Stellar structure and evolution
Computer assignment for advanced students

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Part 1. Solve the Lane-Emden equation for $n = 0$, $n = 0.5$, $n = 1$, $n = 1.5$, $n = 2$, $n = 2.5$, $n = 3$, $n = 3.5$, $n = 4$, $n = 4.5$, and $n = 5$. Plot the results.

Solution

The function for solving the Lane-Emden equation for given n:

```
;-------------------------------------------------------------------------;
; lane_emden.pro
;-------------------------------------------------------------------------;
; Description:
; Function for solving the Lane-Emden equation for given polytropic
; index n. Creates vector ksi and solves theta for those values.
; Also added: return dTheta/dKsi

; Input: n
; Output: array result=[[ksi], [theta],[dtheta_dksi]]
;-------------------------------------------------------------------------;

function lane_emden,n
;-------------------------------------------------------------------------;
; Polytropic index
;-------------------------------------------------------------------------;
; Polytropic index n is taken from the input
n=n

;-------------------------------------------------------------------------;
; Vector ksi
;-------------------------------------------------------------------------;
; Ksi has values in the range [0,100]
ksi=findgen(100000)*0.001d0

; Vectors dTheta/dKsi and Theta have the same amount of components as Ksi
dtheta_dksi=ksi*0.d0
theta=ksi*0.d0

;-------------------------------------------------------------------------;
; Lane-Emden equation
;-------------------------------------------------------------------------;
;Loop that solves theta[i] from ksi[i]

;Starting values
i=1
dtheta_dksi[0]=0.d0
ksi[0]=0.d0
theta[0]=1.d0

;Integration loop
while theta[i-1] gt 0.d0 do begin
if i ge n_elements(ksi) then break
;Theta
   theta[i]=theta[i-1]+(ksi[i]-ksi[i-1])*dtheta_dksi[i-1]
;Derivative of theta
```

```
   dtheta_dksi[i]=dtheta_dksi[i-1]-(ksi[i]-ksi[i-1])*(theta[i-1]^n+2.d0/ksi[i]*dtheta_dksi[i-1])
;Increase index i
   i=i+1
endwhile

;-------------------------------------------------------------------------;
;Cut the zeroes away from the end of the theta -vector, and
;make ksi to be the same lenght as theta.
;-------------------------------------------------------------------------;

;Make new vectors ksi0 and theta0
ksi0=0.d0
theta0=1.d0
dtheta_dksi0=0.d0

j=1
while theta[j-1] gt 0.d0 do begin
if j ge n_elements(ksi) then break
theta0=[theta0,theta[j]]
ksi0=[ksi0,ksi[j]]
dtheta_dksi0=[dtheta_dksi0,dtheta_dksi[j]]
j=j+1
endwhile

;Result array is [[ksi0],[theta0],[dtheta_dksi0]]
result=[[ksi0],[theta0],[dtheta_dksi0]]

return,result
end
```

The main program, that calls the previously defined function for solving the Lane-Emden equation, and plots the results as (ξ, θ) for all requested values of n:

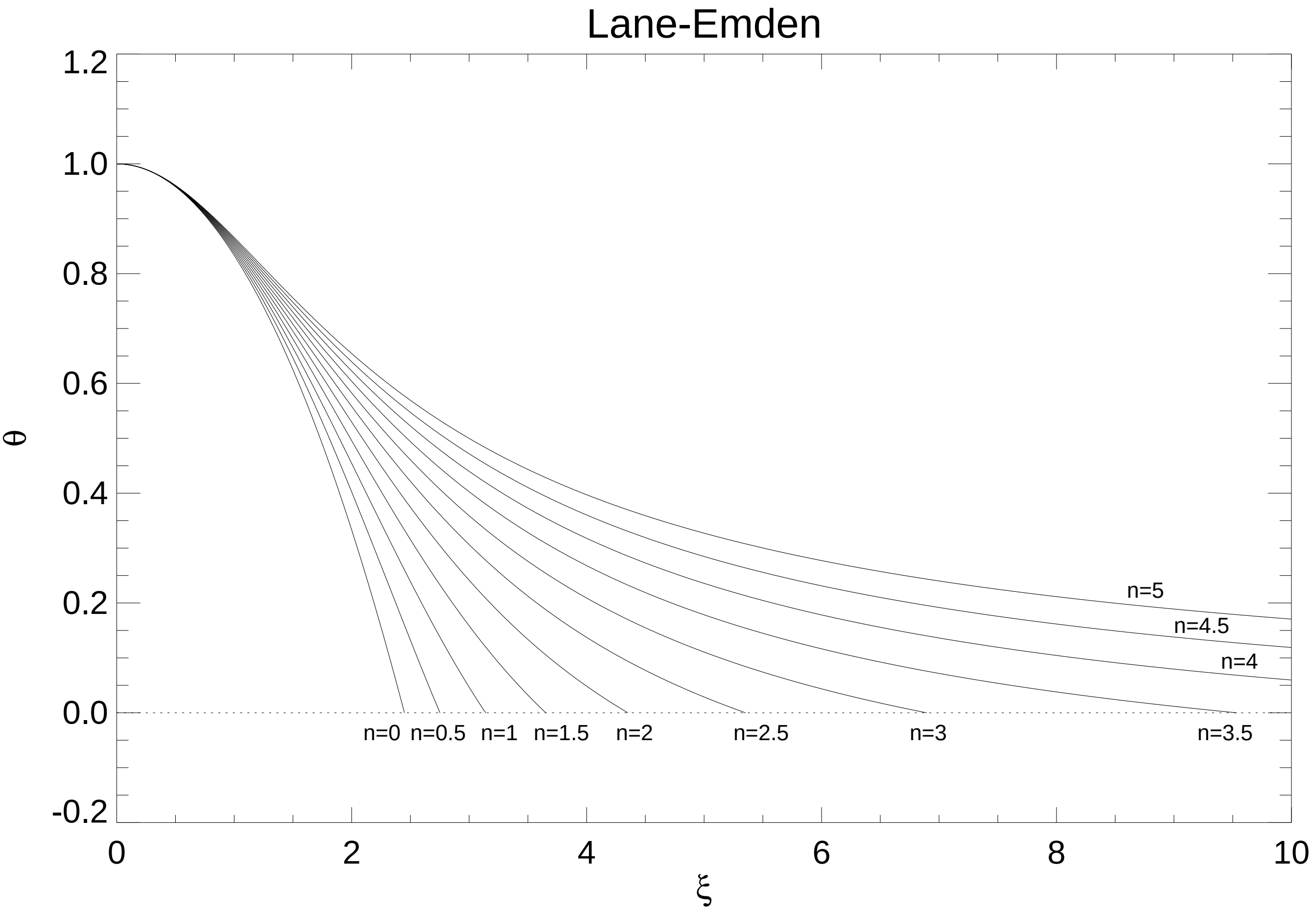
```
;-------------------------------------------------------------------------;
; Stellar structure and evolution
; Computer assignment for advanced students
;-------------------------------------------------------------------------;
;  PART I
;-------------------------------------------------------------------------;
; Use the subroutine PsPlot to save results in a postscript plot
; (written by Heikki Salo)
;-------------------------------------------------------------------------;

pro PsPlot,routine,filename
thisdir=getenv('PWD')+'/'
psopen,/color,dir=thisdir,filename
call_procedure,routine
psclose
end
;-------------------------------------------------------------------------;

;-------------------------------------------------------------------------;
;  MAIN PROGRAM starts here
;-------------------------------------------------------------------------;
; Solve the Lane-Emden equation for
; n=[0,0.5,1,1.5,2,2.5,3,3.5,4,4.5,5]
; and plot the results (ksi,theta).

pro stellar_part1

;Use the function lane_emden for solving the Lane-Emden equation

;-------------------------------------------------------------------------;
;For n=0
;-------------------------------------------------------------------------;
n1=0
results1=lane_emden(n1)
;Separate the results
ksi1=results1[*,0]
theta1=results1[*,1]

;Plot the results as (ksi,theta) for all values of n
plot,ksi1,theta1,xrange=[0,10],yrange=[-0.1,1.1],title='Lane-Emden',xtitle='!9x!X!N',ytitle='!9q!X!N'
oplot,[0,10],[0,0],linestyle=1
```

```
xyouts,2.1,-0.05,'n=0'

;------------------------------------------------------------------------;
;For n=0.5
;------------------------------------------------------------------------;
n2=0.5
results2=lane_emden(n2)
;Separate the results
ksi2=results2[*,0]
theta2=results2[*,1]

;Plot the results as (ksi,theta) for all values of n
oplot,ksi2,theta2
xyouts,2.5,-0.05,'n=0.5'

;------------------------------------------------------------------------;
;For n=1
;------------------------------------------------------------------------;
n3=1
results3=lane_emden(n3)
;Separate the results
ksi3=results3[*,0]
theta3=results3[*,1]

;Plot the results as (ksi,theta) for all values of n
oplot,ksi3,theta3
xyouts,3.1,-0.05,'n=1'

;------------------------------------------------------------------------;
;For n=1.5
;------------------------------------------------------------------------;
n4=1.5
results4=lane_emden(n4)
;Separate the results
ksi4=results4[*,0]
theta4=results4[*,1]

;Plot the results as (ksi,theta) for all values of n
oplot,ksi4,theta4
xyouts,3.55,-0.05,'n=1.5'

;------------------------------------------------------------------------;
;For n=2
;------------------------------------------------------------------------;
n5=2
results5=lane_emden(n5)
;Separate the results
ksi5=results5[*,0]
theta5=results5[*,1]

;Plot the results as (ksi,theta) for all values of n
oplot,ksi5,theta5
xyouts,4.25,-0.05,'n=2'

;------------------------------------------------------------------------;
;For n=2.5
;------------------------------------------------------------------------;
n6=2.5
results6=lane_emden(n6)
;Separate the results
ksi6=results6[*,0]
theta6=results6[*,1]

;Plot the results as (ksi,theta) for all values of n
oplot,ksi6,theta6
xyouts,5.25,-0.05,'n=2.5'

;------------------------------------------------------------------------;
;For n=3
;------------------------------------------------------------------------;
n7=3
results7=lane_emden(n7)
;Separate the results
ksi7=results7[*,0]
theta7=results7[*,1]
```

```
;Plot the results as (ksi,theta) for all values of n
oplot,ksi7,theta7
xyouts,6.75,-0.05,'n=3'

;-------------------------------------------------------------------------;
;For n=3.5
;-------------------------------------------------------------------------;
n8=3.5
results8=lane_emden(n8)
;Separate the results
ksi8=results8[*,0]
theta8=results8[*,1]

;Plot the results as (ksi,theta) for all values of n
oplot,ksi8,theta8
xyouts,9.2,-0.05,'n=3.5'

;-------------------------------------------------------------------------;
;For n=4
;-------------------------------------------------------------------------;
n9=4
results9=lane_emden(n9)
;Separate the results
ksi9=results9[*,0]
theta9=results9[*,1]

;Plot the results as (ksi,theta) for all values of n
oplot,ksi9,theta9
xyouts,9.4,0.08,'n=4'

;-------------------------------------------------------------------------;
;For n=4.5
;-------------------------------------------------------------------------;
n10=4.5
results10=lane_emden(n10)
;Separate the results
ksi10=results10[*,0]
theta10=results10[*,1]

;Plot the results as (ksi,theta) for all values of n
oplot,ksi10,theta10
xyouts,9,0.145,'n=4.5'

;-------------------------------------------------------------------------;
;For n=5
;-------------------------------------------------------------------------;
n11=5
results11=lane_emden(n11)
;Separate the results
ksi11=results11[*,0]
theta11=results11[*,1]

;Plot the results as (ksi,theta) for all values of n
oplot,ksi11,theta11
xyouts,8.6,0.21,'n=5'

end

;-------------------------------------------------------------------;
; Save the results to a PostScript file using PsPlot
;-------------------------------------------------------------------;

pro Plot_everything
PsPlot, 'stellar_part1', 'stellar_part1.ps'
end
```

Results

The final results as (ξ, θ) for all requested values of n.



Part 2. In the two cases where we have found analytical solutions to the Lane-Emden equation, compare them to your numerical derivations, so you can check that everything is correct.

Answer:

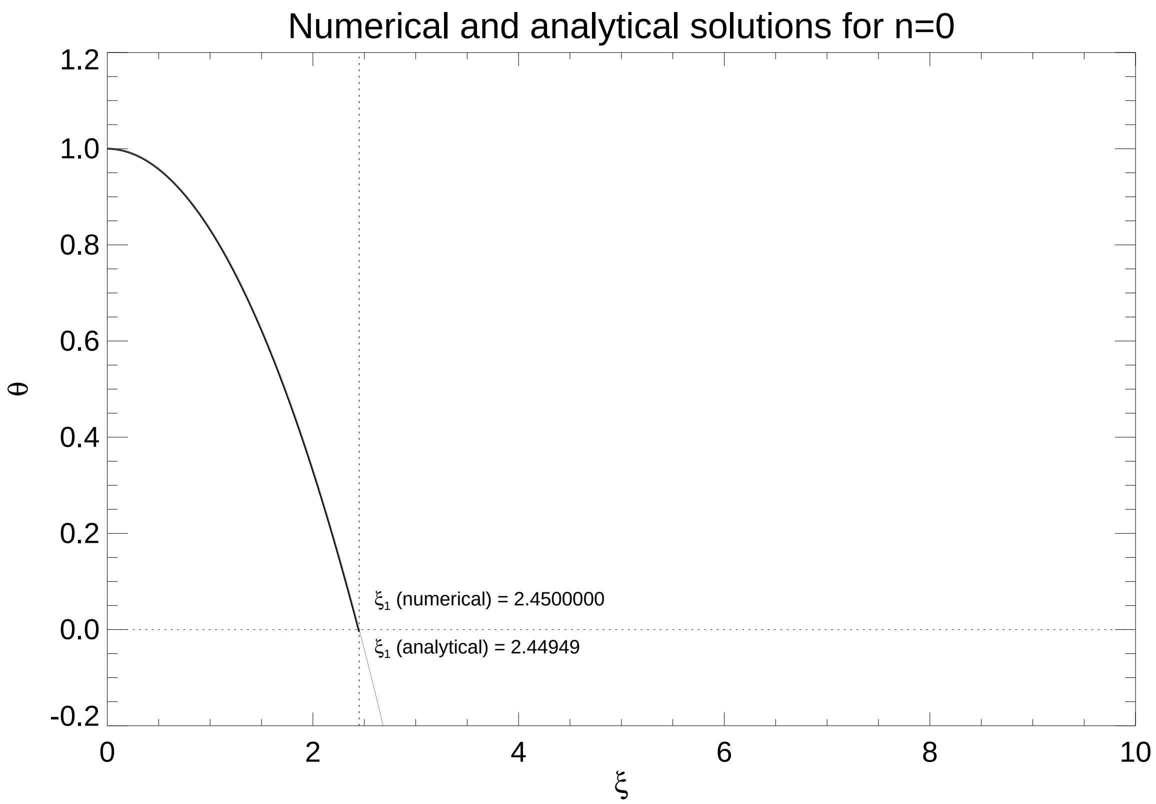
In the exercises we solved the Lane-Emden equation analytically for $n = 0$ and $n = 1$.

For $n = 0$ the solution was

$$\theta(\xi) = -\frac{1}{6}\xi^2 + 1\,, \tag{1}$$

which is a downwards opening parabola that crosses the θ axis at $\theta = 1$ and the ξ axis at $\xi_1 = \sqrt{6} \approx 2.44949$.

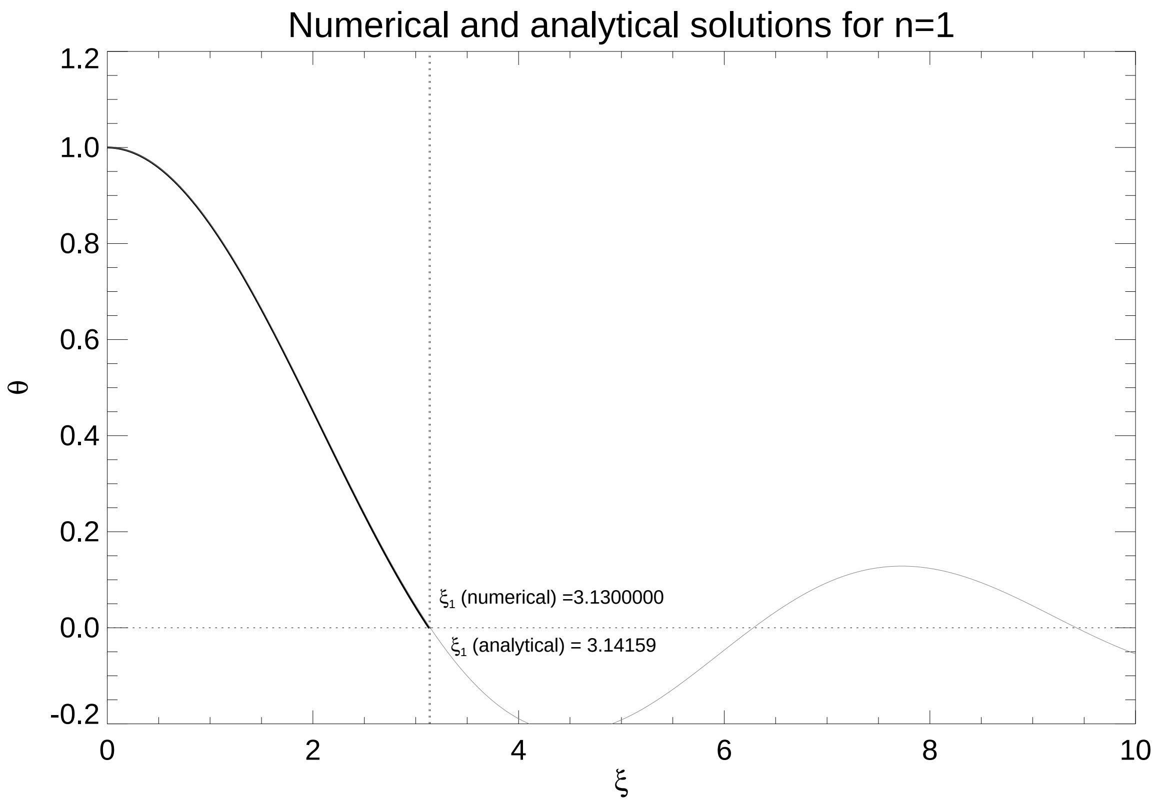
In IDL the numerical (black line) and analytical (grey line) solutions look like this:



For $n = 1$ the analytical solution is

$$\theta(\xi) = \frac{\sin \xi}{\xi}\,. \tag{2}$$

In IDL the numerical (black line) and analytical (grey line) solutions for $n = 1$ look like this:


Numerical and analytical solutions for n=1
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
θ
0
2
4
6
8
10
ξ
ξ_1 (numerical) =3.1300000
ξ_1 (analytical) = 3.14159

Part 3. Compute D_n, M_n, R_n and B_n for each n.

Solution

From lecture notes we get that

$$R_n = \xi_1 \,, \tag{3}$$

where ξ_1 corresponds to the stellar radius (where $\theta = 0$).

Other constants are (again, from the lecture notes):

$$B_n = \frac{(3D_n)^{\frac{3-n}{n}}}{(n+1) M_n^{\frac{n-1}{n}} R_n^{\frac{3-n}{n}}} \tag{4}$$

$$D_n = -\left[\frac{3}{\xi_1}\left(\frac{d\theta}{d\xi}\right)_{\xi_1}\right]^{-1} \tag{5}$$

$$M_n = -\xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi_1} \tag{6}$$

These are solved in the program written in IDL:

```
;-----------------------------------------------------------------------;
; Stellar structure and evolution
; Computer assignment for advanced students
;-----------------------------------------------------------------------;

;  PART III - Polytropic constants

;-----------------------------------------------------------------------;
; Use the subroutine PsPlot to save results in a postscript plot
; (written by Heikki Salo)
;-----------------------------------------------------------------------;

pro PsPlot,routine,filename
thisdir=getenv('PWD')+'/'
psopen,/color,dir=thisdir,filename
call_procedure,routine
psclose
end
;-----------------------------------------------------------------------;

;-----------------------------------------------------------------------;
;  MAIN PROGRAM starts here
;-----------------------------------------------------------------------;
;Compute Dn, Mn, Rn and Bn for each value of n.

pro stellar_part3

;-----------------------------------------------------------------------;
; Solve Lane-Emden for n=0
;-----------------------------------------------------------------------;
n1=0.d0
results1=lane_emden(n1)
;Separate the results
ksi1=results1[*,0]
theta1=results1[*,1]
dtheta_dksi1=results1[*,2]

;-----------------------------------------------------------------------;
; Rn (for n=0)
;-----------------------------------------------------------------------;
;The program lane_emden cuts the results at theta=0, which means that
;the last term of the vector ksi is Rn
```

```
limit1=n_elements(ksi1)-1
Rn1=ksi1(limit1)
;print,theta1(limit1)
print,'Rn1(n=0)'
print,Rn1

;-----------------------------------------------------------------------;
; Dn (for n=0)
;-----------------------------------------------------------------------;
Dn1=-1.d0/(3.d0/Rn1*dtheta_dksi1(limit1))
print,'Dn1(n=0)'
print,Dn1

;-----------------------------------------------------------------------;
; Mn (for n=0)
;-----------------------------------------------------------------------;
Mn1=-Rn1^2*dtheta_dksi1(limit1)
print,'Mn1(n=0)'
print,Mn1

;-----------------------------------------------------------------------;
; Bn (for n=0)
;-----------------------------------------------------------------------;
;The equation would normally be:
;Bn1=(3.d0*Dn1)^((3-n1)/(3*n1))/((n1+1)*Mn^((n1-1)/n1)*Rn^((3-n1)/n1))
;n=0 is a special case, so for n->0 we define
Bn1=1.d0
print,'Bn1(n=0)'
print,Bn1

;-----------------------------------------------------------------------;
; Solve Lane-Emden for n=0.5
;-----------------------------------------------------------------------;
n2=0.5d0
results2=lane_emden(n2)
;Separate the results
ksi2=results2[*,0]
theta2=results2[*,1]
dtheta_dksi2=results2[*,2]

;-----------------------------------------------------------------------;
; Rn (for n=0.5)
;-----------------------------------------------------------------------;
limit2=n_elements(ksi2)-1
Rn2=ksi2(limit2)
print,'Rn2(n=0.5)'
print,Rn2

;-----------------------------------------------------------------------;
; Dn (for n=0.5)
;-----------------------------------------------------------------------;
Dn2=-1.d0/(3.d0/Rn2*dtheta_dksi2(limit2))
print,'Dn2(n=0.5)'
print,Dn2

;-----------------------------------------------------------------------;
; Mn (for n=0.5)
;-----------------------------------------------------------------------;
Mn2=-Rn2^2*dtheta_dksi2(limit2)
print,'Mn2(n=0.5)'
print,Mn2

;-----------------------------------------------------------------------;
; Bn (for n=0.5)
;-----------------------------------------------------------------------;
Bn2=(3.d0*Dn2)^((3.d0-n2)/(3.d0*n2))/((n2+1.d0)*Mn2^((n2-1.d0)/n2)*Rn2^((3.d0-n2)/n2))
print,'Bn2(n=0.5)'
print,Bn2

;-----------------------------------------------------------------------;
; Solve Lane-Emden for n=1
;-----------------------------------------------------------------------;
n3=1.d0
results3=lane_emden(n3)
;Separate the results
```

```
ksi3=results3[*,0]
theta3=results3[*,1]
dtheta_dksi3=results3[*,2]

;-------------------------------------------------------------------------;
; Rn (for n=1)
;-------------------------------------------------------------------------;
limit3=n_elements(ksi3)-1
Rn3=ksi3(limit3)
print,'Rn3(n=1)'
print,Rn3

;-------------------------------------------------------------------------;
; Dn (for n=1)
;-------------------------------------------------------------------------;
Dn3=-1.d0/(3.d0/Rn3*dtheta_dksi3(limit3))
print,'Dn3(n=1)'
print,Dn3

;-------------------------------------------------------------------------;
; Mn (for n=1)
;-------------------------------------------------------------------------;
Mn3=-Rn3^2*dtheta_dksi3(limit3)
print,'Mn3(n=1)'
print,Mn3

;-------------------------------------------------------------------------;
; Bn (for n=1)
;-------------------------------------------------------------------------;
Bn3=(3.d0*Dn3)^((3.d0-n3)/(3.d0*n3))/((n3+1.d0)*Mn3^((n3-1.d0)/n3)*Rn3^((3.d0-n3)/n3))
print,'Bn3(n=1)'
print,Bn3

;-------------------------------------------------------------------------;
; Solve Lane-Emden for n=1.5
;-------------------------------------------------------------------------;
n4=1.5d0
results4=lane_emden(n4)
;Separate the results
ksi4=results4[*,0]
theta4=results4[*,1]
dtheta_dksi4=results4[*,2]

;-------------------------------------------------------------------------;
; Rn (for n=1.5)
;-------------------------------------------------------------------------;
limit4=n_elements(ksi4)-1
Rn4=ksi4(limit4)
print,'Rn4(n=1.5)'
print,Rn4

;-------------------------------------------------------------------------;
; Dn (for n=1.5)
;-------------------------------------------------------------------------;
Dn4=-1.d0/(3.d0/Rn4*dtheta_dksi4(limit4))
print,'Dn4(n=1.5)'
print,Dn4

;-------------------------------------------------------------------------;
; Mn (for n=1.5)
;-------------------------------------------------------------------------;
Mn4=-Rn4^2*dtheta_dksi4(limit4)
print,'Mn4(n=1.5)'
print,Mn4

;-------------------------------------------------------------------------;
; Bn (for n=1.5)
;-------------------------------------------------------------------------;
Bn4=(3.d0*Dn4)^((3-n4)/(3*n4))/((n4+1.d0)*Mn4^((n4-1)/n4)*Rn4^((3.d0-n4)/n4))
print,'Bn4(n=1.5)'
print,Bn4

;-------------------------------------------------------------------------;
; Solve Lane-Emden for n=2
;-------------------------------------------------------------------------;
n5=2.d0
```

```
results5=lane_emden(n5)
;Separate the results
ksi5=results5[*,0]
theta5=results5[*,1]
dtheta_dksi5=results5[*,2]

;-----------------------------------------------------------------------;
; Rn (for n=2)
;-----------------------------------------------------------------------;
limit5=n_elements(ksi5)-1
Rn5=ksi5(limit5)
print,'Rn5(n=2)'
print,Rn5

;-----------------------------------------------------------------------;
; Dn (for n=2)
;-----------------------------------------------------------------------;
Dn5=-1.d0/(3.d0/Rn5*dtheta_dksi5(limit5))
print,'Dn5(n=2)'
print,Dn5

;-----------------------------------------------------------------------;
; Mn (for n=2)
;-----------------------------------------------------------------------;
Mn5=-Rn5^2*dtheta_dksi5(limit5)
print,'Mn5(n=2)'
print,Mn5

;-----------------------------------------------------------------------;
; Bn (for n=2)
;-----------------------------------------------------------------------;
Bn5=(3.d0*Dn5)^((3.d0-n5)/(3.d0*n5))/((n5+1.d0)*Mn5^((n5-1)/n5)*Rn5^((3.d0-n5)/n5))
print,'Bn5(n=2)'
print,Bn5

;-----------------------------------------------------------------------;
; Solve Lane-Emden for n=2.5
;-----------------------------------------------------------------------;
n6=2.5d0
results6=lane_emden(n6)
;Separate the results
ksi6=results6[*,0]
theta6=results6[*,1]
dtheta_dksi6=results6[*,2]

;-----------------------------------------------------------------------;
; Rn (for n=2.5)
;-----------------------------------------------------------------------;
limit6=n_elements(ksi6)-1
Rn6=ksi6(limit6)
print,'Rn6(n=2.5)'
print,Rn6

;-----------------------------------------------------------------------;
; Dn (for n=2.5)
;-----------------------------------------------------------------------;
Dn6=-1.d0/(3.d0/Rn6*dtheta_dksi6(limit6))
print,'Dn6(n=2.5)'
print,Dn6

;-----------------------------------------------------------------------;
; Mn (for n=2.5)
;-----------------------------------------------------------------------;
Mn6=-Rn6^2*dtheta_dksi6(limit6)
print,'Mn6(n=2.5)'
print,Mn6

;-----------------------------------------------------------------------;
; Bn (for n=2.5)
;-----------------------------------------------------------------------;
Bn6=(3.d0*Dn6)^((3.d0-n6)/(3.d0*n6))/((n6+1.d0)*Mn6^((n6-1)/n6)*Rn6^((3.d0-n6)/n6))
print,'Bn6(n=2.5)'
print,Bn6

;-----------------------------------------------------------------------;
; Solve Lane-Emden for n=3
```

```
;-----------------------------------------------------------------------;
n7=3.d0
results7=lane_emden(n7)
;Separate the results
ksi7=results7[*,0]
theta7=results7[*,1]
dtheta_dksi7=results7[*,2]

;-----------------------------------------------------------------------;
; Rn (for n=3)
;-----------------------------------------------------------------------;
limit7=n_elements(ksi7)-1
Rn7=ksi7(limit7)
print,'Rn7(n=3)'
print,Rn7

;-----------------------------------------------------------------------;
; Dn (for n=3)
;-----------------------------------------------------------------------;
Dn7=-1.d0/(3.d0/Rn7*dtheta_dksi7(limit7))
print,'Dn7(n=3)'
print,Dn7

;-----------------------------------------------------------------------;
; Mn (for n=3)
;-----------------------------------------------------------------------;
Mn7=-Rn7^2*dtheta_dksi7(limit7)
print,'Mn7(n=3)'
print,Mn7

;-----------------------------------------------------------------------;
; Bn (for n=3)
;-----------------------------------------------------------------------;
Bn7=(3.d0*Dn7)^((3.d0-n7)/(3.d0*n7))/((n7+1.d0)*Mn7^((n7-1)/n7)*Rn7^((3.d0-n7)/n7))
print,'Bn7(n=3)'
print,Bn7

;-----------------------------------------------------------------------;
; Solve Lane-Emden for n=3.5
;-----------------------------------------------------------------------;
n8=3.5d0
results8=lane_emden(n8)
;Separate the results
ksi8=results8[*,0]
theta8=results8[*,1]
dtheta_dksi8=results8[*,2]

;-----------------------------------------------------------------------;
; Rn (for n=3.5)
;-----------------------------------------------------------------------;
limit8=n_elements(ksi8)-1
Rn8=ksi8(limit8)
print,'Rn8(n=3.5)'
print,Rn8

;-----------------------------------------------------------------------;
; Dn (for n=3.5)
;-----------------------------------------------------------------------;
Dn8=-1.d0/(3.d0/Rn8*dtheta_dksi8(limit8))
print,'Dn8(n=3.5)'
print,Dn8

;-----------------------------------------------------------------------;
; Mn (for n=3.5)
;-----------------------------------------------------------------------;
Mn8=-Rn8^2*dtheta_dksi8(limit8)
print,'Mn8(n=3.5)'
print,Mn8

;-----------------------------------------------------------------------;
; Bn (for n=3.5)
;-----------------------------------------------------------------------;
Bn8=(3.d0*Dn8)^((3.d0-n8)/(3.d0*n8))/((n8+1.d0)*Mn8^((n8-1)/n8)*Rn8^((3.d0-n8)/n8))
print,'Bn8(n=3.5)'
print,Bn8
```

```
;----------------------------------------------------------------------;
; Solve Lane-Emden for n=4
;----------------------------------------------------------------------;
n9=4.d0
results9=lane_emden(n9)
;Separate the results
ksi9=results9[*,0]
theta9=results9[*,1]
dtheta_dksi9=results9[*,2]

;----------------------------------------------------------------------;
; Rn (for n=4)
;----------------------------------------------------------------------;
limit9=n_elements(ksi9)-1
Rn9=ksi9(limit9)
print,'Rn9(n=4)'
print,Rn9

;----------------------------------------------------------------------;
; Dn (for n=4)
;----------------------------------------------------------------------;
Dn9=-1.d0/(3.d0/Rn9*dtheta_dksi9(limit9))
print,'Dn9(n=4)'
print,Dn9

;----------------------------------------------------------------------;
; Mn (for n=4)
;----------------------------------------------------------------------;
Mn9=-Rn9^2*dtheta_dksi9(limit9)
print,'Mn9(n=4)'
print,Mn9

;----------------------------------------------------------------------;
; Bn (for n=4)
;----------------------------------------------------------------------;
Bn9=(3.d0*Dn9)^((3.d0-n9)/(3.d0*n9))/((n9+1.d0)*Mn9^((n9-1)/n9)*Rn9^((3.d0-n9)/n9))
print,'Bn9(n=4)'
print,Bn9

;----------------------------------------------------------------------;
; Solve Lane-Emden for n=4.5
;----------------------------------------------------------------------;
n10=4.5d0
results10=lane_emden(n10)
;Separate the results
ksi10=results10[*,0]
theta10=results10[*,1]
dtheta_dksi10=results10[*,2]

;----------------------------------------------------------------------;
; Rn (for n=4.5)
;----------------------------------------------------------------------;
limit10=n_elements(ksi10)-1
Rn10=ksi10(limit10)
print,'Rn10(n=4.5)'
print,Rn10

;----------------------------------------------------------------------;
; Dn (for n=4.5)
;----------------------------------------------------------------------;
Dn10=-1.d0/(3.d0/Rn10*dtheta_dksi10(limit10))
print,'Dn10(n=4.5)'
print,Dn10

;----------------------------------------------------------------------;
; Mn (for n=4.5)
;----------------------------------------------------------------------;
Mn10=-Rn10^2*dtheta_dksi10(limit10)
print,'Mn10(n=4.5)'
print,Mn10

;----------------------------------------------------------------------;
; Bn (for n=4.5)
;----------------------------------------------------------------------;
Bn10=(3.d0*Dn10)^((3.d0-n10)/(3.d0*n10))/((n10+1.d0)*Mn10^((n10-1)/n10)*Rn10^((3.d0-n10)/n10))
print,'Bn10(n=4.5)'
```

```
print,Bn10

;-----------------------------------------------------------------------;
; Solve Lane-Emden for n=5
;-----------------------------------------------------------------------;
n11=5.d0
results11=lane_emden(n11)
;Separate the results
ksi11=results11[*,0]
theta11=results11[*,1]
dtheta_dksi11=results11[*,2]

;-----------------------------------------------------------------------;
; Rn (for n=5)
;-----------------------------------------------------------------------;
limit11=n_elements(ksi11)-1
Rn11=ksi11(limit11)
print,'Rn11(n=5)'
print,Rn11

;-----------------------------------------------------------------------;
; Dn (for n=5)
;-----------------------------------------------------------------------;
Dn11=-1.d0/(3.d0/Rn11*dtheta_dksi11(limit11))
print,'Dn11(n=5)'
print,Dn11

;-----------------------------------------------------------------------;
; Mn (for n=5)
;-----------------------------------------------------------------------;
Mn11=-Rn11^2*dtheta_dksi11(limit11)
print,'Mn11(n=5)'
print,Mn11

;-----------------------------------------------------------------------;
; Bn (for n=5)
;-----------------------------------------------------------------------;
Bn11=(3.d0*Dn11)^((3.d0-n11)/(3.d0*n11))/((n11+1.d0)*Mn11^((n11-1)/n11)*Rn11^((3.d0-n11)/n11))
print,'Bn11(n=5)'
print,Bn11

end

;-------------------------------------------------------------------;
; Save the results to a PostScript file using PsPlot
;-------------------------------------------------------------------;

pro plot_everything
PsPlot, 'stellar_part3', 'stellar_part3.ps'
end
```

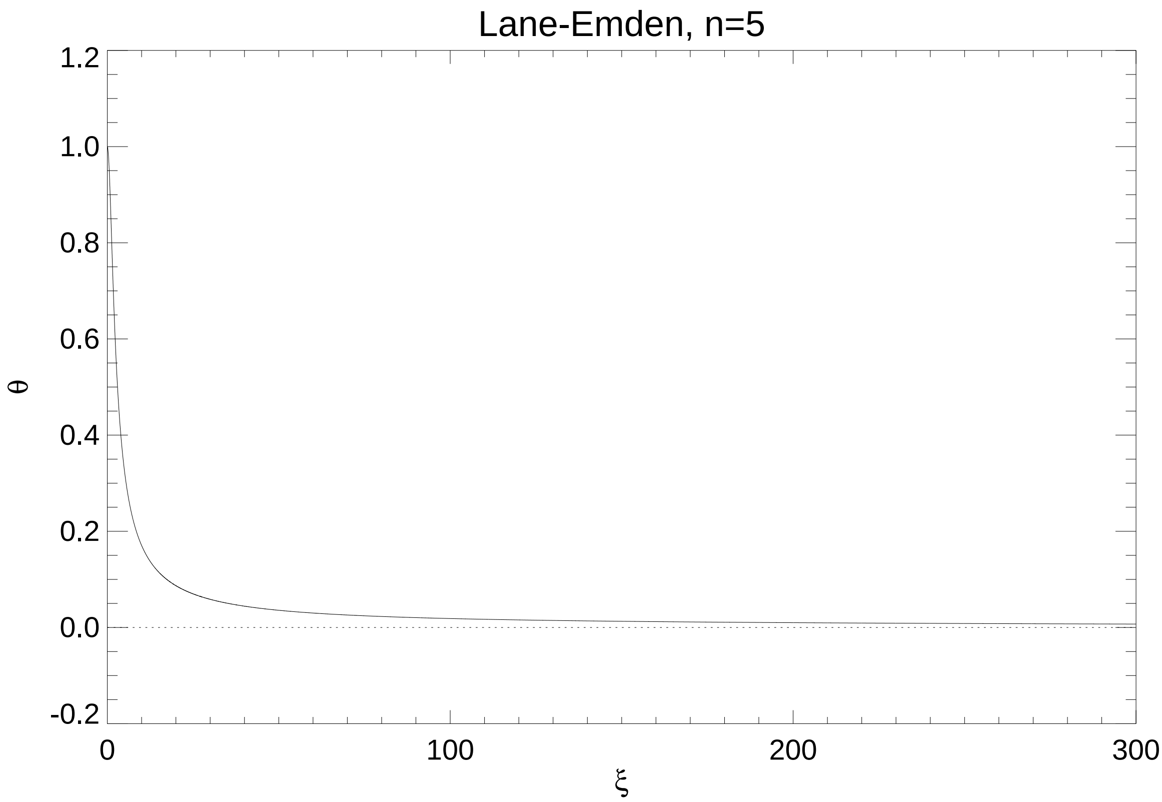
Results

Table of the final results:

n	D_n	M_n	R_n	B_n
0.0	0.99959184	4.8980408	2.4490000	1.0000000
0.5	1.8338162	3.7885081	2.7520000	0.27432365
1.0	3.2890093	3.1406349	3.1410000	0.23314443
1.5	5.9851064	2.7126838	3.6520000	0.20564929
2.0	11.395406	2.4094405	4.3510000	0.18546755
2.5	23.395297	2.1854552	5.3530000	0.16965657
3.0	54.164361	2.0164105	6.8940000	0.15663448
3.5	152.94777	1.8886915	9.5340000	0.14543875
4.0	623.61190	1.7953612	14.976000	0.13539276
4.5	6230.1945	1.7359810	31.895000	0.12587534
5.0	?	?	?	?

Solving the polytropic constants for all the other polytropes except $n = 5$ was rather straightforward, but in the $n = 5$ case the results didn't make any sense.

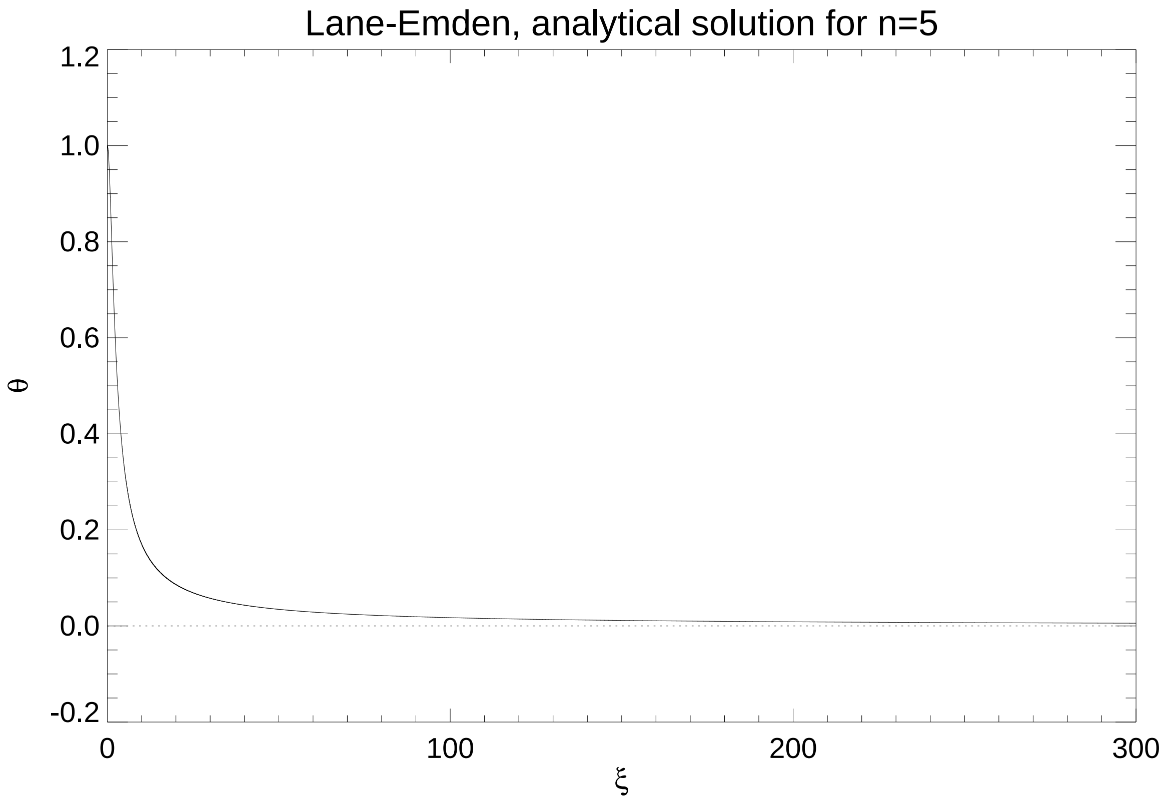
If we solve the Lane-Emden equation separately for $n = 5$ between $0 \leq \xi \leq 300$ and plot the results, we see that $\theta = 0$ is never reached (increasing ξ doesn't seem to help, the $\theta = 0$ limit is never crossed). This must mean that $R_n \to \infty$ when $\theta \to 0$.



The analytical solution for $n = 5$

$$\theta(\xi) = (1 + \frac{\xi^2}{3})^{-\frac{1}{2}} \tag{7}$$

yields similar results



The failed iteration for $\mathbf{n \to 5}$

This method didn't succeed, but I'll document the efforts anyway for the purpose of maybe gaining some sympathy points.

I thought I could solve R_n for different values of $n = 4.9$, $n = 4.99$, $n = 4.999$ and so on with a handy iteration loop (as long as $n < 5$, R_n should be a finite number), and then use the results for solving the other constants. I would then see what values the polytropic constants are approaching as $R_n \to \infty$. However, this method would require that I would be able to increase the integration range ridiculously high.

Only the $n = 4.9$ graph reached $\theta = 0$ before $\xi = 300$, none of the following ones did. The magical $\xi = 300$ limit is a result of trial and error. If I increased the number of points in the $\theta(\xi)$ -vector to over 30000, the computer freezed while trying to solve the Lane-Emden equation. Increasing the range of ξ could only be achieved by increasing the step size, but that led me nowhere, as the accuracy dropped wildly.

I have concluded that my efforts were pointless and stupid, as I could just have set $n = 5$ and R_n as incredibly large.

Solution: The program in IDL

```
;-----------------------------------------------------------------------;
; Stellar structure and evolution
; Computer assignment for advanced students
;-----------------------------------------------------------------------;
; Solving the polytropic constants for n=5

pro constants_n5

n=5

;-----------------------------------------------------------------------;
;  Rn
;-----------------------------------------------------------------------;
;Instead of solving the Rn from Lane-Emden equation like in the
;previous versions, just set Rn to be ridiculously large

;Rn=10.d0^38
Rn=10.d0^99

;-----------------------------------------------------------------------;
; (dtheta/dksi)_ksi1
;-----------------------------------------------------------------------;
;The derivative of the analytical solution (at ksi=Rn) is:

dtheta_dksi=-0.5d0*(1.d0+Rn^2/3.d0)^(-1.5d0)*2.d0/3*Rn

;-----------------------------------------------------------------------;
; Dn
;-----------------------------------------------------------------------;
Dn=-1.d0/(3.d0/Rn*dtheta_dksi)
print,'Dn'
print,Dn

print,'Compare with: infinity'

;-----------------------------------------------------------------------;
; Mn
;-----------------------------------------------------------------------;
Mn=-Rn^2*dtheta_dksi
print,'Mn'
print,Mn

print,'Compare with:'
print,sqrt(3.d0)

;-----------------------------------------------------------------------;
; Bn
```

```
;--------------------------------------------------------------------------;
Bn=(3.d0*Dn)^((3.d0-n)/(3.d0*n))/((n+1.d0)*Mn^((n-1.d0)/n)*Rn^((3.d0-n)/n))
print,'Bn'
print,Bn

print,'Compare with:'
print,(3.d0^(1.d0/3)*6.d0)^(-1.d0)

end
```

Final results for n=5

Using the analytical solution, and setting R_n as very large (and checking if the values change when R_n increases), I got the following results:

$$D_n = 1.9245009 \cdot 10^{296}$$

This value increased every time I increased R_n, so it is safe to assume that $D_n \to \infty$.

$$M_n = 1.7320508$$

I compared this with Mikko's analytical solution $M_n = \sqrt{3}$ and it was the same result.

$$B_n = 0.11556021$$

Mikko got the result $M_n = (3^{\frac{1}{3}} \cdot 6)^{-1}$ which is the same thing, although mathematically more beautiful.

Updated table of the final results:

n	D_n	M_n	R_n	B_n
0.0	0.99959184	4.8980408	2.4490000	1.0000000
0.5	1.8338162	3.7885081	2.7520000	0.27432365
1.0	3.2890093	3.1406349	3.1410000	0.23314443
1.5	5.9851064	2.7126838	3.6520000	0.20564929
2.0	11.395406	2.4094405	4.3510000	0.18546755
2.5	23.395297	2.1854552	5.3530000	0.16965657
3.0	54.164361	2.0164105	6.8940000	0.15663448
3.5	152.94777	1.8886915	9.5340000	0.14543875
4.0	623.61190	1.7953612	14.976000	0.13539276
4.5	6230.1945	1.7359810	31.895000	0.12587534
5.0	$1.9245009 \cdot 10^{296}$ (∞)	$M_n = 1.7320508$	∞	0.11556021

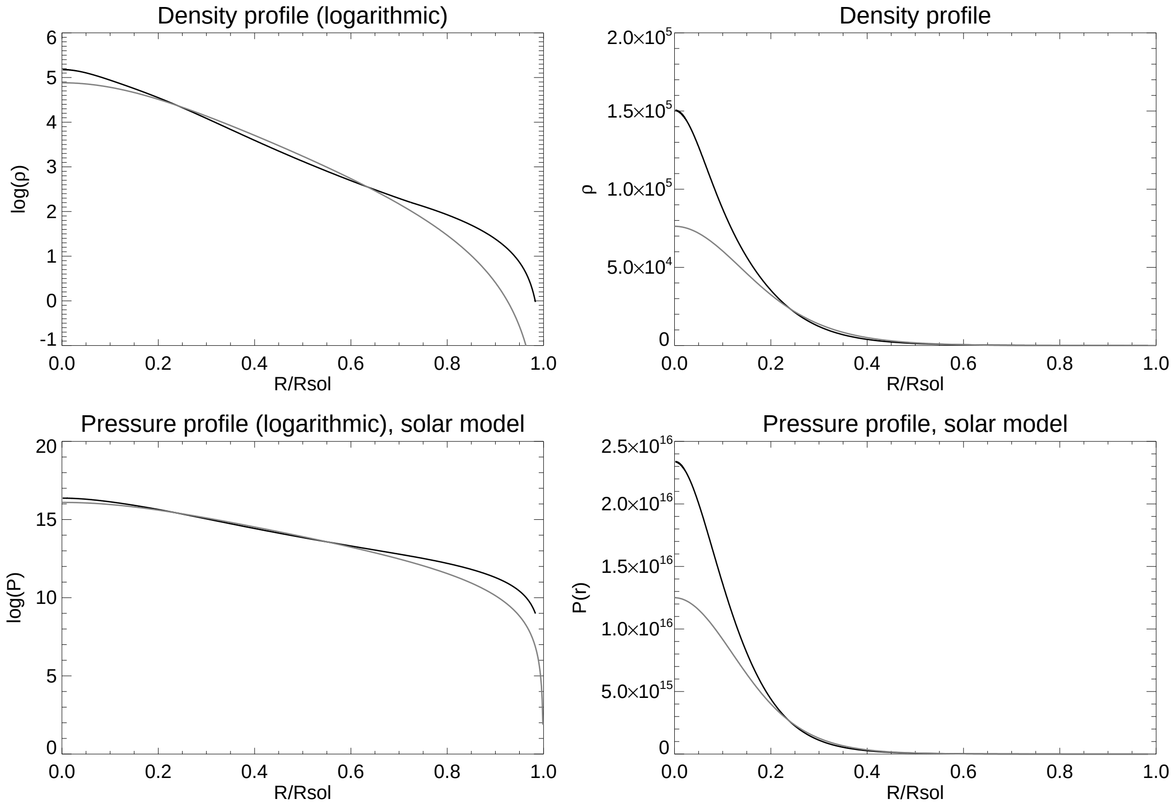
Part 4 A sophisticated solar model can be found at

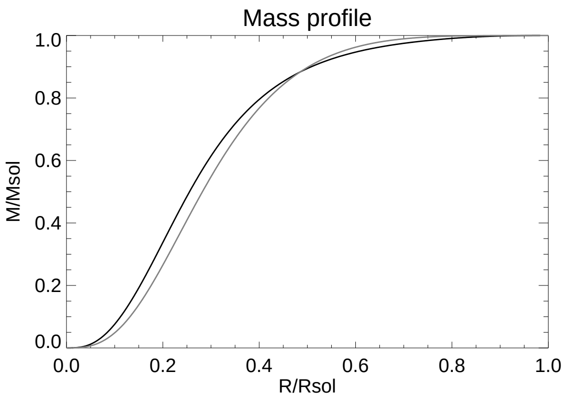
`http://www.sns.ias.edu/~jnb/SNdata/Export/BS2005/bs05_agsop.dat`
`(Bachall et al.2005).`

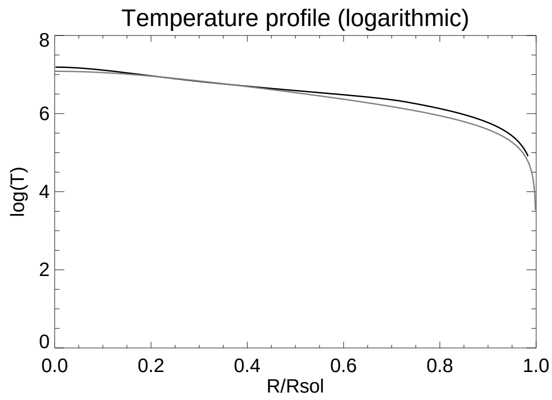
Compare the profiles $\log\rho$, $M/M_\odot$, $\log P$ and $\log T$ as a function of $R/R_\odot$ of this complicated model with those of an $n=3$ polytrope considering that the composition of the Sun is uniform with $\mu=0.61$.

Results

IDL plots of the profiles for $n=3$ polytrope (gray line) and the solar model (black line):







Central pressure, temperature and density for the models

From the lecture notes I found that the pressure at the center of a polytropic star is

$$P_c = (4\pi)^{1/3} B_n G M^{2/3} \rho_c^{4/3} \,, \tag{8}$$

and I got $P_c = 1.2503143 \cdot 10^{16}$ Pa as a result. In the solar model the pressure was $P_c = 2.33800 \cdot 10^{16}$ Pa at the point closest to $R = 0$ in the data file.

The central density is

$$\rho_c = D_n \frac{M_\odot}{\frac{4\pi}{3} R_\odot^3} \,, \tag{9}$$

which was $\rho_c = 76277.871 \mathrm{kg/m}^3$ according to my model. In the solar model the central density was $\rho_c = 150500.00 \mathrm{kg/m}^3$.

Finally, the central temperature is

$$T_c = \frac{\mu P_c}{R \rho_c} \tag{10}$$

which was $T_c = 1.2122072 \cdot 10^7$K according to my model, and $T_c = 1.54800 \cdot 10^7$K according to the solar model.

IDL code for comparing the profiles

```
;-----------------------------------------------------------------------;
; Stellar structure and evolution
; Computer assignment for advanced students
;-----------------------------------------------------------------------;
; Final part - compare to a more realistic solar model
;-----------------------------------------------------------------------;
; Use the subroutine PsPlot to save results in a postscript plot
; (written by Heikki Salo)
;-----------------------------------------------------------------------;
pro PsPlot,routine,filename
thisdir=getenv('PWD')+'/'
psopen,/color,dir=thisdir,filename
call_procedure,routine
psclose
end
;-----------------------------------------------------------------------;

;-----------------------------------------------------------------------;
; MAIN PROGRAM starts here
;-----------------------------------------------------------------------;
pro compare

;-----------------------------------------------------------------------;
; Read data from bs05_agsop.dat -file
; 12 columns, 1284 rows
;-----------------------------------------------------------------------;

;create empty array
data=fltarr(12,1284)
;open the data file
openr,lun,'bs05_agsop.dat',/get_lun
;read data
readf,lun,data
;close the file
close,/all
;check the data array
help,data
;print,data

;Separate the columns that are needed

mass=reform(data(0,*))
;help,mass
;print,mass

radius=reform(data(1,*))
;help,radius
;print,radius


temperature=alog10(reform(data(2,*)))
;help,temperature
;print,temperature
;Central temperature:
temperature_c=data(2,0)
print,'Central temperature (K)'
print,temperature_c

;Central density:
density_c=data(3,0)*1000.d0
print,'Central density (kg/m^3)'
print,density_c
;Density:
density=reform(data(3,*))*1000.d0
density_log=alog10(density)
;help,density
;print,density

pressure=reform(data(4,*))*0.1d0
pressure_log=alog10(pressure)
;Central pressure
pressure_c=data(4,0)*0.1d0
print,'Central pressure (Pa)'
print,pressure_c
```

```
;help,pressure
;print,pressure

;-----------------------------------------------------------------------;
; Solar model:
; log(rho) as a function of R/Rsol
;-----------------------------------------------------------------------;
;!p.multi=[0,2,2]
;nwin
;plot,radius,density_log,xtitle='R/Rsol',ytitle='log(!9r!X!N)',title='Density
profile (logarithmic), solar model';,yrange=[-10,0]

;Non-logarithmic density
;nwin
;plot,radius,density,xtitle='R/Rsol',ytitle='!9r!X!N(r)',title='Density
profile, solar model';,yrange=[-10,0]

;-----------------------------------------------------------------------;
; Solar model:
; log(P) as a function of R/Rsol
;-----------------------------------------------------------------------;
;nwin
;plot,radius,pressure_log,xtitle='R/Rsol',ytitle='log(P)',title='Pressure
profile (logarithmic), solar model'

;Non-logarithmic pressure
;nwin
;plot,radius,pressure,xtitle='R/Rsol',ytitle='P(r)',title='Pressure profile,
solar model'

;-----------------------------------------------------------------------;
; Solar model:
; M/Msol as a function of R/Rsol
;-----------------------------------------------------------------------;
;nwin
;plot,radius,mass,xtitle='R/Rsol',ytitle='M/Msol',title='Mass profile, solar
model'

;-----------------------------------------------------------------------;
; Solar model:
; log(T) as a function of R/Rsol
;-----------------------------------------------------------------------;
;nwin
;plot,radius,temperature,xtitle='R/Rsol',ytitle='log(T)',title='Temperature
profile (logarithmic), solar model'

;-----------------------------------------------------------------------;
; Profiles for n=3 polytrope
;-----------------------------------------------------------------------;

; Solve the Lane-Emden equation for n=3
n=3.d0
results=lane_emden(n)
ksi=results[*,0]
theta=results[*,1]
dtheta_dksi=results[*,2]

; Solve Rn
Rn=ksi(n_elements(ksi)-1)
print,'Rn'
print,Rn

; Solve Mn
Mn=-Rn^2*dtheta_dksi(n_elements(ksi)-1)
print,'Mn'
print,Mn

; Solve Dn
Dn=-(3.d0/Rn*dtheta_dksi(n_elements(ksi)-1))^(-1.d0)
print,'Dn'
print,Dn

; Solve Bn
Bn=(3.d0*Dn)^((3.d0-n)/(3*n))/((n+1.d0)*Mn^((n-1.d0)/n)*Rn^((3.d0-n)/n))
print,'Bn'
print,Bn
```

```
;-----------------------------------------------------------------------;
; n=3: Density profile
;-----------------------------------------------------------------------;
; Transform theta(ksi) -> rho(r/R)
; r/R = ksi/Rn and rho/rho_c = theta^n

; Solar mass (in kg)
Msol=1.9891d0*10.d0^30.d0
; Solar radius (in m)
Rsol=6.955d0*10.d0^8
; Solar volume (in m^3)
Vsol=4.d0/3*!dpi*Rsol^3
; Central density
rho_c=Dn*Msol/Vsol
print,'Central density from n3 (kg/m^3)'
print,rho_c

; Relative radius r/R
radius_n3=ksi/Rn

; Density rho
densityn3=theta^n*rho_c
; log(rho)
density_n3=alog10(densityn3)

; Plot (r,rho)
;nwin
;plot,radius_n3,density_n3,title='Density profile (logarithmic), n=3 polytrope',
xtitle='r/Rsol',ytitle='log(!9r!X!N)';,yrange=[-10,0]

;----------------------------------------------------------------------;
; ABSOLUTE density rho (kg/m^3)
;----------------------------------------------------------------------;
;Absolute density = (rho/rho_c)*rho_c
rho=densityn3;*rho_c

;Plot rho and compare with lecture notes
;nwin
;plot,radius_n3,densityn3,title='Density profile, n=3 polytrope',
xtitle='r/Rsol',ytitle='!9r!X(r)!N'

;-----------------------------------------------------------------------;
; ABSOLUTE radius (m)
;-----------------------------------------------------------------------;
Rabs=radius_n3*Rsol

;-----------------------------------------------------------------------;
; n=3: Mass profile
;-----------------------------------------------------------------------;

;Integrate from density profile
;dM=rho(r)*4*pi*r^2*dr

mass_n3=radius_n3*0.d0
i=1
while i lt n_elements(radius_n3) do begin
  dr=Rabs[i]-Rabs[i-1]
  mass_n3[i]=mass_n3(i-1)+rho[i]*4.d0*!dpi*Rabs[i]^2*dr
  i=i+1
endwhile

; Scale the mass to M/Msol
massn3=mass_n3/max(mass_n3)

;print,mass_n3
;Plot this after the pressure profiles
;plot,radius_n3,massn3,title='Mass profile, n=3 polytrope',xtitle='r/Rsol',ytitle='M/Msol'

;-----------------------------------------------------------------------;
; n=3: Pressure profile
;-----------------------------------------------------------------------;
;Gravitational constant (m^3/(kg*s^2))
G=6.67408d0*10^(-11.d0)

;Central pressure
```

```
press_c=(4.d0*!dpi)^(1.d0/3)*Bn*G*Msol^(2.d0/3)*rho_c^(4.d0/3)
print,'Central pressure (Pa)'
print,press_c

;----------------------------------------------------------------------;
; Solve the pressure profile from polytropic eq. of state  P=K*rho^gamma,
; where gamma=1+1/n is the adiabatic index
;----------------------------------------------------------------------;

;Adiabatic index
gamma=1.d0+1.d0/n

;Constant K
K=(4.d0*!dpi*G/(n+1.d0)^n*(G*Msol/Mn)^(n-1.d0)*(Rsol/Rn)^(3.d0-n))^(1.d0/n)

;Polytropic eq. of state
press_n3=K*rho^gamma

;log(P)
pressn3=alog10(press_n3)

;nwin
;plot,radius_n3,pressn3,title='Pressure profile (logarithmic), n=3 polytrope',xtitle='r/Rsol',ytitle='log(P)'

;nwin
;plot,radius_n3,press_n3,title='Pressure profile, n=3 polytrope',xtitle='r/Rsol',ytitle='P(r)'

;Plot the mass profile now
;nwin
;plot,radius_n3,massn3,title='Mass profile, n=3 polytrope',xtitle='r/Rsol',ytitle='M/Msol'

;----------------------------------------------------------------------;
; n=3: Temperature profile
;----------------------------------------------------------------------;
;Assume that the gas in Sun is non-degenerate -> ideal gas
;P=RR/mu*rho*T -> T=P*mu/(RR*rho)

;Specific ideal gas constant RR (J/(mol*K))
;RR=kB/mH (from the lectures)
RR=1.3806488*10^(-23.d0)/(1.008d0*1.660539*10.d0^(-27.d0))
print,'RR'
print,RR

;Ratio of ions and electrons
mu=0.61d0

;Temperature
temp_n3=press_n3*mu/(RR*rho)

;log(T)
tempn3=alog10(temp_n3)

;plot,radius_n3,tempn3,title='Temperature profile (logarithmic), n=3
;polytrope',xtitle='r/Rsol',ytitle='log(T)'

;Central temperature Tc
mu=0.61d0
Tc_n3=mu*press_c/(RR*rho_c)
print,'Central temperature from n3'
print,Tc_n3

;-----------------------------------------------------------------------;
; COMBINED FINAL PLOTS
;-----------------------------------------------------------------------;

;-----------------------------------------------------------------------;
; log(rho) as a function of R/Rsol
;-----------------------------------------------------------------------;
!p.multi=[0,2,2]
nwin
plot,radius,density_log,xtitle='R/Rsol',ytitle='log(!9r!X!N)',title='Density
profile (logarithmic)',thick=3;,yrange=[-10,20]
oplot,radius_n3,density_n3,col=130,thick=3

nwin
plot,radius,density,xtitle='R/Rsol',ytitle='!9r!X!N',title='Density profile',
```

```
thick=3;,yrange=[-10,20]
oplot,radius_n3,densityn3,col=130,thick=3

;-------------------------------------------------------------------------;
; log(P) as a function of R/Rsol
;-------------------------------------------------------------------------;
nwin
plot,radius,pressure_log,xtitle='R/Rsol',ytitle='log(P)',title='Pressure profile
(logarithmic), solar model',thick=3;,yrange=[0,30]
oplot,radius_n3,pressn3,col=130,thick=3

;Non-logarithmic pressure
nwin
plot,radius,pressure,xtitle='R/Rsol',ytitle='P(r)',title='Pressure profile,
solar model',thick=3;,yrange=[0,2*10^19]
oplot,radius_n3,press_n3,col=130,thick=3

;-------------------------------------------------------------------------;
; M/Msol as a function of R/Rsol
;-------------------------------------------------------------------------;
nwin
plot,radius,mass,xtitle='R/Rsol',ytitle='M/Msol',title='Mass profile',thick=3
oplot,radius_n3,massn3,col=130,thick=3

;-------------------------------------------------------------------------;
; log(T) as a function of R/Rsol
;-------------------------------------------------------------------------;
nwin
plot,radius,temperature,xtitle='R/Rsol',ytitle='log(T)',title='Temperature
profile (logarithmic)',thick=3;,yrange=[0,10]
oplot,radius_n3,tempn3,col=130,thick=3;,title='Temperature profile (logarithmic),
n=3 polytrope',xtitle='r/Rsol',ytitle='log(T)'

end

;-------------------------------------------------------------------------;
; Save the results to a PostScript file using PsPlot
;-------------------------------------------------------------------------;
pro plot_everything
PsPlot, 'compare', 'compare.ps'
end
```