The Wishart Distribution

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Before seeing the math, what is Wishart Distribution?

- Wishart: multivariate extension of the gamma distribution

Chi-square distribution	Wishart distribution
Sums of squares of n draws from a <u>univariate</u> normal distribution	Sums of squares (and cross-products) of n draws from a <u>multivariate</u> normal distribution

Sample mean of n i.i.d. chi-squared variable:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim \text{Gamma}\left(\alpha = n\,k/2, \theta = 2/n\right) \qquad \text{where} \quad X_i \sim \chi^2(k)$$

vcov of MVN

Let $\mathrm{S} \sim \mathrm{Wish}_{p}(\boxed{\Sigma}, \nu)$ DF

dimension

We must have *ν>p*-1 v>p+1

pdf: $$f(\mathrm{S}) = \frac{|\mathrm{S}|^{\boxed{\frac{\nu-p-1}{2}}}}{2^{\frac{\nu p}{2}}|\Sigma|^{\frac{\nu}{2}}\Gamma_p\left(\frac{\nu}{2}\right)} \exp\left[-\frac{1}{2}\mathrm{tr}(\Sigma^{-1}\mathrm{S})\right]$$

Gamma function: $$\Gamma_p(x) = \pi^{\frac{1}{2}\binom{p}{2}} \prod_{j=1}^{p} \Gamma[x + (1-j)/2]$$

<u>Draws from a Wishart represents sums of squares not variances.</u>

Sum of individual Wishart variables:

Let $\mathrm{S}_1, \mathrm{S}_2, \ldots, \mathrm{S}_i, \ldots, \mathrm{S}_k$ be independent from *k* Wishart: $\mathrm{S}_i \sim \mathrm{Wish}(\Sigma, \nu_i)$.

Then we have: $$\mathrm{SS} = \sum_{i=1}^{k} \mathrm{S}_i \sim \mathrm{Wish}\left(\Sigma, \sum_{i=1}^{k} \nu_i\right)$$

Expected Value: $\mathrm{E}(\mathbf{S}) = \nu\Sigma$

Chi-Square Distribution

The expected value of $\chi^2(\nu)$ is ν

Differences are the underlying dimensionality of the data and a scale component.

Variance: $\mathrm{Var}(\mathrm{S}_{ij}) = \nu(\sigma_{ij}^2 + \sigma_{ii}\sigma_{jj})$

X~$\chi^2(\nu)$, p=1, Var(X) = $\nu(1+1*1)$ as $\sigma_{11} = \sigma_{11}^2 = 1$

Note that Var($\mathbf{S}_{ij}$) is a set of variances.

Covariance:

$$
\begin{aligned}
\mathrm{Cov}(\mathbf{S}) &= \mathrm{Cov}\left(\sum_{i=1}^{\nu} \mathbf{x}_i\mathbf{x}_i^T\right) \\
&= \sum_{i=1}^{\nu} \mathrm{Cov}(\mathbf{x}_i\mathbf{x}_i^T) \\
&= \nu\mathrm{Cov}(\mathbf{C}\mathbf{z}_i\mathbf{z}_i^T\mathbf{C}^T)
\end{aligned}
$$

Wishart's Relationship to Normal Distribution

$$\mathbf{x}_i \overset{\text{iid}}{\sim} N(\mathbf{0}, \mathbf{\Sigma}) \qquad i = 1, 2, \ldots, n$$

Stack $\boldsymbol{X_i}$ as rows

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \\ \vdots \\ \mathbf{x}_n^T \end{bmatrix}$$

Wishart describes the pdf of this p-by-p matrix

$$\mathbf{S} = \boxed{\mathbf{X}^T\mathbf{X}} \sim \text{Wish}_p(\mathbf{\Sigma}, n)$$

Wishart's Relationship to the χ^2 Distribution

We have $\mathbf{S} \sim \mathrm{Wish}_p(\boldsymbol{\Sigma}, \nu)$.

Given any $\boldsymbol{\lambda} \in \mathbb{R}^p$

$$\boxed{\boldsymbol{\lambda}^T \mathbf{S} \boldsymbol{\lambda}} \sim \boldsymbol{\lambda}^T \boldsymbol{\Sigma} \boldsymbol{\lambda} \times \chi^2(\nu)$$

This quadratic form is a scaled $\chi^2(\nu)$