

Project 3: A semilinear elliptic equation

Spring 2023

Question 1

We want to find the weak formulation of the problem

$$\begin{cases} -\Delta u_{n+1} + \alpha u_n^2 u_{n+1} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where u_n is the current iterate.

We multiply by a test function $v \in H_0^1(\Omega)$ and integrate to obtain:

$$\int_{\Omega} f v = \int_{\Omega} (-\Delta u_{n+1} v + \alpha u_n^2 u_{n+1} v).$$

We use integration by parts to rewrite the right hand side:

$$\begin{aligned} \int_{\Omega} (-\Delta u_{n+1} v) &= - \int_{\Omega} \operatorname{div}(\nabla u_{n+1}) v \\ &= - \int_{\Omega} \operatorname{div}(\nabla u_{n+1} v) + \int_{\Omega} \nabla u_{n+1} \nabla v \\ &= - \oint_{\partial\Omega} v(\nabla u_{n+1} \cdot \vec{n}) + \int_{\Omega} \nabla u_{n+1} \nabla v && \text{by the divergence theorem} \\ &= \int_{\Omega} \nabla u_{n+1} \nabla v && \text{by } v \in H_0^1(\Omega), \end{aligned}$$

where we denote \vec{n} the outer normal vector on the boundary $\partial\Omega$.

The weak formulation is:

$$\int_{\Omega} (\nabla u_{n+1} \nabla v + \alpha u_n^2 u_{n+1} v) = \int_{\Omega} f v \quad \forall v \in H_0^1(\Omega). \quad (1)$$

Question 2

Question 3