Project 3: A semilinear elliptic equation

Spring 2023

Question 1

We want to find the weak formulation to the problem of finding u_{n+1} satisfying

$$\begin{cases} -\Delta u_{n+1} + \alpha u_n^2 u_{n+1} = f & \text{in } \Omega, \\ u_{n+1} = 0 & \text{on } \partial\Omega, \end{cases}$$

where u_n is the current iterate, with $u_n = 0$ on $\partial\Omega$.

We multiply by a test function $v \in H_0^1(\Omega)$ and integrate to obtain:

$$\int_\Omega fv = \int_\Omega \left(-\Delta u_{n+1} v + \alpha u_n^2 u_{n+1} v \right).$$

We use integration by parts to rewrite the right hand side:

$$\begin{aligned} \int_\Omega (-\Delta u_{n+1} v) &= -\int_\Omega \operatorname{div}(\nabla u_{n+1} v) + \int_\Omega \nabla u_{n+1} \nabla v && \text{integration by part} \\ &= -\oint_{\partial\Omega} v(\nabla u_{n+1} \cdot \vec{n}) + \int_\Omega \nabla u_{n+1} \nabla v && \text{by the divergence theorem} \\ &= \int_\Omega \nabla u_{n+1} \nabla v && \text{by } v \in H_0^1(\Omega), \end{aligned}$$

where we denote $\vec{n}$ the outer normal vector on the boundary $\partial\Omega$.

The weak formulation is:

$$\int_\Omega \nabla u_{n+1} \nabla v + \int_\Omega \alpha u_n^2 u_{n+1} v = \int_\Omega fv \quad \forall v \in H_0^1(\Omega). \tag{1}$$

We have a stiffness integral, and a mass integral with the reaction term αu_n^2, as well as a Poisson right hand side.

Question 2

Question 3