CELESTIAL MECHANICS (Fall 2012): COMPUTER EXERCISES II

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5. Calculation of R and V from orbital elements:

6. Calculation of orbital elements from R and V

7. Playing with orbits

Example routines for the solution of the exercises can be copied from

`/wrk/hsalo/TM2012_idl.dir/TM2012_DEMO2.dir`

5. Calculation of $\vec{R}$ and $\vec{V}$ from orbital elements (elem_to_rv.pro)

The previous examples have dealt with elliptic motion in a coordinate system aligned with the orbital plane, with the x-axis pointing to the pericenter. Let's denote the Cartesian coordinates in this system by $\tilde{x}$ and $\tilde{y}$, and the velocities by $\tilde{v}_x$ and $\tilde{v}_y$. The calculation of these coordinates for a given time t required just a, ϵ and τ: the shape and size of the orbit is given by ϵ and a, while τ gave the instant of pericenter passage.

Here we calculate the position $\vec{R}$ and velocity $\vec{V}$ vectors with respect to a reference coordinate system (e.g. Earth's orbital plane, with the x-axis pointing to the vernal equinox), starting from the full set of orbital elements $(a, \epsilon, i, \Omega, \omega, \tau)$. The orientation of the planet's orbital plane is defined with Ω and i, the longitude of node with respect to a fixed direction along the xy-reference plane, and the inclination of the orbit, respectively. The orientation of the pericenter in the orbital plane is defined with ω, the argument of pericenter, giving the angle of the pericenter from the nodal line (along the orbital plane).

ELLIPTIC ORBIT: The following steps are required (steps 1-3 were already needed in previous exercises, but are collected here to list a complete set of formulas):

1) Calculate the period, and the mean anomaly M

$$\begin{aligned} P &= 2\pi\sqrt{a^3/\mu} \\ M &= \frac{2\pi}{P}(t-\tau) \end{aligned} \tag{1}$$

2) Solve the eccentric anomaly E from Kepler's equation

$$M = E - \epsilon \sin(E) \tag{2}$$

3) Calculate the orbital plane coordinates $\tilde{x}, \tilde{y}$ and $\tilde{v}_x, \tilde{v}_y$:

$$\begin{aligned} \tilde{x} &= a(\cos E - \epsilon) \\ \tilde{y} &= b \sin E \\ \tilde{v}_x &= -a \sin E \ \sqrt{\mu/a^3}\ (1-\epsilon\cos E)^{-1} \\ \tilde{v}_y &= \ \ b \cos E \ \sqrt{\mu/a^3}\ (1-\epsilon\cos E)^{-1} \end{aligned} \tag{3}$$

4) All the extra which needs to be done is to rotate the components of position and velocity to the reference system (of course $\tilde{z} = 0$ and $\tilde{v}_z = 0$ along the orbit):

$$\begin{aligned} \vec{R} &= \tilde{x}\left(\frac{\vec{A}}{a}\right) + \tilde{y}\left(\frac{\vec{B}}{b}\right) + \tilde{z}\vec{N} \\ \vec{V} &= \tilde{v}_x\left(\frac{\vec{A}}{a}\right) + \tilde{v}_y\left(\frac{\vec{B}}{b}\right) + \tilde{v}_z\vec{N} \end{aligned} \tag{4}$$

where the directions of $\vec{A}$, $\vec{B}$, and $N = \left(\frac{\vec{A}}{a}\right) \times \left(\frac{\vec{B}}{b}\right)$ in the reference system (obtained by a rotation of the orbital plane- coordinate system by $-\omega$ with respect to the z-axis, $-i$ with respect to the new x-axis, and by $-\Omega$ around the new z-axis) are given by:

$$\frac{\vec{A}}{a} = \begin{pmatrix} \cos\omega\cos\Omega - \sin\omega\sin\Omega\cos i \\ \cos\omega\sin\Omega + \sin\omega\cos\Omega\cos i \\ \sin\omega\sin i \end{pmatrix} \tag{5}$$

$$\frac{\vec{B}}{b} = \begin{pmatrix} -\sin\omega\cos\Omega - \cos\omega\sin\Omega\cos i \\ -\sin\omega\sin\Omega + \cos\omega\cos\Omega\cos i \\ \cos\omega\sin i \end{pmatrix} \tag{6}$$

$$\vec{N} = \begin{pmatrix} \sin\Omega\sin i \\ -\cos\Omega\sin i \\ \cos i \end{pmatrix} \tag{7}$$

HYPERBOLIC ORBIT: The treatment of hyperbolic orbit can be incorporated to the same procedure: the only changes are in steps 2 and 3. In the case of hyperbolic orbit ($\epsilon > 1$), the Kepler's equation is replaced by

$$M = \epsilon \sinh E - E \tag{8}$$

which can be solved with similar methods as in the elliptic case (see **hkepler.pro**).

Instead of step 3 we have:

$$\begin{aligned}
\tilde{x} &= a(\epsilon - \cosh E) \\
\tilde{y} &= b \sinh E \\
\tilde{v}_x &= -a \sinh E \ \ \sqrt{\mu/a^3} \ (\epsilon \cosh E - 1)^{-1} \\
\tilde{v}_y &= \ \ b \cosh E \ \ \sqrt{\mu/a^3} \ (\epsilon \cosh E - 1)^{-1},
\end{aligned} \tag{9}$$

obtained from the vectorial representation of hyperbolic orbit:

$$\vec{R} = \vec{A}(\epsilon - \cosh E) + \vec{B} \sinh E \tag{10}$$

$$\vec{V} = \dot{\vec{R}} = (-\vec{A} \sinh E + \vec{B} \cosh E)\dot{E}, \tag{11}$$

where $\dot{E}$ is get from differentiating Eq. 8, giving

$$\dot{E} = \frac{\sqrt{\mu/a^3}}{(\epsilon \cosh E - 1)} \tag{12}$$

Also remember that for hyperbolic orbit $b = a\sqrt{\epsilon^2 - 1}$

WHAT TO DO: Make an IDL-procedure performing the above steps, taking as input variables the set of orbital elements $(a, \epsilon, i, \Omega, \omega, \tau)$ and time t. The units of time and distances can all be specified by the variable μ. For example, if we study the motion in the Solar system, then we know that the orbital period equals one year at the distance of Earth's orbit. According to Eq. 2 we must thus use $\mu = 4\pi^2$ if time is measured in years and distances in Astronomical units. Other choices are also possible: for example if we choose $\mu = 1$, then according to Eq. 2, the orbital period is 2π at the unit distance. This is a very natural choice, as then the circular orbit velocity and angular velocity also equal unity at the unit distance.

6. Calculation of orbital elements from $\vec{R}$ and $\vec{V}$ (rv_to_elem.pro)

Here we look at the inverse problem, i.e. the calculation of orbital elements from the position and velocity vectors, specified in the reference system (e.g. Earth's orbital plane).

ELLIPTIC ORBIT:

1) Calculate the unit vector in the direction perpendicular to the orbital plane, from the vector $\vec{K} = \vec{R} \times \vec{V}$:

$$\vec{N} = \frac{\vec{R} \times \vec{V}}{|\vec{R} \times \vec{V}|}, \tag{13}$$

from which we get (from Eq. 7)

$$\begin{aligned} \Omega &= \operatorname{atan}(N_x, -N_y) \\ i &= \operatorname{acos}(N_z) \end{aligned} \tag{14}$$

2) Calculate the eccentricity vector $\vec{e}$,

$$\vec{e} = -\frac{1}{\mu}\vec{K} \times \vec{V} - \vec{R}/r. \tag{15}$$

Then rotate $\vec{e}$ from the reference system to the orbital plane-coordinates (involves rotation by Ω around the z-axis, followed by rotation by i around the new x-axis):

$$\begin{aligned} \tilde{e}_x &= \cos\Omega\ e_x + \sin\Omega\ e_y \\ \tilde{e}_y &= -\sin\Omega\cos i\ e_x + \cos\Omega\cos i\ e_y + \sin i\ e_z, \end{aligned} \tag{16}$$

and finally

$$\begin{aligned} \omega &= \operatorname{atan}(\tilde{e}_y, \tilde{e}_x) \\ \epsilon &= \sqrt{\tilde{e}_x^2 + \tilde{e}_y^2} \end{aligned} \tag{17}$$

3) Semi-major axis a is obtained from the energy equation, $h = \frac{1}{2}v^2 - \mu/r = -\frac{\mu}{2a}$ for elliptic orbit, giving

$$a = -\frac{1}{v^2/\mu - 2/r}. \tag{18}$$

4) Time of pericenter passage is obtained by calculating first the eccentric anomaly and then using Kepler's equation. Since $r = a(1 - \epsilon\cos E)$ we obtain

$$\cos E = \frac{(1 - r/a)}{\epsilon} \tag{19}$$

This alone does not determine E uniquely. The correct sign of E is determined by the sign of $\vec{R}\cdot\vec{V}$ (for $0^\circ < E < 180^\circ$ we have $\vec{R}\cdot\vec{V} > 0$)

$$E = \mathrm{acos}(\cos E)\ \ \mathrm{sign}(\vec{R}\cdot\vec{V}) \tag{20}$$

Then calculate M and finally τ,

$$\begin{aligned} M &= E - \epsilon\sin E \\ \tau &= t - M\sqrt{a^3/\mu} \end{aligned} \tag{21}$$

HYPERBOLIC ORBIT: The treatment of hyperbolic orbit (the type of orbit is identified in the step 2 above if $\epsilon > 1$) is again very similar to elliptic orbit. Only the steps 3 and 4 are different. Since in the case of hyperbolic orbit the total energy is positive, $h = \frac{\mu}{2a}$, the sign in Eq. 18 needs to be reversed. Thus both elliptic and hyperbolic case can be treated if we use

$$a = \frac{1}{|v^2/\mu - 2/r|} \tag{22}$$

The step 4) needs to replaced by

$$\begin{aligned} \cosh E &= \frac{(1 + r/a)}{\epsilon} \\ E &= \mathrm{acosh}(coshE)\ \ \mathrm{sign}(\vec{R}\cdot\vec{V}) \\ M &= \epsilon\sinh E - E \\ \tau &= t - M\sqrt{a^3/\mu} \end{aligned} \tag{23}$$

WHAT TO DO: Make an IDL-procedure performing the above steps, taking as input variables the vectors $\vec{R}$ and $\vec{V}$ and time, and returning the set of orbital elements $(a, \epsilon, i, \Omega, \omega, \tau)$. The units of time and distances are specified by the variable μ, as discussed earlier in the case of obtaining $\vec{R}$ and $\vec{V}$ from orbital elements.

7. Playing with orbits

Copy the example IDL procedures for the solution of exercises 5 and 6 to your own directory:

```
cp /wrk/hsalo/TM2012_DEMO2.dir/* .
```

These routines include: (give the name without arguments to obtain an info message)

- **elem_to_rv.pro** – calculate $\vec{R}$ and $\vec{V}$ from orbital elements (Exercise 5)
- **rv_to_elem.pro** – calculate orbital elements from $\vec{R}$ and $\vec{V}$ (Exercise 6)
- **elem_to_example.pro** – Check the above two routines:
  first calculate $\vec{R}$ and $\vec{V}$ for a given time from orbital elements,
  then transform $\vec{R}$ and $\vec{V}$ back to orbital elements (hopefully the same!)
- **kepler.pro** – solves Kepler's equation $(M, \epsilon \Rightarrow E)$
- **kepler_array.pro** – solves Kepler's equation for an array of M values
- **hkepler.pro** – as kepler.pro, except for a hyperbolic orbit ($\epsilon > 1$

+ auxillary mathematical routines missing from basic IDL-routines

- **cross_product.pro** – calculate vector product of two 3-element vectors
- **dot_product.pro** – calculate scalar product of two 3-element vectors
- **acosh.pro** – function returning inverse hyperbolic cosine

Use the above routines in the next exercises (or better, write your own equivalent routines!)

a) Procedure for calculating + plotting an orbit

Make a subroutine-type procedure (call it for example `elem_orbit_oma.pro`) for calculating/plotting an orbit with given orbital elements (`elem=`$[a, \epsilon, i, \Omega, \omega, \tau]$), over an interval of time t_1, t_2.

Hint: make repeated calls to **elem_to_rv**, after checking what type of input/output it uses. Pay attention to the units you use.

You may also have a look at the example procedure **elem_orbit.pro**

b) Satellite launched from Earth orbit

Make a main-program type procedure, where a velocity increment is given to a satellite at Earth orbit:

- in the tangential direction

- in the radial direction

- in the vertical direction

Calculate the orbital elements of the satellite. Assume that Earth orbit is circular with speed 30 km/sec, and use different velocity increments in the range 0-30 km/sec. Use the procedure **elem_orbit.pro** (or the equivalent procedure you have written)

For examples of solutions, see

- **elem_orbit_demo_tangential.pro**
- **elem_orbit_demo_radial.pro**
- **elem_orbit_demo_vertical.pro**
- **elem_orbit_demo_vertical.pro** – illustrates the effect of vertical velocity increment using IDL's interactive xplot3d-plotting procedure

Check also other example routines:

- **elem_rv_demo1.pro** – Solves the Exercise II.1 (satellite launced with a velocity vector making different angles with the tangential direction

- **elem_rv_demo2.pro** – shows how a 'swarm of satellites' launched from Earth disperses with time.