FUN WITH POLYTROPES

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Abstract. In this report we solve the Lane-Emden equation using the differential transformation method (DTM). We utilise the Adomian decomposition method (ADM) in transforming the nonlinear terms of the equation and calculate Padé approximants of the obtained solutions. We then proceed to calculate the values of the polytropic constants.

The semi-analytical method used here, in solving the equation, is both convenient and accurate (even exact). Unlike purely numerical methods, it provides a smooth, functional form of the solution, allowing efficient computation of data for arbitrary input values.

I. Introduction

I.I. **The Lane-Emden Equation.** The Lane-Emden equation is, in general form,

$$u''(x) + \frac{\alpha}{x}u'(x) + f(x,u) = h(x), \tag{I.1}$$

where $f(x,u)$ and $h(x)$ are nonlinear analytic functions. We assume $u(x)$ is analytic for $x > 0$ and $\mathbb{R} \ni \alpha > 0$. The equation, subject to initial values $u(0); u'(0) \in \mathbb{R}$, is encountered in various occasions in physics and astronomy.

A particular form of eq. (I.1),

$$u''(x) + \frac{2}{x}u'(x) + u^a = 0 \tag{I.2}$$

where $a \in [0,5]$; $u(0) = 1$ and $u'(0) = 0$, is an important equation in the theory of stellar structure. The polytropic equation of state relates the star's pressure P and its density ρ:

$$P = K\rho^{1+\frac{1}{a}}, \tag{I.3}$$

where $K \in \mathbb{R}$ and $\rho = \rho_{\mathrm{c}}u^a$, ρ_{c} being the central density. Thus, the solutions of eq. (I.2), *polytropes*, describe the variation of a star's pressure and density (through u) as a function of its dimensionless radius x. In this report we solve eq. (I.2).

Remark 1. For eq. (I.2), an analytical solution exists for $a \in \{0, 1, 5\}$.

I.II. **Polytropic Constants.** A relation between the stellar mass M and the stellar radius R,

$$\left(\frac{GM}{M_a}\right)^{a-1}\left(\frac{R}{R_a}\right)^{3-a} = \frac{[(a+1)K]^a}{4\pi G}, \tag{I.4}$$

where G is the gravitational constant, is expressed in terms of polytropic constants M_a and R_a. The polytropic constant R_a is the root of the solution of eq. (I.2) i.e. $R_a = x(u = 0)$, while $M_a = -R_a^2\,u'(R_a)$.

A linear relation between the star's central density ρ_{c} and it's average density ρ_{avg} is written using the polytropic constant D_a:

$$D_a = \frac{\rho_{\mathrm{c}}}{\rho_{\mathrm{avg}}} = \frac{R_a^3}{3M_a}. \tag{I.5}$$

The polytropic constant

$$B_a = \frac{(3D_a)^{\frac{3-a}{3a}}}{(a+1)M_a^{\frac{a-1}{a}} R_a^{\frac{3-a}{a}}} \tag{I.6}$$

is used in expressing the central pressure of a star, P_c:

$$P_c = (4\pi)^{1/3} B_a G M^{2/3} \rho_c^{4/3}. \tag{I.7}$$

II. Methods

II.I. **The Differential Transformation Method.** The differential transformation method was introduced by Zhou (1986) to solve initial value problems in electric circuit analysis. Khan et al. (2012) applied DTM to Lane-Emden type equations.

The method provides Taylor series as solutions – in theory, the solutions are exact. In practise, however, the solutions take the form of Taylor polynomials.

Definition 2. The differential transformation of $u^{(k)}(x)$ is

$$U(k) = \frac{1}{k!}\, u^{(k)}(x)\Big|_{x=x_0},$$

where $u(x)$ is the original function and $U(k)$ the transformed function.

Definition 3. The differential inverse transformation of $U(k)$ is

$$u(x) = \sum_{k=0}^{\infty} U(k)(x-x_0)^k.$$

Theorem 4. *Let f, g and h be functions and F, G and H be the differential transformations of these functions, respectively. If*

$$f(x) = g^{(n)}(x), \textit{ then } F(k) = \frac{(k+n)!}{k!} G(k+n); \tag{II.1}$$

$$f(x) = g(x)h(x), \textit{ then } F(k) = \sum_{l=0}^{k} G(l)H(k-l); \tag{II.2}$$

$$f(x) = x^n, \textit{ then } F(k) = \delta(k-n). \tag{II.3}$$

Proof. See Jang et al. (2000) for proof.

II.II. **The Adomian Decomposition Method.** The differential transformation method, while applicable to eq. (I.2) with $a \in \mathbb{N}^0$, requires refinements to allow solving the equation for $a \in \mathbb{R}$. Ŝmarda & Khan (2015) presented the usage of the Adomian decomposition method in conjunction with DTM to overcome the aforementioned limitations, also greatly simplifying the transformations of nonlinear functions altogether. Ŝmarda & Khan (2015) call DTM refined with this approach the *improved differential transformation method* (IDTM).

A nonlinear but analytic $f(u)$ has the following decomposition:

$$f(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, ..., u_n), \tag{II.4}$$

where A_n are Adomian polynomials. The general formula for calculating the Adomian polynomials is

$$A_n(u_0, u_1, ..., u_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left(\sum_{k=0}^{\infty}\lambda^k u_k\right)\right]_{\lambda=0},\ n \geq 0. \tag{II.5}$$

Theorem 5. *Let f(u) be a known analytic function with u given by a convergent series*

$$u = \sum_{n=0}^{\infty} c_n x^n.$$

Then

$$f(u) = f\left(\sum_{n=0}^{\infty} c_n x^n\right) = \sum_{n=0}^{\infty} A_n(c_0, c_1, ..., c_n)x^n,$$

where $\mathbb{R} \ni c_i \,\forall\, i \in \mathbb{N}^0_{\leq n}$.

Proof. See Adomian & Rach (1991) for proof.

By theorem 5 and definition 3 with $x_0 = 0$

$$f(u) = \sum_{n=0}^{\infty} A_n(U(0), U(1), ..., U(n))x^n. \tag{II.6}$$

Taking the differential transformation of eq. (II.6) we get

$$F(k) = \sum_{n=0}^{\infty} A_n(U(0), U(1), ..., U(n))\delta(k-n) = A_k(U(0), U(1), ..., U(k))$$

since $A_n(U(0), U(1), \cdots, U(n))$ are independent of x. Therefore, by eq. (II.5),

$$F(k) = \frac{1}{k!}\frac{d^k}{dx^k}\left[f\left(\sum_{i=0}^{\infty} U(i)x^i\right)\right]_{x=0}, \; k \geq 0. \tag{II.7}$$

If we apply Faà di Bruno's formula,

$$(f \circ u)^{(k)}(x) = \sum_{n=1}^{k} f^{(n)}(u(x))B_{k,n},$$

to eq. (II.7) we get

$$F(k) = \frac{1}{k!}\sum_{n=1}^{k} f^{(n)}\left(\sum_{i=0}^{\infty} U(i)x^i\right) B_{k,n}\Bigg|_{x=0}. \tag{II.8}$$

Here $B_{k,n}$ are Bell polynomials, defined by a recurrence relation (Wheeler, 1987)

$$B_{k,n} = \sum_{i=1}^{k-n+1} \binom{k-1}{i-1} u^{(i)}(x) B_{k-i,n-1} \tag{II.9}$$

subject to conditions $B_{k,n} = 0 \;\forall\; k < n$ and $B_{k,k} = (u'(x))^k$.

Now,

$$f^{(n)}\left(\sum_{i=0}^{\infty} U(i)x^i\right)\Bigg|_{x=0} = f^{(n)}(U(0))$$

and, by definition 2,

$$u^{(i)}(x)\Big|_{x=0} = i!\,U(i).$$

Therefore

$$F(k) = \frac{1}{k!}\sum_{n=1}^{k} f^{(n)}(U(0))B^0_{k,n}, \tag{II.10}$$

where

$$B^0_{k,n} = \sum_{i=1}^{k-n+1} \binom{k-1}{i-1} i!\,U(i)B^0_{k-i,n-1}. \tag{II.11}$$

II.III. **Padé Approximants.** Padé approximants are rational approximations of functions, denoted by

$$[m/n] \equiv \frac{P_m(x)}{Q_n(x)}, \tag{II.12}$$

where $P_m(x)$ and $Q_n(x)$ are polynomials with $\deg P_m \leq m$ and $\deg Q_n \leq n$. The classical definition,

$$u(x) - [m/n] = \mathcal{O}(x^{m+n+1}), \tag{II.13}$$

where $u(x)$ is a formal power series, determines the coefficients of $P_m(x)$ and $Q_n(x)$. A conventional normalization of eq. (II.13) is the *Baker definition*, $Q_n(0) = 1$.

The Cramer's rule solution to eq. (II.13) is

$$P_m(x) = \begin{vmatrix} u_{m-n+1} & u_{m-n+2} & \cdots & u_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ u_m & u_{m+1} & \cdots & u_{m+n} \\ \sum\limits_{j=n}^{m} u_{j-n} x^j & \sum\limits_{j=n-1}^{m} u_{j-n+1} x^j & \cdots & \sum\limits_{j=0}^{m} u_j x^j \end{vmatrix} \tag{II.14}$$

and

$$Q_n(x) = \begin{vmatrix} u_{m-n+1} & u_{m-n+2} & \cdots & u_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ u_m & u_{m+1} & \cdots & u_{m+n} \\ x^n & x^{n-1} & \cdots & 1 \end{vmatrix}. \tag{II.15}$$

Remark 6. $[m/0]$ correspond to truncated Maclaurin series.

Remark 7. $[n/n]$ is called the *diagonal Padé approximant.*

III. Results

III.I. **The Maclaurin Series Solutions.** Multiplying eq. (I.2) by x and taking the differential transformation we have

$$\begin{aligned} 0 &= \sum_{i=0}^{k} \delta(i-1)(k+2-i)(k+1-i)U(k+2-i) + 2(k+1)U(k+1) \\ &+ \sum_{i=0}^{k} \delta(i-1)F(k-1) \\ &= k(k+1)U(k+1) + 2(k+1)U(k+1) + F(k-1), \end{aligned}$$

where F is the differential transformation of $f(u) = u^a$. Hence the recurrence relation is

$$U(k+2) = \frac{F(k)}{(k+2)(k+3)}, \ k \geq 0. \tag{III.1}$$

From initial conditions, we have $U(0) = 1$ and $U(1) = 0$.

Now, $F(k)$ is defined by eq. (II.10). By definition 3 the differential inverse transformation of $U(k)$ is the solution of eq. (I.2).

The obtained Maclaurin series solutions, truncated to $\mathcal{O}(100)$, are presented in fig. 1. For nonlinear eq. (I.2) ($a \notin \{0, 1\}$), the rate of convergence of the obtained solutions is low enough to make the method inefficient.

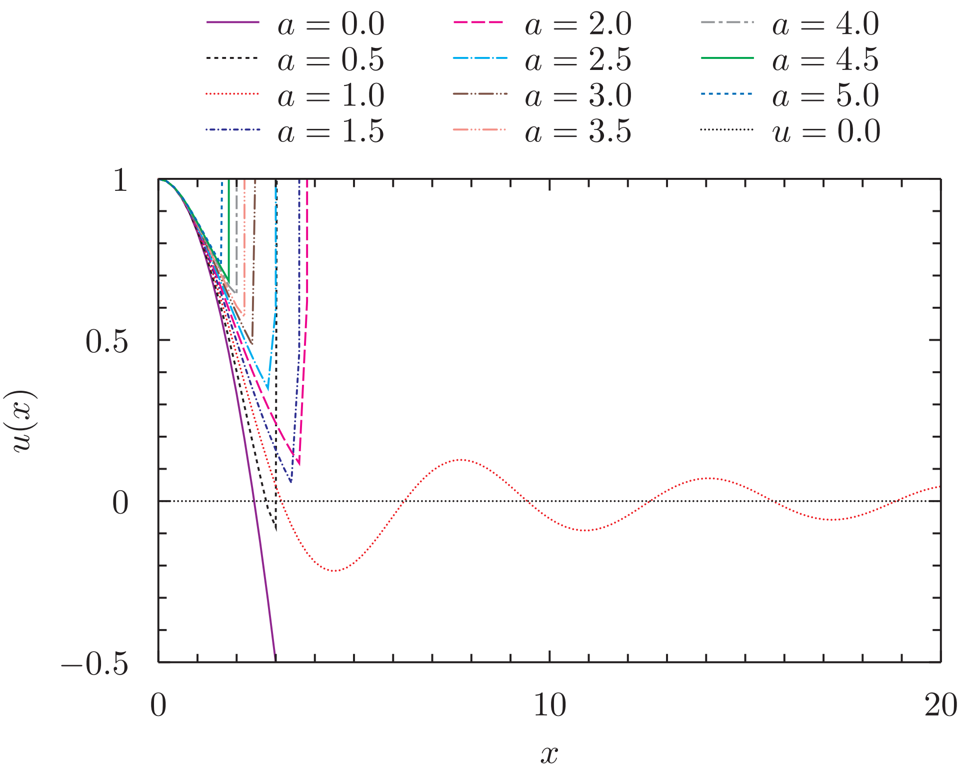

FIGURE 1. The Maclaurin series solutions of eq. (I.2), truncated to $\mathcal{O}(100)$

III.II. **The Padé Approximant Solutions.** In order to accelerate the convergence of the obtained Maclaurin series for the nonlinear eq. (I.2), we calculated the diagonal Padé approximants using the Cramer's rule solution (eq. (II.14) and eq. (II.15)).

The solutions for the nonlinear eq. (I.2), plotted with $\mathcal{O}(100)$ Maclaurin series, or Padé approximant [100/0], solutions of linear eq. (I.2), are presented in fig. 2.

Remark 8. For actual implementation of the calculations, see appendix A for the used code.

III.III. **Values of the Polytropic Constants.** In calculating the values of the polytropic constants, we used Padé approximant [100/0] solutions for the linear eq. (I.2) and Padé approximant [50/50] solutions for the nonlinear eq. (I.2). The values, calculated using the equations presented in section I.II, are listed in table 1.

IV. DISCUSSION

IV.I. **Accuracy.** The accuracy of the results depends on the order of the obtained series. Also, since the solutions are truncated Maclaurin series, the accuracy decays as a function of x. However, in the case of $a = 0$, the exact result is obtained already at $\mathcal{O}(2)$. It is also noteworthy that in the cases $a \in \{1, 5\}$, the obtained formal series have a closed-form expression.

To illustrate the accuracy of the method used, the difference between our results and analytical solutions is plotted in fig. 3.

IV.II. **Padé Approximants.** It should be noted diagonal Padé approximants were used without justification. After serious effort, it became clear that finding the best Padé approximants (BPA) is beyond the scope of this report – especially since the results obtained with diagonal Padé approximants are highly accurate.

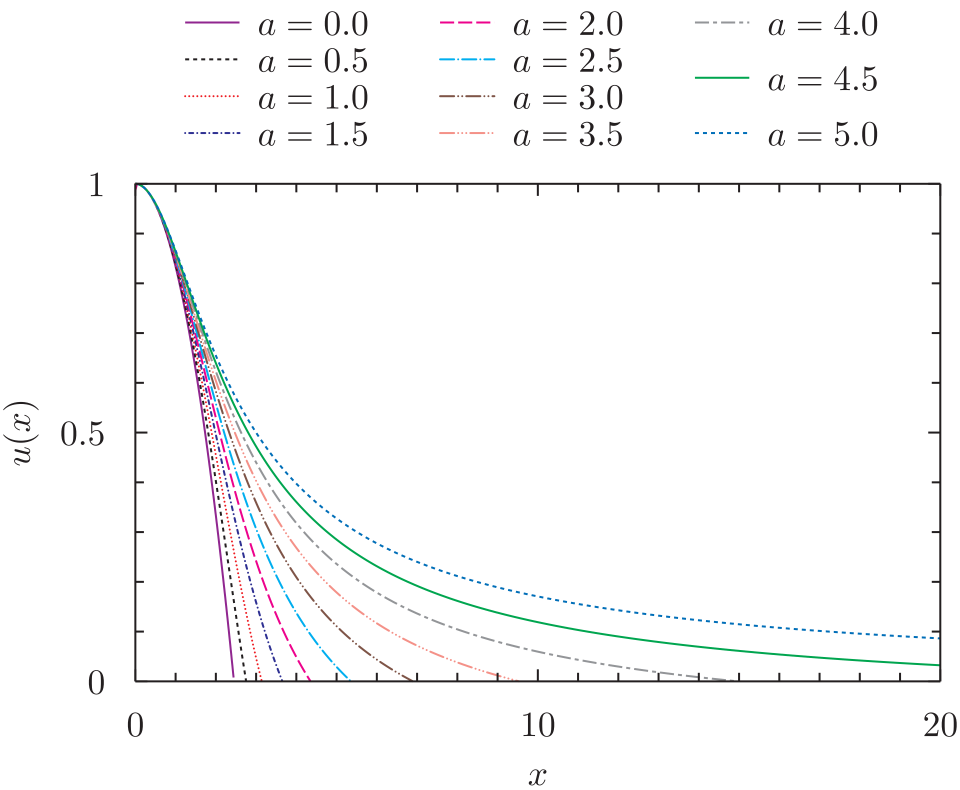

FIGURE 2. The diagonal Padé approximant [50/50] solutions of nonlinear eq. (I.2) and the $\mathcal{O}(100)$ Maclaurin series, or Padé approximant [100/0], solutions of linear eq. (I.2). The solutions are plotted until $u(x) < 0$ for $x \in [0, 20]$.

TABLE 1. The values of the polytropic constants.

a	R_a	M_a	D_a	B_a
0.0	2.449490	4.898979	1.000000	1^i
0.5	2.752697	3.789147	1.834902	0.274293
1.0	3.141593	3.141593	3.289868	0.233097
1.5	3.653754	2.714046	5.990725	0.205580
2.0	4.352875	2.411046	11.40254	0.185385
2.5	5.355275	2.187200	23.40645	0.169566
3.0	6.896849	2.018236	54.18248	0.156540
3.5	9.535806	1.890555	152.8838	0.145343
4.0	14.97160	1.797152	622.4411	0.135303
4.5	32.00346	1.692663	6455.040	0.128014
5.0	∞^i	$\sqrt{3}^{\,i}$	∞^i	$^{-3}\!\sqrt{3}/6^{\,i}$

Notes. The values, calculated to great accuracy, are presented here with a precision of seven digits; the superscript i marks an analytical result.

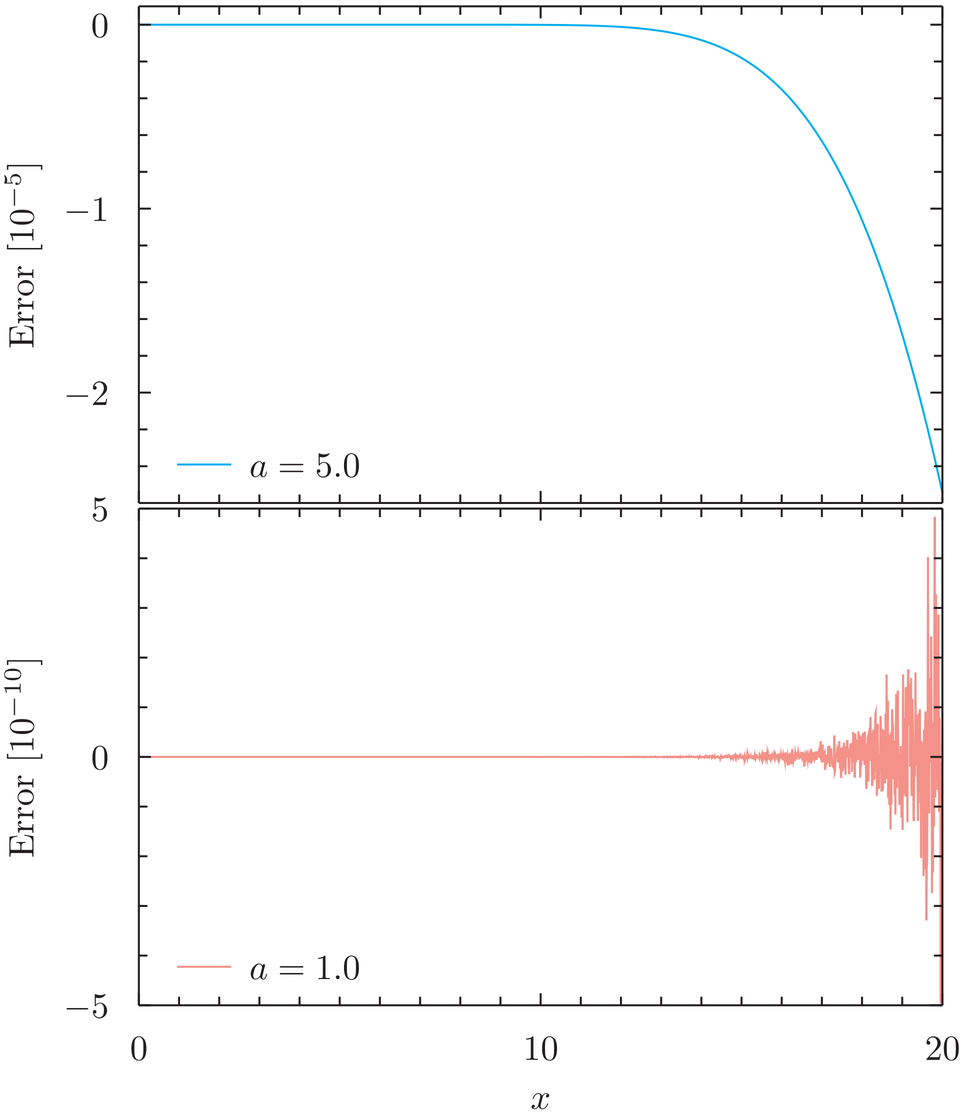

FIGURE 3. The error of the diagonal Padé approximant [50/50] for $a \in \{1, 5\}$. For $a = 0$ DTM produces an exact solution at $\mathcal{O}(2)$.

V. Conclusions

In this report we solved the Lane-Emden equation, eq. (I.2), using the improved differential transformation method. The solutions were used to calculate the values of the polytropic constants for various exponents a.

The differential transformation method, improved with the application of the Adomian decomposition method for nonlinear functions, is a powerful and accurate method in solving singular initial value problems. While, at first sight, the actual calculations did look cumbersome to implement in code, that was not the case. Applying Faà di Bruno's formula, in conjunction with Bell polynomials, to the recurrence relation obtained for eq. (I.2) simplified the problem considerably.

In order to accelerate the convergence of the truncated Maclaurin series obtained as solutions of the nonlinear eq. (I.2), we used diagonal Padé approximants with excellent results.

Appendix A. code

Written in Julia, version 0.4.0.

```
using Memoize


function lane_emden(a,ord)
  c=[U(a,k)::BigFloat for k=0:ord]
  i=floor(Integer,ord/2)
  m,n=(a==0 || a==1) ? (ord,0) : (i,i)
  pade(c,m,n)
end


function fact(k)
  !isinteger(k) ? BigFloat(factorial(k)) :
  BigFloat(factorial(BigInt(k)))
end


@memoize function U(a,k)
  k==0 ? one(BigFloat)  :
  k==1 ? zero(BigFloat) :
  -F(a,k-2)/(k^2+k)
end


function F(a,k)
  k==0 ? U(a,0)^a :
  one(BigFloat)/fact(k)*sum([dfdu(a,n)*B(a,k,n) for n=1:k])
end


function dfdu(a,n)
  isinteger(a) && n>a>=0 ? zero(BigFloat) :
  fact(a)/fact(a-n)*U(a,0)^(a-n)
end


@memoize function B(a,k,n)
  bin(x,y)=BigFloat(binomial(BigInt(x),y))
  n==k ? U(a,1)^k :
  n==1 ? U(a,k)*fact(k) :
  sum([bin(k-1,i-1)*fact(i)*U(a,i)*B(a,k-i,n-1) for i=1:k-n+1])
end
```

```
function pade(c,m,n)
  w=[m+i+j>n ? c[m-n+i+j] : zero(BigFloat) for i=1:n, j=1:n+1]
  fmt(v)=det(vcat(w,transpose(v)))
  P(x)=fmt([m+j+2>n ? sum([c[i+j-n+1]*x^i for i=n-j:m]) :
    zero(BigFloat) for j=0:n])
  Q(x)=fmt([x^(n-i) for i=0:n])
  u(x)=(x!=0) ? Float64(P(x)/Q(x)) : 1.0
end
```

Appendix B. Acknowledgements

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