Simple Threshold Fully Homomorphic Encryption from LWE with Polynomial Modulus

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Simple Threshold **Fully Homomorphic Encryption** from LWE with Polynomial Modulus

Fully Homomorphic Encryption (FHE)

FHE scheme:

- $\mathsf{KGen}(1^\lambda) \to (\mathsf{pk}, \mathsf{sk})$ — λ security parameter
- $\mathsf{Enc}(\mathsf{pk}, m) \to \mathsf{ct}$ — $\mathcal{M}$ message space
- $\mathsf{Eval}(\mathsf{pk}, f, \mathsf{ct}_1, \mathsf{ct}_2) \to \mathsf{ct}'$ — $f : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$
- $\mathsf{Dec}(\mathsf{sk}, \mathsf{ct}') \to m'$

Properties:

- Correctness — $\mathsf{Dec}(\mathsf{Eval}(f, \mathsf{Enc}(m_1), \mathsf{Enc}(m_2))) = f(m_1, m_2)$
- Semantic security — IND-CPA

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Problem:

- sk single point of failure

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1

Simple **Threshold** Fully Homomorphic Encryption from LWE with Polynomial Modulus

Threshold FHE

t-out-of-n Threshold FHE scheme:

- $\mathsf{KGen}(1^\lambda) \to (\mathsf{pk}, \mathsf{sk}_1, \ldots, \mathsf{sk}_n)$
- $\mathsf{Enc}(\mathsf{pk}, m) \to \mathsf{ct}$
- $\mathsf{Eval}(\mathsf{pk}, f, \mathsf{ct}_1, \mathsf{ct}_2) \to \mathsf{ct}'$
- $\mathsf{PartDec}(\mathsf{sk}_i, \mathsf{ct}') \to d_i$ $\quad i \in \{1, \ldots, n\}$
- $\mathsf{Combine}(\{ d_i \}_{i \in S}) \to m'$ $\quad S \subset \{1, \ldots, n\}$

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Properties:

- Correctness — for $|S| > t$ recover correct message
- Partial decryption security — for $|S| \leq t$ no information is leaked
- Semantic security — FHE is IND-CPA

Applications:

- Storing sensitive data
- Electronic voting protocols
- Multiparty computations

Simple Threshold Fully Homomorphic Encryption from LWE with Polynomial Modulus

Simple Threshold FHE, First Trial

linear function in
enc randomness and sk

Ingredients:

1) FHE with β nearly linear decryption:

- $\langle \mathsf{ct}, \mathsf{sk} \rangle = f(m_1, m_2) + e_{\mathsf{ct}}$
- $\|e_{\mathsf{ct}}\|_\infty \leq \beta$

Simple Threshold FHE, First Trial

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2) t-out-of-n linear secret sharing scheme:
   - $\mathsf{Share}(\mathsf{sk}) \to \mathsf{sk}_1, \ldots, \mathsf{sk}_n$
   - $\mathsf{Rec}(\{ \langle y, \mathsf{sk}_i \rangle \}_{i \in S}) = \langle y, \mathsf{sk} \rangle$ $\quad |S| > t$

Simple Threshold FHE, First Trial

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1st Trial construction:

- KGen: compute $\mathsf{Share}(\mathsf{sk}) \to (\mathsf{sk}_1, \ldots, \mathsf{sk}_n)$
- PartDec: compute $d_i = \langle \mathsf{ct}, \mathsf{sk}_i \rangle$
- Combine: compute $\mathsf{Rec}(\{ d_i \})$

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- Combine: compute $\mathsf{Rec}(\{ d_i \}) = \mathsf{Rec}(\{ \langle \mathsf{ct}, \mathsf{sk}_i \rangle \}) =^{(2)} \langle \mathsf{ct}, \mathsf{sk} \rangle =^{(1)} f(m_1, m_2) + e_{\mathsf{ct}}$

Problem:

- Corrupted parties learn e_{ct}
- After enough decryptions, can recover sk

Simple Threshold FHE, Second Trial

Ingredients:

1) FHE with nearly linear decryption:

- $\langle \mathsf{ct}, \mathsf{sk} \rangle = f(m_1, m_2) + e_{\mathsf{ct}}$ e_{ct} ciphertext noise

2) t-out-of-n linear secret sharing scheme:

- $\mathsf{Rec}(\{ \langle y, \mathsf{sk}_i \rangle \}) = \langle y, \mathsf{sk} \rangle$

2nd Trial Construction:

- PartDec: compute $d_i = \langle \mathsf{ct}, \mathsf{sk}_i \rangle + e_{flood,i}$ e_{flood} flooding noise
- Combine: compute $\mathsf{Rec}(\{ d_i \})$

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- Combine: compute $\mathsf{Rec}(\{ d_i \})$
  $= \mathsf{Rec}(\{ \langle \mathsf{ct}, \mathsf{sk}_i \rangle + e_{flood,i} \}) = f(m_1, m_2) + e_{\mathsf{ct}} + \mathsf{Rec}(\{ e_{flood,i} \})$

Problem:

- Is $\mathsf{Rec}(\{ e_{flood,i} \})$ still small? Needed for correctness!
- For Shamir secret sharing: no

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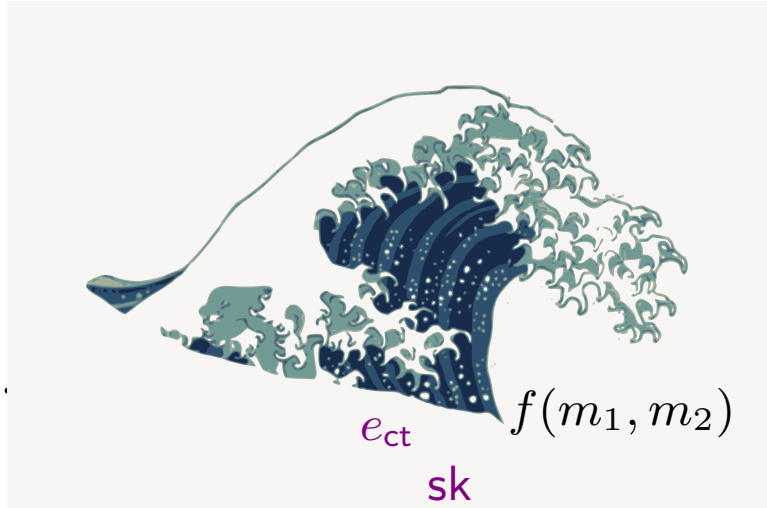

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Simple Threshold FHE, Final [BD10]

Ingredients:

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   - $\langle \mathsf{ct}, \mathsf{sk} \rangle = f(m_1, m_2) + e_{\mathsf{ct}}$ e_{ct} ciphertext noise

2) t-out-of-n linear secret sharing scheme with small reconstruction:
   - $\mathsf{Rec}(\{ \langle y, \mathsf{sk}_i \rangle \}) = \langle y, \mathsf{sk} \rangle$
   - if e_i small, then $\mathsf{Rec}(\{ e_i \})$ also small

Construction:
   - PartDec: compute $d_i = \langle \mathsf{ct}, \mathsf{sk}_i \rangle + e_{flood,i}$ e_{flood} flooding noise
   - Combine: compute $\mathsf{Rec}(\{ d_i \})$

Simple Threshold FHE, Final [BD10]

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Partial Decryption Security

Two worlds:

- Real: e_{ct} and e_{flood}
- Simulated: only e_{flood}

How close are they? [BD10] measures with statistical distance Δ

$$\Delta(\mathsf{Real}, \mathsf{Sim}) \leq \Delta(e_{flood} + e_{\mathsf{ct}}, e_{flood}) \leq \mathsf{negl}(\lambda)$$

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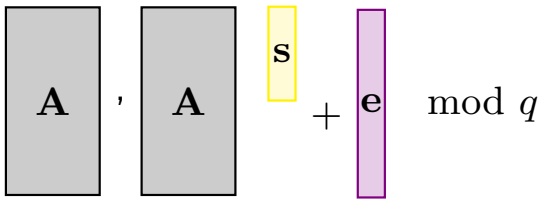
Problem:

- $\|e_{flood}\|$ needs to be super-polynomially larger than $\|e_{\mathsf{ct}}\|$
- LWE-based constructions: $\|e_{flood}\| \sim$ LWE modulus q and $\|e_{\mathsf{ct}}\| \sim$ LWE noise $\mathbf{e}$, thus super-polynomial modulus-noise ratio
  - Larger parameters
  - Easier problem



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💡 Idea: change the measure!

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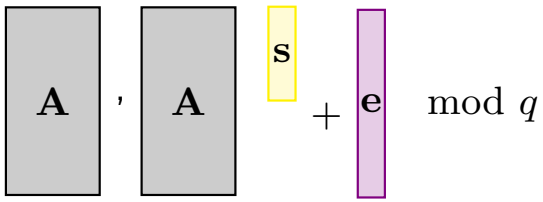

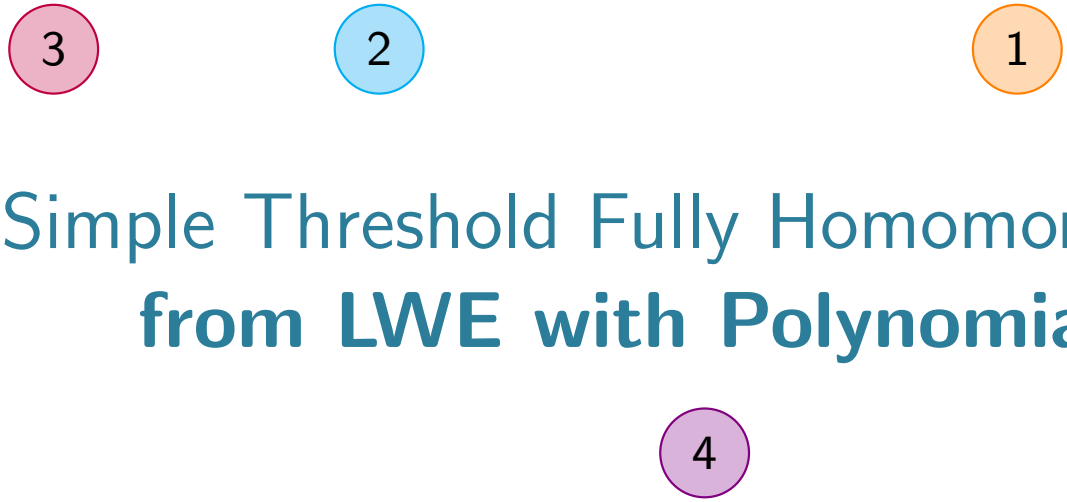
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from LWE with Polynomial Modulus
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Improved Noise Flooding via Rényi Divergence 1/2

Let P, Q be discrete probability distributions

In [BD10]: Statistical Distance $\Delta(P, Q) = \frac{1}{2} \sum_{x \in \mathrm{Supp}(P)} |P(x) - Q(x)|$

In our work: Rényi Divergence

$$\mathrm{RD}(P, Q) = \sum_{x \in \mathrm{Supp}(P)} \frac{P(x)^2}{Q(x)}$$

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Both fulfill the **probability preservation property** for an event E:

[BD10]:	$P(E)$	$\leq$	$\Delta(P,Q) + Q(E)$	(additive)
Our work:	$P(E)^2$	$\leq$	$\mathrm{RD}(P,Q) \cdot Q(E)$	(multiplicative)

- $Q(E)$ negligible $\Rightarrow$ $P(E)$ negligible
- $\Delta(P,Q) =^!$ negligible and $\mathrm{RD}(P,Q) =^!$ **constant**

Improved Noise Flooding via Rényi Divergence 2/2

Two worlds:

- Real: e_{ct} and e_{flood}
- Simulated: only e_{flood}

How close are they?

$$\Delta(\mathsf{Real}, \mathsf{Sim}) \leq \Delta(e_{flood} + e_{\mathsf{ct}}, e_{flood}) \leq \mathsf{negl}(\lambda)$$

$$\mathsf{RD}(\mathsf{Real}, \mathsf{Sim}) \leq \mathsf{RD}(e_{flood} + e_{\mathsf{ct}}, e_{flood}) \leq \mathsf{constant}$$

Advantage:

- $\|e_{flood}\|$ only needs to be polynomially larger than $\|e_{\mathsf{ct}}\|$
- LWE-based constructions: polynomial modulus-noise ratio

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Disadvantage:

1) Rényi divergence depends on the number of issued partial decryptions
   $\rightarrow$ from simulation-based to game-based security notion
2) Works well with search problems, not so well with decision problems

3 2 1

Simple Threshold Fully Homomorphic Encryption from LWE with Polynomial Modulus

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5 **Challenges on the way**

Our Approaches

1) Rényi divergence depends on the number of issued partial decryptions
   - → use a IND-CPA security notion for Threshold FHE [JRS17]
   - → add a priori bound on partial decryption queries

2) Works well with search problems, not so well with decision problems
   - → define a one-way security notion for Threshold FHE
   - → show how to lift one-way security to IND-CPA security for Threshold FHE use Goldreich-Levin hardcore bits

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Game-Based Security for t-out-of-n Threshold FHE

Challenger $\mathcal{C}$ | Adversary $\mathcal{A}$

$(\mathsf{pk}, \mathsf{sk})$

pk

m_0, m_1 ←

$b \leftarrow \{0, 1\}$

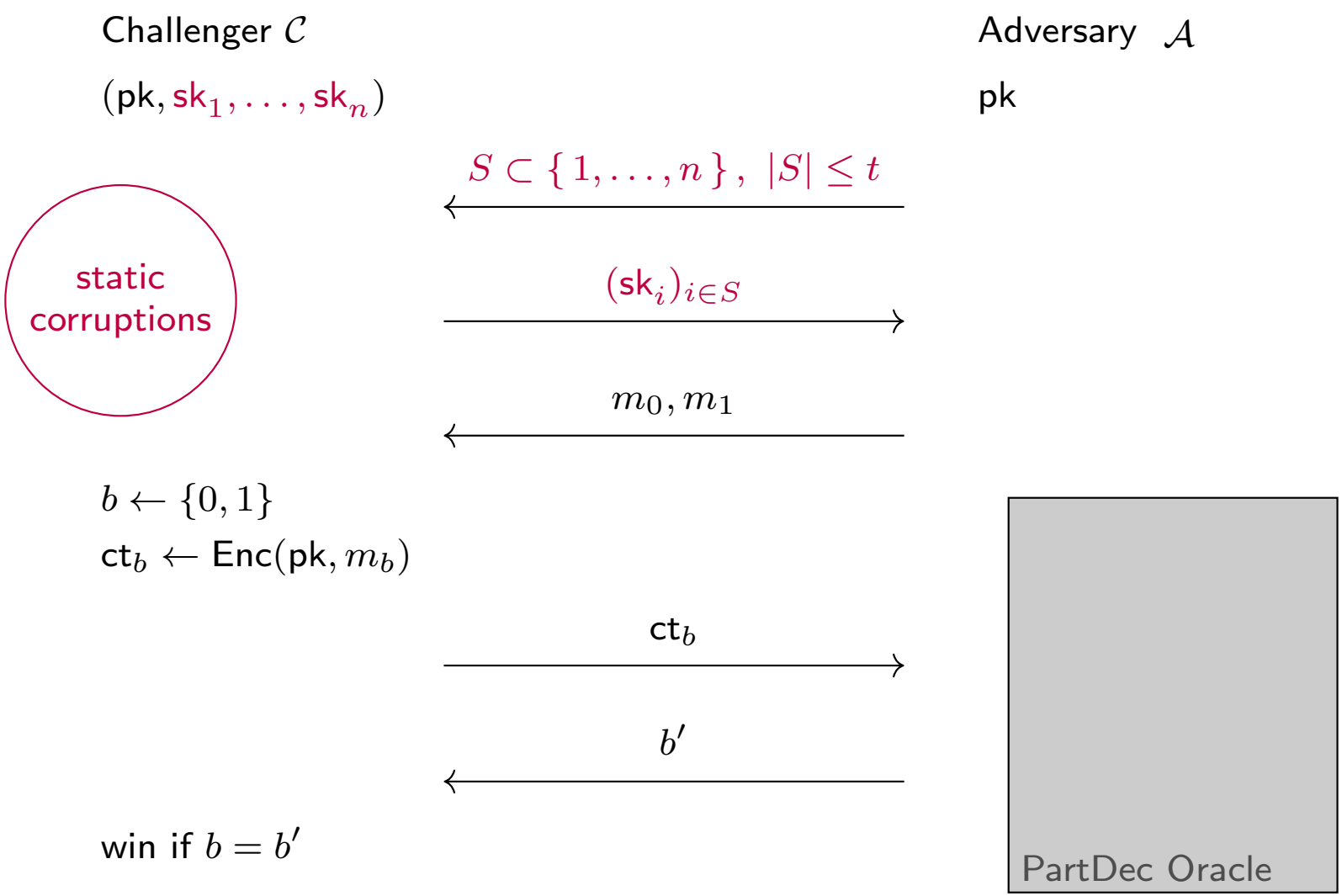
$\mathsf{ct}_b \leftarrow \mathsf{Enc}(\mathsf{pk}, m_b)$

ct_b →

b' ←

win if $b = b'$

Game-Based Security for t-out-of-n Threshold FHE



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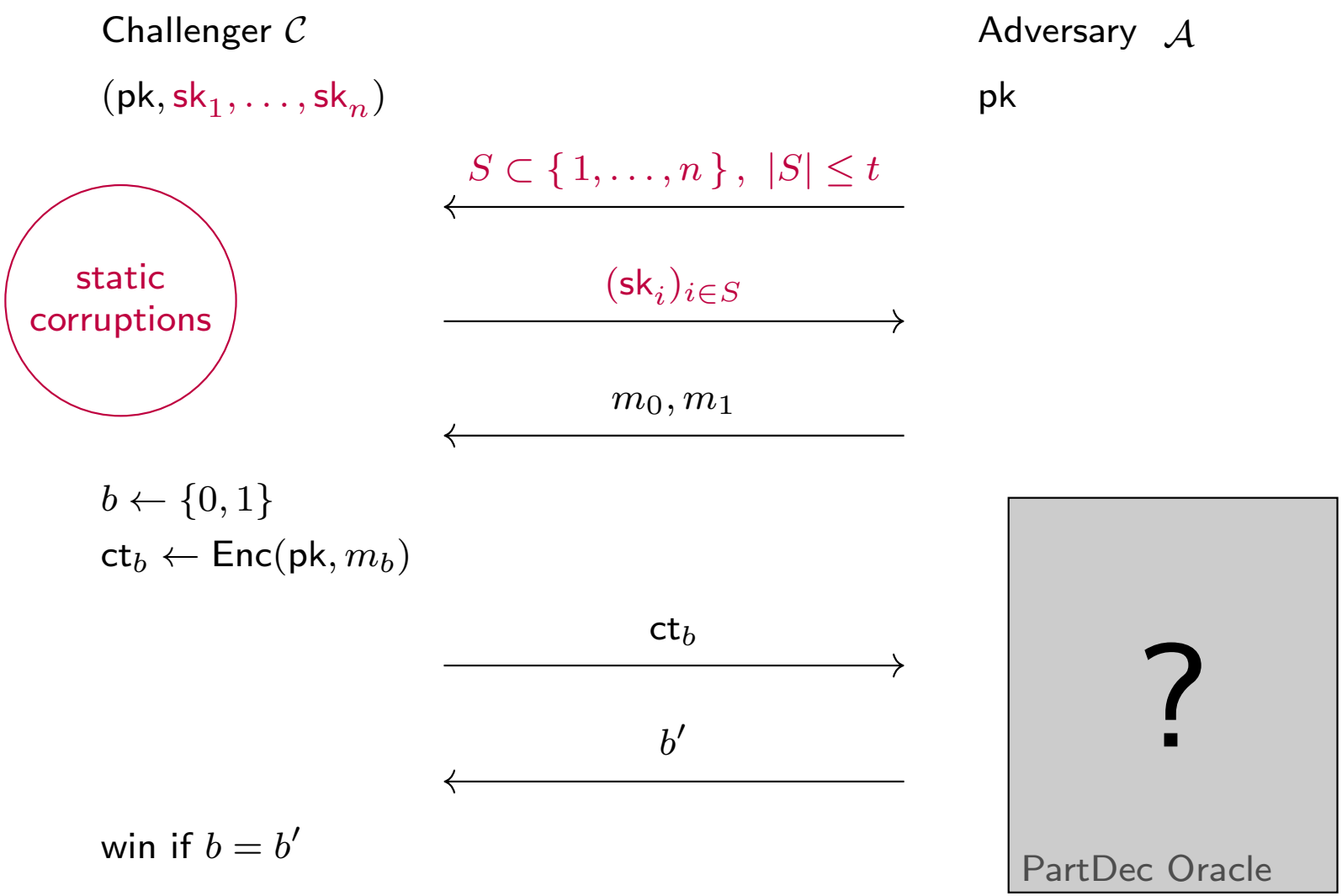

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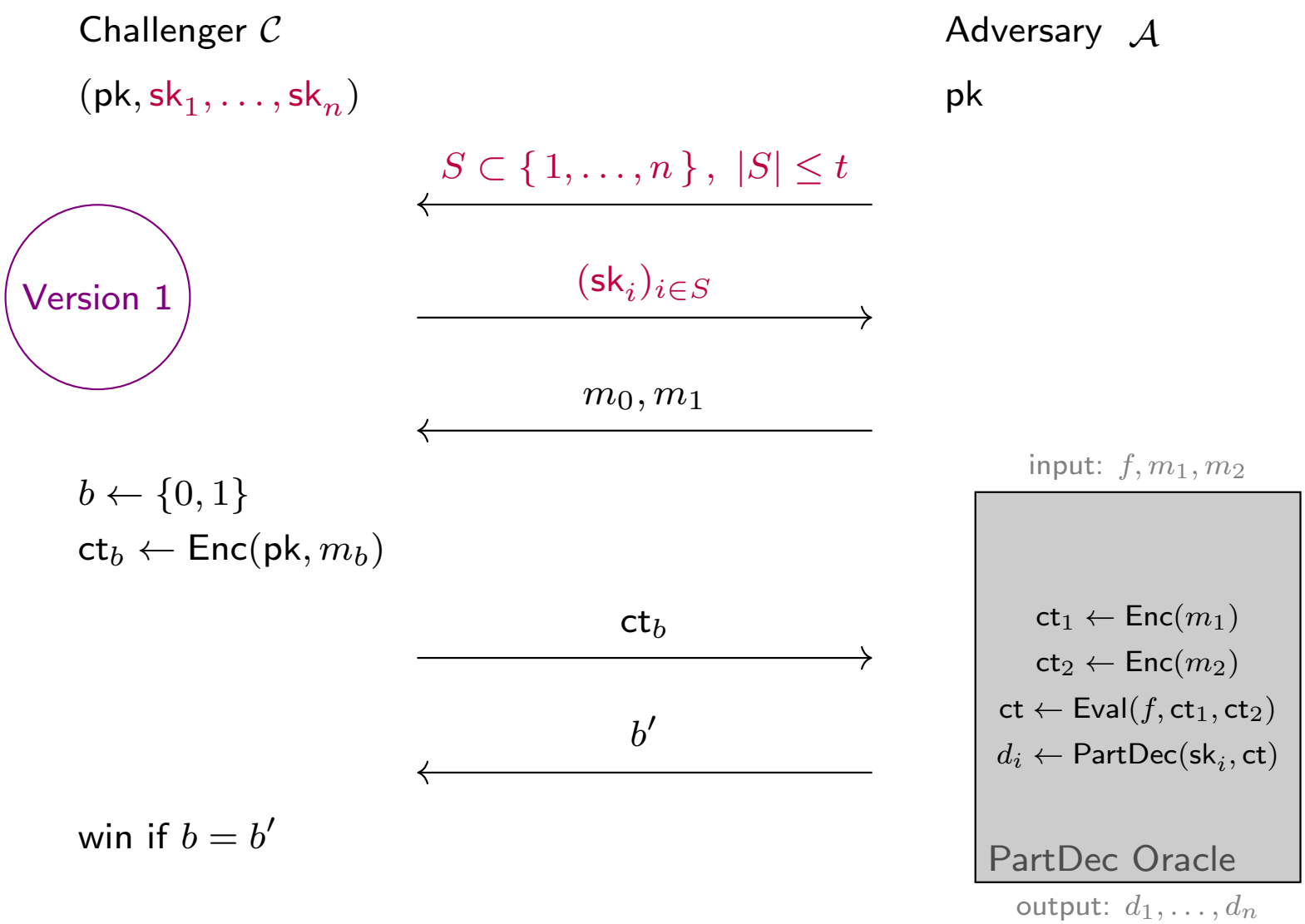

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$(\mathsf{pk}, \mathsf{sk}_1, \ldots, \mathsf{sk}_n)$

Adversary $\mathcal{A}$

pk

$S \subset \{1, \ldots, n\},\ |S| \leq t$

Version 1

$(\mathsf{sk}_i)_{i \in S}$

m_0, m_1

$b \leftarrow \{0, 1\}$

$\mathsf{ct}_b \leftarrow \mathsf{Enc}(\mathsf{pk}, m_b)$

ct_b

b'

win if $b = b'$

input: f, m_1, m_2

$\mathsf{ct}_1 \leftarrow \mathsf{Enc}(m_1)$

$\mathsf{ct}_2 \leftarrow \mathsf{Enc}(m_2)$

$\mathsf{ct} \leftarrow \mathsf{Eval}(f, \mathsf{ct}_1, \mathsf{ct}_2)$

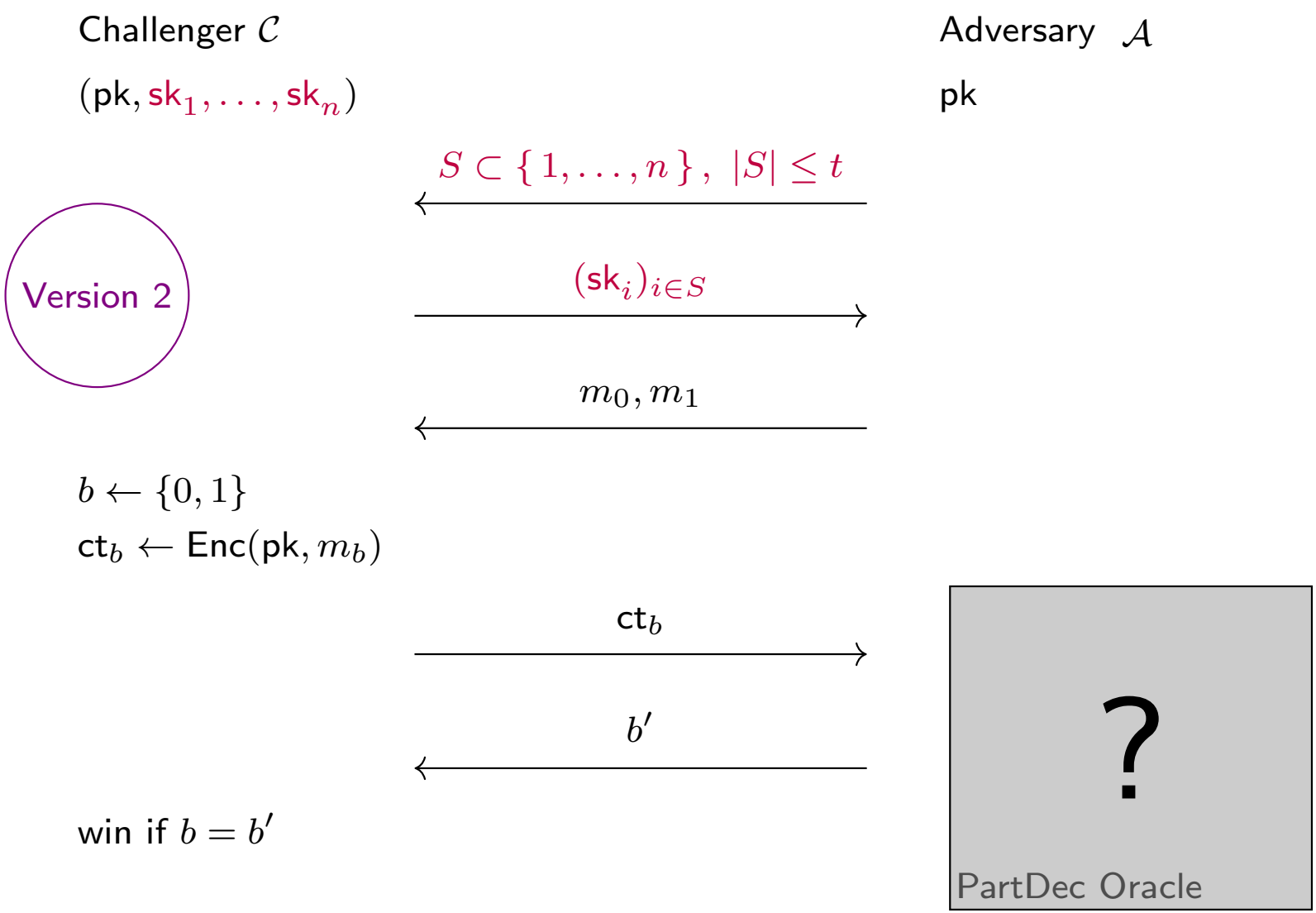
$d_i \leftarrow \mathsf{PartDec}(\mathsf{sk}_i, \mathsf{ct})$

PartDec Oracle

output: $d_1, \ldots, d_n$

Problem: queries to PartDec are independent of ct_b

ℓ-IND-CPA Security for t-out-of-n Threshold FHE [JRS17]



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Version 2

Enc_b Oracle

$b \leftarrow \{0, 1\}$

b'

win if $b = b'$

PartDec Oracle

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input: m_0, m_1

$\mathsf{ct}_b \leftarrow \mathsf{Enc}(m_b)$

Enc_b Oracle

output: ct_b

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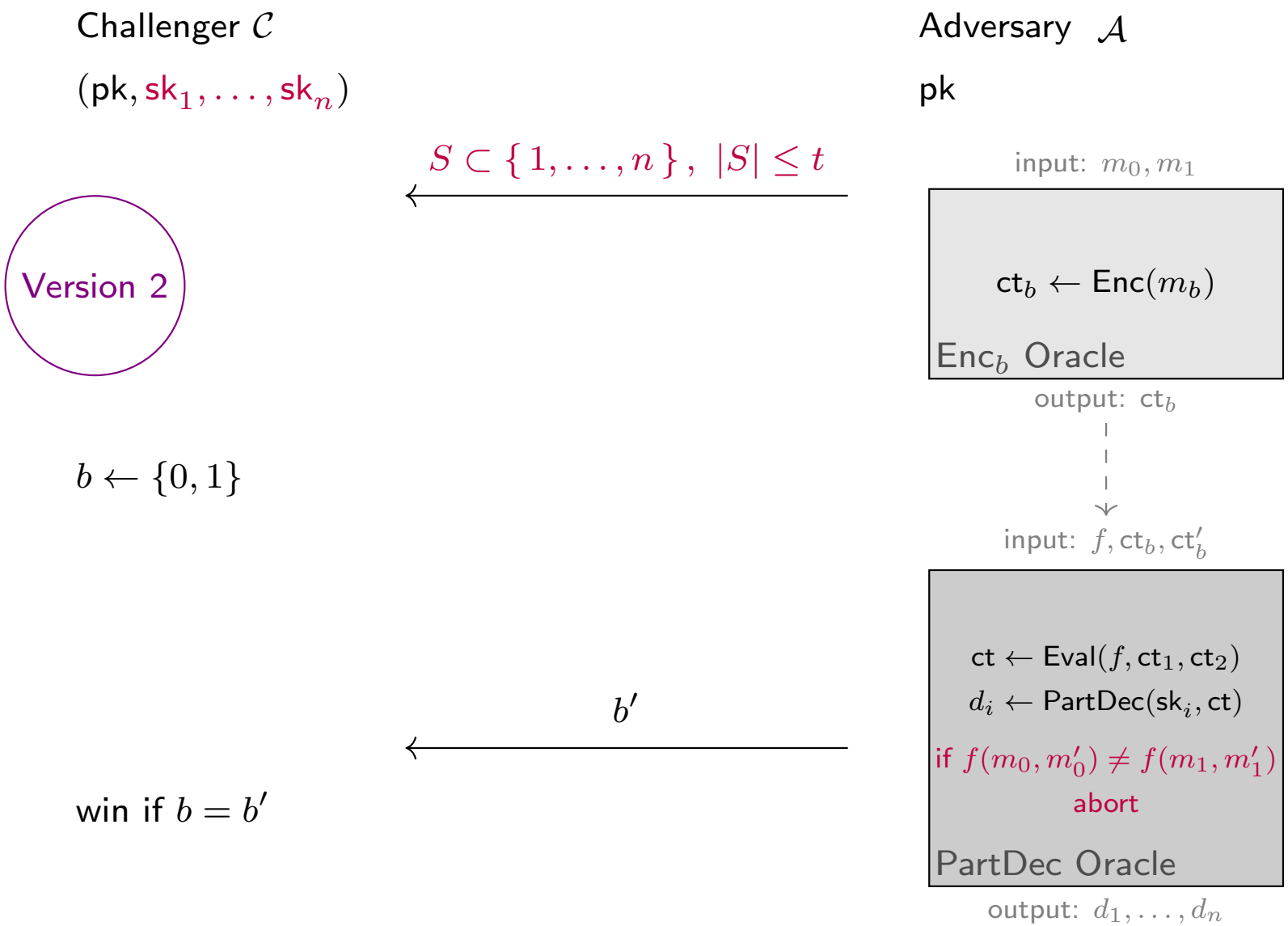

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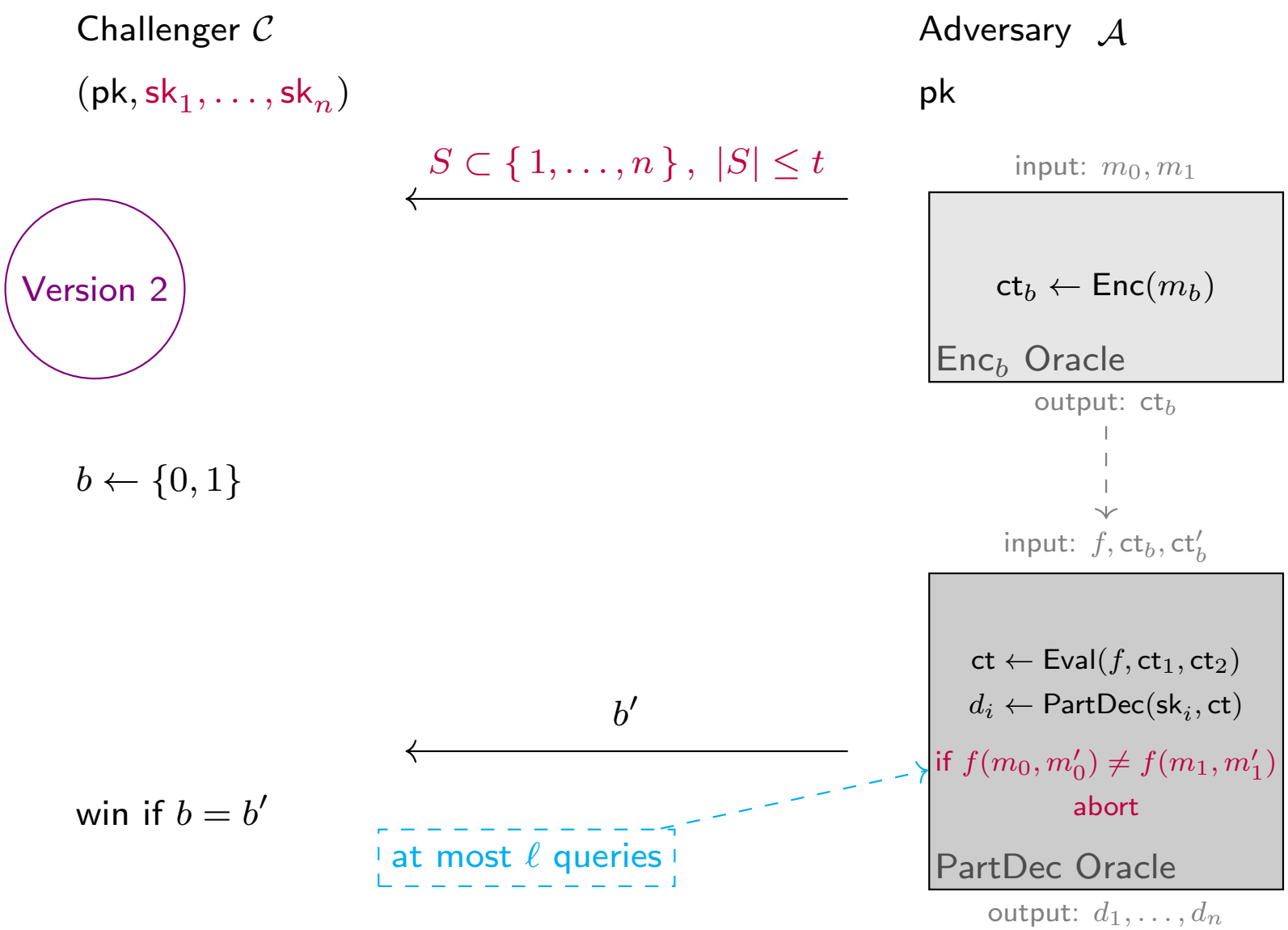

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Conclusion

Wrapping Up

My intuition on Rényi divergence:

- In search problems always beneficial
- In decision problems depends on your setting

Related Works:

- [CSS+22] Use Rényi divergence directly for IND-CPA, but much weaker model
- [LMSS22] Use Rényi differential privacy for IND-CPA, but much worse parameters

Open Problems:

- Adaptive corruptions $\rightarrow$ Master project Michael & Magdalena
- Alternative noise flooding?

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Thank you.


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