Threshold Fully Homomorphic Encryption from LWE

Challenges and Perspectives

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CNRS, Univ Montpellier, LIRMM, France

Context

👉 In asymmetric cryptography there is a public key and a secret key. The secret key is used for a **critical operation** and thus needs to be protected.

- 🔒 Encryption: secret key allows to decrypt ciphertexts
- ✏️ Signature: secret key allows to sign messages

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- 🔒 Encryption: secret key allows to decrypt ciphertexts
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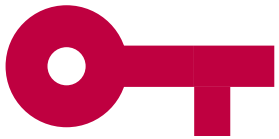
👉 The secret key can be seen as a **single point of failure**.

- Someone else learns it: security issue
- I loose it: operability issue



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Cryptocurrency | News

Youtuber Loses $60,000 In Crypto and NFTs After Exposing His Private Key While Live Streaming

By **Newton Gitonga** - September 2, 2023

DARRYN POLLOCK NOV 30, 2017

Infamous Discarded Hard Drive Holding 7,500 Bitcoins Would be Worth $80 Million Today

Cryptonews • Altcoin News • LHV Bank Founder Has Lost Private Key to ETH Stash Worth $470 Million

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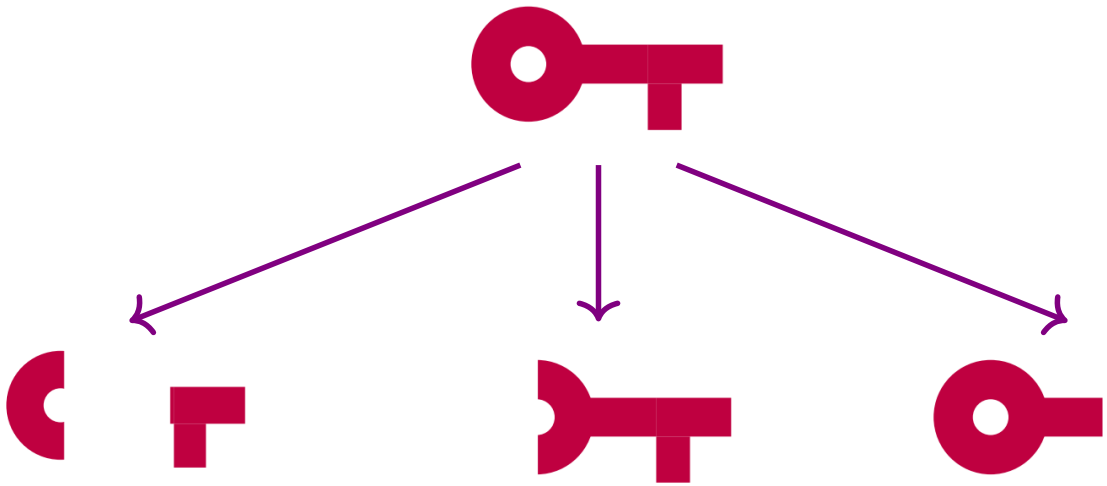
Ruholamin Haqshanas

Last updated: **November 7, 2023 02:36 EST** | 2 min read

Motivation Threshold Cryptography [DF89]

The secret key can be seen as a **single point of failure**.

Idea: divide the secret key into multiple shares



Better security: multiple secret key shares needed

Better operability: not necessarily all secret key shares needed

Today: Threshold Fully Homomorphic Encryption

FHE scheme:

- $\mathsf{KGen} \rightarrow (\mathsf{pk}, \mathsf{sk})$
- $\mathsf{Enc}(\mathsf{pk}, m) \rightarrow \mathsf{ct}$ $\quad m \in \{0,1\}$
- $\mathsf{Eval}(\mathsf{pk}, f, \mathsf{ct}_1, \mathsf{ct}_2) \rightarrow \widehat{\mathsf{ct}}$ $\quad f : \{0,1\} \times \{0,1\} \rightarrow \{0,1\}$
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t-out-of-n Threshold FHE scheme:

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Properties:

- Correctness — t parties can recover the message
- Security — less than t parties learn nothing about message

Applications:

- Electronic voting protocols
- Universal thresholdizer [BGG+18]

Overview of Today's Talk

🚩 Structure:

- Part 1: *Basic Blueprint of Threshold FHE*
- Part 2: *Suitable Secret Sharings*
- Part 3: *Different Noise Floodings*
- Part 4: *Defining Security*

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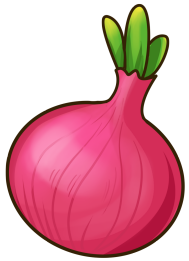
This talk: overview
Christian's talk: details

Part 1:

Basic Blueprint

Ingredients for Threshold FHE based on LWE

FHE with nearly linear decryption



Linear secret sharing

FHE from LWE with nearly linear decryption

FHE scheme:

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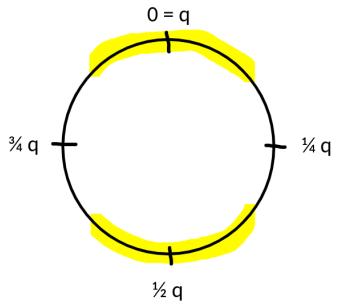
q modulus

Nearly linear decryption:

- sk and ct vectors over $\mathbb{Z}_q$
- $\langle \mathsf{ct}, \mathsf{sk} \rangle \bmod q = \frac{q}{2} \cdot f(m_1, m_2) + e_{\mathsf{ct}}$
- e_{ct} encryption noise
- $\|e_{\mathsf{ct}}\|_\infty < q/4$



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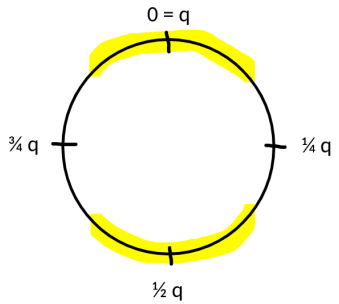
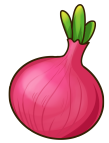


⚠ Damien's talk: decryption failure should be small enough!

Linear secret sharing



t-out-of-n secret sharing:

- $\mathsf{Share}(\mathsf{sk}) \to (\mathsf{sk}_1, \ldots, \mathsf{sk}_n)$
- $\mathsf{Rec}(\{\, \mathsf{sk}_i \,\}_{i \in S}) \to \mathsf{sk}$ $\qquad S \subseteq \{\, 1, \ldots, n \,\}$

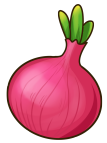
Properties:

- if $|S| < t$ no information about sk leaked
- if $|S| \geq t$ successful reconstruction of sk

Linearity:

- $\mathsf{Rec}(\{\, \langle y, \mathsf{sk}_i \rangle \,\}_{i \in S}) = \langle y, \mathsf{Rec}(\{\, \mathsf{sk}_i \,\}_{i \in S} \rangle$

Blueprint for Threshold FHE, Trial



linear secret sharing

$\approx$ linear decrypt

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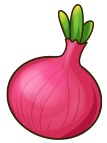

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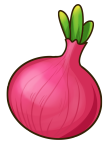

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⚠ Problem:

- Ciphertext noise e_{ct} depends on sk
- After "enough" partial decryptions, recover sk

Blueprint for Threshold FHE [BD10]

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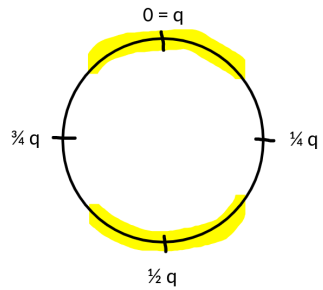

Research Directions

- Part 1: Different approach than noise flooding?

Part 2:

Suitable Secret Sharings

Recall: Blueprint for Threshold FHE [BD10]

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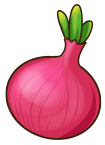
Shamir's Secret Sharing over $\mathbb{Z}_q$, Trial

n parties
t threshold



Share(sk):

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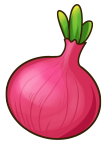

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Plug into Threshold FHE:

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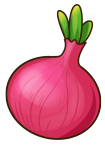

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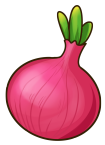
⚠ Problem: Lagrange coefficient λ_i are rationals, not integers

Shamir's Secret Sharing over $\mathbb{Z}_q$, Approaches



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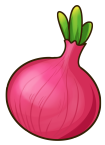

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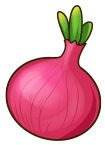
Approaches:

- Move λ_i to PartDec [GKS23, MBH23] ⚠ different model

Shamir's Secret Sharing over $\mathbb{Z}_q$, Approaches



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- Clearing out denominators, multiply by $n!$ [Sho00, BGG+18] ⚠ $\log q > n \log n$

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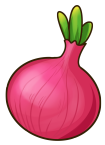

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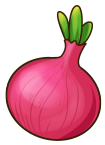
- Recursive 2-out-of-3 Shamir secret sharing [CCK23] many shares per party

Shamir's Secret Sharing over $\mathbb{Z}_q$, Approaches



$\mathsf{Rec}(\{\mathsf{sk}_i\}_{i\in S})$:

- compute Lagrange coefficients $\lambda_i = \prod_{k\in S\setminus\{i\}} \frac{k}{k-i} \in \mathbb{Q}$
- output $\sum_{i\in S} \lambda_i \mathsf{sk}_i$

Plug in Threshold FHE:

- PartDec: $d_i = \langle \mathsf{sk}_i, \mathsf{ct}' \rangle + n! \cdot e_{flood,i}$
- Combine: $\sum_{i\in S} \lambda_i d_i = \langle \mathsf{sk}, \mathsf{ct}' \rangle + \sum_{i\in S} (\lambda_i \cdot n!) e_{flood,i}$
  $\in \mathbb{Z}$

Approaches:

- Move λ_i to PartDec [GKS23, MBH23] — ⚠ different model
- Clearing out denominators, multiply by $n!$ [Sho00, BGG+18] — ⚠ $\log q > n \log n$
- Recursive 2-out-of-3 Shamir secret sharing [CCK23] — many shares per party
- ✖ Bit-decomposition of λ_i — insecure!

Alternative Approaches for Linear Secret Sharing

- $\{0,1\}$-LSSS [BGG+18] — many shares per party
  - from Monotone Boolean formulas
  - Naive secret sharing
  - Replicated secret sharing

- Pseudorandom secret sharing of bounded values over $\mathbb{Z}$ [BD10] — requires setup

Research Directions

- Part 1: Different approach than adding noise?
- Part 2: Different approach for linear secret sharing?

Part 3:

Different Noise Floodings

Recall: Blueprint for Threshold FHE [BD10]

t-out-of-n Threshold FHE scheme:

- KGen $\rightarrow (\mathsf{pk}, \mathsf{sk})$ and Share$(\mathsf{sk}) \rightarrow (\mathsf{sk}_1, \ldots, \mathsf{sk}_n)$
- Enc and Eval unchanged
- PartDec : compute $d_i = \langle \mathsf{ct}, \mathsf{sk}_i \rangle + e_{flood,i}$ flooding noise
- Combine : compute $\mathsf{Rec}(\{ d_i \}_{i \in S})$

$$\begin{aligned}
\mathsf{Rec}(\{ d_i \}_{i \in S}) &= \mathsf{Rec}(\{ \langle \mathsf{ct}, \mathsf{sk}_i \rangle \}_i + e_{flood,i}) \\
&= \langle \mathsf{ct}, \mathsf{sk} \rangle + \mathsf{Rec}(\{ e_{flood,i} \}) \\
&= \frac{q}{2} \cdot f(m_1, m_2) + \underbrace{e_{\mathsf{ct}} + \mathsf{Rec}(\{ e_{flood,i} \})}_{\text{small}}
\end{aligned}$$

Recall: Blueprint for Threshold FHE [BD10]

t-out-of-n Threshold FHE scheme:

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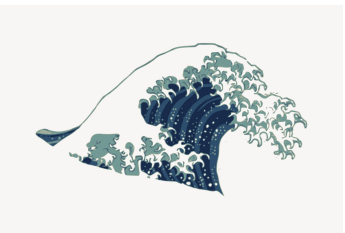
\end{aligned}
$$



Partial Decryption Security

Two worlds:

- Real: e_{ct} and $e_{flood} := \mathsf{Rec}(\{ e_{flood,i} \}_{i \in S})$
- Simulated: only e_{flood}

How close are they? [BD10] measures with statistical distance Δ

$$\Delta(\mathsf{Real}, \mathsf{Sim}) \leq \Delta(e_{flood} + e_{\mathsf{ct}}, e_{flood}) \leq \mathsf{negl}(\lambda)$$

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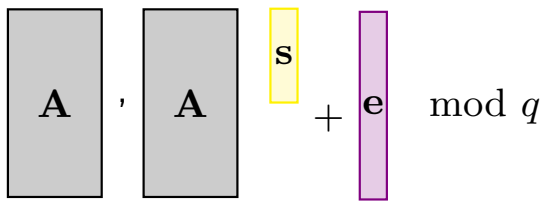
Problem:

- $\|e_{flood}\|$ needs to be super-polynomially larger than $\|e_{\mathsf{ct}}\|$
- LWE-based constructions: $\|e_{flood}\| \sim$ LWE modulus q and $\|e_{\mathsf{ct}}\| \sim$ LWE noise $\mathbf{e}$, thus super-polynomial modulus-noise ratio
  - Larger parameters
  - Easier problem



Partial Decryption Security

Can be avoided for Gaussian distributions in full threshold PKE setting! [MS23]

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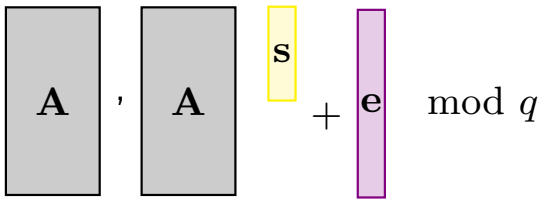

Partial Decryption Security

Idea: change the measure! [BLR+18]

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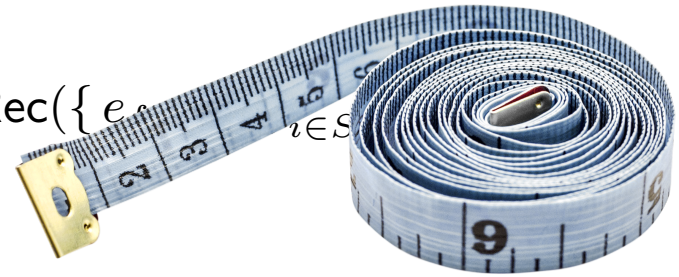
- Simulated: only e_{flood}



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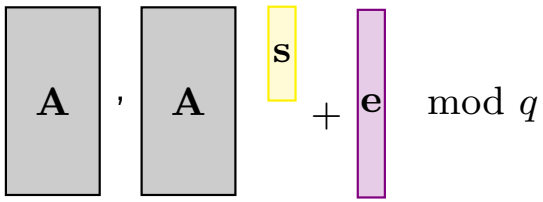

Improved Noise Flooding via Rényi Divergence 1/2

Let P, Q be discrete probability distributions

In [BD10]: Statistical Distance $\Delta(P, Q) = \frac{1}{2} \sum_{x \in \mathrm{Supp}(P)} |P(x) - Q(x)|$

In [BS23]: Rényi Divergence

$$\mathrm{RD}(P, Q) = \sum_{\substack{x \in \mathrm{Supp}(P) \\ \subset \mathrm{Supp}(Q)}} \frac{P(x)^2}{Q(x)}$$

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Both fulfill the **probability preservation property** for an event E:

[BD10]:	$P(E)$	$\leq$	$\Delta(P,Q) + Q(E)$	(additive)
Our work:	$P(E)^2$	$\leq$	$\mathrm{RD}(P,Q) \cdot Q(E)$	(multiplicative)

- $Q(E)$ negligible $\Rightarrow$ $P(E)$ negligible
- $\Delta(P,Q) =^!$ negligible and $\mathrm{RD}(P,Q) =^!$ **constant**

Improved Noise Flooding via Rényi Divergence 2/2

Two worlds:

- Real: e_{ct} and e_{flood}
- Simulated: only e_{flood}

How close are they?

$$\Delta(\mathsf{Real}, \mathsf{Sim}) \leq \Delta(e_{flood} + e_{\mathsf{ct}}, e_{flood}) \leq \mathsf{negl}(\lambda)$$

$$\mathsf{RD}(\mathsf{Real}, \mathsf{Sim}) \leq \mathsf{RD}(e_{flood} + e_{\mathsf{ct}}, e_{flood}) \leq \text{constant}$$

Advantage:

- $\|e_{flood}\|$ only needs to be polynomially larger than $\|e_{\mathsf{ct}}\|$
- LWE-based constructions: polynomial modulus-noise ratio

Improved Noise Flooding via Rényi Divergence 2/2

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Advantage:

- $\|e_{flood}\|$ only needs to be polynomially larger than $\|e_{\mathsf{ct}}\|$
- LWE-based constructions: polynomial modulus-noise ratio

Disadvantage:

1) Rényi divergence depends on the number of issued partial decryptions
   $\rightarrow$ from simulation-based to game-based security notion
2) Works well with search problems, not so well with decision problems

Research Directions

- Part 1: Different approach than adding noise?
- Part 2: Different approach for linear secret sharing?
- Part 3: Optimal noise analysis?

Part 4:

Defining Security

Goal: Game-Based Security for t-out-of-n Threshold FHE

Challenger $\mathcal{C}$ | Adversary $\mathcal{A}$

$(\mathsf{pk}, \mathsf{sk})$

pk

m_0, m_1 ←

$b \leftarrow \{0, 1\}$

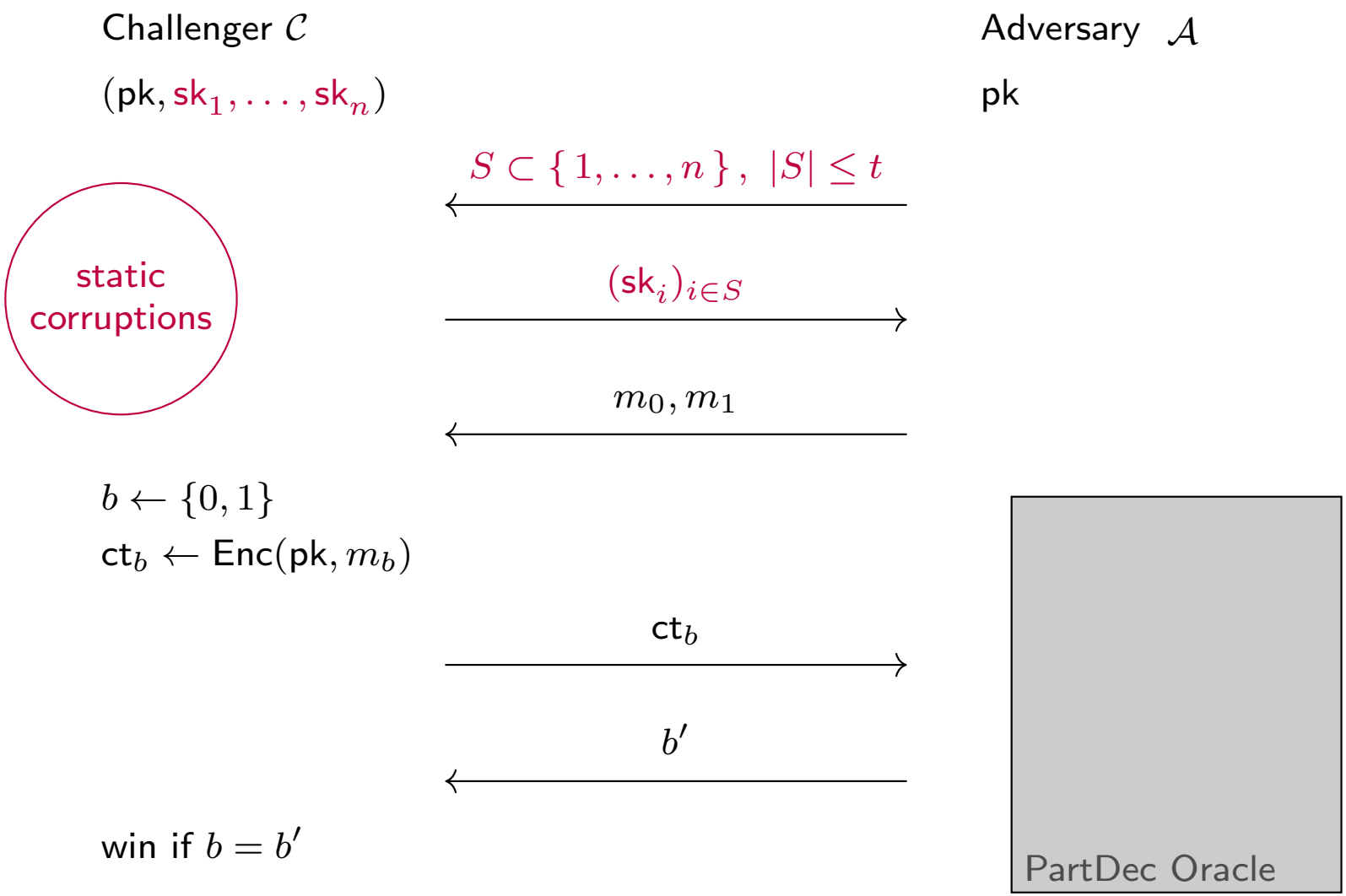
$\mathsf{ct}_b \leftarrow \mathsf{Enc}(\mathsf{pk}, m_b)$

ct_b →

b' ←

win if $b = b'$

Goal: Game-Based Security for t-out-of-n Threshold FHE



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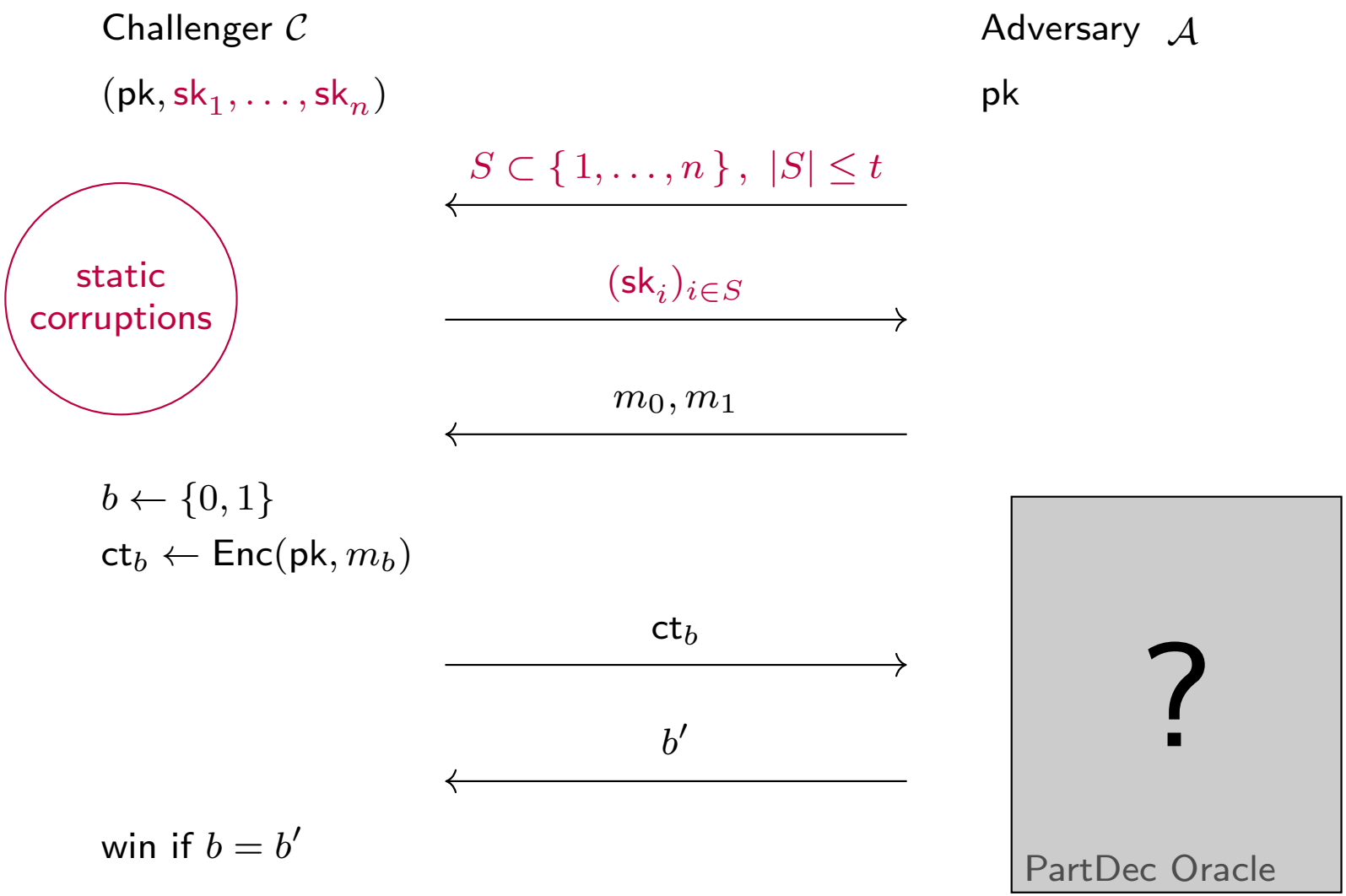

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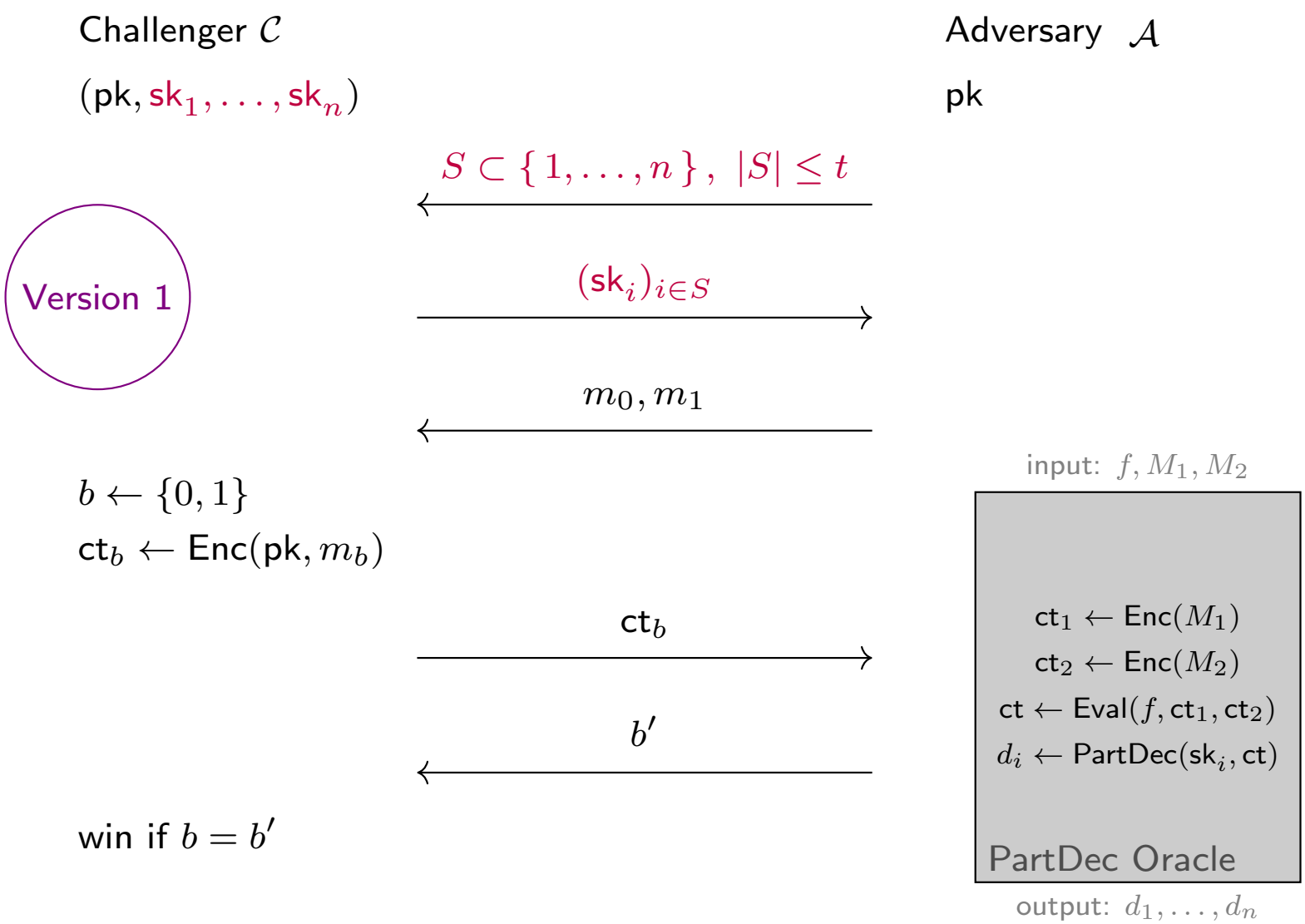

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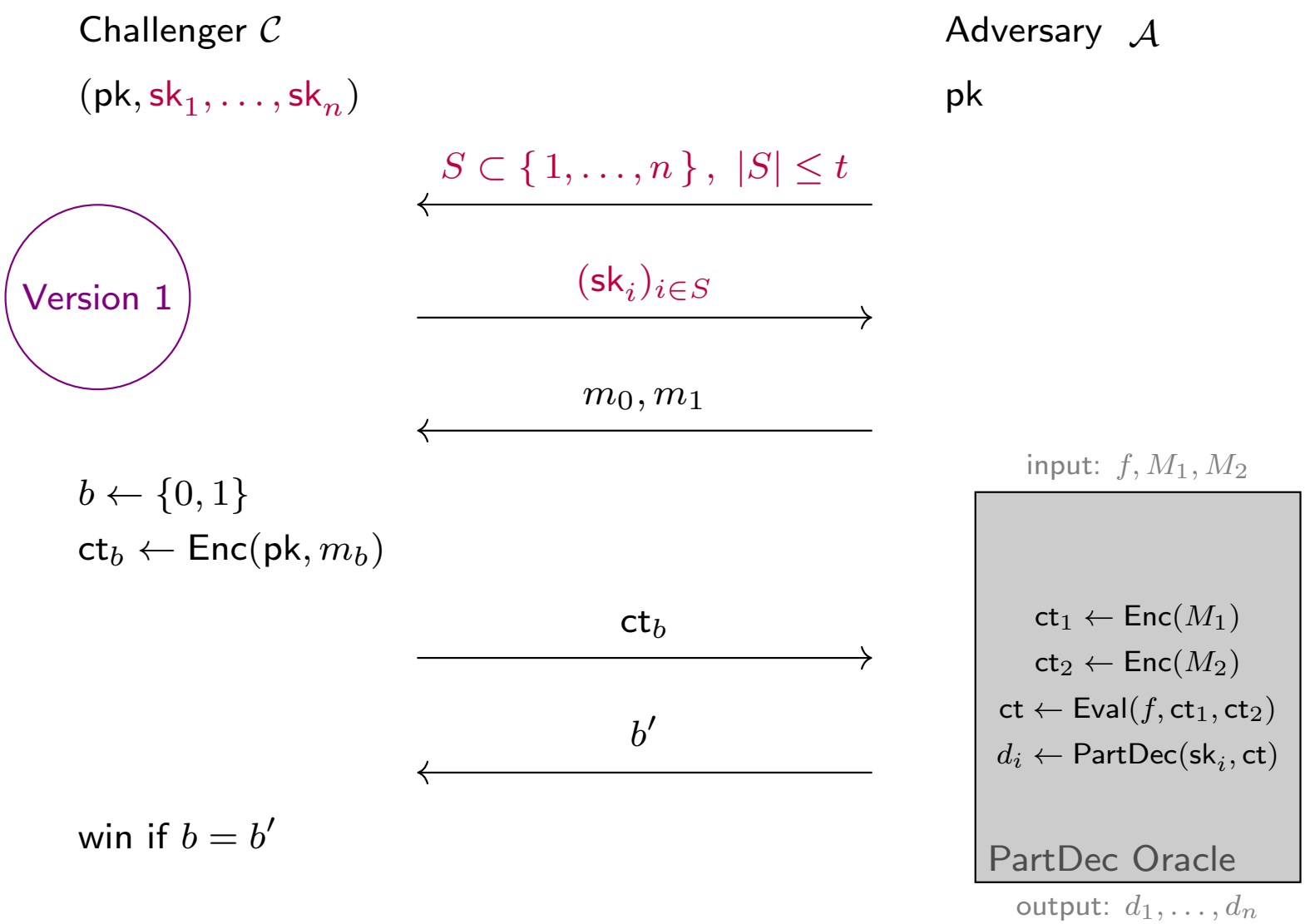
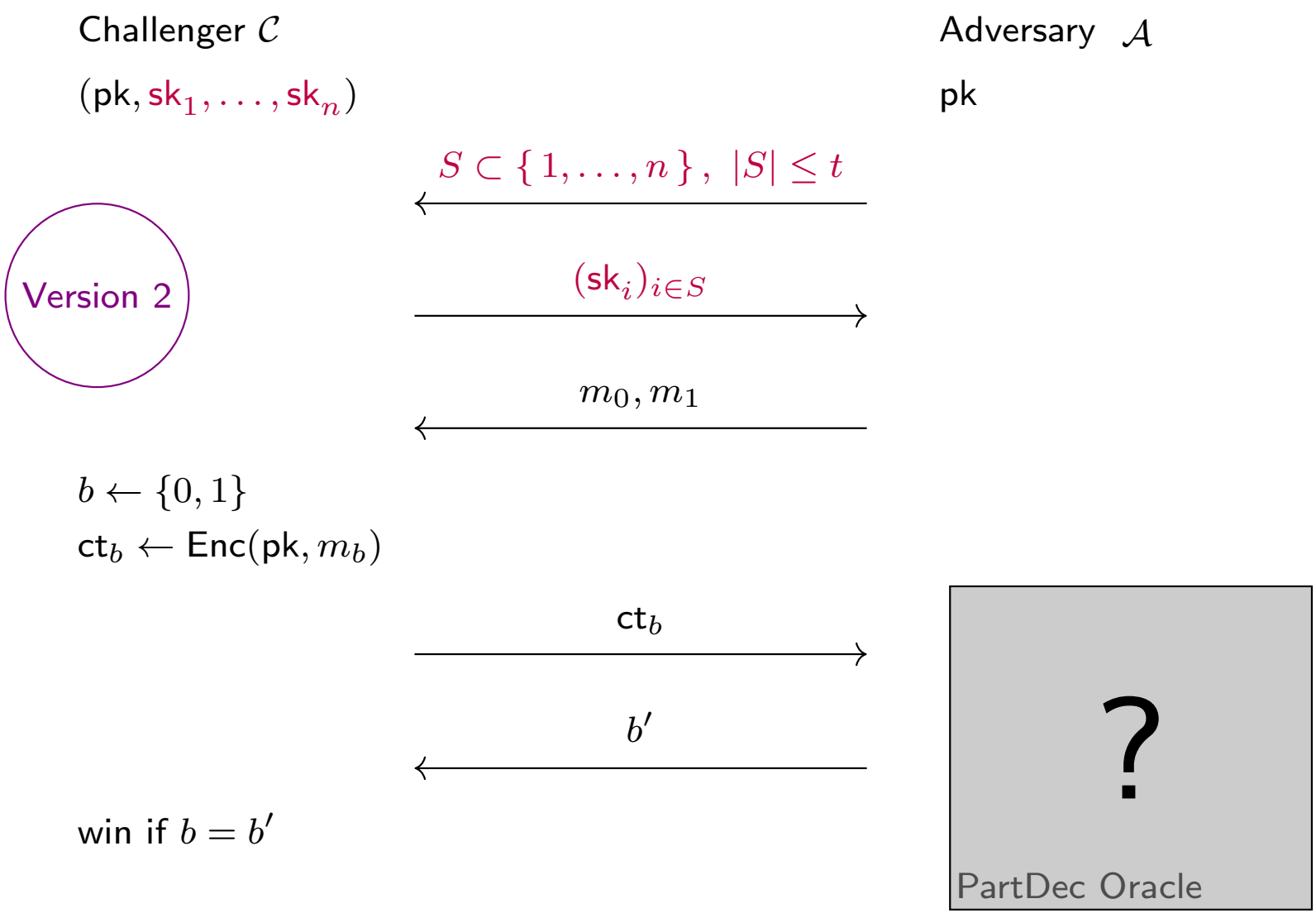


Very weak: queries to PartDec oracle are independent of ct_b

IND-CPA Security for t-out-of-n Threshold FHE [JRS17, BS23, CCP+24]



IND-CPA Security for t-out-of-n Threshold FHE [JRS17, BS23, CCP+24]

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$(\mathsf{pk}, \mathsf{sk}_1, \ldots, \mathsf{sk}_n)$

Adversary $\mathcal{A}$

pk

$S \subset \{1, \ldots, n\}, \ |S| \leq t$

Version 2

Enc_b Oracle

$b \leftarrow \{0, 1\}$

b'

win if $b = b'$

PartDec Oracle

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input: m_0, m_1

$\mathsf{ct}_b \leftarrow \mathsf{Enc}(m_b)$

Enc_b Oracle

output: ct_b

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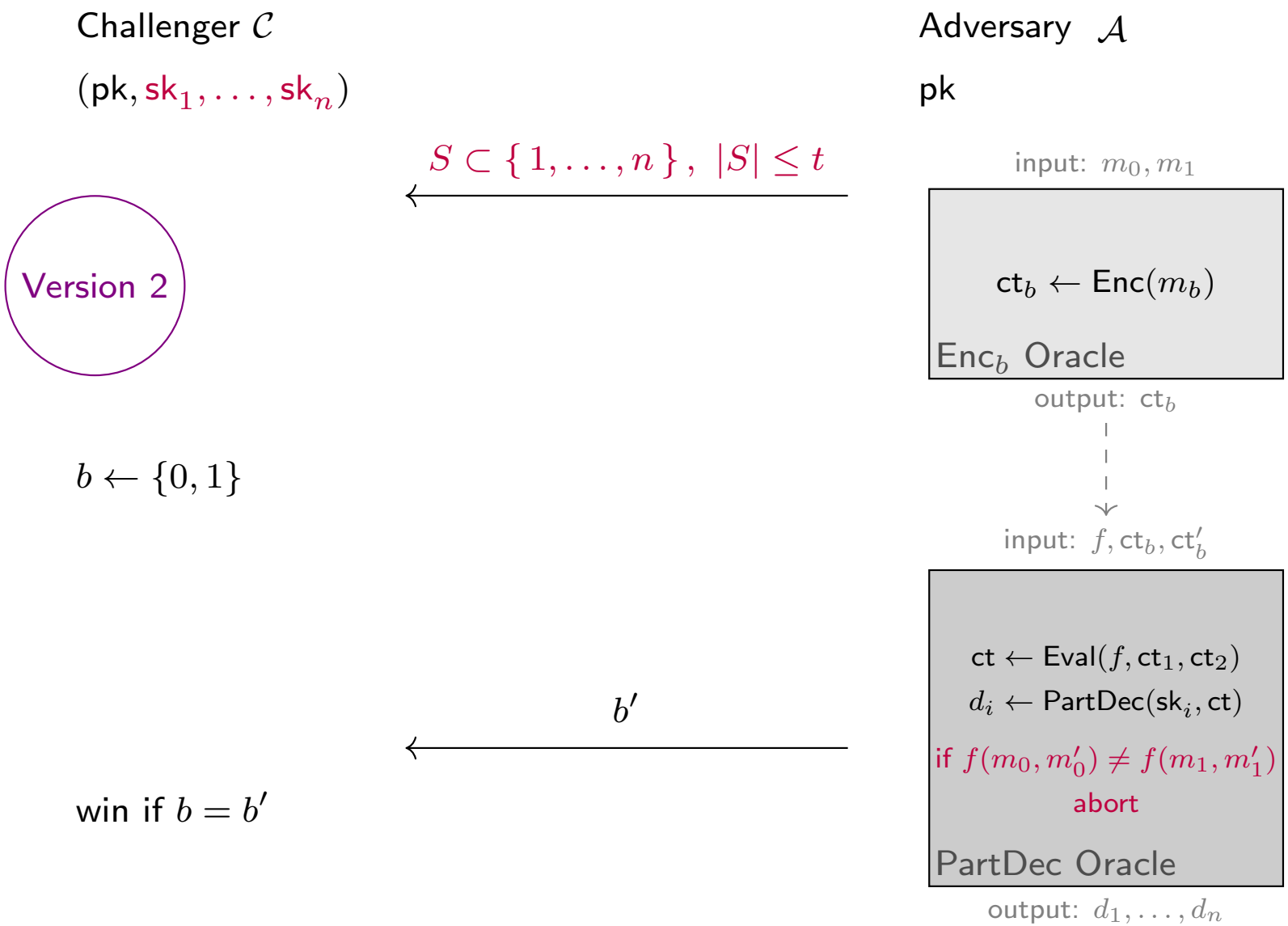

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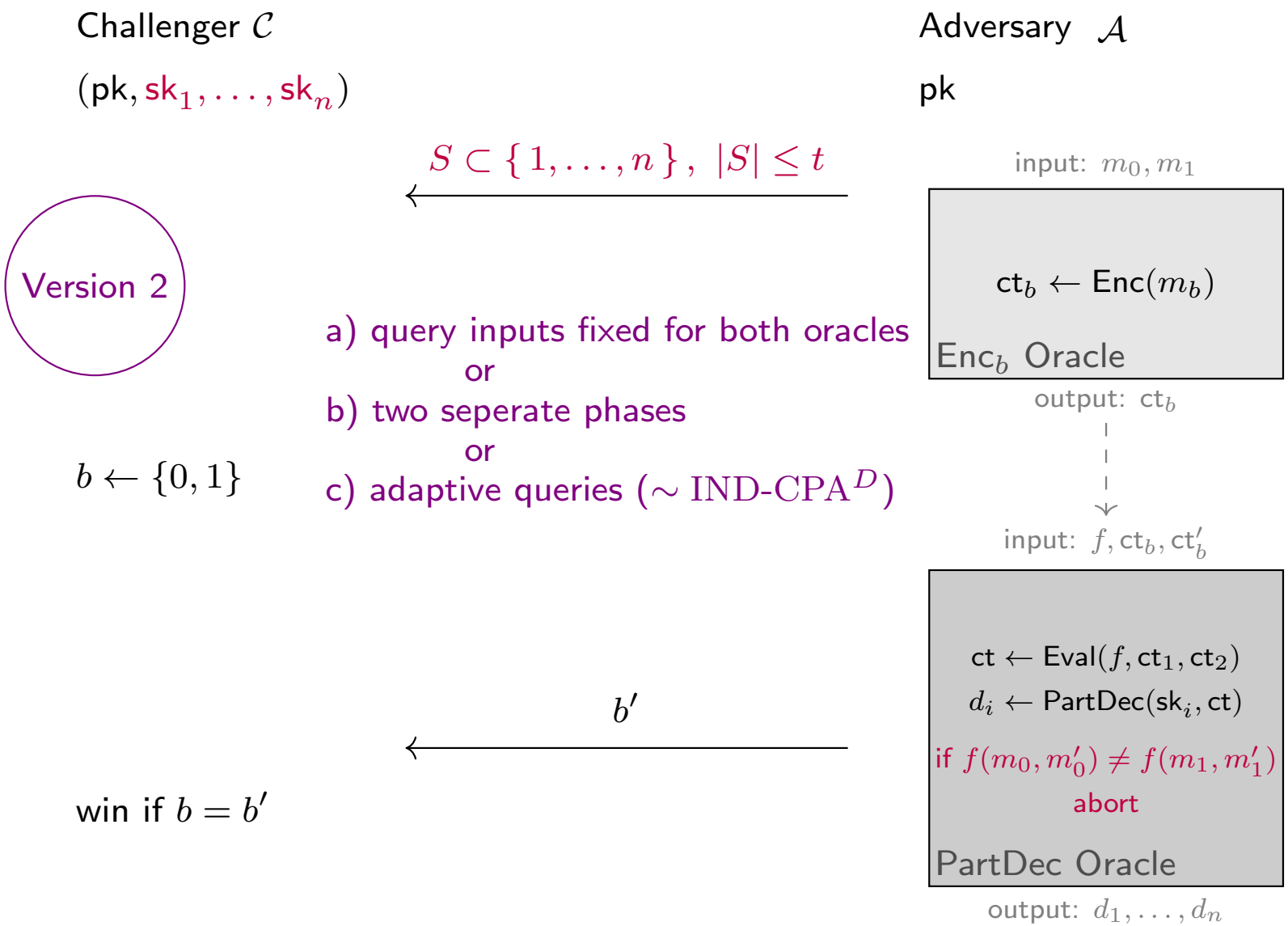

Research Directions

- Part 1: Different approach than adding noise?
- Part 2: Different approach for linear secret sharing?
- Part 3: Different noise analysis?
- Part 4: Best efficiency-security trade-off?

Wrap-Up

🏳 Hopefully you have now a rough idea:

- Part 1: *What the blueprint of ThFHE is!*
- Part 2: *What suitable secret sharings are!*
- Part 3: *How to use flooding noise!*
- Part 4: *How to define security!*
- **What research directions there are :-)**

Any questions or interested in my research?

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Thanks!


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