Efficient algorithm for the detection of parabolic curves

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ABSTRACT

A new approach for parabolic curve detection is presented based on Hough transforms. This approach uses the vertices of the curve as the detecting parameters which also indicate the position of the maximum curvature. Evidence for the transformation is gathered from the edge gradient obtained from an edge enhancement operator. The Sobel operator is used for edge enhancement. For the detection of parabolic curves in any orientation, a coordinate transformation matrix is used to derive a new parabolic equation which involves the edge gradient information. In the proposed algorithm, parabolic curves in any orientation are detected by using a 3D accumulator array. The new algorithm, therefore reduces the accumulator size from 4D accumulator array if an ordinary Hough transform for parabola detection is used. The reduced parameter space of the proposed algorithm improves the processing time and decreases storage requirement for the detection of parabolic curves. The paper reports on the accuracy obtained when this approach is used on natural and synthetic images containing parabolic and other curves.

Keywords: Hough transform, curve detection, parabolic curve, Sobel operator, image geometry

1. INTRODUCTION

Many image processing problems require algorithms for the detection of curves. This is because most natural scenes and industrial objects contain curved edges, such as arcs of circles, ellipses, parabolas or combination of straight edges and curves. Hough transforms have been used increasingly for the detection of lines and curves in images. In Hough transform techniques, first the parameters which specify the shape must be identified. Then the significant edge elements of the image are isolated using an edge detection algorithm. Finally the locus of the geometric shape is computed from the parameter space generated by using every candidate of the edge elements.[1]

The parameters of the geometric shapes must be identified and simplified for the detection of the shapes in digital images. Parameters can be simplified by using all relevant information from the image characteristics such as the local edge gradient. The basic parameters of any geometric curve can be identified from the equation of the curve. For example, the parameters for a circle are its centre and radius.[2,3] For more complex curves, many parameters will be involved to specify the shape in the image. For shapes with high dimentionality a large accumulator array is needed, thus increasing exponentially the computing time and the storage requirements.

The parametric clustering method of the Hough transform is the common method used for the detection of shapes with many parameters. Clustering the parameters will reduce the computing time and storage requirement. Tsuji and Matsumoto[4] first used the clustering technique for the detection of an ellipse. The centre of an ellipse was detected by a two dimensional accumulator array using the information from the two parallel tangents of the ellipse edge. Then a one dimensional accumulator array was used to test and find the other parameter of the ellipse. Finally, the least means square method was used to evaluate the five parameters of the ellipse. Tsukune and Goto[5] introduced the edge vector and used it for the detection of the centre of an ellipse as a set of two dimensional accumulator arrays of the parameter cluster. The local edge information was used to detect the other cluster of ellipse parameters and a one dimensional histogram was then used to determine the last parameter from the edge location data. Yip *et al.*[6] used several pairs of a two dimensional accumulator array to locate the positions of the four vertices of an ellipse and a circle, and from these positions all the other parameters can be calculated. Yoo and Sethi[7] used the method of defining the polar and pole of the conic for the detection of an ellipse. The generalized Hough transform is another technique for describing the parameters of a geometrical shape[8]. This technique makes full use of the local edge gradient direction information, where all the gradient direction information is transformed into the parameter space of the shape. This technique can be used in the detection of circles[9] and ellipses[10], however, it is popular for the detection of arbitrary shapes with more complex edges.

In the detection of a parabolic curve, four parameters are involved. They are the vertex (x_o,y_o) or the focal point (f_x,f_y), the orientation (θ), and the coefficient which carries the information about the curvature of the parabola. Some existing algorithms used in the detection of parabolic curves will be discussed in this paper, followed by our proposed algorithm. The accuracy of the proposed algorithm will then be evaluated. The performance of the algorithms will be compared with the algorithm used by Wechsler and Sklansky[11] and the application of the algorithm to the detection of general curves will also be included.

2. PARAMETERIZATION OF PARABOLIC CURVE

Geometric shapes that can be obtained from intersecting a double-napped right circular cone with a plane are called conic sections or simply conics. These are circles, ellipses, parabolas and hyperbolas. In polar coordinates (r,β), the equation of the conic satisfies one of the following equations:

$$r=\frac{de}{1\pm e\cos\beta} \quad \text{or} \quad r=\frac{de}{1\pm e\sin\beta} \tag{1}$$

where e is the eccentricity and d is the shortest distance of the focal points to the directrix. The conic is a parabola if $e = 1$, an ellipse if $0 < e < 1$, or a hyperbola if $e > 1$. In Cartesian coordinates the conic equation can be written as

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0 \tag{2}$$

where the coefficients (A,B,C,D,E,F) are real numbers. The shape of the conic depends on the values of the coefficients. There are various ways to identify a parabola. The popular parameters are the vertex position (x_o,y_o), the focus position (f_x,f_y), the curvature (k_v), the orientation (θ_o), and the coefficients of the conic in Cartesian or polar coordinates as in equations (1) and (2).

A parabola is defined as a locus of points which are equidistant to a fixed point called the focus and a fixed straight line called the directrix. A simple representation of a parabola is one with either the y-axis or the x-axis as the directrix and being symmetrical along the other axis. A parabola with directrix $x=x_d$ and symetrical axis $y=y_o$ is illustrated in Fig. 1.

The parabola in Fig. 1 can be represented by equation (3)

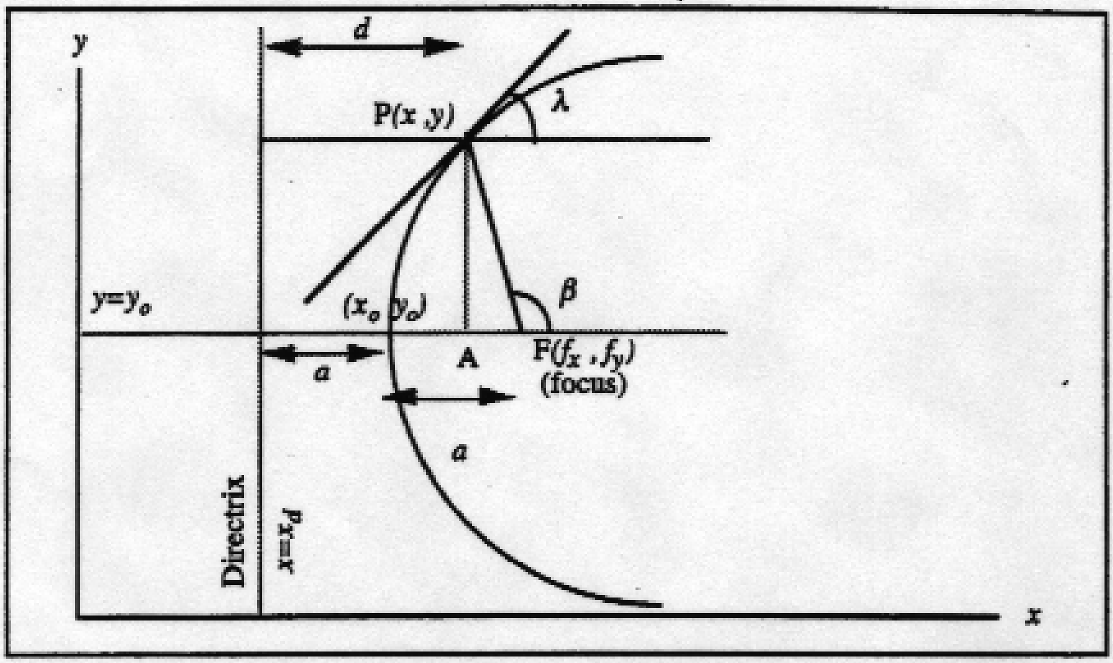


Fig. 1. A parabola.

$$(y - y_0)^2 = 4a(x - x_0) \tag{3}$$

The differentiation of this equation with respect to x gives

$$\frac{dy}{dx} = \sqrt{\frac{a}{x - x_0}} = \tan\lambda \tag{4}$$

This is the equation of the tangent of a parabola. Wechsler and Sklansky[11] had derived the relation of the tangent angle (λ) and angle β as

$$\beta = 2\lambda \tag{5}$$

From the triangle FAP in Fig. 1, the focal point of the parabola can be obtained as

$$f_x = x + d\cos\beta \tag{6}$$
$$f_y = y - d\sin\beta \tag{7}$$

The angle β can be calculated using the relation in equation (5), $\tan\lambda$ can be derived from the gradient direction of the local edge information. The value of a can be calculated from equation (4).

For the parabolic curve in the orientation parallel with the x-axis, equation (3) can be rearranged as

$$y - y_0 = p\,(x - x_0)^2 \tag{8}$$

where (x_0,y_0) is the vertex position and $p=1/4a$ is the coefficient representing the size and the curvature of the parabola. Based on equation (8), Ballard[9] introduced the use of the term dy/dx for the selection of suitable edge points. The relation of dy/dx and the directional information associated with the edge is given by

$$\frac{dy}{dx} = \tan\left(\phi(x,y) - \frac{\pi}{2}\right) \tag{9}$$

where $\phi(x,y)$ is the direction of the gradient. The parabolic parameters are detected by incrementing the accumulator array which contains all the parameters $A(x_0,y_0,p,\theta)$. This approach is the same as the ordinary Hough transform except for the use of directional information to reduce the number of edge pixels selected.

Casacent and Krishnapuram[12] suggested the inversion of the Hough transform for locating curved objects. The straight line Hough transform was used to transform the curve into the angle-radius (θ,ρ) Hough space. Then the Hough space was inverted to locate the centre of the curve.

In our proposed technique, the parabolic curve is also represented by equation (8). The differential of this equation is used to replace the coefficient (p) and simplify the representation of the parabolic curve in the digital image. The differentiation of equation (8) gives

$$\frac{dy}{dx} = 2p(x-x_0) \tag{10}$$

By replacing equation (10) into equation (8) and rearranging the result, we obtain

$$y_0 = y + \frac{1}{2}\frac{dy}{dx}(x - x_0) \tag{11}$$

From equation (11), parabolic curves parallel with the y-axis or x-axis can be detected by using only a 2D accumulator array. Edge gradient information is used to estimate the dy/dx term.

2.1. Local edge information

Edges carry information about object boundaries which is very important for image analysis, object identification, shape detection, and many other applications. Edge detection is a preliminary task for the detection of parabolic curves. In our proposed algorithm, the edge detector computes a digital approximation of the magnitude and the direction of the local edge gradient. The magnitude of the local edge gradient is used to select significant points for the transform. The direction of the local edge gradient is used to estimate the dy/dx of the pixels of the curve boundaries.

The Sobel operator is used for the enhancement of object edges. The operator estimates the differences of the gradient function along the x-direction and y-direction. Assume that $g(x,y)$ is the array of digitised values of a continuous image. Let $\Delta g_x(x,y)$ and $\Delta g_y(x,y)$ be the differences of grey level $g(x,y)$ in x-direction and y-direction respectively. The magnitude of the local edge gradient is given by

$$G(x,y) = \sqrt{\Delta g_x(x,y)^2 + \Delta g_y(x,y)^2} \tag{12}$$

and the local edge direction is given as

$$\phi(x,y) = \tan^{-1}\left(\frac{\Delta g_y(x,y)}{\Delta g_x(x,y)}\right) \tag{13}$$

Based on the Sobel operator the formulae for $\Delta g_x(x,y)$ and $\Delta g_y(x,y)$ are given as follows

$$\begin{aligned}\Delta g_x(x,y) = {} & 2[g(x,y+1) - g(x,y-1)] + [g(x+1,y+1) - g(x+1,y-1)] \\ & + [g(x-1,y+1) - g(x-1,y-1)]\end{aligned} \tag{14}$$

$$\begin{aligned}\Delta g_y(x,y) = {} & 2[g(x-1,y) - g(x+1,y)] + [g(x-1,y-1) - g(x+1,y-1)] \\ & + [g(x-1,y+1) - g(x+1,y+1)]\end{aligned} \tag{15}$$

2.2. Parabolic curves in any orientation

In real images, curves can be in any orientation. Parabolic curves in any orientation are detected by using an algorithm which is based on the coordinate transformation of the parabolic equation. In this method the edge pixels are transformed to the new coordinates. This will reduce the computing time. Fig. 2 shows how the algorithm is derived. Coordinates x'-y' in Fig. 2 are the rotation of the coordinates x-y by θ degrees with the centre of the coordinate system as the axis of rotation. The vertex of the parabola is (x_o', y_o') in x'-y' coordinate or (x_o, y_o) in x-y coordinate. The equation of the parabola in x'-y' coordinate can be written as

$$(y' - y_o') = p\,(x' - x_o')^2 \tag{16}$$

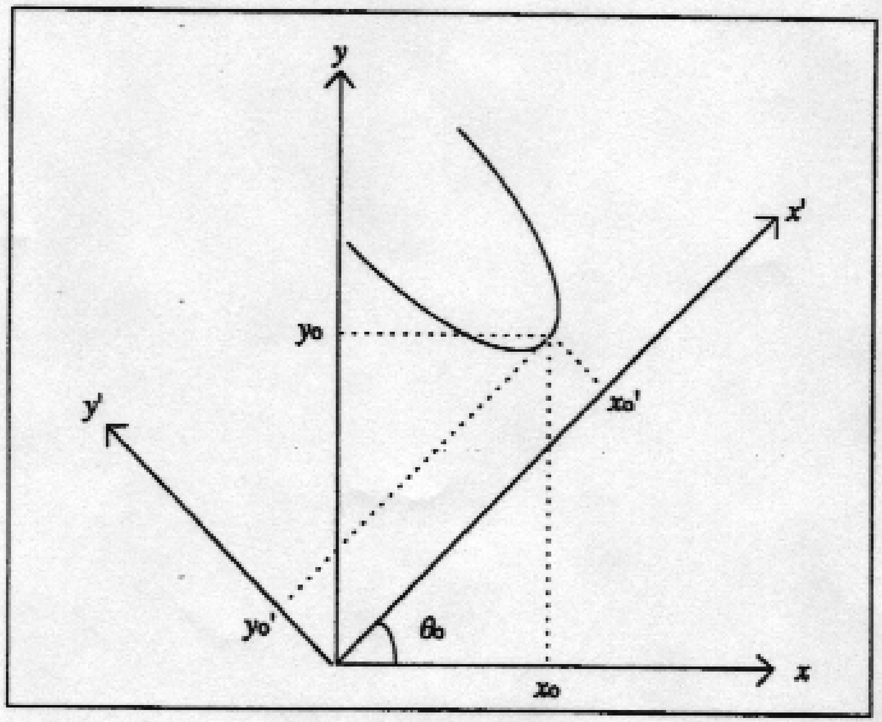


Fig. 2. Tilted parabolic curve.

The standard two dimensional geometric transformation matrice for anticlockwise rotation by an angle θ is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{17}$$

where θ is the transformation angle. By substituting x, y, x_0, and y_0 in equation (16) with the transformation relation from equation (17), the parabolic equation (16) can be written as;

$$\begin{aligned}&(-x\sin\theta + y\cos\theta) - (-x_0\sin\theta + y_0\cos\theta)\\ &= p[(x\cos\theta + y\sin\theta) - (x_0\cos\theta + y_0\sin\theta)]^2\end{aligned} \tag{18}$$

and the differentiation of this equation results in the following equation

$$\begin{aligned}&-\sin\theta + \frac{dy}{dx}\cos\theta\\ &= 2p\,[(x\cos\theta + y\sin\theta) - (x_0\cos\theta + y_0\sin\theta)]\,.\,[\cos\theta + \frac{dy}{dx}\sin\theta]\end{aligned} \tag{19}$$

By substituting equation (19) into equation (18), a new relation for the parabolic vertex and its orientation can be derived to produce the relation of y_0, x_0, and θ_0 as

$$\begin{aligned}y_0 &= \left[\frac{k_1(x\cos\theta_0 + y\sin\theta_0) + (x\sin\theta_0 - y\cos\theta_0)}{(k_1\sin\theta_0 - \cos\theta_0)}\right]\\ &\quad - \frac{(k_1\cos\theta_0 + \sin\theta_0)}{(k_1\sin\theta_0 - \cos\theta_0)}\,x_0\end{aligned} \tag{20}$$

where
$$k_1 = \frac{-\sin\theta_0 + \frac{dy}{dx}\cos\theta_0}{2(\cos\theta_0 + \frac{dy}{dx}\sin\theta_0)} \tag{21}$$

From this relation it can be seen that a 3D-accumulator array is sufficient for the detection of parabolic curves in any orientation. The parameters are x_0, y_0 which specify the vertex position and θ_0 the orientation. This may be compared with a 4D accumulator required when the ordinary Hough transform[9] is used. The reduced parameter space of the proposed algorithm improves the processing time and decreases storage requirement necessary for the detection of parabolic curves.

3. IMPLEMENTATION AND RESULTS

The block diagram in Fig. 3 illustrates how the algorithm was implemented. 256 x 256 digital images were used as input images. Synthetic binary images and real images of 256 grey level were used as testing images. The Sobel operation was applied to calculate the edge gradient magnitude and direction. The edge gradient magnitude was used to detect the edges of the shape, by introducing a threshold value to filter the gradient magnitude. The gradient direction was calculated for every pixel of the detected edges. The edge gradient direction was used to estimate the local differentiation of the edge shape, dy/dx. The detected edge pixels of the shape and its associated dy/dx were then transformed into the Hough space $H(x_o,y_o)$ if the parabolic curves in the image were parallel with x-axis or y-axis, or had known orientation. But if the orientation of the parabolic curve was unknown the transformation was done by using three dimensional accumulator $H(x_o,y_o,\theta_o)$. The Hough space was then averaged by using 3x3 template and thresholded to get the best estimation of the parabolic curve vertex. The peak value without everaging was also detected for the estimation of the parabolic vertex, but the averaged value gave a more accurate position. Fig. 4(a) shows the synthetic binary image of multiple parabolic curves. Fig. 4(b) shows the transformed Hough space (x_o, y_o) as a grey scale image. The highest grey level in the image represents the peak.

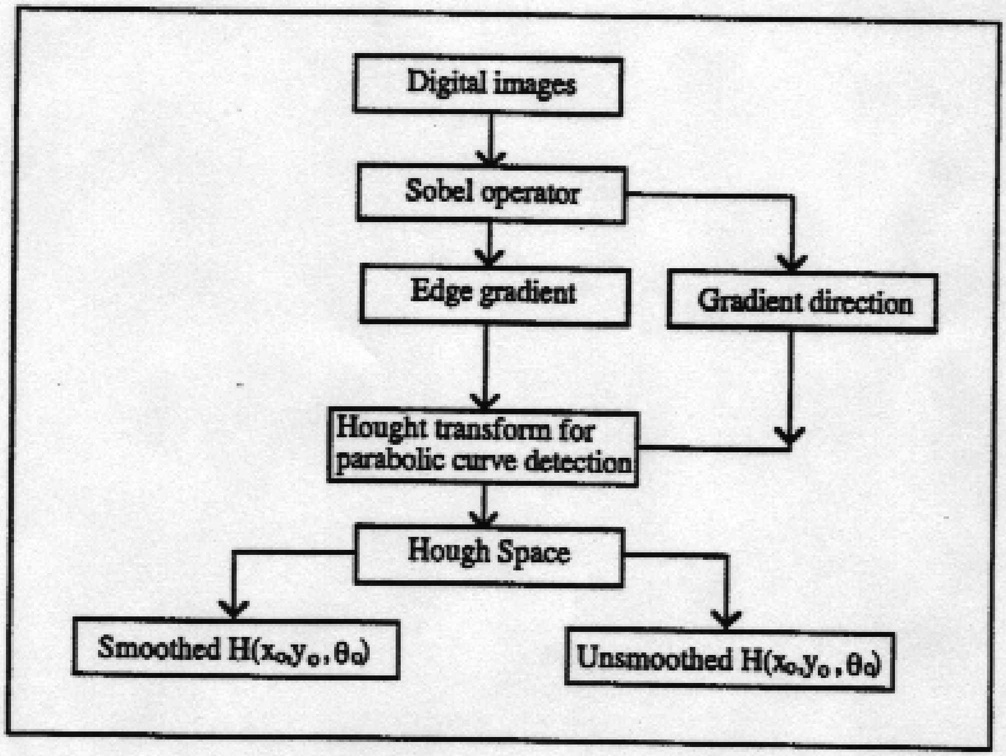
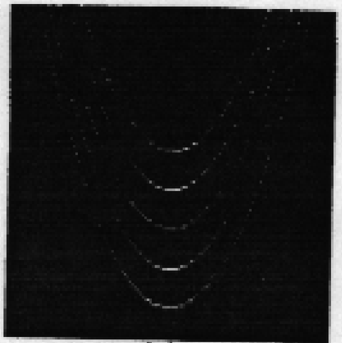


Fig. 3. Block diagram of algorithm implementation.

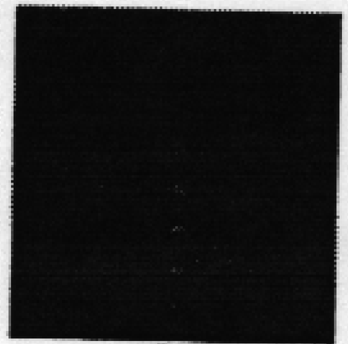
In order to evaluate the performance of the algorithm three types of images had been created. They are a series of synthetic binary images of parabolic curves with different curvatures, a series of synthetic binary images of parabolic curves with different orientation, and a real image which is a grey scale image of the moon's crescent.

The algorithm was then tested for the detection of synthetic parabolic curves with various curvatures and orientations. The inaccuracy (in pixel differences) in the position of the vertex had then been measured for the different curvatures and orientations. The inaccuracies by taking the maximum peak position and the average position of the peaks as the vertex location were also compared. The results are shown in Fig. 5 and Fig. 6. From the graph it can be seen that the average position method gives the better accuracy, and the accuracy is also not too dependent on the curvature and the orientation of the curve. The inaccuracy observed in Fig. 6 can be attributed to poor estimation of edge gradient of binary image obtained from the simple Sobel operator[13]. This may be improved by using a more accurate gradient estimation method. However, by comparing the results produced in Fig. 5 and Fig. 6 where the Sobel operator was used to estimate the edge gradients for both experiments, the algorithm gives a better accuracy in the detection of the parabolic curve with known orientation. This may be explained by inaccuracies produced if more parameters are involved.

The same images had also been tested using the algorithm proposed by Wechsler and Sklansky[11] for locating the focal point (f_x, f_y) of the parabolic curve. The accuracies of the detected positions were compared and the comparison results are shown in Fig. 7. From the graph, it appears that our proposed algorithm can detect the parabolic parameters more accurately.



(a)



(b)

Fig. 4. (a) Binary image of parabolic curves in 90° orientation. (b) Hough transform of (a).

The algorithm has also been tested on a real image, for the detection of the width of the moon's crescent. This image is not strictly parabolic. In this test the moon's crescent image was detected by using the algorithm for the detection of parabolic curve in any orientation. The algorithm provides a sufficiently accurate result for the detection and measurement of the moon's crescent width. Fig. 8(b) shows the parameter space for the moon crescent image of a 256 x 256 grey scale image. Fig. 8(c) shows the count in the parameter space along the line through the positions of (x_{01}, y_{01}) and (x_{02}, y_{02}), the vertices of the crescent's curves. The crescent's width can be measured from the two highest peaks.

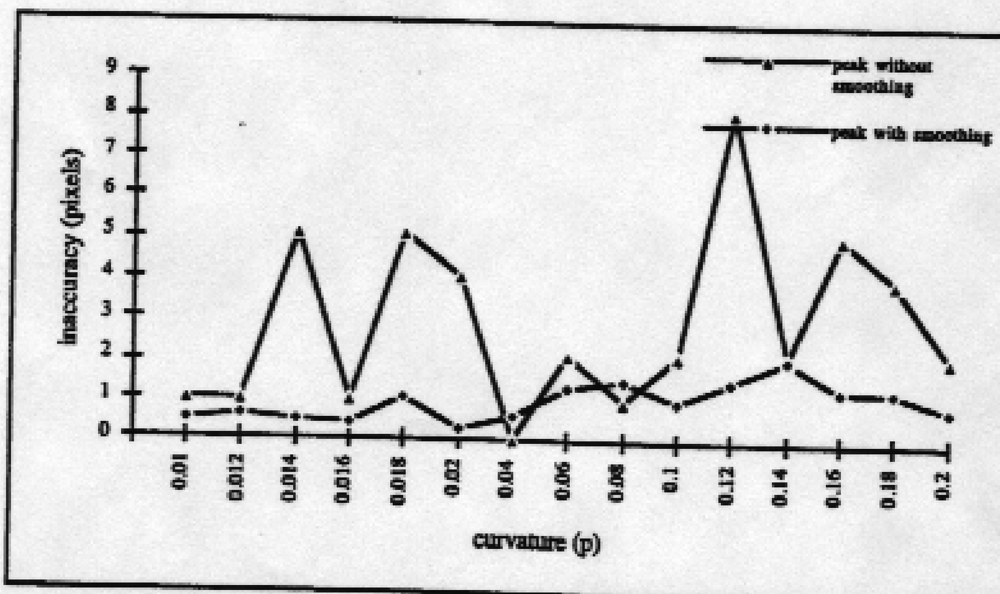


Fig. 5. Graph of inaccuracy in the estimated vertex position against the parabolic curvature.

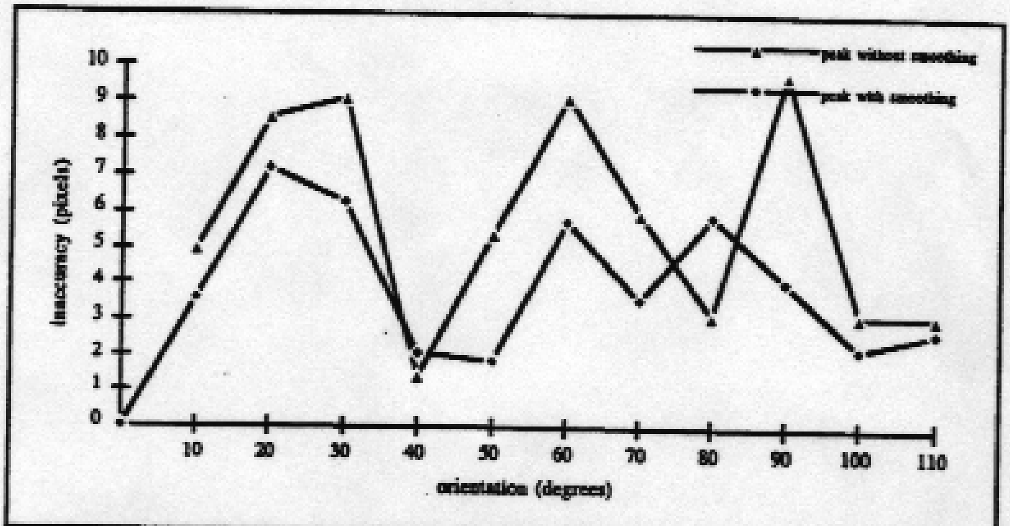


Fig. 6. Graph of the inaccuracy in the estimated vertex position against the orientation for p=0.06

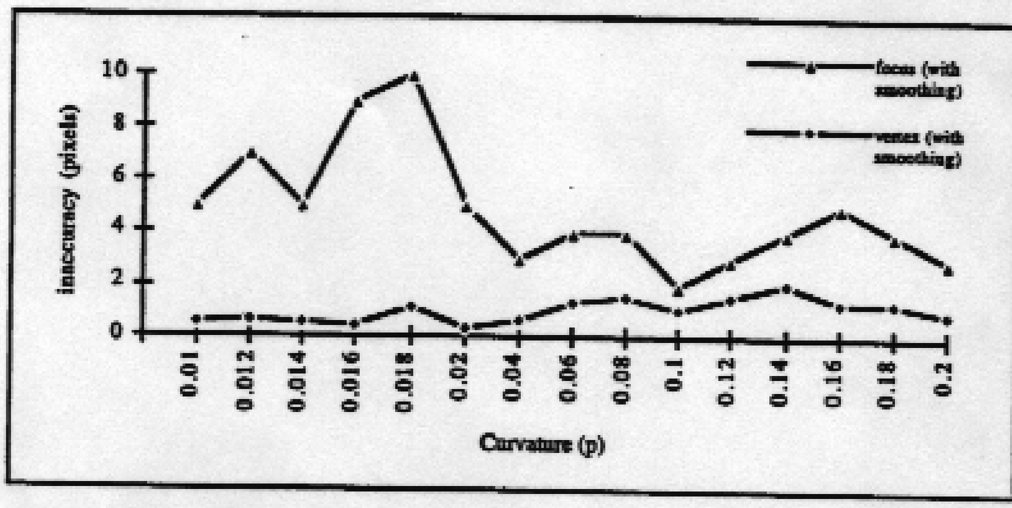
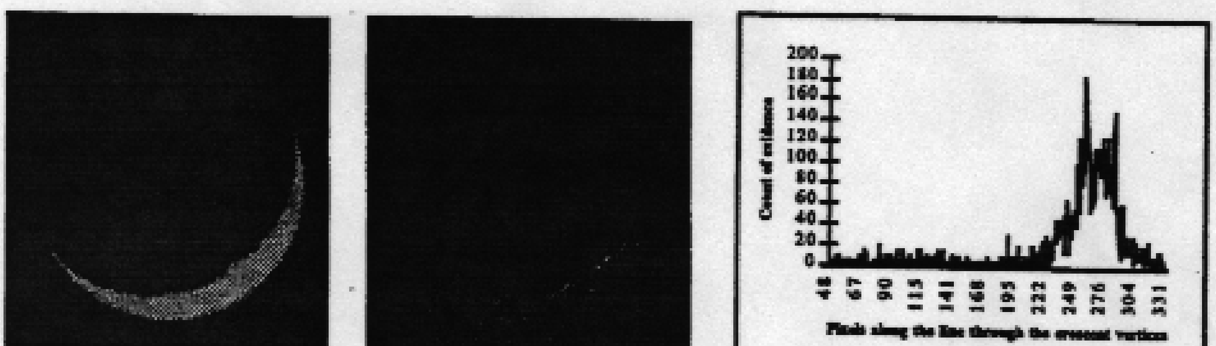


Fig. 7. Comparison of the inaccuracy of Wechsler and Sklansky method and our proposed method.



(a) (b) (c)

Fig. 8. (a) Moon crescent image, 256 grey levels. (b) Hough transform of (a). (c) Total count of evident along the line through the position of (x_{o1}, y_{o1}) and (x_{o2}, y_{o2}) of the crescent vertices.

4. CONCLUSION

A new approach for the detection of parabolic curves using Hough transforms has been suggested. The new algorithm reduces the accumulator array from 4D to 3D if general parabolic parameters are used. However, if the orientation of the parabolic curve is fixed and known the accumulator array is reduced to 2D and the accuracy of the detection result is also better. The algorithm gives the vertex positions of parabolic curves and their orientations. The algorithm may also be used for the detection and measurement of some curves which are not strictly parabolic. Very good accuracy is obtained if the orientation of the parabolic curve is parallel with *x* or *y*-axes or has a fixed known orientation. The higher error in the detection of parabolic curves in any orientation may be due to poor estimation of edge gradient direction especially for binary images.

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