

# Homework 4 - Continuous Optimization

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We want to solve the following problem:

$$\min_{x,y \in \mathbb{R}^n} f(x,y) \text{ subject to } h(x,y) = 0, \quad (\text{P})$$

where  $f$  and  $h$  are specified on the homework sheet.

## Question 1

The feasible set is  $S = \{x, y \in \mathbb{R}^n \mid h(x, y) = 0\} = \{x, y \in \mathbb{R}^n \mid 1 - x^\top x = 0, 1 - y^\top y = 0, x^\top y = 0\}$ . It is not convex. Indeed, we will give two points  $z_1$  and  $z_2 \in S$  such that  $z = \lambda z_1 + (1 - \lambda)z_2 \notin S$  for a given  $\lambda \in ]0, 1[$ . We will work with  $n = 2$ .

We will take  $x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $y_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  and write  $z_1 = (x_1, y_1)$ . First we check that  $z_1 \in S$ .

- $1 - x_1^\top x_1 = 1 - (1 \ 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 - 1 = 0$
- $1 - y_1^\top y_1 = 1 - (0 \ 1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 - 1 = 0$
- $x_1^\top y_1 = (1 \ 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$

And we will take  $z_2 = (x_2, y_2) = \left( \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$ . We also check that  $z_2 \in S$ .

- $1 - x_2^\top x_2 = 1 - (0 \ 1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 - 1 = 0$
- $1 - y_2^\top y_2 = 1 - (1 \ 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 - 1 = 0$
- $x_2^\top y_2 = (0 \ 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$

Lastly, we will take  $\lambda = \frac{1}{2}$ . We can compute

$$z = \lambda z_1 + (1 - \lambda)z_2 = \frac{1}{2} \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) + \frac{1}{2} \left( \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = \left( \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \right) = (x, y)$$

But now, we have  $x^\top y = \begin{pmatrix} 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = (\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{2} \neq 0$ . So  $h_3(x, y) \neq 0$  and  $z \notin S$ . Hence the set  $S$  is not convex.

By definition, LICQ holds at  $x \in S$  if and only if  $\nabla h_1(x), \dots, \nabla h_p(x)$ , and  $\nabla g_i(x)$  for  $i \in \mathcal{I}(x)$  are linearly independent.

Here, we do not have any inequality constraint function  $g_i$  but we have three equality constraints functions  $h_i$ :

- $h_1(x, y) = 1 - x^\top x$
- $h_2(x, y) = 1 - y^\top y$
- $h_3(x, y) = x^\top y$

If we compute their gradients, we get:

- $\nabla_x h_1(x, y) = \nabla_x (1 - x^\top x) = \nabla_x (1 - \sum_{i=1}^n x_i^2) = -2x$  and  $\nabla_y h_1(x, y) = 0$

then  $\nabla h_1(x, y) = \begin{bmatrix} -2x \\ \vec{0} \end{bmatrix}$  where  $\vec{0}$  is the vector  $\vec{0} = (0, \dots, 0)^\top \in \mathbb{R}^n$ .

- $\nabla_x h_2(x, y) = 0$  and  $\nabla_y h_2(x, y) = \nabla_y (1 - y^\top y) = \nabla_y (1 - \sum_{i=1}^n y_i^2) = -2y$

then  $\nabla h_2(x, y) = \begin{bmatrix} \vec{0} \\ -2y \end{bmatrix}$

- $\nabla_x h_3(x, y) = \nabla_x \sum_{i=1}^n x_i \cdot y_i = y$  and  $\nabla_y h_3(x, y) = \nabla_y \sum_{i=1}^n x_i \cdot y_i = x$

then  $\nabla h_3(x, y) = \begin{bmatrix} y \\ x \end{bmatrix}$

We show that these three gradients are linearly independent:

$$\lambda_1 \nabla h_1(x, y) + \lambda_2 \nabla h_2(x, y) + \lambda_3 \nabla h_3(x, y) = 0 \iff \begin{cases} -2\lambda_1 x + \lambda_3 y = 0 \\ -2\lambda_2 y + \lambda_3 x = 0 \end{cases}$$

We have that this is true without having all the  $\lambda_i = 0$ , if and only if  $x = \lambda y$  (for some  $\lambda \neq 0$  which can be deduced from the previous equations), or  $y = 0$ . But if  $y = 0$ , we have  $h_2(x, y) = 1 \neq 0$  so our point is not feasible. Same with  $h_1$  if  $x = 0$ . Lastly, if  $x = \lambda y$  for  $\lambda \neq 0, x, y \neq 0$ , we get that  $h_3(x, y) = x^\top y = \lambda y^\top y = \lambda \|y\|^2 \neq 0$ , so our point is not feasible either.

Hence  $\lambda_1 \nabla h_1(x, y) + \lambda_2 \nabla h_2(x, y) + \lambda_3 \nabla h_3(x, y) = 0$  implies  $\lambda_1 = \lambda_2 = \lambda_3 = 0$ .

To conclude, we get that for all of our feasible points, i.e. the points in the set  $S$ ,  $\nabla h_1(x, y), \nabla h_2(x, y), \nabla h_3(x, y)$  are linearly independent, which means by definition that LICQ holds.

## Question 2

Our two first constraints  $h_1(x, y) = 1 - x^\top x = 0$  and  $h_2(x, y) = 1 - y^\top y = 0$  can be rephrased  $\|x\| = \|y\| = 1$ . We notice that there is no constraint on  $x$  and  $y$  at the same time, i.e. a constraint that tells us what  $x$  must be related to  $y$  and vice-versa. Similarly, we can notice that our target function doesn't have a part where  $x$  and  $y$  are mixed.

It means that if we optimized  $x$  and  $y$  separately, the optimum found will be the optimum for our relaxed problem as well. So we rewrite our relaxed problem as:

$$\min_{\substack{x \in \mathbb{R}^n \\ \|x\|=1}} \frac{1}{2} x^\top A x + \min_{\substack{y \in \mathbb{R}^n \\ \|y\|=1}} \frac{1}{2} y^\top B y$$

The two problems are solved identically as the only thing changing is the matrix. To solve them, we first recall example 2.14 from the notes:

If  $A$  is a symmetric linear map with eigenvalues  $\lambda_1, \dots, \lambda_n$ , then  $\forall u \in \mathcal{E}$ , we have

$$\lambda_{\min} \|u\|^2 \leq \langle u, A(u) \rangle \leq \lambda_{\max} \|u\|^2$$

and by rewriting the scalar product we get

$$\lambda_{\min} \|u\|^2 \leq u^\top A u \leq \lambda_{\max} \|u\|^2$$

Applied to our problems (let's look at the first as they are similar), knowing that we optimize on vector of norm 1, we get that for every feasible  $x$ , we have

$$\frac{1}{2} \lambda_{\min} \leq \frac{1}{2} x^\top A x \leq \frac{1}{2} \lambda_{\max}$$

It tells us that our optimal value can't be lower than  $\frac{1}{2} \lambda_{\min}$ . If we find an  $x$  that attains this bound, we will know that it is the optimal value. So let's find it.

As  $\lambda_{\min}$  is an eigenvalue, it means that  $\exists v_{\min} \in \mathbb{R}^n, v_{\min} \neq 0$  such that  $A v_{\min} = \lambda_{\min} v_{\min}$ . We take  $x = \frac{v_{\min}}{\|v_{\min}\|}$ , which is feasible since it has norm 1, and then we have

$$x^\top A x = \frac{1}{\|v_{\min}\|^2} v_{\min}^\top (A v_{\min}) = \frac{1}{\|v_{\min}\|^2} v_{\min}^\top (\lambda_{\min} v_{\min}) = \frac{\lambda_{\min}}{\|v_{\min}\|^2} v_{\min}^\top v_{\min} = \frac{\lambda_{\min}}{\|v_{\min}\|^2} \cdot \|v_{\min}\|^2 = \lambda_{\min}$$

Therefore our lower bound is attained for this  $x$ , which means it is the optimal value.

We denote  $\lambda_{\min}(A)$  and  $\lambda_{\min}(B)$  the minimal eigenvalues for the matrices  $A$  and  $B$ . From our previous reasoning, we have that  $\frac{\lambda_{\min}(A)}{2}$  and  $\frac{\lambda_{\min}(B)}{2}$  are our optimal values, and so the optimal value for our relaxed problem is  $\frac{\lambda_{\min}(A) + \lambda_{\min}(B)}{2}$ .

Now, what does that say about the target problem?

Because the relaxed problem is our target problem without a constraint, we know its optimal value must be lower than the optimal value of the target problem, as every feasible solution of our target problem is feasible in our relaxed problem (and we work with a minimization problem). Therefore, we can say that our target problem has a lower bound of  $\frac{\lambda_{\min}(A) + \lambda_{\min}(B)}{2}$ . However, we don't know (yet) if it is the optimal value or not.

### Question 3

Let's find an expression for the Lagrangian function  $L(x, y, \mu)$ . We denote  $I_n$  for the identity matrix in  $\mathbb{R}^{n \times n}$ .

$$\begin{aligned} L(x, y, \mu) &= f(x, y) + \mu^\top h(x, y) \\ &= \frac{1}{2} x^\top A x + \frac{1}{2} y^\top B y + [\mu_1 \quad \mu_2 \quad \mu_3] \begin{bmatrix} 1 - x^\top x \\ 1 - y^\top y \\ x^\top y \end{bmatrix} \\ &= \frac{1}{2} x^\top A x + \frac{1}{2} y^\top B y + \mu_1 (1 - x^\top x) + \mu_2 (1 - y^\top y) + \mu_3 x^\top y \\ &= \frac{1}{2} (x^\top A x - 2\mu_1 x^\top x + y^\top B y - 2\mu_2 y^\top y + 2\mu_3 x^\top y) + \mu_1 + \mu_2 \\ &= \frac{1}{2} (x^\top (A - 2\mu_1 I_n) x + y^\top (B - 2\mu_2 I_n) y + x^\top (\mu_3 I_n) y + y^\top (\mu_3 I_n) x) + \mu_1 + \mu_2 \\ &= \frac{1}{2} \begin{bmatrix} x^\top & y^\top \end{bmatrix} \begin{bmatrix} A - 2\mu_1 I_n & \mu_3 I_n \\ \mu_3 I_n & B - 2\mu_2 I_n \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \mu_1 + \mu_2 \end{aligned}$$

As  $A$  and  $B$  are symmetric by assumptions, and  $I_n$  is also symmetric, we see that the above matrix is symmetric.

## Question 4

By definition,  $L_D(\mu) = \inf_{x,y \in \mathbb{R}^n} L(x,y,\mu)$ . Using the previous question, and denoting  $M_\mu := \begin{bmatrix} A - 2\mu_1 I_n & \mu_3 I_n \\ \mu_3 I_n & B - 2\mu_2 I_n \end{bmatrix}$ , we find:

$$\begin{aligned} L_D(\mu) &= \inf_{x,y \in \mathbb{R}^n} L(x,y,\mu) \\ &= \inf_{x,y \in \mathbb{R}^n} \left\{ \frac{1}{2} \begin{bmatrix} x^\top & y^\top \end{bmatrix} M_\mu \begin{bmatrix} x \\ y \end{bmatrix} + \mu_1 + \mu_2 \right\} \\ &= \inf_{x,y \in \mathbb{R}^n} \left\{ \frac{1}{2} \begin{bmatrix} x^\top & y^\top \end{bmatrix} M_\mu \begin{bmatrix} x \\ y \end{bmatrix} \right\} + \mu_1 + \mu_2 \\ &= \begin{cases} \mu_1 + \mu_2 & \text{if } M_\mu \succeq 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

Indeed, if  $M_\mu$  is not positive semidefinite, there exists a negative eigenvalue  $\lambda < 0$ . We can consider its associated eigenvector  $v = (v_x, v_y) \in \mathbb{R}^{2n}$  (with  $v_x, v_y \in \mathbb{R}^n$ ; by simplicity we take  $v$  such that  $\|v\| = 1$ ), i.e. we have  $M_\mu v = \lambda v$ . Then for all  $n \in \mathbb{N}$ , we have  $(nv)^\top M_\mu (nv) = n^2 v^\top \lambda v = n^2 \lambda$ , so by looking at the sequence  $(nv)_{n \in \mathbb{N}}$ , we have that  $L(nv_x, nv_y, \mu)$  converges to  $-\infty$ .

Hence we can write the dual as:

$$\max_{\mu \in \mathbb{R}^3} \{\mu_1 + \mu_2\} \quad \text{subject to} \quad \begin{bmatrix} A - 2\mu_1 I_n & \mu_3 I_n \\ \mu_3 I_n & B - 2\mu_2 I_n \end{bmatrix} \succeq 0 \quad (\text{D})$$

## Question 5

We know that  $M_\mu$  is symmetric for all  $\mu \in \mathbb{R}^3$ . If  $\mu$  is a solution of the dual problem, then the matrix  $M_\mu$  associated with the values of  $\mu$  should be positive semidefinite. This implies in particular that the diagonal blocks would be positive semidefinite too, i.e.  $A - 2\mu_1 I_n \succeq 0$  and  $B - 2\mu_2 I_n \succeq 0$ . The two latter conditions are equivalent to  $2\mu_1 \leq \lambda_{\min}(A)$  and  $2\mu_2 \leq \lambda_{\min}(B)$ , where  $\lambda_{\min}(A)$  and  $\lambda_{\min}(B)$  are the smallest eigenvalues of  $A$  and  $B$  respectively.

Hence, if we focus only on satisfying the conditions  $A - 2\mu_1 I_n \succeq 0$  and  $B - 2\mu_2 I_n \succeq 0$ , the optimal value for (D) is  $\frac{\lambda_{\min}(A) + \lambda_{\min}(B)}{2}$ .

In fact, this is the optimal value even for the whole condition  $M_\mu \succeq 0$ . Indeed, let's focus on the relaxed problem for a bit. Call  $P'$  the relaxed primal, and  $D'$  the relaxed dual. By question 2, we know that  $\frac{\lambda_{\min}(A) + \lambda_{\min}(B)}{2}$  is the lower bound on the optimal value for  $P'$ . Now observe that the relaxed dual function is in fact

$$L_{D'}(\mu) = \inf_{x,y \in \mathbb{R}^n} \left\{ \frac{1}{2} \begin{bmatrix} x^\top & y^\top \end{bmatrix} \begin{bmatrix} A - 2\mu_1 I_n & 0 \\ 0 & B - 2\mu_2 I_n \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\} + \mu_1 + \mu_2,$$

This comes directly from the fact that if we remove the third constraint  $h_3(x,y) = 0$ , then we no longer have any  $\mu_3$ .

By weak duality for the relaxed problem, and denoting  $S' = \{(x,y) \mid h_1(x,y) = 0, h_2(x,y) = 0\}$  the feasible set of  $P'$ , we get:

$$\max_{\mu \in \mathbb{R}^3} L_{D'}(\mu) \leq \min_{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n} L_{P'}(x,y) = \min_{(x,y) \in S'} f(x,y) = \frac{\lambda_{\min}(A) + \lambda_{\min}(B)}{2}$$

Moreover, we know  $\max_{\mu \in \mathbb{R}^3} L_D(\mu) \leq \max_{\mu \in \mathbb{R}^3} L_{D'}(\mu)$  as if we go from  $D'$  to  $D$  we add a constraint, and the feasible maximum can only go lower. Then, putting this together with the previous equality gives:

$$\max_{\mu \in \mathbb{R}^3} L_D(\mu) \leq \frac{\lambda_{\min}(A) + \lambda_{\min}(B)}{2}$$

Finally, we see that taking  $\mu = \left( \frac{\lambda_{\min}(A)}{2}, \frac{\lambda_{\min}(B)}{2}, 0 \right)$  in the dual  $D$  gives  $M_\mu \succeq 0$  as both diagonal blocks will be positive semidefinite (and the anti-diagonal blocks are zero), and  $L_D(\mu) = \frac{\lambda_{\min}(A) + \lambda_{\min}(B)}{2}$ . So the upper bound is attained and is indeed the optimal value of the dual problem.

## Question 6

To use the strong duality theorem, we need to satisfy the two following assumptions:

- (A1) the primal problem (P) admits a KKT point  $(x^*, y^*) \in S$  with valid Lagrange multipliers  $\mu^* \in \mathbb{R}^3$
- (A2) the function  $(x, y) \mapsto L(x, y, \mu^*)$  is convex

First, let's prove that (A1) holds.

We will use the theorem 8.26 of the lecture notes, as we know by question 1 that LICQ holds at all feasible points. It remains to show that there exists a local minimum of  $f$  in  $S$ .

Observe that  $S = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid h(x, y) = 0\}$  is a closed set as a preimage of the closed set  $\{0\} \subset \mathbb{R}$  by the continuous function  $h$ . Moreover, we have  $S \subset \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid h_1(x, y) = 0, h_2(x, y) = 0\} = \partial B_1(0) \times \partial B_1(0)$ , where  $B_1(0) = \{z \in \mathbb{R}^n \mid \|z\| < 1\}$  is the unit ball in  $\mathbb{R}^n$ . Hence  $S$  is bounded. So we deduce that it is compact (as it is closed and bounded in  $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ ).

Now,  $f$  is continuous on the compact set  $S$ , thus it must attain its (global) minimum in it, say at a point  $(x^*, y^*)$ . The theorem 8.26 implies that  $(x^*, y^*)$  is a KKT point (as LICQ holds in particular at this point). Let  $\mu^* \in \mathbb{R}^3$  be valid Lagrange multipliers for  $(x^*, y^*)$ .

Consider now the condition (A2). It is equivalent to  $\begin{bmatrix} A - 2\mu_1^* I_n & \mu_3^* I_n \\ \mu_3^* I_n & B - 2\mu_2^* I_n \end{bmatrix}$  being positive semi-definite. Now for this latter condition to be satisfied, we would need information about  $\mu^*$ . But we can't expect to have those informations before solving the problem, as we need to solve it to find  $(x^*, y^*)$ . Hence we do not know if (A2) holds.

## Question 7

Here, we will calculate the gradient of the augmented Lagrangian function. In the first section, we present unrequested details, and then we present the results.

### Details

We will first compute the gradient with respect to  $x$ .  $\nabla_y$  will be done similarly. First of all, we know, by linearity, that

$$\nabla_x L_\beta(x, y, \mu) = \nabla_x f(x, y) + \nabla_x \mu^\top h(x, y) + \frac{\beta}{2} \nabla_x \|h(x)\|^2$$

We will compute each of these expressions:

We know that  $\nabla_x f(x, y) = Ax$

Now for the second one, we can develop  $\mu^\top h(x, y) = \mu_1(1 - x^\top x) + \mu_2(1 - y^\top y) + \mu_3 x^\top y$ , so we have:

$$\begin{aligned} \nabla_x \mu^\top h(x, y) &= \nabla_x (\mu_1(1 - x^\top x)) + \nabla_x (\mu_2(1 - y^\top y)) + \nabla_x (\mu_3 x^\top y) \\ &= \mu_1 \nabla_x (1 - x^\top x) + \mu_3 \nabla_x (x^\top y) \\ &= \mu_1 (-2x) + \mu_3 y \\ &= -2\mu_1 x + \mu_3 y \end{aligned}$$

As  $\frac{\partial}{\partial x_i}(x^\top x) = \frac{\partial}{\partial x_i} \sum_{i=1}^n x_i^2 = 2x_i$  and  $\frac{\partial}{\partial x_i}(x^\top y) = \frac{\partial}{\partial x_i} \sum_{i=1}^n x_i y_i = y_i$

Now for the third part we will use the fact that  $\nabla ||x||^2 = 2x$  as  $x^\top x = ||x||^2$  and we've just computed it before. We also know the chain rule:  $\frac{\partial}{\partial x_i} f \circ g(x) = \nabla f(g(x)) \cdot \frac{\partial}{\partial x_i} g(x)$ . We can use that to compute our  $\nabla$  :

$$\frac{\partial}{\partial x_i} \frac{\beta}{2} ||h(x, y)||^2 = \beta h(x, y) \cdot \frac{\partial}{\partial x_i} (h(x, y))$$

which gives us the formula

$$\begin{aligned} \nabla_x \frac{\beta}{2} ||h(x, y)||^2 &= \beta \nabla_x h(x, y) \cdot h(x, y) \text{ where } \nabla_x h(x, y) \text{ is a } n \times 3 \text{ matrix} \\ &= \beta \begin{bmatrix} -2x & \vec{0} & y \end{bmatrix} \cdot \begin{bmatrix} 1 - x^\top x \\ 1 - y^\top y \\ x^\top y \end{bmatrix} \\ &= \beta \begin{bmatrix} 2(x^\top x - 1)x_1 + (x^\top y)y_1 \\ \vdots \\ 2(x^\top x - 1)x_n + (x^\top y)y_n \end{bmatrix} \\ &= \beta (2(x^\top x - 1)x + (x^\top y)y) \end{aligned}$$

## Results

Putting together our three calculations, we get

$$\begin{aligned} \nabla_x L_\beta(x, y, \mu) &= Ax - 2\mu_1 x + \mu_3 y + \beta (2(x^\top x - 1)x + (x^\top y)y) \\ &= Ax + 2(\beta(x^\top x - 1) - \mu_1)x + (\beta \cdot x^\top y + \mu_3)y \end{aligned}$$

Similarly, we get

$$\begin{aligned} \nabla_y L_\beta(x, y, \mu) &= By - 2\mu_2 y + \mu_3 x + \beta (2(y^\top y - 1)y + (x^\top y)x) \\ &= (\beta \cdot x^\top y + \mu_3)x + By + 2(\beta(y^\top y - 1) - \mu_2)y \end{aligned}$$

## Question 8

We wrote code that takes as input  $z = [x^\top, y^\top]^\top$ ,  $\mu$ ,  $\beta$  (and also  $A$  and  $B$ ) and returns  $L_\beta(z, \mu)$  and  $\nabla_z L_\beta(z, \mu)$ . Our function is named `LBetaValAndGrad` as it calls the two sub-functions we are showing below, `LBeta` and `LBetaGrad`.

```
1 function [val, grad] = LBetaValAndGrad(z, mu, beta, A, B)
2     val = LBeta(z, mu, beta, A, B);
3     grad = LBetaGrad(z, mu, beta, A, B);
4 end
```

```
1 function val = LBeta(z, mu, beta, A, B)
2     hz = h(z);
3     val = f(z, A, B) + mu' * hz;
4     if beta ≠ 0 % Just to avoid calculating the norm if not necessary...
5         val = val + beta * sum(hz .* hz) / 2;
6         % = val + beta * vecnorm(hz)^2 / 2;
7     end
8 end
```

```

1 function grad = LBetaGrad(z, mu, beta, A, B)
2     n = length(z)/2;
3     x = z(1:n);
4     y = z(n+1:end);
5
6     C1 = 2 * (beta * (x'*x - 1) - mu(1));
7     C2 = 2 * (beta * (y'*y - 1) - mu(2));
8     C3 = beta * x' * y + mu(3);
9
10    grad = [A*x + C1*x + C3*y; C3*x + B*y + C2*y];
11 end

```

The second function uses the functions  $f$  and  $h$  which are a direct implementation of their definition.

## Question 9

We have used the Matlab `fminunc` function. We set it to use the 'Quasi-Newton' sub-algorithm as it is the default parameter, it performs as well as Trust-Region in our situation, and it is a little quicker. We have created a function `minXY` which calls the precedent function with our choices for the parameters. Here is our implementation.

```

1 function [z, zval] = minXY(mu, beta, A, B, z0, verbose)
2 % Our function that runs fminunc with desired parameters,
3 % to minimize LBeta with fixed mu and beta.
4 % Input:
5 % mu           : Fixed vector of size (3, 1)
6 % beta         : Fixed cost factor.
7 % A and B      : The defined matrices, size (n, n)
8 % z0           : Initial point.
9 % verbose      : Boolean indicating what the algorithm should print.
10 % Output:
11 % z            : The final z found with the given algorithm.
12 % zval         : It's value.
13 options = optimoptions('fminunc');
14 options.SpecifyObjectiveGradient = true; % indicate gradient is provided
15 %options.Algorithm = 'trust-region';    % We could also use TR...
16 if verbose % add display
17     options.Display = 'iter-detailed';
18 else
19     options.Display = 'none';
20 end
21 [z, zval] = fminunc(@(z) LBetaValAndGrad(z, mu, beta, A, B), z0, options);
22 end

```

This function with  $\beta = 1.42$ ,  $\mu = [1, 2, -3]^\top$  and initial guess  $z_0 = [1, 0, -1, 2, 1, 1, 2, 0, 1, 2]^\top$  gives after 50 iterations

```

1 Found x and y are
2   -1.3695   -1.2693
3   -0.2535    0.9636
4   -0.0107   -1.0786
5    0.0825    0.7410
6    1.4721    0.5569
7
8 with value f(x, y) = -28.9565 and LBeta(x, y, mu) = -26.4171
9 and h(x, y), mu: which have norms 5.2991, 3.7417
10   -3.1138    1.0000
11   -3.5622    2.0000
12    2.3865   -3.0000

```

One can see the full listing in the Appendix.

Note that the results would have been little bit different if we were using 'trust-region' as a sub-algorithm for the function `fminunc`.

In the 19-th line of the `main.m`, we defined a function handle which allows us to invoke `minXY` easily, and which does not output anything in the console:

```
silentMinXY = @(mu, beta, z0)minXY(mu, beta, A, B, z0, 0);
```

This is the function we will use for now on.

## Question 10

The part of the main running the quadratic penalty method is the following:

```
1 mu = [0; 0; 0];
2 z0 = randn(2*n, 1);
3 z = z0;
4 beta = 1;
5 for i = 1:N
6     [z, zval] = silentMinXY(mu, beta, z);
7
8     % Display
9     disp(newline + "Iteration " + i + " with beta = " + beta + ". We found x and y:")
10    disp([z(1:n), z(n+1:end)])
11    disp("with value f(x, y) = " + f(z) + "    and LBeta(x, y, mu) = " + LBeta(z, mu, beta, ...
12          A, B))
13    hz = h(z);
14    disp("and h(x, y), beta*h(x, y):          which have norms " + vecnorm(hz) + ", " + beta * ...
15          vecnorm(hz))
16    disp([hz, beta * hz])
17    beta = 2 * beta; % Updating
18 end
```

We listed the complete output in the Appendix, but here is what the final iteration leaves:

```
1 Iteration 9 with beta = 256. We found x and y:
2    0.6036    -0.4636
3    0.2996     0.3649
4   -0.3370   -0.6272
5    0.1038     0.4992
6   -0.6591     0.1458
7
8 with value f(x, y) = -6.41    and LBeta(x, y, mu) = -6.3694
9 and h(x, y), beta*h(x, y):    which have norms 0.01781, 4.5593
10   -0.0128   -3.2785
11   -0.0119   -3.0504
12   -0.0033   -0.8566
```

We notice that the norm of  $h(x, y)$  is divided by two at each iterations, since  $\beta h(x, y)$  stays almost constant. This reassures us that we could hope, as we have seen in the lecture notes, that  $h(x_k) \approx \frac{\mu^*}{\beta_k}$ . However, we will indeed have a convergence of this sort, as we will see in question 11, but maybe not to  $\mu^*$ , as we will see in question 12.

We see that after 9 iterations, the points  $x$  and  $y$  are still *far* from  $h(x, y) = 0$ . That's why we consider the augmented Lagrangian method.

## Question 11

The part of the main running the augmented Lagrangian method is the following:



```

1 mu = [0; 0; 0];
2 % z0 = randn(2*n, 1); % We start with the same z0 as for QPM.
3 z = z0;
4 beta = 1;
5 for i = 1:N
6     [z, zval] = silentMinXY(mu, beta, z);
7
8     hz = h(z);
9     mu = mu + beta * hz; % Updating
10
11     % Display
12     disp(newline + "Iteration " + i + " with beta = " + beta + ". We found x and y:")
13     disp([z(1:n), z(n+1:end)])
14     disp("with value f(x, y) = " + f(z) + " and LBeta(x, y, mu) = " + LBeta(z, mu, beta, ...
15         A, B))
16     disp("and h(x, y), new mu: which have norms " + vecnorm(hz) + ", " + vecnorm(mu))
17     disp([hz, mu])
18     beta = 2 * beta; % Updating
19 end

```

Note that the only difference is the update of  $\mu$  at line 9. Also, we decided not to choose a new  $z_0$  randomly, but to reuse the same starting point as with the QPM of question 10. The final iteration leaves:

```

1 Iteration 9 with beta = 256. We found x and y:
2     0.5988    -0.4605
3     0.2988     0.3626
4    -0.3372    -0.6237
5     0.1041     0.4965
6    -0.6539     0.1447
7
8 with value f(x, y) = -6.3288 and LBeta(x, y, mu) = -6.3288
9 and h(x, y), new mu: which have norms 3.2811e-08, 4.5595
10    -0.0000    -3.2774
11     0.0000    -3.0515
12     0.0000    -0.8582

```

Again, we listed the complete outputs in the Appendix.

This time,  $h(x, y)$  goes to 0 really quickly. Moreover, we see that our estimation of  $\mu$  quickly stabilizes to a fixed value which is indeed the one  $\beta h(x, y)$  was converging to with the quadratic penalty method.

Note that depending on the starting points, the sign of the values may differ. But with all our tests, the numbers were more or less the same.

## Question 12

Our program finds  $\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} -3.2774 \\ -3.0515 \\ \pm 0.8582 \end{pmatrix}$ . The sign of the last value varies, but this is not important for the rest of our calculations.

We know from question 3 that the Lagrangian is

$$L(x, y, \mu) = \frac{1}{2} \begin{bmatrix} x^\top & y^\top \end{bmatrix} \begin{bmatrix} A - 2\mu_1 I_n & \mu_3 I_n \\ \mu_3 I_n & B - 2\mu_2 I_n \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \mu_1 + \mu_2.$$

We know that for a fixed  $\mu$ , this function is convex if only if the matrix—that we named  $M_\mu$  in question 4—is positive semi-definite. In our main, the following lines are calculating the minimum eigenvalue of  $M_\mu$  and may output a warning in the console.

```

1 I = eye(5);
2 M = [A - 2 * mu(1) * I, mu(3) * I; mu(3) * I, B - 2 * mu(2) * I];
3 lambdaM = min(eig(M));
4 disp("hence, lambda_min(M_mu) = " + lambdaM + ".")
5 if lambdaM < 0
6     disp("And therefore, M_mu is not semi-positive definite,")
7     disp("and mu is not feasible in (D).")
8 else
9     disp(">= M_mu is semi-positive definite.")
10 end

```

As this warning is triggered by our value of  $\mu$ , we conclude that  $L(x, y, \mu)$  is not convex for this  $\mu$ , and also that  $\mu$  is not feasible in (D). That is bad news, since our hope was that  $\mu^k$  converges to  $\mu^*$  as our algorithm progresses; but instead we have that  $\mu$ , which is the limit<sup>1</sup> of  $\mu^k$ , is not feasible in (D) and therefore is not equal to  $\mu^*$ . Note that if  $\mu$  was feasible, then  $\mu_1 + \mu_2 = -6.3288$  would be a lower bound for the value  $f(x, y)$  from the weak duality theorem, and we could conclude that our found  $(x, y)$  (which reached this value) is a global minimum. Moreover, the optimal value for (D) is  $\frac{\lambda_{\min}(A) + \lambda_{\min}(B)}{2} = -6.7355$ . Therefore there is no  $\mu$  feasible in the dual that will show that our found  $(x, y)$  is a global minimum of (P).

Now we consider the strong duality theorem. We have seen in question 6 that the first assumption holds. However, we can't know if the second one holds before actually solving the problem. We also know that if the theorem does not hold, then there will be a gap between the optimal values for (P) and (D). With our found  $(x, y)$ , we see that there is a gap with the optimal value of (D), but we can't know if we truly found an optimal point for (P), or if there is a better solution—hiding somewhere and really difficult to find—with no gap and a feasible  $\mu^*$ , which would make the Lagrangian convex. Thus, we can't know that the theorem holds, nor be sure that it does not hold. However, we are pretty close of a no-gap solution, and the fact that every execution of our algorithm leaves to the same (four, considering axial symmetries) solutions is reassuring in thinking that no better solution exists.

---

<sup>1</sup> $\mu$  should be the limit of our calculated sequence  $\mu^k$ . In practice, we stop at  $\mu^9$ . Two things motivate this choice. First, the value of  $\mu^k$  rapidly converged, and, as the associated value  $h(x^k)$  is really close to zero at that point, the value of  $\mu^k$  should stay unchanged. But secondly, as  $\beta$  increases if we do too much iterations, small calculation residues in  $h(x^k)$  will be drastically amplified and will change  $\mu^k$  to reach extreme values.

# Appendix

## Complete listing of the call of fminunc in question 9

```

1 =====
2 Start of one fminunc with fixed mu and beta
3
4 Iteration  Func-count      f(x)      Step-size      First-order
5           0           1      178.82      0.0121625      optimality
6           1           2      10.1162      0.0121625      82.2
7           2           3     -4.20962      1          17.7
8           3           4    -15.0808      1          12.7
9           4           5    -17.2193      1          5.64
10          5           6    -19.7316      1          6.58
11          6           7    -21.3684      1          6.67
12          7           8    -21.9965      1          5.65
13          8           9    -22.4082      1          2.56
14          9          10    -22.9733      1          2.17
15         10          11    -23.0828      1          1.42
16         11          12    -23.1327      1          0.947
17         12          13    -23.1999      1          0.782
18         13          14    -23.2755      1          0.884
19         14          15    -23.3273      1          1.05
20         15          16    -23.3533      1          0.745
21         16          17    -23.3677      1          0.522
22         17          18    -23.3878      1          0.486
23         18          19    -23.4311      1          0.485
24         19          20    -23.5459      1          0.81
25                                     1.7
26                                     First-order
27 Iteration  Func-count      f(x)      Step-size      optimality
28         20          22    -23.6747      0.402734      3.25
29         21          23    -23.9004      1          3.18
30         22          24    -24.136      1          2.7
31         23          26    -24.3296      0.5          1.35
32         24          27    -24.4815      1          0.818
33         25          28    -24.5246      1          0.755
34         26          29    -24.6162      1          0.793
35         27          30    -24.7419      1          1.06
36         28          31    -24.925      1          1.33
37         29          32    -25.0605      1          2.47
38         30          33    -25.2445      1          2.3
39         31          34    -25.481      1          3.21
40         32          35    -25.7183      1          1.71
41         33          36    -25.9075      1          1.9
42         34          37    -26.1991      1          1.26
43         35          38    -26.341      1          0.724
44         36          39    -26.3684      1          0.644
45         37          40    -26.3901      1          0.254
46         38          41    -26.3943      1          0.249
47         39          42    -26.4027      1          0.219
48                                     First-order
49 Iteration  Func-count      f(x)      Step-size      optimality
50         40          43    -26.4078      1          0.229
51         41          44    -26.411      1          0.185
52         42          45    -26.4127      1          0.123
53         43          46    -26.4145      1          0.119
54         44          47    -26.4161      1          0.106
55         45          48    -26.4169      1          0.0649
56         46          49    -26.4171      1          0.0159
57         47          50    -26.4171      1          0.00199
58         48          51    -26.4171      1          0.000491
59         49          52    -26.4171      1          0.000166
60         50          53    -26.4171      1          2.59e-05

```

```

60
61 Optimization completed: The first-order optimality measure, 3.112651e-07, is less
62 than options.OptimalityTolerance = 1.000000e-06.
63
64 Found x and y are
65     -1.3695    -1.2693
66     -0.2535     0.9636
67     -0.0107    -1.0786
68      0.0825     0.7410
69      1.4721     0.5569
70
71 with value f(x, y) = -28.9565    and LBeta(x, y, mu) = -26.4171
72 and h(x, y), mu:    which have norms 5.2991, 3.7417
73     -3.1138     1.0000
74     -3.5622     2.0000
75      2.3865    -3.0000

```

## Complete listing of the QPM of question 10

```

1  =====
2  Start of the Quadratic penalty method
3
4  Iteration 1 with beta = 1. We found x and y:
5      1.3349    -0.9717
6      0.4912     0.7564
7     -0.4391    -1.2122
8      0.1089     0.9557
9     -1.4509     0.3235
10
11 with value f(x, y) = -27.0433    and LBeta(x, y, mu) = -16.69
12 and h(x, y), beta*h(x, y):    which have norms 4.5504, 4.5504
13     -3.3330    -3.3330
14     -3.0037    -3.0037
15     -0.7585    -0.7585
16
17
18 Iteration 2 with beta = 2. We found x and y:
19      1.0365    -0.7603
20      0.4026     0.5928
21     -0.3792    -0.9649
22      0.1000     0.7630
23     -1.1276     0.2499
24
25 with value f(x, y) = -16.6912    and LBeta(x, y, mu) = -11.5126
26 and h(x, y), beta*h(x, y):    which have norms 2.2757, 4.5513
27     -1.6619    -3.3239
28     -1.5051    -3.0102
29     -0.3890    -0.7779
30
31
32 Iteration 3 with beta = 4. We found x and y:
33      0.8481    -0.6284
34      0.3517     0.4911
35     -0.3503    -0.8132
36      0.0978     0.6450
37     -0.9235     0.2035
38
39 with value f(x, y) = -11.5137    and LBeta(x, y, mu) = -8.9226
40 and h(x, y), beta*h(x, y):    which have norms 1.1382, 4.5528
41     -0.8281    -3.3124
42     -0.7548    -3.0192
43     -0.2002    -0.8006

```

```

44
45
46 Iteration 4 with beta = 8. We found x and y:
47     0.7350    -0.5507
48     0.3248     0.4314
49    -0.3393    -0.7252
50     0.0989     0.5764
51    -0.8012     0.1761
52
53 with value f(x, y) = -8.9232    and LBeta(x, y, mu) = -7.6266
54 and h(x, y), beta*h(x, y):      which have norms 0.56935, 4.5548
55    -0.4126    -3.3009
56    -0.3786    -3.0291
57    -0.1027    -0.8216
58
59
60 Iteration 5 with beta = 16. We found x and y:
61     0.6708    -0.5075
62     0.3114     0.3984
63    -0.3364    -0.6767
64     0.1007     0.5383
65    -0.7318     0.1610
66
67 with value f(x, y) = -7.6269    and LBeta(x, y, mu) = -6.9781
68 and h(x, y), beta*h(x, y):      which have norms 0.28479, 4.5566
69    -0.2057    -3.2916
70    -0.1899    -3.0376
71    -0.0523    -0.8371
72
73
74 Iteration 6 with beta = 32. We found x and y:
75     0.6360    -0.4845
76     0.3050     0.3809
77    -0.3362    -0.6508
78     0.1022     0.5180
79    -0.6941     0.1530
80
81 with value f(x, y) = -6.9781    and LBeta(x, y, mu) = -6.6536
82 and h(x, y), beta*h(x, y):      which have norms 0.14243, 4.5579
83    -0.1027    -3.2853
84    -0.0951    -3.0436
85    -0.0265    -0.8468
86
87
88 Iteration 7 with beta = 64. We found x and y:
89     0.6177    -0.4727
90     0.3019     0.3718
91    -0.3365    -0.6374
92     0.1031     0.5074
93    -0.6743     0.1489
94
95 with value f(x, y) = -6.6536    and LBeta(x, y, mu) = -6.4912
96 and h(x, y), beta*h(x, y):      which have norms 0.071229, 4.5586
97    -0.0513    -3.2816
98    -0.0476    -3.0473
99    -0.0133    -0.8523
100
101
102 Iteration 8 with beta = 128. We found x and y:
103     0.6083    -0.4666
104     0.3003     0.3673
105    -0.3368    -0.6306
106     0.1036     0.5020
107    -0.6642     0.1468
108

```

```

109 with value f(x, y) = -6.4912    and LBeta(x, y, mu) = -6.41
110 and h(x, y), beta*h(x, y):      which have norms 0.035618, 4.5591
111     -0.0256    -3.2795
112     -0.0238    -3.0493
113     -0.0067    -0.8552
114
115
116 Iteration 9 with beta = 256. We found x and y:
117     0.6036    -0.4636
118     0.2996     0.3649
119    -0.3370    -0.6272
120     0.1038     0.4992
121    -0.6591     0.1458
122
123 with value f(x, y) = -6.41    and LBeta(x, y, mu) = -6.3694
124 and h(x, y), beta*h(x, y):      which have norms 0.01781, 4.5593
125     -0.0128    -3.2785
126     -0.0119    -3.0504
127     -0.0033    -0.8566

```

## Complete listing of the ALM of question 11

```

1  =====
2  Start of the augmented Lagrangian method
3
4  Iteration 1 with beta = 1. We found x and y:
5      1.3349    -0.9717
6      0.4912     0.7564
7     -0.4391    -1.2122
8      0.1089     0.9557
9     -1.4509     0.3235
10
11 with value f(x, y) = -27.0433    and LBeta(x, y, mu) = 4.0166
12 and h(x, y), new mu:      which have norms 4.5504, 4.5504
13     -3.3330    -3.3330
14     -3.0037    -3.0037
15     -0.7585    -0.7585
16
17
18 Iteration 2 with beta = 2. We found x and y:
19     0.5927    -0.4849
20     0.2931     0.3805
21    -0.3230    -0.6199
22     0.0962     0.4859
23    -0.6474     0.1617
24
25 with value f(x, y) = -6.339    and LBeta(x, y, mu) = -6.3265
26 and h(x, y), new mu:      which have norms 0.052296, 4.5535
27     0.0302    -3.2727
28    -0.0264    -3.0566
29    -0.0336    -0.8256
30
31
32 Iteration 3 with beta = 4. We found x and y:
33     0.6017    -0.4603
34     0.2962     0.3623
35    -0.3318    -0.6228
36     0.1020     0.4960
37    -0.6569     0.1446
38
39 with value f(x, y) = -6.3353    and LBeta(x, y, mu) = -6.3287
40 and h(x, y), new mu:      which have norms 0.0079021, 4.5591

```

```

41     -0.0019    -3.2803
42       0.0020    -3.0485
43     -0.0074    -0.8552
44
45
46 Iteration 4 with beta = 8. We found x and y:
47     0.5987    -0.4609
48     0.2988     0.3629
49    -0.3371    -0.6237
50     0.1041     0.4963
51    -0.6538     0.1449
52
53 with value f(x, y) = -6.329    and LBeta(x, y, mu) = -6.3288
54 and h(x, y), new mu:    which have norms 0.00063213, 4.5594
55     0.0004    -3.2773
56    -0.0004    -3.0516
57    -0.0003    -0.8578
58
59
60 Iteration 5 with beta = 16. We found x and y:
61     0.5988    -0.4605
62     0.2988     0.3626
63    -0.3372    -0.6237
64     0.1041     0.4965
65    -0.6539     0.1447
66
67 with value f(x, y) = -6.3289    and LBeta(x, y, mu) = -6.3288
68 and h(x, y), new mu:    which have norms 2.5921e-05, 4.5595
69    -0.0000    -3.2774
70     0.0000    -3.0515
71    -0.0000    -0.8582
72
73
74 Iteration 6 with beta = 32. We found x and y:
75     0.5988    -0.4605
76     0.2988     0.3626
77    -0.3372    -0.6237
78     0.1041     0.4965
79    -0.6539     0.1447
80
81 with value f(x, y) = -6.3288    and LBeta(x, y, mu) = -6.3288
82 and h(x, y), new mu:    which have norms 3.6134e-07, 4.5595
83     0.0000    -3.2774
84    -0.0000    -3.0515
85    -0.0000    -0.8582
86
87
88 Iteration 7 with beta = 64. We found x and y:
89     0.5988    -0.4605
90     0.2988     0.3626
91    -0.3372    -0.6237
92     0.1041     0.4965
93    -0.6539     0.1447
94
95 with value f(x, y) = -6.3288    and LBeta(x, y, mu) = -6.3288
96 and h(x, y), new mu:    which have norms 8.9348e-08, 4.5595
97    -0.0000    -3.2774
98     0.0000    -3.0515
99    -0.0000    -0.8582
100
101
102 Iteration 8 with beta = 128. We found x and y:
103     0.5988    -0.4605
104     0.2988     0.3626
105    -0.3372    -0.6237

```

```

106      0.1041      0.4965
107     -0.6539      0.1447
108
109 with value f(x, y) = -6.3288      and LBeta(x, y, mu) = -6.3288
110 and h(x, y), new mu:      which have norms 7.7183e-08, 4.5595
111      0.0000     -3.2774
112     -0.0000     -3.0515
113      0.0000     -0.8582
114
115
116 Iteration 9 with beta = 256. We found x and y:
117      0.5988     -0.4605
118      0.2988      0.3626
119     -0.3372     -0.6237
120      0.1041      0.4965
121     -0.6539      0.1447
122
123 with value f(x, y) = -6.3288      and LBeta(x, y, mu) = -6.3288
124 and h(x, y), new mu:      which have norms 3.2811e-08, 4.5595
125     -0.0000     -3.2774
126      0.0000     -3.0515
127      0.0000     -0.8582

```

**The informations and comments we made our program output, according to the final results of ALM.**

```

1  =====
2  Final result's interpretation
3  Our (hopefully feasible) obtained value for the Dual problem (D) is mu_1 + mu_2 = -6.3288
4  But 2 * mu(1) > lambda_min(A): -6.5548 > -7.1291
5  But 2 * mu(2) > lambda_min(B): -6.1029 > -6.3419
6  hence, lambda_min(M_mu) = -1.121.
7  And therefore, M_mu is not semi-positive definite,
8  and mu is not feasible in (D).
9
10 Note that the theoretical maximal value for (D) is:
11 (lambda_min(A) + lambda_min(B))/2 = -6.7355

```