Partial Vandermonde Problems and PASS Encrypt

Katharina Boudgoust[1] Amin Sakzad[2] Ron Steinfeld[2]

[1]Aarhus University, Denmark

[2]Faculty of Information Technology, Monash University, Australia

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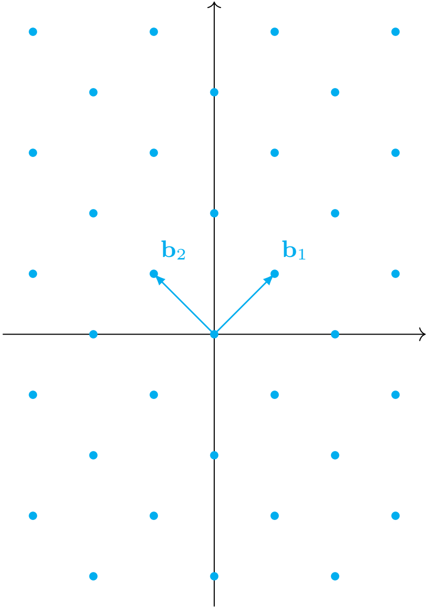
Lattice-Based Cryptography

Provably secure public-key cryptography needs well-defined assumptions in the form of mathematical problems.

(Main) Lattice Problems for Crypto:

- Short Integer Solution [Ajt96]
- NTRU [HPS98]
- Learning With Errors [Reg05]



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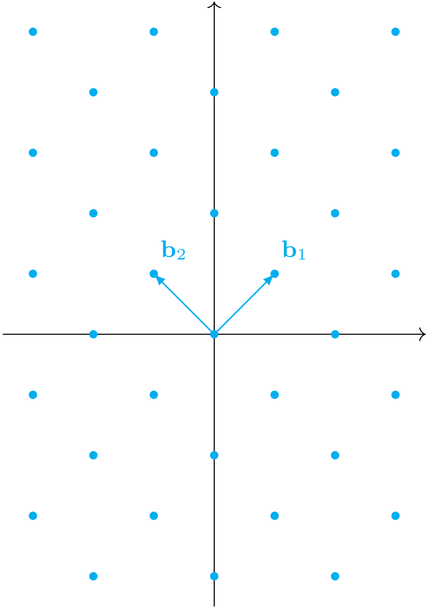
today



Partial Vandermonde Problems

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Write $\{\omega_j\}_{j=1,\ldots,n}$ for $\{\omega^k : k \in \mathbb{Z}_{2n}^\times\}$. This defines the **Vandermonde transform** $\mathbf{V} \colon R \to \mathbb{Z}_q^n$

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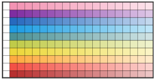

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Note: For $\Omega \subseteq \{\omega_j\}_{j=1,\dots,n}$ write $\mathbf{V}_\Omega \cdot a = b$. **(partial Vandermonde transform)**

Partial Vandermonde Problems

Choose a random subset $\Omega \subseteq \{\omega_j\}_{j=1,\ldots,n}$ of size $|\Omega| = t$.

Partial Vandermonde knapsack problem (PV-Knap): Sample $\mathsf{e} \sim$ DistrE over $\mathbb{Z}^n$ defining

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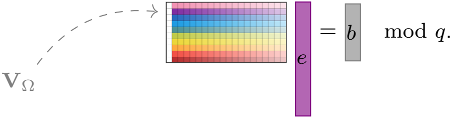

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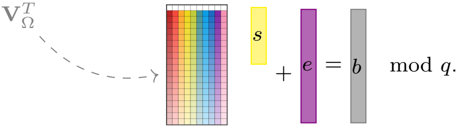

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Conjecture: Hard to solve if DistrE provides elements of small norm.

Equivalence of PV-Knap and PV-LWE

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$$\mathbf{V}_{\Omega'} \cdot \mathbf{V}_\Omega^T = 0 \in \mathbb{Z}_q^{t \times t}.$$

(parity check matrix, ⚠ only for power-of-two cyclotomics)

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Lemma (Adapted [MM11, Sec. 4.2])

Let ψ denote a distribution over $\mathbb{Z}^n \cong R$. There is an efficient reduction from PV-LWE$_\psi$ to PV-Knap$_\psi$, and vice versa.

Idea: Given $(\mathbf{V}_\Omega, b)$, with $b = \mathbf{V}_\Omega^T s + e$. Compute Ω' such that $\mathbf{V}_{\Omega'} \cdot \mathbf{V}_\Omega^T = 0$.
Then, $b' := \mathbf{V}_{\Omega'} b = \mathbf{V}_{\Omega'} e$ is an instance of PV-Knap.

Hidden Ideal Lattice 1/2

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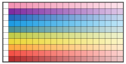

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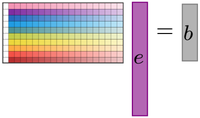

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The matrix $\mathbf{V}_\Omega$ defines an **ideal lattice**:

$$\Lambda_q^{\perp}(\mathbf{V}_\Omega) = \{a \in R \colon \mathbf{V}_\Omega a = 0 \bmod q\}$$

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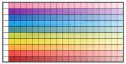

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Idea:

1) Solve $\mathbf{V}_\Omega y = b \bmod q$ for the unknown y in R (in general not in the support of DistrE)
2) Find a **closet vector** v of y in $\Lambda_q^\perp(\mathbf{V}_\Omega)$, i.e., $\|y - v\|$ smallest
3) The element $e := y - v$ is a solution to PV-Knap

⚠ Promise variant of the closest vector problem, called **Bounded Distance Decoding (BDD)**

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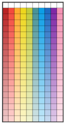

$$s + e = b \mod q.$$

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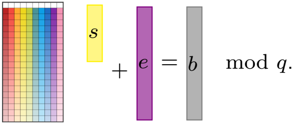

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This is an instance of **BDD** in the **ideal lattice**

$$\Lambda_q(\mathbf{V}_\Omega) = \{a \in R \colon a = \mathbf{V}_\Omega^T s \bmod q \text{ for some } s \in \mathbb{Z}_q^t\}$$

PASS Encrypt

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$$e_1 = (\mathsf{pk} \circ \mathbf{V}_\Omega r') + \mathbf{V}_\Omega s'$$
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$$\mathbf{V}^{-1}(e_1 || c') = \mathbf{V}^{-1}(\mathbf{V}(f \cdot r' + s')) = f \cdot pr + ps + m = m \bmod p$$

if f, r and s are small enough

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⚠ PASS problem is tailored to PASS Encrypt!
? Reduce it from some more general problem?

Properties of PASS Encrypt

Homomorphic properties:

Addition: $\mathsf{Enc}(\mathsf{pk}, m_1) + \mathsf{Enc}(\mathsf{pk}, m_2) = \mathsf{Enc}(\mathsf{pk}, m_1 + m_2)$

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Scheme	NTRU [HPS98]	P-LWE Regev [LP11]	PASS Encrypt
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Concrete Security:

Known: key recovery and randomness recovery attacks [HS15, DHSS20]

New: plaintext recovery using hints attacks

💡 make use of leaky LWE estimator of Dachman-Soled et al. [DDGR20]

Conclusion and Perspectives

Open Questions and Perspectives

Follow-ups

- Construct encryption scheme based only on PV-LWE / PV-Knap

Questions **?**

- Hardness of partial Vandermonde problems
  - Cryptanalysis?
  - Worst-case to average-case reductions as for LWE?
- More cryptographic applications

Thank you.

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