Numerical Approximation of PDEs

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Project 3: A semilinear elliptic equation

The goal of this project is to compute the solution of the following nonlinear problem:

$$\begin{cases} -\Delta u + \alpha u^3 = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $f = 100$ and $\Omega = [0,1]^2$ and $\alpha > 0$. Equation (1) is a "semilinear" equation, and cannot be analyzed with the usual Lax–Milgram's lemma due to the nonlinear term αu^3.
As a solution strategy, we use a so-called *fixed-point* iteration: we seek u as the limit $n \to \infty$ of the recursive sequence

$$\begin{cases} -\Delta u_{n+1} + \alpha u_n^2 u_{n+1} = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

Here, given some current iterate u_n, the new iterate u_{n+1} is computed by solving (2). The iteration is terminated once $\|u_n - u_{n+1}\|$ is deemed sufficiently small.
Note that this method solves the nonlinear problem by solving a sequence of linear auxiliary problems.

Answer the following points

1. Given some u_n and $\alpha > 0$, derive the weak form of (2).
2. Implement the fixed-point scheme on a mesh containing at least 100 vertices. Use $\mathbb{P}_1$ basis functions in combination with a *Gauss quadrature scheme* of order 7. For this, you can use the solutions of exercise 2 of session 8 from the exercise class as a starting point for your Matlab code. Run the scheme for $\alpha = 0.1$ and $\alpha = 2$. Plot the solution and count the number of iterations required for $\|u_n - u_{n+1}\|_\infty$ to drop below 10^{-6}. As an initial guess, use $u_0 = 0$.

   Obviously, the required number of iterations for $\alpha = 2$ is unacceptable. Therefore, we now try a different strategy. We now seek u as the limit $n \to \infty$ of the recursive Newton-sequence

   $$u_{n+1} = u_n + \partial u_n,$$

   where ∂u_n satisfies

   $$\int_\Omega \nabla\phi \cdot \nabla \partial u_n + 3\alpha u_n^2 \phi \partial u_n \mathrm{d}\Omega = \int_\Omega -\nabla\phi \cdot \nabla u_n - \phi(\alpha u_n^3 - f)\mathrm{d}\Omega, \quad \forall \phi \in H_0^1(\Omega)$$
   $$\partial u_n = 0, \text{ on } \partial\Omega.$$

3. Approximate u by implementing the newton scheme on a mesh comprised of at least 100 mesh vertices. Use $\mathbb{P}_1$ basis functions. Terminate the iteration as soon as $\|u_{n+1} - u_n\|_\infty < 10^{-6}$. Use the same initial guess as in the fixed-point iteration. Do this for $\alpha \in \{0.1, 2, 5\}$. Does the Newton-scheme fare better for larger values of α? *HINT: for the implementation of the Newton-scheme, you can slightly adjust and re-use some of the assembly routines from the fixed-point scheme.*