STELLAR DYNAMICS (SPRING 2014)
COMPUTER EXERCISES: Plummer-sphere

(Heikki Salo 04.05.2010/25.03.2014

- Create particle positions and velocities from the Plummer-sphere distribution function.

- Check that the created N-body model is an equilibrium solutions, via integrations with Hernquist's TREECODE.

1. Plummer-sphere

- Plummer-sphere density distribution is (b = core radius, M is the total mass)

$$\rho(r) \;=\; \frac{3M}{4\pi b^3}\left(1+\frac{r^2}{b^2}\right)^{-5/2}$$

This density distribution, with the associated velocity distribution, provides one *equilibrium solution* to

1) the collisionless time-independent Boltzman equation

$$\left(\vec{v}\cdot\frac{\partial}{\partial\vec{r}} - \vec{\nabla}\Phi\cdot\frac{\partial}{\partial\vec{v}}\right) f(\vec{r},\vec{v}) = 0$$

2) combined with Poisson equation

$$\nabla^2\Phi(\vec{r}) = 4\pi G\rho(\vec{r}) = 4\pi G\int_{\substack{velocity\\ space}} f(\vec{r},\vec{v})d^3\vec{v}$$

The solution is obtained by assuming a phase-space density distribution

$$f(\epsilon) = \begin{cases} F\ \epsilon^{7/2} & \text{if } \epsilon > 0 \\ 0 & \text{if } \epsilon \leq 0 \end{cases}$$

where ϵ denotes the relative energy ($\epsilon = -\Phi - \frac{1}{2}v^2 + \Phi_0$), and F is a normalization constant. (Since f is a function of energy, and energy is a constant of motion in the time-independent case $\Rightarrow$ this f is also a solution of Boltzman equation. Fulfilling Poisson's equation is a bit trickier.)

- The goal of this exercise is to create $N = 10^4$ position and velocity vectors drawn from the above distribution function.

The results will be checked with a fully self-gravitating N-body simulation: if the created coordinates are correctly calculated $\Rightarrow$ the system will retain its density and velocity distribution constant during the evolution of the system.

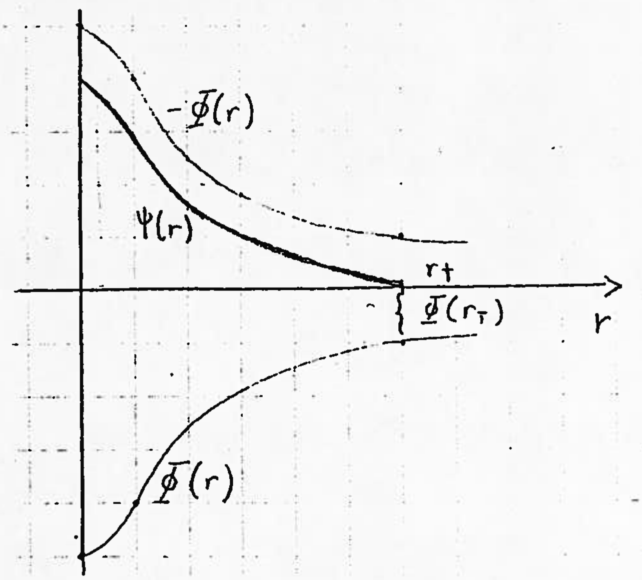
2. Background:

Relative potential, relative energy

Assume: Gravity potential $\Phi(r)$

Relative potential is defined by $\Psi(r) = -\left[\Phi(r) - \Phi(r_{\mathrm{trunc}})\right]$, so that $\Psi(r_{\mathrm{trunc}}) = 0$

Usually one denotes $\Phi_0 \equiv \Phi(r_{\mathrm{trunc}})$



$\boxed{\Psi = -\Phi + \Phi_0}$ relative potential

$\boxed{\epsilon = -E + \Phi_0 = \Psi - \frac{1}{2}v^2}$ relative energy

- The choice $f = 0$ for $\epsilon \leq 0$ corresponds to the maximum extent of the system $r < r_{\mathrm{trunc}}$ ('tidal radius')

  At each distance $v^2 < {v_{max}}^2 = 2\Psi$

  $\Rightarrow\ v_{max} = 0$ when $\Psi = 0$ at $r = r_{\mathrm{trunc}}$.

- A system with an infinite extent: $\Phi_0 = \lim_{r\to\infty} \Phi(r) = 0$

  (in realistic case where $\rho(r)$ drops faster than $1/r^2$)

Phase-space density distribution $\quad f = f(\epsilon)$

- Use spherical coordinates $r, \phi, \theta, v_r, v_\phi, v_\theta$

  $f = f(\epsilon) = f(\Psi - \frac{1}{2}[{v_r}^2 + {v_\phi}^2 + {v_\theta}^2])$

  $\Rightarrow$ isotropic velocity distribution

  $< {v_r}^2 >= \int f(r, v_r, v_\phi, v_\theta)\, {v_r}^2 dv_r dv_\phi dv_\theta \; = \; < {v_\phi}^2 > \; = \; < {v_\theta}^2 >$

- Isotropic velocity distribution $\quad \Rightarrow$

$$\int_{\substack{velocity\\space}} f(\vec{v}) d^3\vec{v} = \int_0^\infty f(v) 4\pi v^2 dv \qquad \text{where} \quad v = |\vec{v}|$$

- Sperically symmetric density distribution $\quad \Rightarrow$

$$\nabla^2 \Phi(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi(r)}{\partial r}\right)$$

- Since $\Psi = -\Phi + \Phi_0$, it satisfies Poisson equation in the form (note minus sign)

$$\nabla^2 \Psi(r) = -4\pi G \rho$$

Derivation of Plummer-solution

- Assume

$$f(\epsilon) = \begin{cases} F\ \epsilon^{7/2} & \text{if } \epsilon > 0 \\ 0 & \text{if } \epsilon \leq 0 \end{cases}$$

$$\begin{aligned} \rho(r) \; &= \; \int f(r, \vec{v}) d^3\vec{v} = 4\pi \int_0^\infty f(\Psi - \frac{1}{2}v^2) v^2 dv \\ &= \; 4\pi F \int_0^{\sqrt{2\Psi(r)}} \left[\Psi(r) - \frac{1}{2}v^2\right]^{\frac{7}{2}} v^2 dv \qquad (1) \end{aligned}$$

Insert: $v^2 = 2\Psi(r)\cos^2\theta$ (Note: this θ is just an auxillary variable, not the spherical coordinate used below!)

$\Rightarrow \; v dv = -2\Psi \cos\theta \sin\theta d\theta \qquad$ Limits: $v = 0 \rightarrow \theta = \pi/2$

$\Rightarrow \; dv = -\sqrt{2\Psi} \sin\theta d\theta \qquad\qquad\qquad\quad v = \sqrt{2\Psi} \rightarrow \theta = 0$

Substitution $\Rightarrow$

$$
\begin{aligned}
\rho(r) &= 4\pi F \int_{\pi/2}^{0} \underbrace{\Psi^{\frac{7}{2}} \sin\theta^7}_{f} \underbrace{2\Psi\cos^2\theta}_{v^2} \underbrace{(-\sqrt{2\Psi})\sin\theta d\theta}_{dv} \\
&= 4\pi F \Psi^5 2^{3/2} \int_0^{\pi/2} (\sin^8\theta - \sin^{10}\theta) d\theta \\
&= c_5 \Psi^5
\end{aligned}
$$

where $c_5 = \frac{2\pi^{3/2}(5-3/2)!\ F}{5!} \approx 1.5266F$ (BT Eq. 4-107b)

- Insert ρ into Poisson equation:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\Psi(r)}{\partial r}\right) = -4\pi G c_5 \Psi^5$$

(a special case of *Lane-*Emden equation)

Trial solution $\Psi = A(1 + r^2/b^2)^{-1/2}$, with A and b constants $\Rightarrow$

$$
\begin{aligned}
\frac{\partial\Psi(r)}{\partial r} &= A\, 2r/b^2\, (-1/2)(1 + r^2/b^2)^{-3/2} \\
r^2\frac{\partial\Psi(r)}{\partial r} &= -Ar^3/b^2\, (1 + r^2/b^2)^{-3/2} \\
\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi(r)}{\partial r}\right) &= -A\left[3r^2/b^2\, (1 + r^2/b^2)^{-3/2} - (3/2)\, r^3/b^2\, 2r/b^2\, (1 + r^2/b^2)^{-5/2}\right] \\
&= -A\left[3r^2/b^2 + 3r^4/b^4 - 3r^4/b^4\right](1 + r^2/b^2)^{-5/2}
\end{aligned}
$$

Left-hand side: $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi(r)}{\partial r}\right) = \frac{-3A}{b^2}(1 + r^2/b^2)^{-5/2}$

Right-hand side $-4\pi G c_5 \Psi^5 = -4\pi G c_5 A^5 (1 + r^2/b^2)^{-5/2}$

Fulfilled if $\frac{-3A}{b^2} = -4\pi G c_5 A^5$

$\Rightarrow$ $\boxed{\rho = c_5\Psi^5 = c_5 A^5 (1 + r^2/b^2)^{-5/2}}$ = Plummer sphere

constants determined by the total mass M $\Rightarrow$ ... $A = GM/b$

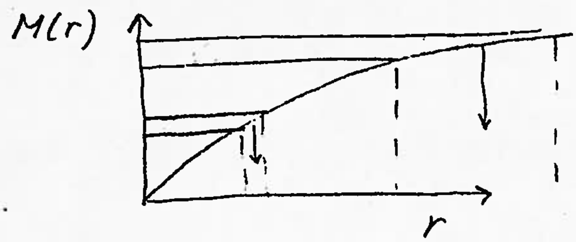
3.Create positions and velocities:

3.1. Radial distribution

- We want to represent each mass increment ΔM with a similar number of particles

$\rho = C(1 + r^2/b^2)^{-5/2}$ $\quad$ C=constant

$dM(r) = \rho(r) 4\pi r^2 dr$ $\quad$ mass increment corresponding to radius increment



Standard solution in terms of uniformly distributed random numbers

$$\frac{\int_0^r dM}{\int_0^\infty dM} = \mathrm{rnd}(0,1) \qquad \longrightarrow \text{solve for } r$$

Denote $s_1 \sim \mathrm{rnd}(0,1)$ $\quad \Rightarrow$ we have for the corresponding r_i

$$\frac{\int_0^{r_i} (1 + r^2/b^2)^{-5/2}\, r^2 dr}{\int_0^\infty (1 + r^2/b^2)^{-5/2}\, r^2 dr} = s_1$$

Solve $\int (1 + r^2/b^2)^{-5/2}\, r^2 dr$ $\qquad$ [insert $t^2 = r^2/b^2$]

$$= b^3 \int (1 + t^2)^{-5/2}\, t^2 dt$$

$$= \frac{b^3}{3} \frac{t^3}{(1+t^2)^{3/2}}$$

Denote $t_i = r_i/b \quad \Rightarrow$

$$\frac{\int_0^{r_i}(1+r^2/b^2)^{-5/2}\, r^2 dr}{\int_0^{\infty}(1+r^2/b^2)^{-5/2}\, r^2 dr} = \frac{\frac{b^3}{3}\left(\frac{t_i^3}{(1+t_i^2)^{3/2}} - 0\right)}{\frac{b^3}{3}(1-0)} = s_1$$

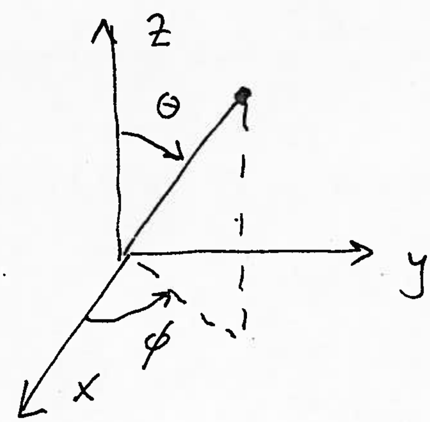
$\Rightarrow\ t_i^3 = s_1(1+t_i^2)^{3/2}$

$\Rightarrow\ t_i^2 = s_1^{2/3}(1+t_i^2)$

$\Rightarrow\ t_i^2(1-s_1^{2/3}) = s_1^{2/3}$

$\Rightarrow$ Lenght of the radius vector determined from

$$\boxed{r_i = bt_i = b\sqrt{\frac{s_1^{2/3}}{(1-s_1^{2/3})}}}$$



- Components of the radius vector:

  $x = r\sin\theta_r \cos\phi_r$

  $y = r\sin\theta_r\ \sin\phi_r$

  $z = r\cos\theta_r$

Spherical symmetry $\quad\Rightarrow$ directions of radius vectors random

ϕ_r uniformly distributed between $[0, 2\pi]$

$\Rightarrow\ (\phi_r)_i = 2\pi s_2$

$\cos\theta_r$ uniformly distributed between $[-1, 1]$

$\Rightarrow\ (\cos\theta_r)_i = 2s_3 - 1$

3.1. Velocity distribution

- Speeds $v_i = |\vec{v}_i|$ created from the distribution function

  For a particle with a distance r_i we have

  $f(r_i, v) = F(\Psi_i - \frac{1}{2}v^2)^{7/2}$

  where Ψ_i is the relative potential corresponding to distance r_i

  $\Psi_i = GM/b \;\; (1 + r_i{}^2/b^2)^{-1/2}$

  Maximum velocity at distance r_i determined by $v_{max} = \sqrt{2\Psi_i}$

The same standard procedure to create v_i as was used for radial distances

Number of particles in the interval $[v, dv]$ equals $f(v)4\pi v^2 dv \quad \Rightarrow$

$$\frac{\int_0^{v_i} f 4\pi \; v^2 dv}{\int_0^{v_{max}} f 4\pi \; v^2 dv} = s_4 \sim \mathrm{rnd}(0,1)$$

The integral $\int_0^{v_i} f 4\pi \; v^2 dv$ was already solved

$= 4\pi F \int_0^{v_i} \; [\Psi_i - \frac{1}{2}v^2]^{\frac{7}{2}} \, v^2 dv$

$= 4\pi F \Psi_i^5 2^{3/2} \int_{\theta_i}^{\pi/2} (\sin^8\theta - \sin^{10}\theta) d\theta$

where $v_i{}^2 = 2\Psi_i \cos^2\theta_i$

Therefore,

$$\frac{\int_0^{v_i} f 4\pi \; v^2 dv}{\int_0^{v_{max}} f 4\pi \; v^2 dv} = \frac{\int_{\theta_i}^{\pi/2} (\sin^8\theta - \sin^{10}\theta) d\theta}{\int_0^{\pi/2} (\sin^8\theta - \sin^{10}\theta) d\theta} = s_4 \qquad (\ldots \rightarrow \theta_i \rightarrow v_i)$$

- It is possible to calculate the integral analytically (yields $\sin 2\theta$, $\sin 4\theta$,... terms).

  But: not possible to invert it to give θ_i explicitely in terms of s_4.

  This has to be done iteratively.

  $\Rightarrow$ Easiest to tabulate the integral itself, and invert the equation using these tabulated values

Denote

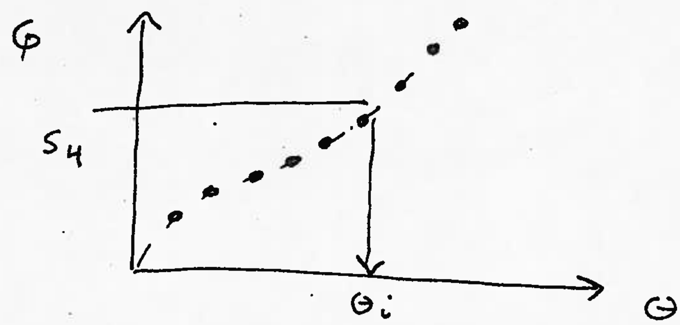
$$\mathcal{G}(\theta) = \frac{\int_{\theta_i}^{\pi/2}(\sin^8\theta - \sin^{10}\theta)d\theta}{\int_0^{\pi/2}(\sin^8\theta - \sin^{10}\theta)d\theta}$$

this function is monotonically decreasing in the interval $[0, \pi/2]$, with

$$\mathcal{G}(0) = 1$$

$$\mathcal{G}(\pi/2) = 0$$

Tabulate $\mathcal{G}(\theta)$ [for example using spacing $0.01\pi/2$]



Choose s_4, find the nearest $\mathcal{G}$ values from the table

(Higher accuracy; interpolate between two values).

$\Rightarrow$ v_i corresponding to random number s_4

- Components of velocity vector

$$v_x = v\sin\theta_v \cos\phi_v$$

$$v_y = v\sin\theta_v \sin\phi_v$$

$$v_z = v\cos\theta_v$$

Isotropic distribution $\Rightarrow$ directions of velocity vectors random

ϕ_v uniformly distributed between $[0, 2\pi]$

$$\Rightarrow (\phi_v)_i = 2\pi s_5$$

$\cos\theta_v$ uniformly distributed between $[-1, 1]$

$$\Rightarrow (\cos\theta_v)_i = 2s_6 - 1$$

4. WHAT TO DO:

- Choose the number of particles N and the value of the core radius b
- Create N particle positions, using random numbers

  $r_i = b\sqrt{\frac{s_1{}^{2/3}}{(1-s_1{}^{2/3})}}$

  $\phi_i = 2\pi s_2$

  $\cos\theta_i = 2s_3 - 1$

  $\Rightarrow \ \vec{r} = [r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta]$

In IDL: `randomu` function gives uniform random numbers between $[0, 1]$

Note: s_1, s_2, s_3 different for each particle

- Check the particle positions:

  - by plotting the $x - y$, $x - z$ projections (try also 3D plot: xplot3d)

  - by plotting the radial density distribution

    (tabulate the number of r_i's in different bins; use `histo_f`)

- Create the velocity vector for each particle, using random numbers

First tabulate the function (needs to be just once)

$$G(\theta) = \frac{\int_{\theta_i}^{\pi/2} (\sin^8\theta - \sin^{10}\theta)\, d\theta}{\int_0^{\pi/2} (\sin^8\theta - \sin^{10}\theta)\, d\theta}$$

For each particle, choose a random number s_4 and solve $G(\theta) = s_4$

(in practice: find the value θ_i for which $G(\theta_i) - s$ is the smallest).

find the values $\Psi_i = GM/b \ (1 + r_i{}^2/b^2)^{-1/2}$ for the particle

$\Rightarrow v_i = \sqrt{2\Psi_i} \cos(\theta_i)$ Gravitational constant

Isotropic direction of velocity

$$(\phi_v)_i = 2\pi s_5$$

$$(\cos\theta_v)_i = 2s_6 - 1$$

$$\Rightarrow \vec{v} = [v \sin\theta_v \cos\phi_v, v \sin\theta_v \sin\phi_v, v \cos\theta_v]$$

- Check the velocity distribution:

Plot $G(\theta)$ against $\cos\theta = v/v_{max}$ = fractional number of particles in the velocity interval $[v, v + dv]$