CELESTIAL MECHANICS (Fall 2006): COMPUTER EXERCISES III

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8. Numerical integration of 2-body relative orbit

Using I and II order Taylor series

Using Runge-Kutta 4 method

8. Numerical integration of 2-body relative orbit

- Dynamical equation for two-body relative motion,

$$\ddot{\vec{R}} = -G(m_1 + m_2)\frac{\vec{R}}{r^3} = -\mu\frac{\vec{R}}{r^3}$$

- General principle: divide the second order differential equation for radius vector $\vec{R}$ into two first order differential equations for vectors $\vec{R}$ and $\vec{V}$

$$\begin{aligned} \dot{\vec{R}} &= \vec{V} \\ \dot{\vec{V}} &= \ddot{\vec{R}} \end{aligned}$$

- Choose a Cartesian coordinate system $\vec{R} = [x, y, z]$, $\vec{V} = [v_x, v_y, v_z]$
  In component form: $(r = \sqrt{x^2 + y^2 + x^2}))$

$$\begin{aligned} \dot{x} &= v_x \\ \dot{y} &= v_y \\ \dot{z} &= v_z \\ \dot{v}_x &= \ddot{x} &&= -\mu\ x/r^3 \\ \dot{v}_y &= \ddot{y} &&= -\mu\ y/r^3 \\ \dot{v}_z &= \ddot{z} &&= -\mu\ z/r^3 \end{aligned}$$

- This is an *initial value problem (alkuarvoproblеema)* : position, velocity known at a given instant of time $\Rightarrow$ orbit for any desired time interval.
  6 initial values needed

- We next look at some simple methods for the solution of this problem: truncated Taylor expansions, Leap-frog method, RK4

A. First order truncated Taylor expansion: $f(t+\Delta t) = f(t) + f'(t)\Delta t$

- Simplest possible method of solution for any first order differential equation

  i) at time $t = t_o$ the position $[x_0, y_0, z_0]$ and velocity $[(v_x)_0, (v_y)_0, (v_z)_0]$

  ii) Approximation at time $t_1 = t_0 + \Delta t$:

$$
\begin{aligned}
x_1 &= x_0 + \dot{x}\ \Delta t & \dot{x} &= (v_x)_0 \\
y_1 &= y_0 + \dot{y}\ \Delta t & \dot{y} &= (v_y)_0 \\
z_1 &= z_0 + \dot{z}\ \Delta t & \dot{z} &= (v_z)_0 \\
(v_x)_1 &= (v_x)_0 + \dot{v_x}\ \Delta t & \dot{v_x} &= \ddot{x}_0 = -\mu\ x_0/{r_0}^3 \\
(v_y)_1 &= (v_y)_0 + \dot{v_y}\ \Delta t & \dot{v_y} &= \ddot{y}_0 = -\mu\ y_0/{r_0}^3 \\
(v_z)_1 &= (v_z)_0 + \dot{v_z}\ \Delta t & \dot{v_z} &= \ddot{z}_0 = -\mu\ z_0/{r_0}^3
\end{aligned}
$$

  (NOTE: derivatives evaluated at t_0 $\Rightarrow$ must use $r_0 = \sqrt{{x_0}^2 + {y_0}^2 + {z_0}^2}$) in velocity derivatives

  iii) Repeat the step ii) for the next time steps $t_i \rightarrow t_i + \Delta t$,
  until the whole desired time interval is covered

- Easy to program, extremely inaccurate

  error/step proportional to $(\Delta t)^2$, at the time interval t_{tot} one needs $t_{tot}/\Delta t$ steps

  $\Rightarrow$ (in the worst case) the total error proportional to Δt

  $\Rightarrow$ NEVER USED! (except in initial tests)

B. Second order Taylor expansion: $f(t+\Delta t) = f(t) + f'(t)\Delta t + \frac{1}{2}f''(t)(\Delta t)^2$

Approximation at time $t_1 = t_0 + \Delta t$:

$$\begin{aligned}
x_1 &= x_0 + \dot{x}\ \Delta t + \frac{1}{2}\ddot{x}\ (\Delta t)^2 \\
y_1 &= y_0 + \dot{y}\ \Delta t + \frac{1}{2}\ddot{y}\ (\Delta t)^2 \\
z_1 &= z_0 + \dot{z}\ \Delta t + \frac{1}{2}\ddot{z}\ (\Delta t)^2 \\
(v_x)_1 &= (v_x)_0 + \dot{v_x}\ \Delta t + \frac{1}{2}\ddot{v}_x\ (\Delta t)^2 \\
(v_y)_1 &= (v_y)_0 + \dot{v_y}\ \Delta t + \frac{1}{2}\ddot{v}_y\ (\Delta t)^2 \\
(v_z)_1 &= (v_z)_0 + \dot{v_z}\ \Delta t + \frac{1}{2}\ddot{v}_z\ (\Delta t)^2
\end{aligned}$$

Here $\ddot{v}_x = \dddot{x}$, $\ddot{v}_y = \dddot{y}$ and $\ddot{v}_z = \dddot{z}$.

- How to construct $\dddot{\vec{R}}_0 \equiv [\dddot{x}_0, \dddot{y}_0, \dddot{z}_0]$?

  Obtain $\dddot{\vec{R}}$ by differentiating $-\mu\vec{R}/r^3 \quad \Rightarrow$

$$\dddot{\vec{R}} = d/dt\ (\ddot{\vec{R}}) = -\mu\frac{\dot{\vec{R}}r^3 - 3r^2\dot{r}\vec{R}}{r^6} = -\mu/r^3\left(\dot{\vec{R}} - 3\frac{(\vec{R}\cdot\dot{\vec{R}})\vec{R}}{r^2}\right),$$

,

$\dot{r}$ is calculated using the formula $r^2 = \vec{R}\cdot\vec{R} \quad \Rightarrow d(r^2)/dt = 2r\dot{r} \equiv 2\vec{R}\cdot\dot{\vec{R}}$

- In the component form

$$\begin{aligned}
\dddot{x} &= -\mu/r^3\left(v_x - 3\frac{(\vec{R}\cdot\dot{\vec{R}})}{r^2}x\right) \qquad & \vec{R}\cdot\dot{\vec{R}} &= xv_x + yv_y + zv_z \\
\dddot{y} &= -\mu/r^3\left(v_y - 3\frac{(\vec{R}\cdot\dot{\vec{R}})}{r^2}y\right) \qquad & r &= \sqrt{x^2+y^2+z^2} \\
\dddot{z} &= -\mu/r^3\left(v_z - 3\frac{(\vec{R}\cdot\dot{\vec{R}})}{r^2}z\right)
\end{aligned}$$

- The error/step is proportional to $(\Delta t)^3$ (worst case total error proportional to $(\Delta t)^2$)

  $\Rightarrow$ allows longer steps than the first order Taylor expansion, still not very good

C. Higher order Taylor expansions: $f(t+\Delta t) = f(t) + \sum \frac{1}{k!} f^{(k)}(t)(\Delta t)^k$

- In practice the II order method is far too inaccurate
  - $\Rightarrow$ higher order methods (e.g. 8th order etc.)
- Calculation of higher order derivatives of $\ddot{\vec{R}}$ gets rapidly very cumbersome

$\Rightarrow$ tabulated f and g series:

$$\vec{R}(\tau) = f(\tau, \vec{R}_0, \vec{R}'_0)\ \vec{R}_o + g(\tau, \vec{R}_0, \vec{R}'_0)\ \vec{R}'_0$$

$$\vec{V}(\tau) = \dot{f}(\tau, \vec{R}_0, \vec{R}'_0)\ \vec{R}_o + \dot{g}(\tau, \vec{R}_0, \vec{R}'_0)\ \vec{R}'_0$$

where

$\tau \equiv \sqrt{\mu}(t - t_0)$ and

$'$ denotes differentiation with respect to normalized time $(d/dt = (d/d\tau)(d\tau/dt) = \sqrt{\mu}\ d/d\tau)$

$f = 1 - \frac{1}{2}\tau^2 {r_0}^3 + \frac{1}{2}\tau^3 {r_0}^{-4} {r_0}' \ldots$

$g = \tau - \frac{1}{6}\tau^3 {r_0}^{-3} \ldots$

(here truncated with the accuracy that corresponds to II order Taylor expansion

D 'TIME-CENTERED LEAPFROG'

- A method sometimes used in N-body simulations: the accuracy corresponds to second degree Taylor expansion, but the method does not require the calculation of $\dddot{\vec{R}} = \dot{\vec{F}}$
- The positions and velocities are approximated with formulas which are symmetric with respect to time (*time-centered*)

  For example, the x component (y and z are treated in a similar manner)

$$
\begin{aligned}
v_x(t+\Delta t/2) &= v_x(t-\Delta t/2) + f_x(t)\Delta t \\
x(t+\Delta t) &= x(t) + v_x(t+\Delta t/2)\Delta t
\end{aligned}
$$

It is easy to see that the accuracy corresponds to II degree Taylor:
write the Taylor expansions

$$v_x(t+\Delta t/2) = v_x(t) + f_x(t)\Delta t/2 + \frac{1}{2}\dot{f}_x(t)(\Delta t/2)^2$$
$$v_x(t-\Delta t/2) = v_x(t) - f_x(t)\Delta t/2 + \frac{1}{2}\dot{f}_x(t)(\Delta t/2)^2$$

subtract the latter from the first $\Rightarrow$

$$v_x(t+\Delta t/2) - v_x(t-\Delta t/2) = f_x(t)\Delta t \; + \; ...\Delta t^3$$

leads to leapfrog formula, with an error term proportional to Δt^3

Same accuracy holds for the position vector components

E. Fourth order Runge-Kutta method (RK4)

- A method accurate to $(\Delta t)^4$, without the need to calculate explicitly the the higher order derivatives of force.
- The accuracy is achieved by evaluating the forces in 4 different intermediate steps.
- Assume that $dx/dt = f(t,x)$, then

$$x(t+\Delta t) = x(t) + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4) \text{ where}$$

$$k_1 = f(t, x(t))$$
$$k_2 = f(t + \frac{\Delta t}{2}, x(t) + \frac{\Delta t}{2}k_1)$$
$$k_3 = f(t + \frac{\Delta t}{2}, x(t) + \frac{\Delta t}{2}k_2)$$
$$k_4 = f(t + \Delta t, x(t) + \Delta t k_3)$$

In practice most programming packages have subroutines for the above steps

- IDL contains a library-routine RK4: to use it one needs to
  - collect the components of $\vec{R}$ and $\vec{V}$ into a single variable to be integrated, say, $yy = [x, y, z, vx, vy, vz]$
  - make a loop over time steps, calling RK4 at each step for updating the variable yy
  - provide a function routine for returning the derivatives of yy

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```
        variable yy=[x,y,z,vx,vy,vz] integrated over the step dt
        func(time,yy) returns the derivate of yy (=[dx/dt,dy/dt,dz/dt,d(vx)/dt,d(vy)/dt, d(vz)/dt])
```

```
time=0.
for i=0,n do begin
        dydx=func(time,yy)
        res=rk4(yy,dydx,time,dt,'func',/double)
        time=time+dt
        yy=res
endfor
```

```
function func,t,yy
        x=yy(0) & y=yy(1) & z=zz(2) & vx=yy(3) & vy=yy(4) & vz=yy(5)
        ...
        dydx=yy*0.
        dydx(0)= ...
return,dydx

end
```

WHAT TO DO?

- Make a program to numerically integrate the 2-body relative motion. Also compare to analytical solution, and check the conservation of energy and angular momentum.

- Analytical solution: Choose values for the orbital elements: a, ϵ, t_0. Also, set the units by choosing $\mu = G(m_1 + m_2) = 1$. The orbital period $P = 2\pi\sqrt{a^3/\mu}$ (therefore if $\mu = 1$ $\Rightarrow$ $P = 2\pi$ at the unit distance)

  Solve Kepler's equation t:

  $M = 2\pi\frac{t-t_0}{P}$

  $M = E - \epsilon \sin E$

  Position and velocity ($b = a\sqrt{1-\epsilon^2}$)

  $x = a(\cos E - \epsilon)$

  $y = b \sin E$

  $z = 0$

  $v_x = -a \sin E \ \sqrt{\mu/a^3} \ (1 - \epsilon \cos E)^{-1}$

  $v_y = \quad b \cos E \ \sqrt{\mu/a^3} \ (1 - \epsilon \cos E)^{-1}$

  $v_z = 0$

  At $t = t_0 \quad \Rightarrow \quad M = 0, E = 0 \quad \Rightarrow \quad [x_0, y_0, 0]$, $[v_{x0}, v_{y0}, 0]$

- Numerical solution:

  Integrate the orbit numerically for times $t = t_0 + i \quad \Delta t, \quad i = 0, 1, 2, ...$

  Use the analytical solution for the calculation of initial values.

  Compare to analytical solution evaluated for the same times

  Check the energy and angular momentum

  $h = \frac{1}{2}v^2 - \mu/r$

  $|\vec{k}| = (\vec{R} \times \vec{V})_z = xv_y - yv_x$

- Compare I and II order Taylor expansions, RK4,

  using different time steps (for example $\Delta t = 0.1, 0.01, 0.001$ orbital periods).