# Theoretical Hardness of Algebraically Structured Learning With Errors

PhD Defense

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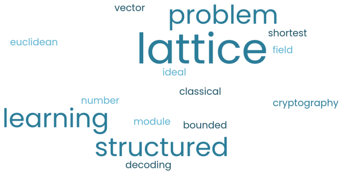
16th November 2021





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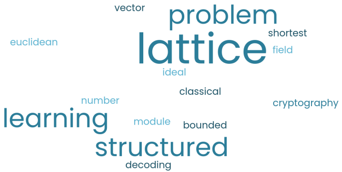



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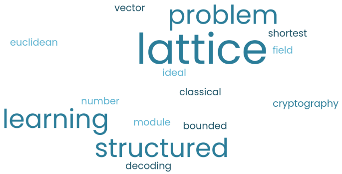
- Factoring





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⚠ $\exists$ poly-time quantum algorithm [Sho97].

Sources for assumedly quantum-resistant problems:

- Euclidean Lattices
- Codes
- Isogenies
- Multivariate Systems
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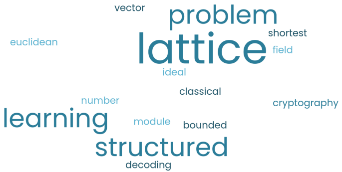



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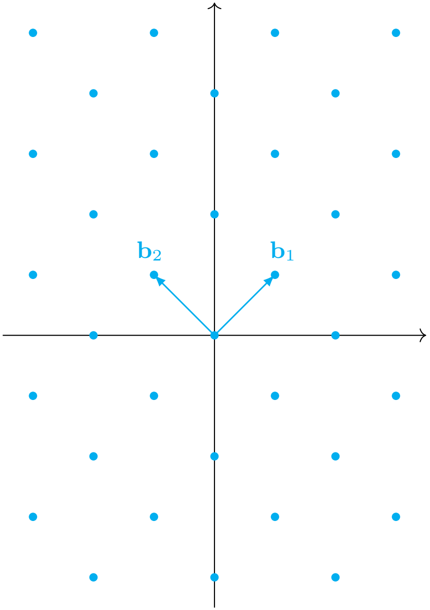
my research

# Hard Lattice Problems

An **Euclidean lattice** $\Lambda$ of rank $n$ with a basis $\mathbf{B} = (\mathbf{b}_j)_{1 \le j \le n}$ is given by

$$\Lambda(\mathbf{B}) = \left\{ \sum_{j=1}^{n} z_j \mathbf{b}_j : z_j \in \mathbb{Z} \right\}.$$



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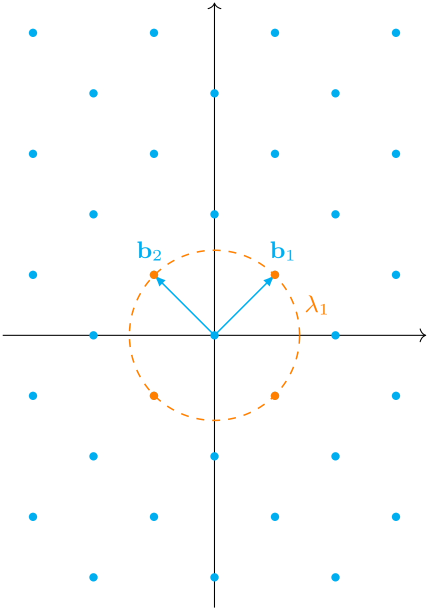
The **minimum** of $\Lambda$ is

$$\lambda_1(\Lambda) := \min_{\mathbf{v} \in \Lambda \setminus \{\mathbf{0}\}} \|\mathbf{v}\|.$$

The **shortest vector problem** (SVP) asks to find a vector $\mathbf{w}$ such that $\|\mathbf{w}\| = \lambda_1(\Lambda)$.



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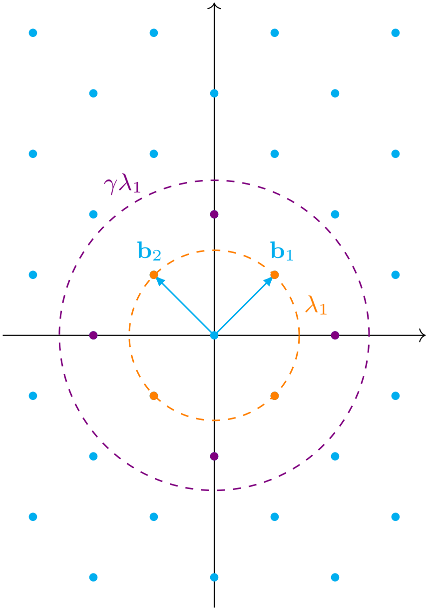

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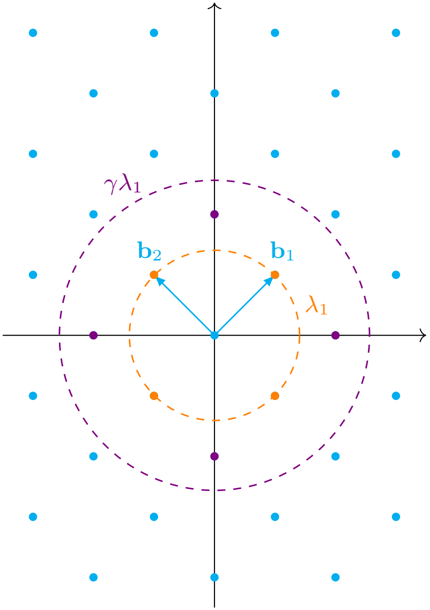

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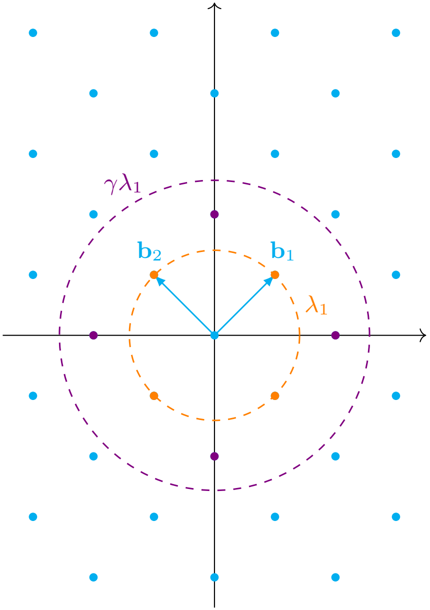
⚠ Hard to build cryptography on top of $\mathsf{SVP}_\gamma$.



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Idea: use intermediate problems!

(Main) Mathematical Problems:

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- NTRU [HPS98]
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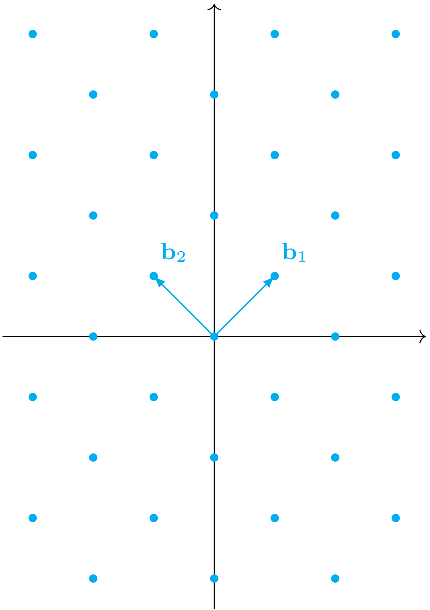

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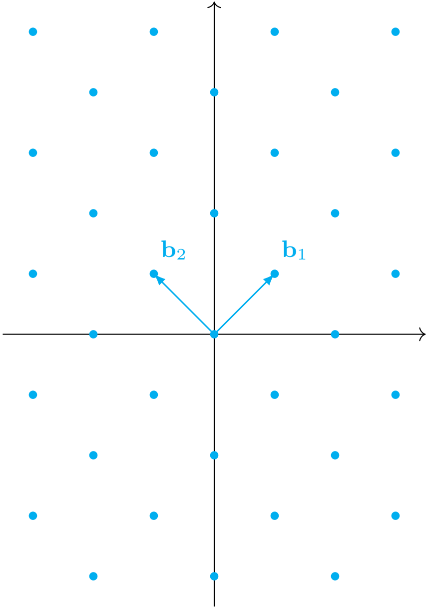

# Lattice-Based Cryptography

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(Main) Mathematical Problems:

- Short Integer Solution [Ajt96]
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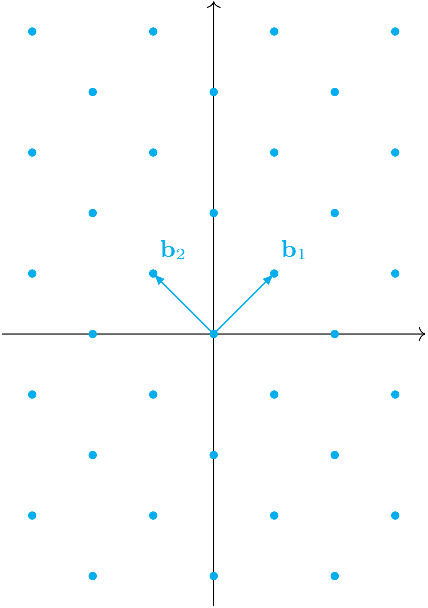
- Learning With Errors [Reg05]
  - Strong security guarantees
    At least as hard as variants of $\mathrm{SVP}_\gamma$ for any Euclidean lattice
  - Efficiency
    Linear algebra & parallelizable
  - Many cryptographic applications
    Fully Homomorphic Encryption, E-Voting, Zero-Knowledge Proofs, ...



# NIST Competition

Started in 2016: NIST project to define new standards for post-quantum cryptography.

A majority (5 out of 7) of the finalist candidates are based on **lattice problems**.

Several among them (3 out of 5) are based on (variants of) **Learning With Errors**.

Public Key Encryption

- Kyber: (module variant of) Learning With Errors
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Digital Signature

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**Observation**

*Lattice-based cryptography, and in particular Learning With Errors, plays a* ***key role*** *in designing post-quantum cryptography.*

# Outline

# Learning With Errors

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| ⚠ Storage | $m(d+1)\log_2 q$ bits | $\sim \tilde{O}(\lambda^2)$ |
| ⚠ Computation | $O(md)$ | $\sim O(\lambda^2)$ |

$\lambda$ security parameter

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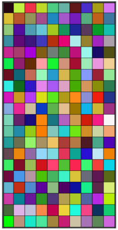

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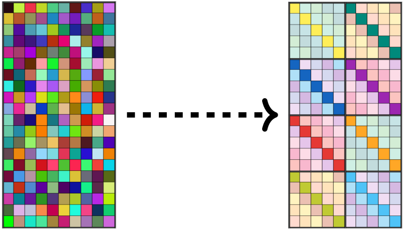

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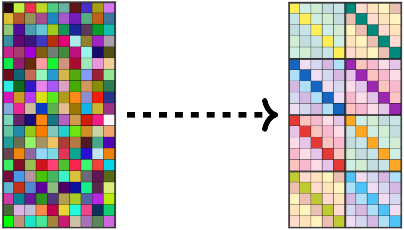

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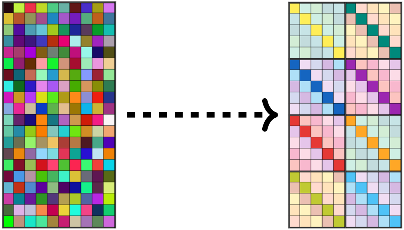

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my research

# My Contributions

**Asiacrypt'20**
**CT-RSA'21**

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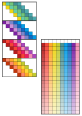
1. Module Learning With Errors with a binary secret
2. Classical hardness of Module Learning With Errors

$$\left(A\ ,\ A\,s + e = b\right)$$



II. Proposing new structured variants

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4. Partial Vandermonde Learning With Errors

**Asiacrypt'19**
**& under submission**

III. Building $\overbrace{\text{public}}^{\text{provable secure}}$ key encryption

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6. PASS Encrypt, related to (4.)

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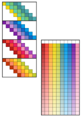

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# Hardness of Module Learning With Errors

Joint work with C. Jeudy, A. Roux-Langlois and W. Wen

# Ring of Integers over a Number Field

Idea: replace $\boxed{\mathbb{Z}}$ by the ring of integers $\boxed{R}$ of some number field $K$ of degree $n$.

Think of $R = \mathbb{Z}[x]/\langle x^n + 1\rangle$ and $K = \mathbb{Q}[x]/\langle x^n + 1\rangle$ with $n = 2^\ell$.

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Note: defines matrix-vector multiplication over $\mathbb{Z}$, denoted by $\mathrm{Rot}(a) \cdot s$ .

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Example: $n = 4$ thus $R = \mathbb{Z}[x]/\langle x^4 + 1\rangle$

$$m = 4 \left\{ \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix}}_{d = 2} \right. \cdot \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \in R$$

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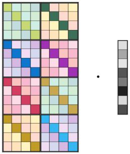

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# Module Learning With Errors (Module-LWE, M-LWE) [BGV12, LS15]

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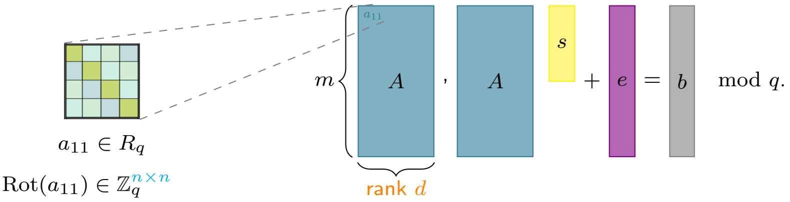

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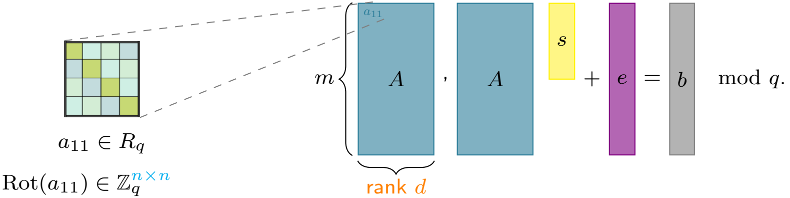


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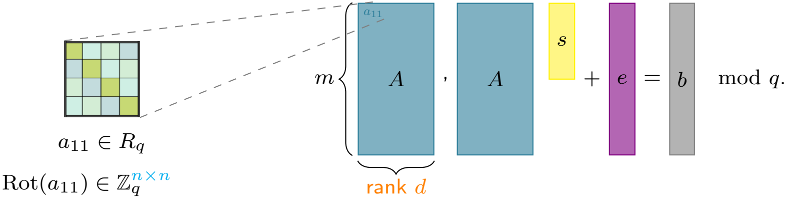


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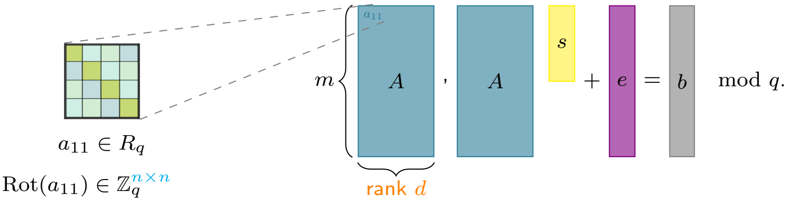


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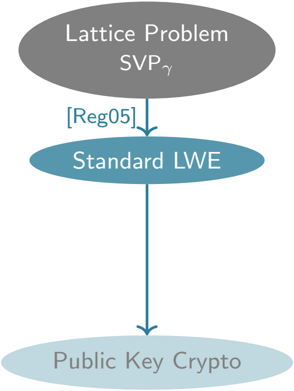
For $d = 1$, we call this Ring-LWE [SSTX09, LPR10].

# Motivation: Theory



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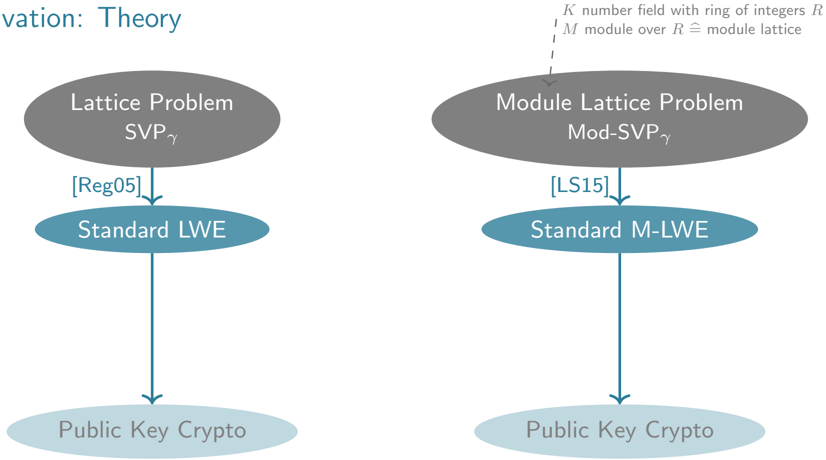
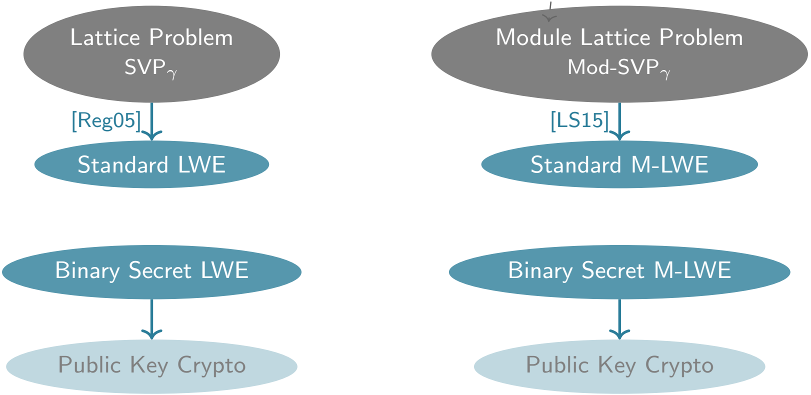


# Motivation: Theory vs. Praxis

$K$ number field with ring of integers $R$
$M$ module over $R \mathrel{\widehat{=}}$ module lattice




- Efficiency
- Functionality (e.g., Fully Homomorphic Encryption)
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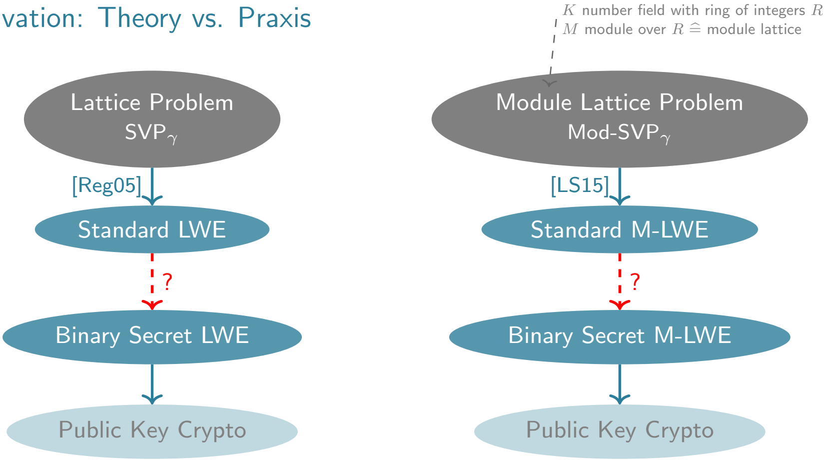


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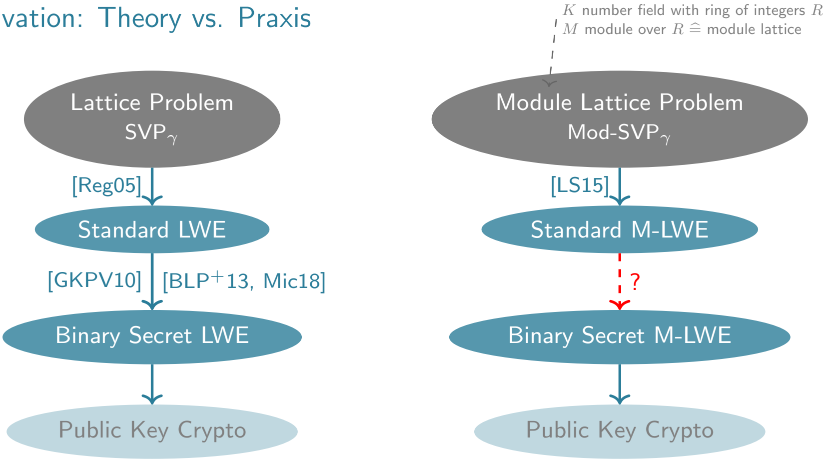


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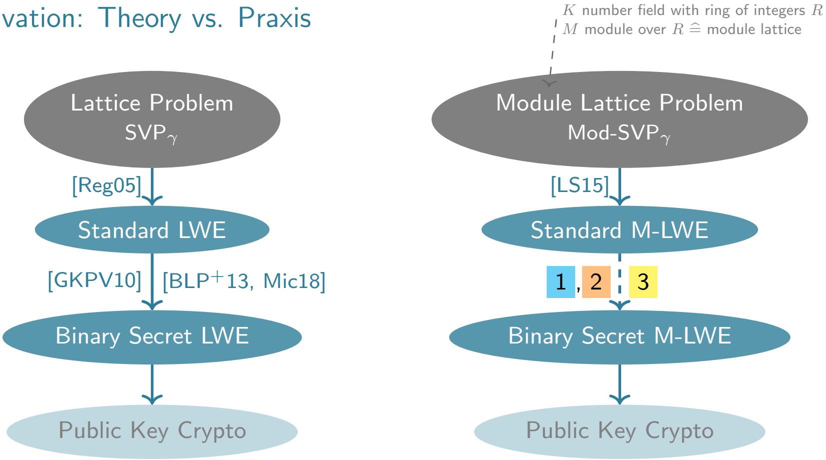


Contributions:

1 Extending and Improving [GKPV10] to M-LWE — [BJRW20] Asiacrypt'20

2 Extending [BLP+13] to M-LWE — [BJRW21] CT-RSA'21

3 Generalizing both proofs to bounded secrets — [Bou21] PhD Thesis

# Hardness of Module-LWE with Binary Secrets (Cyclotomics)

| Standard M-LWE | $\rightarrow$ | Binary Secret M-LWE |
|---|---|---|
| modulus $q$ | | modulus $q$ |
| ring degree $n$ | | ring degree $n$ |
| secret $\mathbf{s}' \bmod q$ | | secret $\mathbf{s} \bmod 2$ |
| Gaussian width $\alpha$ | | Gaussian width $\beta$ |
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| LWE analogue | [GKPV10] using RD* | [BLP+13] |
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⚠ $\gg$ **practice**

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# Proof of Hardness of Module-LWE with Binary Secrets

The secret $\mathbf{s} \in R_2^d$ is binary and the secret $\mathbf{s}' \in R_q^k$ is modulo $q$.

$$m\left\{\underbrace{\mathbf{A}}_{d}\right., \quad \mathbf{A}\mathbf{s} + \mathbf{e}$$

**M-LWE with binary secret**

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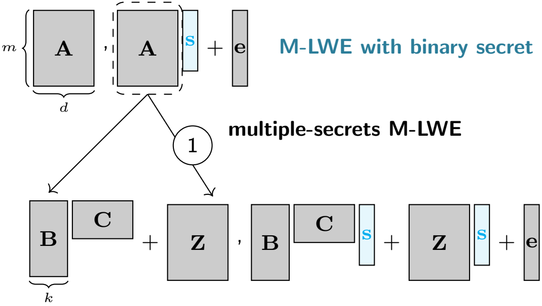

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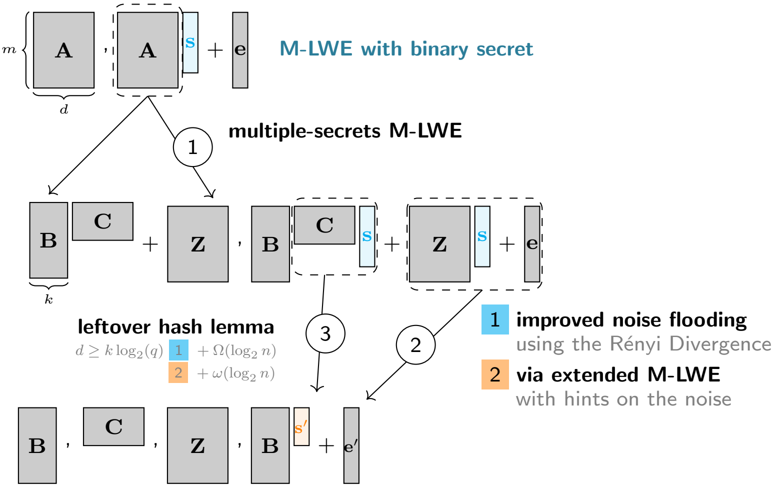

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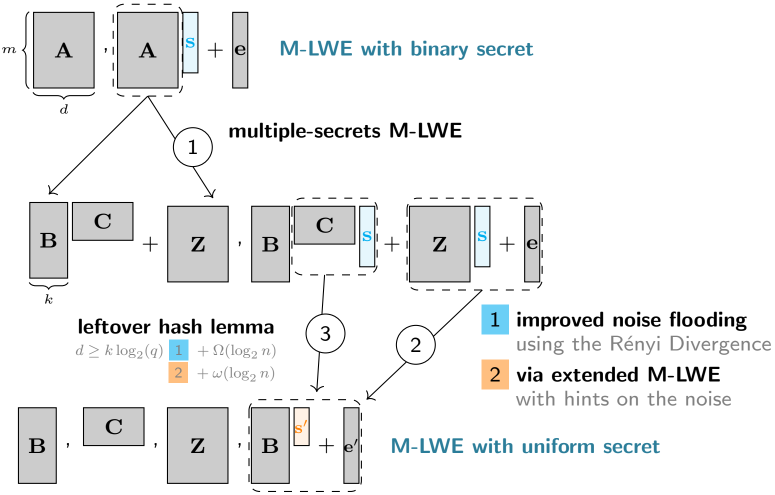

# Improved Noise Flooding via Rényi Divergence 1/2

Let $P, Q$ be discrete probability distributions.

In [GKPV10]: Statistical Distance

$$\mathrm{SD}(P,Q) = \frac{1}{2} \sum_{x \in \mathrm{Supp}(P)} |P(x) - Q(x)|$$

**In our work:** Rényi Divergence

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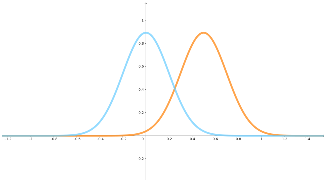

Example: two Gaussians $D_\beta$ and $D_{\beta,s}$

$$\mathrm{RD}(D_\beta, D_{\beta,s}) = \exp\left(\frac{2\pi\|s\|^2}{\beta^2}\right)$$

$$\mathrm{SD}(D_\beta, D_{\beta,s}) = \frac{\sqrt{2\pi}\|s\|}{\beta}$$

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Both fulfill the **probability preservation property** for an event $E$:

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We need: $Q(E)$ negligible $\Rightarrow P(E)$ negligible

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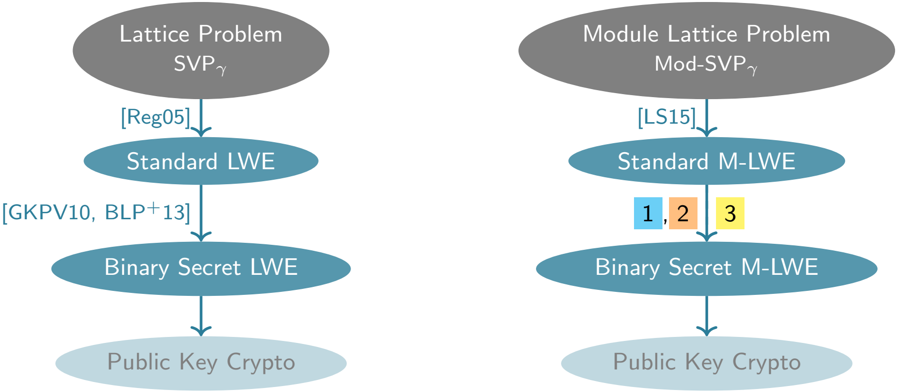
⚠ Rényi Divergence only for search problems.

# Motivation (Continued)




* *Pseudorandomness of ring-LWE for any ring and modulus* C. Peikert, O. Regev and N. Stephen-Davidowitz

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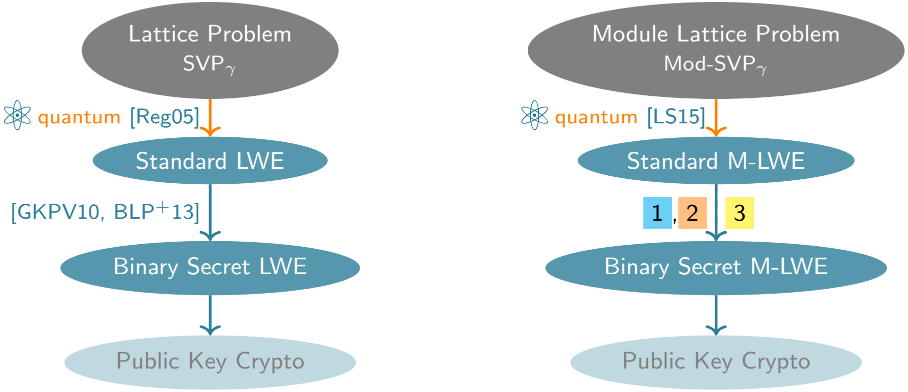


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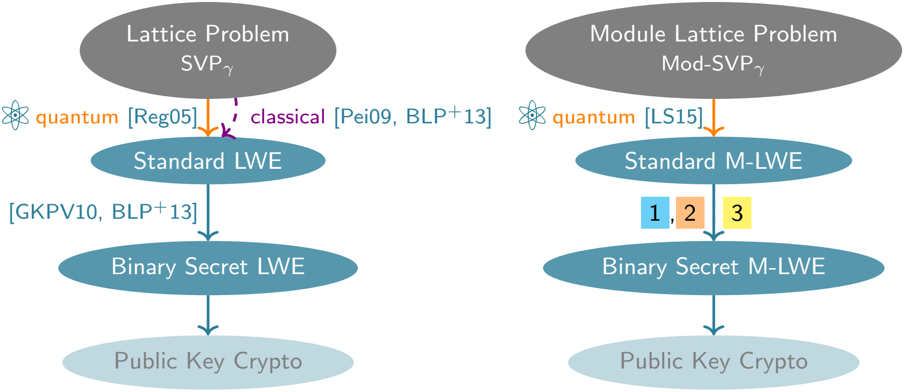


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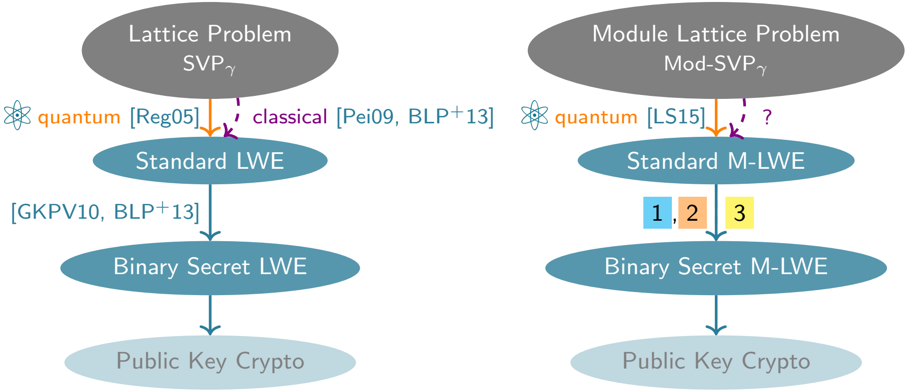


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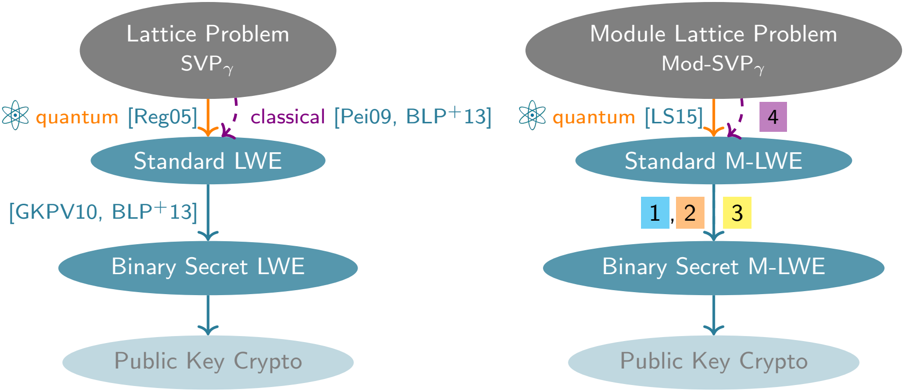


Contributions:

4 Classical reduction, modulus $q$ is poly-small, but linear rank [BJRW20]
extending [Pei09] and [BLP$^+$13] to M-LWE
and combining them with [PRS17]*

---

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# Classical Hardness of Module-LWE

High level idea following [BLP$^+$13]:

- Step 1: Classical reduction from decision Mod-SVP$_\gamma$ to decision Module-LWE with exponentially large modulus $q$

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- Step 3: Modulus reduction from exponentially large to polynomially small modulus for Module-LWE with binary secret

  Using [AD17], computing bounds on singular values of rotation matrix, loss in the reduction depends on the norm of the secret.

# Partial Vandermonde Learning With Errors

Joint work with A. Sakzad and R. Steinfeld

Under submission

# Partial Vandermonde Transform [HPS+14, LZA18]

Again: Let $R$ be the ring of integers of a number field $K$ of degree $n$.
Think of $R = \mathbb{Z}[x]/\langle x^n + 1 \rangle$ and $K = \mathbb{Q}[x]/\langle x^n + 1 \rangle$ with $n = 2^\ell$.

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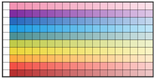

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Note: For $\Omega \subseteq \{\omega_j\}_{j=1,\ldots,n}$ write $\mathbf{V}_\Omega \cdot a = b$. **(partial Vandermonde transform)**

# Partial Vandermonde Problems

Choose a random subset $\Omega \subseteq \{\omega_j\}_{j=1,\ldots,n}$ of size $|\Omega| = t$.

**Partial Vandermonde knapsack problem (PV-Knap):** Sample e $\sim$ DistrE over $\mathbb{Z}^n$ defining

$$\mathbf{V}_\Omega \, e = b \mod q.$$

Search: find e

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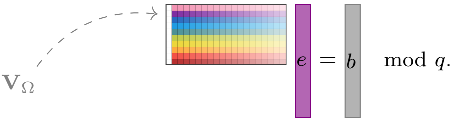
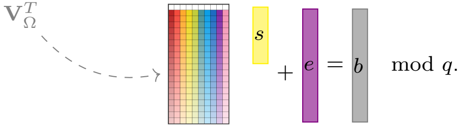



Search: find e

**Partial Vandermonde Learning With Errors (PV-LWE):** Sample s $\sim$ DistrS over $\mathbb{Z}^t$ and e $\sim$ DistrE over $\mathbb{Z}^n$ defining




Search: find e (and secret s)

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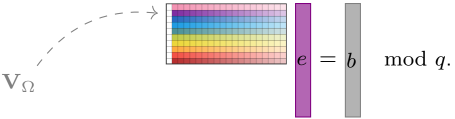


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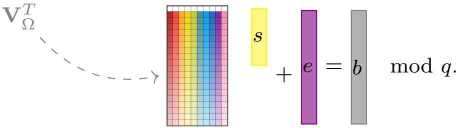


Search: find e (and secret s)

**Conjecture:** Hard to solve if DistrE provides elements of small norm.

# Equivalence of PV-Knap and PV-LWE

Let $t = n/2$ and set $\mathcal{P}_t = \{\Omega \subseteq \{\omega_j\}_{j=1,\ldots,n} \colon |\Omega| = t\}$.

Property 1: $\mathbf{V}_\Omega$ defines a ring homomorphism from $R$ to $\mathbb{Z}_q^t$:

$$\mathbf{V}_\Omega(a \cdot b) = (\mathbf{V}_\Omega a) \circ (\mathbf{V}_\Omega b)$$

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Property 2: $\Omega^c = \{\omega_j\}_j \setminus \Omega$ defines the **complement** partial Vandermonde transform $\mathbf{V}_{\Omega^c}$.
Given $\mathbf{V}_\Omega a$ and $\mathbf{V}_{\Omega^c} a$, we can recover $a$.

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(parity check matrix, ⚠ only for power-of-two cyclotomics)

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## Lemma (Adapted [MM11, Sec. 4.2])

*Let $\psi$ denote a distribution over $\mathbb{Z}^n \cong R$. There is an efficient reduction from PV-LWE$_\psi$ to PV-Knap$_\psi$, and vice versa.*

**Idea:** Given $(\mathbf{V}_\Omega, b)$, with $b = \mathbf{V}_\Omega^T s + e$. Compute $\Omega'$ such that $\mathbf{V}_{\Omega'} \cdot \mathbf{V}_\Omega^T = 0$.
Then, $b' := \mathbf{V}_{\Omega'} b = \mathbf{V}_{\Omega'} e$ is an instance of PV-Knap.

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| [HS15] | Our work |
|---|---|
| deterministic | randomized |
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$$\mathbf{V}^{-1}(e_1 || c') = \mathbf{V}^{-1}(\mathbf{V}(f \cdot r' + s')) = f \cdot pr + ps + m = m \bmod p$$

if $f, r$ and $s$ are small enough

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$\Rightarrow$ leaky variant of **PV-Knap**, that we call the **PASS problem**.

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$\Rightarrow$ leaky variant of **PV-Knap**, that we call the **PASS problem**.

⚠ PASS problem is tailored to PASS Encrypt!
? Reduce it from some more general problem?

# Properties of PASS Encrypt

**Homomorphic properties:**

Addition: $\mathsf{Enc}(\mathsf{pk}, m_1) + \mathsf{Enc}(\mathsf{pk}, m_2) = \mathsf{Enc}(\mathsf{pk}, m_1 + m_2)$

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**Efficiency:**

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**Concrete Security:**

Known: key recovery and randomness recovery attacks [HS15, DHSS20]

New: plaintext recovery using hints attacks

make use of leaky LWE estimator of Dachman-Soled et al. [DDGR20]

# Conclusion and Perspectives

# Conclusion

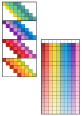
**Asiacrypt'20**
**CT-RSA'21**

I. Study of existing structured variants

**1. Module Learning With Errors with a binary secret**

**2. Classical hardness of Module Learning With Errors**

$$A \cdot A \, s + e = b$$



II. Proposing new structured variants

3. Middle-Product Learning With Rounding

**4. Partial Vandermonde Learning With Errors & Knapsack**

**Asiacrypt'19**
**& under submission**

III. Building $\overbrace{\text{public key encryption}}^{\text{provable secure}}$

5. Based on MP-LWR (3.)

**6. PASS Encrypt, related to (4.)**

# Open Questions and Perspectives

## I. Module LWE

Follow-ups

- General secret distributions (Entropic Secret Module-LWE)
- Small noise distributions (extending [MP13])

Open questions ?

- Classical and binary hardness for smaller ranks, in particular rank equals 1 (Ring-LWE)
  - Avoid leftover hash lemma in the reduction?
  - Avoid exponentially large modulus in [Pei09]?
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## II. Partial Vandermonde LWE

Follow-ups

- Construct encryption scheme based only on PV-LWE / PV-Knap

Questions ?

- Hardness of partial Vandermonde problems
  - Cryptanalysis?
  - Worst-case average-case reductions as for LWE?
- More cryptographic applications

# Contributions

Published:

CT-RSA'21 **On the Hardness of Module-LWE with Binary Secret** [HAL]
Katharina Boudgoust, Corentin Jeudy, Adeline Roux-Langlois & Weiqiang Wen.

Asiacrypt'20 **Towards Classical Hardness of Module-LWE: The Linear Rank Case** [HAL]
Katharina Boudgoust, Corentin Jeudy, Adeline Roux-Langlois & Weiqiang Wen.

Asiacrypt'19 **Middle-Product Learning with Rounding Problem and its Applications** [HAL]
Shi Bai, Katharina Boudgoust, Dipayan Das, Adeline Roux-Langlois, Weiqiang Wen & Zhenfei Zhang.

Under Submission:

- **Vandermonde meets Regev: Public Key Encryption Schemes Based on Partial Vandermonde Problems.** Katharina Boudgoust, Amin Sakzad and Ron Steinfeld.

E-Print:

- **Compressed Linear Aggregate Signatures Based on Module Lattices** [IACR ePrint] Katharina Boudgoust and Adeline Roux-Langlois.

Thank you.

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Katharina Boudgoust, Corentin Jeudy, Adeline Roux-Langlois, and Weiqiang Wen.
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In *ASIACRYPT (2)*, volume 12492 of *Lecture Notes in Computer Science*, pages 289–317. Springer, 2020.

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On the hardness of module-lwe with binary secret.
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