3.5 1D Poisson Equation

Solve

$$\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} + e^{-x^2} = 0 \tag{3.9}$$

on $[-5, 5]$ with boundary conditions $f(-5) = 1$ and $f'(5) = 0$ using a second order finite difference scheme with $\Delta x = 0.01$.

Plot the curve and check visually that the boundary conditions are satisfied and that the curvature of the curve is highest at $x = 0$ (which is what the ODE specifies).

3.6 Diffusion–Advection

The one-dimensional Diffusion-Advection equation is

$$D\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} - \frac{\mathrm{d}}{\mathrm{d}x}\left(v(x) f(x)\right) = 0, \tag{3.10}$$

where $v(x)$ is some velocity field that *advects* particles around. The first term is the diffusion term that tends to spread particles.

(a) Use a second order finite difference scheme to solve the diffusion-advection with $D = 2$ and $v(x) = -\sin x$ on $[0, 25]$ with boundary conditions $f(0) = 1$ and $f(25) = 0$. Use $N = 1000$ grid points.

It is always a good idea to 'sanity check' the result of a numerical solution.

(b) Explain the shape of $f(x)$. Does it make sense compared to the physical interpretation of the diffusion-advection equation?

(c) Explain what happens for $D \to 0$ and $D \to \infty$. Plot for instance $D = 0.5$ and $D = 15$ and compare to $D = 2$. Would your code work for $D = 0$?

3.7 Beam Equation

The bending $y(x)$ of an elastic rod subject to a load $w(x)$ along its length is described by the Euler–Bernoulli beam equation:

$$y''''(x) = w(x). \tag{3.11}$$

We will consider this ODE on $x \in [0, 1]$. We fix the left end of the rod to a wall giving the boundary conditions $y(0) = 0$ and $y'(0) = 0$. The other end hangs free giving the boundary conditions $y''(1) = 0$ and $y'''(1) = 0$.

(a) Solve the ODE with $w(x) = -1.0$ using finite differences over $N = 100$ grid points and plot the result.

We now move all the load to a single point along the rod:

(b) Solve the ODE with $w(x) = -\delta(x - x_0)$ for $x_0 = 0.2$, $x_0 = 0.5$ and $x_0 = 0.9$, and plot the results.
Note: On a finite grid the Dirac delta function becomes e.g. $(0, 0, \cdots, 0, 0, 1/\Delta x, 0, 0, \cdots, 0, 0)$.

(c) Do the curves that you obtain make sense physically?