CELESTIAL MECHANICS (Fall 2012): COMPUTER EXERCISES I

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1. **Solution of Kepler's equation**
2. **Study elliptical orbits**
3. **Compare different anomalies:** E, f, M
4. **Numerical evaluation of time averages on elliptical orbit**

1. Solution of Kepler's equation (kepler.pro, kepler_demo.pro)

The position $\vec{R}$ and velocity $\vec{V}$ in an elliptic orbit can not be expressed in a closed form as a function of time t. However, $\vec{R}$ and $\vec{V}$ can calculated either by using the true anomaly f or the eccentric anomaly E. To obtain either f or E one needs to first solve the Kepler's equation, connecting E and the mean anomaly M:

$$M = \sqrt{\mu/a^3}(t - \tau) = E - \epsilon \sin E \tag{1}$$

where

$$\begin{aligned} \mu &= G(m_1 + m_2), \ G = \text{gravitational constant}, \ m_1, m_2 = \text{masses} \\ \tau &= \text{pericenter time}, \ a = \text{semimajor axis}, \ \epsilon = \text{eccentricity} \end{aligned}$$

To solve E as a function of M from Kepler's equation, $M = E - \epsilon \sin E$, one can use for example substitution iteration. Denote $x = \epsilon \sin E$, so that $M = E - x$. From this E can be expressed in the form $E = M + x$, and one can set up an iteration

$$E_{i+1} = M + \epsilon \sin E_i, \quad i = 0, 1, 2, ... \tag{2}$$

using for example $E_0 = M$ as the first approximation. The iteration is continued until $|E_{i+1} - E_i|$ is sufficiently small, say less than 10^{-10}.

Another possible way to solve Kepler's equation is to use Newton's method for finding the root of the equation $g(E) = E - \epsilon \sin E - M = 0$. This is done by the iteration schema

$$E_{i+1} = E_i - g(E_i)/g'(E_i), \quad i = 0, 1, 2, ... \tag{3}$$

where $g'(E_i) = dg(E_i)/dE_i = 1 - \epsilon \cos E_i$. Using here $E_0 = M$ as a first guess gives $E_1 = M + \epsilon \sin M/(1 - \epsilon \cos M)$, which can also be used as an improved starting value in the schema of Eq. **??**.

WHAT TO DO: a) Write an IDL-procedure that solves E for a given M and ϵ, using either of the above described methods. b) Solve the eccentric anomaly E corresponding to mean anomaly $M = 45°$, for $\epsilon = 0.01, \ 0.05, \ 0.50, \ 0.90, \ 0.99$. How many iteration steps are required to reach an accuracy of $0.01°$ in each case? Try also what happens for $M = 359°$!

2. Study elliptical orbits (elliptic_demo.pro)

a) In terms of eccentric anomaly E the position in elliptic orbit can be represented as

$$\vec{R} = \vec{A}(\cos E - \epsilon) + \vec{B} \sin E \tag{4}$$

where $\vec{A}$ and $\vec{B}$ are perpendicular vectors in the plane of the orbit, and $\vec{A}$ points to the pericenter of the orbit ($|\vec{A}| = a,\ |\vec{B}| = b = a\sqrt{1-\epsilon^2}$). Choose the x-axis parallel to $\vec{A}$ and plot elliptic orbits with eccentricity $\epsilon = 0.0, 0.1, ..., 0.8$. Use $a = 1$ for the semimajor axis, and use for example 200 uniformly chosen E values from 0 to 2π to calculate (x, y) points defining the orbit.

b) In terms of the true anomaly f (= polar angle from the pericenter), the length of the radius vector is given by

$$r = |\vec{R}| = \frac{a(1-\epsilon^2)}{1+\epsilon \cos f} \tag{5}$$

As above, take 200 f values from 0 to 2π, and calculate (x, y) points defining the orbit.

c) Uniformly chosen E or f values, while easy to use for plotting the orbit, do not represent the movement of the particle in its orbit during equal time intervals. To do this, one has to take equally spaced points in time, or what is equivalent, equally spaced values of mean anomaly. Repeat what was done in a) and b) by choosing 200 uniformly chosen M values from 0 to 2π. For each of these, solve Kepler's equation to get E.

d) Make also a plot with fewer points (say 50), for $\epsilon = 0.0$ and 0.8, illustrating the change in speed at various parts of the orbit.

e) Make also plots of the length of the radius vector as a function of time. Choose equally spaced M values, calculate E, and use the equation

$$r = |\vec{R}| = a(1 - \epsilon \cos E) \tag{6}$$

Also check how the central force varies with time (proportional to $1/r^2$). Plot also $1/r^3$ versus time: $2D/r^3$ represents the magnitude of the tidal force a particle with a diameter D would experience (tidal force = difference in the force felt by the different parts of the particle – differentiate $1/r^2$ with respect to r)

f) Play also with velocity. In terms of E,

$$\vec{V} = \dot{\vec{R}} = (-\vec{A}\sin E + \vec{B}\cos E)\dot{E}, \tag{7}$$

where the derivative $\dot{E}$ is obtained by differentiating Kepler's equation,

$$\dot{E} = \frac{\sqrt{\mu/a^3}}{(1-\epsilon\cos E)} \tag{8}$$

You can choose the units by setting $\mu = 1,\ a = 1$ (what is the orbital period with these values?)

From the velocity components one can evaluate numerically the speed $v = |\vec{V}|$ as a function of time (via E), as well as the radial velocity $v_r = \vec{V}\cdot\vec{R}/r$, and the tangential velocity $v_t = |\vec{V} - v_r\vec{R}/r|$. Compare these with the analytical formulas derived in the lectures (exercise I.6),

$$\begin{aligned} v_r &= v_{circ}\ a/r\ \epsilon\ \sin E \\ v_t &= v_{circ}\ b/r \\ v &= v_{circ}\sqrt{\frac{1+\epsilon\cos E}{1-\epsilon\cos E}} \end{aligned} \tag{9}$$

where $v_{circ} = \sqrt{\mu/a}$ is the speed on a circular orbit.

g) Finally, study how the kinetic energy and potential energy (of the relative orbit per unit mass) vary on different positions in the eccentric orbit, $E_{kin} = \frac{1}{2}v^2$ and $E_{pot} = -\mu/r$. Also verify that their sum equals the total energy $h = -\frac{\mu}{2a}$. Also check that the angular momentum $\vec{R}\times\vec{V}$ is constant, with absolute value of $\sqrt{\mu a(1-\epsilon^2)}$.

3. Compare different anomalies: E, f, M (ano_demo.pro)

Using the previous results, you can choose equally spaced M values, and calculate the corresponding E. From E you can obtain f, with the formulas derived in lectures

$$\cos f = \frac{\cos E - \epsilon}{1 - \epsilon \cos E}, \tag{10}$$

$$\sin f = \frac{\sqrt{1-\epsilon^2}\sin E}{1 - \epsilon \cos E}, \tag{11}$$

and using the IDL arctan-function with two arguments, f =**atan(sin f, cos f)**. Make plots showing E and f vs. M for different eccentricities.

Plot also $E - M$ and $f - M$ vs. M, and compare the differences to the first terms of series expansions

$$\begin{aligned} E &= M + \epsilon \sin M + \epsilon^2(\frac{1}{2}\sin 2M) + \epsilon^3(\frac{3}{8}\sin 3M - \frac{1}{8}\sin M) + ..., \quad (12) \\ f &= M + 2\epsilon \sin M + \epsilon^2(\frac{5}{4}\sin 2M) + \epsilon^3(\frac{13}{12}\sin 3M - \frac{1}{4}\sin M) + ... \end{aligned}$$

4. Numerical evaluation of time averages (aver_demo.pro)

In the lectures (exercise I.7) the time averages of various functions (e.g. $< r >$, $< v^2 >$) were evaluated analytically, by utilizing a change of variables from time to either eccentric anomaly or to true anomaly. Make the same numerically, utilizing the fact that

$$< x >= \frac{1}{2\pi} \int_0^{2\pi} x(m)\ dM \tag{13}$$

which can be numerically evaluated as

$$< x >= \frac{1}{n} \sum x(M_i), \quad M_i = i\ (2\pi/n), \quad i = 0, ..., n-1 \tag{14}$$

Do this for quantities $r, r^2, 1/r, 1/r^{-2}, 1/r^{-3}, v^2$, for different eccentricities, and compare to analytic formulas derived in exercise I.7:

$$\begin{aligned}
< r > &= a(1 + \epsilon^2/2), \\
< r^2 > &= a^2(1 + 3\epsilon^2/2), \\
< 1/r > &= 1/a, \\
\\
< 1/r^2 > &= 1/a^2(1 - \epsilon^2)^{-1/2}, \\
< 1/r^3 > &= 1/a^3(1 - \epsilon^2)^{-3/2}, \\
\\
< v^2 > &= \mu/a
\end{aligned}$$