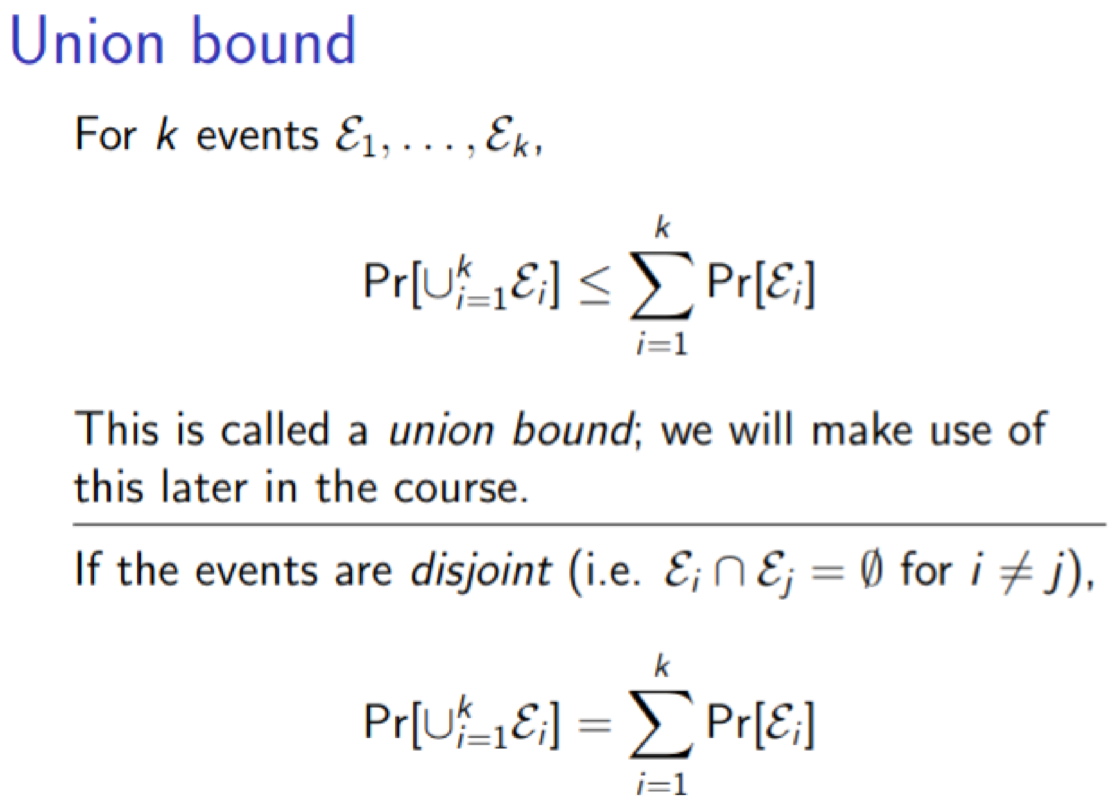
RAD (2-3) Moments and Deviation

Union bound



Union bound

For k events $\mathcal{E}_1, \ldots, \mathcal{E}_k$,

$$\Pr[\cup_{i=1}^{k} \mathcal{E}_i] \leq \sum_{i=1}^{k} \Pr[\mathcal{E}_i]$$

This is called a *union bound*; we will make use of this later in the course.

If the events are *disjoint* (i.e. $\mathcal{E}_i \cap \mathcal{E}_j = \emptyset$ for $i \neq j$),

$$\Pr[\cup_{i=1}^{k} \mathcal{E}_i] = \sum_{i=1}^{k} \Pr[\mathcal{E}_i]$$

Fremlæggelse

Occupancy problems

Proof

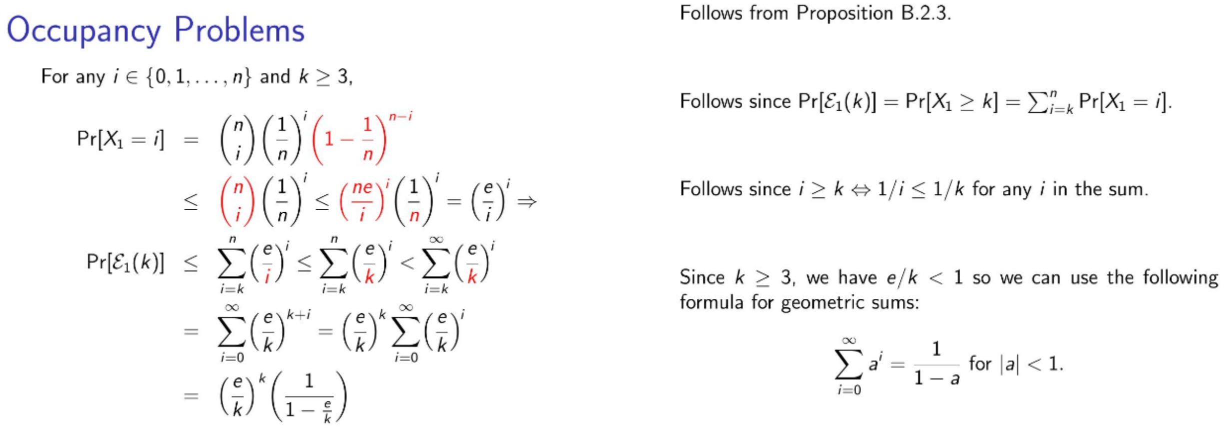
vi at med sandsyndlighed $1-\frac{1}{n}$, ingen spand får mere end $k^\star = \lceil 3\frac{\ln n}{\ln \ln n} \rceil$ bolde.

lad $m = n \geq 3$, og for $i = 1, \ldots, n$ lad X_i antallet af bolde i den i'te spand.

We study the following occupancy problem: find k så at, med høj sandsynlighed (sandsynlighed mindst $1-1/n$), ingen spand har mere end k bolde.

Definer et event:

Lad $\mathcal{E}_j(k)$ være eventet hvor spand j har mindst k bold (Med andre ord $X_j \geq k$). Vi kigger først på $\mathcal{E}_1(k)$, eventet hvor spand 1 har mindst k bolde. De andre spande er symmetriske



Occupancy Problems

For any $i \in \{0, 1, \ldots, n\}$ and $k \geq 3$,

$$\begin{aligned}
\Pr[X_1 = i] &= \binom{n}{i}\left(\frac{1}{n}\right)^i\left(1 - \frac{1}{n}\right)^{n-i} \\
&\leq \binom{n}{i}\left(\frac{1}{n}\right)^i \leq \left(\frac{ne}{i}\right)^i\left(\frac{1}{n}\right)^i = \left(\frac{e}{i}\right)^i \Rightarrow \\
\Pr[\mathcal{E}_1(k)] &\leq \sum_{i=k}^{n}\left(\frac{e}{i}\right)^i \leq \sum_{i=k}^{n}\left(\frac{e}{k}\right)^i < \sum_{i=k}^{\infty}\left(\frac{e}{k}\right)^i \\
&= \sum_{i=0}^{\infty}\left(\frac{e}{k}\right)^{k+i} = \left(\frac{e}{k}\right)^k\sum_{i=0}^{\infty}\left(\frac{e}{k}\right)^i \\
&= \left(\frac{e}{k}\right)^k\left(\frac{1}{1 - \frac{e}{k}}\right)
\end{aligned}$$

Follows from Proposition B.2.3.

Follows since $\Pr[\mathcal{E}_1(k)] = \Pr[X_1 \geq k] = \sum_{i=k}^{n} \Pr[X_1 = i]$.

Follows since $i \geq k \Leftrightarrow 1/i \leq 1/k$ for any i in the sum.

Since $k \geq 3$, we have $e/k < 1$ so we can use the following formula for geometric sums:

$$\sum_{i=0}^{\infty} a^i = \frac{1}{1-a} \text{ for } |a| < 1.$$

For et $i \in \{0, 1, \cdots, n\}$ og $k \geq 3$

sandsynligheden for at der er i bolde i spand 1:

Det her er binomial fordelingen. Det er fordi vi har gentagne bernoulli forsøg, enten er bolden i spanden eller ej. Her er $p = 1/n$ fordi at sandsynligheden for at en bold rammer i spand 1 er 1/n

Binomial koefficent: antallet af måder du kan vælge i bolde ud af de af n bolde.

$\left(\frac{1}{n}\right)^i$ er sandsynligheden for at vores i udvalgte bolde rammer i spanden

$\left(1 - \frac{1}{n}\right)^{n-i}$ sandsynligheden for at de resterende bolde ryger ned i andre spande.

$$\Pr\left[X_1 = i\right] = \binom{n}{i}\left(\frac{1}{n}\right)^i\left(1 - \frac{1}{n}\right)^{n-i}$$

Vi smider den sidste faktor væk og får en øvre grænse. Visig

$$\leq \binom{n}{i}\left(\frac{1}{n}\right)^i$$

Omskriver binomial koeficent pga propasition B. 2. 3. N'er går ud med hianden og vi får $(e/i)^i$

$$\leq \left(\frac{ne}{i}\right)^i \left(\frac{1}{n}\right)^i = \left(\frac{e}{i}\right)^i \Rightarrow$$

Sandsynligheden for at spand 1 får mindst k bolde er sandsynligheden for at spand 1 får k, k+1, k+2 og helt op til k+n bolde.

Øvre grænse på at spand 1 modtager mindst i bolde er $(e/i)^i$. så hvis vi summer fra k til n får vi en øvre grænse på at spand 1 får mindst k bolde.

$$\Pr\left[\mathcal{E}_1(k)\right] \leq \sum_{i=k}^{n} \left(\frac{e}{i}\right)^i$$

Nævner må altid være mindst k helt op til n, derfor når vi istedet bare vælger den til at være k får vi den mindste mulige nævner kan have. da alle led nu har en mindre værdi, resulterer det i at brøkken giver et større tal og derfor bliver summen større.

$$\leq \sum_{i=k}^{n} \left(\frac{e}{k}\right)^i$$

mange flere led, derfor er sum strengt større

$$< \sum_{i=k}^{\infty} \left(\frac{e}{k}\right)^i$$

Prøv at sæt k = 3 i begge udtryk, dertil bliver det klart at de er ens.

$$= \sum_{i=0}^{\infty} \left(\frac{e}{k}\right)^{k+i}$$

$(\frac{e}{k})^k$ er en konstant og kan trækkes ud.

$$= \left(\frac{e}{k}\right)^k \sum_{i=0}^{\infty} \left(\frac{e}{k}\right)^i$$

Summen er en geometrisk sum, og har udtrykket:

$$= \left(\frac{e}{k}\right)^k \left(\frac{1}{1-\frac{e}{k}}\right)$$

Letting $k^\star = \left\lceil 3\frac{\ln n}{\ln \ln n} \right\rceil$, it can be shown that
vi viser ikke denne ulighed

$$\left(\frac{e}{k^\star + 1}\right)^{k^\star+1} \left(\frac{1}{1-\frac{e}{k^\star+1}}\right) \leq n^{-2}$$

Combining with the previous slide, the probability that bin 1 receives more than $k^\star$ balls is the probability that it receives at least $k^\star + 1$ balls:

$$\Pr\left[\mathcal{E}_1\left(k^\star + 1\right)\right] \leq \left(\frac{e}{k^\star + 1}\right)^{k^\star+1} \left(\frac{1}{1-\frac{e}{k^\star+1}}\right) \leq n^{-2}$$

This obviously generalizes to any bin i pga symmetri:

$$\Pr\left[\mathcal{E}_i\left(k^\star + 1\right)\right] \leq n^{-2}$$

We have shown that for each bin i,

$$\Pr\left[\mathcal{E}_i\left(k^\star + 1\right)\right] \leq \frac{1}{n^2}$$

By a union bound (see slides from lecture 1),

evnetet siger en eller anden spand blandt de n spande der modtager skrapt mere end k* bolde. Til det bruger vi et union bound

$$\Pr\left[\cup_{i=1}^n \mathcal{E}_i\left(k^\star + 1\right)\right] \leq \sum_{i=1}^n \Pr\left[\mathcal{E}_i\left(k^\star + 1\right)\right] \leq \sum_{i=1}^n \frac{1}{n^2} = \frac{1}{n}$$

We have shown:
With probability at least $1 - \frac{1}{n}$, no bin receives more than

$k^\star = \left\lceil 3\frac{\ln n}{\ln\ln n} \right\rceil$ balls.

Markov's Inequality

Let Y be a random variable taking only non-negative values. Then for all $t > 0$:

$$Pr[Y \geq t] \leq \frac{E[Y]}{t}$$

Equivalently, for k > 0 and if $E[Y] > 0$:

$$Pr[Y \geq kE[Y]] \leq \frac{1}{k}$$

Kan bruges til at give bounds på sandsynligheder for algoritme køretider. Den er ret generel, men giver tit ikke gode bounds

Y kunne være køretid, så sandsynligheden for at den er langsom den er højst forventingen divideret med t. i køretids eksempel får vi en postiv forventing, derfor kan vi sige at sandsynligheden for at køretiden er større eller lig dens forventing gange k er 1/k.

proof

$$\mathbb{E}[Y] = \sum_y y \operatorname{Pr}[Y = y] \geq$$

we the sum a subset of y, therefore we take the sum of a smaller set, since y is positive and t > 0, we get a sum that is smaller. The values we didn't include in this sum couldn't be negative.

$$\sum_{y \geq t} y \operatorname{Pr}[Y = y] \geq$$

now we replace y with t in the sum, we can do this since we know that all the values of y in the previous sum must be bigger than t. We sum a bunch of factors that all are smaller than the y factors in the previous sum.

$$\sum_{y \geq t} t \Pr[Y = y] =$$

Since t is not dependent on the values we sum we can pull it out of the sum.

$$t \sum_{y \geq t} \Pr[Y = y] = t \Pr[Y \geq t]$$

This sum is by definition the probability that y is bigger or equal to t.

Now we can divide by t on both sides in:

$$\mathbb{E}[Y] \geq \sum_{y \geq t} t \Pr[Y = y] = t \Pr[Y \geq t]$$

Thus, $\Pr[Y \geq t] \leq \frac{\mathbb{E}[Y]}{t}$.

Anden ulighed

For the second part, set $t = k\mathbb{E}[Y] > 0$ and apply the first part:

$$\Pr[Y \geq k\mathbb{E}[Y]] = \Pr[Y \geq t] \leq \frac{\mathbb{E}[Y]}{t} = \frac{\mathbb{E}[Y]}{k\mathbb{E}[Y]} = \frac{1}{k}$$

Chebyshev's Inequality

Given a random variable X with expectation $\mathbb{E}[X] = \mu_X$, define its variance ($\mathrm{Var}[X]$ or σ_X^2) as $\sigma_X^2 := \mathbb{E}\left[(X - \mu_X)^2\right]$, and its standard deviation as $\sigma_X := \sqrt{\mathbb{E}\left[(X - \mu_X)^2\right]}$.

Theorem

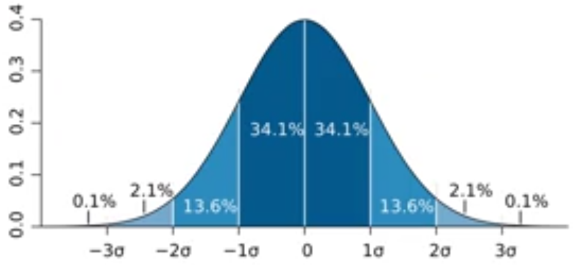
Let X be a random variable with expectation μ_X and standard deviation $\sigma_X > 0$. Then for all $t > 0$:

$$\Pr\left[|X - \mu_X| \geq t\sigma_X\right] \leq \frac{1}{t^2}$$

Sandsynligheden for X's afvigelse (forskellen) fra dens expectation er mindst t standard afvigelser væk fra expectation, er mindre eller lig $\frac{1}{t^2}$.

Se en normalfordeling:

Illustration of the standard deviation for a normal distribution with expectation 0 (from wikipedia):



dens expectation er 0 som ses da toppen ligger i 0. langs x-aksen er der t standard afvigelser, sandsynligheden for at vi er mindst 1 standafvigelse fra expectation t=1 (husk det er numerisk), er $2 \cdot 13.6 + 2 \cdot 2.1 + 2 \cdot 0.1 = 31.6$. det er sandsyndlighederne ude fra $|\sigma|$.

Her siger Chebyshev's ikke noget brugbart, eftersom vi vælger t til at være 1. Dvs. at sandsynligheden er 100%. sandsynligheden for X's afvigelse (forskellen) fra dens expectation er mindst 1 standard afvigelser væk fra expectation er mindre eller lig 100% (duh)

Proof

Here we use that $k > 0$ and $\mathbb{E}[Y] = \sigma_X^2 > 0$ so that we can use the second version of Markov's inequality.

Let $k = t^2$ and $Y = (X - \mu_X)^2$.

Then $\sigma_X^2 = \mathbb{E}[Y]$

Since numerical signs are used we can take the power of 2 through the whole inequality and it's the same. This makes it positive.

$$\begin{aligned} \Pr\left[|X - \mu_X| \geq t\sigma_X\right] &= \Pr\left[(X - \mu_X)^2 \geq t^2\sigma_X^2\right] \\ &= \Pr[Y \geq k\mathbb{E}[Y]] \\ &\leq \frac{1}{k} \\ &= \frac{1}{t^2} \end{aligned}$$

Theorem

Let X be a random variable with expectation μ_X and standard deviation $\sigma_X > 0$. Then for all $t > 0$:

$$\Pr\left[|X - \mu_X| \geq t\sigma_X\right] \leq \frac{1}{t^2}$$

Proof

Letting $Y = (X - \mu_X)^2$,

$$\begin{aligned}\Pr\left[|X - \mu_X| \geq t\right] &= \Pr\left[(X - \mu_X)^2 \geq t^2\right] \\ &= \Pr\left[Y \geq t^2\right] \\ &\leq \frac{\mathbb{E}[Y]}{t^2} = \frac{\sigma_X^2}{t^2}\end{aligned}$$

where the inequality follows from the first version of Markov's inequality.

Theorem

Let X be a random variable with expectation μ_X and standard deviation $\sigma_X > 0$. Then for all $t > 0$:

$$\Pr\left[|X - \mu_X| \geq t\right] \leq \frac{\sigma_X^2}{t^2} = \frac{\mathrm{Var}[X]}{t^2}$$

Ekstra

Balls and bins

Let $m = n \geq 3$, and for $i = 1, \ldots, n$ let X_i be the number of balls in the ith bin.

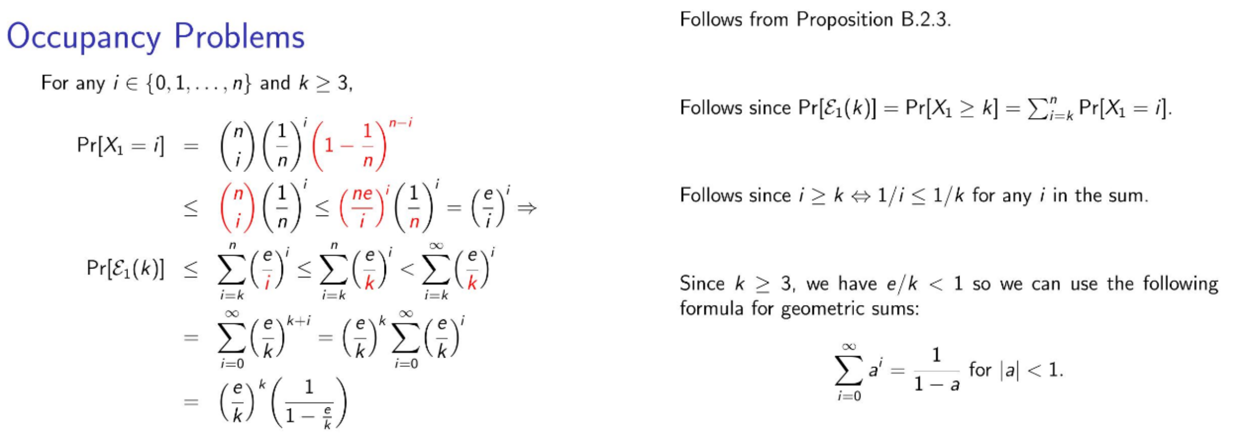
We study the following occupancy problem: find k such that, with high probability (probability at least $1 - 1/n$), no bin contains more than k

balls.

Let $\mathcal{E}_j(k)$ be the event that bin j contains at least k balls $(X_j \geq k)$. First consider $\mathcal{E}_1(k)$.

For any $i \in \{0, 1, \ldots, n\}$ and $k \geq 3$,

$$\begin{aligned}
\Pr[X_1 = i] &= \binom{n}{i}\left(\frac{1}{n}\right)^i\left(1-\frac{1}{n}\right)^{n-i} \\
&\leq \binom{n}{i}\left(\frac{1}{n}\right)^i \leq \left(\frac{ne}{i}\right)^i\left(\frac{1}{n}\right)^i = \left(\frac{e}{i}\right)^i \Rightarrow \\
\Pr[\mathcal{E}_1(k)] &\leq \sum_{i=k}^{n}\left(\frac{e}{i}\right)^i \leq \sum_{i=k}^{n}\left(\frac{e}{k}\right)^i < \sum_{i=k}^{\infty}\left(\frac{e}{k}\right)^i \\
&= \sum_{i=0}^{\infty}\left(\frac{e}{k}\right)^{k+i} = \left(\frac{e}{k}\right)^k \sum_{i=0}^{\infty}\left(\frac{e}{k}\right)^i \\
&= \left(\frac{e}{k}\right)^k\left(\frac{1}{1-\frac{e}{k}}\right)
\end{aligned}$$



Occupancy Problems

For any $i \in \{0, 1, \ldots, n\}$ and $k \geq 3$,

$$\begin{aligned}
\Pr[X_1 = i] &= \binom{n}{i}\left(\frac{1}{n}\right)^i\left(1-\frac{1}{n}\right)^{n-i} \\
&\leq \binom{n}{i}\left(\frac{1}{n}\right)^i \leq \left(\frac{ne}{i}\right)^i\left(\frac{1}{n}\right)^i = \left(\frac{e}{i}\right)^i \Rightarrow \\
\Pr[\mathcal{E}_1(k)] &\leq \sum_{i=k}^{n}\left(\frac{e}{i}\right)^i \leq \sum_{i=k}^{n}\left(\frac{e}{k}\right)^i < \sum_{i=k}^{\infty}\left(\frac{e}{k}\right)^i \\
&= \sum_{i=0}^{\infty}\left(\frac{e}{k}\right)^{k+i} = \left(\frac{e}{k}\right)^k \sum_{i=0}^{\infty}\left(\frac{e}{k}\right)^i \\
&= \left(\frac{e}{k}\right)^k\left(\frac{1}{1-\frac{e}{k}}\right)
\end{aligned}$$

Follows from Proposition B.2.3.

Follows since $\Pr[\mathcal{E}_1(k)] = \Pr[X_1 \geq k] = \sum_{i=k}^{n} \Pr[X_1 = i]$.

Follows since $i \geq k \Leftrightarrow 1/i \leq 1/k$ for any i in the sum.

Since $k \geq 3$, we have $e/k < 1$ so we can use the following formula for geometric sums:

$$\sum_{i=0}^{\infty} a^i = \frac{1}{1-a} \text{ for } |a| < 1.$$

Letting $k^\star = \lceil 3\frac{\ln n}{\ln\ln n} \rceil$, it can be shown that

$$\left(\frac{e}{k^\star+1}\right)^{k^\star+1}\left(\frac{1}{1-\frac{e}{k^\star+1}}\right) \leq n^{-2}$$

Combining with the previous slide, the probability that bin 1 receives more than $k^\star$ balls is the probability that it receives at least $k^\star + 1$ balls:

$$\Pr\left[\mathcal{E}_1\left(k^\star+1\right)\right] \leq \left(\frac{e}{k^\star+1}\right)^{k^\star+1}\left(\frac{1}{1-\frac{e}{k^\star+1}}\right) \leq n^{-2}$$

This obviously generalizes to any bin i :

$$\Pr\left[\mathcal{E}_i\left(k^\star+1\right)\right] \leq n^{-2}$$

We have shown that for each bin i,

$$\Pr\left[\mathcal{E}_i\left(k^\star+1\right)\right] \leq \frac{1}{n^2}$$

By a union bound (see slides from lecture 1),

$$\Pr\left[\cup_{i=1}^{n}\mathcal{E}_i\left(k^\star+1\right)\right] \leq \sum_{i=1}^{n}\Pr\left[\mathcal{E}_i\left(k^\star+1\right)\right] \leq \sum_{i=1}^{n}\frac{1}{n^2} = \frac{1}{n}$$

We have shown:
With probability at least $1-\frac{1}{n}$, no bin receives more than
$k^\star = \left\lceil 3\frac{\ln n}{\ln\ln n}\right\rceil$ balls.

Birthday problem

Suppose m balls are randomly assigned to n bins. What is the probability that all balls land in distinct bins?

For $n = 365$ a related question is "how large must a group of people be before it is likely that two people have the same birthday"?

Let $\mathcal{E}_i$ be the event that the ith ball lands in an empty bin. From the first lecture, we have:

$$\Pr\left[\cap_{i=1}^{m}\mathcal{E}_i\right] = \prod_{i=2}^{m}\Pr\left[\mathcal{E}_i \mid \cap_{j=1}^{i-1}\mathcal{E}_j\right] = \prod_{i=2}^{m}\left(1-\frac{i-1}{n}\right)$$

$$\leq \prod_{i=2}^{m} e^{-\frac{i-1}{n}} = e^{\left(\sum_{i=2}^{m}-\frac{i-1}{n}\right)} = e^{-\frac{1}{n}\sum_{i=2}^{m}(i-1)} = e^{-\frac{m(m-1)}{2}} < e^{-\frac{(m-1)^2}{2n}}.$$

For $m \geq \lceil \sqrt{2n} + 1 \rceil$, the probability that all m balls end in distinct bins is less than $1/e$.

$$1 + x \leq e^x \forall x \in \mathbb{R} \text{ (proposition B.3.1)}$$

$$\sum_{j=1}^{m-1} = \frac{m(m-1)}{2}$$

For $n = 365$, this is $m \geq 29$
Recall that m is the number of balls and n is the number of bins. With $m = 29$ people, the probability that at least two of them have the same birthday is at least $1 - 1$ /e which is roughly 63%.

Independent $/k$-independent events

Events $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n$ are (mutually) independent if for every subset $I \subseteq [1, n]$,

$$\Pr\left[\cap_{i \in I} \mathcal{E}_i\right] = \prod_{i \in I} \Pr\left[\mathcal{E}_i\right]$$

These events are k-independent if for every subset $I \subseteq [1, n]$ of size at most k, If $k = 2$, the events are said to be pairwise independent.

Equivalently, $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n$ are (mutually) independent if for every $j \in [1, n]$ and every subset $I \subseteq [1, n] \backslash \{j\}$,

$$\Pr\left[\mathcal{E}_j \mid \cap_{i \in 1} \mathcal{E}_i\right] = \Pr\left[\mathcal{E}_j\right]$$

provided that none of the intersections are empty

Independent variables

A set of random variables $X_1, X_2, \ldots, X_n$ is (mutually) independent if for every subset $I \subseteq [1, n]$ and for any set of real values $\{x_i\}_{i \in 1}$,

$$\Pr\left[\cap_{i \in I} X_i = x_i\right] = \prod_{i \in I} \Pr\left[X_i = x_i\right]$$

This is similar to the definition on the previous examples for events $\{X_i = x_i\}$ but has to hold for all choices of the values $\{x_i\}_{i \in 1}$

k-independent variables

A set of random variables $X_1, X_2, \ldots, X_n$ is k-independent if for any subset $I \subseteq [1, n]$ with $|I| \leq k$ and for any set of real values $\{x_i\}_{i \in l}$,

$$\Pr\left[\cap_{i \in I} X_i = x_i\right] = \prod_{i \in I} \Pr\left[X_i = x_i\right]$$

If $k = 2$, the random variables are said to be pairwise independent: for every distinct pair of indices (i, j) and any values a, b,

$$\Pr\left[X_i = a \wedge X_j = b\right] = \Pr\left[X_i = a\right] \cdot \Pr\left[X_j = b\right]$$

linearity of variance

Let $X_1, X_2, \ldots, X_m$ be pairwise independent random variables, and $X = \sum_{i=1}^m X_i$. We will show that the linearity of variance for independent variables is also true for pairwise independent variables. As follows from the course book the variance of X is given by

$$E[(X - \mu)^2] = E[(\sum_{i=1}^m X_i - \mu_i)^2]$$

where $\mu_i = E[X_i]$ and $\mu = \sum_{i=1}^m \mu_i$. Using linearity of expectation the expression can be expanded

$$E[(X - \mu)^2] = \sum_{i=1}^m E[(X_i - \mu_i)^2] + 2 \sum_{i<j} E[(X_i - \mu_i)(X_j - \mu_j)].$$

Since all pairs X_i, X_j are pairwise independent, so are the pairs $(X_i - \mu_i), (X_j - \mu_j)$ and by (3.2), the expectation of the product can be replaced by the product of the expectations. Since

$E[(X_i - \mu_i)] = E[X_i] - \mu_i = 0$, the latter summations vanishes and it follows that

$$E[(X - \mu)^2] = \sum_{i=1}^{m} E[(X_i - \mu_i)^2] = \sum_{i=1}^{m} \sigma_{X_i}^2.$$

Two-Point sampling

truly random numbers are hard to obtain. Two-point sampling is a way to take just two random independent values and turn them into many pairwise independent values.

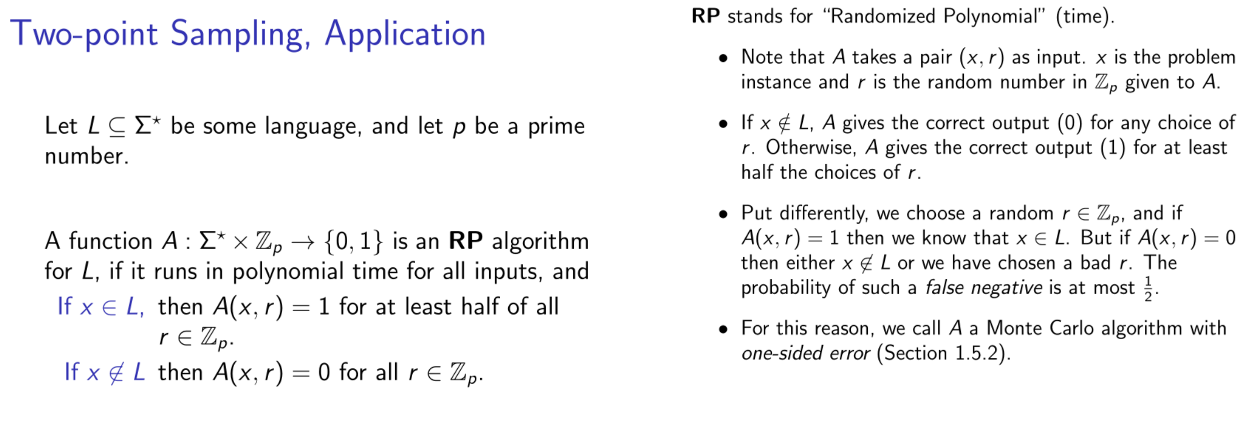
Let p be prime, and let a, b be independent random variables uniformly chosen from

$\mathbb{Z}_p = \{0, \cdots, p-1\}$.
For $i = 0, 1, \cdot, p - 1$, let

$$r_i = (a \cdot i + b) \; mod \; p$$

Then for any $i \neq j$ (mod p), r_i and r_j are independent and uniform in $\mathbb{Z}_p$

Thus, $r_0, r_1, \cdots, r_p - 1$ are pairwise independent.



Two-point Sampling, Application

Let $L \subseteq \Sigma^\star$ be some language, and let p be a prime number.

A function $A : \Sigma^\star \times \mathbb{Z}_p \to \{0, 1\}$ is an **RP** algorithm for L, if it runs in polynomial time for all inputs, and

If $x \in L$, then $A(x, r) = 1$ for at least half of all $r \in \mathbb{Z}_p$.

If $x \notin L$ then $A(x, r) = 0$ for all $r \in \mathbb{Z}_p$.

RP stands for "Randomized Polynomial" (time).

- Note that A takes a pair (x, r) as input. x is the problem instance and r is the random number in $\mathbb{Z}_p$ given to A.
- If $x \notin L$, A gives the correct output (0) for any choice of r. Otherwise, A gives the correct output (1) for at least half the choices of r.
- Put differently, we choose a random $r \in \mathbb{Z}_p$, and if $A(x, r) = 1$ then we know that $x \in L$. But if $A(x, r) = 0$ then either $x \notin L$ or we have chosen a bad r. The probability of such a *false negative* is at most $\frac{1}{2}$.
- For this reason, we call A a Monte Carlo algorithm with *one-sided error* (Section 1.5.2).

Antag at algoritme A bruger $\lg n$ tilfældige bits repræsenteret som et tal $r \in \{0, \ldots, n-1\}$ hvor n er et primtal. I følgende bruger vi notationen $A(x, r)$ for at beskrive outputtet af A på input x, hvor A

vælger den tilfældige bitstreng r. Og lad os i fejlsandsynlighederne antage, at vores konkrete $x \in L$ så det korrekte svar er 1.

Algoritme 1 - $t \lg n$ random bits

Vælg t tal $r_0, \ldots, r_{t-1} \in [n]$ uafhængigt og uniformt tilfældigt.
Beregn $A(x, r_0), \ldots, A(x, r_{t-1})$. Hvis vi en enkelt gang ser tallet 1 er det bevis på $x \in L$, ellers hvis vi \emph{alle} gange får 0 vælger vi det som output.

Så vil fejlsandsynligheden være $< \frac{1}{2}^t = 1/2^t$.

Problemet ved denne tilgang er, at vi skal vælge $t \lg n$ random bits.
Hvis vi f.eks. vælger $t = 2$ skal vi bruge $2 \ln n$ random bits for en fejlsandsynlighed $< 1/4$.

Algoritme 2 - $2 \lg n$ random bits}

Vælg $a, b \in [n]$ uafhængigt og uniformt tilfældigt.

Da vi antager n er et primtal, så ved vi at såfremt vi lades $r_i = (a * i + b) \mod n$, så vil r_i og r_j hvor $i \neq j$ være uniformt distribueret i $[n]$ og parvist uafhængige (kan blot antages, skal ikke bevises).

Igen beregner vi $A(x, r_0), \ldots, A(x, r_{t-1})$ og vælger 1 såfremt den optræder bare én gang, ellers 0.Algoritme 2 - $2 \lg n$ random bits}

Nu bruger vi kun $2 \lg n$ random bits.

Sandsynlighed for at algoritme 2 fejler

For $i = 0, \ldots, t-1$ lader vi $Y_i = A(x, r_i)$. Lad nu $Y = \sum_{i \in [t]} Y_i$.

Da kan vi beregne den forventede værdi:

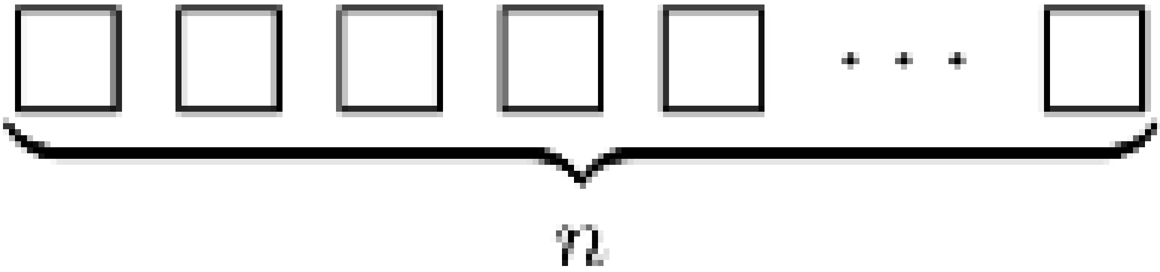
$$\mathbb{E}[Y] = \sum_{i \in [t]} \mathbb{E}[Y_i] = tp \geq \frac{t}{2}$$

Idet vi lader symbolet $p = \mathbb{P}[Y_i = 1] \geq \frac{1}{2}$.

sandsynligheden for den fejler $1/t$

Coupon collector

Betragt følgende eksperiment. Vi har n unikke unikke kupontyper:



I hver runde vælges en kupon-type uafhængigt og uniformt tilfældigt. Vi stopper når alle kupon-typer er valgt. Hvor mange runder vil der være i dette eksperiment?\

For at besvare dette skal vi først definere hvad en epoke er. For $i = 0, \ldots, n-1$ består den i'te epoke af de runder, der starter lige efter den i'te succes og slutter i runden med $(i+1)$'te succes, hvor en succes er defineret som at vælge en kupontype vi ikke har set før. Eksempelvis kunne vi have:

$$\underbrace{C_2,}_{\text{Epoke } 0} \underbrace{C_2, C_1,}_{\text{Epoke } 1} \underbrace{C_2, C_2, C_3,}_{\text{Epoke } 2} \ldots$$

For $i = 0, \ldots, n-1$ lader vi Y_i være længden af epoke i. Lad nu $Y = \sum_{i=0}^{n-1} Y_i$. Vi har, at sandsynligheden i den i'te epoke for at finde en ny kupon er antallet af ufundne kuponer $n - i$ over alle de forskellige kupontyper n:

$$p_i = \frac{n-i}{n}$$

Bruger vi, at dette er geometrisk distribueret får vi:

$$\mathbb{E}[Y_i] = \frac{1}{p_i} = \frac{n}{n-i}$$

Da kan vi beregne:

$$\mu_Y = \sum_{i=0}^{n-1} \mathbb{E}[Y_i] = \sum_{i=0}^{n-1} \frac{n}{n-i} = n \sum_{i=1}^{n} \frac{1}{i} = nH_n = n \ln n + \Theta(n) = O(n \ln n)$$