Physics 3926 Project 1

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1 Introduction

The purpose of this investigation was to explore the relationship between mass and radius for white dwarf stars. A white dwarf is the final evolutionary stage for a small star, the incredibly dense core which remains after the star collapses. The internal structure of these stars is governed by two coupled ordinary differential equations (ODEs), which define how the density and mass evolve with radius:

$$\frac{d\boldsymbol{\rho}}{d\boldsymbol{r}} = -K_1 \frac{3\boldsymbol{m}\boldsymbol{\rho}}{\boldsymbol{r}^2} \frac{\sqrt{1+\boldsymbol{\rho}^{2/3}}}{\boldsymbol{\rho}^{2/3}} \tag{1}$$

$$\frac{d\boldsymbol{m}}{d\boldsymbol{r}} = K_2 \boldsymbol{r}^2 \boldsymbol{\rho} \tag{2}$$

where the bold symbols represent dimensionless quantities given by:

$$\boldsymbol{\rho} = \rho/\rho_0,\ \rho_0 = (9.74 \times 10^5)\mu_e \text{ gcm}^{-3}$$

$$\boldsymbol{r} = r/r_0,\ r_0 = (7.72 \times 10^8)/\mu_e \text{ cm}$$

$$\boldsymbol{m} = m/m_0,\ m_0 = (5.67 \times 10^3 3)/(\mu_e^2) \text{ g}$$

with μ_e the number of nucleons per electron, taken to be $\mu_e = 2$ in this case. The quantities above were chosen specifically in order to force the constants K_1 and K_2 to be unity.

This system of ODEs was numerically solved to calculate the approximate mass and radius of white dwarfs with different center densities. This allowed for the estimation of the Chandrasekhar limiting mass - the largest possible mass of a white dwarf, beyond which a star can no longer be supported by electron degeneracy pressure.

2 Program

2.1 Part 1 and 2

Firstly, the function `odes(r,x)` defined the system of coupled ODEs which model the density and mass of white dwarfs (equations 1 and 2). This function takes an initial radius, as well as an initial density and mass (in a two-element array). Note that the variables used in these equations are the dimensionless quantities. Next, the main function of the program, `solve_system(odes, x0, method)`, was defined. This function uses the scipy `solve_ivp` method to solve the system of ODEs defined in the first function. The `x0` parameter gives the density and mass at the center of the star (the initial conditions), and `method` gives the method of numerical integration to be used by `solve_ivp`.

To perform the numerical simulation, an initial radius of 10^{-5} m was chosen as this was sufficiently close to zero without raising a `ZeroDivisionError`. In order to use this as the initial point for the integration, it had to be converted to a dimensionless quantity by dividing it by r_0. Similarly, a density of 10^{-8} cm^{-3} was chosen as being near enough to zero to end the numerical integration. This value was also converted to a dimensionless quantity by dividing by ρ_0. The `events` parameter in `solve_ivp` was used to terminate the integration at this value of density. Solving the system until $\rho \approx 0$ will cause the final mass and radius given to be that of the simulated white dwarf.

After calling `solve_ivp`, the radius, mass, and density components of the solution were extracted and converted to physical quantities by multiplying by r_0, m_0 and ρ_0, respectively. The `astropy.units` package was also used in this step to multiply each result with the corresponding unit. Although these solutions are given as arrays of each r-value of the numerical solution and the corresponding m and ρ values at each point, the actual mass and radius of the simulated white dwarfs are the *final* r and m values. For this reason, the `solve_system` function returns only the last values of the mass and radius arrays.

To generate the solutions, `solve_ivp` was called for 10 values of ρ_c ranging from 0.1 to 2.5×10^6 cm^{-3}, using Runge-Kutta method of order 5 (RK5). These were plotted as a scatter plot (see figure 1). It is evident in this figure that the simulated white dwarfs approach a limiting mass of approximately 2.85×10^{33} g (as represented by the vertical line on figure 1). Converted to be multiples of the solar mass (which is done by the code), the estimated Chandrasekhar limit is $M_c = \frac{5.733}{\mu_e^2}$. This is in close agreement with the literature value of $M_c = \frac{5.836}{\mu_e^2}$.

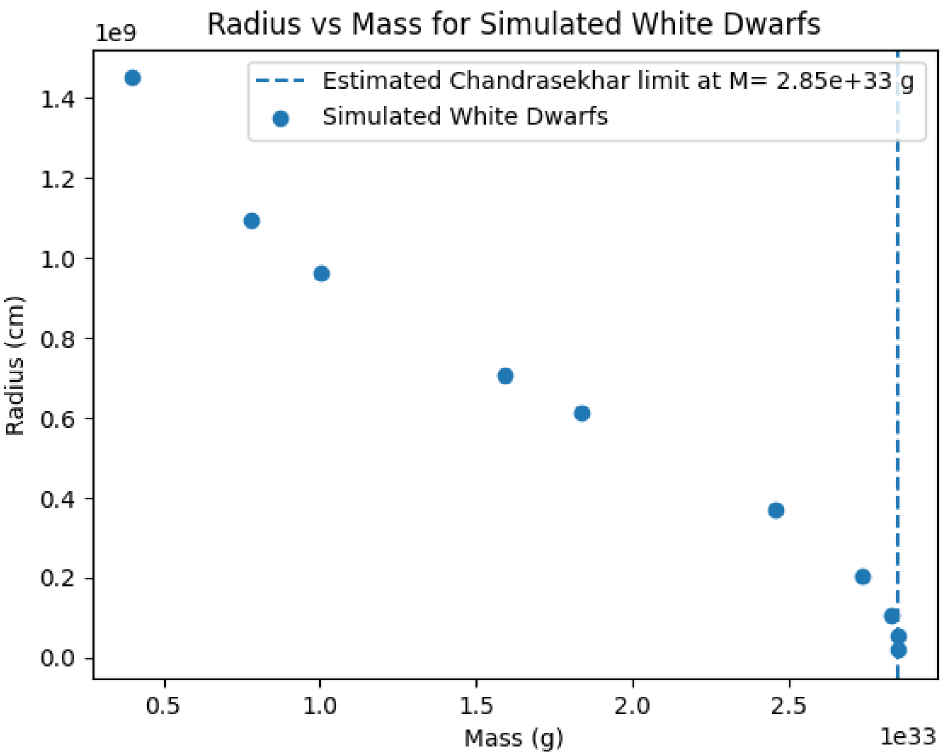

Figure 1: The simulated masses and radii for white dwarfs of varying center densities. As center density increases (and approaches 2.5×10^6) the mass of the stars approach $\sim 2.85 \times 10^{33}$, marked by the vertical line. This mass is the estimated Chandrasekhar limit.

2.2 Part 3

To compare the original method of numerical integration (RK5) with Runge-Kutta method of order 3 (RK3), the `solve_system` function was called using both 'RK45' and 'RK23' for three center densities (0.1, 100, and 2.5×10^6 cm^{-3}). The radii and masses calculated for each are displayed in table 1, as are the absolute and relative differences between the solutions of the two methods. While the absolute differences are significant, the relative differences show that the variation between the two methods is actually relatively small due to the scale of the radius and mass.

	Radius (cm)			**Mass (g)**		
Center Density (cm^{-3})	$\rho_c = 0.1$	$\rho_c = 100$	$\rho_c = 2.5 \times 10^6$	$\rho_c = 0.1$	$\rho_c = 100$	$\rho_c = 2.5 \times 10^6$
Calculated Value With RK45	1.451×10^9	3.682×10^8	1.909×10^7	3.966×10^{33}	2.459×10^{33}	2.850×10^{33}
Calculated Value With RK	1.448×10^9	3.672×10^8	1.901×10^7	3.973×10^{33}	2.461×10^{33}	2.859×10^{33}
Absolute Difference	2.87×10^6	1.02×10^6	7.80×10^4	7.34×10^{29}	1.97×10^{30}	8.91×10^{30}
Relative Difference	1.98×10^{-3}	2.76×10^{-3}	4.09×10^{-3}	1.85×10^{-3}	8.01×10^{-4}	3.13×10^{-3}

Table 1: Simulated radii and masses compared between the RK5 and RK3 methods of numerical integration.

2.3 Part 4

The white dwarf data from the *Gaia* satellite was imported into the program in order to be compared with the simulated results. The data was first converted from units of solar mass and radius to physical quantities by multiplying by `M_sun` and `R_sun` from the `astropy.constants` package. The `.to()` method of the `astropy.units` package was then used to convert from kg and m to g and cm. These radii were plotted against the masses, with the measured uncertainties used to produce error bars. These simulated data points (from part 2) were also plotted for comparison. The result is shown in figure 2.

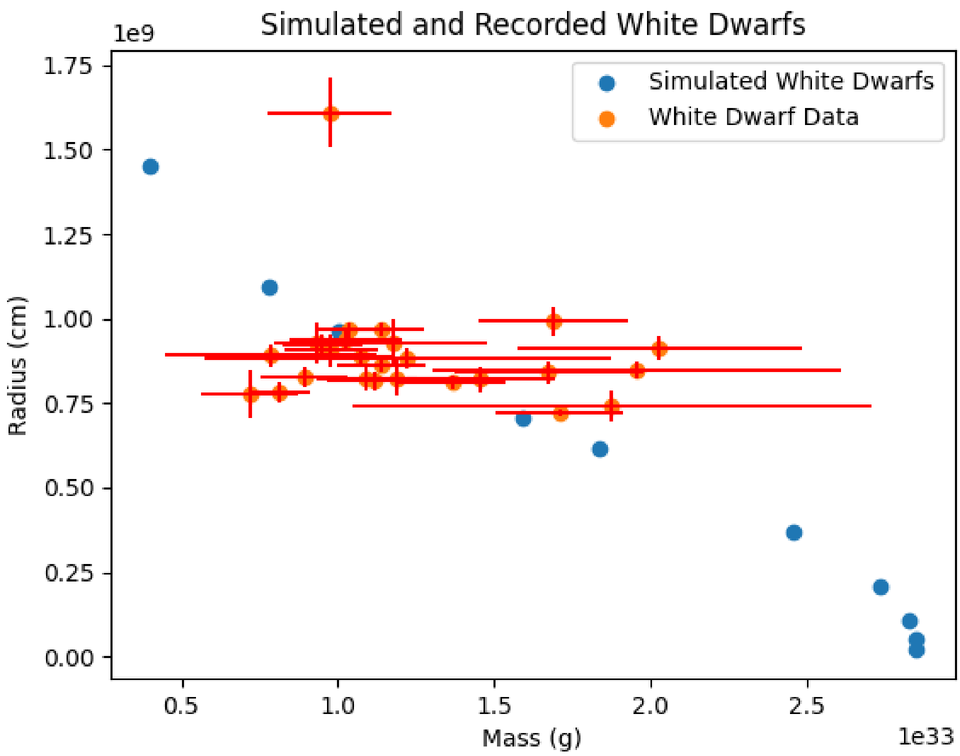

Figure 2: White dwarf radius vs mass for real data (orange) and simulation results (blue).

As is evident in figure 2, there is little correlation between the simulated and recorded white dwarf data. While the two results are on the same order of magnitude, the real data does not appear to follow the trend in the radius-mass relation predicted by the simulation. It is possible that the large uncertainties in mass (represented by the large error-bars in figure 2) contribute to the unexpected behaviour observed. It would be beneficial to compare data of white dwarfs over a greater range of masses to observe more clearly whether they follow the predicted trend.

3 Conclusion

The purpose of this simulation was to investigate the properties of white dwarf stars, specifically the relationship between mass and radius in conjunction with center density. It was found that as the center densities increased, so would the final mass of the star, resulting in a decreased radius. Furthermore, the simulated white dwarfs appeared to approach a limiting maximum mass of $M \approx 2.85 \times 10^{33}$ kg, which is consistent with the literature value for the Chandrasekhar limiting mass. These results were compared to recorded data, and there was not a clear correlation between the two. More data-points, particularly across a greater range of masses/radii, should be used to accurately investigate the radius-mass relation in real white dwarfs.