Clarifying the Relationship Between Faster-Than-Light Travel and Time Travel

Summary of Research Project, Summer 2021

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Contents

1 Constant Velocity

Two spaceships moving at a constant relative velocity to each other send each other tachyons. The ship sending the "first" tachyon is known as "ship 1" and is the basis for the S frame. The other ship is travelling to the right at a constant velocity in the S frame and is known as "ship 2" with its frame of reference called S' frame. Once the "first" tachyon from ship 1 reaches ship 2, ship 2 will then send a tachyon back to ship 1. Both tachyons will move at a constant velocity. Time travel is said to occur when the tachyon sent by ship 2 reaches ship 1 at a time before ship 1 sends the "first" tachyon. Since ship 1 sends the tachyon at $t = 0$ in the S frame, if the tachyon sent by ship 2 reaches ship 1 at any time with $t < 0$ time travel has occurred.

1.1 Equations of Motion and Transformations

The two ships are moving at a relative velocity of v and at $t = 0$ in the S frame they are x_c away from each other. From this the transformations between the S and S' frames are as follows (where γ represents $\frac{1}{\sqrt{1-v^2}}$).

$$x' = \gamma(x - x_c - vt)$$
$$t' = \gamma(t - v(x - x_c))$$
$$x = \gamma(x' + vt') + x_c$$
$$t = \gamma(t' + vx')$$

In the S frame ship 1 is not moving in space, ship 2 is moving at a constant velocity to the right v and is at a distance of x_c to ship 1 at $t = 0$, and the "first" tachyon (called tach 1 from now on) also moves at a constant velocity to the right u_1 and starts at $(0, 0)$. So the equations of motion in the S frame are: ship 1$\rightarrow x = 0$, ship 2$\rightarrow t = \frac{x-x_c}{v}$, tach 1$\rightarrow t = \frac{x}{u_1}$ In the S' frame ship 2 is not moving in space, ship 1 is moving at a constant velocity to the left v and is at a distance of $\sqrt{1-v^2}x_c$ (found using the above transformations) to ship 2 at $t = 0$, and the "second" tachyon (called tach 2 from now on) also moves at a constant velocity to the left u_2 and starts at $(\theta, 0)$ (where θ is the t value in the S' frame that tach 1 intersects with ship 2). So the equations of motion in the S' frame are: ship 1$\rightarrow t = \frac{-x-\sqrt{1-v^2}x_c}{v}$, ship 2$\rightarrow x = 0$, and tach 2 $\rightarrow t = -\frac{x}{u_2} + \theta$

1.2 Tachyon 1 Meeting Ship 2

The point where tach 1 will meet ship 2 will be name (x_0, t_0) and can be found using the intersection of the equations of motion for tach 1 and ship 2 in the S frame. We can set the right side of both equations equal to each other since the left sides of both are t which will both be $t = t_0$ when they intersect. After setting them equal to each other, we can replace x with x_0 and find an expression for x_0 since this is where they will meet.

$$\frac{x - x_c}{v} = \frac{x}{u_1}$$
$$\frac{x_0 - x_c}{v} = \frac{x_0}{u_1}$$
$$x_0 - x_c = \frac{vx_0}{u_1}$$
$$-x_c = x_0\left(\frac{v}{u_1} - 1\right)$$
$$x_0 = \frac{x_c}{1 - \frac{v}{u_1}}$$
$$\boxed{x_0 = \frac{x_c u_1}{u_1 - v}}$$

Using the value for x_0 we can fill it into the expression for tach 1 in the S frame to find an expression for

t_0.

$$t = \frac{x}{u_1}$$
$$t_0 = \frac{x_0}{u_1}$$
$$t_0 = \frac{1}{u_1}\left(\frac{x_c u_1}{u_1 - v}\right)$$
$$\boxed{t_0 = \frac{x_c}{u_1 - v}}$$

With the expressions for both x_0 and t_0 we can transform (x_0, t_0) into the S' frame in order to find the value for θ which will be the t coordinate of the transformed values. So (x_0, t_0) becomes $(\gamma(x_0 - x_c - vt_0), \gamma(t_0 - v(x_0 - x_c))) \rightarrow \left(\frac{1}{\sqrt{1-v^2}}\left(\frac{x_c u_1}{u_1-v} - x_c - \frac{v x_c}{u_1-v}\right), \frac{1}{\sqrt{1-v^2}}\left(\frac{x_c}{u_1-v} - v\left(\frac{x_c u_1}{u_1-v} - x_c\right)\right)\right)$. Now setting the t portion of the coordinate to be θ we can simplify it to find θ:

$$\theta = \frac{1}{\sqrt{1-v^2}}\left(\frac{x_c}{u_1 - v} - v\left(\frac{x_c u_1}{u_1 - v} - x_c\right)\right)$$
$$\theta = \frac{x_c}{\sqrt{1-v^2}}\left(\frac{1}{u_1 - v} - \frac{v u_1}{u_1 - v} + v\right)$$
$$\theta = \frac{x_c}{\sqrt{1-v^2}}\left(\frac{1 - v u_1 + v u_1 - v^2}{u_1 - v}\right)$$
$$\boxed{\theta = \frac{x_c\sqrt{1-v^2}}{u_1 - v}}$$

1.3 Tachyon 2 Meeting Ship 1

With an expression for θ we can now include it in the equation for tach 2 in the S' frame which yields $t = -\frac{x}{u_2} + \frac{x_c\sqrt{1-v^2}}{u_1-v}$. With this equation we can find the intersection of tach 2 and ship 1 in the S' frame. This point will be called (x_1, t_1) and like before when finding the intersection between tach 1 and ship 2, we can set the left side of the equations for tach 2 and ship 1 equal to each other and solve for x_1.

$$-\frac{x}{u_2} + \frac{x_c\sqrt{1-v^2}}{u_1 - v} = \frac{-x - \sqrt{1-v^2}x_c}{v}$$
$$-\frac{v x_1}{u_2} + \frac{v x_c\sqrt{1-v^2}}{u_1 - v} = -x_1 - \sqrt{1-v^2}x_c$$
$$x_1\left(1 - \frac{v}{u_2}\right) = -\sqrt{1-v^2}x_c - \frac{v x_c\sqrt{1-v^2}}{u_1 - v}$$
$$x_1 = \frac{u_2}{u_2 - v}\left(-x_c\sqrt{1-v^2} - \frac{v x_c\sqrt{1-v^2}}{u_1 - v}\right)$$
$$x_1 = \frac{u_2}{u_2 - v}\left(-\frac{x_c u_1\sqrt{1-v^2}}{u_1 - v} + \frac{v x_c\sqrt{1-v^2}}{u_1 - v} - \frac{v x_c\sqrt{1-v^2}}{u_1 - v}\right)$$
$$\boxed{x_1 = -\frac{x_c u_1 u_2\sqrt{1-v^2}}{(u_2 - v)(u_1 - v)}}$$

We can now find t_1 with the expression for x_1 and the equation for ship 1:

$$t_1 = -\frac{x_1}{u_2} + \frac{x_c\sqrt{1-v^2}}{u_1 - v}$$
$$t_1 = -\frac{1}{u_2}\left(-\frac{x_c u_1 u_2\sqrt{1-v^2}}{(u_2-v)(u_1-v)}\right) + \frac{x_c\sqrt{1-v^2}}{u_1 - v}$$
$$t_1 = \left(\frac{x_c u_1\sqrt{1-v^2}}{(u_2-v)(u_1-v)}\right) + \frac{x_c\sqrt{1-v^2}}{u_1 - v}$$
$$t_1 = \frac{x_c u_1\sqrt{1-v^2}}{(u_2-v)(u_1-v)} + \frac{x_c u_2\sqrt{1-v^2}}{(u_2-v)(u_1-v)} - \frac{x_c v\sqrt{1-v^2}}{(u_2-v)(u_1-v)}$$
$$\boxed{t_1 = \frac{x_c\sqrt{1-v^2}}{(u_2-v)(u_1-v)}\,(u_1+u_2-v)}$$

1.4 Final Result

To see if time travel has occurred, we need to know the t coordinate in the S frame of the intersection between tach 2 and ship 1. We can find this value using the transformations from above and the values we found for x_1 and t_1. Calling the resulting t value T we get:

$$T = \gamma(t' + vx')$$
$$T = \gamma(t_1 + vx_1)$$
$$T = \frac{1}{\sqrt{1-v^2}}\left(\frac{x_c\sqrt{1-v^2}}{(u_2-v)(u_1-v)}\,(u_1+u_2-v) + v\left(-\frac{x_c u_1 u_2\sqrt{1-v^2}}{(u_2-v)(u_1-v)}\right)\right)$$
$$T = \boxed{x_c\left(\frac{u_1+u_2-v-vu_1u_2}{(u_2-v)(u_1-v)}\right)}$$

1.5 Requirements of the Variables

We can find some constraints on the variables that will need to be met in order for time travel to occur. Some assumptions made are that $x_c > 0$, $u_1 > 0$, $u_2 > 0$ and $0 < v < 1$. With these assumptions, in order to get $T < 0$ we first look at the numerator.

$$x_c(u_1+u_2-v-vu_1u_1) < 0$$
$$u_1+u_2 < v + vu_1u_2$$
$$\frac{u_1+u_2}{1+u_1u_2} < v$$
$$\text{Using the assumption that } v < 1$$
$$\frac{u_1+u_2}{1+u_1u_2} < 1$$
$$u_1+u_2 < 1+u_1u_2$$
$$u_1 - 1 < u_1u_2 - u_2$$
$$u_1 - 1 < u_2(u_1-1)$$
$$\boxed{1 < u_2}$$

Now to make T overall negative with a negative numerator, the denominator must be positive. For $(u_2 - v)(u_1 - v)$ if $u_2 > 1$ then it is also $u_2 > v$ so $(u_2 - v)$ is positive. To make the total denominator be positive, this means $u_1 - v > 0$ or $u_1 > v$. We can also add another restriction on the value of u_1 since we know tachyon 1 must reach ship 2 for anything to happen. Due to this $u_1 > \frac{1}{v}$ must hold or else tach 1 will never meet ship 2. With $v < 1$, $1 < \frac{1}{v}$ so we also have:

$$u_1 > \frac{1}{v} > 1$$
$$\boxed{u_1 > 1}$$

To summarize, the above requirements prove that both u_1 and u_2 must be above the speed of light for time travel to occur if the assumptions that $x_c > 0$ and $0 < v < 1$ hold.

2 Accelerating Warp Drive

In this case again there are two spaceships moving at constant relative velocities, but instead of sending tachyons they utilize a warp drive. A warp drive allows objects placed inside the "warp bubble" to travel faster than the speed of light. The warp drives have a constant acceleration (note that this is simple non-relativistic acceleration – the limitations of relativistic acceleration do not apply here) and need to start with zero velocity and end with zero velocity at their destination. Like before, time travel occurs when the object sent by ship 2 using a warp drive (called warp 2) reaches ship 1 before the "first" object sent using a warp drive (called warp 1) leaves ship 1. The transformations remain the same as the constant velocity case as the two ships move at velocity v and start at a distance of x_c in the S frame in this case also.

2.1 Equations of Motion for Warp Drive 1

In order to start and end with zero velocity, the warp drive will spend the first half of its journey with a constant acceleration and the second half with the same value as a deceleration. $a_1(t)$, $v_1(t)$, and $x_1(t)$ represent the acceleration, velocity, and position respectively for the first half of the trip of warp 1 whereas $a_2(t)$, $v_2(t)$, and $x_2(t)$ represent acceleration, velocity, and position respectively for the second half of warp 1's trip. The velocity and position equations are determined using integration, so v_a, v_b, x_a, and x_b are all constants of integration that are calculated using the initial conditions below.

Acceleration:

$$a_1(t) = a$$
$$a_2(t) = -a$$

Velocity:

$$v_1(t) = at + v_a$$
$$v_2(t) = -at + v_b$$

Position:

$$x_1(t) = \frac{at^2}{2} + v_a t + x_a$$
$$x_2(t) = -\frac{at^2}{2} + v_b t + x_b$$

$v_a = 0$ and $x_a = 0$ because the velocity and position both start at 0 when $t = 0$. The point where warp 1 will meet ship 2 is named (x_0, t_0). Since the functions of velocity and position are functions of time, the halfway point will be $\frac{t_0}{2}$. So, to find v_b and x_b we need $v_1\left(\frac{t_0}{2}\right) = v_2\left(\frac{t_0}{2}\right)$ and $x_1\left(\frac{t_0}{2}\right) = x_2\left(\frac{t_0}{2}\right)$ since the position and velocity of the first half of the trip will be equal to the second half of the trip at the halfway point:

$$v_1\left(\frac{t_0}{2}\right) = v_2\left(\frac{t_0}{2}\right)$$
$$\frac{at_0}{2} = -\frac{at_0}{2} + v_b$$
$$\boxed{v_b = at_0}$$

$$x_1\left(\frac{t_0}{2}\right) = x_2\left(\frac{t_0}{2}\right)$$
$$\frac{a\left(\frac{t_0}{2}\right)^2}{2} = \frac{-a\left(\frac{t_0}{2}\right)^2}{2} + at_0\left(\frac{t_0}{2}\right) + x_b$$
$$\frac{at_0^2}{4} - \frac{at_0^2}{2} = x_b$$
$$\boxed{x_b = -\frac{at_0^2}{4}}$$

The result is these position equations for warp drive 1 in the S frame:

$$x_1(t) = \frac{at^2}{2}$$
$$x_2(t) = -\frac{at^2}{2} + at_0 t - \frac{at_0^2}{4}$$

2.2 Warp Drive 1 Meeting Ship 2

(x_0,t_0) is the name given to the point where warp drive 1 meets ship 2. This point will be on the line $x_2(t)$, knowing this we can plug this point into that equation to find an expression for t_0.

$$x_0 = -\frac{at_0^2}{2} + at_0t_0 - \frac{at_0^2}{4}$$
$$x_0 = -\frac{2at_0^2}{4} + \frac{4at_0^2}{4} - \frac{at_0^2}{4}$$
$$x_0 = \frac{at_0^2}{4}$$
$$\frac{4x_0}{a} = t_0^2$$
$$\boxed{t_0 = 2\sqrt{\frac{x_0}{a}}}$$

(x_0,t_0) transformed into S' frame will be the starting point for warp 2. To make calculations easier to read the transformed t coordinate will be referred to as θ. Using the transformations from section 1.1 $(x_0, t_0) \to (\gamma(x_0 - x_c - vt_0), \gamma(t_0 - v(x_0 - x_c))) \to (\gamma(x_0 - x_c - vt_0), \theta)$

2.3 Equations of Motion for Warp Drive 2

The position equations for warp drive 2 in ship 2's frame of reference will be similar to warp drive 1 in ship 1's frame but will be moving in the opposite direction (so x becomes $-x'$) and it will have a t-intercept added to both equations because it does not start at $t' = 0$, and instead $x'(0) = \theta$ (with $\theta = \gamma(t_0 - v(x_0 - x_c))$). $(x_3, t_3 + \theta)$ is the point where warp 2 meets ship 1 where t_3 is the time taken for warp 2 to reach ship 1 from ship 2, which is equivalent to t_0 from the previous section. This means t_0 is replaced by t_3 in the following equations. The last difference is $a \to a_2$ as the returning warp drive may have a different acceleration to the first one.

$$-x_1'(t') = \frac{a_2(t' - \theta)^2}{2}$$
$$-x_2'(t') = -\frac{a_2(t' - \theta)^2}{2} + a_2(t' - \theta)t_3 - \frac{a_2 t_3^2}{4}$$

The intersection point of warp 2 and ship 1 will be called $(0, T)$ in the S frame, or $(x_3,t_3 + \theta)$ in the S' frame. This will be a point on the line $x_2'(t')$ so plugging this point into that equation gives:

$$-x_3 = -\frac{a_2(t_3+\theta-\theta)^2}{2} + a_2(t_3+\theta-\theta)t_3 - \frac{a_2t_3^2}{4}$$
$$-x_3 = -\frac{a_2(t_3)^2}{2} + a_2(t_3)t_3 - \frac{a_2t_3^2}{4}$$
$$-x_3 = a_2t_3^2\left(-\frac{1}{2}+1-\frac{1}{4}\right)$$
$$\boxed{x_3 = -\frac{a_2t_3^2}{4}}$$

2.4 Final Result

$(0,T)$ is $(x_3,t_3+\theta)$ transformed into the S frame. So by transforming $(x_3,t_3+\theta)$ using the transformations from seciotn 1.1 we can get an expression for T:

$$0 = \gamma(x_3 + v(t_3+\theta)) + x_c$$
$$T = \gamma(t_3+\theta+vx_3)$$

Then using those expressions to solve for $t_3+\theta$.

$$0 = \gamma(x_3+vt_3+v\theta)+x_c$$
$$-\frac{x_c}{\gamma} = x_3+vt_3+v\theta$$
$$v\theta = -\frac{x_c}{\gamma} - x_3 - vt_3$$
$$\theta = -\frac{x_c}{\gamma v} - \frac{x_3}{v} - t_3$$
$$\boxed{t_3+\theta = \frac{-x_c-\gamma x_3}{v\gamma}}$$

Next, putting the expression for $t_3+\theta$ above into the equation for T yields:

$$T = \gamma(t_3+\theta+vx_3)$$
$$T = \gamma\left(\frac{-x_c-\gamma x_3}{v\gamma}+vx_3\right)$$
$$\boxed{T = \frac{\gamma}{v}\left(-\frac{x_c}{\gamma}+x_3(v^2-1)\right)}$$

We want to replace x_3 in terms of a, a_2, v, and x_c. To start we want the path of ship 1 in ship 2's frame. It will have the same speed (but in the opposite direction so $v \to -v$) but the t-intercept will be different. To find it we can transform $(0,0)$ in ship 1's frame to its equivalent in ship 2's frame which is $(-\gamma x_c, \gamma v x_c)$. With this point and knowing the slope to be $-1/v$ we can find the t-intercept (called A here):

$$t = -\frac{x}{v} + A$$
$$\gamma v x_c = \frac{\gamma x_c}{v} + A$$
$$A = \gamma x_c\left(v - \frac{1}{v}\right)$$
$$\boxed{t = \frac{-x+\gamma x_c(v^2-1)}{v}}$$

Now rearranging the equation we just got above for the path of the first ship in the S' frame we can set it equal to the equation for the second half of the path of warp 2 since it will intersect with ship 1 at $(x_3, t_3 + \theta)$. Plugging in this point we can then find an expression for t_3 (using the quadratic formula and choosing the positive result since t_3 must be positive):

$$\begin{aligned}
vt' - \gamma x_c(v^2 - 1) &= -\frac{a_2(t' - \theta)^2}{2} + a_2(t' - \theta)t_3 - \frac{a_2 t_3^2}{4} \\
v(t_3 + \theta) - \gamma x_c(v^2 - 1) &= -\frac{a_2(t_3 + \theta - \theta)^2}{2} + a_2(t_3 + \theta - \theta)t_3 - \frac{a_2 t_3^2}{4} \\
vt_3 + v\theta - \gamma x_c(v^2 - 1) &= -\frac{a_2 t_3^2}{2} + a_2 t_3^2 - \frac{a_2 t_3^2}{4} \\
0 &= \frac{a_2 t_3^2}{4} - vt_3 - v\theta + \gamma x_c(v^2 - 1) \\
&\boxed{t_3 = \frac{2v + 2\sqrt{v^2 + a_2 v\theta - \gamma a_2 x_c(v^2 - 1)}}{a_2}}
\end{aligned}$$

Using the expression we found in section 2.3 for x_3 we can now substitute in the above result for t_3 to get

$$\begin{aligned}
x_3 &= -\frac{a_2 t_3^2}{4} \\
x_3 &= -\frac{a_2}{4}\left(\frac{2v + 2\sqrt{v^2 + a_2 v\theta - \gamma a_2 x_c(v^2 - 1)}}{a_2}\right)^2 \\
x_3 &= -\frac{1}{4a_2}\left(4v^2 + 8v\sqrt{v^2 + a_2 v\theta - \gamma a_2 x_c(v^2 - 1)} + 4(v^2 + a_2 v\theta - \gamma a_2 x_c(v^2 - 1))\right) \\
x_3 &= -\frac{v^2}{a_2} - \frac{2v\sqrt{v^2 + a_2 v\theta - \gamma a_2 x_c(v^2 - 1)}}{a_2} - \frac{v^2}{a_2} - v\theta + \gamma x_c(v^2 - 1) \\
&\boxed{x_3 = -\frac{2v^2}{a_2} - \frac{2v\sqrt{v^2 + a_2 v\theta - \gamma a_2 x_c(v^2 - 1)}}{a_2} - v\theta + \gamma x_c(v^2 - 1)}
\end{aligned}$$

Putting this new expression for x_3 into the equation for T yields:

$$T = \frac{\gamma}{v}\left(-\frac{x_c}{\gamma} + (v^2 - 1)\left(-\frac{2v^2}{a_2} - \frac{2v\sqrt{v^2 + a_2 v\theta - \gamma a_2 x_c(v^2 - 1)}}{a_2} - v\theta + \gamma x_c(v^2 - 1)\right)\right)$$

Now we just want an expression for θ which can be found by first finding an expression for t_0 then plugging that into the equation for θ that comes from the transformation of the point (x_0, t_0). To find t_0 first we use the intersection of the path of ship 2 and the second half of warp 1's path, then use the quadratic formula to solve.

$$\begin{aligned}
vt + x_c &= -\frac{at^2}{2} + at_0 t - \frac{at_0^2}{4} \\
vt_0 + x_c &= -\frac{at_0^2}{2} + at_0^2 - \frac{at_0^2}{4} \\
0 &= \frac{at_0^2}{4} - vt_0 - x_c \\
&\boxed{t_0 = \frac{2v + 2\sqrt{v^2 + ax_c}}{a}}
\end{aligned}$$

Next using $x_0 = \frac{at_0^2}{4}$ and the expression we just found for t_0 we can solve for θ:

$$\theta = \gamma(t_0 - v(x_0 - x_c))$$
$$\theta = \gamma\left(t_0 - \frac{avt_0^2}{4} + vx_c\right)$$
$$\theta = \gamma\left(\frac{2v + 2\sqrt{v^2 + ax_c}}{a} - \frac{av}{4}\left(\frac{2v + 2\sqrt{v^2 + ax_c}}{a}\right)^2 + vx_c\right)$$
$$\theta = \gamma\left(\frac{2v + 2\sqrt{v^2 + ax_c}}{a} - \frac{v}{4a}\left(8v^2 + 4ax_c + 8v\sqrt{v^2 + ax_c}\right) + vx_c\right)$$
$$\boxed{\theta = \frac{2\sqrt{1 - v^2}}{a}\left(v + \sqrt{v^2 + ax_c}\right)}$$

Lastly, we plug this expression for θ back into the equation for T which gives an expression relying only on x_c, v, a, and $a2$

$$T = \frac{1}{v}\Bigg(- x_c - \sqrt{1 - v^2}\Bigg(- \frac{2v^2}{a_2} - \frac{2v}{a_2}\sqrt{v^2 + \frac{2a_2 v\sqrt{1 - v^2}}{a}\left(v + \sqrt{v^2 + ax_c}\right) + a_2 x_c\sqrt{1 - v^2}} \\ - \frac{2v\sqrt{1 - v^2}}{a}\left(v + \sqrt{v^2 + ax_c}\right) - x_c\sqrt{1 - v^2}\Bigg)\Bigg)$$

2.5 Visualizing the Result

There is a Mathematica notebook in this Github repository (called simulations.nb) that can be used to visualize the trips of the two ships that use warp drives. The user can control the dependent variables mentioned in the result (x_c, v, and a, both warp drives have the same acceleration in the simulation) and see the effect it has on the paths of the ships. The variables v and a can be controlled with sliders whereas x_c can be changed by dragging the circle on the x-axis. Whether or not time travel will occur based on the values of the variables can be seen in "Arrival time", because if that value is negative it means the second ship arrived before the first one left. The user can also see on the position-time graph if the path of the returning ship intersects the time axis of S below where the first ship leaves.

2.6 Requirements of the Variables

As either v or x_c go up, the minimum acceleration needed for time travel goes down. This makes sense as the further away the two spaceships start, the longer the warp drive has to accelerate. x_c also must be > 0 for time travel to occur, if it is less than or equal to 0 it will be impossible. Same for v, the speed of the second ship, it must be > 0, if it is less than or equal to 0 time travel cannot occur. Due to the expression for T having so many square roots with a and a_2 under them, it is difficult if not impossible to create an expression for the minimum acceleration required for time travel as a function of x_c and v. Due to this, there is not an expression included for the minimum acceleration, but it can be determined by setting the equation for $T = 0$ and plugging in values for x_c and v.

3 Accelerating Tachyons

In this case again there are two spaceships moving at constant relative velocities, but this time they send tachyons that start with infinite velocity (as that is the velocity of a tachyons at rest) and are decelerated to speeds above the speed of light. Note that we are using non-relativistic acceleration here, which means our result are only a crude approximation; precise result would require taking into account the relativistic 4-acceleration. We leave that for future work. The tachyons have a constant deceleration and need to start with infinite velocity and end with infinite velocity at their destination, that is, start and end at "rest". Like before, time travel occurs when the tachyon sent by ship 2 (called tach 2) reaches ship 1 before the "first" tachyon (called tach 1) leaves ship 1. The transformations remain the same as the constant velocity case as the two ships move at velocity v and start at a distance of x_c in the S frame in this case also.

3.1 Equations of Motion for Tachyon 1

In order to start and end with infinite velocity, the tachyon will spend the first half of its journey with a constant deceleration and the second half with the same value but as an acceleration. a_1, v_1, and t_1 represent the deceleration, velocity, and position respectively for the first half of the trip of tach 1 whereas a_2, v_2, and t_2 represent acceleration, velocity, and position respectively for the second half of tach 1's trip. The position equations were determined by switching the place of x and t in the warp 1's equations of motion as tachyons act similarly to a warp drive but with the t and x axis switched. (x_0, t_0) is again the point where tach 1 meets ship 2, and here t_0 was replaced with x_0 as compared to the warp drive equations.

Acceleration:

$$a_1 = a$$
$$a_2 = -a$$

Velocity:

$$v_1 = ax$$
$$v_2 = -ax + ax_0$$

Position:

$$t_1 = \frac{ax^2}{2}$$
$$t_2 = -\frac{ax^2}{2} + ax_0x - \frac{ax_0^2}{4}$$

3.2 Tachyon 1 Meeting Ship 2

Like before, (x_0, t_0) is the place where tachyon 1 will meet ship 2. So this point will be on the line $t_2(x)$, which we can plug into that equation to find the value for t_0.

$$t_0 = -\frac{ax_0^2}{2} + ax_0^2 - \frac{ax_0^2}{4}$$
$$\boxed{t_0 = \frac{ax_0^2}{4}}$$

(x_0, t_0) will be transformed into the S' frame as above so $(x_0, t_0) \to (\gamma(x_0 - x_c - vt_0), \gamma(t_0 - v(x_0 + x_c)) \to (\gamma(x_0 - x_c - vt_0), \theta)$.

3.3 Equations of Motion for Tachyon 2

The position equations for tachyon 2 in ship 2's frame of reference will be similar to tachyon 1 in ship 1's frame, but will be moving in the opposite direction and it will have a t-intercept added to both equations because it does not start at $(0, 0)$, but at $(0, \theta)$. As well as changing $t_0 \to t_3$ in the equations where t_3 represents the time taken to move from ship 2 to ship 1 in the S' frame.

$$t_1'(x') = \frac{a_2x'^2}{2} + \theta$$
$$t_2'(x') = -\frac{a_2x'^2}{2} + a_2x_3x' - \frac{a_2x_3^2}{4} + \theta$$

3.4 Tachyon 2 Meeting Ship 1

The intersection point of tachyon 2 and ship 1 is $(0, T)$ in the S frame or called $(x_3, t_3 + \theta)$ in the S' frame. This will be a point on the line $t'_2(x')$ so plugging this point into the equation gives an expression for t_3.

$$t_3 = -\frac{a_2 x_3^2}{2} + a_2 x_3^2 - \frac{a_2 x_3^2}{4}$$
$$\boxed{t_3 = \frac{a_2 x_3^2}{4}}$$

3.5 Final Result

$(0, T)$ is $(x_3,t_3+\theta)$ transformed into ship 1's frame. So by transforming $(x_3,t_3+\theta)$ using the transformations from section 1.1 we can get an expression for T.

$$0 = \gamma(x_3 + v(t_3 + \theta)) + x_c$$
$$T = \gamma(t_3 + \theta + vx_3)$$

Then using the same replacement as used for warp drives for $t_3 + \theta$:

$$T = \gamma\left(-\frac{x_c}{v\gamma} - \frac{x_3}{v} + vx_3\right)$$
$$T = \gamma\left(-\frac{x_c}{v\gamma} + x_3\left(\frac{v^2 - 1}{v}\right)\right)$$
$$\boxed{T = \frac{\gamma}{v}\left(-\frac{x_c}{\gamma} + x_3(v^2 - 1)\right)}$$

Again we want to replace x_3 in terms of a, v, and x_c. So using the expression for ship 1 in the S' frame to find where it will intersect with tach 2 gives:

$$\frac{-x_3 - x_c\sqrt{1 - v^2}}{v} = \frac{a_2 x_3^2}{4} + \theta$$
$$-x_3 - x_c\sqrt{1 - v^2} = \frac{a_2 v x_3^2}{4} + v\theta$$
$$0 = \frac{a_2 v x_3^2}{4} + x_3 + x_c\sqrt{1 - v^2} + v\theta$$
$$\boxed{x_3 = \frac{-2 + 2\sqrt{1 - a_2 v(v\theta + x_c\sqrt{1 - v^2})}}{a_2 v}}$$

Next we need to find θ again, in a similar way as for warp drives. First we need to find x_0 by looking for the intersection of tach 1 and ship 2 by setting the t values of each of their equations equal to each other.

$$\frac{x_0 - x_c}{v} = -\frac{a x_0^2}{4}$$
$$0 = \frac{a v x_0^2}{4} - x_0 + x_c$$
$$\boxed{x_0 = \frac{2 - 2\sqrt{1 - a v x_c}}{a v}}$$

Plugging this into an expression for θ and using the fact that $t_0 = \frac{ax_0^2}{4}$:

$$\theta = \gamma(t_0 - v(x_0 - x_c))$$
$$\theta = \gamma\left(\frac{ax_0^2}{4} - v(x_0 - x_c)\right)$$
$$\theta = \gamma\left(\frac{a(\frac{2-2\sqrt{1-avx_c}}{av})^2}{4} - v\left(\frac{2-2\sqrt{1-avx_c}}{av} - x_c\right)\right)$$
$$\theta = \gamma\left(\frac{a}{4}\left(\frac{4-8\sqrt{1-avx_c}+4-4avx_c}{a^2v^2}\right) - \frac{2-2\sqrt{1-avx_c}}{a} + vx_c\right)$$
$$\theta = \gamma\left(\frac{2-2\sqrt{1-avx_c}-avx_c}{av^2} - \frac{2-2\sqrt{1-avx_c}}{a} + vx_c\right)$$
$$\boxed{\theta = \frac{\sqrt{1-v^2}\left(2-avx_c-2\sqrt{1-avx_c}\right)}{av^2}}$$

Finally plugging x_3 and θ into the expression above for T we can get the final expression in terms of x_c, v, and a only.

$$T = \frac{1}{v}\left(-x_c - \sqrt{1-v^2}\left(\frac{-2+2\sqrt{1-a_2v(v\theta+x_c\sqrt{1-v^2})}}{a_2v}\right)\right)$$
$$T = \frac{1}{v}\left(-x_c + \frac{2\sqrt{1-v^2}}{a_2v}\left(1-\sqrt{1-a_2v\sqrt{1-v^2}\left(\frac{\left(2-avx_c-2\sqrt{1-avx_c}\right)}{av}+x_c\right)}\right)\right)$$
$$\boxed{T = \frac{1}{v}\left(-x_c + \frac{2\sqrt{1-v^2}}{a_2v}\left(1-\sqrt{1-\frac{2a_2\sqrt{1-v^2}}{a}\left(1-\sqrt{1-avx_c}\right)}\right)\right)}$$

3.6 Visualizing the Result

There is a Mathematica notebook in this Github repository (called simulations.nb) that can be used to visualize the trips of the two tachyons. The user can control the dependent variables mentioned in the result (x_c, v, a, and a_2) and see the effect it has on the paths of the warp drives. The variables v, a, and a_2 can be controlled with sliders whereas x_c can be changed by dragging the circle on the x-axis. The acceleration of the tachyons are actually deceleration since they start from infinite velocity and decelerate to a finite velocity. Whether or not time travel will occur based on the values of the variables can be seen in "Arrival time", because if that value is negative it means the second tachyon arrived before the first one left. The user can also see on the position-time graph if the path of the returning tachyon intersects the time axis of S below where the first tachyon leaves.

3.7 Requirements of the Variables

In the expression for T, there is a term $\sqrt{1-avx_c}$ which means $avx_c < 1$ or else the square root will be imaginary. So $a < \frac{1}{vx_c}$ in order for tach 2 to meet ship 1. Since tachyons cannot go slower than the speed of light the minimum velocity it can decelerate to has to be 1. The minimum velocity happens at the halfway point each time, or $\frac{x_0}{2}$ on the first trip and $\frac{x_3}{2}$ on the way back. So on the first trip $v = \frac{ax_0}{2} < 1$ which we can then plug in the value for x_0 to find

$$\frac{ax_0}{2} < 1$$
$$\frac{a}{2}\left(\frac{2-2\sqrt{1-avx_c}}{av}\right) < 1$$
$$\frac{1-\sqrt{1-avx_c}}{v} < 1$$
$$1-\sqrt{1-avx_c} < v$$
$$1-v < \sqrt{1-avx_c}$$
$$1-2v+v^2 < 1-avx_c$$
$$-2+v < -ax_c$$
$$\frac{v-2}{x_c} < -a$$
$$\boxed{a < \frac{2-v}{x_c}}$$

We can then do the same for the return trip

$$\frac{-a_2x_3}{2} < 1$$
$$\frac{-a_2}{2}\left(\frac{-2+2\sqrt{1-a_2v(v\theta+x_c\sqrt{1-v^2})}}{a_2v}\right) < 1$$
$$\frac{1-\sqrt{1-a_2v(v\theta+x_c\sqrt{1-v^2})}}{v} < 1$$
$$1-\sqrt{1-a_2v(v\theta+x_c\sqrt{1-v^2})} < v$$
$$1-v < \sqrt{1-a_2v(v\theta+x_c\sqrt{1-v^2})}$$
$$1-2v+v^2 < 1-a_2v(v\theta+x_c\sqrt{1-v^2})$$
$$-2v+v^2 < -a_2v(v\theta+x_c\sqrt{1-v^2})$$
$$v-2 < -a_2(v\theta+x_c\sqrt{1-v^2})$$
$$\frac{v-2}{v\theta+x_c\sqrt{1-v^2}} < -a_2$$
$$\frac{2-v}{v\theta+x_c\sqrt{1-v^2}} > a_2$$
$$\frac{2-v}{\left(\frac{\sqrt{1-v^2}\left(2-avx_c-2\sqrt{1-avx_c}\right)}{av}\right)+x_c\sqrt{1-v^2}} > a_2$$
$$\frac{2-v}{\sqrt{1-v^2}\left(\frac{2-2\sqrt{1-avx_c}}{av}\right)} > a_2$$
$$\frac{2-v}{\frac{2\sqrt{1-v^2}}{av}\left(1-\sqrt{1-avx_c}\right)} > a_2$$
$$\boxed{\frac{av(2-v)}{2\sqrt{1-v^2}\left(1-\sqrt{1-avx_c}\right)} > a_2}$$

So the constraints are for the tachyons to remain faster than the speed of light and take a path that will intersect with the path of the first ship are:

$$a < \frac{2-v}{x_c}$$
$$a < \frac{1}{vx_c}$$
$$a_2 < \frac{av(2-v)}{2\sqrt{1-v^2}\left(1-\sqrt{1-avx_c}\right)}$$

Also similar to accelerating warp drives, x_c and v must both be > 0 or else time travel cannot occur.

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