$$\lambda e^{\pi} + \sum_{i=0}^{n} \frac{\phi^{n}}{n^{n}}$$

Katleho Nyoni

SEQUENCE AND SERIES

1 SEQUENCE AND SERIES

The idea of sequences and series' go a long way back into the past. One of the earliest encounters could be seen around 5th BCE where the famous Greek philosopher Zeno of Elea introduced a number of problems where only Nine survived, which came to be known as *Zeno's Paradoxes*[1], these were his proposals to the questions concerning space and time at their time. On one of them, Zeno argued that a man who wanted to walk across a room, had to walk half the distance of the room first, but before travelling that distance they had to travel halfway of that half, then half of that, and repeatedly infinitely many times producing the sequence

$$\dots;\frac{1}{16};\frac{1}{8};\frac{1}{4};\frac{1}{2};1$$

Which he claimed was impossible since it requires a number of infinite tasks and wouldn't have the first distance to travel, meaning the journey wouldn't even start. Traveling across the same room would give the sequence

$$\frac{1}{2};\frac{1}{4};\frac{1}{8};\dots;\frac{1}{2^n}$$

This sequence is popularly known as the Geometric sequence. More about that later.

In present day, sequence and series has become increasingly important as it is widely used in almost every field. In finance it is used to calculate the return of loans and investments, in physics it is used to study waves and their properties. In Chemistry it is used to see how chemical reactions end, in Statistics it is used to study trends, in ecology and epidemiology it is used to model populations of the same and different species. In astronomy and cosmology it is used to study the emission of radiation/energy of celestial bodies; and it is used by TikTok and Netflix to recommend you videos and movies you may like.

First and foremost, what's the difference between a sequence and a series?
Sequence : a list of ordered numbers separated by a comma

e.g. $1,2,3,4,\dots,n$ where 1 is the 1st term denoted by T_1

$2,4,6,8,\dots,2n$ where $2n$ is the formula for the sequence denoted T_n

$3,6,9,12,\dots,2n+1$

Thus, a sequence is denoted by

[1] "Many of these paradoxes argue that contrary to the evidence of one's senses, motion is nothing but an illusion" supporting *monism* source : https://en.wikipedia.org/wiki/Zeno%27s_paradoxes

$T_1, T_2, T_3, \dots, T_n$

Series : the sum of individual terms that form a sequence, separated by the + operator, sometimes the -, or even both

e.g. $1 + 2 + 3 + 4 + \cdots$

$2 + 4 + 6 + 8 + \cdots$

$3 + 6 + 9 + 12 + \cdots$

Thus, a series is denoted by

$S_n = T_1 + T_2 + T_3 + \cdots + T_n = \sum_{i=1}^{n} T_i$

Where

$$S_1 = T_1$$
$$S_2 = T_1 + T_2 = \sum_{i=1}^{2} T_i$$
$$S_3 = T_1 + T_2 + T_3 = \sum_{i=1}^{3} T_i$$

There are *recursive* and *non-recursive* sequences.
RECURSIVE : these are sequences which whose terms are dependent on the previous term. They are a *'regress'*. A very famous one is the Fibonacci sequence that appears a lot in nature, especially in the rows of corn or becomes the sequence of the petals on a flower.

Fibonacci sequence : 0,1,1,2,3,5,8,13, …

Another interesting example is the logistic equation, which is often a realistic approach used to model the population of species.

DEPAND

NON-RECURSIVE : sequences that are not recursive. These are the 'normal' ones. We'll only be dealing with the Arithmetic(linear), Quadratic and Geometric sequences and series'.

1.1 ARITHMETIC SEQUENCE

An Arithmetic sequence can be seen as a linear function where the c in

$$\mathrm{y} = \mathrm{mx} + \mathrm{c}$$

can either be negative or positive. This is a constant(fixed) number that is added to each previous term to find the next.
For the sake of formality, in this chapter we use the notation

$$T_n = \phi n + \gamma \qquad 1.1.1$$

But more importantly, to obtain the equation in 1.1 , we use the equation

$$T_n = a + (n-1)d \qquad 1.1.2$$

Where $\phi = d$ and $\gamma = a - d$. Here T_n denotes the general term, a the first term of the sequence and d is the common difference. Since a sequence is defined by

$$T_1, T_2, T_3, \dots, T_n$$

To obtain the next term in the arithmetic sequence you have to add the common difference d, thus

$$\begin{aligned}
T_2 &= T_1 + d \\
T_3 &= T_2 + d = (T_1 + d) + d = T_1 + 2d \\
T_4 &= T_3 + d = T_1 + 2d + d = T_1 + 3d \\
&\dots \\
T_n &= T_1 + (n-1)d
\end{aligned}$$

■

Then to get the common difference d we get it by subtracting from the previous term

$$\begin{aligned}
d &= T_2 - T_1 \\
d &= T_3 - T_2 \\
d &= T_4 - T_3 \\
&\dots \\
d &= T_{k+1} - T_k
\end{aligned}$$

■

EXAMPLE 1
Suppose you're given the arithmetic sequence $5, 9, 13, 17, \dots$ and you're asked to find:

a) the common difference d
b) the next three terms
c) the general term T_n
d) using the general term, find the 100th term

e) which term in the sequence is equal to 1181

SOLUTION

a) To find d, we calculate the difference of two numbers that are next to each other, preferably from right to left. Thus selecting any two would give $d = 9 - 5 = 4$. Which means that the terms of our sequences are increasing by 4 each time, that is added to the previous term, since the term is positive.

b) Using d as 4 as found in the previous question we get

$$T_5 = 17 + 4 = 21$$
$$T_6 = 21 + 4 = 25$$
$$T_7 = 25 + 4 = 29$$

c) Using equation 1.1.2 we obtain

$$T_n = 5 + (n - 1)(4)$$
$$T_n = 5 + 4n - 4$$
$$T_n = 4n + 1$$

Or alternatively, we could approach it as

$$T_n = (4)n + (5) - (4)$$

Which yields the same results.

d) Since we now have the general term, we find the 100th term by substituting 100 where there is n on our equation. This yields

$$T_{100} = 4(100) + 1 = 401$$

e) Since we now have the general term, to find the nth term that is equal to 1181, we solve for n wwhere $T_n = 1181$ thus we get

$$T_n = 4n + 1$$
$$1181 = 4n + 1$$
$$1181 - 1 = 4n$$
$$4n = 1180$$
$$n = \frac{1180}{4} = 295$$

EXAMPLE 2

The first three terms of an arithmetic sequence are

$$2x + 1 \,, x - 3 \,, 3x - 4$$

a) Determine the value of x and then write the values of the first three terms.

SOLUTION

What we know about the arithmetic sequence is that they have the same constant difference, thus

$$3x - 4 - (x - 3) = x - 3 - (2x + 1)$$

$$2x - 1 = -x - 4$$
$$2x + x = -4 + 1$$
$$3x = -3$$
$$x = -\frac{3}{3} = -1$$

Thus the sequence is $-1, -4, -7$ ■

EXAMPLE 3

$x, y, 8, \ldots$ and $1, 4x, 3y, \ldots$ are arithmetic sequences.

a) Determine the values of x and y .

SOLUTION

What we know about the arithmetic sequence is that they have the same constant difference, thus

$8 - y = y - x$ (1) $\Rightarrow$ $2y - 8 = x$

$3y - 4x = 4x - 1$ (2) $\Rightarrow$ $3y - 8x + 1 = 0$

Which now becomes

$$3y - 8(2y - 8) + 1 = 0$$
$$3y - 16y + 64 + 1 = 0$$
$$-13y + 65 = 0$$
$$y = \frac{65}{13} = 5$$

Thus $x = 2(5) - 8 = 2$ ■

EXAMPLE 4

Determine the arithmetic sequences if the 4th term is -6 and the 11th is -34

SOLUTION

What we know is that $T_4 = -6$ and $T_{11} = -34$, thus using our equation, we obtain

$$T_4 = -6 = a + (4 - 1)d \qquad (1)$$
$$T_{11} = -34 = a + (11 - 1)d \qquad (2)$$

Then we solve the equations simultaneously.

$\Rightarrow a = -3d - 6$ and

$$-34 = (-3d - 6) + 10d$$
$$-34 = 7d - 6$$
$$-34 + 6 = 7d$$

$d = -\frac{28}{7} = -4$

Then solving for a we get

$$a = -3(-4) - 6 = 6$$

■

1.2 ARITHMETIC SERIES

From the previous chapter, we showed that a sequence is defined as

$$T_1, T_2, T_3, \dots, T_n$$

Thus the series is

$$S_1 = T_1$$

$$S_2 = T_1 + T_2 = \sum_{i=1}^{2} T_i$$

$$S_3 = T_1 + T_2 + T_3 = \sum_{i=1}^{3} T_i$$

$$\dots$$

$$S_n = T_1 + T_2 + \cdots + T_n = \sum_{i=1}^{n} T_i$$

Where S_n is the sum of all the n terms[2].

An arithmetic series is the sum of the arithmetic sequence terms. The formula for the sum of n consecutive terms is given by

$$S_n = \frac{n}{2}[2a + (n-1)d] \qquad (1.2.1)$$

Or by the following equation if the last term, denoted by L is known

$$S_n = \frac{n}{2}(a + L) \qquad (1.2.2)$$

where $L = a + (n-1)d$.

Before we proof that the sum of the arithmetic terms are given by (1.2.1/2), let me tell you a story :

It is said that back in the 19th century, a teacher was lazy and his learners kept making noise, so he gave them the task to sum num the natural numbers 1 to 100. He knew this would keep them busy and would take them a long time. By surprise, one learner had completed within five minutes or less giving the correct answer of 5050, the teacher was perplexed and asked for the script of the young boy to see what he had done. The young man had used a trick. Suppose S_{100} denotes the sum of these numbers

$$S_{100} = 1 + 2 + 3 + \cdots 98 + 99 + 100$$

[2] Notations such as $\sum a_k$ were introduced by Leonard Euler who was a prominent polymath and whose work is estimated to be 800 pages of yearly manuscripts

Reversing this would give the same answer

$$S_{100} = 100 + 99 + 98 + \cdots + 3 + 2 + 1$$

Then summing the two you'd get twice the amount

$$2S_{100} = (1 + 100) + (2 + 99) + (3 + 98) + \cdots + (98 + 3) + (99 + 2) + (100 + 1)$$
$$2S_{100} = 101 + 101 + 101 + \cdots + 101 + 101 + 101$$
$$2S_{100} = 100(101)$$
$$S_{100} = \frac{100(101)}{2} = \frac{10100}{2} = 5050.$$

Now we're going to use this fact to prove the equation that gives the sum of arithmetic terms

PROOF

Suppose you have an arithmetic sequence

$a, a + d, a + 2d, \ldots, a + (n - 1)d$

and let the sum of these terms be denoted by

$$S_n = a + a + d + a + 2d + \cdots a + (n - 3)d + a + (n - 2)d + a + (n - 1)d$$
$$S_n = a + a + d + a + 2d + \cdots + a + (n - 1)d$$

now reversing these terms we get

$$S_n = a + (n - 1)d + \cdots + a + 2d + a + d + a$$

Now adding the two you get

$$S_n + S_n = (a + a + (n - 1)d) + \cdots + (2a + (n - 1)d) + [2a + (n - 1)d])$$
$$2S_n = n[2a + (n - 1)d]$$
$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

and where $l = a + (n - 1)d$ is the last term in the sequence/series

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}[a + a + (n - 1)d] = \frac{n}{2}[a + l]$$

EXAMPLE 1

Determine the sum of the arithmetic series $4 + 11 + 18 + 25 + \cdots + 368$

SOLUTION

What we know is that $l = 368$ and the common difference $d = T_{n+1} - T_n = 7$, thus using our equation, we obtain

$$368 = (7)n + (4) - (7)$$
$$7n = 368 + 3 = 371$$

$$n = \frac{371}{7} = 53$$

Then solving for S_n we get

$$S_{53} = \frac{53}{2}[4 + 368]$$
$$S_{53} = 9858$$

EXAMPLE 2

How many terms of the arithmetic series $8 + 13 + 18 + \cdots$ will add up to 1700

SOLUTION

What we know is that $S_n = 1700$ and the common difference $d = T_{n+1} - T_n = 5$, thus using our equation, we obtain

$$1700 = \frac{n}{2}[2(8) + (n-1)(5)]$$
$$1700 \times 2 = n[16 + 5n - 5]$$
$$3400 = n[5n + 11] = 5n^2 + 11n$$
$$5n^2 + 11n - 3400 = 0$$
$$n = \frac{\left(-b \pm \sqrt{b^2 - 4ac}\right)}{2a}$$
$$n = -\frac{11 \pm \sqrt{11^2 - 4(5)(-3400)}}{2(5)}$$
$$n = -\frac{136}{5} \text{ or } n = 25$$

Thus $n = 25$ since $S\colon \mathbb{N} \to \mathbb{R}$ in simple.

EXAMPLE 3

Given the arithmetic series $-4 - 1 + 2 + \cdots$

a) Calculate the smallest number of terms where $S_n > 300$
b) Determine the greatest number of terms that can be added together if the answer must be less than 100

SOLUTION

a) Using our formula, we get

$$\frac{n}{2}[2(-4) + (n-1)(3)] > 300$$

$$n[-8+3n-3] > 600$$
$$3n^2 - 11n - 600 > 0$$
$$n = \frac{11 \pm \sqrt{11^2 - 4(3)(-600)}}{2(3)}$$
$$n > -12.43 \text{ or } n > 16.09$$

$$\text{Thus } n = \lceil 16.09 \rceil = 17$$

b) Almost similarly, we have

$$\frac{n}{2}[2(-4) + (n-1)(3)] < 100$$
$$3n^2 - 11n - 200 < 0$$
$$= \frac{11 \pm \sqrt{11^2 - 4(3)(-200)}}{2(3)}$$
$$n < -6.53 \text{ or } n < 10.20$$

Thus $n = \lfloor 10.20 \rfloor = 10$

EXAMPLE 4

Determine the sum of natural numbers from 1 to 1000 that are not divisible by 5

SOLUTION

The sequence of natural numbers is $1,2,3,4,\ldots,1000$ up to a 1000 and the sequence of numbers that **ARE** divisible by 5 is $5,10,15,20,\ldots,1000$. So to know the sum of $\mathbb{N}$ that are **NOT** divisible we have to subtract those that **are** divisible from the sum of the first sequence. Thus

$$S_{1000} = \frac{1000}{2}[1+1000]$$
$$S_{1000} = 500500$$

Is the sum of the first sequence. Now we have

$$1000 = 5n + (5-5)$$
$$n = \frac{1000}{5} = 200$$

Thus

$$S_{200} = \frac{200}{2}[5+1000]$$
$$S_{200} = 100500$$

Which is the sum of the series of terms divisible by 5. Now the sum of those that are **not** divisible by 5 is

$$S_{1000} - S_{200} = 500\,500 - 100\,500 = 400\,000$$

1.3 GEOMETRIC SEQUENCE

A geometric sequence is a sequence where the succeeding term is obtained by multiplying the previous term with a constant ratio. Given a sequence

$$T_1, T_2, T_3, \dots, T_n$$

Then

$$T_2 = rT_1$$
$$T_3 = rT_2 = r(rT_1) = r^2T_1$$
$$T_4 = rT_3 = r(r^2T_1) = r^3T_1$$
$$\dots$$
$$T_n = r^{n-1}T_1 = ar^{n-1}$$

■

$$T_n = ar^{n-1} \qquad (1.3.1)$$

Where $a = T_1$, r is the constant ratio and the general/nth term is given by T_n.

To obtain the constant ratio r from the previous derivation

$$r = \frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_4}{T_3} = \dots = \frac{T_{k+1}}{T_k}$$

■

EXAMPLE 1

Consider the geometric sequence

$$\frac{1}{2}; \frac{1}{4}; \frac{1}{8}; \frac{1}{32}; \dots$$

And determine

a) The nth term.

b) What term is equal to $\frac{1}{1024}$.

c) The value of the 7th term.

SOLUTION

a) The common ratio is $\frac{T_{k+1}}{T_k} = \frac{1}{2}$ thus the general term is

$$T_n = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n-1}$$

b) The question is asking for n , thus

$$\frac{1}{1024} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^{n}$$
$$\ln\left(\frac{1}{1024}\right) = \ln\left(\frac{1}{2}\right)^{n}$$
$$\ln\left(\frac{1}{1024}\right) = n\ln\left(\frac{1}{2}\right)$$
$$n = \frac{\ln\left(\frac{1}{1024}\right)}{\ln\left(\frac{1}{2}\right)} = 10$$

c) $T_7 = \left(\frac{1}{2}\right)^7 = \frac{1}{128}$

EXAMPLE 2

The first three terms of a geometric sequence are $x + 2; 3x + 1; 7x - 1$. Determine the value(s) of x.

SOLUTION

We know that the common ratio is

$$\frac{7x - 1}{3x + 1} = \frac{3x + 1}{x + 2}$$

Then follows

$$(3x + 1)^2 = (x + 2)(7x - 1)$$
$$9x^2 + 6x + 1 = 7x^2 + 13x - 2$$
$$2x^2 - 7x + 3 = 0$$
$$(2x - 1)(x - 3) = 0$$
$$x = \frac{1}{2} \text{ or } x = 3$$

EXAMPLE 3

Determine the geometric sequence if the second term is $\frac{1}{9}$ and the sixth term is $\frac{1}{729}$.

SOLUTION

We know

$$T_2 = \frac{1}{9} = ar^{(2)-1} = ar$$
$$T_6 = \frac{1}{729} = ar^{(6)-1} = ar^5$$

Now solving simultaneously, we get

$$\frac{1}{729} = \left(\frac{1}{9r}\right)r^5$$
$$\frac{1}{729} = \left(\frac{1}{9}\right)r^4$$
$$r^4 = \frac{1}{81}$$
$$\sqrt[4]{r^4} = \sqrt[4]{\frac{1}{81}}$$
$$r = \pm\frac{1}{3}$$

Then a is

$$a = \left(\frac{1}{9}\right)\left(\pm\frac{1}{3}\right)^{-1} = \pm\frac{1}{3}$$

And the sequence is

$\pm\frac{1}{3};\frac{1}{9};\pm\frac{1}{27}$ depending on whether a is positive or negative.

1.4 GEOMETRIC SERIES

1.4.1 FINITE GEOMETRIC SERIES

From the previous derivation of the nth term of the geometric sequence, we have observed that a geometric sequence is defined as

$$T_1, rT_1, r^2T_1, r^3T_1, r^4T_1, \dots, r^{n-1}T_1$$

Thus the geometric series is obtained by summing these individual terms. Let the sum of these terms be denoted by

$$S_n = T_1 + rT_1 + r^2T_1 + r^3T_1 + r^4T_1 + \cdots + r^{n-1}T_1$$

$$S_n = T_1 + rT_1 + r^2T_1 + r^3T_1 + r^4T_1 + \cdots + r^{n-3}T_1 + r^{n-2}T_1 + r^{n-1}T_1$$

$$rS_n = rT_1 + r^2T_1 + r^3T_1 + r^4T_1 + \cdots + r^{n-2}T_1 + r^{n-1}T_1 + r^nT_1$$

$$S_n - rS_n = T_1 + \cdots + r^{n-1}T_1 - (rT_1 + \cdots + r^nT_1)$$

$$S_n - rS_n = T_1 - r^nT_1$$

$$S_n(1-r) = T_1(1-r^n)$$

$$S_n = \frac{T_1(1-r^n)}{1-r} \quad \blacksquare$$

Thus the sum of geometric terms is given by

$$S_n = \frac{T_1(1-r^n)}{1-r} = \frac{T_1(r^n-1)}{r-1}, \ r \neq 1 \qquad (1.4.1)$$

EXAMPLE 1

Consider the geometric series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{32} + \cdots$$

And

a) Calculate how many terms will be needed for the sum to be 0.8 .
b) Calculate the sum of the first 200 terms.
c) Determine the least number of terms for which the sum will be greater or equal to 1.

SOLUTION

a) The common ratio is $\frac{T_{k+1}}{T_k} = \frac{1}{2}$ thus the general term is

$$T_n = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n-1}$$

$$S_n = 0.8 = \frac{\frac{1}{2}\left(\left(\frac{1}{2}\right)^n - 1\right)}{\frac{1}{2} - 1}$$

$$-0.8 \times 0.5 = 0.5((0.5)^{\mathrm{n}} - 1)$$

$$-0.8 + 1 = 0.5^{\mathrm{n}}$$

$$\ln\left(\frac{1}{2}\right)^n = \ln(0.2)$$

$$\mathrm{n}\ln\left(\frac{1}{2}\right) = \ln(0.2)$$

$$\mathrm{n} = \frac{\ln(0.2)}{\ln(0.5)} = \lfloor 2.32 \rfloor = 2$$

b) The question is asking for S_{200} , thus

$$S_{200} = \frac{\frac{1}{2}\left(\left(\frac{1}{2}\right)^{200} - 1\right)}{\frac{1}{2} - 1} = 1$$

c) **CONVERGENCE**

$$S_n = 1 = \frac{\frac{1}{2}\left(\left(\frac{1}{2}\right)^n - 1\right)}{\frac{1}{2} - 1}$$

$$-0.5 = 0.5((0.5)^{\mathrm{n}} - 1)$$

$$-1 + 1 = 0.5^{\mathrm{n}}$$

$$\ln\left(\frac{1}{2}\right)^n = \ln(0)$$

$$\mathrm{n} = ERROR$$

1.4.2 INFINITE GEOMETRIC SERIES

An infinite geometric sequence would be denoted as

$$T_1, rT_1, r^2T_1, \ldots$$

And the infinite geometric series would be

$$S_\infty = a + ar + ar^2 + \cdots = \sum_{n=1}^{\infty} ar^{n-1}.$$

Since

$$\lim_{n\to\infty} r^n = \begin{cases} \infty\ , & r > 1 \\ 0\ , & |r| < 1 \end{cases}$$

Equation (1.4.1) becomes

$$S_\infty = \frac{a}{1-r}\ ; r \neq 1 \qquad (1.4.2)$$

Furthermore, if $|r| < 1$ for a Geometric Series, then the series is ***convergent***

EXAMPLE 1

Consider the geometric series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{32} + \cdots$$

And calculate S_∞

SOLUTION

$$\boldsymbol{S}_\infty = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

EXAMPLE 2

Does $0.\overline{9}$ converge or diverge?

SOLUTION

$$0.999999\ldots = 0.9 + 0.09 + 0.009 + 0.0009 + \cdots$$

$$0.999999\ldots = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \frac{9}{10^4} + \cdots$$

$$0.999999\ldots = \frac{\frac{9}{10}}{1-\frac{1}{10}} = \frac{\frac{9}{10}}{\frac{9}{10}} = 1$$

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