$$\lambda e^{\pi} + \sum_{i=0}^{n} \frac{\phi^{n}}{n^{n}}$$

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PROBABILITY THEORY

1 SETS

1.1 HISTORY

The idea of grouping things has existed for the longest of time in the history of the Modern Homo Sapiens specie. The study of grouping objects was later called set theory, where a set is a collection/grouping of objects. The study of modern set theory is often attributed to one of its founders, a prominent German mathematician—George Cantor along with another German mathematician Richard Dedekind, which they developed in the 1870s. As the topic was relatively new and hot at the time, it received backlashes often from older mathematicians who did not embrace new idea. A famous mathematician and philosopher, Bertrand Russell also attacked some of these ideas which gave birth to a concepts like Russell's Paradoxes or fuzzy set theory. However, as I've outline that this isn't a new idea, you have also used sets in your daily life.

Informally, a **set** is a collection of objects or elements. These objects are also called members. A classroom is a set made of students, and these students are the members of this set. In that same classroom, we can have a set which consists of only boys. Remember that the art of grouping can group things that have similarities or are entirely different. The classroom set has both girls and boys. The similarity is that they're all human. The set grouping all boys in that classroom share the similarity being that they're boys, but they're most likely different in many other ways, from names to their Bloop Pressure. Thus a set is often denoted with a capital letter, while the elements are denoted by small letters. Let

$$\mathrm{V} = \{a, e, i, o, u\}$$

Be an example where the set V is the set of all vowels. Where a, e, i, o, u are the members of the set V. The curly braces means "*the set of*". Since u and a, e, i, o, u are elements of V, we denote it as

$u \epsilon \mathrm{V}$ and $(a, e, i, o, u) \epsilon \mathrm{V} \equiv a, e, i, o, u \epsilon \mathrm{V}$

Respectively. It oath to be noted that the *order* of elements in the set and *repetition* are *not* important, thus a set with elements

$\{o, e, u, a, i\}$ and $\{a, a, i, i, i, i, i, o, u, e, e\}$

Are the same as the set V. The number of elements within a set are referred to as a **Cardinality**, denoted by the absolute bars $|X|$ where X is a set. Thus the cardinality of the set of vowels V is 5 denoted

$$|\mathrm{V}| = 5.$$

Another simple example would be as follows, lets call it the *Black Box* example :

Suppose you have a *Black Box* in a class, then ask each and every student to put a paper inside of the box(more like voting). The papers could be different colors and shapes, or just all plain White square papers.
The Box will act as a set while the papers of the students inside the box are regarded as the elements of the set(or elements of the box).
If the box was empty, then we'd say the box is an **empty set/null set**. This is a special case of a set and it is denoted by

$$\phi \text{ or empty curly braces } \{\}.$$

In turn, $\phi = \{\}$ with a cardinality of 0.

EXAMPLE 1

Suppose there are three sets

$$A = \phi \qquad B = \{\phi\} \qquad C = \{\}$$

1. Which of these are equal?
   a) A and B
   b) A and C
   c) B and C
2. What are the cardinalities of each set?

SOLUTION

1. For the first question :
   a. A and B are not equal since A is a null set-without elements while B is a set that has a null set as an element. Thus, $A \neq B$
   b. A and C are equal since A is a null set-without elements and C is also set that has no elements in it. Thus $A = C$
   c. B and C are not equal since B is a set that has a null set as an element while C is a null set-without elements . Thus, $B \neq C$
2. As for the cardinalities

$$|A| = |C| = 0$$
$$|B| = 1.$$

1.2 SUBSETS

A **subset** can informally be defined as *a set within a set*. As per the name suggests, it a portion of a set and it is denoted by $\subseteq$. Formally, we say the set X_1 is the subset of X_2 if every element of X_1 is also an element in X_2.
Suppose you have three sets $A = \{2,4,6,8,10\}$; $B = \{2,6,8\}$ and $C = \{2,6\}$.
C is the subset of A and B since 2 & 6 are in both the sets A and B,that is

$C \subseteq A$ and $C \subseteq B$.

Set B is a subset of A but not of C. That is

$B \subseteq A$ and $B \nsubseteq C$.

And lastly A is not a subset of either sets.

Every set is a subset of itself.

Common basic sets which are also subsets of others include :

- **Integers**: all set of numbers without a decimal. These are all set of integers which are either positive or negative. Represented by the symbol $\mathbb{Z}$
  denoted by $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.
- **Natural Numbers**: all positive integers *EXCEPT* 0. Represented by the symbol $\mathbb{N}$
  denoted by $\mathbb{N} = \{1,2,3, \dots\}$.
- **Whole Numbers**: all positive integers *INCLUDING* 0, denoted by $\mathbb{N}_0 = \{0,1,2,3, \dots\}$.
- **Rational Numbers**: Represented by the symbol $\mathbb{Q}$. All fractions of the form
  $\mathbb{Q} = \left\{\frac{a}{b} \,\middle|\, a, b \in \mathbb{Z}, b \backslash\neq 0\right\}$.
- **Irrational numbers**: numbers that are thought not to be rational are said to be irrational.
- **Imaginary Numbers**: imagined numbers that may not exist where $\sqrt{-1} = i$
- **Real Numbers**: all numbers that can be quantified and thought not to be imaginary. These numbers are made up by both rational and irrational numbers Represented by the symbol $\mathbb{R}$
- **Complex Numbers**: a combination of both real and imaginary numbers. Represented by the symbol $\mathbb{C}$ and denoted as $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R} \; and \; i^2 = -1\}$

1.3 SET OPERATIONS

Suppose you`re given two sets $S = \{a, b, c\}$ and $T = \{1, b, 2\}$

A **Union** contains all elements of sets, denoted by ∪

$$\text{S or T} = \text{S} \cup T = \{a, b, c, 1,2\}$$

the notation $\text{S} \cup T$ is called S union T which is equivalent to $S\ or\ T$.

Intersection are common elements within the sets, denoted by ∩

$$S\ and\ T = S \cap T = \{b\}$$

the notation $S \cap T$ is called S intersection T which is equivalent to $S\ and\ T$.

Universal set: A set that contains every element denoted $\mathcal{u}$.

Complement : all elements within a universal set *THAT IS NOT* within the given set, denoted as $not\ A = A' = A^C = \bar{A}$.

Difference/Relative complement : these are elements in the set S BUT NOT in T where

$$S - T = S \cap \bar{T} = \{a, c\}$$

There are also addition, subtraction and multiplication for sets

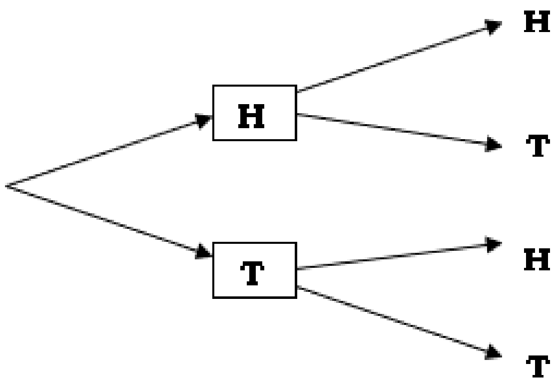
2 PROBABILITY

Probability came from studying games from many individuals of the past and can be dated to the emergence of the Homo Sapien specie. In modeling, we say that there are two kinds of events, namely, a deterministic and random event. Inherently from the name, a deterministic relation is one that can be easily modelled and determined precisely since it has an underlying mechanism. The later is known for not having a mechanism, this meaning it cannot be known precisely. However, random events can be approximated. Probability deals with modelling random events of all the possible outcomes that exist.

In probability, the universal set is called the sample space. This is the set of all possible outcomes denoted by Ω. Let us solidify this definition by an example.

EXAMPLE 2.1

Suppose you want to determine the sample space of flipping a fair coin twice. We'll first use a tree to represent this to make it easy to understand. Suppose we denote Head by H and Tail by T.



Which would give the sample space

$$\Omega = \{hh, ht, th, tt\}.$$

Probability then is the chance that an event will occur. How *likely* will an event happen, informally. The probability of any event is a ratio or a number that is between 0 and 1, with the probability of 0 implying that an event is impossible to happen and 1 being certain the event will happen. Probabilities can also be represented in terms of percentages from 0% to 100%. The probability of an event A where A is a particular set, is denoted as $P(A)$ and is said to be the probability of A happening. A probability is a ratio of a particular event happening over all possible events that could happen while an **odd** is the ratio of a particular event happening over all those that wont happen as the other one is happening.

EXAMPLE 2.2

To revisit the previous example, suppose that a fair coin is flipped twice.

Let H_1 and H_2 denote that the event of a head is on the first and second toss respectively.

We can clearly see that these two sets are both subsets of the sample space

$$\Omega = \{hh, ht, th, tt\}$$

Making them to be

$$H_1 = \{hh, ht\} \text{ and } H_2 = \{hh, th\} \text{ respectively.}$$

Now the probability of getting Head on the first flip is

$$P(H_1) = \frac{2}{4} = \frac{1}{2} = 0.5 = 50\%$$

And *loosely so*, one would define the probability of this event as

$$P(H_1) = \frac{|H_1|}{|\Omega|}$$

But this is Illegal, DON'T SAY IT to a mathematician or statistician, they'll kill you.

2.2 PROBABILISTIC PROPERTIES

We assume that the probability of the sample space and the null set are as follows respectively

$P(\Omega) = 1$ and $P(\phi) = 0$.

And inherently from these axioms, we have

Property A : $P(A) + P(A^c) = 1$

Property B : $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(Still working on the other parts that come after this)