Solutions

University of Oulu
Astronomy and Space Physics
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Celestial Mechanics II, Fall 2015
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Exercise Problems — Part 2

(Due Date: Tuesday, 06.10.15, *before* the lecture. You can also put the solutions in the red box outside office FY277.)

1. Atmospheric Drag

(a) Check from a source of your choice what is the structure of the *upper* part of Earth's atmosphere. Describe briefly the different parts and altitude ranges. What densities are expected there, roughly?

(b) Derive the equation given in the lecture

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -\frac{C_\mathrm{d}\,A\,\rho}{m}\,v\,(e+\cos f)\;.$$

for the slow evolution of the eccentricity under the influence of atmospheric drag.

(c) Solve the equation for the slow evolution of the semi-major axis

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{C_\mathrm{d}\,A\,\rho}{m}\frac{a^2}{\mu}v^3$$

for the case of constant density ρ and $e = 0$.

2. *Advanced*: Slow evolution of the longitude of the ascending node

Show that the general evolution equation for the longitude of the ascending node reads

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \sqrt{\frac{a(1-e^2)}{\mu}}\,N\,\frac{\sin\varphi}{1+e\cos f}\,.$$

Hint: Use the relations given in the lecture and express the longitude of the ascending node in terms of components of the angular momentum vector as

$$\tan\Omega = -\frac{h_x}{h_y}\,.$$

Also, you must express the unit vectors normal to the orbital plane and in azimuthal direction, $\vec{e}_N$ and $\vec{e}_\varphi$, respectively, in terms of Ω, i, and $\varphi = \varpi + f$ as

$$\vec{e}_N = \begin{pmatrix} \sin i\,\sin\Omega \\ -\sin i\,\cos\Omega \\ \cos i \end{pmatrix}, \quad \text{and} \quad \vec{e}_\varphi = \begin{pmatrix} -\sin\varphi\,\cos\Omega - \cos i\,\sin\Omega\,\cos\varphi \\ -\sin\varphi\,\sin\Omega + \cos i\,\cos\Omega\,\cos\varphi \\ \sin i\,\cos\varphi \end{pmatrix}.$$

Keep in mind that the angular momentum vector points in the direction of $\vec{e}_N$.

Exercise Problems Part 2

(1)

1. Atmospheric Drag

(a) Exosphere – 700 km to 10,000 km
molecules on ballistic trajectories, no collisions, practically
$\rho \sim 10^{-14} \frac{kg}{m^3}$

Thermosphere – 80 km to ~~1000~~ 700 km or 1000 km
height depends on solar activity

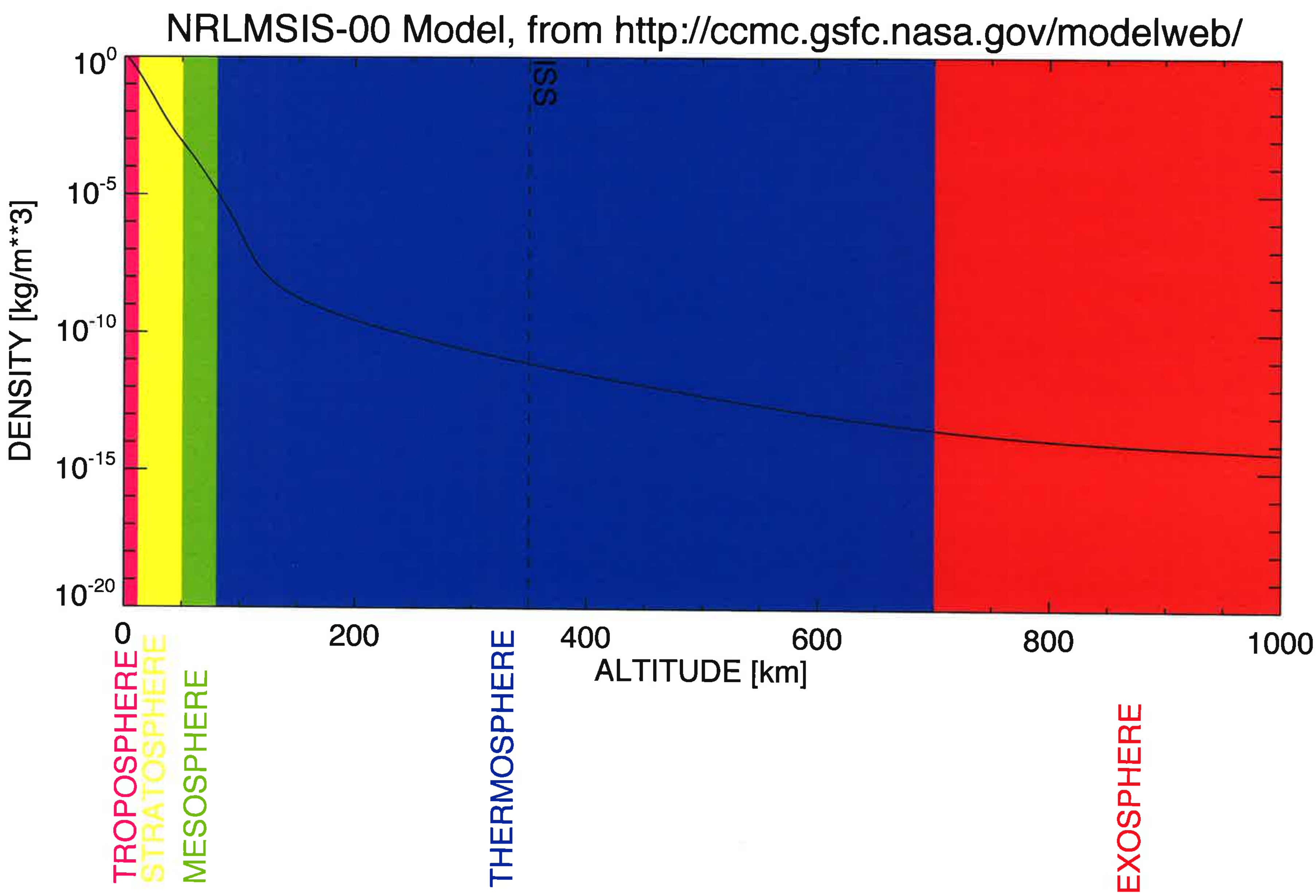
80 – 500 km: the Ionosphere

Mesosphere – 50 km to 80 km, too high for ~~spacecraft~~ planes, too low for spacecraft

Stratosphere – 12 km to 50 km;

Troposphere – 0 km to 12 km
contains about 80% of mass of the atmosphere

c2)



/Users/jschmidt/Teaching/CelestialMechanicsSpecialFall2015/Exercise2/NRLMSISE

jschmidt Tue Oct 6 16:00:45 2015

(15) Lecture:

$$\frac{de}{dt} = \sqrt{\frac{a(1-e^2)}{\mu}} \left[\underbrace{R}_{\text{radial component of force}} \sin f + \underbrace{T}_{\text{azimuthal component of force}} \cos f + \frac{T}{e}\left(1 - \frac{r}{a}\right)\right]$$

For atmospheric drag (lecture):

$$R = -\frac{C_d A \rho}{2m} v \frac{h}{a(1-e^2)} e \sin f$$

$$T = -\frac{C_d A \rho}{2m} v \frac{h}{r}$$

$$\Rightarrow \frac{de}{dt} = \sqrt{\frac{a(1-e^2)}{\mu}} \left(-\frac{C_d A \rho}{2m}\right) v \frac{h}{a(1-e^2)} \left\{ e \sin^2 f + \cos f(1+e\cos f) + \frac{1}{e}\left(1 - \frac{1-e^2}{1+e\cos f}\right) * (1+e\cos f) \right\}$$

$$= -\frac{C_d A \rho}{2m} v \left\{ e \sin^2 f + \underbrace{\cos f + e\cos^2 f}_{} + \frac{1}{e}\underbrace{\left(1+e\cos f - 1 + e^2\right)}_{= \cos f + e} \right\}$$

$h = \sqrt{\mu a(1-e^2)}$

$$= -\frac{C_d A \rho}{2m} v \left\{ e + \cos f + \cos f + e \right\}$$

$$= -\frac{C_d A \rho}{m} v \left(e + \cos f\right)$$

(c) $$\frac{da}{dt} = -\frac{C_d A \rho}{m}\frac{a^2}{\mu} v^3$$

$\rho = \text{const}$; $e = 0$

$$\Rightarrow v = \Omega a = \sqrt{\frac{\mu}{a}}$$

$$\Rightarrow \frac{da}{dt} = -\frac{C_d A \rho}{m}\frac{a^2}{\mu}\left(\frac{\mu}{a}\right)^{3/2}$$

$$= -\frac{C_d A \rho}{m}\sqrt{a\mu}$$

let $a = \alpha \cdot a_0$

a_0 characteristic scale (e.g. initial semimajor axis)

$t = \tau \cdot \theta$

τ characteristic timescale

$$\Rightarrow \frac{a_0}{\theta}\frac{d\alpha}{d\tau} = -\frac{C_d A \rho}{m}\sqrt{a_0\mu}\sqrt{\alpha}$$

$$\frac{d\alpha}{dt} = \underbrace{-\theta\frac{C_d A \rho}{m}\sqrt{\frac{\mu}{a_0}}}_{\text{choose } \frac{1}{\theta} = \frac{C_d A \rho}{m}\sqrt{\frac{\mu}{a_0}}}\sqrt{\alpha}$$

$$\Rightarrow \boxed{\frac{d\alpha}{d\tau} = -\sqrt{\alpha}}$$

$$\frac{d\alpha}{\sqrt{\alpha}} = -d\tau$$

$$\Rightarrow 2\, d\sqrt{\alpha} = -d\tau$$

$$\Rightarrow d\left(\sqrt{\alpha} + \frac{\tau}{2}\right) = 0$$

$$\Rightarrow \sqrt{\alpha} + \frac{\tau}{2} = \text{const}$$

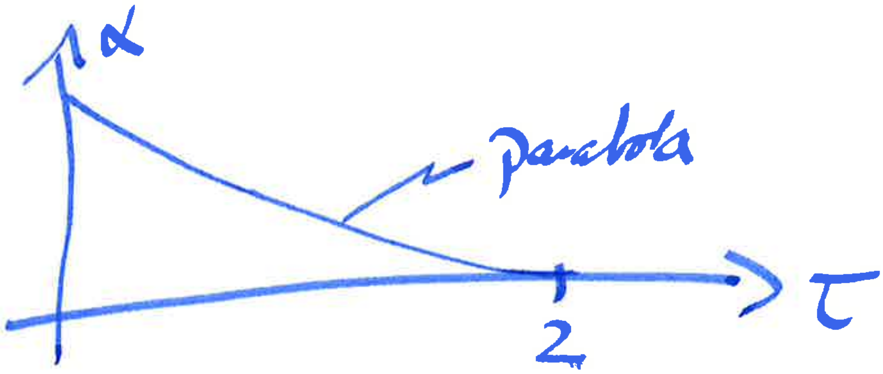
$$\equiv \sqrt{\alpha_0} = \sqrt{\alpha(\tau=0)} = 1$$

if we take $a_0 = a(t=0)$

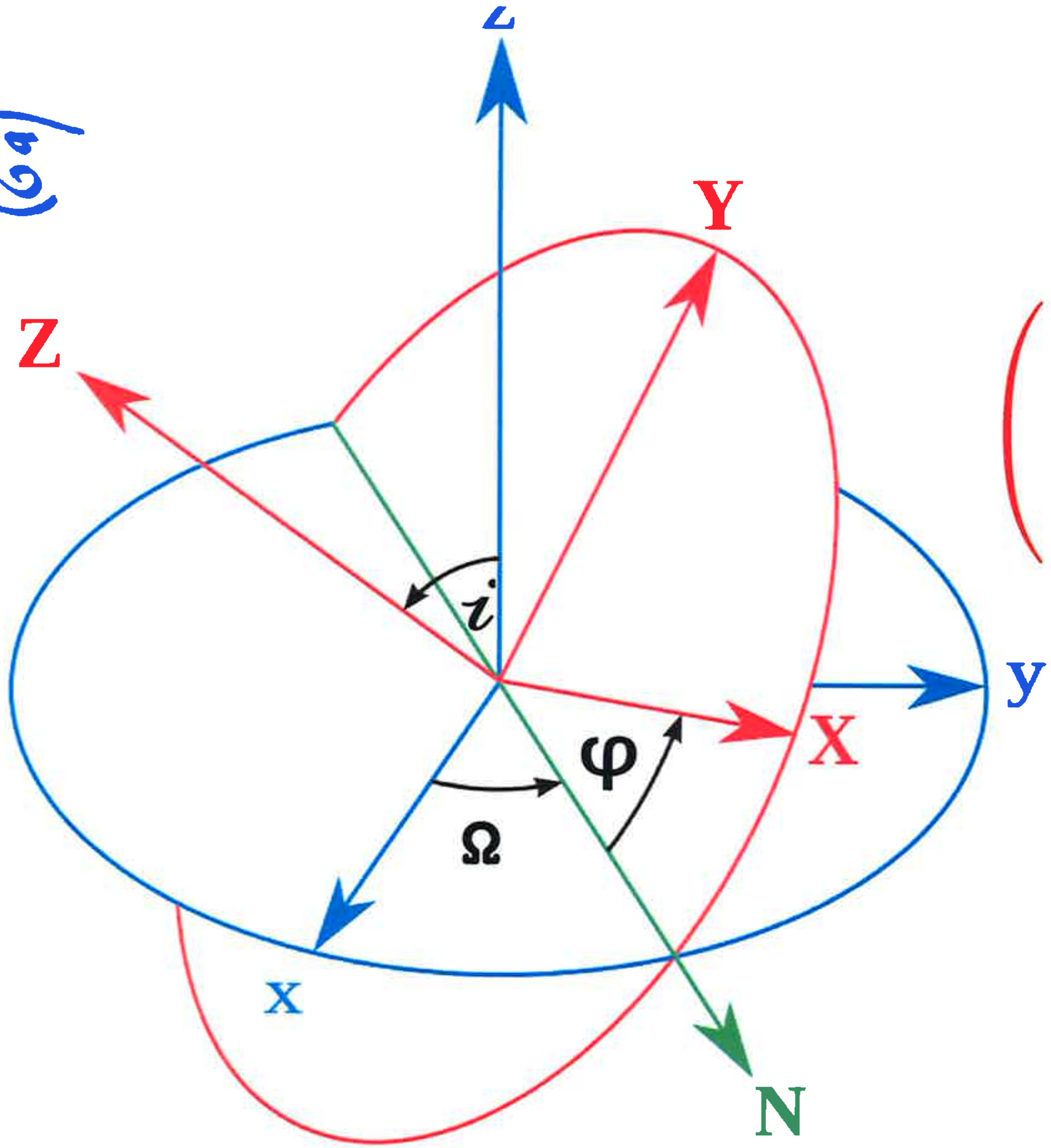
$$\Rightarrow \alpha = \left(1 - \frac{\tau}{2}\right)^2$$

$$a = a_0\left(1 - \frac{1}{2}\frac{t}{\theta}\right)^2$$

$$a = a_0\left[1 - \frac{1}{2}\frac{C_d A \rho}{m}\sqrt{\frac{\mu}{a_0}}\, t\right]^2$$



(6a)



xyz - inertial frame

XYZ - cartesian frame with XY = orbit plane

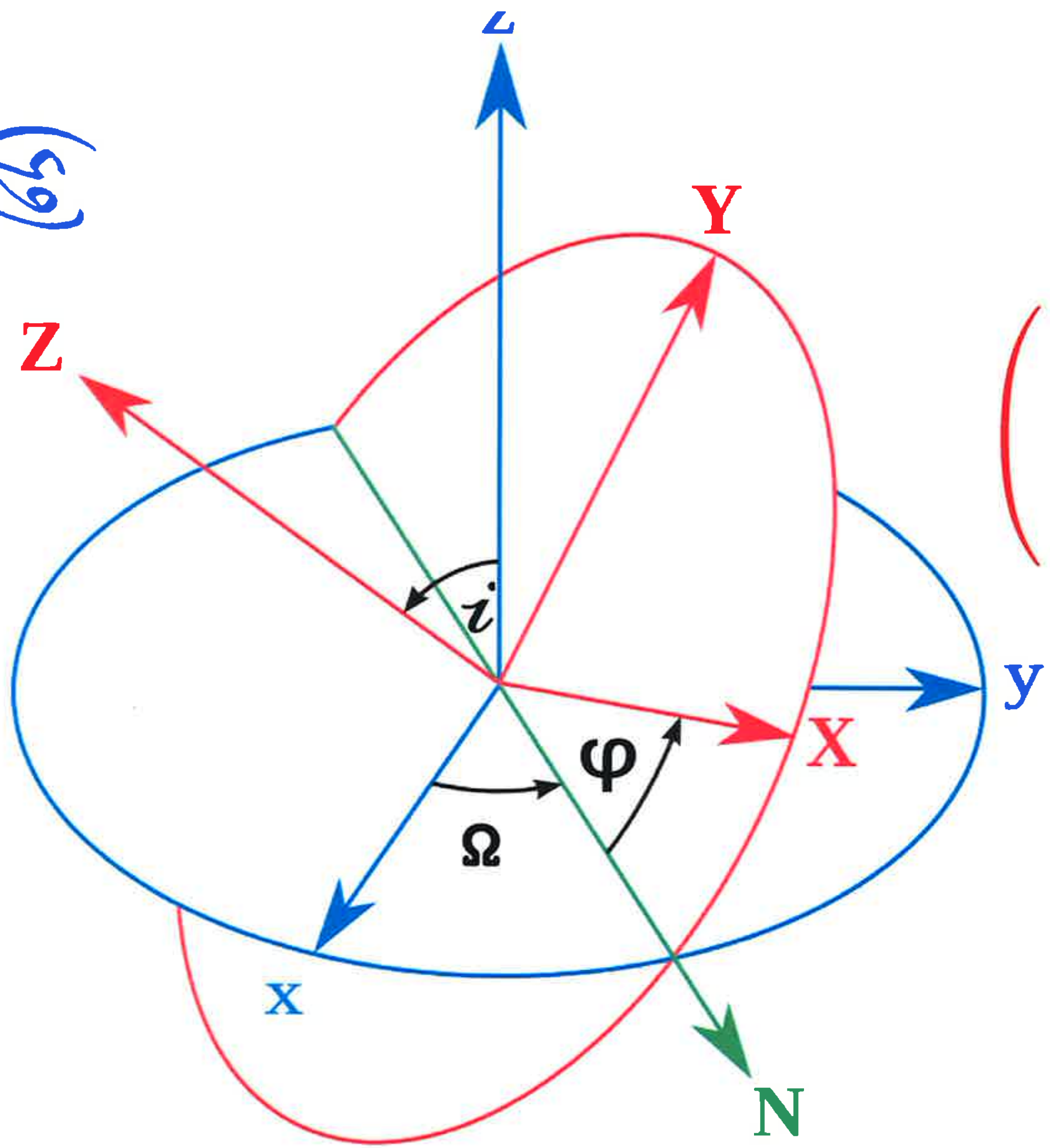
$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = B\,C\,D \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$D = \begin{pmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{pmatrix}$$

$$B = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(65)



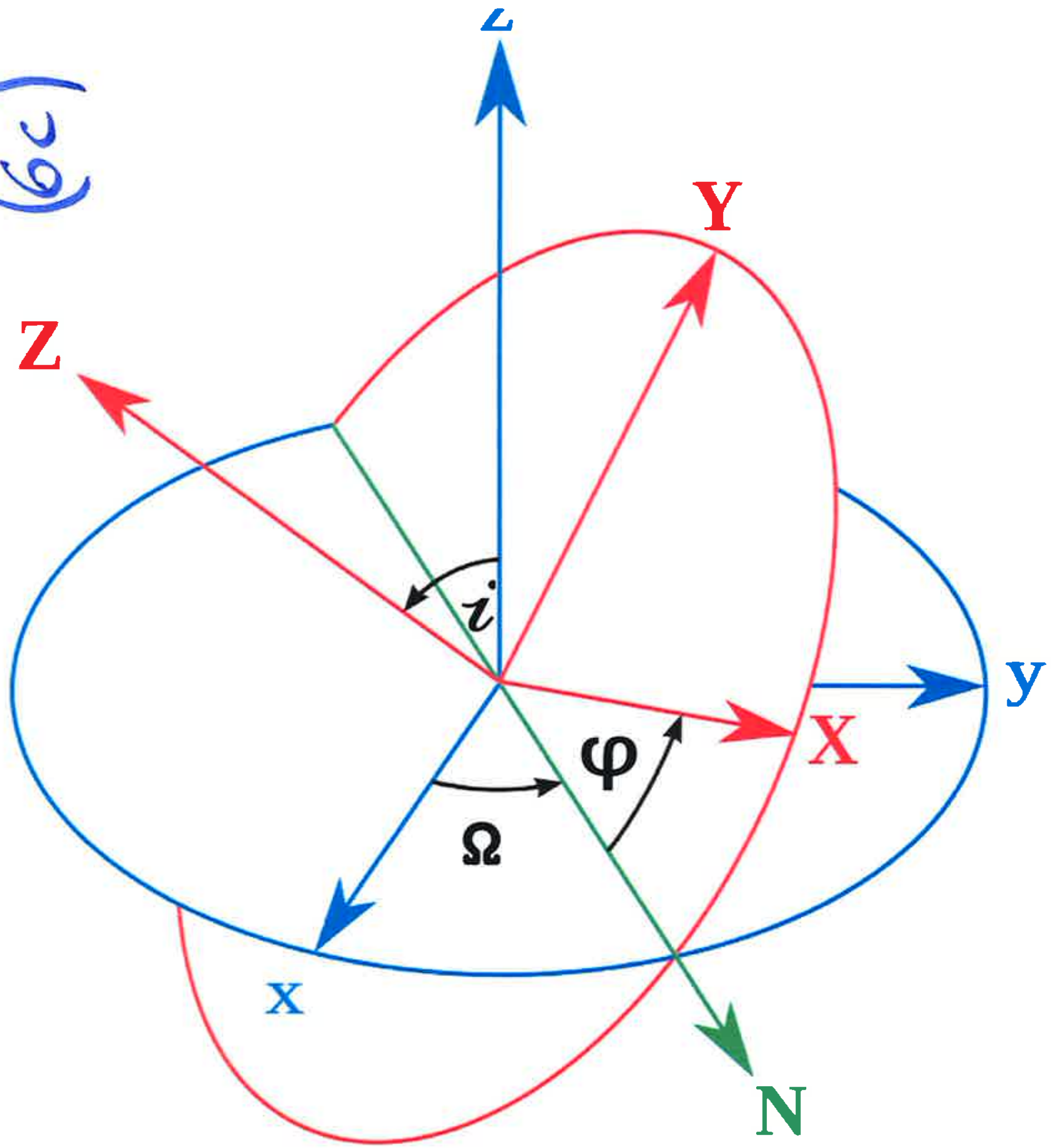
xyz - inertial frame

XYZ - cartesian frame with XY = orbit plane

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = B\,C\,D \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$B\,C\,D = \begin{pmatrix} \cos\varphi\cos\Omega - \cos i\sin\Omega\sin\varphi & \cos\varphi\sin\Omega + \cos i\cos\Omega\sin\varphi & \sin i\sin\varphi \\ -\sin\varphi\cos\Omega - \cos i\sin\Omega\cos\varphi & -\sin\varphi\sin\Omega + \cos i\cos\Omega\cos\varphi & \sin i\cos\varphi \\ \sin i\sin\Omega & -\sin i\cos\Omega & \cos i \end{pmatrix}$$

(6c)



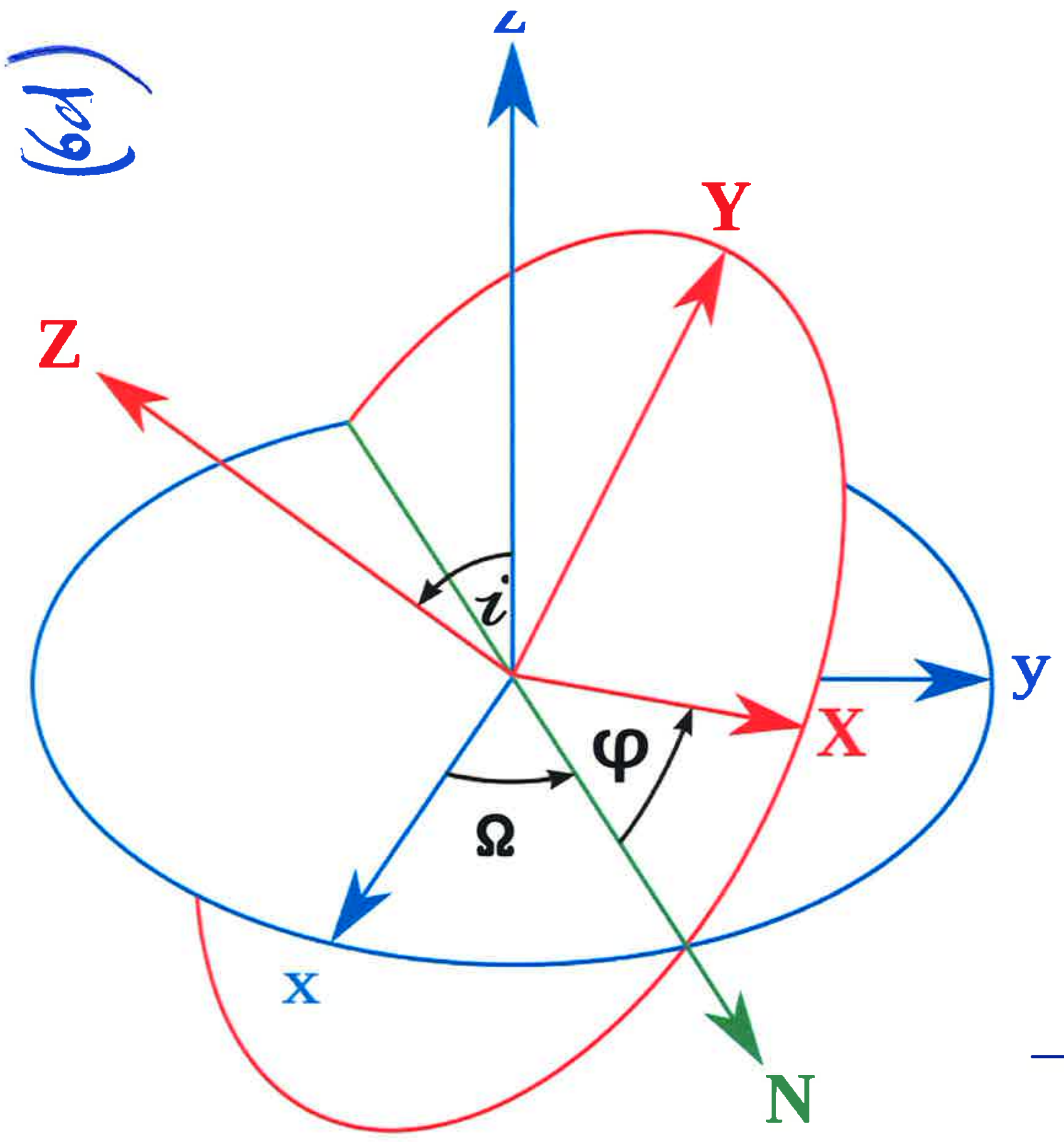
xyz - inertial frame

XYZ - cartesian frame with XY = orbit plane

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (B\,C\,D)^{-1} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$(B\,C\,D)^{-1} = \begin{pmatrix} \cos\varphi\cos\Omega - \cos i\sin\Omega\sin\varphi & -\sin\varphi\cos\Omega - \cos i\sin\Omega\cos\varphi & \sin i\sin\Omega \\ \cos\varphi\sin\Omega + \cos i\cos\Omega\sin\varphi & -\sin\varphi\sin\Omega + \cos i\cos\Omega\cos\varphi & -\sin i\cos\Omega \\ \sin i\sin\varphi & \sin i\cos\varphi & \cos i \end{pmatrix}$$

(6d)



xyz - inertial frame

XYZ - cartesian frame with XY = orbit plane

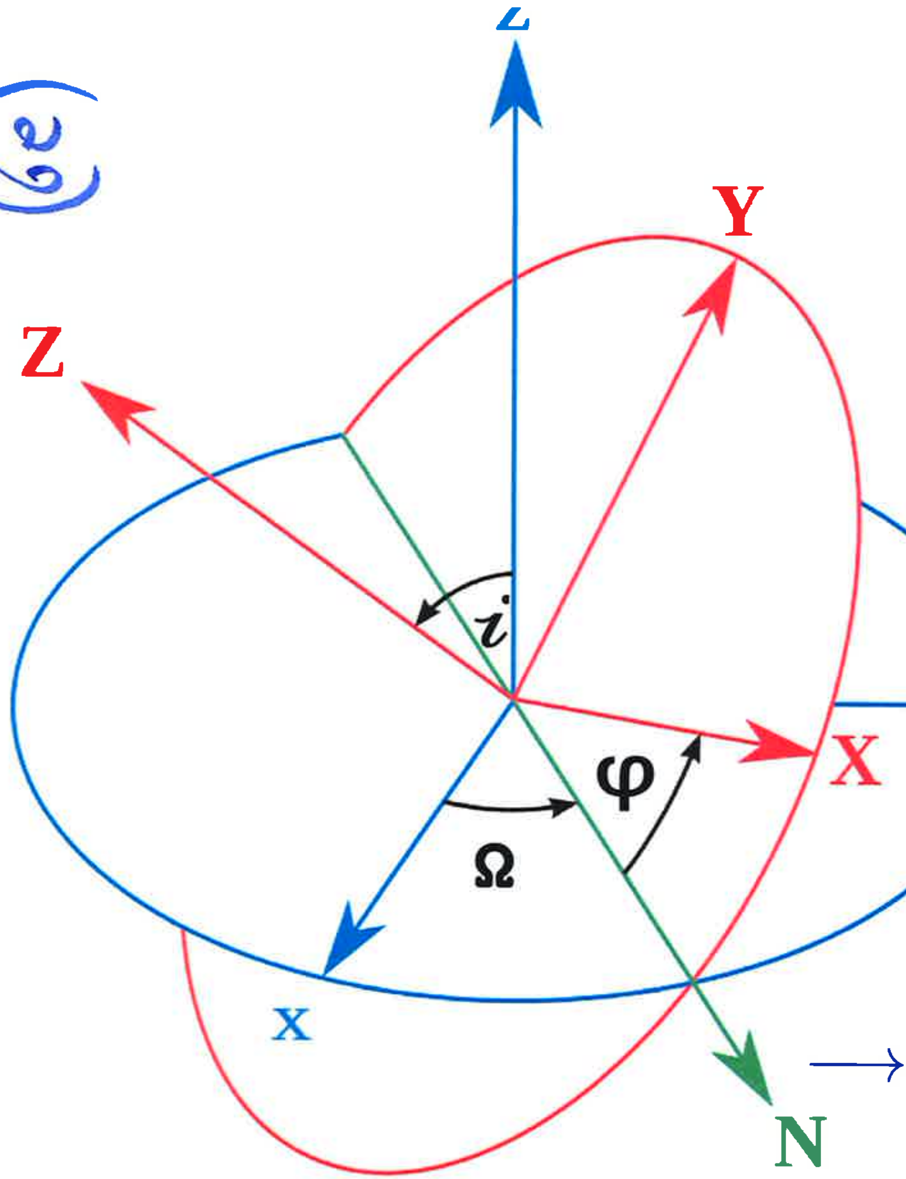
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (B\,C\,D)^{-1} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$\vec{e}_N = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\longrightarrow \quad \vec{e}_N = \begin{pmatrix} \sin i \sin\Omega \\ -\sin i \cos\Omega \\ \cos i \end{pmatrix}$$

$$(B\,C\,D)^{-1} = \begin{pmatrix} \cos\varphi\cos\Omega - \cos i \sin\Omega \sin\varphi & -\sin\varphi\cos\Omega - \cos i \sin\Omega\cos\varphi & \sin i \sin\Omega \\ \cos\varphi \sin\Omega + \cos i \cos\Omega \sin\varphi & -\sin\varphi\sin\Omega + \cos i \cos\Omega\cos\varphi & -\sin i \cos\Omega \\ \sin i \sin\varphi & \sin i \cos\varphi & \cos i \end{pmatrix}$$

(6e)



xyz - inertial frame

XYZ - cartesian frame with XY = orbit plane

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (B\,C\,D)^{-1} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$\vec{e}_\varphi = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\longrightarrow \quad \vec{e}_\varphi = \begin{pmatrix} -\sin\varphi\cos\Omega - \cos i\sin\Omega\cos\varphi \\ -\sin\varphi\sin\Omega + \cos i\cos\Omega\cos\varphi \\ \sin i\cos\varphi \end{pmatrix}$$

$$(B\,C\,D)^{-1} = \begin{pmatrix} \cos\varphi\cos\Omega - \cos i\sin\Omega\sin\varphi & -\sin\varphi\cos\Omega - \cos i\sin\Omega\cos\varphi & \sin i\sin\Omega \\ \cos\varphi\sin\Omega + \cos i\cos\Omega\sin\varphi & -\sin\varphi\sin\Omega + \cos i\cos\Omega\cos\varphi & -\sin i\cos\Omega \\ \sin i\sin\varphi & \sin i\cos\varphi & \cos i \end{pmatrix}$$

(2) To show: $\frac{d\Omega}{dt} = \sqrt{\frac{a(1-e^2)}{\mu}}\, N \frac{\sin f}{1+e\cos f}$

starting from $\frac{d\underset{\sim}{h}}{dt} = r\left(T\underset{\sim}{e_N} - N\underset{\sim}{e_f}\right)$

unit vector ~~tan~~ in azimuthal direction

unit vector normal to the plane: $\underset{\sim}{h} = h\,\underset{\sim}{e_N}$

and $\tan\Omega = -\frac{h_x}{h_y}$

$\Rightarrow \frac{d}{dt}\tan\Omega = -\frac{d}{dt}\frac{h_x}{h_y}$; $\frac{d}{dt}\tan\Omega = \frac{d}{dt}\frac{\sin\Omega}{\cos\Omega}$

$= \frac{\cos\Omega\cos\Omega + \sin\Omega\sin\Omega}{\cos^2\Omega}\,\frac{d\Omega}{dt}$

$\Rightarrow \frac{1}{\cos^2\Omega}\frac{d\Omega}{dt} = -\frac{\frac{dh_x}{dt}h_y - h_x\frac{dh_y}{dt}}{h_y^2}$ $= \frac{1}{\cos^2\Omega}\frac{d\Omega}{dt}$

$$\Rightarrow \boxed{\frac{d\Omega}{dt} = \frac{h_x\frac{dh_y}{dt} - h_y\frac{dx}{dt}}{h_y^2}\cos^2\Omega}$$

from $\frac{d\underset{\sim}{h}}{dt} = r\left(T\underset{\sim}{e}_N - N\underset{\sim}{e}_\varphi\right)$

we have

$$\frac{dh_x}{dt} = r\left(T\underbrace{e_N^x - N e_\varphi^x}\right)$$

x-component of these vectors

$$\frac{dh_y}{dt} = r\left(Te_N^y - N e_\varphi^y\right)$$

and $h_x = h e_N^x$; $h_y = h e_N^y$; from $\underset{\sim}{h} = h\underset{\sim}{e}_N$

$$\Rightarrow \frac{d\Omega}{dt} = \frac{\omega^2\Omega}{h^2 (e_N^y)^2}\left[h e_N^x r\left(Te_N^y - Ne_\varphi^y\right) - h e_N^y r\left(Te_N^x - Ne_\varphi^x\right)\right]$$

these cancel

$$= \frac{\omega^2\Omega}{h (e_N^y)^2} r N\left[e_N^y e_\varphi^x - e_N^x e_\varphi^y\right]$$

$$e_N^y = -\sin i \cos\Omega \quad ; \quad e_N^x = \sin i \sin\Omega$$

$$e_\varphi^x = -\sin\varphi \cos\Omega - \cos i \sin\Omega \cos\varphi$$

$$e_\varphi^y = -\sin\varphi \sin\Omega + \cos i \cos\Omega \cos\varphi$$

$$\Rightarrow \frac{d\Omega}{dt} = \frac{\cancel{\cos^2\Omega}}{h \sin^2 i \cancel{\cos^2\Omega}} \, r N \sin i \Big[\cos\Omega \{ -\sin\varphi\cos\Omega - \cos i \sin\Omega \cos\varphi \} - \sin\Omega \{ -\sin\varphi \sin\Omega + \cos i \cos\Omega \cos\varphi \} \Big]$$

$$= \frac{1}{h \sin i} \left\{ + \cos^2\Omega \sin\varphi + \sin^2\Omega \sin\varphi \right\} r N$$

$$= \frac{rN}{h \sin i} \sin\varphi \quad ; \quad r = \frac{a(1-e^2)}{1+e\cos\varphi} \; ; \quad h = \sqrt{\mu a (1-e^2)}$$

$$\Rightarrow \boxed{\frac{d\Omega}{dt} = \sqrt{\frac{a(1-e^2)}{\mu}} \, N \frac{\sin\varphi}{1+e\cos\varphi}}$$