_{1,1}$ and $G_{1,2}$ are blocks, then G is a generalized cycle with parts $G_{1,1}$, $G_{1,2}$, and G_2'.

If $G_{1,1}$ and $G_{1,2}$ are not blocks, then the process can be repeated. The graph G has a finite number of edges which implies that this process is finite and a representation of G as a generalized cycle will be obtained in which each part is a block. ■

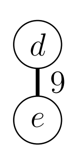
Example 3.14. Let G be the graph

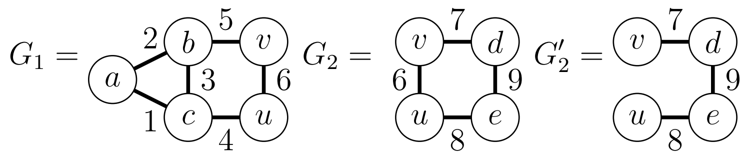


Set $H_1 = abc$, shown here.



Then $H_2 = G - \{u, v\} - V(H_1) = \{d, e\}$.





Lemma 3.15 ([4])**.** *Suppose that G is a block having a representation as a generalized cycle, the parts $G_1, G_2, \ldots, G_k$, of which are blocks. Let H be a graph for which there is an isomorphism θ from $M(G)$ to $M(H)$ and, for each i, let H_i be the subgraph of H induced by $\theta(E(G_i))$. Then H is a generalized cycle with parts $H_1, H_2, \ldots, H_k$.*

Proof. Claim: $|(V(H_j) \cap V(H_{-j}))| \geq 2$.

For any block G_j of the generalized cycle of G, define $G_{-j} = \bigcup_{i \neq j} G_i$ and $H_{-j} = \bigcup_{i \neq j} H_i$. Since $G, G_1, \ldots, G_k$ are blocks, all of $H, H_1, \ldots, H_k$ are also blocks. For any block H_j of H, define $H_{-j} = \bigcup_{i \neq j} H_i$. The graph G has a representation as a generalized cycle which implies that G has a cycle meeting all of $E(G_1), \ldots, E(G_k)$. Therefore the graph H_{-j} has an edge e having an endpoint x in H_j. Choose an edge f in H_j. Then, since H is a block, it has a cycle C that contains both e and f. Follow cycle C from x along e and stop at the first vertex belonging to H_j. Call this vertex y. Let P be this path from x to y. The edge f is in the cycle C and $f \in H_j$ which implies that y exists, $y \in (V(H_j) \cap V(H_{-j}))$, and $y \neq x$. Therefore $|(V(H_j) \cap V(H_{-j}))| \geq 2$.

Claim: $(V(H_j) \cap V(H_{-j})) = \{x, y\}$. Let u be one of the contact verticies of G_j. Let E_u be the set of edges of G_j meeting at u. The subgraph G_j is a block. This implies that E_u is a bond in G_j. The subgraph $\theta(E_u)$ is also a bond in H_j. Therefore $H_j \backslash \theta(E_u)$ has two components, call them H_j' and H_j''.

Suppose, for a contradiction, that x and y are joined by a path in $H_j \backslash \theta(E_u)$. Then the union of this path with P is a cycle, call it C'. Then $E(P) \subseteq E_{-j}$ which implies that $C' \subseteq E(H_{-j}) \cup H_j \backslash \theta(E_u)$ and C' meets both $E(H_{-j})$ and $H_j \backslash \theta(E_u)$. Since $\theta(E_u) \subseteq E(H_j)$, $C' \cap \theta(E_u) = \emptyset$. But every cycle of G that contains an edge of G_j and an edge of G_{-j} contains an edge of E_u since E_u is a vertex bond. This implies that every cycle of H that contains an edge of H_j and an edge of H_{-j} also contains an edge of $\theta(E_u)$. Contradiction.

Therefore either x or y is in H_j' and the other vertex is in H_j''. Assume that $x \in V(H_j')$.

Suppose for another contradiction that $(V(H_j) \cap V(H_{-j})) \neq \{x, y\}$. By the first claim,$|(V(H_j) \cap V(H_{-j}))| \geq 2$. Therefore there exists a vertex $z \in (V(H_j) \cap V(H_{-j}))$

such that $z \notin \{x, y\}$. Let $g \in E(H_{-j})$ and $h \in E(H_j)$ such that g and h meet at z. The graph H is a block which implies that there exists a cycle, call it C'', such that C'' contains both g and h. Follow cycle C'' from z along g and stop at the first vertex belonging to $V(H_j) \cup V(P)$. Call this vertex w. It's clear that $z \neq w$. Let Q be this path from z to w. Then either (i) $w \in V(P)$, (ii) $w \in V(H'_j) - V(P)$, or (iii) $w \in V(H''_j) - V(P)$.

Without loss of generality, assume $z \in H'_j$. (i) If $w \in V(P)$, then H has a cycle, C, obtained by combining Q, part of P from w to x, and a path from x to z. (ii) If $w \in V(H'_j) - V(P)$, then H has a cycle, C, obtained by combining Q, P, and a path from w to z. (iii) If $w \in V(H''_j) - V(P)$, then H has a cycle, C, obtained by combining Q, P, a path from x to z in H'_j, and a path from y to w in H''_j. In all of the cases i, ii, and iii, C is a cycle containing an edge in H_j and and edge in H_{-j} but C does not have an edge in $\theta(E_u)$. Contradiction.

Therefore, $(V(H_j) \cap V(H_{-j})) \neq \{x, y\}$ and $|(V(H_j) \cap V(H_{-j}))| = 2$. The graph H is a block. This implies that H is a generalized cycle having parts $H_1, \dots, H_k$. ■

Theorem 3.16 (Whitney's 2-Isomorphism Theorem,[4]). *Let G and H be graphs having no isolated verticies. Then $M(G)$ and $M(H)$ are isomorphic if and only if G and H are 2-isomorphic.*

Proof. ($\Rightarrow$) Let G^+ be the graph formed from the disjoint union of the blocks of G and H^+ be the graph formed from the disjoint union of the blocks of H. The matroids $M(G^+)$, $M(G)$, $M(H)$, and $M(H^+)$ are congruent which implies that $M(G^+) \cong M(H^+)$. Therefore, the corresponding blocks of G and H have isomorphic cycle matroids. The graph G is 2-isomorphic to the graph G^+ and the graph H is 2-isomorphic to the graph H^+.

Assume G is a block. If G is 3-connected, then by Lemma 3.10, G and H are 2-isomorphic. Assume G is not 3-connected. If $|V(G)| \leq 4$, then G and H are 2-isomorphic.

Assume that for $|V(G)| < n$, G and H are 2-isomorphic. Let $|V(G)| = n \geq 5$. By Lemma 3.13, G has a representation as a generalized cycle, $G_1, G_2, \dots, G_k$, each part of which is a block. By Lemma 3.15, H is a generalized cycle with parts $H_1, H_2, \dots, H_k$. $H_i = H[\theta(E(G_i))]$.

Add e_i to G_i where e_i is an edge joining the contact verticies of G_i and add f_i to H_i where f_i is an edge joining the contact verticies of H_i. Then C is a cycle of

G_i meeting $E(G_i)$ and $E(G_{-i})$ if and only if $\theta(C)$ is a cycle of H meeting $E(H_{-i})$ and $E(H_i) \in \theta(E(G_i) \cap C)$. Therefore, θ can be extended as an isomorphism from $M(G_i + e_i)$ to $M(H_i + f_i)$ by mapping e_i to f_i.

Note that $|V(G_i + e_i)| = |V(G_i)| < |V(G)|$. By the induction assumption, $H_i + f_i$ can be transformed into a graph isomorphic to $G_i + e_i$ by a sequence of twistings. Every 2-element vertex cut in $H_i + f_i$ is also a vertex cut in H. This implies that the same sequence of twistings can transform H_i into H_i'. By repeating the same sequence for each $i \in \{1, 2, \ldots, k\}$, H is transformed into a generalized cycle H' with parts $H_1', H_2', \ldots, H_k'$.

Then $M(H') \cong M(G)$ and for each $i \in \{1, 2, \ldots, k\}$, there exists an isomorphism θ_i from G_i to H_i' such that θ_i maps the contact verticies of G_i to the contact verticies of H_i'. If the cyclic order of G is $G_1, G_2, \ldots, G_k$, then for some permutation, σ of $\{2, 3, \ldots, k\}$, the cyclic order of H' is $H_1', H_{\sigma(2)}', \ldots, H_{\sigma(k)}'$. By twisting, H' can be transformed into a generalized cycle, H'', having cyclic order $H_1', H_2', \ldots, H_k'$. If the graph H'' is not isomorphic to G, then by additional twisings about the contact verticies of H'', H'' can be transformed into a graph which is isomorphic to G. Hence, for $|V(G)| = n$, G and H are 2-isomorphic. By induction, for all $n \in \mathbb{N}$, G and H are 2-isomorphic.

($\Leftarrow$) None of the operations which define a 2-isomorphism, vertex identification, vertex cleaving, or twisting, alter the edge sets of the cycles of a graph. By definition, 2-isomorphism is an equivalence relation. Therefore if G is 2-isomorphic to H, then $G \cong H$. ■

4 Greedy Algorithm

The Greedy algorithm is an algorithm for solving combinatorial optimization problems. For a finite set E, a collection of subsets $\mathcal{I}$ of E, and a weight function w, the Greedy algorithm solves the optimization problem $(\mathcal{I}, w)$ by finding $I \in \mathcal{I} \subseteq E$ such that I is of maximum weight. The Greedy algorithm works if and only if $(E, \mathcal{I})$ has a matroid structure. In fact, for some combinatorial optimization problems which do not have a matroid structure, including some variants of the traveling salesman problem, the Greedy algorithm produces the unique worst possible solution [1].

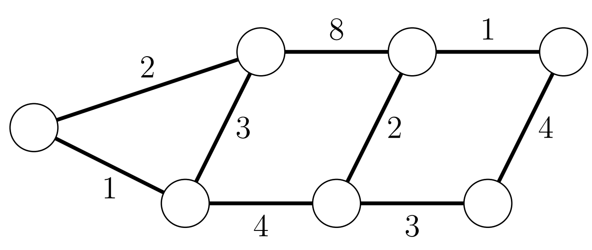
Let E be a finite set and $\mathcal{I}$ a collection of subsets of E such that if $A \in \mathcal{I}$ and $B \subset A$ then $B \in \mathcal{I}$. Let $w\colon E \longrightarrow \mathbb{R}^+$ be a weight function. Extend $w\colon 2^E \longrightarrow \mathbb{R}^+$, for $A \subseteq E$, $w(A) = \sum_{e \in A} w(e)$.

The greedy algorithm solves $(\mathcal{I}, w)$, that is it finds $A \in \mathcal{I}$ such that $w(A) \geq w(B)$ for all $B \in \mathcal{I}$.

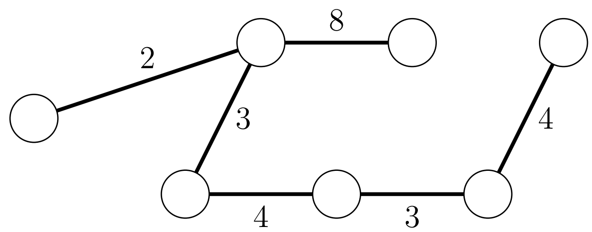
The Greedy Algorithm:

```
Set A = E
Set B = ∅
Loop until done{
Select x = z ∈ A such that w(z) ≥ w(y) for all y ∈ A
and B ∪ {z} ∈ I
If no such x exists, exit.
Otherwise set B = B ∪ {x} and A = A\{x}
}
```

Example 4.1. Let G be a connected graph with the weight function $w : E \to \mathbb{R}^+$. The greedy algorithm finds the spanning tree of G of maximum weight.
G:



The Greedy Algorithm selects B:



as the maximum weight spanning tree of G.

Theorem 4.2 ([2] [4] [5])**.** *The Greedy algorithm maximizes $(\mathcal{I}, w)$ if and only if $\mathcal{I}$ is the collection of independent sets of a matroid on the ground set E.*

Proof. ($\Rightarrow$) Assume that $\mathcal{I}$ is not the collection of independent sets of a matroid on the ground set E. By construction, $(E, \mathcal{I})$ satisfies conditions (1) and (2) in the definition of a matroid. Assume $(E, \mathcal{I})$ does not satisfy condition (3). Then there exists I', $I'' \in \mathcal{I}$ such that $|I'| < |I''|$ and $I' \cup \{e\} \notin \mathcal{I}$ for all $e \in I'' \backslash I'$.

Define $w \colon E \longrightarrow \mathbb{R}^+$ by

$$w(e) = \begin{cases} 1, & \text{if } e \in I' \\ \alpha, & \text{if } e \in I'' \backslash I' \\ 0, & \text{otherwise} \end{cases}$$

where $\frac{|I' \backslash I''|}{|I'' \backslash I'|} < \alpha < 1$.

Let B be the set selected by the Greedy Algorithm. The Greedy Algorithm will first choose all of the elements of I', so $I' \subseteq B$. By assumption if $e \in I'' \backslash I'$, then $I' \cup \{e\} \notin \mathcal{I}$. That implies that the remaining elements of B all have weight 0. Then $w(B) = |I'|$ but

$$w(I'') = \sum_{e \in I''} w(e) = |I' \cap I''| + |I'' \backslash I'|\alpha > |I' \cap I''| + |I'' \backslash I'| \frac{|I' \backslash I''|}{|I'' \backslash I'|} = |I'|.$$

Therefore, if $\mathcal{I}$ is not the collection of independent sets of a matroid on the ground set E, then the Greedy algorithm does not maximize $(\mathcal{I}, w)$.

($\Leftarrow$) Let $(E, \mathcal{I})$ be a matroid M. Let $B = \{e_1, \ldots, e_r\}$ be the base selected by the Greedy Algorithm. Let $T = \{f_1, \ldots, f_r\}$ be another base of M such that $f_1 \geq f_2 \geq \cdots \geq f_r$.

Assume by way of contradiction that $w(T) > w(B)$. Then there exists some i, $1 \leq i \leq r$ such that $w(e_i) < w(f_i)$. Let k be the smallest such i. Let $I' = \{e_1, \ldots, e_{k-1}\}$ and $I'' = \{f_1, \ldots, f_k\}$. Then $|I'| < |I''|$. By condition (3) of the definition, there exists some $x \in (I'' \backslash I')$ such that $I' \cup \{x\} \in \mathcal{I}$. By assumption, $w(x) \geq w(f_k) > w(e_k)$ but the Greedy Algorithm chose e_k instead of x which implies that $w(e_k) \geq w(x)$. Contradiction. Therefore, $w(B) \geq w(T)$ for all $T \in \mathcal{B}(M)$. $\blacksquare$

5 Appendix A

The Greedy Algorithm function was written in Sage. The function takes a graphic matroid, m1, and a weight function, weightfn. The weight function is written as [[edge 1,weight 1],...,[edge n, weight n]] where edge i is an edge in the ground set of m1 and weight i is a nonnegative real number. The function returns a set in the independent sets of m1 of maximum weight.

```
def greedyAlg(m1, weightfn):
    B=Set()
    wfn=weightfn[:]
    for i in range(len(wfn)):
        if wfn[i][1]<0:
            return 'Weight values must be nonnegative.'
    done=0
    totalWt=0
    while(done==0):
        d=len(wfn)
        wt=-1
        for i in range(d):
            if wfn[i][1]>wt and\
               m1.is_independent(B.union(Set([wfn[i][0]]))):
                e=wfn[i]
                wt=wfn[i][1]
        if wt>-1:
            B=B.union(Set([e[0]]))
            totalWt=totalWt+e[1]
            wfn.remove(e)
        else:
            done=1
    print 'Weight = '
    print totalWt
    return B
```

Create a connected graph and label the edges.

```
G2 = Graph({1:[2,3,4],2:[4,5],3:[4,5],4:[5]})
```

```
G2.set_edge_label(1,2,'a')
G2.set_edge_label(1,3,'b')
G2.set_edge_label(1,4,'c')
G2.set_edge_label(2,4,'d')
G2.set_edge_label(2,5,'e')
G2.set_edge_label(3,4,'f')
G2.set_edge_label(3,5,'g')
G2.set_edge_label(4,5,'h')
```

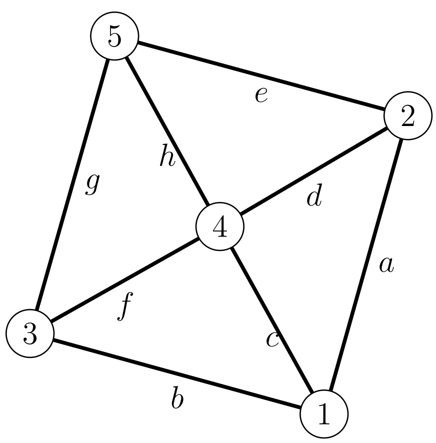
Create a graphic matroid from the graph G2.

```
M2=Matroid(G2)
```

Define a weight function for the matroid M2.

```
wtfn=[['a',6],['b',3],['c',9],['d',6],['e',4],['f',6],\
  ['g',3],['h',5]]
```

```
G2.plot(edge_labels=true)
```



Call the Greedy Algorithm function.

```
greedyAlg(M2,wtfn)
Weight =
26
{'a', 'h', 'c', 'f'}
```

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