Alternative ways of counting

Katrijn Standaert

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Chapter 1

Roman numerals

1.1 Introduction

There are a lot of things we have the ancient Romans to thank for, from the invention of concrete to the basics of law and a whole lot inbetween, but the way we count isn't one of them. The numbers we use in our daily life came to Europe about a thousand years ago from Arabic speakers in Spain and North Africa and are therefore called *Arabic numerals*. Little known fact: the numbers were originally invented in India, but Arabic scientists collaborated with Indian scientists and adopted the numeral system way before the Europeans became aware of it. Asia, the Middle East and North Africa were the place to be for scientists in those days, during the Middle Ages Europe didn't have the most civilized culture. If you ever learn about Mesopotamia or Persia in your history lessons, that region of the globe is now called the Middle East. Arabs adopted the new numeral system, translated the Indian symbols into the symbols we all know and use, our digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, and introduced them to Europe from the tenth century onward. Europeans dubbed them "Arabic numerals" and the name stuck.

Europeans were a bit reluctant to say goodbye to the Roman numerals they grew up with and loved. It was a bit hard to acknowledge that the culture you keep going on crusades against actually has a better way of working with numbers than your own system, so the switch from Roman numerals to Arabic numerals was a gradual one. It is only from the fourteenth century onward that the Arabic numerals took over. The use of Roman numerals persists to this day, e.g., in the film industry. No matter how hard it is to decipher in which year a movie or TV show was made, it is always written in Roman numerals in the credits.

In Ancient Rome, writing numbers and calculating with them was a bit different from what we are used to. The main symbols they used were:

Symbol	*I*	*V*	*X*	*L*	*C*	*D*	*M*
Value	1	5	10	50	100	500	1000

There are two main things about these symbols:

- There is no symbol for 0. This is a bit like discovering a new kind of cookie that is instantly your favourite: before the introduction of Arabic numerals, people didn't know what they were missing.
- The biggest symbol has the value of 1000. Abstract maths weren't invented yet, children weren't forced to do maths with big numbers in schools. In fact, most children never had any formal lessons. Which may sound cool on days where school feels boring or the lesson feels endless, but I bet you still prefer school to long days of doing hard work for barely any money. Education was a privilege for the rich people who were in charge. There are no Roman numerals for numbers like 5000 or 10 000, because there was no need for them. Most people had trouble counting up to 100.

Romans counted by repeating symbols, until they came close to the next symbol. They were fine with repeating a *I, X, C* or *M* symbol up to three times, but adding a fourth one was simply out of the question. I guess they thought it just looks bad and I have to admit, it does get hard to decipher and it would be easier to make mistakes. They solved that problem by using the symbols *V*, *L* and *D*.

The numbers for 1 to 10 look like this.

Arabic	1	2	3	4	5	6	7	8	9	10
Roman	*I*	*II*	*III*	*IV*	*V*	*VI*	*VII*	*VIII*	*IX*	*X*

The number 4 is written is IV, which you should interpret as I less than V. Similarly, 9 is IX, which means you have to subtract I from X. Roman numerals are written from left to right, starting with the biggest symbol and ending with the smallest. If you see a smaller value symbol before a bigger value symbol, then you have to subtract the smaller value from the bigger value. Unfortunately, there are very strict rules regarding what you can place before a larger symbol.

- You can only use the symbol $I(1)$ to subtract from $V(5)$ or $X(10)$.
- You can use the symbol $X(10)$ to subtract from $L(50)$ or $C(100)$.
- You can use the symbol $C(100)$ to subtract from $D(500)$ or $M(1000)$.
- You can never use the symbols for either $V(5)$, $L(50)$ or $D(500)$ to subtract from a bigger number.

- A number can never contain two V's, nor two L's. Two D's in one number are out of the question as well. This is because $V + V = X$, $L + L = C$, $D + D = M$

It takes some getting used to, but after a while it starts to make sense.

Let's look at some more examples before it's your turn.
$29 = 20 + 9 = X + X + IX = XXIX$
$34 = 30 + 4 = XXX + IV = XXXIV$
$43 = 40 + 3 = XL + III = XLIII$
$51 = 50 + 1 = L + I = LI$
$68 = 60 + 8 = LX + VIII = LXVIII$
$76 = 70 + 6 = LXX + VI = LXXVI$
$87 = 80 + 7 = LXXX + VII = LXXXVII$
$99 = 90 + 9 = XC + IX = XCIX$
$115 = 100 + 10 + 5 = C + X + V = CXV$
$438 = 400 + 30 + 8 = CD + XXX + VIII = CDXXXVIII$
$674 = 600 + 70 + 4 = DC + LXX + IV = DCLXXIV$
$777 = 700 + 70 + 7 = DCC + LXX + VII = DCCLXXVII$
$960 = 900 + 60 = CM + LX = CMLX$
$1,594 = 1000 + 500 + 90 + 4 = M + D + XC + IV = MCXCIV$
$2,023 = 2000 + 20 + 3 = MM + XX + III = MMXXIII$
$3,333 = 3000 + 300 + 30 + 3 = MMM + CCC + XXX + III = MMMCCCXXXIII$
$3,721 = 3000 + 700 + 20 + 1 = MMM + DCC + XX + I = MMMDCCXXI$

1.2 Exercises

1. True or false: all seven Roman numerals can repeat in a number, e.g., XX.
2. $LXXXIX$ tiles were made to create a Roman mosaic. IF $LXXII$ of these tiles were stolen how many were left for the mosaic?
3. A Roman man was having a toga made for a party. He took LVI gold coins to the market to buy the material. It cost $XXIV$. How many coins does he have left to pay the seamstress of his toga?
4. Archaeologists expect to find LVI Roman villas in Dewa. They have found $XLII$. How many more are there to find?
5. XI workers are working on painting a fence. If each worker can paint V meters of fence in an hour, how many meters can the workers paint in total over a period of VI hours?

6. Amir is XV years old, his cousin Said is $XXIV$ years old. What is the sum of their ages?

7. There are four Roman numerals that are allowed to repeat. X is one of them. What are the other three?

8. I was born in the year $MCMLXXXIV$. What year is that?

9. Can you write the year you were born in Roman numerals?

10. Rewrite these numbers as Roman numerals.

(a) $3 =$	(f) $81 =$	(k) $987 =$
(b) $8 =$	(g) $140 =$	(l) $1,111 =$
(c) $16 =$	(h) $275 =$	(m) $1,234 =$
(d) $25 =$	(i) $567 =$	(n) $2,345 =$
(e) $36 =$	(j) $789 =$	(o) $3,456 =$

11. Rewrite these numbers in Arabic numerals.

(a) XV=	(f) DLIII=	(k) DLV=
(b) XLIX=	(g) MDCCCX=	(l) MMCDXLVIII=
(c) XXVIII=	(h) LXXVII=	(m) MDCXCIX=
(d) LXXIX=	(i) MMMCDXXI=	(n) MMMCDLXXIV=
(e) MCDXLIV=	(j) MCMLXIV=	(o) MMCMLXXXVII=

12. Jake saw a cornerstone on an old building. It said: *Erected MCMXI.* How old is that building?

13. What is the biggest number you can make with these Roman numerals? What is the value of that number in Arabic numerals?

14. $<$ or $>$

(a) IX XI	(f) CDLXXIX CDXXXIV
(b) LVII XLVIII	(g) D CCCXCVIII
(c) XCLXXVIII CXLV	(h) DCCCLXXXVIII DCCXCLXXX
(d) MMCMXLII MMMCXV	(i) MCMCCCXXXIII MMCXCXI
(e) MMMCDLXXXII MMMDCX	(j) DCCLXVII DCCLXVI

15. Flora made $XIII$ animals with aquabeads. She wants to give II animals to each of her $VIII$ friends. Has she already made enough animals? If not, how many more does she need to make?

16. Write your answer in Roman numerals.

    (a) $15 + 38 =$
    (b) $101 + 48 =$
    (c) $567 + 725 =$
    (d) $73 + 678 =$
    (e) $7,589 - 5,213 =$
    (f) $1699 + 25 =$
    (g) $123 - 25 =$
    (h) $3450 - 508 =$

17. Write your answer in Arabic numerals.

    (a) XXX+MCMXIV=
    (b) MCDXLIV+DXLIX=
    (c) CDXLIV+DLV=
    (d) M-DCLXVI=
    (e) MMDLXX-MCXVIII=
    (f) CXCIX+LXVI=
    (g) DCCLVI+CMXLVIII=
    (h) DCXXVII-XLIX=

18. You may have felt it coming, and here it is: write your answer in Roman numerals. Can you calculate them without writing down the Arabic numerals?

    (a) XXVIII+XXIII=
    (b) XLVI+XCXIV=
    (c) XCIX+LXXXV=
    (d) CCLV+LXXVII=
    (e) CDIX+DCCCLXXXVIII=
    (f) DLV+MDLV=
    (g) MCMXXIX+MCCXXII=
    (h) MCXI+MMCCXXII=
    (i) CMXCIX+VI=
    (j) DCXXVII+MMIII=
    (k) CXC-LVII=
    (l) D-CCXXII=
    (m) M-II=
    (n) D-C=
    (o) C-V=
    (p) MMXXIII-DLV=
    (q) MDCLVI-CMLXXX=
    (r) MCMXL-DCC=
    (s) MMXXIII-XI=
    (t) MDCLXVI-XCIX=

19. Why should we leave it at additions and subtractions? Let's see if you can do some multiplications and divisions as well. You can use Arabic numerals for the calculations, but write your answer in Roman numerals.

(a) CXI x III =

(b) CXXIII x IV =

(c) LXXVII x VII =

(d) DCXLI x VI =

(e) MDLV x II =

(f) MMDCLXXXIV ÷ II =

(g) MMDCVIII ÷ IV =

(h) MMMDCXCVI ÷ III =

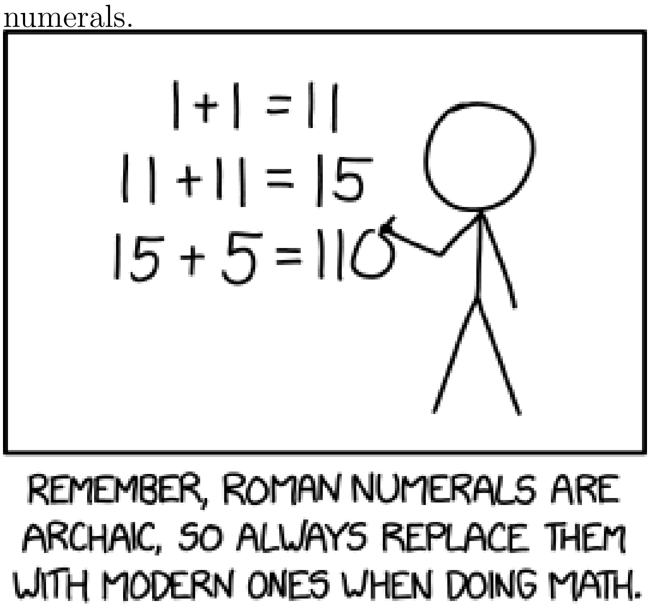
(i) MMCMLXXXVIII ÷ IX =

(j) DLV ÷ V =

20. Complete the Roman numeral magic squares. In a magic square, each column, row and diagonal adds up to the same sum.

VIII		IV	X			IV		
	V		V	VII			V	
VI					IV	VII		VI

21. Randall Monroe is the creator of *xkcd*, a very cool (and nerdy) webcomic. He makes cartoons about all sorts of scientific subjects. He even made one about Roman numerals.



What are the correct sums that are displayed in this comic? Write them down, either in Roman or Arabic numerals.

22. This cartoon got the following subtext.

100he100k out th1s 1nno5at4e str1ng en100o501ng
1'5e been 500e5e50op1ng! 1t's 6rtua100y perfe100t!
...hang on, what's a *virtuacy*?

(a) Can you translate this message?

(b) What was meant with a *virtuacy*?

(c) Can you use this to encode the message *I am loving this*?

(d) Can you write your own message in this way?

23. I have encrypted a message for you. Can you decipher it?

Letter	*A*	*B*	*C*	*D*	*E*	*F*
Roman	III	MCMIX	MDCL	DCVI	CCLXIX	CMXVI
Arabic						

Letter	*G*	*H*	*I*	*J*	*K*	*L*
Roman	DXCV	MCMXVIII	VIII	XLII	XXXVI	MMCCXXII
Arabic						

Letter	*M*	*N*	*O*	*P*	*Q*	*R*
Roman	MDLVII	MMCCCXLII	DXXI	CCCXXVII	MMMDCCXCIX	DXII
Arabic						

Letter	*S*	*T*	*U*	*V*	*W*	*X*
Roman	IX	CDXLVIII	MCDLI	DCCCVI	MMMCCCXL	MMCMLXVI
Arabic						

Letter	*Y*	*Z*
Roman	MDV	MCMXCIX
Arabic		

Now that you managed to complete the encryption key, I am sure that the message won't be a problem for you. Here it goes:

1505/521/1451 3/512/269 595/521/521/606 3/448 448/1918/8/9,

1505/521/1451 3/512/269 606/521/8/2342/595 3/2343

3/3340/269/9/521/1557/269 42/521/1909!

Chapter 2

Binary numbers

You probably learned to count at a young age. You said those magical words *one, two, three* and your parents marveled at how bright their little kid is. A bit later, you managed to count all the way up until ten, without leaving any of the numbers out. By the time you are reading this, I know you can make up an impressive big number. And you know that there is always an even bigger number than whatever your big number is, no matter what it was (even if you used not very specific terms like a million bazillion, there's always a million bazillion and one :P).

Maybe you learned to count in another language at school. Maybe you even learned to count in a different language first. Maybe you have already learned to count in four or five different languages. It's a cool thing to learn to count from one to ten, or a hundred, in another language. But did you know that counting doesn't have to be the way you do it?

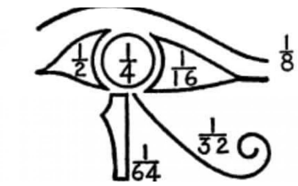
Counting can also be $0, 1, 10, 11, 100...$ and that's just as correct as $0, 1, 2, 3, 4....$ Or at least it is if you are counting in binary. Binary code as we now use it was officially invented by Gottfried Wilhelm Leibniz in the 17th century. The more maths you will learn, the more familiar you will become with the name of this mathematician. He did important work in multiple branches of maths, so his name will keep popping up in your future maths courses.

Leibniz wasn't the very first person who preferred using just 2 digits to calculate. Mathematicians use knowledge of all who came before, explore concepts more deeply, apply ideas in new areas etc. Even Leibniz, the one who invented the framework for binary code as we know and use it now, stands on the shoulders of giants. Maths isn't magic, even though it can definitely be magical.

In some regards, working with binary numbers is a lot easier than working with the decimal numbers we are used to. That is why binary systems were established many times, in many places, long before Leibniz came along. When so many peoples used and formalized a system, independent of each other, it usually means that it is worth the effort. It also helps us to not brush binary off as something that is only useful for computers and modern science, as binary predates computers by thousands of years. So let's take a closer look at some of these older binary systems.

2.1 Ancient Egypt

In ancient Egypt they used a form of binary code with Horus Eye fractions about 4400 years ago, to measure grains and liquids. Horus was the sky god in Egyptian mythology. They believed that the Sun was his right eye and the Moon his left and that they traversed the sky when he, a falcon, flew across it. The sun is so much brighter than the moon, because the left eye was smashed to pieces during one of the many battles between Horus and Seth that was meant to determine who got the throne of Egypt. The Eye of Horus was divided into six parts, each of which got their own hieroglyphic sign.

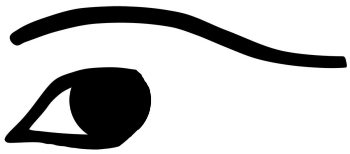


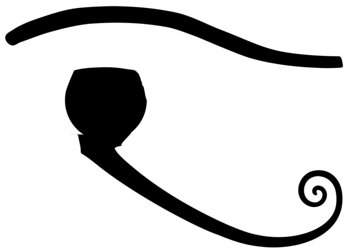
The curious aspect is that these hieroglyphic signs are read as fractions, divided by two from left to right in the eye. This way

- the left half of the eye is 1/2
- the pupil is 1/4
- the eyebrow is 1/8
- the right half of the eye is 1/16
- the lower parts are 1/32 and 1/64

This means that these hieroglyphs could be used for binary calculations, for the first time in recorded history. For those of you who, like me, prefer a happy ending to a myth: the god of science and wisdom, Thoth, had put the pieces back together and gave them back to Horus. Unfortunately, Thoth was unable to put the entire eye back together. Eyes are complicated body parts, even gods find it hard to put them back together as before. The sum of all those fractions is 63/64, leaving 1/64 for the chunk that Thoth was unable to compose.

By combining parts of the Eye of Horus, you can make a visual representation of fractions, of which the nominator is a power of 2 (which means 2, 2×2, $2 \times 2 \times 2$, $2 \times 2 \times 2 \times 2$ etc.).





$\frac{7}{8} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ $\frac{13}{32} = \frac{1}{4} + \frac{1}{8} + \frac{1}{32}$

Exercise Can you draw these fractions?

$\frac{5}{16}$ $\frac{55}{64}$

2.2 China

The I Ching, or Book of Changes, is an ancient Chinese divination text that is among the oldest of the Chinese classics. The text is around 2800-2900 years old. It reveals at its core the binary number system, that arranges meaningful symbols and uses them to communicate messages. They used hexagrams, what in this context means a figure composed of six horizontal lines, where each line is Yang or Yin.

- Yang is an unbroken, solid line
- Yin is a broken line, an open line with a gap in the center

The hexagram lines are traditionally counted from the bottom up, so the lowest line is considered line one while the top line is line six. A hexagram is formed by combining two trigrams, figures containing three horizontal lines each.

The five elements that you can find in these eight trigrams, Water, Wood, Fire, Earth and Metal, are still used in traditional Chinese medicine and in Feng Shui, a practice that tries to bring harmony between individuals and their surrounding environment.

The binary system explained in the I Ching resembles the binary code we use today, because Leibniz himself studied it thoroughly before formalizing his own version of binary code.

Exercise

- In which two numbers do Water and Flame meet?
- In which numbers is Earth joined by Wind?
- Which number is pure Heaven?
- And which one is pure Earth?

2.3 Pingala

Pingala was an Indian scholar who invented a type of binary system around 200 BC, some 2200 years ago. Pingala's binary system had a large number of similarities with the modern binary system. His version of the binary system used short and long syllables to represent 0's and 1's. However, the way you make a Pingala binary number is different from the procedure we follow today.

The procedure of Pingala system is as follows:

- Inspect the number. If it is even, write 1, otherwise write 0. Always add numbers to the right of what you already had.
- If the number was even and bigger than 2, divide by 2. If the number is 2, you are done.
- If the number was odd and bigger than 1, add 1 and then divide by two.
- Repeat these steps.

Using these rules, the numbers from 1 to 16 look as follows

1 = 0	5 = 0 0 1	9 = 0 0 0 1	13 = 0 0 1 1
2 = 1	6 = 1 0 1	10 = 1 0 0 1	14 = 1 0 1 1
3 = 0 1	7 = 0 1 1	11 = 0 1 0 1	15 = 0 1 1 1
4 = 1 1	8 = 1 1 1	12 = 1 1 0 1	16 = 1 1 1 1

Let us look at an example to better understand the Pingala system of Binary Numbers. Let us find the binary equivalent of 122 in Pingala System.

- 122 is an even number, so write 1.
- $122 \div 2 = 61$.
- 61 is an odd number, so write 0. You now have 10.
- $61 + 1 = 62$, $62 \div 2 = 31$
- 31 is an odd number, so write 0. You now have 100.
- $31 + 1 = 32$, $32 \div 2 = 16$.
- 16 is an even number, so write 1. You now have 1001.
- $16 \div 2 = 8$
- 8 is an even number, so write 1, You now have 10011.
- $8 \div 2 = 4$
- 4 is an even number, so write 1. You now have 100111.
- $4 \div 2 = 2$
- 2 is an even number, so write 1. You now have 1001111.
- You got the number 2, so you can stop.

We have found that 122 is equivalent to 1001111 in Pingala binary. We encountered the numbers 122, 61, 31, 16, 8, 4 and 2 along the way. If we stick to the order of these numbers, write 1 for every even number we encounter and 0 for every odd number, we get 1001111. This means that we didn't mix up our 1's and 0's along the way, phew!

We can check that this is the correct representation, but it feels a bit murky.

$$\begin{aligned} 1 \times 1 + 0 \times 2 + 0 \times 4 + 1 \times 8 + 1 \times 16 + 1 \times 32 + 1 \times 64 &= \\ 64 + 32 + 16 + 8 + 1 &= 121 \end{aligned}$$

By adding 1, which we added while dividing 61, to 121 we get 122, which is our desired number.
Don't worry if this verification feels a bit discombobulating, just try to understand the steps that led us to the binary number.

Exercise Can you work out what the binary equivalent is in the Pingala system for the following numbers?

- 25
- 38
- 66
- 2137

2.4 Francis Bacon

In 1605, Francis Bacon invented Bacon's Bilateral Cipher, which has similarities with the binary system. The cipher he created was used to encrypt messages so that only the sender and receiver could read the message. He used the letters 'a' and 'b' instead of 0 and 1 to form a binary code for every letter in a message.

aaaaa A	aaaab B	aaaba C	aaabb D	aabaa E	aabab F
aabba G	aabbb H	abaaa I	abaab K	ababa L	ababb M
abbaa N	abbab O	abbba P	abbbb Q	baaaa R	baaab S
baaba T	baabb V	babaa W	babab X	babba Y	babbb Z

In this cipher, the symbol for I can be used for either the letter I or J. The symbol for V can be used for either U or V. It all makes things just a little more exciting, especially if your message contains words like obviously or djinn.

Exercise Can you decipher the following message?
aaabaabbababbaaaabbabaaaaaaaaabaababaabbababaaaaaabaabaabaaaabbababbaabaaab,

babbaabbabbaabb aabbbaaaaabaabbaabaa

aaabbaabaaaaabaabaaaabbbaaabbbaabaabaaaaaabaaaaabb baabaaabbbaabaa

ababbaabaabaaabbaaabaaaaaaabbaaabaa,

babaaaabaaababaababa aaabbabbababbaaaabaa!

There are more types of binary systems, but after deciphering that message, it's time for us to look at the binary code that Leibniz invented and is now used worldwide.

2.5 Leibniz's binary code

The binary code that was formalized by Gottfried Leibniz in 1689 is the form of binary code that is now used in pretty much every piece of electronics you own. It is used in smartphones and computers, but also in microwaves, digital pictures, tv's, a birthday card with sound, any clock or watch that is not a family heirloom (it doesn't matter if it is digital or analog, all clocks and watches have been using binary for decades). Digital music is stored by using binary, before that CD's also used it. Everything you can find on the internet is the translation of some series of 0's and 1's, whether it's text, pictures, sound of video. Binary code is used so much now, that it is quite impossible to imagine a world without it.

Leibniz's version of binary code is all around us. So much so, that nobody really calls it Leibniz's version of binary code anymore. It's just binary code. It's basically the opposite of what happened to Google. Google used to be just one of many search engines on the internet, but now we use googling as a verb that means "to use a search engine to find information online". When we use this verb we sort of ignore the other things google started doing once their search engine became so popular. Leibniz invented the most used type of binary code, but instead of calling working with binary code "Leibniz-ing", like we did with google, we just call it binary. Leibniz did very important crucial work in multiple branches of maths (you will hear his name again if you ever learn calculus), but he was also a prominent figure in the history of philosophy and he wrote works on ethics, politics, theology, law, philology and history. Scientists have been using his work for centuries. So by the time people started to use his binary code in computers "doing the Leibniz thing" could mean too many different things to know which thing it was. Leibniz's name may not be known by many, but most adults who grew up with computers have heard about his binary code some way or another, which is pretty awesome.

2.5.1 Power of a number

For working with binary numbers, it is useful to know what the power of a number is. So what is that, the power of a number?

Without realising it, you probably have encountered powers before in your maths lessons. This happened every time you had to calculate the square or the cube of a number.
When you are asked to calculate the square of 8, or 8^2, you are asked to work out how much 8×8 is, and your answer would be 64. The number 64 is the square of 8, but you can also say that *8 to the power 2 is 64* or *64 is the 2nd power of 8.*
When you are asked to calculate the cube of 5, or 5^3, you have to work out how much $5 \times 5 \times 5$ is and your answer would 125. The number 125 is the cube of 5, but you can also call it *5 to the power 3 is 125* or *the 3rd power of 5 is 125.*

Definition 1 *The* ***power*** *of a number says how many times to use the number in a multiplication. It is written as a small number to the right and above the base number.*

When you use power 2, you can also call it the square of a number. When you use power

3, you can call it the cube of a number. But there are no special names for higher powers, so after power 3, we just call them power 4, power 5, power 6 and so on.

An other word for the little number that we write, is exponent or index. This word isn't important to you yet, but you will come across it some day in a future maths course. I hope that you'll find working with powers/exponents/indices as much fun as I did when I was in secondary.

There are two special numbers when it comes to powers: 0 and 1.

- The number 0
  - No matter how many times you multiply 0 with 0, the result is always 0. This means that $0^2 = 0^3 = 0^{16} = 0^{1,256} = 0$.
  - For any number that is not 0, the power 0 of that number is 1. This means that $1^0 = 2^0 = 3^0 = 21^0 = 2,012^0 = 1$
  - Even mathematicians don't agree on what the value of 0^0 should be, so in this course we'll pretend that that doesn't exist.
- The number 1
  - No matter how many times you multiply 1 with 1, the result is always 1. This means that $1^2 = 1^3 = 1^{25} = 1^{3,721} = 1$.
  - For any number, the power 1 of that number is the number itself. This means that $1^1 = 1$, $2^1 = 2$, $3^1 = 3$, $3,981^1 = 3,981$ and so on. It means that you can add the power 1 to whichever number you want and it doesn't change a thing. Though I doubt that your teacher will be pleased if you start answering your homework questions with *Juno had saved* 8^1 *euros and* 2^1 *cents*, it is technically correct.

Exercise Calculate these powers.

1. $6^3 =$
2. $4^3 =$
3. $5^4 =$
4. $11^3 =$
5. $3^5 =$
6. $2^4 =$
7. $2^6 =$
8. $10^2 =$
9. $10^3 =$
10. $10^4 =$
11. $25^1 =$
12. $123,456,789^0 =$
13. $39^2 =$
14. $987,654,321^1 =$
15. $93^0 =$

You may have noticed that the only powers that are very straightforward to calculate are powers of 10: you start by writing down 1, and then add as many zeros as the power. So

$10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1,000$ and so on.
This works both ways: to know what power of 10 a number is, you count the number of zeros there are at the end and that number is the power you were searching for. So $1,000,000 = 10^6$, $100,000,000,000,000 = 10^{14}$ etc.

2.5.2 Decimal numbers

Before we delve into binary numbers, we'll take a closer look at the numbers we are most used to. Our normal numbers called *decimal numbers*. The name comes from the Latin word for ten, decem, because we form numbers by using ten digits, from 0 to 9. As humans this feels natural to us, as we have ten fingers to count on (and ten toes, but they are usually hidden away in socks, so they're not all that useful for counting).

When you learned how to count, you might have learned that the right-most digit is the "ones' place", the next is the "tens' place", the next is the "hundreds' place", etc. Another way to say that is that the digit in the right-most position is multiplied by 1, the digit one place to its left is multiplied by 10, and the digit two places to its left is multiplied by 100.

Let's visualize the number 234:

2	3	4
$hundreds'place$	$tens'place$	$ones'place$
100	10	1
10^2	10^1	10^0

When we multiply each digit by its place, we can see that

$$234 = (2 \times 100) + (3 \times 10) + (4 \times 1).$$

When we use the power notation, we get

$$234 = (2 \times 10^2) + (3 \times 10^1) + (4 \times 10^0).$$

We can generalize this for every decimal number we encounter.
For example,

$$\begin{aligned} 321,234,567,890 = \; & (3 \times 10^{11}) + (2 \times 10^{10}) + (1 \times 10^9) + (2 \times 10^8) + (3 \times 10^7) \\ & +(4 \times 10^6) + (5 \times 10^5) + (6 \times 10^4) + (7 \times 10^3) + (8 \times 10^2) \\ & +(9 \times 10^1) + (0 \times 10^0). \end{aligned}$$

Understanding that we implicitly use powers of 10 to write decimal numbers will help us to understand how binary numbers works.

Exercise Complete the table.

number	10^{11}	10^{10}	10^9	10^8	10^7	10^6	10^5	10^4	10^3	10^2	10^1	10^0
$321,234,567,890$	3	2	1	2	3	4	5	6	7	8	9	0
$25,268$												
$20,102,012$												
$198,438$												

2.5.3 Converting decimal numbers to binary numbers

Binary numbers are formed in the same way as decimal numbers. The only differences are that we only use 2 digits, 0 and 1, instead of 10 digits. And instead of multiplying the digit by a power of 10, we multiply it by a power of 2.

Easy peasy, right? Well, to be honest, it probably doesn't feel easy peasy just yet. But I promise that it will all make sense soon.

As mathematicians usually love complicated stuff, but not completely unnecessary complicated stuff, the binary numbers for 0 and 1 are very logical. The binary number for 0 is exactly the same as the decimal number 0, and binary number for 1 is the same as the decimal number 1. It would be completely bonkers if we said that the binary number 0 has a decimal value of 27 and the binary number 1 has a decimal value of 61, or anything other than 0 and 1. I have to admit that it does sound fun, but it would be way too confusing to be useful, so Leibniz decided against it when he made his formal system for binary numbers.

We can only use the digits 0 and 1, so there is no way that the binary number for 2 is the same as the decimal number. But as was mentioned before, the logic behind binary numbers is the same as the logic behind decimal numbers. When we run out of digits in the decimal system, we add an extra digit and start over. Or, easier said, after 9 comes 10, after 99 comes 100 and so on. If we do this when we have only the digits 0 and 1 available to us, we get

- $0 = 0$
- $1 = 1$
- $2 = 10$
- $3 = 11$
- $4 = 100$
- $5 = 101$
- $6 = 110$
- $7 = 111$
- $8 = 1000$
- $9 = 1001$
- $10 = 1010$
- $11 = 1011$

Exercise Can you work out what the next binary numbers will be?

- $12 = 1100$
- $13 =$
- $14 =$
- $15 =$
- $16 =$
- $17 =$
- $18 =$
- $19 =$
- $20 =$
- $21 =$
- $22 =$
- $23 =$
- $24 =$
- $25 =$
- $26 =$
- $27 =$
- $28 =$
- $29 =$
- $30 =$
- $31 =$

Remember, we can write the decimal number 234 as

$$\begin{aligned} 234 &= (2 \times 100) + (3 \times 10) + (4 \times 1) \\ &= (2 \times 10^2) + (3 \times 10^1) + (4 \times 10^0). \end{aligned}$$

Another way to write this is: $\begin{array}{ccc} 2 & 3 & 4 \\ 100 & 10 & 1 \\ 10^2 & 10^1 & 10^0 \end{array}$

We can write the binary number 1011 in the same way, as

$$1011 = (1 \times 2^3) + (0 \times 2^2) + (1 \times 2^1) + (1 \times 2^0),$$

or $\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 2^3 & 2^2 & 2^1 & 2^0 \end{array}$

Writing a binary number in this way makes it easy to check if it is correct. To put your mind at ease, I promise that this check makes a lot more sense than it did with the Pingala binary numbers.

$$\begin{aligned} 1011 &= (1 \times 2^3) + (0 \times 2^2) + (1 \times 2^1) + (1 \times 2^0) \\ &= (1 \times 8) + (0 \times 4) + (1 \times 2) + (1 \times 1) \\ &= 8 + 2 + 1 \\ &= 11 \end{aligned}$$

There's no hocus pocus involved with adding missing 1's or anything. This way of checking works with every binary number that you come across from now on.

It would be a tedious job if you had to make a list of all numbers 0, 1, 10, 11, 100... and keep going like this to convert a random number to it's binary representation, so we won't do that. There are two ways to convert a decimal number to a binary one. The first way is by following a procedure that has some resemblance to how Pingala binary numbers were formed.

The procedure to make convert a decimal number to binary is as follows

1. If the number is 0 or 1, write 0 or 1 and stop.
2. if the number is bigger than 1, there are two options
   - If your number is even, write 0 and divide it by 2.
   - If your number is odd, write 1. Take 1 away from your number to make it even and divide it by 2.

   Always write new digits to the left of what you already had.
3. Repeat as long as is necessary.

Let us use this procedure to find the binary number for 122.

- 122 is an even number, so we write 0 and divide by 2. Our new number is 61 and we have written 0.
- 61 is an odd number, so we write 1 to the left of what we already had, take 1 away so we have 60 and divide that by 2. Our new number is 30 and we have written 10.
- 30 is an even number, so we write 0 and divide by 2. Our new number is 15 and we have written 010.
- 15 is an odd number, so we write 1 to the left of what we already had, take 1 away so we have 14 and divide that by 2. Our new number is 7 and we have written 1010.
- 7 is an odd number, so we write 1 to the left of what we already had, take 1 away so we have 6 and divide that by 2. Our new number is 3 and we have written 11010.
- 3 is an odd number, so we write 1 to the left of what we already had, take 1 away so we have 2 and divide that by 2. Our new number is 1 and we have written 111010.
- 1 is a special case, where we write 1 and stop. We have now written 1111010

The binary number we have found is 1111010. We can check this, by calculating

$$\begin{aligned}(1\times 2^6)+(1\times 2^5)+(1\times 2^4)+(1\times 2^3)+(0\times 2^2)+(1\times 2^1)+(0\times 2^0) &= \\ (1\times 64)+(1\times 32)+(1\times 16)+(1\times 8)+(0\times 4)+(1\times 2)+(0\times 1) &= \\ 64+32+16+8+2 &= 122\end{aligned}$$

Exercise Can you convert these numbers to binary by using this first method?

- 38
- 66
- 125
- 525
- 2137

For the second way of converting numbers to binary, you need to know the values of powers of 2. So let's find those first.

Exercise Powers of 2

- $2^0 = 1$
- $2^1 =$
- $2^2 =$
- $2^3 =$
- $2^4 =$
- $2^5 = 32$
- $2^6 =$
- $2^7 =$
- $2^8 =$
- $2^9 =$
- $2^{10} = 1024$
- $2^{11} =$
- $2^{12} =$
- $2^{13} =$
- $2^{14} =$
- $2^{15} = 32768$
- $2^{16} =$
- $2^{17} =$

Make sure to check your answers to this exercise before proceeding, as we will be using them a lot from now on.

The second way of converting a decimal number to binary is as follows:

1. Start by making a chart, in which you place powers of 2 up until the greatest power that is smaller than or equal to the number you want to convert. Orden these powers from biggest to smallest. 2^0 is always the last one on the right.
2. Subtract the biggest power of two can fit in your number, and mark it with a 1.
3. Continue until the result of your subtraction is 0.
4. Write 0's in the empty spaces.
5. You now have your binary number.

Let us use this procedure to find the binary number for 122.

- The biggest power of two that is smaller than 122 is $64 = 2^6$, so the table we need to make is as follows.

64	32	16	8	4	2	1
2^6	2^5	2^4	2^3	2^2	2^1	2^0

- 64 fits in 122, so we subtract that. Our new number is $122 - 64 = 58$ and we write a 1 underneath 64.

64	32	16	8	4	2	1
2^6	2^5	2^4	2^3	2^2	2^1	2^0
1						

- 32 fits in 58 so we subtract that. Our new number is $58 - 32 = 26$ and we write 1 underneath 32.

64	32	16	8	4	2	1
2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	1					

- 16 fits in 26 so we subtract that. Our new number is $26 - 16 = 10$ and we write 1 underneath 16.

64	32	16	8	4	2	1
2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	1	1				

- 8 fits in 10 so we subtract that. Our new number is $10 - 8 = 2$ and we write 1 underneath 8.

64	32	16	8	4	2	1
2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	1	1	1			

- 2 fits in 2 so we subtract that. Our new number is $2 - 2 = 0$, so we can stop subtracting and we write 1 underneath 2.

64	32	16	8	4	2	1
2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	1	1	1		1	

- We write 0's in the empty spaces.

64	32	16	8	4	2	1
2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	1	1	1	0	1	0

- We now have the binary number 1111010.

We checked before that 1111010 is the correct binary number for 122, so hurray, this method really does lead to the same result.

Exercise Can you convert these numbers to binary by using this second method? To help you out a bit, you get the tables of powers of two that you need.

- 48 =

32	16	8	4	2	1
2^5	2^4	2^3	2^2	2^1	2^0

- 99 =

64	32	16	8	4	2	1
2^6	2^5	2^4	2^3	2^2	2^1	2^0

- 234 =

128	64	32	16	8	4	2	1
2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

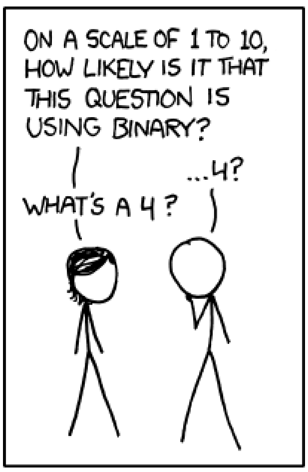
- 579 =

512	256	128	64	32	16	8	4	2	1
2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

- 2173 =

2048	1024	512	256	128	64	32	16	8	4	2	1
2^{11}	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

2.5.4 Converting binary numbers to decimal numbers



xkcd comics cover many maths topics :D

Now that you know how to convert a decimal number to a binary one, we will look at how to convert a binary number to a decimal one. The steps we need to follow are

1. Count the number of digits you have in your binary number. To make sure this works with every possible number, we will call it n. This n can be 1, but it can also be 1067. The steps are always the same.
2. Multiply the left digit with 2^{n-1}, the one next to that with 2^{n-2}, the one next to that with 2^{n-3} and so on. The second to last digit to the right will be multiplied with 2^1, the last one with 2^0. We will have to add everything

later, so it is a good idea to already add a +sign between the multiplications.

3. Convert the powers of 2 you used to their decimal values.

4. Add them all up and you get your decimal number.

We actually already did this, when we looked more closely at the binary number 1011 and when checked the binary number we found for 122.

Let us look at another binary number to make sure we really understand the process. We will convert the binary number 10110010101 to a decimal number.

- The number 10110010101 contains 11 digits. So in this case $n = 11$.
- We multiply the left digit with 2^{n-1}, so in our case this means 1×2^{10}.
- We multiply the next digit with 2^{n-2}, which means 0×2^9.
- We keep going like this and get $1 \times 2^{10} + 0 \times 2^9 + 1 \times 2^8 + 1 \times 2^7 + 0 \times 2^6 + 0 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$.
- It may feel a bit pointless to keep writing all these multiplications in the next step and that is a perfectly valid point. When you multiply by 0 it becomes 0, when you multiply a number by 1 you get the number itself, so you can use this to your advantage and reduce the long expression from before to $2^{10} + 2^8 + 2^7 + 2^4 + 2^2 + 2^0$. When you do maths and you don't want to keep writing stuff that doesn't matter anyway, you can call what you're doing *efficient*. Just make sure that you don't leave out too much, otherwise your result won't be efficient anymore, it will just be wrong.
- Anyway, converting those powers to decimals looks like this in our case $1024 + 256 + 128 + 16 + 4 + 1$.
- The sum of these numbers is 1429.

Exercise Convert these binary numbers to decimal numbers. Because this may be the very first time you have to follow steps with a generalized number n, you get a bit of help with the first exercises. And as a friendly reminder, so you don't have to keep turning your pages to find the values you're looking for:

2048	1024	512	256	128	64	32	16	8	4	2	1
2^{11}	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

1. $101 =$
   The number 101 has 3 digits, so $n = 3$. You have to multiply the left digit with $2^{n-1} = 2^2$, the one next to that with $2^{n-2} = 2^1$ and the right digit with 2^0.
2. $1001 =$
   The number 1001 has 4 digits, so $n = 4$. The left digit is multiplied by $2^{n-1} = 2^3$.
3. $10101 =$
   10101 has 5 digits, so the left digit is multiplied by 2^4.
4. $11011 =$
5. $101000 =$
6. $110011 =$
7. $1111111 =$
8. $1000000 =$
9. $1000011 =$
10. $1100011 =$
11. $1101011 =$
12. $100011000 =$
13. $11101010 =$
14. $101001101 =$
15. $110111100 =$
16. $1111101000 =$
17. $10001010111 =$
18. $10011111100 =$
19. $100010101110 =$
20. $1101111000 =$

2.5.5 Adding binary numbers

Adding binary numbers happens similarly as with decimal numbers. Just as with decimal numbers, you add the digits from right to left and carry over a digit when your result exceeds the highest available digit. As we only use 0 and 1 in binary numbers, there are only 4 possibilities when you add 2 numbers:

	carry over	result
$0+0$	0	0
$0+1=1+0$	0	1
$1+1$	1	0
$1+1+1$	1	1

We will examine one addition in more detail to understand how this works. We will add the binary numbers 100011101 and 10001110.

We start by writing them in a way that makes it easy to see which digits you need to add together.

$$\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
+ & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
\end{array}$$

We start by adding the digits on the right. $1 + 0 = 1$, so we get

$$\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
+ & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
 & & & & & & & & 1
\end{array}$$

We move on to the digits on the left of them. $0 + 1 = 1$, so we get

$$\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
+ & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
 & & & & & & & 1 & 1
\end{array}$$

We move on to the digits on the left of them. $1 + 1 = 10$, so the result is 0 and we carry over 1. We add the 1 that is carried over to the top and get

$$\begin{array}{ccccccccc}
 & & & & & 1 & & & \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
+ & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
 & & & & & & 0 & 1 & 1
\end{array}$$

We move on to the digits on the left of them. $1 + 1 + 1 = 11$, so the result is 1 and we carry over 1. We add the 1 that is carried over to the top and get

$$\begin{array}{ccccccccc}
 & & & & 1 & 1 & & & \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
+ & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
 & & & & & 1 & 0 & 1 & 1
\end{array}$$

We move on to the digits on the left of them. $1 + 1 + 0 = 10$, so the result is 0 and we carry over 1. We add the 1 that is carried over to the top and get

$$\begin{array}{ccccccccc}
 & & & 1 & 1 & 1 & & & \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
+ & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
 & & & & 0 & 1 & 0 & 1 & 1
\end{array}$$

We move on to the digits on the left of them. $1 + 0 + 0 = 1$, so the result is 1 and we don't carry anything over. We get

$$\begin{array}{ccccccccc}
 & & & 1 & 1 & 1 & & & \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
+ & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
 & & & 1 & 0 & 1 & 0 & 1 & 1
\end{array}$$

We move on to the digits on the left of them. $0 + 0 = 0$, so we get

$$\begin{array}{ccccccccc}
 & & & 1 & 1 & 1 & & & \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
+ & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
 & & 0 & 1 & 0 & 1 & 0 & 1 & 1
\end{array}$$

We move on to the digits on the left of them. $0 + 1 = 1$, so we get

$$\begin{array}{ccccccccc}
 & & & 1 & 1 & 1 & & & \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
+ & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1
\end{array}$$

We don't need to add anything to the digit on the left, so we get

$$\begin{array}{ccccccccc}
 & & & 1 & 1 & 1 & & & \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
+ & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1
\end{array}$$

We have now succesfully added the binary numbers and found that $100011101 + 10001110 = 110101011$. In decimal numbers, this sum is $285 + 142 = 427$. Adding in binary numbers may look hard, but as you are only adding 0's and 1's, the calculations involved probably won't give you a headache.

Exercise Adding two binary numbers.

1.
$$\begin{array}{ccccccc}
 & & 1 & 0 & 0 & 1 & 0 & 1 \\
+ & & & 1 & 0 & 1 & 0 & 0 \\
\hline
\end{array}$$

2.
$$\begin{array}{cccccccccc}
 & & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
+ & & & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
\hline
\end{array}$$

3.
$$\begin{array}{cccccccccc}
 & & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
+ & & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
\hline
\end{array}$$

4.
$$\begin{array}{cccccccc}
 & & 1 & 0 & 0 & 1 & 0 & 1 \\
+ & & & 1 & 1 & 1 & 0 & 0 \\
\hline
\end{array}$$

5.
$$\begin{array}{ccccccccc}
 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
+ & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
\hline
\end{array}$$

6.
$$\begin{array}{ccccccccc}
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
+ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array}$$

7.
$$\begin{array}{cccccccccc}
 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
+ & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
\hline
\end{array}$$

8.
$$\begin{array}{cccccccccc}
 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
+ & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
\hline
\end{array}$$

9.

$$\begin{array}{ccccccccc} & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ + & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ \hline \end{array}$$

10.

$$\begin{array}{ccccccccc} & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ + & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline \end{array}$$

Exercise Can you add two binary numbers when they aren't aligned?

1. $110011 + 100010 =$
2. $1111 + 100111 =$
3. $100000 + 101010 =$
4. $100011 + 101001 =$
5. $101111 + 10100 =$
6. $1111101 + 1010100 =$
7. $11010100 + 11000010 =$
8. $1110001 + 1001101 =$
9. $1010111 + 1100110 =$
10. $1110101 + 1100011 =$
11. $11000110 + 110001001 =$
12. $101101011 + 100100001 =$
13. $101010100 + 10000110 =$
14. $10111011 + 110000010 =$
15. $10110111 + 111011 =$
16. $1111101 + 101101101 =$
17. $11101101 + 10110111 =$
18. $1100011 + 1110111 =$
19. $11011100 + 101110111 =$
20. $101100111 + 1001111 =$

When you add three or more numbers, it can get a bit tricky as you may get a digit that has to be carried over 2 spaces. Let's look at one example to see one possible way to deal with this.

$$\begin{array}{cccccc} & 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 1 \\ + & 1 & 1 & 1 & 1 & 1 \\ \hline \end{array}$$

Once again we start with adding the digits on the right. $1 + 1 + 1 = 11$, so we get

$$\begin{array}{cccccc} & & & & 1 & \\ & 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 1 \\ + & 1 & 1 & 1 & 1 & 1 \\ \hline & & & & & 1 \end{array}$$

When we add the next digits, we get $1 + 1 + 1 + 1 = 100$, which is the first time we encounter an intermediate sum that has 3 digits. To see the difference between digits that are carried over 1 space, like before, and digits that are carried over 2 spaces, I write these on a different height.

$$
\begin{array}{cccccc}
 & & 1 & & & \\
 & & & & 1 & \\
 & 1 & 1 & 1 & 1 & 1 \\
 & 1 & 1 & 1 & 1 & 1 \\
+ & 1 & 1 & 1 & 1 & 1 \\
\hline
 & & & & 0 & 1
\end{array}
$$

It is not necessary to write the carried over digits down like this, you'll be grand as long as it is clear to you which digits belong to which column. I have chosen this notation to make it easier to spot mistakes when you check your answers. Digits that are carried over 1 space are right above your sum, digits that are carried over 2 spaces are on the row above, and if you ever encounter sums that have digits carried over 3 spaces, they will be written down on the row above that one.

The next digits we add are, once again, $1 + 1 + 1 = 11$, so we get

$$
\begin{array}{cccccc}
 & & 1 & & & \\
 & & 1 & & 1 & \\
 & 1 & 1 & 1 & 1 & 1 \\
 & 1 & 1 & 1 & 1 & 1 \\
+ & 1 & 1 & 1 & 1 & 1 \\
\hline
 & & & 1 & 0 & 1
\end{array}
$$

When using this notation, you can immediately see that you now have a 1 that was carried over 1 space, as well as a 1 that was carried over 2 spaces. Adding our next digits gives us $1 + 1 + 1 + 1 + 1 = 101$, so we get

$$
\begin{array}{ccccccc}
 & 1 & & 1 & & & \\
 & & & 1 & & 1 & \\
 & & 1 & 1 & 1 & 1 & 1 \\
 & & 1 & 1 & 1 & 1 & 1 \\
+ & & 1 & 1 & 1 & 1 & 1 \\
\hline
 & & & 1 & 1 & 0 & 1
\end{array}
$$

The sum of the next digits is once again $1 + 1 + 1 = 11$, so we get

$$
\begin{array}{ccccccc}
 & 1 & & 1 & & & \\
 & 1 & & 1 & & 1 & \\
 & & 1 & 1 & 1 & 1 & 1 \\
 & & 1 & 1 & 1 & 1 & 1 \\
+ & & 1 & 1 & 1 & 1 & 1 \\
\hline
 & & 1 & 1 & 1 & 0 & 1
\end{array}
$$

The next sum is made purely from carried over digits. We have $1 + 1 = 10$, or

$$
\begin{array}{ccccccc}
 & 1 & & 1 & & & \\
1 & 1 & & 1 & & 1 & \\
 & & 1 & 1 & 1 & 1 & 1 \\
 & & 1 & 1 & 1 & 1 & 1 \\
+ & & 1 & 1 & 1 & 1 & 1 \\
\hline
 & 0 & 1 & 1 & 1 & 0 & 1
\end{array}
$$

Our end result is

$$
\begin{array}{ccccccc}
 & 1 & & 1 & & & \\
1 & 1 & & 1 & & 1 & \\
 & & 1 & 1 & 1 & 1 & 1 \\
 & & 1 & 1 & 1 & 1 & 1 \\
+ & & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 0 & 1 & 1 & 1 & 0 & 1
\end{array}
$$

It's your turn to turn this theory into practice.

Exercise Can you add three numbers? Be very careful with the numbers you carry over when you have to add $1+1+1+1$ or $1+1+1+1+1$! When you encounter these, you have to carry over a digit, not to the next digit on the left, but to the one on the left of that. This is the first time you ever encountered that, and it might help you to write the digits that are carried over 2 spaces on a different height than the digits that are carried over 1 space. When you check your solutions, the top line will contain the digits that are carried over 2 spaces, the line below that will contain the digits that are carried over 1 space. Just don't ruin the fun by simply copying the answers instead of doing the work, because you'll need to be able to work with carried over digits later on and the more you practice the easier it gets. Good luck!

1.
$$
\begin{array}{ccccccc}
 & 1 & 1 & 0 & 0 & 0 & 1 \\
 & 1 & 0 & 0 & 1 & 0 & 0 \\
+ & & 1 & 1 & 0 & 0 & 1 \\
\hline
\end{array}
$$

2.
$$
\begin{array}{ccccccccc}
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
+ & & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
\hline
\end{array}
$$

3.
$$
\begin{array}{ccccccc}
 & 1 & 1 & 1 & 0 & 0 & 0 \\
 & 1 & 0 & 1 & 1 & 0 & 1 \\
+ & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
\end{array}
$$

4.
$$
\begin{array}{cccccccccc}
 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
+ & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
\hline
\end{array}
$$

5.
$$
\begin{array}{ccccccccc}
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
+ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array}
$$

6.
$$
\begin{array}{cccccccccc}
 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
 & & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
+ & & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
\end{array}
$$

7.

$$\begin{array}{r} 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 1 \\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 0 \\ +\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1 \\ \hline \end{array}$$

8.

$$\begin{array}{r} 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1 \\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0 \\ +\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 1 \\ \hline \end{array}$$

9.

$$\begin{array}{r} 1\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 0 \\ +\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 0 \\ \hline \end{array}$$

10.

$$\begin{array}{r} 1\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 1 \\ +\ 1\ 0\ 0\ 1\ 1\ 0 \\ \hline \end{array}$$

Exercise Can you add even more numbers in one go? Keep paying attention to the numbers you carry over.

1.

$$\begin{array}{r} 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0 \\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0 \\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \\ +\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1 \\ \hline \end{array}$$

2.

$$\begin{array}{r} 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1 \\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0 \\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 1 \\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 1 \\ +\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0 \\ \hline \end{array}$$

3.

$$\begin{array}{r} 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1 \\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 0 \\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \\ +\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1 \\ \hline \end{array}$$

4.

$$\begin{array}{r} 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0 \\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 1 \\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 1 \\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \\ +\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \\ \hline \end{array}$$

5.

$$\begin{array}{r} 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1 \\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1 \\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1 \\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 1 \\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0 \\ +\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 1 \\ \hline \end{array}$$

6.

$$\begin{array}{r} 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1 \\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1 \\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0 \\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 1 \\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0 \\ +\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1 \\ \hline \end{array}$$

2.5.6 Subtracting binary numbers

Just like addition, the subtraction of binary numbers happens similarly to what happens with decimal numbers. Just like with decimal numbers, the tricky part is when you have to borrow from the adjecent digit. We will examine one subtraction in more detail to understand how it works. We will subtract 111010010 from 1100101011.

We start by writing them in a way that makes it easy to see which digit you need to be subtract from which.

$$\begin{array}{cccccccccc} 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ - & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline \end{array}$$

We start by subtracting the digits on the right. $1 - 0 = 1$, so we get

$$\begin{array}{cccccccccc} 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ - & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline & & & & & & & & & 1 \end{array}$$

We move on to the digits on the left of them. $1 - 1 = 0$, so we get

$$\begin{array}{cccccccccc} 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ - & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline & & & & & & & & 0 & 1 \end{array}$$

We move on to the digits on the left of them. $0 - 0 = 0$, so we get

$$\begin{array}{cccccccccc} 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ - & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline & & & & & & & 0 & 0 & 1 \end{array}$$

We move on to the digits on the left of them. $1 - 0 = 1$, so we get

$$\begin{array}{cccccccccc} 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ - & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline & & & & & & 1 & 0 & 0 & 1 \end{array}$$

We move on to the digits on the left of them. $0 - 1$ is impossible, so we have to borrow from the digit on the left. That way, the digit on the left gets reduced by 1 and the original 1 now becomes 10. We now have

$$\begin{array}{cccccccccc} & & & & 0 & 10 & & & & \\ 1 & 1 & 0 & 0 & \cancel{1} & \cancel{0} & 1 & 0 & 1 & 1 \\ - & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline & & & & & & 1 & 0 & 0 & 1 \end{array}$$

We are working in binary, so $10 - 1 = 1$, so we get

$$\begin{array}{cccccccccc} & & & & 0 & 10 & & & & \\ 1 & 1 & 0 & 0 & \cancel{1} & \cancel{0} & 1 & 0 & 1 & 1 \\ - & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline & & & & & 1 & 1 & 0 & 0 & 1 \end{array}$$

We move on to the digits to the left. $0 - 0 = 0$, so we get

$$\begin{array}{cccccccccc}
 & & & & 0 & 10 & & & & \\
1 & 1 & 0 & 0 & \cancel{1} & \cancel{0} & 1 & 0 & 1 & 1 \\
- & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
\hline
 & & & & 0 & 1 & 1 & 0 & 0 & 1
\end{array}$$

We move on to the digits to the left. We get $0 - 1$ again, so we have to borrow from the digit on the left. The digit directly on the left is 0 as well, so we have to borrow from even further to the left. We get

$$\begin{array}{cccccccccc}
 & 0 & 1 & 10 & 0 & 10 & & & & \\
1 & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} & 1 & 0 & 1 & 1 \\
- & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
\hline
 & & & & 0 & 1 & 1 & 0 & 0 & 1
\end{array}$$

When we do the actual subtraction, we use $10 - 1 = 1$, so we get

$$\begin{array}{cccccccccc}
 & 0 & 1 & 10 & 0 & 10 & & & & \\
1 & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} & 1 & 0 & 1 & 1 \\
- & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
\hline
 & & & 1 & 0 & 1 & 1 & 0 & 0 & 1
\end{array}$$

We move on to the digits on the left. $1 - 1 = 0$, so we get

$$\begin{array}{cccccccccc}
 & 0 & 1 & 10 & 0 & 10 & & & & \\
1 & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} & 1 & 0 & 1 & 1 \\
- & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
\hline
 & & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1
\end{array}$$

We move to the digits on the left. We encounter $0 - 1$ again, so we have to borrow from the left. We get

$$\begin{array}{cccccccccc}
 & 10 & \cancel{0} & 1 & 10 & 0 & 10 & & & \\
\cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} & 1 & 0 & 1 & 1 \\
- & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
\hline
 & & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1
\end{array}$$

We do the actual subtraction $10 - 1 = 1$ and get

$$\begin{array}{cccccccccc}
 & 10 & \cancel{0} & 1 & 10 & 0 & 10 & & & \\
\cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} & 1 & 0 & 1 & 1 \\
- & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
\hline
 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1
\end{array}$$

There are no more digits to the left, so we found that $1100101011 - 111010010 = 101011001$. Take another good look at how borrowing worked, take a deep breath in and out and be proud of yourself, because now you know how to subtract binary numbers.

Exercise Subtracting binary numbers.

1.
$$\begin{array}{r} 1\;0\;0\;1\;0\;1 \\ -\;1\;0\;1\;0\;0 \\ \hline \end{array}$$

2.
$$\begin{array}{r} 1\;0\;1\;0\;1\;1 \\ -\;1\;0\;0\;1\;1 \\ \hline \end{array}$$

3.
$$\begin{array}{r} 1\;0\;1\;1\;0\;0 \\ -\;1\;1\;0\;1\;1 \\ \hline \end{array}$$

4.
$$\begin{array}{r} 1\;0\;0\;1\;0\;1 \\ -\;1\;1\;1\;0\;0 \\ \hline \end{array}$$

5.
$$\begin{array}{r} 1\;1\;0\;1\;0\;1\;0\;1 \\ -\;1\;0\;1\;0\;1\;0\;0\;0 \\ \hline \end{array}$$

6.
$$\begin{array}{r} 1\;0\;1\;0\;1\;0\;1\;0 \\ -\;1\;0\;1\;0\;1\;0\;1 \\ \hline \end{array}$$

7.
$$\begin{array}{r} 1\;0\;1\;0\;1\;0\;1\;0\;0 \\ -\;1\;0\;1\;1\;0\;0\;0\;0 \\ \hline \end{array}$$

8.
$$\begin{array}{r} 1\;1\;0\;0\;0\;1\;0\;1 \\ -\;1\;0\;0\;0\;1\;1\;0\;1 \\ \hline \end{array}$$

9.
$$\begin{array}{r} 1\;1\;1\;0\;0\;0\;1\;1 \\ -\;1\;0\;0\;1\;1\;1\;0\;1 \\ \hline \end{array}$$

10.
$$\begin{array}{r} 1\;1\;0\;1\;1\;0\;0\;1\;1 \\ -\;1\;0\;1\;1\;0\;1\;0\;1\;1 \\ \hline \end{array}$$

Exercise Subtracting binary numbers, where the borrowing part of the calculations gets a bit trickier.

1.
$$\begin{array}{r} 1\;0\;0\;1\;0\;0 \\ -\;1\;0\;1\;1\;1 \\ \hline \end{array}$$

2.
$$\begin{array}{r} 1\;1\;1\;0\;0\;1 \\ -\;1\;0\;1\;1\;1\;0 \\ \hline \end{array}$$

3.
$$\begin{array}{r} 1\;1\;1\;1\;0\;1\;0 \\ -\;1\;1\;0\;1\;1\;1\;1 \\ \hline \end{array}$$

4.

$$\begin{array}{r} 1\;1\;0\;0\;1\;0\;0\;0 \\ -\;1\;0\;1\;1\;1\;1\;0 \\ \hline \end{array}$$

5.
$$\begin{array}{r} 1\;0\;0\;1\;0\;1\;1\;0\;0 \\ -\;1\;0\;1\;1\;0\;0\;0\;1 \\ \hline \end{array}$$

6.
$$\begin{array}{r} 1\;1\;0\;1\;1\;0\;1\;1\;0 \\ -\;1\;1\;0\;0\;1\;1\;0\;0\;1 \\ \hline \end{array}$$

7.

$$\begin{array}{r} 1\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 0 \\ -\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1 \\ \hline \end{array}$$

8.

$$\begin{array}{r} 1\ 1\ 1\ 0\ 0\ 0\ 1\ 0\ 1 \\ -\ \ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 1 \\ \hline \end{array}$$

9.

$$\begin{array}{r} 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0 \\ -\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 0 \\ \hline \end{array}$$

10.

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0 \\ -\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 1 \\ \hline \end{array}$$

Exercise Can you subtract two binary numbers when they aren't aligned?

1. $10001 - 10 =$
2. $11111111 - 1001101 =$
3. $10000 - 1011 =$
4. $110011 - 100011 =$
5. $111001 - 101 =$
6. $1001000 - 111 =$
7. $101010 - 11001 =$
8. $11001 - 1111 =$
9. $100110 - 1010 =$
10. $100111 - 1101 =$

2.5.7 Multiplication of binary numbers

You may have suspected it already and indeed, multiplication works the same for binary numbers as it does for decimal ones. The down side: multiplying by 0 turns the whole number into 0 and multiplying by 1 simply gives you back your original number, so all useful binary multiplications are long multiplications. The up side: the multiplications involved are very easy:

- $0 \times 0 = 0$
- $0 \times 1 = 1 \times 0 = 0$
- $1 \times 1 = 1$

And that's it. There are no difficult to remember times tables, like $6 \times 7 = 42$ or whichever multiplication it is that still makes you hesitate just that tiny bit longer before you can give the answer. Everyone has at least one multiplication that their mind refuses to remember as easily as the other ones. If you're lucky, you also have at least one multiplication that is really easy for you to remember, like $3 \times 7 = 21$. But all of those are irrelevant when you multiply binary numbers. Aren't machines that use binary, like calculators and computers, lucky that they never had to learn their times tables? Hmmm, they can't think for themselves and their only way of rebelling against human commands is freezing or shutting down, so I guess I still prefer learning those times tables and being human.

Let's first look at some cool things that happen when we multiply binary numbers.

1. To double a binary number, you multiply it by 2, or the binary number 10, so you add an extra 0 at the right. So multiplying by 2 in binary is like multiplying a decimal number with ten.
2. To quadruple a binary number, you have to multiply it by $4 = 2^2$, or the binary number 100, so you add two zeroes at the right.
3. Multiplying a binary number with $8 = 2^3$, or 1000, gives you 3 added zeroes.
4. The same pattern goes on and on and on.

Conclusion Every time you multiply a binary number with a power of 2, you add as many zeroes as the power of 2 you multiply it with. So if you multiply a binary number with 2^n, you add n zeroes to the binary number.

Exercise Calculate these binary multiplications.

1. $10111 \times 100 =$
2. $1011 \times 10 =$
3. $11011 \times 1000 =$
4. $111 \times 10000 =$
5. $1011 \times 10 =$
6. $11111 \times 1000 =$
7. $10101 \times 100 =$
8. $100111 \times 10 =$
9. $11001 \times 100 =$
10. $1001 \times 1000 =$
11. $10110 \times 100000 =$
12. $100000 \times 100 =$
13. $100101 \times 10 =$
14. $1001001 \times 100 =$
15. $110001 \times 1000 =$
16. $10111 \times 100 =$
17. $11010 \times 10000 =$
18. $10010 \times 10 =$
19. $110101 \times 1000 =$
20. $1101100 \times 100 =$

Let us now refresh our memory of how long multiplication works by looking at an example in more detail. We'll multiply the binary numbers 1111001 and 10110.

We start by writing them in a way that makes it easy to see which digit you need to be multiply with which.

$$\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 0 & 0 & 1 \\
\times & & 1 & 0 & 1 & 1 & 0 \\
\hline
\end{array}$$

We start by multiplying with the most right digit of 10110. This is 0, so we get.

$$\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 0 & 0 & 1 \\
\times & & 1 & 0 & 1 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

We move on to the digit on the left of that. Just like when you work with decimal numbers, you write a 0 at the end. All zeroes that are there by default will be bold so you can see the difference with the zeroes you get as a result of the multiplications. The digit we now multiply with is 1, so we get.

$$\begin{array}{cccccccc}
 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0}
\end{array}$$

We move on to the digit on the left, which is another 1. We now have to write two 0's at the end of our number, so we get

$$\begin{array}{ccccccccc}
 & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0}
\end{array}$$

We move on to the digit on the left, which is 0. We now have to write three 0's at the end of our number, so we get

$$\begin{array}{cccccccccc}
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}$$

We move on to the digit on the left, which is the last digit, another 1. We now have to write four 0's at the end of our number, so we get

$$
\begin{array}{ccccccccccc}
 & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
 & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}
$$

By writing down every single multiplication we encounter, it is easy to see a system in what we do. Normally, you wouldn't write down the result when you multiply by zero, because when you do maths it is nice to be ~~lazy~~ efficient and adding a bunch of zeroes is not necessary when you already understand how things work. Just make sure that you don't forget any of the extra zeroes at the end!

We have now finished the multiplication part of the calculation, but we aren't done yet. We still have to add all of our numbers. Because there are so many numbers you have to add, I strongly advise you to leave some room at the top, so you have some space to write down digits that are carried over. As we will be adding more than two binary numbers, we will be looking at this addition in more detail as well. I know you already know how to do this, as you have done it in exercises before. This more of a friendly reminder of where to place the carried over digits when your sum exeeds 11 so it doesn't get confusing.

We start by adding some space at the top for carried over digits.

$$
\begin{array}{cccccccccccc}
 & & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & & & & & & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
+ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
\end{array}
$$

No digits are carried over in the digits on the right, or when we add any of the three columns to the left of those, so we get

$$
\begin{array}{cccccccccccc}
 & & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & & & & & & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
+ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
 & & & & & & & & 0 & 1 & 1 & 0
\end{array}
$$

We move to the digits on the left of what we already have. Now we add $0+1+0+0+1 = 10$, so we have to carry over 1 and we get

$$
\begin{array}{cccccccccccc}
 & & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & & & & & 1 & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
+ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
 & & & & & & & 0 & 0 & 1 & 1 & 0
\end{array}
$$

We move to the digits on the left of what we already have. Now we add $1+0+1+1+0+0 = 11$, so we have to carry over 1 and we get

$$
\begin{array}{cccccccccccc}
 & & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & & & & 1 & 1 & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
+ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
 & & & & & & 1 & 0 & 0 & 1 & 1 & 0
\end{array}
$$

We move to the digits on the left of what we already have. Again, we add $1+0+1+1+0+0 = 11$, so we have to carry over 1 and we get

$$
\begin{array}{cccccccccccc}
 & & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & & & & & & & & & & \\
 & & & & 1 & 1 & 1 & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
+ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
 & & & & & 1 & 1 & 0 & 0 & 1 & 1 & 0
\end{array}
$$

We move to the digits on the left of what we already have. This time we add $1+1+1+0+1 = 100$, so we have to carry over 1, but a space further to the left than usually and we get

$$
\begin{array}{cccccccccccc}
 & & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & 1 & & & & & & & & & \\
 & & & & 1 & 1 & 1 & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
+ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
 & & & & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0
\end{array}
$$

We move to the digits on the left of what we already have. We add $1 + 0 + 1 = 10$, so we have to carry over 1 and we get

$$
\begin{array}{cccccccccccc}
 & & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & 1 & & & & & & & & & \\
 & & 1 & & 1 & 1 & 1 & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
+ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
 & & & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0
\end{array}
$$

We move to the digits on the left of what we already have. We add $1 + 1 + 0 + 1 = 11$, so we have to carry over 1 and we get

$$\begin{array}{cccccccccccc}
 & & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & 1 & & & & & & & & & \\
 & 1 & 1 & & 1 & 1 & 1 & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
+ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
 & & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0
\end{array}$$

We move to the digits on the left of what we already have. We add $1 + 1 = 10$, so we have to carry over 1 and we get

$$\begin{array}{cccccccccccc}
 & & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & 1 & & & & & & & & & \\
1 & 1 & 1 & & 1 & 1 & 1 & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
+ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0
\end{array}$$

We add this last digit to get the complete result and get

$$\begin{array}{cccccccccccc}
 & & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & 1 & & & & & & & & & \\
1 & 1 & 1 & & 1 & 1 & 1 & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
+ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0
\end{array}$$

If you convert the binary numbers to decimal numbers, you would find that this multiplication is correct and no mistakes were made in the process. Hurrah! So now it's your turn.

Exercise Multiply these binary numbers. You don't have to write anything down when you multiply by zero, just be mindful of the zeroes at the end. And make sure that you leave enough space at the top to carry over digits. Good luck!

1.
$$\begin{array}{rrrrrrr} & 1 & 0 & 0 & 1 & 0 & 1 \\ \times & 1 & 0 & 1 & 0 & 0 \\ \hline \\ \hline \end{array}$$

2.
$$\begin{array}{rrrrrrr} & 1 & 0 & 1 & 0 & 1 & 1 \\ \times & 1 & 0 & 0 & 1 & 1 \\ \hline \\ \hline \end{array}$$

3.
$$\begin{array}{rrrrrr} 1 & 0 & 1 & 1 & 0 & 1 \\ \times & & & 1 & 1 & 1 \\ \hline \\ \hline \end{array}$$

4.
$$\begin{array}{rrrrrrr} 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ \times & & & 1 & 0 & 1 & 0 \\ \hline \\ \hline \end{array}$$

5.
$$\begin{array}{rrrrrr} 1 & 0 & 0 & 1 & 0 & 1 \\ \times & 1 & 1 & 1 & 0 & 0 \\ \hline \\ \hline \end{array}$$

6.
$$\begin{array}{rrrrrr} 1 & 1 & 0 & 1 & 1 & 1 \\ \times & & 1 & 1 & 0 & 1 \\ \hline \\ \hline \end{array}$$

7.
$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \\ \times \qquad\quad 1\ 1\ 0\ 0\ 1 \\ \hline \\ \\ \hline \end{array}$$

8.
$$\begin{array}{r} 1\ 0\ 1\ 1\ 0\ 0 \\ \times\ 1\ 1\ 0\ 1\ 1 \\ \hline \\ \\ \hline \end{array}$$

9.
$$\begin{array}{r} 1\ 1\ 1\ 0\ 1\ 1\ 1 \\ \times \quad 1\ 1\ 1\ 0\ 1\ 1 \\ \hline \\ \\ \hline \end{array}$$

10.
$$\begin{array}{r} 1\ 0\ 1\ 1\ 0\ 1\ 0 \\ \times \qquad 1\ 1\ 1\ 1\ 1 \\ \hline \\ \\ \hline \end{array}$$

11.
$$\begin{array}{r} 1\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 1 \\ \times \qquad\quad 1\ 0\ 0\ 1\ 1 \\ \hline \\ \\ \hline \end{array}$$

12.
$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1 \\ \times \quad 1\ 1\ 1\ 0\ 0\ 1 \\ \hline \\ \\ \hline \end{array}$$

13.
$$\begin{array}{r} 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1 \\ \times \qquad 1\ 1\ 1\ 1\ 1 \\ \hline \\ \\ \hline \end{array}$$

14.
$$\begin{array}{r} 1\ 1\ 0\ 1\ 0\ 1\ 1 \\ \times \quad 1\ 1\ 0\ 1\ 1\ 1 \\ \hline \\ \\ \hline \end{array}$$

Exercise These multiplications are a bit harder, because while adding you might have to carry over more than 1 digit (like when your sum is 111) or the digit you have to carry over is several places away (like when your sum is 1001). But I know you can do it.

1.
$$\begin{array}{r} 1\;0\;1\;1\;1\;1\;0\;1 \\ \times\;1\;1\;1\;1\;0\;1 \\ \hline \end{array}$$

4.
$$\begin{array}{r} 1\;1\;1\;1\;1\;1 \\ \times\;1\;1\;1\;1\;1 \\ \hline \end{array}$$

2.
$$\begin{array}{r} 1\;0\;1\;1\;1\;1\;1\;1 \\ \times\;1\;1\;0\;1\;1\;1\;1 \\ \hline \end{array}$$

5.
$$\begin{array}{r} 1\;1\;1\;0\;1\;1\;1 \\ \times\;1\;1\;1\;1\;1\;1 \\ \hline \end{array}$$

3.
$$\begin{array}{r} 1\;1\;1\;1\;1\;1\;1 \\ \times\;1\;1\;1\;1\;1\;1 \\ \hline \end{array}$$

6.
$$\begin{array}{r} 1\;1\;1\;1\;1\;1\;1 \\ \times\;1\;1\;1\;1\;1\;1\;1 \\ \hline \end{array}$$

2.5.8 Division of binary numbers

Dividing by 0 isn't going to happen any time soon and dividing by 1 isn't very informative... So we're stuck with long divisions when we want to divide binary numbers and gain new information. But before you get discouraged by that prospect, let me remind you that you're only working with 0's and 1's, which makes it a lot easier than when you have to do long divisions of decimal numbers.

When you are asked to divide a binary number by a power of two, you don't even have to do any calculations at all. Binary numbers give us that information very easily, so let us take a look at that first. We will look at the binary numbers from 1 to 32 to get an intuitive feeling of this.

1	1001	10001	11001
10			
11	1011		
100			
101			
110		10110	
111			11111
1000			

Exercise Patterns in binary numbers

1. Complete the table.
2. Underline all even numbers with a blue colour pencil.

3. Underline all multiples of 4 with a red colour pencil.
4. Underline all multiples of 8 with a yellow colour pencil.
5. Underline all multiples of 16 with a green colour pencil.
6. Study the numbers you have underlined.
   (a) What do all multiples of 2 have in common?
   (b) What do all multiples of 4 have in common?
   (c) What do all multiples of 8 have in common?
   (d) What do all multiples of 16 have in common?

By doing the previous exercise, we hope that is has become clear to you that you can determine if a binary number is a multiple of 2, 4, 8 or 16 simply by looking at it.

1. If a binary number ends with 0, it is an even number. If it ends with 1, it is odd. The value of the last digit is crucial to determine whether or not a number is a multiple of 2.
2. A binary number is a multiple of $4 = 2^2$ if the number ends in 00. If it doesn't end in 00, the last two digits give you the remainder you get when you divide the number by 4.
3. If a binary number ends with 000, it is a multiple of $8 = 2^3$. If not, the last three digits give you the remainder you get when you divide the binary number by 8.

Conclusion If you want to determine whether or not a binary number is a multiple of a power of 2, look at as many digits on the right as the power of 2 you are interested in. So if you want to know if it is a multiple of 2^n, you look at the last n digits of the binary number. If they are all zero, then your number is a multiple of 2^n. If not, your last n digits give you the remainder you get when you divide your binary number by 2^n.

Let us look at some more examples, because explaining maths with words sometimes makes it seem more complicated than it actually is. There's a life lesson in there for you: even when it looks hard and sounds even harder, once you start and get the hang of it, you might be surprised at how quickly it became an easy peasy thing to do.

1. $100101 \div 10 = 10010$ r 1
2. $100101 \div 100 = 1001$ r 1
3. $100101 \div 1000 = 100$ r 101
4. $11010 \div 10 = 1101$ r 0
5. $11010 \div 100 = 110$ r 10
6. $11010 \div 1000 = 11$ r 10

7. $111111 \div 10 = 11111$ r 1
8. $111111 \div 100 = 1111$ r 11
9. $111111 \div 1000 = 111$ r 111
10. $111111 \div 10000 = 11$ r 1111
11. $101010100 \div 10 = 10101010$ r 0
12. $101010100 \div 100 = 1010101$ r 0
13. $101010100 \div 1000 = 101010$ r 100
14. $101010100 \div 10000 = 10101$ r 100

Exercise Divide these binary numbers by powers of 2. If there is a remainder, write it down as well. I know you can do it!

1. $110111010 \div 10 =$
2. $110111010 \div 100 =$
3. $110111010 \div 1000 =$
4. $110111010 \div 10000 =$
5. $11001000 \div 10 =$
6. $11001000 \div 100 =$
7. $11001000 \div 1000 =$
8. $11001000 \div 10000 =$
9. $10011001 \div 10 =$
10. $10011001 \div 100 =$
11. $10111011 \div 1000 =$
12. $1101100 \div 100 =$
13. $110110 \div 100 =$
14. $1001100 \div 10 =$
15. $101100 \div 1000 =$
16. $100100 \div 100 =$
17. $10110 \div 10000 =$
18. $11000 \div 100 =$
19. $1101110 \div 1000 =$
20. $11010110 \div 10 =$

Now that you know how to divide a binary number by a power of 2, we will look at what happens when you divide by a random binary number. We will divide 110000101 by 101 as an example for how binary long division works.

We start by writing it down as usual.

$$101 \quad)\overline{110000101}$$

We start by writing 101 underneath it. We subtract it from the number above, and note that our solution starts with 1.

$$\begin{array}{rl} & \underline{\;1\qquad\qquad\quad} \\ 101 &)\ 110000101 \\ & \underline{-101} \\ & \quad\;\;1 \end{array}$$

After the subtraction we get 1, which is smaller than 101. So we add another digit and get 10, which still is smaller. This means that we have to add a 0 to our result. We get

```
      10
101 ) 110000101
     -101
       10
```

We add another digit and get 100, so it stays smaller than 101. We have to add another 0 to our result so we get

```
      100
101 ) 110000101
     -101
       100
```

It is only when we add a fourth digit and get 1000, that we finally have a number that is bigger than 101. So we add 1 to our result, subtract 101 from this number and get.

```
      1001
101 ) 110000101
     -101
       1000
       -101
         11
```

We move on to the next digit. We now have 111, which is bigger than 101. So we add 1 to our result, subtract 101 from this number and get.

```
      10011
101 ) 110000101
     -101
       1000
       -101
         111
        -101
          10
```

We move on to the next digit. We now have 100, which is smaller than 101. So we add 0 to our result and get.

```
      100110
101 ) 110000101
     -101
       1000
       -101
         111
        -101
          100
```

We move on to the last digit. We now have 1001, which is smaller than 101. So we add 1 to our result, subtract 101 from this number and get.

$$
\begin{array}{ll}
 & \underline{\quad 1001101 \quad} \\
101 &)\ 110000101 \\
 & \underline{-101} \\
 & \quad\ \ 1000 \\
 & \quad\ \underline{-101} \\
 & \qquad\ \ 111 \\
 & \qquad \underline{-101} \\
 & \qquad\quad 1001 \\
 & \qquad\quad \underline{-101} \\
 & \qquad\qquad 100
\end{array}
$$

We found that $110000101 \div 101 = 1001101$ with remainder 100.

Exercise It is your turn now. Calculate the result of these divisions, don't forget to write down the remainder. You can use the rest of this page and the next one for your calculations.

1. $110111001 \div 101 =$
2. $1000101011 \div 100 =$
3. $11100111101 \div 110 =$
4. $10111001 \div 111 =$
5. $11111111 \div 100 =$
6. $11111111 \div 101 =$
7. $11111111 \div 110 =$
8. $11111111 \div 111 =$
9. $1011011101 \div 1010 =$
10. $110010010001 \div 1100 =$
11. $10010111011 \div 1101 =$
12. $1011100111 \div 1011 =$
13. $110110110110 \div 1111 =$
14. $100101101 \div 11101 =$
15. $110011010100 \div 11010 =$
16. $1101001101001 \div 11011 =$

2.5.9 Binary fractions

We started our adventure in the wonderful world of binary numbers by looking at how the hieroglyphs of the Eye of Horus were used in Ancient Egypt, as a way to count in binary fractions. By now you have learned more about working with binary numbers than most computer programmers know, but we still haven't encountered anything that connects your current knowledge of binary numbers to the Horus Eye fractions. It would be so frustrating to leave you guessing for a link and stop learning about binary right now, so we won't do that.

Let us go above and beyond, and learn about the binary numbers that come after the decimal point. The point that shows you which part of a number is bigger than 1 and which part of a number is smaller is called the **decimal point** and yes, that is a very confusing name when you aren't working with decimal numbers.

I feel like I keep repeating myself, but here we go again: binary numbers work in the same way as decimal numbers. So we will take a closer look at a decimal number first, and then we will look at the equivalent system for binary numbers.

For this, we will take a closer look at the decimal number 12,345.6789. We can write this as

$$\begin{aligned} 12,345.6789 &= (1\times 10,000)+(2\times 1,000)+(3\times 100)+(4\times 10)+(5\times 1)\\ &\quad +(6\times \frac{1}{10})+(7\times \frac{1}{100})+(8\times \frac{1}{1,000})+(9\times \frac{1}{10,000}). \end{aligned}$$

We can also write this in terms of powers of 10, as follows:

$$\begin{aligned} 12,345.6789 &= (1\times 10^4)+(2\times 10^3)+(3\times 10^2)+(4\times 10^1)+(5\times 10^0)\\ &\quad +(6\times \frac{1}{10^1})+(7\times \frac{1}{10^2})+(8\times \frac{1}{10^3})+(9\times \frac{1}{10^4}). \end{aligned}$$

Once you understand this, you might suspect that the binary number 11010.01011 can also be seen as

$$\begin{aligned} 11010.01011 &= (1\times 2^4)+(1\times 2^3)+(0\times 2^2)+(1\times 2^1)+(0\times 2^0)\\ &\quad +(0\times \frac{1}{2^1})+(1\times \frac{1}{2^2})+(0\times \frac{1}{2^3})+(1\times \frac{1}{2^4})+(1\times \frac{1}{2^5}). \end{aligned}$$

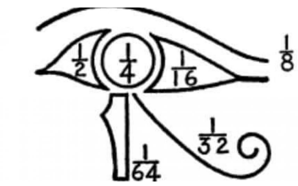
When we calculate the powers of two that we used, we find that

$$\begin{aligned} 11010.01011 &= (1\times 16)+(1\times 8)+(0\times 4)+(1\times 2)+(0\times 1)\\ &\quad +(0\times \frac{1}{2})+(1\times \frac{1}{4})+(0\times \frac{1}{8})+(1\times \frac{1}{16})+(1\times \frac{1}{32}). \end{aligned}$$

This means that the decimal number that is equivalent to the binary number 11010.01011 is

$$\begin{aligned}
11010.01011 &= 16+8+2+\frac{1}{4}+\frac{1}{16}+\frac{1}{32} \\
&= 26\frac{11}{32} \\
&= 26+0.25+0.0625+0.03125 \\
&= 26.34375
\end{aligned}$$

With this understanding of binary fractions, we can convert parts of the Eye of Horus to binary numbers and get the link we were looking for.

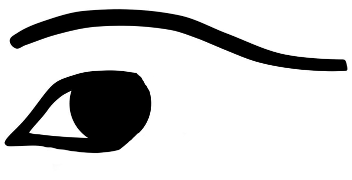


When we convert the parts to binary numbers, we get

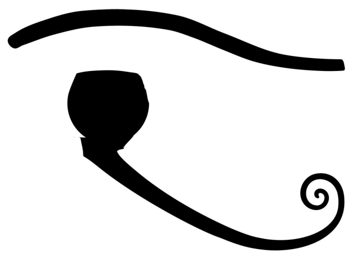
- the left half of the eye is 1/2, or 0.1 in binary numbers
- the pupil is 1/4, or 0.01 in binary.
- the eyebrow is 1/8, or 0.01 in binary.
- the right half of the eye is 1/16, or 0.001 in binary
- the lower parts are 1/32 and 1/64, or 0.0001 in binary.

This means that the hieroglyph that represents the entire Eye of Horus has a binary value of 0.11111. It is less than 1, because even after putting the eye back together, there still was a part that the god Thoth was unable to compose.

We can now convert the pictures that contain only parts of the Eye of Horus to binary numbers.

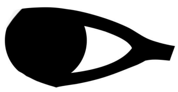


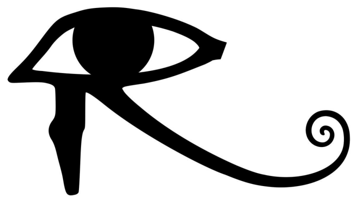
$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8}$
0.111



$\frac{1}{4}+\frac{1}{8}+\frac{1}{32}=\frac{13}{32}$
0.01101

Exercise Write the binary value of these hieroglyphs.





Exercise Can you draw the hieroglyphs that correspond to these binary values?

0.10011

0.1101

2.5.10 Final words

You can now give yourself a pat on the back, followed by a thumbs up, because you deserve it!

It took you years and years to get to know the decimal system and to be able to do operations on the numbers. Counting, adding, subtracting, multiplying and dividing numbers, none of that was knowledge you just woke up with one day. You learned to count and learned those operations one by one. Even if you learned them at an incredible speed, it still took you years to go from $1,\ 2,\ 3...$ to $16,042 \div 13 = 1,234$.

But now, in just a short period of time, you learned to do all those things with binary numbers and that is simply amazing! You should be proud of yourself!

To end this part, I give you two binairos to solve. They don't really have anything to do with binary numbers, but they are logical puzzles so you use the maths part of your brain to solve them, they only use 0's and 1's and I think they are fun, so I hope you enjoy solving them.

The rules:

1. Each box should contain either a zero or a one.

2. It is not allowed to have more than two equal numbers immediately next to or below each other. So every 00 has to be preceded and followed by a 1, and every 11 has to be preceded and followed by a 0. When you have 0?0, the ? can not be 0 because that would give three zeroes in a row.

3. Each row and each column should contain an equal amount of zeroes and ones.

4. Each row is unique and each column is unique. Thus, any row cannot be exactly equal to another row, and any column cannot be exactly equal to another column.

The first binairo can be completed without using the fourth rule, but you probably need it for the second one. If you have some columns or rows with just a few missing numbers and you have no idea what else you can do, check if you can find another column or row that is already complete and that has all the same numbers as the one that is almost completed.

		1		1							
				0		1					1
0										0	1
		1		1							
0		1					1			1	
				0			1				
1					1			0			1
		0			1				0		
		0							0		
							0				
		0		1			0			0	
					0	0				0	

			1	0		1	1			1	1
	0							1			1
	1				1			1	1		
							1			0	1
0		0									
	0	0				1		0			
	1					0	0				
		1								0	0
					1		0				0
0			0				0				
1	1			1	0				1		
		0		1		1	0				

In conclusion:

Here is our list of
top 10 binary numbers:

- 1
- 0

Chapter 3

Hexadecimal numbers

3.1 Introduction

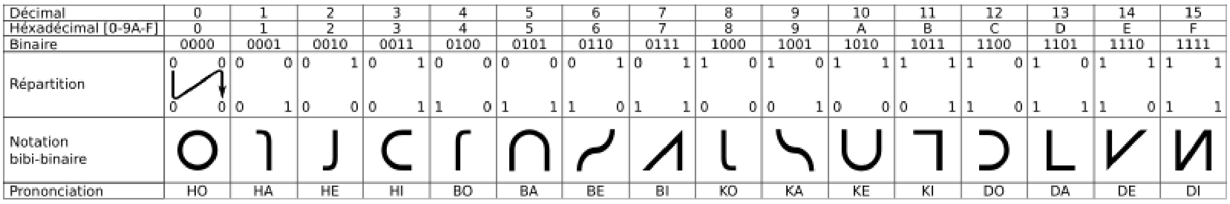
Now that you have learned that it is possible to do maths with just 2 digits instead of our usual 10, it may be less surprising to learn that it is also possible to do maths with more than 10 digits. The most commonly used number system that does this, is the hexadecimal system. The prefix hexa means six, which you may already have figured out from the name of a polygon with six sides, a hexagon. Combine that with the knowledge that decimal numbers use ten digits, and you would be correct in thinking that hexadecimal stands for the number sixteen.

Where binary number systems have been around for literally thousands of years, hexadecimal numbers aren't even a century old. People only started working with them somewhere in the 1950s and the system we use now was introduced by IBM in 1963, so only about 60 years ago.

The fact that you learn about them now is all thanks to technological developments and computers. Once scientists figured out that computers are amazing with binary, that became the starting point for all computer science. Computers became quicker and better at making all kinds of difficult calculations, but big binary numbers are quite challenging to read and interpret for humans. Do you know what the binary value for 111010110111100110100010110001 stands for? Can you estimate how long it would take you to calculate all the powers of two you need to find its decimal value? It would take quite some time to find that the decimal value is $987,654,321$. The decimal value of that very long binary number still is less than 1 billion and it already feels like a big jumble of 0's and 1's... And that's why people invented hexadecimal numbers. Well, we say invented, but it's more a matter of "take the principles we use for decimal and binary numbers, but work with 16 digits instead of 10 or 2".

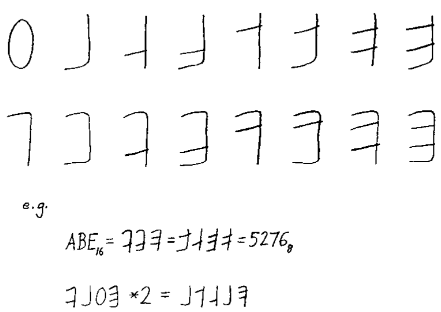
There is a challenge when you want to write numbers with 16 different digits: you run out of digits. Deciding what the first ten digits should be was simple: we still use 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, and they get the same values as what we are used to with decimal numbers.

Deciding what the next 6 digits should be was a lot trickier, because what should you do? Several groups of mathematicians worked on this problem at about the same time. Some agreed to use $\overline{0}$, $\overline{1}$, $\overline{2}$, $\overline{3}$, $\overline{4}$ and $\overline{5}$, but others disagreed. I think they made the right call there, can you imagine how often quickly scribbled down numbers would lead to mistakes? Even when the numbers are printed neatly here, I can see how easy it would be to confuse $\overline{1}$ with 7, or $\overline{5}$ with a 5 where the top line isn't perfectly attached to the rest of the digit.



Décimal	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Héxadécimal [0-9A-F]	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
Binaire	0000	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	1110	1111
Répartition	0 0 / 0 0	0 0 / 0 1	0 1 / 0 0	0 1 / 0 1	0 0 / 1 0	0 0 / 1 1	0 1 / 1 0	0 1 / 1 1	1 0 / 0 0	1 0 / 0 1	1 1 / 0 0	1 1 / 0 1	1 0 / 1 0	1 0 / 1 1	1 1 / 1 0	1 1 / 1 1
Notation bibi-binaire																
Prononciation	HO	HA	HE	HI	BO	BA	BE	BI	KO	KA	KE	KI	DO	DA	DE	DI

A new set of symbols for hexadecimal numbers was called the Bibi=binary system for numeric notation, first described in 1968 by Robert Lapointe. Each Bibi digit is formed from a square arranging the 1-bits in its binary representation. If only a single bit is 1 a vertical line runs through the centre and ends in that bit's corner; otherwise it relies on the order of the positions of the 1-bits. When there are exactly two 1-bits, the line passes round the centre. The forms are rounded when there are less than three 1-bits, and use sharp corners when three or four of the bits are 1. This notation is quite nice, but it did not become very popular and it is not how we write down hexadecimal numbers now.



The proposed new number symbols source: wikipedia

Bruce Alan Martin of Brookhaven National Laboratory suggested to use another completely new set of symbols for hexadecimal numbers, also in 1968. This idea might have stuck if the people involved were young, as a big part of learning to read and write is learning the meanings of specific squiggles, so sixteen more wouldn't be that big of a deal. However, the scientists involved were adults and they weren't all too keen on learning new symbols. Especially because by then, other groups had already found a way of dealing with the problem of the last six digits without introducing new symbols: they used letters.

One group used the letters u to z for the remaining digits, as $u = 10$, $v = 11$, $w = 12$, $x = 13$, $y = 14$ and $z = 15$. This system was used in 1956 by the Bendix G-15 computer.

source: wikipedia

Nothing will happen if you forget this information immediately, I just thought it was a good opportunity to show you what a computer looked like less than 70 years ago. This specific computer is described as one of the first personal computers, meaning that before this one most computers needed an operator: someone who was trained specifically in working with the machines. This machine was 1.52m high and weighed about 440 kg. Everything that thing could do, and a lot more, you can now do on every laptop, pc or smarthphone. Isn't it amazing how quickly technology changes?

Mathematicians liked the idea of using letters for the last six digits, but not everyone agreed on using the letters u to z. Some of the alternatives are listed here. It is not a complete list, but it gives you some idea of how many different systems have been tried out.

- The ORDVAC an ILLIAC I computers, from 1952, used the letters K, S, N, J, F and L for the values 10 to 15.
- The Librascope LGP-10, from 1956, used the letters F, G, J, K, Q and W for the values 10 to 15.
- The Monrobot XI, from 1960, usted the letters S, T, U, V, W and X for the values 10 to 15.
- The Pacific Data Systems 1020, from 1964, used the letters L, C, A, S, M and D for the values 10 to 15.

The team at IBM used the letters A to F, and that is the hexadecimal system that trumped all other systems and is widely used today. It makes sense, because it is easier to start with a and count from there, than it is to count backwards from z. You can check this for yourself, by covering the text above and timing how long it takes for you to answer the next question.

If you know that u has the value 10, what value does w have? And y? v? z?

Write down how long it took you to find all four values and reset your timer.

If you know that a has the value 10, what value does c have? And e? b? f?

Write down how long it took you to find the answers this time. In general, this took just a bit less time than before. This is because we are so used to saying and using the alphabet in order, that the beginning of it comes almost automatically, whereas we have to think about it more consciously when it come to the last letters of the alphabet. With systems like the ORDVAC and ILLIAC I computers, that used six letters that seem randomly picked out of the alphabet, it takes even longer to determine what value a letter represents. No wonder those hexadecimal systems didn't survive the 1960s.

The hexadecimal system that was introduced by IBM in 1963 is the one that is now used worldwide. The only variations that still exist is in using uppercase letters (capital letters) or lowercase letters (small letters). Both are correct, but as uppercase letters are a bit more common, we'll be using those when we write down hexadecimal numbers.

We conclude this introduction with an overview of the hexadecimal digits and their decimal values

decimal value	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
hexadecimal digit	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F

3.2 Converting decimal numbers to hexadecimal and vice versa

The nice thing about learning hexadecimal numbers after learning binary numbers. is that you already know all the mechanisms that are involved. Without realising it, you did all necessary ground work while learning about binary. In what follows, very little of what we encounter will actually be new to you, even though you will once again be working with a completely different numbering system.

Let us start with one of the snippets of actual new information. When we work with binary numbers, it is usually pretty obvious to see whether a number is binary or decimal. But when you work with hexadecimal numbers, this is a lot harder to know. Because both use all ten decimal digits, hexadecimal numbers quite often look like regular decimal numbers, but they have a different value. As long as you stick with one system, be it decimal, hexadecimal or binary, there is no problem because you know what to expect. But when you convert numbers from one system to another, it is useful to know what type of numbers you are dealing with. Mathematicians came up with 2 solutions for this problem.

1. Add the notation *dec* to decimal numbers, and *hex* to hexadecimal numbers, so you get $10_{dec} = A_{hex}$.
2. Add the number of digits your numbering system uses to the number, which is 10 for decimal numbers, 16 for hexadecimal numbers and 2 for binary numbers. This way, you get $10_{10} = A_{16} = 1010_2$.

Both methods of distinguishing between number systems are correct and are used frequently. In this course we will be using the notations *dec* and *hex* to show the difference between decimal and hexadecimal numbers.

Exercise 1 Practice using these notations by converting the first 40 decimal numbers to hexadecimal ones.

- $0_{dec} = 0_{hex}$
- $1_{dec} = 1_{hex}$
- $2_{dec} =$
- $3_{dec} =$
- $4_{dec} =$
- $5_{dec} = 5_{hex}$
- $6_{dec} =$
- $7_{dec} =$
- $8_{dec} =$
- $9_{dec} =$
- $10_{dec} = A_{hex}$
- $11_{dec} =$
- $12_{dec} =$
- $13_{dec} =$
- $14_{dec} =$
- $15_{dec} = F_{hex}$
- $16_{dec} = 10_{hex}$
- $17_{dec} =$
- $18_{dec} =$
- $19_{dec} =$
- $20_{dec} = 14_{hex}$
- $21_{dec} =$
- $22_{dec} =$
- $23_{dec} =$
- $24_{dec} =$
- $25_{dec} = 19_{hex}$
- $26_{dec} =$
- $27_{dec} =$
- $28_{dec} =$
- $29_{dec} =$
- $30_{dec} = 1E_{hex}$
- $31_{dec} =$
- $32_{dec} = 20_{hex}$
- $33_{dec} =$
- $34_{dec} =$
- $35_{dec} = 23_{hex}$
- $36_{dec} =$
- $37_{dec} =$
- $38_{dec} =$
- $39_{dec} =$

To convert a random decimal number to a hexadecimal one, we will need to divide by 16, possibly multiple times. So the first thing we'll do is revise our 16 times table, to make these divisions just a tiny bit easier.

Exercise 2 Fill in the 16 times table. As these will be very useful for converting numbers from decimal to hexadecimal and from hexadecimal to decimal, check your answers before proceeding.

1. $1 \times 16 =$
2. $2 \times 16 =$
3. $3 \times 16 = 48$
4. $4 \times 16 =$
5. $5 \times 16 =$
6. $6 \times 16 = 96$
7. $7 \times 16 =$
8. $8 \times 16 =$
9. $9 \times 16 = 144$
10. $10 \times 16 =$
11. $11 \times 16 = 176$
12. $12 \times 16 =$
13. $13 \times 16 = 208$
14. $14 \times 16 =$
15. $15 \times 16 =$

Terminology reminder: in $77 \div 9 = 8\ r\ 5$

- 77 is called the dividend.
- 9 is called the divisor.
- 8 is called the quotient.
- 5 is called the remainder.

To convert a decimal number to a hexadecimal one, you need to follow these steps:

1. Divide your number by 16. Write the remainder.
2. Divide the quotient you calculated by 16. Write the remainder to the left of what you already have.
3. Repeat step 2 until your quotient is 0.

Let us use this procedure to find the hexadecimal number for $37,285_{dec}$.

- $37,285 \div 16 = 2,330$ r5. We write 5.
- $2,330 \div 16 = 145$ r 10. In hexadecimal numbers 10 is written as A, so we write A to the left of what we already have. We now have $A5$.
- $145 \div 16 = 9$ r 1. We write 1 to the left of what we already have, so we now have $1A5$.
- $9 \div 16 = 0$ r9. Our quotient is 0 so this is our final step. We write 9 to the left of what we already have and get $91A5$.

We found that $37,285_{dec} = 91A5_{hex}$.

We write the hexadecimal number we found in a different way to check whether or not our conversion was succesful.

digit	9	1	A	5
value	16^3	16^2	16^1	16^0
decimal value	4,096	256	16	1

By writing our digits this way, it is easier to see that we can also write our number as

$$\begin{aligned}
91A5_{hex} &= (9_{hex} \times 16^3_{dec}) + (1_{hex} \times 16^2 dec) + (A_{hex} \times 16^1_{dec}) + (5_{hex} \times 16^0_{dec}) \\
& \quad \text{starting here, everything is written in decimal numbers} \\
&= (9 \times 16^3) + (1 \times 16^2) + (10 \times 16^1) + (5 \times 16^0) \\
&= (9 \times 4,096) + (1 \times 256) + (10 \times 16) + (5 \times 1) \\
&= 36,864 + 256 + 160 + 5 \\
&= 37,285.
\end{aligned}$$

We found that $91A5_{hex}$ corresponds to $37,285_{dec}$, so we are now completely sure that no mistakes were made and we can give ourselves a pat on the back for that.

While we were checking the result, we converted a hexadecimal number to a decimal one. To convert a random hexadecimal number, we might need more powers of 16. The following table might come in handy.

16^8	16^7	16^6	16^5	16^4	16^3	16^2	16^1	16^0
$4,294,967,296$	$268,435,456$	$16,777,216$	$1,048,576$	$65,536$	$4,096$	256	16	1

Table 3.1: Powers of 16

There is another way of converting decimal numbers to hexadecimal ones, that might be quicker and easier when you are working with relatively small numbers, but might take more time when you are asked to convert big numbers. The steps of this second way of converting a decimal number to a hexadecimal are

1. Determine which power of 16 is smaller or equal to your decimal number. Call this power n.
2. Calculate how many times this power of 16, 16^n fits in you decimal number. Write this number down and subtract this multiple of the power from your original number.
3. Calculate how many times the power 16^{n-1} fits in your remaining decimal number. Write this number down to the right of what you already found and subtract this multiple of the power from your remaining number.
4. Keep calculating how often the powers of 16 fits in your remaining number and always write it down at the right of what you already had.
5. You stop when you have done this for $16^0 = 1$.

When we use these steps to convert $10,010_{dec}$ to a hexadecimal number, we get

- The biggest power of 16 that fits in $10,010_{dec}$ is $16^3 = 4,096$. So in our case $n = 3$.
- $16^3 = 4,096$ fits twice in our original number. So we write down 2 and we subtract $2 \times 4,096 = 8,192$ from our original number $10,010$. We get $10,010 - 8,192 = 1,818$ and we have written 2.

- $16^2 = 256$ fits seven times in the remaining number $1,818$. We estimated the number 7 by rounding 256 to 250, remembering that $1,000 = 4 \times 250$ and estimating that it would fit another 3 times in the remaining 808. So we write down 7 to the right of what we already had and we subtract $7 \times 256 = 1,792$ from our remaining number $1,818$. We get $1,818 - 1,792 = 26$ and we have written 27. If we had gotten a result that is bigger than 256, we should have increased our estimate. And if our multiplication had given us a number bigger than our remaining number, we should have decreased our estimate. But this time it was perfect, so hurray, we can move on to the next step.

- $16^1 = 16$ fits once in the remaining number 26. So we write down 1 to the right of what we already had and we subtract $1 \times 16 = 16$ from our remaining number 26. We get $26 - 16 = 10$ and we have written 271.

- $16^0 = 1$ fits ten times in our remaining number 10. So we write down 10, which in hexadecimal numbers is A, to the right of what we already had. We now have $271A$ and this is the hexadecimal number we were looking for.

To make sure that we haven't made any mistakes and that the statement $10,010_{dec} = 271A_{hex}$ is correct, we rewrite the hexadecimal number as

$$\begin{aligned}
271A_{hex} &= (2 \times 16^3) + (7 \times 16^2) + (1 \times 16^1) + (A \times 16^0) \\
&= (2 \times 16^3) + (7 \times 16^2) + (1 \times 16^1) + (10 \times 16^0) \\
&= (2 \times 4,096) + (7 \times 256) + (1 \times 16) + (10 \times 1) \\
&= 8,192 + 1,792 + 16 + 10 \\
&= 10,010
\end{aligned}$$

We can conclude that we didn't make any mistakes and this second method leads to the correct result as well. Choosing a method to convert decimal numbers to binary depends on what type of calculations you prefer. The first method is centered around division, where the second method requires multiplications, subtractions and estimates. As long as you get the desired result, you can choose the method that leads you there.

By checking the correctness of the hexadecimal results, we converted these numbers back to binary. We can generalize the method we used to convert any hexadecimal number to its decimal counterpart.

The steps we need to follow to do this are:

1. Count the number of digits you have in your hexadecimal number. To make sure this works with every possible number, we will call it n. This n can be 1, but it can also be 1067. The steps are always the same.

2. Multiply the left digit with 16^{n-1}, the one next to that with 16^{n-2}, the one next to that with 16^{n-3} and so on. The second to last digit to the right will be multiplied with 16^1, the last one with 16^0. We will have to add everything later, so it is a good idea to already add a +sign between the multiplications.
3. Convert the powers of 16 you used with their decimal values.
4. Work out the multiplications.
5. Add them all up and you get your decimal number.

Let us use these steps to find the decimal number that corresponds with $1A2B3C_{hex}$.

- The number $1A2B3C_{hex}$ contains 6 digits, so in this case $n = 6$.
- We multiply the left digit with 16^{n-1}, so in our case this means 1×16^5.
- We multiply the next digit with 16^{n-2}, so in our case this means $A \times 16^4 = 10 \times 16^4$.
- We keep going like this and get 2×16^3, $B \times 16^2 = 11 \times 16^2$, 3×16^1 and $C \times 16^0 = 12 \times 16^0$.
- By converting the powers to their decimal values, we get $(1 \times 1,048,576) + (10 \times 65,536) + (2 \times 4,096) + (11 \times 256) + (3 \times 16) + (12 \times 1)$.
- By calculating these multiplications, we get $1,048,576 + 655,360 + 8,192 + 2,816 + 48 + 12$.
- By adding all the terms, we get $1,715,004_{dec}$.

Exercises

3. Convert these decimal numbers to hexadecimal numbers.

(a) $21_{dec} =$
(b) $50_{dec} =$
(c) $80_{dec} =$
(d) $99_{dec} =$
(e) $123_{dec} =$
(f) $165_{dec} =$
(g) $250_{dec} =$
(h) $255_{dec} =$
(i) $345_{dec} =$
(j) $505_{dec} =$
(k) $627_{dec} =$
(l) $750_{dec} =$
(m) $806_{dec} =$
(n) $918_{dec} =$
(o) $1000_{dec} =$
(p) $1111_{dec} =$
(q) $1,506_{dec} =$
(r) $2,433_{dec} =$
(s) $2,137_{dec} =$
(t) $3,721_{dec} =$
(u) $7,213_{dec} =$

(v) $4,000_{dec} =$

(w) $16,884_{dec} =$

(x) $123,456_{dec} =$

(y) $345,678_{dec} =$

(z) $567,890_{dec} =$

4. Convert these hexadecimal numbers to decimal ones.

(a) $16_{hex} =$

(b) $21_{hex} =$

(c) $2C_{hex} =$

(d) $39_{hex} =$

(e) $5D_{hex} =$

(f) $7B_{hex} =$

(g) $8E_{hex} =$

(h) $99_{hex} =$

(i) $AB_{hex} =$

(j) $AF_{hex} =$

(k) $BA_{hex} =$

(l) $BF_{hex} =$

(m) $EE_{hex} =$

(n) $FF_{hex} =$

(o) $ABC_{hex} =$

(p) $123_{hex} =$

(q) $BFF_{hex} =$

(r) $CAB_{hex} =$

(s) $FAB_{hex} =$

(t) $FEE_{hex} =$

(u) $2A7_{hex} =$

(v) $5B7_{hex} =$

(w) $234_{hex} =$

(x) $ABCDE_{hex} =$

(y) $111111_{hex} =$

(z) $1010011_{hex} =$

3.3 Converting binary numbers to hexadecimal and vice versa

We owe the current use of hexadecimal numbers to computer science. Computers use binary themselves, but they get their commands from humans (shhh, I know that some computers nowadays can make up their own commands, but to make this point we will pretend that artificial intelligence is irrelevant, mainly because AI simply didn't exist yet when hexadecimal numbers became a thing). There is a saying in computer science "Computers don't make mistakes, only people make mistakes." This means that if your computer doesn't do what you wanted it to do, you can't blame the machine, because it's always the fault of a human. Maybe you made a spelling mistake, or typed in a wrong number, or someone somewhere made a mistake in one of the thousands of lines of code that make up the program or website you are using. The machines themselves are never to blame, though grunting at a screen and letting out a heartfelt sigh or scream "aaargh" does help to relieve the frustration you feel when they don't work the way you want them to.

If you learn to code now, you can use builing blocks in a programming language like scratch, or words that mostly make sense in programming languages like java, python or go. Years of computer science made it possible to use computers the way we do now, without any direct contact between us and the binary that the computer uses. But when computer coding was in its infancy, a lot of the work had to be translated into actual binary. The larger a binary number is, the easier it is for us to make a mistake and leave out a 0 or 1 somewhere, or add one too many. It's not easy to spot the difference between "11010101110100000111010001111101", "110101011110100000111010001111101" and "1101010111010000111010001111101", and those are just three random binary numbers, so it doesn't come close to an actual program. Making just one mistake could give you a completely different result, so getting it right was crucial... and that's where those hexadecimal numbers come in.

Because $16 = 2^4$, there is a very neat and easy way to convert binary numbers to hexadecimal ones, or the other way around. Every power of 16 corresponds to a power of 2 and this is the key to converting numbers from one system to the other. Four digits of a binary number can be converted into exactly one digit of a hexadecimal number, and every digit of a hexadecimal number can be converted into a group of four digits of a binary number. It is perfectly normal if this doesn't make sense yet, it will get less confusing once we actually look at examples.

If we agree to always use 4 digits for the binary numbers, by adding as many zeroes to the left as is necessary, than we can convert binary numbers to hexadecimal digits as follows:

- $0 = 0000$
- $1 = 0001$
- $2 = 0010$
- $3 = 0011$
- $4 = 0100$
- $5 = 0101$
- $6 = 0110$
- $7 = 0111$
- $8 = 1000$
- $9 = 1001$
- $A = 1010$
- $B = 1011$
- $C = 1100$
- $D = 1101$
- $E = 1110$
- $F = 1111$

This is all the information we need to convert numbers.

To convert a binary number to a hexadecimal one, we need to follow these steps:

1. Divide your binary number into groups of 4 digits. Start making these groups with the lowest digits, so at the right side of your number.
2. Convert each group of 4 binary digits into a hexadecimal number, by using the conversions above.
3. You are done.

When we use these steps to convert the binary number 110101011101000001110100011111101 to a hexadecimal one, we get

- 110101011101000001110100011111101 = 1 1010 1011 1010 0000 1110 1000 1111 1101
- 1 1010 1011 1010 0000 1110 1000 1111 1101 = $1ABA0EFD$

- The hexadecimal number we were searching for is $1ABA0EFD$.

To convert a hexadecimal number to a binary one, we need to follow these steps:

1. Convert each hexadecimal digit to a group of 4 binary digits, by using the conversions above.
2. You can leave out the zeroes on the left that proceed the first 1. You are done.

When we do this to convert the hexadecimal number $1A2B3C4D5E6F$ to binary, we get

- $1A2B3C4D5E6F = 0001\ 1010\ 0010\ 1011\ 0011\ 1100\ 0100\ 1101\ 0101\ 1110\ 0110\ 1111$
- The corresponding binary number is 110100010101100111100010011010101111001101111.

Exercises

1. Match the hexadecimal numbers with their corresponding binary representations

(a) $ABBA$	(i) 1011011101011101
(b) $3E4$	(j) 1111100100
(c) $7C4$	(k) 11011101001
(d) $AB96$	(l) 1010101110010110
(e) $6E9$	(m) 11111000100
(f) $3F5$	(n) 1011111010101101
(g) $BEAD$	(o) 1010101110111010
(h) $B75D$	(p) 1111110101

2. Convert these binary numbers to hexadecimal ones. This may be easier if you divide the long binary numbers into groups first. You can do this by using — or \ symbols, or whatever works for you.

(a) 10010110101110100101 =	(f) 1000010101001010001010 =
(b) 11110010001010101011 1 =	(g) 1111010111111010111110 =
(c) 10101001111000111001 =	(h) 11110011110101011110 =
(d) 11110010111001110010 1 =	(i) 1000110110011111010110 =
(e) 11100101110100011001 1 =	(j) 10101010111011001010101 =

(k) 1010010101010101001010101000101010101001110 =

(l) 1101011111110101110110111110111011101111111 =

(m) 1000101011000001101010110100000100101010000010 =

(n) 10010101010010010010010110010001001010010001 =

(o) 1010010101001010100100010100111010010100101110 =

(p) 111111111000100000100000100000101010010010000 =

(q) 10100100101001001001000100001000000111111010011 =

(r) 1010011001010010100101001001010100101010101 0001 =

(s) 1010010100101010001011010101010110010010101110 =

(t) 101001010010010101010010101001010101010100101 =

3. Convert these hexadecimal numbers to binary ones.

(a) 60

(b) $BEEF =$

(c) $C0DE =$

(d) $FACE =$

(e) $DECAF =$

(f) $CAFE =$

(g) $98765 =$

(h) $2137 =$

(i) $F37EA =$

(j) $FE715 =$

(k) $F104A =$

(l) $B0D1CE =$

(m) $D06FACE =$

(n) $ABCDEF =$

(o) $57160 =$

(p) $14E7A8D =$

(q) $5A7AD =$

(r) $7E771CE =$

(s) $1234567 =$

(t) $C705E4 =$

3.4 Adding hexadecimal numbers

Adding hexadecimal numbers is similar to doing this operation with decimal numbers. We will only be working with hexadecimal numbers, so there is no need to add $_{hex}$ to them.

The only difference while adding numbers, is that you can add up to 15 (not 9) before you have to carry over a digit.

We will examine one addition in more detail to understand how this works. We will add the hexadecimal numbers $59A3651$ and $93E2A78$. Where necessary, we will write $\stackrel{dec}{=}$ to make it clear that we are converting or adding the digits in the decimal system. We will write $\stackrel{hex}{=}$ to make it clear that we are converting decimal numbers to hexadecimal ones. As usual, this sounds more confusing than it actually is and it will become clear by following the steps of this example.

We start by writing them in a way that makes it easy to see which digits you need to add together.

$$\begin{array}{cccccccc} & 5 & 9 & A & 3 & 6 & 5 & 1 \\ + & 9 & 3 & E & 2 & A & 7 & 8 \\ \hline \end{array}$$

We start by adding the digits on the right. $1 + 8 = 9$, so we get

$$\begin{array}{cccccccc} & 5 & 9 & A & 3 & 6 & 5 & 1 \\ + & 9 & 3 & E & 2 & A & 7 & 8 \\ \hline & & & & & & & 9 \end{array}$$

We move on to the digits on the left of them. $5+7 \stackrel{dec}{=} 12 \stackrel{hex}{=} C$. In time, you might be able to work out the answers without converting anything to decimal numbers while working it out. In that case, you will be better trained in working with hexadecimal numbers than I am, because I still go back and forth between hexadecimal and decimal while calculating with hexadecimal numbers. Anyway, as $5 + 7 = C$, we get

$$\begin{array}{cccccccc} & 5 & 9 & A & 3 & 6 & 5 & 1 \\ + & 9 & 3 & E & 2 & A & 7 & 8 \\ \hline & & & & & & C & 9 \end{array}$$

We move on to the digits on the left of them. $6 + A \stackrel{dec}{=} 6 + 10 = 16 \stackrel{hex}{=} 10$, so we carry over 1 and we write 0. We get

$$\begin{array}{cccccccc} & & & & 1 & & & \\ & 5 & 9 & A & 3 & 6 & 5 & 1 \\ + & 9 & 3 & E & 2 & A & 7 & 8 \\ \hline & & & & & 0 & C & 9 \end{array}$$

We move on to the digits on the left of them. $1 + 3 + 2 = 6$, so we get

$$\begin{array}{cccccccc} & & & & 1 & & & \\ & 5 & 9 & A & 3 & 6 & 5 & 1 \\ + & 9 & 3 & E & 2 & A & 7 & 8 \\ \hline & & & & 6 & 0 & C & 9 \end{array}$$

We move on to the digits on the left of them. $A + E \stackrel{dec}{=} 10 + 14 = 24 = 16 + 8 \stackrel{hex}{=} 18$, so we carry over 1 and we write 8. We get

$$\begin{array}{cccccccc} & & 1 & & 1 & & & \\ & 5 & 9 & A & 3 & 6 & 5 & 1 \\ + & 9 & 3 & E & 2 & A & 7 & 8 \\ \hline & & & & 8 & 6 & 0 & C & 9 \end{array}$$

We move on to the digits on the left of them. $1 + 9 + 3 \overset{dec}{=} 13 \overset{hex}{=} D$, so we get

$$\begin{array}{cccccccc} & & 1 & & 1 & & & \\ & 5 & 9 & A & 3 & 6 & 5 & 1 \\ + & 9 & 3 & E & 2 & A & 7 & 8 \\ \hline & & D & 8 & 6 & 0 & C & 9 \end{array}$$

We move on to the digits on the left of them. $5 + 9 \overset{dec}{=} 14 \overset{hex}{=} E$, so we get

$$\begin{array}{cccccccc} & & 1 & & 1 & & & \\ & 5 & 9 & A & 3 & 6 & 5 & 1 \\ + & 9 & 3 & E & 2 & A & 7 & 8 \\ \hline & E & D & 8 & 6 & 0 & C & 9 \end{array}$$

We found that $59A3651 + 93E2A78 = ED860C9$.

Now it is your turn. As there is so much more that can happen when you add hexadecimal digits than was the case with binary, or even decimal digits, we will start with the basics. We won't encounter situations where you have to carry over a digit for more than one space like we did with binary numbers, but you will have to be able to switch a lot between decimal and hexadecimal numbers, so that you carry over the right digit. You have to pay really close attention, for example when your result is $24_{dec} = 18_{hex}$ and you have to carry over 1, not 2, or $42_{dec} = 2A_{hex}$, where you have to carry over 2 instead of 4. As usual, it looks more confusing than it actually is and we'll take it one step at a time. We all need to warm up and do some stretches before we are ready to do hard exercises, it doesn't matter if you are exercising your body or your brain.

Exercises Pro tip: write down the decimal values of A to F on a piece of paper and keep that close while you work on these exercises. It's perfectly normal to consult it at least once a minute while you are doing these exercises. I sure did.

Also: expect to make mistakes. Mistakes while adding hexadecimal numbers, mistakes while subtracting them. And probably (a lot) more mistakes while multiplying and dividing them. If you are human, you are bound to make some. Even if, like me, you already know how to work with hexadecimal numbers, you are still destined to make them. And if by some miracle you can get through all these exercises without a single mistake, please tell me, how did you manage to do that? Are you secretly a computer in disguise? Did you encounter a witch or wizard who made it easier for you to work with hexadecimal numbers? Have you unlocked a maths superpower, and if so, can you tell me if I could get it too?

1. Add these hexadecimal numbers. Even though they look like decimal numbers, make sure you treat them as hexadecimal numbers.

(a) $5+7=$
(b) $4+6=$
(c) $6+8=$
(d) $4+4=$
(e) $7+8=$
(f) $7+7=$
(g) $9+4=$
(h) $8+4=$
(i) $3+4=$
(j) $4+5=$
(k) $5+6=$
(l) $6+7=$
(m) $9+5=$
(n) $2+9=$
(o) $3+8=$
(p) $7+4=$
(q) $3+9=$
(r) $5+5=$
(s) $6+9=$
(t) $6+6=$
(u) $8+3=$
(v) $5+8=$
(w) $9+1=$
(x) $7+3=$

2. Add these hexadecimal numbers. This time, the numbers are quite unmistakebly hexadecimal ones.

(a) $1+A=$
(b) $A+5=$
(c) $2+B=$
(d) $B+3=$
(e) $3+C=$
(f) $C+1=$
(g) $2+D=$
(h) $A+4=$
(i) $D+1=$
(j) $2+A=$
(k) $1+B=$
(l) $B+4=$
(m) $E+1=$
(n) $3+A=$
(o) $C+2=$
(p) $1+D=$

3. Add these hexadecimal numbers. It keeps getting slightly harder. From now on, most of your results will contain more than one digit.

(a) $F+1=$
(b) $4+E=$
(c) $E+2=$
(d) $5+B=$
(e) $A+6=$
(f) $6+E=$
(g) $C+5=$
(h) $3+D=$
(i) $A+4=$
(j) $2+C=$
(k) $F+6=$
(l) $5+E=$
(m) $F+2=$
(n) $6+D=$
(o) $C+4=$
(p) $3+B=$
(q) $F+5=$
(r) $3+E=$
(s) $B+6=$
(t) $5+D=$
(u) $F+4=$
(v) $6+C=$
(w) $D+4=$
(x) $3+F=$

4. Add these hexadecimal numbers. Okay, I admit it, maybe this exercise isn't really harder than the previous one. But there are not enough letters in the alphabet to keep that one going...

(a) $F+9=$
(b) $8+E=$
(c) $D+7=$
(d) $9+C=$
(e) $B+8=$
(f) $7+A=$
(g) $9+9=$
(h) $F+8=$
(i) $8+7=$
(j) $7+E=$
(k) $A+9=$
(l) $8+8=$
(m) $B+7=$
(n) $9+7=$
(o) $C+8=$
(p) $9+E=$
(q) $F+7=$
(r) $8+9=$
(s) $B+9=$
(t) $8+E=$
(u) $D+9=$
(v) $7+7=$
(w) $B+8=$
(x) $7+C=$

5. Add these letters to form new words. Nope, just kidding. Add these hexadecimal numbers.

   (a) $A + A =$
   (b) $B + B =$
   (c) $C + C =$
   (d) $D + D =$
   (e) $E + E =$
   (f) $F + F =$
   (g) $A + B =$
   (h) $B + C =$
   (i) $C + D =$
   (j) $D + E =$
   (k) $E + F =$
   (l) $F + A =$
   (m) $A + E =$
   (n) $B + D =$
   (o) $C + F =$
   (p) $D + A =$
   (q) $E + B =$
   (r) $F + B =$
   (s) $A + C =$
   (t) $B + F =$
   (u) $C + E =$
   (v) $D + F =$
   (w) $E + A =$
   (x) $F + C =$

6. Can you add 3 hexadecimal digits?

   (a) $4 + 4 + 4 =$
   (b) $5 + 5 + 5 =$
   (c) $6 + 6 + 6 =$
   (d) $7 + 7 + 7 =$
   (e) $8 + 8 + 8 =$
   (f) $9 + 9 + 9 =$
   (g) $A + A + A =$
   (h) $B + B + B =$
   (i) $C + C + C =$
   (j) $D + D + D =$
   (k) $E + E + E =$
   (l) $F + F + F =$
   (m) $B + F + F =$
   (n) $B + E + D =$
   (o) $A + D + D =$
   (p) $B + A + D =$
   (q) $F + A + B =$
   (r) $D + A + D =$
   (s) $9 + A + 8 =$
   (t) $8 + 6 + F =$
   (u) $5 + E + C =$
   (v) $4 + F + C =$
   (w) $6 + B + 8 =$
   (x) $3 + F + 7 =$

7. Can you add even more in one go?

   (a) $D + A + D + A =$
   (b) $F + A + 9 + 5 =$
   (c) $D + E + A + D =$
   (d) $2 + 7 + 6 + 8 =$
   (e) $F + 4 + A + 9 =$
   (f) $3 + C + A + 5 =$
   (g) $C + 4 + B + B =$
   (h) $C + 8 + 6 + E =$
   (i) $A + B + C + D =$
   (j) $E + F + 9 + 8 =$
   (k) $5 + F + A + 6 =$
   (l) $F + A + C + E =$
   (m) $D + A + B + B + E =$
   (n) $9 + C + 8 + F + 7 =$
   (o) $F + F + F + F + F =$
   (p) $E+F+F+E+C+7 =$
   (q) $B+A+B+B+E+1 =$
   (r) $5 + 7 + A + E + D =$

8. Find the answers to these additions. Can you find a pattern in what happens?

   (a) $50 + F =$
   (b) $41 + F =$
   (c) $B2 + F =$
   (d) $23 + F =$
   (e) $C4 + F =$
   (f) $E5 + F =$
   (g) $76 + F =$
   (h) $17 + F =$
   (i) $A8 + F =$
   (j) $39 + F =$
   (k) $9A + F =$
   (l) $3B + F =$
   (m) $CC + F =$
   (n) $8D + F =$
   (o) $DE + F =$
   (p) $2F + F =$

Pattern:

9. Find the answers to these additions. Is there a clear pattern here as well?

(a) $50 + A =$
(b) $41 + A =$
(c) $B2 + A =$
(d) $23 + A =$
(e) $C4 + A =$
(f) $E5 + A =$
(g) $76 + A =$
(h) $17 + A =$
(i) $A8 + A =$
(j) $39 + A =$
(k) $9A + A =$
(l) $3B + A =$
(m) $CC + A =$
(n) $8D + A =$
(o) $DE + A =$
(p) $2F + A =$

Pattern:

10. Add these hexadecimal numbers of 2 digits each, without the need to carry over digits.

(a) $47 + 85 =$
(b) $AB + 42 =$
(c) $56 + 67 =$
(d) $2C + A0 =$
(e) $58 + 96 =$
(f) $7A + 44 =$
(g) $25 + 52 =$
(h) $55 + 56 =$
(i) $69 + 56 =$
(j) $75 + 75 =$
(k) $48 + A4 =$
(l) $C4 + 2B =$
(m) $C0 + 3F =$
(n) $21 + DE =$
(o) $67 + 67 =$
(p) $88 + 55 =$
(q) $66 + 99 =$
(r) $21 + 37 =$
(s) $73 + 37 =$
(t) $BA + 35 =$
(u) $38 + 87 =$
(v) $CC + 21 =$
(w) $EE + 10 =$
(x) $97 + 68 =$

11. Add these hexadecimal numbers of 2 digits each, where sometimes you do need to carry over digits.

(a) $23 + 9E =$
(b) $38 + 39 =$
(c) $79 + 5F =$
(d) $2C + 76 =$
(e) $48 + AF =$
(f) $5A + 9E =$
(g) $AE + 44 =$
(h) $BA + 21 =$
(i) $6E + 5F =$
(j) $CA + 27 =$
(k) $25 + FE =$
(l) $63 + 7D =$
(m) $68 + DA =$
(n) $DE + AF =$
(o) $CA + FE =$
(p) $B9 + 87 =$
(q) $C1 + AA =$
(r) $FE + E1 =$
(s) $AB + BA =$
(t) $BA + 79 =$
(u) $99 + 57 =$
(v) $87 + CD =$
(w) $AB + CD =$
(x) $FF + FF =$

12. Can you add 3 hexadecimal numbers containing 2 digits each?

(a) $AA + BB + CC =$

(b) $DD + EE + FF =$

(c) $38 + AB + E0 =$

(d) $21 + 98 + 54 =$

(e) $75 + BC + 25 =$

(f) $CA + 88 + 53 =$

(g) $98 + 6A + 3E =$

(h) $FA + BC + 56 =$

(i) $A9 + 8B + 46 =$

(j) $37 + CD + 2A =$

(k) $61 + 58 + 74 =$

(l) $1A + 2B + 3C =$

(m) $4D + 5E + 6F =$

(n) $A4 + A5 + A6 =$

(o) $BA + BB + E1 =$

13. Add these hexadecimal numbers.

(a)
$$\begin{array}{ccccccc} & 2 & A & 3 & B & 7 & F \\ + & A & B & C & D & E & F \\ \hline \end{array}$$

(b)
$$\begin{array}{cccccccc} & 5 & C & 8 & 9 & 3 & 5 & B \\ + & 9 & A & 2 & 4 & D & 7 & 9 \\ \hline \end{array}$$

(c)
$$\begin{array}{ccccccccc} & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 \\ + & 8 & 9 & A & B & C & D & E & F \\ \hline \end{array}$$

(d)
$$\begin{array}{ccccccccc} & 7 & 8 & 9 & 7 & 8 & 9 & 7 & 8 \\ + & A & B & C & 5 & 6 & A & B & C \\ \hline \end{array}$$

(e)
$$\begin{array}{ccccccc} & 2 & 8 & D & 5 & 1 & E \\ + & A & 9 & 5 & 7 & E & 6 \\ \hline \end{array}$$

(f)
$$\begin{array}{ccccccccc} & 1 & 3 & 5 & 7 & 9 & B & D & F \\ + & E & C & A & 8 & 6 & 4 & 2 & 0 \\ \hline \end{array}$$

(g)
$$\begin{array}{ccccccccc} & 9 & 9 & A & A & 5 & 5 & E & E \\ + & B & B & 5 & 5 & 9 & 9 & 5 & 5 \\ \hline \end{array}$$

(h)
$$\begin{array}{ccccccccc} & 3 & 7 & 2 & 1 & 5 & 5 & 2 & 5 \\ + & 9 & 9 & 0 & F & E & E & 9 & 5 \\ \hline \end{array}$$

(i)
$$\begin{array}{ccccccc} & B & 0 & D & 1 & E & 5 \\ + & F & 0 & 0 & D & 1 & E \\ \hline \end{array}$$

(j)
$$\begin{array}{cccccccc} & D & 0 & 6 & F & A & C & E \\ + & 2 & B & A & B & 1 & E & 5 \\ \hline \end{array}$$

14. I feel like I am repeating myself here, but I promise that this is the very last time: add these hexadecimal numbers. Any resemblance between the hexadecimal numbers and real words is purely coincidental, because everyone knows that $0 \neq O$, $1 \neq l$,$3 \neq E$, $5 \neq S$, $6 \neq G$ and $7 \neq T$. Right?

(a)
$$\begin{array}{cccccccc} & E & 1 & E & C & 7 & 3 & D \\ & F & A & C & A & D & E & 5 \\ + & D & E & F & 1 & E & C & 7 \\ \hline \end{array}$$

$$\begin{array}{ccccccccc} & & C & A & B & 0 & 0 & 5 & E \\ & B & A & 5 & E & B & A & 1 & 1 \\ + & F & 0 & 1 & D & A & B & 1 & E \\ \hline \end{array}$$

(b)

(c)

$$\begin{array}{ccccccccc} & B & 0 & 1 & D & F & A & C & E \\ & B & E & 5 & 0 & 7 & 7 & E & D \\ + & C & A & 5 & 5 & E & 7 & 7 & E \\ \hline \end{array}$$

(e)

$$\begin{array}{cccccccccc} & E & 5 & 7 & A & F & E & 7 & 7 & E \\ & F & 0 & 0 & 7 & B & A & 1 & 1 & 5 \\ + & D & A & 7 & A & B & A & 5 & 3 & 5 \\ \hline \end{array}$$

(d)

$$\begin{array}{cccccccc} B & E & D & 5 & 7 & E & A & D \\ D & E & 7 & E & C & 7 & E & D \\ + & D & E & C & 0 & D & E & D \\ \hline \end{array}$$

(f)

$$\begin{array}{cccccccccc} & 5 & 7 & 3 & A & D & F & A & 5 & 7 \\ & 7 & A & 5 & 7 & E & 1 & E & 5 & 5 \\ + & 7 & E & 5 & 5 & E & 1 & A & 7 & E \\ \hline \end{array}$$

(g)

$$\begin{array}{ccccccc} & A & C & C & E & 5 & 5 \\ & B & A & 1 & 1 & A & D \\ & C & A & B & 1 & E & 5 \\ + & F & A & B & 1 & E & 5 \\ \hline \end{array}$$

(h)

$$\begin{array}{ccccccc} & F & 1 & 3 & 3 & C & 3 \\ & 5 & A & F & 3 & 5 & 7 \\ & 7 & 3 & A & 5 & 3 & 5 \\ & C & 0 & A & 7 & 3 & D \\ + & 1 & A & 5 & 5 & 0 & 5 \\ \hline \end{array}$$

(i)

$$\begin{array}{cccccccccc} & E & 5 & C & A & 1 & A & 7 & E & D \\ & F & 1 & 0 & A & 7 & A & B & 1 & E \\ & A & C & C & 0 & 1 & A & D & E & 5 \\ + & F & 0 & 0 & 7 & 1 & 0 & 0 & 5 & E \\ \hline \end{array}$$

(j)

$$\begin{array}{cccccc} & 1 & 0 & A & D & 5 \\ & 5 & 7 & A & 1 & 3 \\ & 7 & A & B & 1 & 3 \\ & B & A & 5 & 3 & D \\ + & C & A & B & 1 & 3 \\ \hline \end{array}$$

3.5 Subtracting hexadecimal numbers

Subtracting hexadecimal numbers is also similar to doing this with decimal numbers. The only difference while subtracting numbers, is that whenever you have to borrow a digit, you have to add 16 (not 10)to the digit you already had.

We will now look at one subtraction in more detail so you can feel confident that you understand how borrowing a digit works before you do it yourself. We will subtract $15AB8E7$ from $33205A8$

We start by writing them in a way that makes it easy to see which digit you need to subtract from which.

$$\begin{array}{cccccccc} & 3 & 3 & 2 & 0 & 5 & A & 8 \\ - & 1 & 5 & A & B & 8 & E & 7 \\ \hline \end{array}$$

We start with the digits on the right. $8-7=1$, so we get

$$\begin{array}{cccccccc} & 3 & 3 & 2 & 0 & 5 & A & 8 \\ - & 1 & 5 & A & B & 8 & E & 7 \\ \hline &&&&&&& 1 \end{array}$$

We move on to the digits on the left of them. $E > A$, so we have to borrow 1 from the next digit, which is 5. Our 5 becomes a 4 and A becomes $1A$. Our adjusted subtraction is $1A - E \overset{dec}{=} 26 - 14 = 12 \overset{hex}{=} C$, so we get

$$\begin{array}{cccccccc} &&&&& 4 & 1A & \\ & 3 & 3 & 2 & 0 & \cancel{5} & \cancel{A} & 8 \\ - & 1 & 5 & A & B & 8 & E & 7 \\ \hline &&&&&& C & 1 \end{array}$$

We move to the digits on the left of them. $8 > 4$, so once again we have to borrow from the next digit, which is 0. We can't borrow from 0, so we have to borrow from the next one, which is 2. Our 2 becomes 1. Our 0 becomes 10, but as we need to borrow a digit from here, our 0 actually becomes F. Our 4 becomes 14. The adjusted subtraction is $14 - 8 \overset{dec}{=} 20 - 8 = 12 \overset{hex}{=} C$, so we get

$$\begin{array}{cccccccc} &&&&& 14 && \\ &&& 1 & F & \cancel{4} & 1A & \\ & 3 & 3 & \cancel{2} & \cancel{0} & \cancel{5} & \cancel{A} & 8 \\ - & 1 & 5 & A & B & 8 & E & 7 \\ \hline &&&&& C & C & 1 \end{array}$$

We move on to the digits on the left of them. $F - B = 4$, so we get

$$\begin{array}{cccccccc} &&&&& 14 && \\ &&& 1 & F & \cancel{4} & 1A & \\ & 3 & 3 & \cancel{2} & \cancel{0} & \cancel{5} & \cancel{A} & 8 \\ - & 1 & 5 & A & B & 8 & E & 7 \\ \hline &&&& 4 & C & C & 1 \end{array}$$

We move on to the digits on the left of them. $A > 1$, so we have to borrow a digit from the next digit, which is 3. This 3 becomes 2 and 1 becomes 11. Our adjusted subtraction is $11 - A \overset{dec}{=} 17 - 10 = 7 \overset{hex}{=} 7$, so we get

$$\begin{array}{cccccccc} &&& 11 && 14 && \\ && 2 & \cancel{1} & F & \cancel{4} & 1A & \\ & 3 & \cancel{3} & \cancel{2} & \cancel{0} & \cancel{5} & \cancel{A} & 8 \\ - & 1 & 5 & A & B & 8 & E & 7 \\ \hline &&& 7 & 4 & C & C & 1 \end{array}$$

We move on to the digits on the left of them. $5 > 2$, so we have to borrow a digit from the next digit, which is another 3. This 3 becomes 2 and 2 becomes 12. Our adjusted subtraction is $12 - 5 \overset{dec}{=} 18 - 5 = 13 \overset{hex}{=} D$, so we get

$$
\begin{array}{cccccccc}
 & & 12 & 11 & & 14 & & \\
 & 2 & \cancel{2} & \cancel{1} & F & \cancel{4} & 1A & \\
 & \cancel{3} & \cancel{3} & \cancel{2} & \cancel{0} & \cancel{5} & \cancel{A} & 8 \\
- & 1 & 5 & A & B & 8 & E & 7 \\
\hline
 & & D & 7 & 4 & C & C & 1
\end{array}
$$

We move on to the digits on the left of them. $2 - 1 = 1$, so we get

$$
\begin{array}{cccccccc}
 & & 12 & 11 & & 14 & & \\
 & 2 & \cancel{2} & \cancel{1} & F & \cancel{4} & 1A & \\
 & \cancel{3} & \cancel{3} & \cancel{2} & \cancel{0} & \cancel{5} & \cancel{A} & 8 \\
- & 1 & 5 & A & B & 8 & E & 7 \\
\hline
 & 1 & D & 7 & 4 & C & C & 1
\end{array}
$$

We have found that $33205A8 - 15AB8E7 = 1D74CC1$.

Now it is your turn. Just like with the addition of hexadecimal numbers, we will build up the exercises until you can do anything that gets thrown at you.

Exercises Remember that piece of paper with the decimal values of A to F? It's just as useful for subtraction as it was for addition. Don't expect perfection from yourself. Correcting your exercises would be so boring if you never made mistakes.

1. Subtract these hexadecimal numbers.

   (a) $F - 9 =$
   (b) $B - 5 =$
   (c) $C - 1 =$
   (d) $E - 6 =$
   (e) $A - 7 =$
   (f) $D - 8 =$
   (g) $A - 3 =$
   (h) $B - 9 =$
   (i) $C - 7 =$
   (j) $D - 4 =$
   (k) $E - 2 =$
   (l) $F - 1 =$
   (m) $B - 1 =$
   (n) $D - 6 =$
   (o) $C - 9 =$
   (p) $F - 5 =$
   (q) $A - 9 =$
   (r) $E - 5 =$
   (s) $F - 2 =$
   (t) $E - 9 =$
   (u) $D - 3 =$
   (v) $C - 5 =$
   (w) $B - 4 =$
   (x) $A - 6 =$

2. Subtract these hexadecimal numbers. You can consider this part two of the previous exercise, because there still aren't enough letters in the alphabet to keep going.

   (a) $E - 8 =$
   (b) $A - 2 =$
   (c) $B - 8 =$
   (d) $F - 8 =$
   (e) $F - 3 =$
   (f) $C - 2 =$
   (g) $A - 8 =$
   (h) $B - 2 =$
   (i) $C - 3 =$
   (j) $D - 1 =$
   (k) $E - 4 =$
   (l) $F - 6 =$
   (m) $D - 5 =$
   (n) $F - 7 =$
   (o) $A - 5 =$
   (p) $E - 1 =$
   (q) $B - 6 =$
   (r) $C - 8 =$
   (s) $A - 1 =$
   (t) $B - 3 =$
   (u) $C - 6 =$
   (v) $D - 2 =$
   (w) $E - 3 =$
   (x) $F - 4 =$

3. Subtract these hexadecimal numbers. It's full on letter time!

(a) $F - A =$
(b) $E - C =$
(c) $D - B =$
(d) $C - B =$
(e) $B - A =$
(f) $A - A =$
(g) $F - C =$
(h) $E - A =$
(i) $D - B =$
(j) $C - A =$
(k) $F - F =$
(l) $D - A =$
(m) $B - B =$
(n) $E - D =$
(o) $F - B =$
(p) $F - E =$
(q) $E - A =$
(r) $D - C =$
(s) $F - D =$
(t) $E - B =$

4. Work out these subtractions.

(a) $AB - 9 =$
(b) $27 - 16 =$
(c) $FF - 35 =$
(d) $CA - B9 =$
(e) $EA - 99 =$
(f) $F6 - A4 =$
(g) $CA - 89 =$
(h) $FF - CD =$
(i) $9A - 78 =$
(j) $8B - 7A =$
(k) $FA - 82 =$
(l) $1E - 13 =$
(m) $AA - 32 =$
(n) $BB - 56 =$
(o) $CC - 2A =$
(p) $DD - 79 =$
(q) $EE - 40 =$
(r) $FF - 1A =$
(s) $2C - 7 =$
(t) $37 - 21 =$
(u) $CD - BA =$
(v) $FE - 9C =$
(w) $9B - 69 =$
(x) $DC - 45 =$

5. Find the answers to these subtractions. Remember that they're all hexadecimal numbers, so be careful when you have to borrow a digit. Can you find a pattern in what happens?

(a) $50 - F =$
(b) $41 - F =$
(c) $B2 - F =$
(d) $23 - F =$
(e) $C4 - F =$
(f) $E5 - F =$
(g) $76 - F =$
(h) $17 - F =$
(i) $A8 - F =$
(j) $F9 - F =$
(k) $9A - F =$
(l) $3B - F =$
(m) $CC - F =$
(n) $8D - F =$
(o) $DE - F =$
(p) $2F - F =$

Pattern:

6. Find the answers to these subtractions. Is there a clear pattern here as well?

(a) $50 - A =$
(b) $41 - A =$
(c) $B2 - A =$
(d) $23 - A =$
(e) $C4 - A =$
(f) $E5 - A =$
(g) $76 - A =$
(h) $17 - A =$
(i) $A8 - A =$
(j) $F9 - A =$
(k) $9A - A =$
(l) $3B - A =$
(m) $CC - A =$
(n) $8D - A =$
(o) $DE - A =$
(p) $2F - A =$

Pattern:

7. Work out these subtractions. Be mindful while borrowing a digit.

(a) $11 - 9 =$
(b) $14 - B =$
(c) $22 - F =$
(d) $56 - 3A =$
(e) $78 - 4B =$
(f) $9A - 7C =$
(g) $15 - 8 =$
(h) $23 - D =$
(i) $4F - 19 =$
(j) $DC - CD =$
(k) $56 - 4E =$
(l) $21 - F =$
(m) $37 - 2C =$
(n) $50 - 3E =$
(o) $DA - AD =$
(p) $CD - BE =$
(q) $7E - 5F =$
(r) $5A - 4F =$
(s) $A1 - 93 =$
(t) $BA - AB =$
(u) $10 - 1 =$
(v) $F1 - 34 =$
(w) $D3 - A5 =$
(x) $16 - F =$

8. Let us try some subtractions with 3 digits.

(a) $123 - 45 =$
(b) $456 - AB =$
(c) $789 - 5A =$
(d) $ABC - 96 =$
(e) $DEF - F5 =$
(f) $DAD - FF =$
(g) $D06 - CA7 =$
(h) $CA7 - DA =$
(i) $709 - EF =$
(j) $FAB - 9D =$
(k) $AC7 - 60 =$
(l) $6E1 - B1 =$
(m) $370 - 21 =$
(n) $C1A - B0 =$
(o) $555 - 1E =$

9. Subtractions with hexadecimal numbers don't get any harder than this.

(a)
$$\begin{array}{rcccccc} & 9 & A & 5 & B & C & 4 \\ - & 6 & F & 2 & E & A & 9 \\ \hline \end{array}$$

(b)
$$\begin{array}{rcccccc} & F & 0 & 0 & 1 & E & D \\ - & 7 & 6 & 9 & 8 & 5 & A \\ \hline \end{array}$$

(c)
$$\begin{array}{rcccccc} & B & 9 & 5 & 0 & 8 & A \\ - & 9 & A & B & E & A & 3 \\ \hline \end{array}$$

(d)
$$\begin{array}{rcccc} & D & 0 & 6 & 5 \\ - & C & A & 7 & 5 \\ \hline \end{array}$$

(e)
$$\begin{array}{rcccccccc} & 9 & 9 & 0 & F & E & E & 9 & 5 \\ - & 3 & 7 & 2 & 1 & 5 & 5 & 2 & 5 \\ \hline \end{array}$$

(f)
$$\begin{array}{rccccc} & 9 & 8 & 7 & 6 & 5 \\ - & 5 & 7 & 9 & B & D \\ \hline \end{array}$$

(g)
$$\begin{array}{rcccccc} & F & E & D & C & B & A \\ - & A & B & C & D & E & F \\ \hline \end{array}$$

(h)
$$\begin{array}{rcccccc} & C & 7 & 5 & E & 9 & F \\ - & B & A & 8 & 0 & E & 1 \\ \hline \end{array}$$

(i)
$$\begin{array}{rcccccccc} & E & C & A & 8 & 6 & 4 & 2 & 0 \\ - & 1 & 3 & 5 & 7 & 9 & B & D & F \\ \hline \end{array}$$

(j)
$$\begin{array}{rcccccccc} & B & B & 5 & 5 & 9 & 9 & 5 & 5 \\ - & 9 & 9 & A & A & 5 & 5 & E & E \\ \hline \end{array}$$

10. Solve these subtractions. Again, I claim that any resemblance to existing words is purely coincidental. But I will admit that making these exercises was fun!

(a)
$$\begin{array}{r} D\ 0\ 6\ 6\ 1\ E\ 5 \\ -\quad F\ 1\ F\ 1 \\ \hline \end{array}$$

(b)
$$\begin{array}{r} F\ 0\ 0\ D\ 1\ E \\ -\ B\ 0\ D\ 1\ E\ 5 \\ \hline \end{array}$$

(c)
$$\begin{array}{r} D\ 0\ 6\ F\ A\ C\ E \\ -\ 2\ B\ A\ B\ 1\ E\ 5 \\ \hline \end{array}$$

(g)
$$\begin{array}{r} F\ 0\ 0\ 7\ B\ A\ 1\ 1 \\ -\ B\ A\ 5\ E\ B\ A\ 1\ 1 \\ \hline \end{array}$$

(d)
$$\begin{array}{r} B\ 0\ 5\ 5\ E\ 5 \\ -\quad B\ 0\ 5\ 5 \\ \hline \end{array}$$

(h)
$$\begin{array}{r} D\ E\ F\ E\ A\ 7 \\ -\quad 6\ 0\ A\ 1 \\ \hline \end{array}$$

(e)
$$\begin{array}{r} C\ A\ 5\ 7\ 1\ E \\ -\quad C\ E\ 1\ 1 \\ \hline \end{array}$$

(i)
$$\begin{array}{r} F\ 1\ 0\ 5\ 5 \\ -\ 6\ 0\ 0\ D \\ \hline \end{array}$$

(f)
$$\begin{array}{r} C\ 1\ E\ F \\ -\ B\ A\ 5\ E \\ \hline \end{array}$$

(j)
$$\begin{array}{r} D\ E\ B\ A\ 7\ E \\ -\ C\ 1\ 0\ 5\ E\ D \\ \hline \end{array}$$

3.6 Multiplication of hexadecimal numbers

Multiplication of hexadecimal numbers is once again very very similar to the multiplication of decimal numbers. Though you may need to go a bit higher up your times tables than what you are used to, as instead of encountering multiples from 0 to 9, you can encounter multiples up to 15 times. Which means that, once you have finished this section, you can confidently say that you have reviewed long multiplication and even went beyond what you are expected to know in that regard.

As 16 is such an important number for the hexadecimal system, we will start with writing out all the 16 times tables you might need. You did this once before, so you'll fly through it this time!

Exercise 1 Work out the 16 times table in decimal numbers. Convert your results to hexadecimal numbers.

1. $1 \times 16 =$
2. $2 \times 16 =$
3. $3 \times 16 = 48_{dec} = 30_{hex}$
4. $4 \times 16 =$
5. $5 \times 16 =$
6. $6 \times 16 =$
7. $7 \times 16 =$
8. $8 \times 16 = 128_{dec} = 80_{hex}$
9. $9 \times 16 =$
10. $10 \times 16 =$
11. $11 \times 16 =$
12. $12 \times 16 =$
13. $13 \times 16 = 208_{dec} = D0_{hex}$
14. $14 \times 16 =$
15. $15 \times 16 =$
16. $16 \times 16 =$

Because the 16 times table is so important, the best thing you can do this very minute is check your answers and, if necessary, correct any mistakes.

We will look at one long multiplication in more detail, to get to know the system in hexadecimal and learn the trickier things that need some extra attention. For this purpose, we will multiply $C978A3 \times 5B2$.

We start by writing them in a way that makes it easier to see which digit you need to multiply with which.

$$\begin{array}{cccccc} C & 9 & 7 & 8 & A & 3 \\ \times & & & 5 & B & 2 \\ \hline \end{array}$$

To make it a bit more obvious which digits we are multiplying in each step, they will be written in bold (which means they are a bit thicker than the rest). Just like with decimal long multiplications, we start by multiplying the most right digits. We have $3 \times 2 = 6$, so we get

$$\begin{array}{cccccc} C & 9 & 7 & 8 & A & \mathbf{3} \\ \times & & & 5 & B & \mathbf{2} \\ \hline & & & & & 6 \end{array}$$

We move on to the digit on the left in our first number. We have $A \times 2 \stackrel{dec}{=} 10 \times 2 = 20 \stackrel{hex}{=} 14$, so we have to carry over 1 and we write down 4.

$$\begin{array}{cccccc} & & & 1 & & \\ C & 9 & 7 & 8 & \mathbf{A} & 3 \\ \times & & & 5 & B & \mathbf{2} \\ \hline & & & & 4 & 6 \end{array}$$

We move on to the next digit on the left. We have $8 \times 2 \stackrel{dec}{=} 16 \stackrel{hex}{=} 10$. We have to add the 1 we carried over to this so we get $10 + 1 = 11$, so we have to carry over 1 and we write down 1.

$$
\begin{array}{cccccc}
 & & 1 & 1 & & \\
C & 9 & 7 & \mathbf{8} & A & 3 \\
\times & & & 5 & B & \mathbf{2} \\
\hline
 & & & 1 & 4 & 6
\end{array}
$$

We move on to the next digit. We have $7 \times 2 \overset{dec}{=} 14 \overset{hex}{=} E$. We have to add the 1 we carried over to this so we get $E + 1 = F$, so we have nothing to carry over and we write down F.

$$
\begin{array}{cccccc}
 & & 1 & 1 & & \\
C & 9 & \mathbf{7} & 8 & A & 3 \\
\times & & & 5 & B & \mathbf{2} \\
\hline
 & & F & 1 & 4 & 6
\end{array}
$$

We move on to the next digit. We have $9 \times 2 \overset{dec}{=} 18 \overset{hex}{=} 12$. We have to carry over 1 and we write down 2.

$$
\begin{array}{cccccc}
1 & & 1 & 1 & & \\
C & \mathbf{9} & 7 & 8 & A & 3 \\
\times & & & 5 & B & \mathbf{2} \\
\hline
 & 2 & F & 1 & 4 & 6
\end{array}
$$

We move on to the next digit, the last one of our first number. We have $C \times 2 \overset{dec}{=} 12 \times 2 = 24 \overset{hex}{=} 18$. We have to add the 1 we carried over to this so we get $18 + 1 = 19$ and we write this down

$$
\begin{array}{ccccccc}
 & 1 & & 1 & 1 & & \\
 & \mathbf{C} & 9 & 7 & 8 & A & 3 \\
 & \times & & & 5 & B & \mathbf{2} \\
\hline
1 & 9 & 2 & F & 1 & 4 & 6
\end{array}
$$

We will now start multiplying with the second digit of the second number, so B. Before we do anything else, we write down 0.

$$
\begin{array}{ccccccc}
 & 1 & & 1 & 1 & & \\
 & C & 9 & 7 & 8 & A & 3 \\
 & \times & & & 5 & \mathbf{B} & 2 \\
\hline
1 & 9 & 2 & F & 1 & 4 & 6 \\
 & & & & & & 0
\end{array}
$$

Once again we start multiplying with the right digit of the first number. Every time that the 16 times table is used to convert a decimal number to a hexadecimal one, the consulted multiplication is written down. This might give you an idea of how useful it is to write down the 16 times table and keep it close.

We have $3 \times B \overset{dec}{=} 3 \times 11 = 33 \overset{hex}{=} 21$, because $2 \times 16 = 32$. We have to carry over 2 and we write down 1. To avoid confusion, the carried over digits that stem from multiplying with the second digit are placed on a new line, above the carried over digits we encountered before.

$$\begin{array}{ccccccc}
 & & & & & 2 & \\
 & 1 & & 1 & 1 & & \\
 & C & 9 & 7 & 8 & A & \mathbf{3} \\
 & \times & & & 5 & \mathbf{B} & 2 \\
\hline
1 & 9 & 2 & F & 1 & 4 & 6 \\
 & & & & & 1 & 0
\end{array}$$

We move on to the digit to the left of that. We have $A \times B \overset{dec}{=} 10 \times 11 = 110 \overset{hex}{=} 6E$, because $6 \times 16 = 96$. We have to add the 2 we carried over. We get $6E + 2 = 70$, so we have to carry over 7 and we write down 0.

$$\begin{array}{ccccccc}
 & & & & 7 & 2 & \\
 & 1 & & 1 & 1 & & \\
 & C & 9 & 7 & 8 & \mathbf{A} & 3 \\
 & \times & & & 5 & \mathbf{B} & 2 \\
\hline
1 & 9 & 2 & F & 1 & 4 & 6 \\
 & & & & 0 & 1 & 0
\end{array}$$

We move on to the digit to the left of that. We have $8 \times B \overset{dec}{=} 8 \times 11 = 88 \overset{hex}{=} 58$, because $5 \times 16 = 80$. We have to add the 7 we carried over. We get $58 + 7 = 5F$, so we have to carry over 5 and we write down F.

$$\begin{array}{ccccccc}
 & & & 5 & 7 & 2 & \\
 & 1 & & 1 & 1 & & \\
 & C & 9 & 7 & \mathbf{8} & A & 3 \\
 & \times & & & 5 & \mathbf{B} & 2 \\
\hline
1 & 9 & 2 & F & 1 & 4 & 6 \\
 & & & F & 0 & 1 & 0
\end{array}$$

We move on to the digit to the left of that. We have $7 \times B \overset{dec}{=} 7 \times 11 = 77 \overset{hex}{=} 4D$, because $4 \times 16 = 64$. We have to add the 5 we carried over. We get $4D + 5 = 52$, so we have to carry over 5 and we write down 2.

$$\begin{array}{ccccccc}
 & & 5 & 5 & 7 & 2 & \\
 & 1 & & 1 & 1 & & \\
 & C & 9 & \mathbf{7} & 8 & A & 3 \\
 & \times & & & 5 & \mathbf{B} & 2 \\
\hline
1 & 9 & 2 & F & 1 & 4 & 6 \\
 & & 2 & F & 0 & 1 & 0
\end{array}$$

We move on to the digit to the left of that. We have $9 \times B \overset{dec}{=} 9 \times 11 = 99 \overset{hex}{=} 63$, because $6 \times 16 = 96$. We have to add the 5 we carried over. We get $63 + 5 = 68$, so we have to carry over 6 and we write down 8.

$$
\begin{array}{ccccccc}
 & 6 & 5 & 5 & 7 & 2 & \\
 & 1 & & 1 & 1 & & \\
 & C & \mathbf{9} & 7 & 8 & A & 3 \\
 & \times & & & 5 & \mathbf{B} & 2 \\
\hline
1 & 9 & 2 & F & 1 & 4 & 6 \\
 & 8 & 2 & F & 0 & 1 & 0
\end{array}
$$

We move on to the digit to the left of that, the last one of our first number. We have $C \times B \overset{dec}{=} 12 \times 11 = 132 \overset{hex}{=} 84$, because $8 \times 16 = 128$. We have to add the 6 we carried over. We get $84 + 6 = 8A$ and we write this down.

$$
\begin{array}{cccccccc}
 & & 6 & 5 & 5 & 7 & 2 & \\
 & & 1 & & 1 & 1 & & \\
 & & \mathbf{C} & 9 & 7 & 8 & A & 3 \\
 & & \times & & & 5 & \mathbf{B} & 2 \\
\hline
 & 1 & 9 & 2 & F & 1 & 4 & 6 \\
8 & A & 8 & 2 & F & 0 & 1 & 0
\end{array}
$$

We will now start multiplying with the third digit of our second number, 5. Before we do anything else, we write down 00.

$$
\begin{array}{cccccccc}
 & & 6 & 5 & 5 & 7 & 2 & \\
 & & 1 & & 1 & 1 & & \\
 & & C & 9 & 7 & 8 & A & 3 \\
 & & \times & & & \mathbf{5} & B & 2 \\
\hline
 & 1 & 9 & 2 & F & 1 & 4 & 6 \\
8 & A & 8 & 2 & F & 0 & 1 & 0 \\
 & & & & & & 0 & 0
\end{array}
$$

We start multiplying with the right digit of the first number. We get $3 \times 5 \overset{dec}{=} 15 \overset{hex}{=} F$. We have nothing to carry over and we write down F.

$$
\begin{array}{cccccccc}
 & & 6 & 5 & 5 & 7 & 2 & \\
 & & 1 & & 1 & 1 & & \\
 & & C & 9 & 7 & 8 & A & \mathbf{3} \\
 & & \times & & & \mathbf{5} & B & 2 \\
\hline
 & 1 & 9 & 2 & F & 1 & 4 & 6 \\
8 & A & 8 & 2 & F & 0 & 1 & 0 \\
 & & & & & F & 0 & 0
\end{array}
$$

We move on to the digit on the left. We get $A \times 5 \overset{dec}{=} 10 \times 5 = 50 \overset{hex}{=} 32$, because $3 \times 16 = 48$. We have to carry over 3 and we write down 2. To avoid confusion, the carried over digits that come from multiplying with 5 also get a new line.

$$\begin{array}{cccccccc}
 & & & & & 3 & & \\
 & & 6 & 5 & 5 & 7 & 2 & \\
 & & 1 & & 1 & 1 & & \\
 & & C & 9 & 7 & 8 & \mathbf{A} & 3 \\
 & & \times & & & \mathbf{5} & B & 2 \\
\hline
 & 1 & 9 & 2 & F & 1 & 4 & 6 \\
8 & A & 8 & 2 & F & 0 & 1 & 0 \\
 & & & & 2 & F & 0 & 0
\end{array}$$

We move on to the digit on the left. We get $8 \times 5 \overset{dec}{=} 40 \overset{hex}{=} 28$, because $2 \times 16 = 32$. We have add the carried over 3 so we get $28 + 3 = 2B$. We have to carry over 2 and we write down B.

$$\begin{array}{cccccccc}
 & & & & 2 & 3 & & \\
 & & 6 & 5 & 5 & 7 & 2 & \\
 & & 1 & & 1 & 1 & & \\
 & & C & 9 & 7 & \mathbf{8} & A & 3 \\
 & & \times & & & \mathbf{5} & B & 2 \\
\hline
 & 1 & 9 & 2 & F & 1 & 4 & 6 \\
8 & A & 8 & 2 & F & 0 & 1 & 0 \\
 & & & B & 2 & F & 0 & 0
\end{array}$$

We move on to the digit on the left. We get $7 \times 5 \overset{dec}{=} 35 \overset{hex}{=} 23$, because $2 \times 16 = 32$. We have add the carried over 2 so we get $23 + 2 = 25$. We have to carry over 2 and we write down 5.

$$\begin{array}{cccccccc}
 & & & 2 & 2 & 3 & & \\
 & & 6 & 5 & 5 & 7 & 2 & \\
 & & 1 & & 1 & 1 & & \\
 & & C & 9 & \mathbf{7} & 8 & A & 3 \\
 & & \times & & & \mathbf{5} & B & 2 \\
\hline
 & 1 & 9 & 2 & F & 1 & 4 & 6 \\
8 & A & 8 & 2 & F & 0 & 1 & 0 \\
 & & 5 & B & 2 & F & 0 & 0
\end{array}$$

We move on to the digit on the left. We get $9 \times 5 \overset{dec}{=} 45 \overset{hex}{=} 2D$, because $2 \times 16 = 32$. We have add the carried over 2 so we get $2D + 2 = 2F$. We have to carry over 2 and we write down F.

$$
\begin{array}{cccccccc}
 & & 2 & 2 & 2 & 3 & & \\
 & & 6 & 5 & 5 & 7 & 2 & \\
 & & 1 & & 1 & 1 & & \\
 & & C & \mathbf{9} & 7 & 8 & A & 3 \\
 & & \times & & & \mathbf{5} & B & 2 \\
\hline
 & 1 & 9 & 2 & F & 1 & 4 & 6 \\
8 & A & 8 & 2 & F & 0 & 1 & 0 \\
 & F & 5 & B & 2 & F & 0 & 0
\end{array}
$$

We move on to the very last digits we have to multiply. We get $C \times 5 \overset{dec}{=} 12 \times 5 = 60 \overset{hex}{=} 3C$, because $3 \times 16 = 48$. We have add the carried over 2 so we get $3C + 2 = 3E$. We write this down.

$$
\begin{array}{ccccccccc}
 & & & 2 & 2 & 2 & 3 & & \\
 & & & 6 & 5 & 5 & 7 & 2 & \\
 & & & 1 & & 1 & 1 & & \\
 & & & C & 9 & 7 & 8 & A & 3 \\
 & & & \times & & & \mathbf{5} & B & 2 \\
\hline
 & & 1 & 9 & 2 & F & 1 & 4 & 6 \\
 & 8 & A & 8 & 2 & F & 0 & 1 & 0 \\
3 & E & F & 5 & B & 2 & F & 0 & 0
\end{array}
$$

Now that we have done all multiplications, we have to add the numbers to get the final result. We've cleaned up what we've found this far and added some space for the digits that get carried over by addition.

$$
\begin{array}{cccccccccc}
 & & & & C & 9 & 7 & 8 & A & 3 \\
 & & & & \times & & & 5 & B & 2 \\
\hline
 & & & & & & & & & \\
 & & & 1 & 9 & 2 & F & 1 & 4 & 6 \\
 & & 8 & A & 8 & 2 & F & 0 & 1 & 0 \\
+ & 3 & E & F & 5 & B & 2 & F & 0 & 0 \\
\hline
\end{array}
$$

We start by adding the digits on the right. We get $6 + 0 + 0 = 6$, so we write down 6. We get

$$
\begin{array}{cccccccccc}
 & & & & C & 9 & 7 & 8 & A & 3 \\
 & & & & \times & & & 5 & B & 2 \\
\hline
 & & & & & & & & & \\
 & & & 1 & 9 & 2 & F & 1 & 4 & 6 \\
 & & 8 & A & 8 & 2 & F & 0 & 1 & 0 \\
+ & 3 & E & F & 5 & B & 2 & F & 0 & 0 \\
\hline
 & & & & & & & & & 6
\end{array}
$$

We move on to the digits on the left of them. We get $4 + 1 + 0 = 5$, so we write down 5. We get

$$\begin{array}{cccccccccc}
 & & & & C & 9 & 7 & 8 & A & 3 \\
 & & & & \times & & & 5 & B & 2 \\
\hline
 & & & & & & & & & \\
 & & & 1 & 9 & 2 & F & 1 & 4 & 6 \\
 & & 8 & A & 8 & 2 & F & 0 & 1 & 0 \\
+ & 3 & E & F & 5 & B & 2 & F & 0 & 0 \\
\hline
 & & & & & & & & 5 & 6
\end{array}$$

We move on to the digits on the left of them. We get $1 + 0 + F = 10$, so we carry over 1 and we write down 0. We get

$$\begin{array}{cccccccccc}
 & & & & C & 9 & 7 & 8 & A & 3 \\
 & & & & \times & & & 5 & B & 2 \\
\hline
 & & & & & & 1 & & & \\
 & & & 1 & 9 & 2 & F & 1 & 4 & 6 \\
 & & 8 & A & 8 & 2 & F & 0 & 1 & 0 \\
+ & 3 & E & F & 5 & B & 2 & F & 0 & 0 \\
\hline
 & & & & & & & 0 & 5 & 6
\end{array}$$

We move on to the digits on the left of them. We get $1 + F + F + 2 = 21$, so we carry over 2 and we write down 1. We get

$$\begin{array}{cccccccccc}
 & & & & C & 9 & 7 & 8 & A & 3 \\
 & & & & \times & & & 5 & B & 2 \\
\hline
 & & & & & 2 & 1 & & & \\
 & & & 1 & 9 & 2 & F & 1 & 4 & 6 \\
 & & 8 & A & 8 & 2 & F & 0 & 1 & 0 \\
+ & 3 & E & F & 5 & B & 2 & F & 0 & 0 \\
\hline
 & & & & & & 1 & 0 & 5 & 6
\end{array}$$

We move on to the digits on the left of them. We get $2 + 2 + 2 + B = 11$, so we carry over 1 and we write down 1. We get

$$\begin{array}{cccccccccc}
 & & & & C & 9 & 7 & 8 & A & 3 \\
 & & & & \times & & & 5 & B & 2 \\
\hline
 & & & & 1 & 2 & 1 & & & \\
 & & & 1 & 9 & 2 & F & 1 & 4 & 6 \\
 & & 8 & A & 8 & 2 & F & 0 & 1 & 0 \\
+ & 3 & E & F & 5 & B & 2 & F & 0 & 0 \\
\hline
 & & & & & 1 & 1 & 0 & 5 & 6
\end{array}$$

We move on to the digits on the left of them. We get $1 + 9 + 8 + 5 \stackrel{dec}{=} 23 \stackrel{hex}{=} 17$, so we carry over 1 and we write down 7. We get

$$\begin{array}{cccccccccc}
 & & & & C & 9 & 7 & 8 & A & 3 \\
 & & & & \times & & & 5 & B & 2 \\
\hline
 & & & 1 & 1 & 2 & 1 & & & \\
 & & & 1 & 9 & 2 & F & 1 & 4 & 6 \\
 & & 8 & A & 8 & 2 & F & 0 & 1 & 0 \\
+ & 3 & E & F & 5 & B & 2 & F & 0 & 0 \\
\hline
 & & & & 7 & 1 & 1 & 0 & 5 & 6
\end{array}$$

We move on to the digits on the left of them. We get $1+1+A+F \overset{dec}{=} 1+1+10+15 = 27 \overset{hex}{=} 1B$, so we carry over 1 and we write down B. We get

$$\begin{array}{cccccccccc}
 & & & & C & 9 & 7 & 8 & A & 3 \\
 & & & & \times & & & 5 & B & 2 \\
\hline
 & & 1 & 1 & 1 & 2 & 1 & & & \\
 & & & 1 & 9 & 2 & F & 1 & 4 & 6 \\
 & & 8 & A & 8 & 2 & F & 0 & 1 & 0 \\
+ & 3 & E & F & 5 & B & 2 & F & 0 & 0 \\
\hline
 & & & B & 7 & 1 & 1 & 0 & 5 & 6
\end{array}$$

We move on to the digits on the left of them. We get $1+8+E \overset{dec}{=} 1+8+14 = 23 \overset{hex}{=} 17$, so we carry over 1 and we write down 7. We get

$$\begin{array}{cccccccccc}
 & & & & C & 9 & 7 & 8 & A & 3 \\
 & & & & \times & & & 5 & B & 2 \\
\hline
 & 1 & 1 & 1 & 1 & 2 & 1 & & & \\
 & & & 1 & 9 & 2 & F & 1 & 4 & 6 \\
 & & 8 & A & 8 & 2 & F & 0 & 1 & 0 \\
+ & 3 & E & F & 5 & B & 2 & F & 0 & 0 \\
\hline
 & & 7 & B & 7 & 1 & 1 & 0 & 5 & 6
\end{array}$$

We move on to the last digits we have to add. We get $1+3=4$, so we write down 4. We get

$$\begin{array}{cccccccccc}
 & & & & C & 9 & 7 & 8 & A & 3 \\
 & & & & \times & & & 5 & B & 2 \\
\hline
 & 1 & 1 & 1 & 1 & 2 & 1 & & & \\
 & & & 1 & 9 & 2 & F & 1 & 4 & 6 \\
 & & 8 & A & 8 & 2 & F & 0 & 1 & 0 \\
+ & 3 & E & F & 5 & B & 2 & F & 0 & 0 \\
\hline
 & 4 & 7 & B & 7 & 1 & 1 & 0 & 5 & 6
\end{array}$$

We have found that $C978A3 \times 5B2 = 47B711056$. If you convert these hexadecimal numbers to decimal, you would find that this multiplication is the equivalent of

$$13,203,619 \times 1,458 = 19,250,876,502.$$

Fortunately for us, you get the same result if you check this with a calculator, so we have not made any mistakes along the way. Yay us!

Exercises Pro tip: the piece of paper that has the decimal values of A to F still comes in handy here. On top of that, the 16 times table is really handy to consult whenever you have to convert numbers to hexadecimal and to be sure that you are carrying over the right digit. Writing it down as well will be worth the effort. Writing down some powers of 16 might be useful as well.

16^4	16^3	16^2	16^1	16^0
$65,536$	$4,096$	256	16	1

2. Before we do anything else, we'll review our times tables and convert all our decimal results to hexadecimal numbers. You can use the 16 times table from earlier on to convert the numbers. Doing this work now will save us a lot of time later on.
   We start with the 2 times table. Make sure that you write down both the decimal and hexadecimal results. You don't have to write $_{dec}$ and $_{hex}$ after every number, but make sure you write everything down in a very orderly way so that it's very clear which is which.

   (a) $1 \times 2 =$
   (b) $2 \times 2 =$
   (c) $3 \times 2 = 6_{dec} = 6_{hex}$
   (d) $4 \times 2 =$
   (e) $5 \times 2 =$
   (f) $6 \times 2 =$
   (g) $7 \times 2 =$
   (h) $8 \times 2 = 16_{dec} = 10_{hex}$
   (i) $9 \times 2 =$
   (j) $10 \times 2 =$
   (k) $11 \times 2 =$
   (l) $12 \times 2 =$
   (m) $13 \times 2 = 26_{dec} = 1A_{hex}$
   (n) $14 \times 2 =$
   (o) $15 \times 2 =$
   (p) $16 \times 2 =$

3. The 3 times table is next.

   (a) $1 \times 3 =$
   (b) $2 \times 3 =$
   (c) $3 \times 3 = 9_{dec} = 9_{hex}$
   (d) $4 \times 3 =$
   (e) $5 \times 3 =$
   (f) $6 \times 3 =$
   (g) $7 \times 3 =$
   (h) $8 \times 3 = 24_{dec} = 18_{hex}$
   (i) $9 \times 3 =$
   (j) $10 \times 3 =$
   (k) $11 \times 3 =$
   (l) $12 \times 3 =$
   (m) $13 \times 3 = 39_{dec} = 27_{hex}$
   (n) $14 \times 3 =$
   (o) $15 \times 3 =$
   (p) $16 \times 3 =$

4. Do the 4 times table now.

   (a) $1 \times 4 =$
   (b) $2 \times 4 =$
   (c) $3 \times 4 = 12_{dec} = C_{hex}$
   (d) $4 \times 4 =$
   (e) $5 \times 4 =$
   (f) $6 \times 4 =$
   (g) $7 \times 4 =$
   (h) $8 \times 4 = 32_{dec} = 20_{hex}$
   (i) $9 \times 4 =$
   (j) $10 \times 4 =$
   (k) $11 \times 4 =$
   (l) $12 \times 4 =$
   (m) $13 \times 4 = 52_{dec} = 34_{hex}$
   (n) $14 \times 4 =$
   (o) $15 \times 4 =$
   (p) $16 \times 4 =$

5. Next up is the 5 times table.

   (a) $1 \times 5 =$
   (b) $2 \times 5 =$
   (c) $3 \times 5 = 15_{dec} = F_{hex}$
   (d) $4 \times 5 =$
   (e) $5 \times 5 =$
   (f) $6 \times 5 =$
   (g) $7 \times 5 =$
   (h) $8 \times 5 = 40_{dec} = 28_{hex}$
   (i) $9 \times 5 =$
   (j) $10 \times 5 =$
   (k) $11 \times 5 =$
   (l) $12 \times 5 =$
   (m) $13 \times 5 = 65_{dec} = 41_{hex}$
   (n) $14 \times 5 =$
   (o) $15 \times 5 =$
   (p) $16 \times 5 =$

6. The 6 times table is next. Can you find the decimal 6 times tables up to $10 \times 6 = 60$ in your hexadecimal results?

   (a) $1 \times 6 =$
   (b) $2 \times 6 =$
   (c) $3 \times 6 = 18_{dec} = 12_{hex}$
   (d) $4 \times 6 =$
   (e) $5 \times 6 =$
   (f) $6 \times 6 =$
   (g) $7 \times 6 =$
   (h) $8 \times 6 = 48_{dec} = 30_{hex}$
   (i) $9 \times 6 =$
   (j) $10 \times 6 =$
   (k) $11 \times 6 =$
   (l) $12 \times 6 =$
   (m) $13 \times 6 = 78_{dec} = 4E_{hex}$
   (n) $14 \times 6 =$
   (o) $15 \times 6 =$
   (p) $16 \times 6 =$

7. Let's do the 7 times table.

(a) $1 \times 7 =$
(b) $2 \times 7 =$
(c) $3 \times 7 = 21_{dec} = 15_{hex}$
(d) $4 \times 7 =$
(e) $5 \times 7 =$
(f) $6 \times 7 =$
(g) $7 \times 7 =$
(h) $8 \times 7 = 56_{dec} = 38_{hex}$
(i) $9 \times 7 =$
(j) $10 \times 7 =$
(k) $11 \times 7 =$
(l) $12 \times 7 =$
(m) $13 \times 7 = 91_{dec} = 5B_{hex}$
(n) $14 \times 7 =$
(o) $15 \times 7 =$
(p) $16 \times 7 =$

8. At this point you are over halfway with the times tables. Next up is the 8 times table, which has a pretty easy conversion from decimal to hexadecimal.

(a) $1 \times 8 =$
(b) $2 \times 8 =$
(c) $3 \times 8 = 24_{dec} = 18_{hex}$
(d) $4 \times 8 =$
(e) $5 \times 8 =$
(f) $6 \times 8 =$
(g) $7 \times 8 =$
(h) $8 \times 8 = 64_{dec} = 40_{hex}$
(i) $9 \times 8 =$
(j) $10 \times 8 =$
(k) $11 \times 8 =$
(l) $12 \times 8 =$
(m) $13 \times 8 = 104_{dec} = 68_{hex}$
(n) $14 \times 8 =$
(o) $15 \times 8 =$
(p) $16 \times 8 =$

9. The 9 times table is next.

(a) $1 \times 9 =$
(b) $2 \times 9 =$
(c) $3 \times 9 = 27_{dec} = 1B_{hex}$
(d) $4 \times 9 =$
(e) $5 \times 9 =$
(f) $6 \times 9 =$
(g) $7 \times 9 =$
(h) $8 \times 9 = 72_{dec} = 48_{hex}$
(i) $9 \times 9 =$
(j) $10 \times 9 =$
(k) $11 \times 9 =$
(l) $12 \times 9 =$
(m) $13 \times 9 = 117_{dec} = 75_{hex}$
(n) $14 \times 9 =$
(o) $15 \times 9 =$
(p) $16 \times 9 =$

10. The 10 times table is next. At least finding you decimal answers is easy!

(a) $1 \times 10 =$
(b) $2 \times 10 =$
(c) $3 \times 10 = 30_{dec} = 1E_{hex}$
(d) $4 \times 10 =$
(e) $5 \times 10 =$
(f) $6 \times 10 =$
(g) $7 \times 10 =$
(h) $8 \times 10 = 80_{dec} = 50_{hex}$
(i) $9 \times 10 =$
(j) $10 \times 10 =$
(k) $11 \times 10 =$
(l) $12 \times 10 =$
(m) $13 \times 10 = 130_{dec} = 82_{hex}$
(n) $14 \times 10 =$
(o) $15 \times 10 =$
(p) $16 \times 10 =$

11. Next up is the 11 times table. Do you notice any patterns in your decimal results?

(a) $1 \times 11 =$
(b) $2 \times 11 =$
(c) $3 \times 11 = 33_{dec} = 21_{hex}$
(d) $4 \times 11 =$
(e) $5 \times 11 =$
(f) $6 \times 11 =$
(g) $7 \times 11 =$
(h) $8 \times 11 = 88_{dec} = 58_{hex}$
(i) $9 \times 11 =$
(j) $10 \times 11 =$
(k) $11 \times 11 =$
(l) $12 \times 11 =$
(m) $13 \times 11 = 143_{dec} = 8F_{hex}$
(n) $14 \times 11 =$
(o) $15 \times 11 =$
(p) $16 \times 11 =$

12. The 12 times table is next.

(a) $1 \times 12 =$
(b) $2 \times 12 =$
(c) $3 \times 12 = 36_{dec} = 24_{hex}$
(d) $4 \times 12 =$
(e) $5 \times 12 =$
(f) $6 \times 12 =$
(g) $7 \times 12 =$
(h) $8 \times 12 = 96_{dec} = 60_{hex}$
(i) $9 \times 12 =$
(j) $10 \times 12 =$
(k) $11 \times 12 =$
(l) $12 \times 12 =$
(m) $13 \times 12 = 156_{dec} = 9C_{hex}$
(n) $14 \times 12 =$
(o) $15 \times 12 =$
(p) $16 \times 12 =$

13. The 13 times table is next.

(a) $1 \times 13 =$
(b) $2 \times 13 =$
(c) $3 \times 13 = 39_{dec} = 27_{hex}$
(d) $4 \times 13 =$

(e) $5 \times 13 =$
(f) $6 \times 13 =$
(g) $7 \times 13 =$
(h) $8 \times 13 = 104_{dec} = 68_{hex}$
(i) $9 \times 13 =$
(j) $10 \times 13 =$
(k) $11 \times 13 =$
(l) $12 \times 13 =$
(m) $13 \times 13 = 169_{dec} = A9_{hex}$
(n) $14 \times 13 =$
(o) $15 \times 13 =$
(p) $16 \times 13 =$

14. The second to last times table you need is the 14 times table.

(a) $1 \times 14 =$
(b) $2 \times 14 =$
(c) $3 \times 14 = 42_{dec} = 2A_{hex}$
(d) $4 \times 14 =$
(e) $5 \times 14 =$
(f) $6 \times 14 =$
(g) $7 \times 14 =$
(h) $8 \times 14 = 112_{dec} = 70_{hex}$
(i) $9 \times 14 =$
(j) $10 \times 14 =$
(k) $11 \times 14 =$
(l) $12 \times 14 =$
(m) $13 \times 14 = 182_{dec} = B6_{hex}$
(n) $14 \times 14 =$
(o) $15 \times 14 =$
(p) $16 \times 14 =$

15. The last times table we'll do is the 15 times table.

(a) $1 \times 15 =$
(b) $2 \times 15 =$
(c) $3 \times 15 = 45_{dec} = 2D_{hex}$
(d) $4 \times 15 =$
(e) $5 \times 15 =$
(f) $6 \times 15 =$
(g) $7 \times 15 =$
(h) $8 \times 15 = 120_{dec} = 78_{hex}$
(i) $9 \times 15 =$
(j) $10 \times 15 =$
(k) $11 \times 15 =$
(l) $12 \times 15 =$
(m) $13 \times 15 = 195_{dec} = C3_{hex}$
(n) $14 \times 15 =$
(o) $15 \times 15 =$
(p) $16 \times 15 =$

16. Now that you have reviewed all your times tables, we'll start multiplying hexadecimal numbers. Be very careful when you convert numbers from decimal to hexadecimal and the other way around.

(a) $8 \times 8 =$
(b) $9 \times 9 =$
(c) $A \times A =$
(d) $B \times B =$
(e) $C \times C =$
(f) $D \times D =$
(g) $E \times E =$
(h) $F \times F =$
(i) $A \times F =$

(j) $B \times C =$

(k) $C \times A =$

(l) $D \times E =$

(m) $E \times B =$

(n) $F \times C =$

(o) $A \times D =$

(p) $B \times F =$

(q) $C \times E =$

(r) $D \times B =$

(s) $E \times A =$

(t) $F \times D =$

(u) $7 \times F =$

(v) $A \times B =$

(w) $C \times D =$

(x) $E \times F =$

17. Now it's time to multiply numbers with more than one digit. Be mindful of conversions and of the digits you carry over! For example, when you calculate 37×5, you start with $7 \times 5 \overset{dec}{=} 35 \overset{hex}{=} 23$, so you'd write down 3 and carry over 2. You continue to $3 \times 5 \overset{dec}{=} 15 \overset{hex}{=} F$¿ We add the carried over 2 to this and get $F + 2 = 11$ and we write this down. When you are working with hexadecimal numbers, $37 \times 5 = 113$, which is not at all the same as the decimal version $37 \times 5 = 185$.

    If you want, you can consult the times tables you worked so hard on to help you with this exercise and the next ones. Remember that making mistakes is a sign of you being challenged and working hard at mastering this completely new skill. Good luck!

    (a) $17 \times 5 =$

    (b) $23 \times 8 =$

    (c) $46 \times A =$

    (d) $67 \times C =$

    (e) $9B \times D =$

    (f) $AB \times F =$

    (g) $C4 \times E =$

    (h) $8E \times 9 =$

    (i) $FF \times 2 =$

    (j) $5D \times B =$

    (k) $123 \times F =$

    (l) $456 \times 6 =$

    (m) $789 \times 7 =$

    (n) $ABC \times A =$

    (o) $DEF \times 4 =$

18. And now, finally, the thing that we've all been waiting for... drumroll... long multiplications with hexadecimal numbers!

    (a)

$$\begin{array}{cccccccc} & & & 2 & 4 & 6 & 8 & A \\ & & & \times & & & 3 & 7 \\ \hline \\ \\ + & & & & & & & \\ \hline \end{array}$$

    (b)

$$\begin{array}{cccccccc} & & & 9 & 1 & 5 & 3 & 7 \\ & & & \times & & & A & 4 \\ \hline \\ \\ + & & & & & & & \\ \hline \end{array}$$

    (c)

$$\begin{array}{cccccccc} & & & A & 1 & 1 & E & 5 \\ & & & \times & & & 9 & C \\ \hline \\ \\ + & & & & & & & \\ \hline \end{array}$$

    (d)

$$\begin{array}{cccccccc} & & & F & 1 & 0 & A & 7 \\ & & & \times & & & D & 8 \\ \hline \\ \\ + & & & & & & & \\ \hline \end{array}$$

    (e)

$$\begin{array}{r} F\ 1\ 0\ 4\ A \\ \times\qquad 2\ 5 \\ \hline \\ + \qquad\qquad\quad \\ \hline \end{array}$$

(f)

$$\begin{array}{r} B\ 2\ E\ 5\ F \\ \times\qquad 9\ 6 \\ \hline \\ + \qquad\qquad\quad \\ \hline \end{array}$$

(g)

$$\begin{array}{r} F\ A\ B\ 1\ E \\ \times\qquad 1\ C \\ \hline \\ + \qquad\qquad\quad \\ \hline \end{array}$$

(h)

$$\begin{array}{r} 6\ 4\ 8\ 8\ 5 \\ \times\qquad 9\ 3 \\ \hline \\ + \qquad\qquad\quad \\ \hline \end{array}$$

(i)

$$\begin{array}{r} 7\ 3\ 0\ 2\ 1 \\ \times\qquad 7\ 2 \\ \hline \\ + \qquad\qquad\quad \\ \hline \end{array}$$

(j)

$$\begin{array}{r} B\ A\ D\ 6\ 3 \\ \times\qquad 6\ E \\ \hline \\ + \qquad\qquad\quad \\ \hline \end{array}$$

(k)

$$\begin{array}{r} 2\ 5\ 1\ 6\ 4 \\ \times\qquad F\ F \\ \hline \\ + \qquad\qquad\quad \\ \hline \end{array}$$

(l)

$$\begin{array}{r} B\ 1\ E\ 5\ 5 \\ \times\qquad 3\ 9 \\ \hline \\ + \qquad\qquad\quad \\ \hline \end{array}$$

(m)

$$\begin{array}{r} C\ 0\ F\ F\ E\ E \\ \times\qquad 2\ 1 \\ \hline \\ + \qquad\qquad\quad \\ \hline \end{array}$$

(n)

$$\begin{array}{r} D\ E\ B\ A\ 7\ E \\ \times\qquad 5\ 4 \\ \hline \\ + \qquad\qquad\quad \\ \hline \end{array}$$

(o)

$$\begin{array}{r} C\ 0\ 0\ 1\ E\ 5\ 7 \\ \times\qquad F\ A \\ \hline \\ + \qquad\qquad\quad \\ \hline \end{array}$$

(p)

$$\begin{array}{r} F\ E\ E\ 7\ 1\ E \\ \times\qquad 3\ 2 \\ \hline \\ + \qquad\qquad\quad \\ \hline \end{array}$$

(q)

$$\begin{array}{r} C\ 0\ B\ B\ 1\ E\ 5 \\ \times \qquad\quad 3\ 8 \\ \hline \\ + \qquad\qquad\qquad \\ \hline \end{array}$$

(r)

$$\begin{array}{r} 6\ 1\ A\ 5\ 5\ E\ 5 \\ \times \qquad\quad C\ 3 \\ \hline \\ + \qquad\qquad\qquad \\ \hline \end{array}$$

(s)

$$\begin{array}{r} 2\ 3\ 5\ 6\ 8\ 9\ B\ C \\ \times \qquad\quad 6\ 2 \\ \hline \\ + \qquad\qquad\qquad \\ \hline \end{array}$$

(t)

$$\begin{array}{r} F\ 5\ B\ 2\ 0\ 3\ A \\ \times \qquad\quad A\ A \\ \hline \\ + \qquad\qquad\qquad \\ \hline \end{array}$$

19. And now some really long long multiplications to end this part with. Don't expect perfection from yourself, I made several mistakes while working on these exercises as well. Figuring out where I had gone wrong was the hardest part of typing out the solutions.

(a)

$$\begin{array}{r} 7\ A\ 5\ 7\ E \\ \times\ F\ 0\ 0\ D \\ \hline \\ + \qquad\qquad\qquad \\ \hline \end{array}$$

(b)

$$\begin{array}{r} 5\ 7\ 0\ F\ F\ 1\ 6\ E \\ \times \qquad F\ 0\ 7\ 0 \\ \hline \\ + \qquad\qquad\qquad \\ \hline \end{array}$$

(c)

$$\begin{array}{r} B\ A\ B\ 1\ E\ 5 \\ \times \qquad B\ E\ D \\ \hline \\ + \qquad\qquad\qquad \\ \hline \end{array}$$

(d)

$$\begin{array}{r} 6\ 1\ A\ D\ D\ E \\ \times \qquad 5\ 1\ A \\ \hline \\ + \qquad\qquad\qquad \\ \hline \end{array}$$

(e)

$$\begin{array}{r} C\ A\ 7\ 5 \\ \times\ D\ 0\ 6\ 5 \\ \hline \\ \\ + \\ \hline \end{array}$$

(f)

$$\begin{array}{r} B\ E\ E\ 7\ 1\ E \\ \times\ 7\ 0\ A\ D\ \\ \hline \\ \\ + \\ \hline \end{array}$$

(g)

$$\begin{array}{r} 5\ A\ 1\ A\ D \\ \times\ 1\ E\ A\ F \\ \hline \\ \\ + \\ \hline \end{array}$$

(h)

$$\begin{array}{r} B\ A\ 5\ E \\ \times\ B\ A\ 1\ 1 \\ \hline \\ \\ + \\ \hline \end{array}$$

(i)

$$\begin{array}{r} C\ 0\ 1\ D\ E\ 5\ 7 \\ \times\ 7\ A\ B\ 1\ E\ 7 \\ \hline \\ \\ + \\ \hline \end{array}$$

(j)

$$\begin{array}{r} C\ A\ 5\ 7\ 1\ E \\ \times\ C\ 1\ 7\ 1\ E\ 5 \\ \hline \\ \\ + \\ \hline \end{array}$$

3.7 Division of hexadecimal numbers

Division of hexadecimal numbers is, once again, very similar to division of decimal numbers. When we were working with binary numbers, we had no choice but to do long divisions.

Here, we'll do the opposite. We'll stick with exercises where you divide by numbers that contain only one digit. We'll look at one example in more detail: $57AB8F \div 9$. We start by writing it down as usual.

$$
\begin{array}{rl}
9 & \overline{)\ 57AB8F}
\end{array}
$$

We start with the first digit on the left. $5 < 9$, so we have to look at the first two digits together, 57. We have to be mindful that we are working with hexadecimal numbers. We can consult the times tables from before, or convert the number. We'll take turns between these two methods. Make sure that you checked your answers if you use your times tables!

We get $57_{hex} = 87_{dec}$. Division by 9 gives us $87 \div 9 = 9\ R6$. To write this down in our long division, we will split the number: $87 = 81 + 6$. Converting this to hexadecimal gives us $57 \div 9 = (51 + 6) \div 9 = 9\ R6$. We now get

$$
\begin{array}{rl}
 & \ \ 9 \\
9 & \overline{)\ 57AB8F} \\
 & \underline{-51} \\
 & \quad\ 6
\end{array}
$$

After subtraction we get 6, which is smaller than 9. So we add another digit, so we get $6A$, which is bigger. This time we will consult the 9 times table to determine the result of the division. We see that 63 and $6C$ are the multiples of 9 closest to $6A$. As $6C > 6A$, we will subtract $63 = B \times 9$. We get

$$
\begin{array}{rl}
 & \ \ 9B \\
9 & \overline{)\ 57AB8F} \\
 & \underline{-51} \\
 & \quad\ 6A \\
 & \quad \underline{-63} \\
 & \qquad 7
\end{array}
$$

After subtraction we get 7, which is smaller than 9. So we add another digit, so we get $7B$, which is bigger. We consult the 9 times tables and see that $75 = D \times 9$ is the biggest multiple of 9 that is smaller than $7B$. We get

$$
\begin{array}{rl}
 & \ \ 9BD \\
9 & \overline{)\ 57AB8F} \\
 & \underline{-51} \\
 & \quad\ 6A \\
 & \quad \underline{-63} \\
 & \qquad 7B \\
 & \qquad \underline{-75} \\
 & \qquad\quad 6
\end{array}
$$

After subtraction we get 6, which is smaller than 9. We add another digit and get 68. We convert this to decimal and get $68_{hex} = 104_{dec}$. As $104 \div 9 = 11\ R5$, we split the number as $104 = 99 + 5$. In hexadecimal numbers this becomes $68 \div 9 = (63 + 5) \div 9 = B\ R5$. We get

$$
\begin{array}{l}
\underline{9BDB} \\
9\ )\ 57AB8F \\
\underline{-51} \\
6A \\
\underline{-63} \\
7B \\
\underline{-75} \\
68 \\
\underline{-63} \\
5
\end{array}
$$

After subtraction we get 5, which is smaller than 9. So we add the last digit and get $5F$. We consult the 9 times table and see that $5A = A \times 9$ is the biggest multiple of 9 that fits. We get

$$
\begin{array}{l}
\underline{9BDB9} \\
9\ )\ 57AB8F \\
\underline{-51} \\
6A \\
\underline{-63} \\
7B \\
\underline{-75} \\
68 \\
\underline{-63} \\
5F \\
\underline{-5\mathrm{A}} \\
5
\end{array}
$$

We get $57AB8F \div 9 = 9BDB9\ R5$.

A very short way of writing all the steps of this division is

$$
\begin{array}{l}
9\ \ )\underline{\ 57^6A^7B^68^5F} \\
9\,B\,D\ B\ 9\ R5
\end{array}
$$

This type of notation will be used in the solutions.

Exercises Accept that you will make mistakes while working on these exercises. It is by making mistakes, understanding where we went wrong and correcting them that we learn. If you don't make any mistakes, that means that it's too easy for you and you're not being challenged. And I don't know about you, but I was completely focused while solving these exercises and I still made several mistakes. It sure was challenging for me!

1. Work out these divisions. Write down the remainder if there is any.

(a) $89356 \div 2 =$
(b) $F48E4 \div 3 =$
(c) $B10B \div 4 =$
(d) $FEEA \div 5 =$
(e) $60EDE \div 6 =$
(f) $87654 \div 7 =$
(g) $51AA \div 8 =$
(h) $7839 \div 9 =$
(i) FEDCBA $\div 2 =$
(j) $7002 \div 3 =$
(k) $B1AB \div 4 =$
(l) $7826 \div 5 =$
(m) $9043 \div 6 =$
(n) $7DB1 \div 7 =$
(o) $2BB8 \div 8 =$
(p) $C1A3 \div 9 =$
(q) $DCCD \div 5 =$
(r) $B105 \div 3 =$

2. Work out these divisions.

(a) $1234 \div A =$
(b) $5678 \div B =$
(c) $5107 \div C =$
(d) $2348 \div D =$
(e) $9102 \div E =$
(f) $8067 \div F =$
(g) $ABBA \div A =$
(h) $2137 \div B =$
(i) $C1FE \div C =$
(j) $8372 \div D =$
(k) $C1A6 \div E =$
(l) $F1E5 \div F =$
(m) BABE50$\div A =$
(n) DEBA7E$\div B =$
(o) B1E55ED$\div C =$
(p) 383940$\div D =$
(q) CADE75$\div E =$
(r) F104A$\div F =$

3. Can you divide by powers of the decimal number 16, just for the fun of it? Everything is written in hexadecimal numbers, so $16_{dec} = 10_{hex}$, $16^2_{dec} = 256_{dec} = 100_{hex}$ etc. Write down the remainder if there is one.

(a) $ABC \div 10 =$
(b) $BEE7 \div 100 =$
(c) $87F4 \div 10 =$
(d) $57AB1E5 \div 100 =$
(e) $C0C0A0 \div 10 =$
(f) $B1AB1AB1A \div 1000 =$
(g) $ABBA \div 10 =$
(h) $B269017 \div 1000 =$
(i) $BABBE1 \div 100 =$
(j) $F10FE \div 100 =$
(k) $F10EF1E \div 10 =$
(l) $3489632DBC \div 10000 =$

4. Can you figure out how to do these long divisions?

(a) $8EDF \div 23 =$
(b) $2B8F \div 15 =$
(c) $1F46A \div 21 =$

3.8 Final words

> A neurologist sees a patient who can't seem to do basic maths anymore.
> "What is 9 plus 9?" "12".
> "What is 8 and 8?" "10".
> The doctor shook her head. "Very interesting. What about 6 times 5?"
> The patient thought for a second, and answered "1E".
> "Aha, I've figured it out!" The doctor said. "Somebody has clearly put a hex on you."

Give yourself a pat on the back, because once again you mastered concepts that you didn't know before and you did it in record time!

You even learned that, with some imagination, you are able to see hexadecimal numbers as actual words. It is crazy how flexible our brains are and what they are capable of.

Did you kn0w that if y0u replace a specific letter in w0rds by a number that vaguely resembles the 0ne y0u replace, that y0ur mind will adapt very quickly and you will still be able t0 c0ntinue reading? It's might seem 1ike a si11y thing t0 d0, but it 1eads y0u to c001 ways 0f writing. Did y0u even n0tice the second 1etter that g0t replaced? TH3R3 AR3 3V3N M0R3 P0SS1B1L1T13S WH3N Y0U US3 CAP1TAL L3TT3RS, D3SP1T3 N0 L0NG3R B31NG ABL3 T0 US3 TH3 NUMB3R 0N3 T0 R3PLAC3 TH3 L3TT3R L. 0UR M1ND 15 V3RY G00D AT ADAPT1NG T0 N3W CIRCUM5TANC35, 35P3C1ALLY 1F 1T 63T5 50M3 T1M3 T0 L3ARN AND D035N'T G3T THR0WN 1N 4T TH3 D33P 3ND FR0M TH3 B361NN1N6. 17 15 4LW4Y5 B3773R 70 61V3 Y0UR53LF 50M3 71M3 70 4D4P7 1N5734D 0F 574R71N6 W17H 4 53N73NC3 L1K3 "7H3 QU1CK 8R0WN F0X JUMP5 0V3R 7H3 L4ZY D06". C0N6R47UL4710N5 T0 Y0U 1F Y0U 571LL 4R3 4BL3 70 R34D 7H15 73X7. Y0U 7RULY 4R3 4M4Z1N6!

Chapter 4

Octal numbers

4.1 Introduction

The reason that we use hexadecimal numbers is because it is an easy and elegant way to convert long strings of binary digits into something that is easier to read and understand for human eyes. We take a group of 4 binary digits and voila, we got 1 hexadecimal digit. Converting a binary number to a hexadecimal one was a simple as splitting the binary digits into groups of 4 and substituting each of them by the corresponding hexadecimal digit. Easy peasy lemon squeezy.

Unfortunately for us, computers don't always use groups of 4 binary digits. They do so most of the time now, but once upon a time certain computers focused on multiples of 3 bits. Some commands still are made by groups of only 3 binary digits, like the file permissions on a Unix system (the part of a file that tells the computer who can read -look at-, write -change- and execute a file). Which means that the long strings of binary digits are just as hard to read as before, but that it gets super confusing when you convert them to hexadecimal numbers. The solution? Octal numbers!

Unlike hexadecimal numbers, a system that uses 8 digits instead of 10 was not invented just because of computers. Several cultures developed octal systems way before computers were invented.

- Do you remember the trigrams that were used in the ancient Chinese I Ching text? Those groups of three lines that, depending on whether or not the line was broken or solid, represented heaven, lake, flame, thunder, wind, water, mountain or earth? As there are eight possible trigrams, you can see the use of trigrams as a type of octal system.

- The Yuki people, original inhabitants of the Eel River area and the Round Valley Reservation of northern California (USA), used an octal system in the Yuki language. This was because the speakers counted using the spaces between their fingers rather than the fingers themselves.
- The Pame languages are a group of languages that is spoken currently by around 12,000 Pame people, an Indigenous people of central Mexico primarily living in the state of San Luis Potosí. The languages use an octal system and this is because their speakers count on the knuckles of a closed fist.
- Several European scientists pleaded for adapting 8 as a base for arithmetic instead of 10. Among them were John Wilkins in 1668, Emanuel Swedenborg in 1716, Hugh Jones in 1745, James Anderson in 1801 and Alfred B. Taylor in the mid-19th century. They saw the number 8 as a more complete number, since it is divisible into halves, quarters and half quarters far more elegantly than is possible with the number 10. Decimal numbers won, but maybe they would be happy to know that some people learn about octal numbers.

Octal numbers use 8 digits: 0, 1, 2, 3, 4, 5, 6 and 7. Which means that it can get super confusing, because every single octal number can be interpreted as a decimal number or as a hexadecimal one. To avoid confusion, we can add $_{oct}$ or $_8$ to our numbers.

4.2 Converting decimal numbers to octal and vice versa

Because all octal numbers look like decimal ones, we will be adding $_{dec}$ and $_{oct}$ to our numbers to distinguish between them.

Exercise 1 Practice using these notations by converting the first 40 decimal numbers to octal ones. Be mindful of the fact that octal numbers only use the digits from 0 to 7.

- $0_{dec} = 0_{oct}$
- $1_{dec} = 1_{oct}$
- $2_{dec} =$
- $3_{dec} =$
- $4_{dec} =$
- $5_{dec} =$
- $6_{dec} =$
- $7_{dec} = 7_{oct}$
- $8_{dec} = 10_{oct}$
- $9_{dec} =$
- $10_{dec} =$
- $11_{dec} =$
- $12_{dec} =$
- $13_{dec} =$
- $14_{dec} =$
- $15_{dec} = 17_{oct}$

- $16_{dec} = 20_{oct}$
- $17_{dec} =$
- $18_{dec} =$
- $19_{dec} =$
- $20_{dec} = 24_{oct}$
- $21_{dec} =$
- $22_{dec} =$
- $23_{dec} =$
- $24_{dec} =$
- $25_{dec} = 31_{oct}$
- $26_{dec} =$
- $27_{dec} =$
- $28_{dec} =$
- $29_{dec} =$
- $30_{dec} = 36_{oct}$
- $31_{dec} =$
- $32_{dec} =$
- $33_{dec} =$
- $34_{dec} =$
- $35_{dec} = 43_{oct}$
- $36_{dec} =$
- $37_{dec} =$
- $38_{dec} =$
- $39_{dec} =$

Similarly to converting decimal numbers to binary or hexadecimal numbers, there are two stategies you can use. Depending on your preference, you can use either strategy. Knowledge of the powers of 8 might come in handy.

8^9	8^8	8^7	8^6	8^5	8^4	8^3	8^2	8^1	8^0
134,217,728	16,777,216	2,097,152	262,144	32,768	4,096	512	64	8	1

Table 4.1: Powers of 16

The strategies for converting numbers from decimal to octal are

1. Divide by 8.
   (a) item Divide your number by 8. Write the remainder.
   (b) Divide the quotient you calculated by 8. Write the remainder to the left of what you already have.
   (c) Repeat step 2 until your quotient is 0.
2. Use powers of 8.
   (a) Determine which power of 8 is smaller or equal to your decimal number. Call this power n.
   (b) Calculate how many times this power of 8, 8^n fits in you decimal number. Write this number down and subtract this multiple of the power from your original number.
   (c) Calculate how many times the power 8^{n-1} fits in your remaining decimal number. Write this number down to the right of what you already found and subtract this multiple of the power from your remaining number.

(d) Keep calculating how often the powers of 8 fits in your remaining number and always write it down at the right of what you already had.

(e) You stop when you have done this for $8^0 = 1$.

We will illustrate both strategies to convert $342,391_{dec}$ to an octal number.

1. Dividing by 8.

   - $342,391 \div 8 = 42,789\ R7$, so we write 7.
   - $42,789 \div 8 = 5,349\ R6$, so we write 6 to the left of what we already have. We now have 67.
   - $5,349 \div 8 = 668\ R5$, so we write 5 to the left of what we already have. We now have 567.
   - $668 \div 8 = 83\ R4$, so we write 4 to the left of what we already have. We now have 4567.
   - $83 \div 8 = 10\ R3$, so we write 3 to the left of what we already have. We now have 34567.
   - $10 \div 8 = 1\ R2$, so we write 2 to the left of what we already have. We now have 234567.
   - $1 \div 8 = 0\ R1$, so we write 1 to the left of what we already have. We now have 1234567.
   - Our quotient is 0, so we can stop. Our octal number is 1234567_{oct}.

2. Using powers of 8.

   - The biggest power of 8 that fits in $342,391_{dec}$ is $8^6 = 262,144$. So in our case $n = 6$.
   - $8^6 = 262,144$ fits once in our original number. So we write down 1 and we subtract $262,144$ from our original number. We get $342,391 - 262,144 = 80,247$ and we have written down 1.
   - $8^5 = 32,768$ fits twice in our remaining number 80,247. So we write down 2 to the right of what we already have and we subtract $2 \times 32,768 = 65,536$ from our remaining number. We get $80,247 - 65,536 = 14,711$ and we have written down 12.
   - $8^4 = 4,096$ fits three times in our remaining number 14,711. So we write down 3 to the right of what we already have and we subtract $3 \times 4,096 = 12,288$. We get $14,711 - 12,288 = 2,423$ and we have written down 123.

- $8^3 = 512$ fits four times in our remaining number $2,423$. So we write down 4 to the right of what we already have and we subtract $4 \times 512 = 2,048$ from our remaining number. We get $2,423 - 2,048 = 375$ and we have written down 1234.
- $8^2 = 64$. As $5 \times 64 = 320$ and $6 \times 64 = 384$, 64 fits 5 times in our remaining number 375. So we write down 5 to the right of what we already have. We get $375 - 320 = 55$ and we have written down 12345.
- $8^1 = 8$ fits 6 times in our remaining number 55, as $55 \div 8 = 6\ R7$, or $55 = 48 + 7 = (6 \times 8) + 7$. We write down 6. We get $55 - 48 = 7$ and we have written down 123456.
- $8^0 = 1$ fits 7 times into our remaining number 7. We write down 7. We get $7 - 7 = 0$ and we have written down 1234567.
- Our remaining number is 0, so we can stop. Our octal number is 1234567_{oct}.

To convert any octal number to a decimal one, we use the following steps (that are incredibly similar to the ones we used to convert hexadecimal or binary numbers to their decimal counterparts):

1. Count the number of digits you have in your octal number. To make sure this works with every possible number, we will call it n. This n can be 1, but it can also be 1067. The steps are always the same.
2. Multiply the left digit with 8^{n-1}, the one next to that with 8^{n-2}, the one next to that with 8^{n-3} and so on. The second to last digit to the right will be multiplied with 8^1, the last one with 8^0. We will have to add everything later, so it is a good idea to already add a +sign between the multiplications.
3. Convert the powers of 8 you used with their decimal values.
4. Work out the multiplications.
5. Add them all up and you get your decimal number.

Let us use these steps to find the decimal number that corresponds with 1234567_{oct}.

- The number 1234567_{oct} contains 7 digits, so in this case $n = 7$.
- We multiply the left digit with 8^{n-1}, so in our case this means 1×8^6.
- We multiply the next digit with 8^{n-2}, so in our case this is 2×8^5.

- We keep going until we get

$$(1 \times 8^6) + (2 \times 8^5) + (3 \times 8^4) + (4 \times 8^3) + (5 \times 8^2) + (6 \times 8^1) + (7 \times 8^0)$$

- By converting the powers to their decimal values, we get

$$(1 \times 262,144) + (2 \times 32,768) + (3 \times 4,096) + (4 \times 512) + (5 \times 64) + (6 \times 8) + (7 \times 1)$$

- We work out the multiplications and get

$$262,144 + 65,536 + 12,288 + 2,048 + 320 + 48 + 7$$

- By adding all the terms, we get $342,391_{dec}$

We are now officially 100% sure that $1234567_{oct} = 342,391_{dec}$.

Exercises

2. Convert these decimal numbers to their octal counterparts.

   (a) $21_{dec} =$
   (b) $54_{dec} =$
   (c) $65_{dec} =$
   (d) $77_{dec} =$
   (e) $80_{dec} =$
   (f) $99_{dec} =$
   (g) $100_{dec} =$
   (h) $187_{dec} =$
   (i) $218_{dec} =$
   (j) $356_{dec} =$
   (k) $437_{dec} =$
   (l) $529_{dec} =$
   (m) $678_{dec} =$
   (n) $769_{dec} =$
   (o) $801_{dec} =$
   (p) $955_{dec} =$
   (q) $1,000_{dec} =$
   (r) $1,111_{dec} =$
   (s) $1,234_{dec} =$
   (t) $2,849_{dec} =$
   (u) $3,566_{dec} =$
   (v) $7,890_{dec} =$
   (w) $32,768_{dec} =$
   (x) $80,888_{dec} =$
   (y) $100,000_{dec} =$
   (z) $1,234,567_{dec} =$

3. Convert these octal numbers to decimal ones.

   (a) $7_{oct} =$
   (b) $17_{oct} =$
   (c) $27_{oct} =$
   (d) $37_{oct} =$
   (e) $47_{oct} =$
   (f) $57_{oct} =$
   (g) $67_{oct} =$
   (h) $77_{oct} =$
   (i) $100_{oct} =$
   (j) $102_{oct} =$
   (k) $213_{oct} =$
   (l) $324_{oct} =$
   (m) $435_{oct} =$
   (n) $546_{oct} =$
   (o) $657_{oct} =$
   (p) $777_{oct} =$
   (q) $1111_{oct} =$
   (r) $2222_{oct} =$

(s) $3333_{oct} =$

(t) $4444_{oct} =$

(u) $5555_{oct} =$

(v) $6666_{oct} =$

(w) $7777_{oct} =$

(x) $12345_{oct} =$

(y) $10770_{oct} =$

(z) $102030_{oct} =$

4.3 Converting binary numbers to octal and vice versa

Nowadays octal numbers are mainly used for computer programming. Like hexadecimal numbers, converting a number from binary to octal or the other way around is pretty straightforward. We use the fact that $8 = 2^3$, so there is a direct way to convert each group of 3 binary digits to an octal digit, or an octal digit to a group of 3 binary numbers.

If we agree to always use 3 digits for the binary numbers, by adding as many zeroes to the left as is necessary, than we can convert binary numbers to octal digits as follows:

- $0 = 000$
- $1 = 001$
- $2 = 010$
- $3 = 011$
- $4 = 100$
- $5 = 101$
- $6 = 110$
- $7 = 111$

This is all the information we need to convert numbers.

To convert a binary number to an octal one, we need to follow these steps:

1. Divide your binary number into groups of 3 digits. Start making these groups with the lowest digits, so at the right side of your number.
2. Convert each group of 3 binary digits into a hexadecimal number, by using the conversions above.
3. You are done.

When we use these steps to convert the binary number 110101011101000001110100011111101 to its octal equivalent, we get

- 110101011101000001110100011111101 = 110 101 011 101 000 001 110 100 011 111 101
- 110 101 011 101 000 001 110 100 011 111 101 = 6530164375
- The octal number we were searching for is 6530164375.

To convert an octal number to a binary one, we need to follow these steps:

1. Convert each octal digit to a group of 3 binary digits, by using the conversions above.
2. You can leave out the zeroes on the left that proceed the first 1. You are done.

When we do this to convert the hexadecimal number 176064253 to binary, we get

- 176064253 = 001 111 110 000 110 100 010 101 011
- The corresponding binary number is 1111110000110100010101011.

Exercises

1. Match the octal numbers with their corresponding binary representations

   (a) 1744
   (b) 3704
   (c) 1765
   (d) 133535
   (e) 125626
   (f) 125672
   (g) 137255
   (h) 3351

   (i) 1011011101011101
   (j) 1111100100
   (k) 11011101001
   (l) 1010101110010110
   (m) 11111000100
   (n) 1011111010101101
   (o) 1010101110111010
   (p) 1111110101

2. Convert these binary numbers to octal ones. Little hint: you can use \ symbols, or something similar, to divide the long binary strings into groups of the right size. Always start at the right hand side when you do this!

   (a) 100101101011110100101 =
   (b) 111100100010101010111 =
   (c) 101010011111000111001 =
   (d) 111100101110011100101 =
   (e) 111001011101000110011 =
   (f) 1000010101001010001010 =
   (g) 1111010111111010111110 =
   (h) 111100111110101011110 =
   (i) 1000110110011111010110 =
   (j) 10101010111011001010101 =
   (k) 1010010101010101001010101000101010101001110 =
   (l) 110101111111010111101101111011101101111111 =
   (m) 1000101011000001101010110100000100101010000010 =
   (n) 1001010101001001001001011001000100101001001 =

(o) 1010010101001010100100010100111010010100101100 =

(p) 11111111100010000001000000100000101010010010 00 =

(q) 101001001010010010010001000010000001111 1010011 =

(r) 101001100101001010010100100101010010101010001 =

(s) 101001010010101000101101010101011001001010110 =

(t) 101001010010010101010010101001010101010100101 =

3. Can you convert these octal numbers to binary?

(a) 1357 =

(b) 2460 =

(c) 2137 =

(d) 1625 =

(e) 3664 =

(f) 75623 =

(g) 52365 =

(h) 23311 =

(i) 12365 =

(j) 34567 =

(k) 1234567 =

(l) 76543210 =

(m) 40176037 =

(n) 65130027123 =

(o) 123562041 =

(p) 5473021465 =

(q) 42705123352 =

(r) 120005265321 =

(s) 21065540 =

(t) 755010013152 =

4.4 Converting hexadecimal numbers to octal and vice versa

If you want to convert a hexadecimal number to its octal equivalent, or the other way around, there is no direct way to do this. You have to make a detour by converting to a different numbering system first.

You can choose to use decimal numbers to do this. Earlier on, you learned how to convert hexadecimal numbers to decimal numbers and back. You learned how to convert octal numbers to decimal numbers and back. Which means that you have all the tools necessary to do this.

To convert a hexadecimal number to octal, you convert the hexadecimal number to decimal and subsequently convert the decimal number you have found to octal. To convert an octal number to hexadecimal, you convert the octal number to its decimal equivalent and proceed to convert this to hexadecimal. As you have already proven that you can do these conversions one at a time, you can be confident that you can do this as well.

So let's go for it!

Or wait, before we dive in, maybe we should stop and think for a bit longer. After all, those conversions from decimal to octal or hexadecimal weren't the easiest things to do, all those powers of 8 and 16 can feel a bit overwhelming. Maybe there is an easier way to do it? We are doing maths after all, so we might as well try to think like mathematicians. Which means that we don't run away from difficult things that require a lot of thinking power, but we will not do a lot of difficult calculations with a high chance of making mistakes along the way if there is an easier way to get the same result. The cool thing is that it's never called laziness in sciences, it's called efficiency :D

There is another numbering system that we learned about as well. And converting hexadecimal and octal numbers to and from this numbering system was pretty straightforward and didn't require any powers of 8 or 16 or multiples of them or divisions. Do you know which one I am talking about?

That's right, I am hinting at the use of binary numbers! There is a straightforward relationship between groups of 4 binary digits and a hexadecimal digit, and between groups of 3 binary digits and an octal digit. So even though we can use the decimal system as the in between to convert numbers from octal to hexadecimal (and vice versa), using binary numbers will give us the results a lot quicker and requires way less calculations.

At this point, it would be nice if this was a lesson where I could see your face and know at what point you realized that binary numbers would be the better choice. Maybe you were surprised by the revelation that using binary numbers would be easier, maybe you were rolling your eyes at the text when I first mentioned using decimal numbers because you immediately knew that you would prefer using binary numbers and you had to keep reading up till this point to be proven right. In any case, I am glad we got there!

To recapitulate, the groups of 4 binary digits that represent a hexadecimal digit are

- $0 = 0000$
- $1 = 0001$
- $2 = 0010$
- $3 = 0011$
- $4 = 0100$
- $5 = 0101$
- $6 = 0110$
- $7 = 0111$
- $8 = 1000$
- $9 = 1001$
- $A = 1010$
- $B = 1011$

- $C = 1100$
- $D = 1101$
- $E = 1110$
- $F = 1111$

The groups of 3 binary digits that represent an octal digit are

- $0 = 000$
- $1 = 001$
- $2 = 010$
- $3 = 011$
- $4 = 100$
- $5 = 101$
- $6 = 110$
- $7 = 111$

The steps you have to follow to convert a hexadecimal number to its octal equivalent are

1. Convert your hexadecimal number to binary by replacing each hexadecimal digit by the corresponding group of 4 binary digits. If there are 0's on the left before the very first 1, ignore them.
2. Regroup the digits of this binary number into groups of 3. Start grouping the digits from the left hand side.
3. Convert the groups of 3 binary digits to their corresponding octal digit. Congratulations, you have now found the octal representation of your hexadecimal number.

We'll use these steps to find the octal representation of the hexadecimal number $B1A5FE307$.

1. $B1A5FE307 \stackrel{bin}{=} 1011\,0001\,1010\,0101\,1111\,1110\,0011\,0000\,0111$
2. $1011\,0001\,1010\,0101\,1111\,1110\,0011\,0000\,0111 = 101\,100\,011\,010\,010\,111\,111\,110\,001\,100\,000\,111$
3. $101\,100\,011\,010\,010\,111\,111\,110\,001\,100\,000\,111 \stackrel{oct}{=} 543227761407$

We found that $543227761407_{oct} = B1A5FE307_{hex}$.

The steps to convert an octal number to its hexadecimal equivalent are so similar you would almost think that the text is just a copy of what came before, with some minor adjustments.

1. Convert your octal number to binary by replacing each octal digit by the corresponding group of 3 binary digits. If there are 0's on the left before the very first 1, ignore them.
2. Regroup the digits of this binary number into groups of 4. Start grouping the digits from the left hand side.

3. Convert the groups of 4 binary digits to their corresponding hexadecimal digit. Congratulations, you have now found the hexadecimal representation of your octal number.

We'll use these steps to convert the octal number 376023401245 to hexadecimal.

1. $376023401245 \overset{bin}{=} 11\,111\,110\,000\,010\,011\,100\,000\,001\,010\,100\,101$
2. $11\,111\,110\,000\,010\,011\,100\,000\,001\,010\,100\,101 = 111\,1111\,0000\,0100\,1110\,0000\,0010\,1010\,0101$
3. $111\,1111\,0000\,0100\,1110\,0000\,0010\,1010\,0101 \overset{hex}{=} 7F04E02A5$

We found that $376023401245_{oct} = 7F04E02A5_{hex}$

Exercises

1. Convert these hexadecimal numbers to octal.
   (a) $D06 =$
   (b) $CA7 =$
   (c) $57A7E =$
   (d) $C01055A1 =$
   (e) $C1A551F1ED =$
   (f) $50C1A1 =$
   (g) $FACED =$
   (h) $BAB1E5 =$
   (i) $C0A57 =$
   (j) $5A1AD =$
   (k) $F0CA1 =$
   (l) $F1EECE =$
   (m) $DEBA7E =$
   (n) $C1A551C =$
2. Convert these octal numbers to hexadecimal.
   (a) $4321 =$
   (b) $623104 =$
   (c) $12760453 =$

(d) 27403262 =

(e) 31362034 =

(f) 40123762 =

(g) 50213407 =

(h) 65730204 =

(i) 12367210 =

(j) 72030124 =

(k) 3567212355 =

(l) 1123663323 =

(m) 7766554433221100 =

(n) 1234567654321 =

(o) 765432101234567 =

4.5 Operations on octal numbers

You have already proven, with both binary numbers and hexadecimal ones, that you can adapt your thinking to a new numbering system and that you can successfully add, subtract, multiply and divide in them. It requires a special kind of logical reasoning to adapt to other systems than the decimal one, and you have proven that you are capable of it.

To add, subtract, multiply and divide octal numbers, you follow the same logic. You get to figure this part out on your own, but I know you can do it. I'll only give you a few hints:

- Be mindful of that number 8 when adding or subtracting. In octal numbers, $5+4 = 11$ and $13 - 5 = 6$.
- Convert partial results from octal to decimal when you are multiplying, to ensure you carry over the correct digit. For example, when you work out 745×5, you encounter $5 \times 5 \overset{dec}{=} 25 \overset{oct}{=} 31$, $4 \times 5 \overset{dec}{=} 20 \overset{oct}{=} 24$ and $7 \times 5 \overset{dec}{=} 35 \overset{oct}{=} 43$.
- Go back and forward between octal and decimal numbers when you are dividing as well.

Now the time has come to shine and show how amazing your mind is!

Exercises As before, assume that you will make mistakes. And know that that is a good thing, because it means that the stuff you are doing is actually challenging you and that is the whole point of this course. It isn't meant to turn you into someone who can do all kinds of crazy operations on numbers perfectly, it's meant for you to explore some of the many ways of thinking outside of what you are used to. It's the learning that is important, the figuring out of what went wrong when you make a mistake and learning how to correct it and learning not to beat yourself up about it, but see it as a part of any learning process. Mistakes say nothing about your worth. The only thing that mistakes do, is prove that you are a person who is willing to step outside of your comfort zone. And that is the definition of bravery.

1. We'll start with the good old 8 times table, because 8 is so important when you are working with octal numbers. We start off with decimal numbers, but please write your answer in both decimal and octal numbers.

   (a) $1 \times 8 = 8_{dec} = 10_{oct}$
   (b) $2 \times 8 =$
   (c) $3 \times 8 =$
   (d) $4 \times 8 =$
   (e) $5 \times 8 =$
   (f) $6 \times 8 =$
   (g) $7 \times 8 =$
   (h) $8 \times 8 =$
   (i) $9 \times 8 =$

2. Work out these sums. While adding octal numbers, always remember that $8_{dec} = 10_{oct}$. So for example, in order to solve $45 + 57$, you encounter $5 + 7 \overset{dec}{=} 12 \overset{oct}{=} 14$. Next you have to work out $4 + 5 \overset{dec}{=} 9 \overset{oct}{=} 11$. With the carried over digit, your answer becomes $45 + 57 = 124$.

   (a) $3 + 3 =$
   (b) $4 + 4 =$
   (c) $5 + 5 =$
   (d) $6 + 6 =$
   (e) $7 + 7 =$
   (f) $3 + 7 =$
   (g) $6 + 5 =$
   (h) $12 + 6 =$
   (i) $17 + 5 =$
   (j) $23 + 7 =$
   (k) $25 + 25 =$
   (l) $14 + 35 =$
   (m) $23 + 54 =$
   (n) $36 + 63 =$
   (o) $72 + 46 =$
   (p) $45 + 66 =$
   (q) $51 + 34 =$
   (r) $65 + 26 =$
   (s) $123 + 456 =$
   (t) $234 + 567 =$
   (u) $306 + 444 =$
   (v) $562 + 370 =$
   (w) $434 + 217 =$
   (x) $3721 + 2516 =$
   (y) $2604 + 1107 =$
   (z) $1234567 + 7654321 =$

3. Can you work out sums with more than two numbers as well? I bet you can!

   (a)
   $$\begin{array}{r} 3\ 1\ 2\ 4\ 6 \\ 4\ 7\ 5\ 2\ 3 \\ +\ 3\ 7\ 2\ 1\ 5 \\ \hline \end{array}$$

   (b)
   $$\begin{array}{r} 3\ 5\ 7\ 0\ 4 \\ 2\ 6\ 7\ 5\ 0 \\ +\ 6\ 0\ 7\ 5\ 4 \\ \hline \end{array}$$

   (c)
   $$\begin{array}{r} 3\ 6\ 1\ 1\ 1\ 4 \\ 5\ 2\ 7\ 7\ 6\ 5 \\ +\ 6\ 0\ 3\ 6\ 5\ 4 \\ \hline \end{array}$$

(d)
$$\begin{array}{rrrrrr} & 2 & 5 & 0 & 3 & 1 \\ & 1 & 1 & 0 & 7 & 2 \\ & 2 & 6 & 0 & 4 & 3 \\ + & 1 & 0 & 0 & 2 & 4 \\ \hline \end{array}$$

(e)
$$\begin{array}{rrrrrrr} & 1 & 6 & 2 & 5 & 3 & 6 \\ & 6 & 4 & 1 & 0 & 0 & 4 \\ & 5 & 1 & 2 & 1 & 4 & 4 \\ + & 2 & 2 & 5 & 3 & 2 & 4 \\ \hline \end{array}$$

(f)
$$\begin{array}{rrrrrr} & 7 & 5 & 3 & 0 & 2 \\ & 6 & 2 & 0 & 4 & 1 \\ & 7 & 6 & 4 & 1 & 3 \\ + & 4 & 6 & 0 & 3 & 6 \\ \hline \end{array}$$

(g)
$$\begin{array}{rrrrr} & 2 & 3 & 4 & 5 \\ & 6 & 7 & 4 & 5 \\ & 7 & 4 & 2 & 5 \\ & 5 & 1 & 7 & 5 \\ + & 3 & 0 & 2 & 5 \\ \hline \end{array}$$

(h)
$$\begin{array}{rrrrrr} & 7 & 3 & 5 & 0 & 3 \\ & 6 & 0 & 3 & 7 & 2 \\ & 2 & 5 & 1 & 0 & 4 \\ & 4 & 5 & 3 & 2 & 7 \\ + & 7 & 5 & 6 & 3 & 4 \\ \hline \end{array}$$

(i)
$$\begin{array}{rrrrrr} & 1 & 0 & 2 & 3 & 4 \\ & 2 & 3 & 6 & 4 & 5 \\ & 7 & 3 & 5 & 0 & 4 \\ & 2 & 7 & 6 & 1 & 3 \\ + & 5 & 0 & 4 & 2 & 7 \\ \hline \end{array}$$

4. Work out these subtractions. Stay mindful of the role of the number 8_{dec}. To solve $43-16$, you encounter $3-6$, so you have to borrow a digit from the left. By doing so, you get $13-6 \overset{dec}{=} 11-6=5 \overset{oct}{=} 5$. Because you had to borrow a digit, you encounter $3-1=2$ next. Your answer becomes $43-16=25$.

(a) $12-7=$
(b) $15-6=$
(c) $23-7=$
(d) $32-5=$
(e) $61-4=$
(f) $52-3=$
(g) $60-2=$
(h) $35-16=$
(i) $44-27=$
(j) $56-34=$
(k) $31-27=$
(l) $52-33=$
(m) $43-34=$
(n) $70-51=$
(o) $205-106=$
(p) $370-21=$
(q) $105-27=$
(r) $463-172=$
(s) $326-175=$
(t) $432-234=$
(u) $753-624=$

5. These subtractions are about as hard as they get.

(a)
$$\begin{array}{rrrrrrr} & 7 & 5 & 3 & 1 & 2 & 4 \\ - & 4 & 3 & 7 & 0 & 6 & 3 \\ \hline \end{array}$$

(b)
$$\begin{array}{rrrrrrr} & 5 & 7 & 0 & 2 & 5 & 3 \\ - & 5 & 5 & 3 & 5 & 2 & 4 \\ \hline \end{array}$$

(c)
$$\begin{array}{rrrrrrr} & 5 & 3 & 0 & 5 & 7 & 2 \\ - & 4 & 1 & 7 & 6 & 5 & 4 \\ \hline \end{array}$$

(d)
$$\begin{array}{rrrrrrr} & 2 & 3 & 5 & 7 & 0 & 1 \\ - & 1 & 7 & 6 & 6 & 3 & 4 \\ \hline \end{array}$$

(e)
$$\begin{array}{rrrrrrr} & 5 & 4 & 6 & 0 & 0 & 2 \\ - & 2 & 4 & 6 & 5 & 4 & 3 \\ \hline \end{array}$$

(f)
$$\begin{array}{rrrrrrr} & 5 & 2 & 0 & 3 & 1 & 4 \\ - & 2 & 4 & 7 & 5 & 6 & 1 \\ \hline \end{array}$$

(g)
$$\begin{array}{r} 6\ 3\ 0\ 2\ 4\ 5 \\ -\ \ 4\ 1\ 3\ 6\ 7\ 2 \\ \hline \end{array}$$

(h)
$$\begin{array}{r} 5\ 0\ 1\ 2\ 3\ 4 \\ -\ \ 4\ 7\ 6\ 5\ 6\ 7 \\ \hline \end{array}$$

(i)
$$\begin{array}{r} 5\ 2\ 7\ 0\ 4\ 1 \\ -\ \ 1\ 3\ 7\ 7\ 5\ 2 \\ \hline \end{array}$$

6. It's time for multiplications! Also know as the main reason why you had to revisit the 8 times table earlier.
   While figuring out how much 456×7 is, you would encounter $6 \times 7 \overset{dec}{=} 42 \overset{oct}{=} 54$, so you would write down 4 and carry over 5. Next up would be $5 \times 7 + 5 \overset{dec}{=} 35 + 5 = 40 \overset{oct}{=} 50$, so you would write down 0 and once again carry over 5. Lastly, you get $4 \times 7 + 5 \overset{dec}{=} 28 + 5 = 33 \overset{oct}{=} 41$. This means that, in octal numbers, $456 \times 7 = 4104$. But you already knew how it works, didn't you?

(a) $5 \times 2 =$	(h) $13 \times 5 =$	(o) $456 \times 3 =$
(b) $4 \times 3 =$	(i) $21 \times 4 =$	(p) $765 \times 2 =$
(c) $2 \times 6 =$	(j) $35 \times 6 =$	(q) $666 \times 6 =$
(d) $6 \times 6 =$	(k) $47 \times 3 =$	(r) $525 \times 7 =$
(e) $7 \times 4 =$	(l) $52 \times 7 =$	(s) $304 \times 5 =$
(f) $3 \times 5 =$	(m) $66 \times 2 =$	(t) $236 \times 4 =$
(g) $7 \times 7 =$	(n) $74 \times 4 =$	(u) $123456 \times 7 =$

7. Can you figure out how to do long multiplications as well?

(a)
$$\begin{array}{r} 3\ 5\ 0\ 7\ 3 \\ \times\ \ \ \ \ \ 6\ 3 \\ \hline \\ +\qquad\qquad \\ \hline \end{array}$$

(b)
$$\begin{array}{r} 4\ 6\ 2\ 1\ 4\ 3 \\ \times\ \ \ \ \ \ \ \ 5\ 4 \\ \hline \\ +\qquad\qquad \\ \hline \end{array}$$

(c)
$$\begin{array}{r} 7\ 6\ 5\ 4\ 3\ 2\ 1 \\ \times\ \ \ \ \ \ \ \ \ \ 7\ 2 \\ \hline \\ +\qquad\qquad \\ \hline \end{array}$$

(d)

(e)
$$\begin{array}{r} 7\ 3\ 0\ 4\ 1 \\ \times\ \ \ \ 5\ 1\ 6 \\ \hline \\ \\ +\qquad\qquad \\ \hline \end{array}$$

(f)

$$\begin{array}{r} 2\ 5\ 3\ 1\ 4 \\ \times \quad 4\ 2\ 7 \\ \hline \\ \\ + \qquad\qquad \\ \hline \end{array} \qquad \begin{array}{r} 3\ 0\ 4\ 1\ 3 \\ \times \quad 2\ 5\ 1 \\ \hline \\ \\ + \qquad\qquad \\ \hline \end{array}$$

(g) (h) (i)

$$\begin{array}{r} 6\ 2\ 5\ 7 \\ \times\ 4\ 3\ 6\ 2 \\ \hline \\ \\ + \qquad\qquad \\ \hline \end{array} \qquad \begin{array}{r} 7\ 1\ 3\ 2 \\ \times\ 4\ 6\ 5\ 4 \\ \hline \\ \\ + \qquad\qquad \\ \hline \end{array} \qquad \begin{array}{r} 3\ 6\ 5\ 4 \\ \times\ 4\ 2\ 7\ 1 \\ \hline \\ \\ + \qquad\qquad \\ \hline \end{array}$$

8. And now, the moment we've all been waiting for... do you feel the excitement building? Do you notice the thrill in the air? And all of that is because of... drumroll... divisions! Woop woop!
   What do you mean you didn't go to sleep last night all excited because today would be the day you get to do divisions of octal numbers? I simply don't understand what you mean.
   Anyway, if you try to divide 736 by 5, you would be pleased at first, because $7 = 5+2$, you write down 1 and have 2 left over. Next, you come across 23, at which point I would very strongly recommend to start switching between octal and decimal numbers, unless you are not like me at all and the octal version of all the times tables already have a solid space in your brain. If you are like me and haven't memorized any of the octal versions of the times tables, it's easier to do $23_{oct} = 19_{dec} = 3 \times 5 + 4$, so you write down another 3 and have 4 left over. Finally, you come across 46. At this point you can once convert it to a decimal number and get $46_{oct} = 38_{dec} = 7 \times 5 + 3$, so you write down 7 and your remainder is 3. This way, you have figured out that $736 \div 6 = 137\ R3$.
   Solve these divisions and write down the remainder if there is one.

(a) $34 \div 2 =$
(b) $56 \div 3 =$
(c) $21 \div 4 =$
(d) $66 \div 5 =$
(e) $53 \div 6 =$
(f) $47 \div 7 =$
(g) $531 \div 2 =$
(h) $672 \div 3 =$
(i) $326 \div 4 =$
(j) $275 \div 5 =$
(k) $704 \div 6 =$
(l) $654 \div 7 =$
(m) $1234 \div 2 =$
(n) $2345 \div 3 =$
(o) $3456 \div 4 =$
(p) $4567 \div 5 =$
(q) $5670 \div 6 =$
(r) $6701 \div 7 =$

9. Can you figure out these long divisions?

(a) $734 \div 21 =$
(b) $3264 \div 12 =$
(c) $3264 \div 13 =$

4.6 Final words

Why don't riddles work in octal?
Because seven ten eleven.

After learning that people can simply decide to only use 2 digits (for binary numbers), 8 digits (for octal numbers) or 16 digits (for hexadecimal numbers), you might suspect that you can follow that same reasoning for however many digits you like. And you would be 100% correct, that is perfectly possible. Binary numbers and hexadecimal numbers happen to be the alternative numbering systems that are most frequently used, but there is no reason whatsoever that prevents you from using 3 digits, or 5, or whatever number you like. The only downside to those numbering systems is that, as far as I know, they aren't used worldwide. But don't let that stop you. You never know what the future will bring. Who knows, maybe at one point someone will be desperately looking for a numbering system that uses 5 digits and will be ever so grateful that you already did the work that goes with it.

Chapter 5

The ASCII table

We have mentioned over and over again that computers only use binary code. But we have come a far way from having to work with 0's and 1's ourselves in order to work with them. We can read words on a screen as easily as words on paper (it's even easier if the handwriting is hard to decipher). It's all thanks to generations of computer scientists who strive to make programs more accessible so more people can work with them.

Internally, the computer still transforms everything into binary code. Letters and other symbols you can type are called characters. Each character gets its own unique sequence of 0's and 1's and every computer uses the same conversions. This means that when you load a website on internet and the computer receives the information as a bunch of 1's and 0's, it can decipher it and show you the website in a way that makes sense to us.

The first table that was used worldwide that converted characters to binary, is ASCII. The American Standard Code for Information Interchange was developed in 1963. This table contained 128 characters, amongst which all lowercase and uppercase letters of the alphabet.

The ASCII table is perfect... until you need more than the normal (Roman) letters. If you want to place an accent on one of your letters, for example because *ar* and *ár* don't have the same meaning in Irish and you want to show which one you mean, or because you have want to meet someone *mañana*, or you want someone to stop asking the same question because *tu as déjà répondu*, then you have a problem. Those characters simple aren't included in the ASCII table. It's a lot worse if you want to use a different type of letters altogether, because you want to write a message in Arabic, Greek, Chinese, Hindu,...

Unicode, formally *The Unicode Standard*, is the solution to this problem. It starts off with the ASCII table, but continues with many more characters, amongst which all the different alphabets you might need. Even smileys are a part of it.

Unicode is what connects characters to binary numbers, which can be converted to octal, hexadecimal and decimal. Because it brings everything we learned about before together, we'll take a closer look at the ASCII table part of it.

It's a great way to encode your messages to keep them secret. Apart from maybe computer programmers, people rarely associate the number 65 with *A*, or 97 with *a*. And even computer programmers would have some trouble with connecting your secret message to the ASCII table if you use the binary, octal or hexadecimal version of the characters.

Don't you just love it when everything comes together and you can have some fun with it?

Binary	*Oct*	*Dec*	*Hex*	*Character*
00100000	40	32	20	[*SPACE*]
00100001	41	33	21	!
00100010	42	34	22	”
00100011	43	35	23	#
00100100	44	36	24	$
00100101	45	37	25	%
00100110	46	38	26	&
00100111	47	39	27	′
00101000	50	40	28	(
00101001	51	41	29	)
00101010	52	42	2*A*	∗
00101011	53	43	2*B*	+
00101100	54	44	2*C*	,
00101101	55	45	2*D*	−
00101110	56	46	2*E*	.
00101111	57	47	2*F*	/
00110000	60	48	30	0
00110001	61	49	31	1
00110010	62	50	32	2
00110011	63	51	33	3
00110100	64	52	34	4
00110101	65	53	35	5
00110110	66	54	36	6
00110111	67	55	37	7
00111000	70	56	38	8
00111001	71	57	39	9
00111010	72	58	3*A*	:
00111011	73	59	3*B*	;
00111100	74	60	3*C*	<
00111101	75	61	3*D*	=
00111110	76	62	3*E*	>
00111111	77	63	3*F*	?
01000000	100	64	40	@

Binary	Oct	Dec	Hex	Character	Binary	Oct	Dec	Hex	Character
01000001	101	65	41	A	01100001	141	97	61	a
01000010	102	66	42	B	01100010	142	98	62	b
01000011	103	67	43	C	01100011	143	99	63	c
01000100	104	68	44	D	01100100	144	100	64	d
01000101	105	69	45	E	01100101	145	101	65	e
01000110	106	70	46	F	01100110	146	102	66	f
01000111	107	71	47	G	01100111	147	103	67	g
01001000	110	72	48	H	01101000	150	104	68	h
01001001	111	73	49	I	01101001	151	105	69	i
01001010	112	74	4A	J	01101010	152	106	6A	j
01001011	113	75	4B	K	01101011	153	107	6B	k
01001100	114	76	4C	L	01101100	154	108	6C	l
01001101	115	77	4D	M	01101101	155	109	6D	m
01001110	116	78	4E	N	01101110	156	110	6E	n
01001111	117	79	4F	O	01101111	157	111	6F	o
01010000	120	80	50	P	01110000	160	112	70	p
01010001	121	81	51	Q	01110001	161	113	71	q
01010010	122	82	52	R	01110010	162	114	72	r
01010011	123	83	53	S	01110011	163	115	73	s
01010100	124	84	54	T	01110100	164	116	74	t
01010101	125	85	55	U	01110101	165	117	75	u
01010110	126	86	56	V	01110110	166	118	76	v
01010111	127	87	57	W	01110111	167	119	77	w
01011000	130	88	58	X	01111000	170	120	78	x
01011001	131	89	59	Y	01111001	171	121	79	y
01011010	132	90	5A	Z	01111010	172	122	7A	z
01011011	133	91	5B	[	01111011	173	123	7B	{
01011100	134	92	5C		01111100	174	124	7C	|
01011101	135	93	5D	]	01111101	175	125	7D	}
01011110	136	94	5E	ˆ	01111110	176	126	7E	˜
01011111	137	95	5F	_	01111111	177	127	7F	[DELETE]
01100000	140	96	60	‘					

Exercises

1. Use the ASCII table to write your own name in binary numbers.
2. Use the ASCII table to write the name of your teacher in octal numbers.
3. Decode this piece of wisdom from The Land of Stories, where the characters have been replaced by the corresponding decimal numbers from the ASCII table.

```
72 97 112 112 105 108 121 32 101 118 101 114 32 97 102 116 101 114 32
```

105 115 32 97 98 111 117 116 32 102 105 110 100 105 110 103 32 104 97

112 112 105 110 101 115 115 32 119 105 116 104 105 110 32 121 111 117

114 115 101 108 102 32 97 110 100 32 104 111 108 1000 105 110 103 32

111 110 32 116 111 32 105 116 32 116 104 114 111 117 103 104 32 97 110

121 32 115 116 111 114 109 32 116 104 97 116 32 99 111 109 101 115 32

121 111 117 114 32 119 97 121 46

4. Decode the following message from Dumbledore, by converting the binary numbers from the ASCII table to characters.

01001001 01110100 00100000 01101001 01110011 00100000 01101111 01110101

01110010 00100000 01100011 01101000 01101111 01101001 01100011 01100101

01110011 00100000 01110100 01101000 01100001 01110100 00100000 01110011

01101000 01101111 01110111 00100000 01110101 01110011 00100000 01110111

01101000 01100001 01110100 00100000 01110111 01100101 00100000 01110100

01110010 01110101 01101100 01111001 00100000 01100001 01110010 01100101

00100000 01100110 01100001 01110010 00100000 01101101 01101111 01110010

01100101 00100000 01110100 01101000 01100001 01101110 00100000 01101111

01110101 01110010 00100000 01100001 01100010 01101001 01101100 01101001

01110100 01101001 01100101 01110011 00101110

5. This message from Gandalf is encrypted by using the hexadecimal numbers from the ASCII table. Can you decipher it?

45 76 65 6E 20 74 68 65 20 73 6D 61 6C 6C 65 73 74 20 70 65 72 73 6F

6E 20 63 61 6E 20 63 68 61 6E 67 65 20 74 68 65 20 63 6F 75 72 73 65

```
20 6F 66 20 74 68 65 20 66 75 74 75 72 65 2E
```

6. The message that got encoded in octal numbers, comes from Vitruvius, one of the master builders from the Lego Movie.

```
104 157 156 140 164 40 167 157 162 162 171 40 141 142 157 165 164 40

167 150 141 164 40 164 150 145 40 157 164 150 145 162 163 40 141 162 145

40 144 157 151 156 147 56 40 131 157 165 40 155 165 163 164 40 145 155

142 162 141 143 145 40 167 150 141 164 40 151 163 40 163 160 145 143 151

141 154 40 141 142 157 165 164 40 171 157 165 56
```

Chapter 6

Revision

6.1 Roman numerals

1. II siblings were born V years apart. Right now, the eldest one is twice as old as the youngest one. How old will each of them be in IX years time?
2. A Roman centurion was told that his own army has CCCLXXVIII soldiers and that the army they have to fight has D soldiers. "That's alright then, we have way more soldiers than they do, so winning this battle will be a piece of cake," was the centurion's reaction.
   Is he correct? Why do you think the centurion thought this?
3. A girl was born in the year MMXII. How old will she be in the year MMLVI?
4. Convert 2345 to Roman numberals.
5. $DCX + LXVIII =$
6. $MMCDXLIX + CCLXVI =$
7. If I take away $DXIX$ from a number, I get MCL. What was the original number?
8. $MCDLIV - DCCCXXV =$
9. $LXII \times V =$
10. $VI \times XXI =$
11. $DCXXVII \div III =$
12. $MMCXXXV \div VII =$

6.2 Binary numbers

There are only 10 kinds of people in the world:
those who understand binary and those who don't.

1. What is binary code useful for?

2. What is the binary representation of the decimal number 39?

3. What is the decimal value of the binary number 110110?

4. $11011 + 101 =$

5. $11010011 + 110101011 =$

6. $1101 - 111 =$

7. $110100110 - 11011001 =$

8. $1011001 \times 1000 =$

9.
$$\begin{array}{rccccc} & 1 & 0 & 1 & 1 & 0 \\ & \times & & 1 & 0 & 1 \\ \hline \\ \hline \end{array}$$

10.
$$\begin{array}{rccccc} & 1 & 1 & 1 & 1 & 1 \\ & \times & & & 1 & 1 \\ \hline \\ \hline \end{array}$$

11.
$$\begin{array}{rcccccc} & 1 & 1 & 0 & 1 & 1 & 1 \\ & \times & 1 & 0 & 1 & 1 & 1 \\ \hline \\ \hline \end{array}$$

12. $1100111 \div 101 =$

13. $10010111001001 \div 110011 =$

14. What is the decimal value of the binary number 0.11?

6.3 Hexadecimal numbers

E times out of F
people will not get jokes in hex.

1. What is the hexadecimal equivalent of 39_{dec}?
2. What is the decimal value of the $5A_{hex}$?
3. What is the binary representation of CD_{hex}?
4. What is the hexadecimal value of the binary number 1100101001110101?
5. $106 + FEE =$
6. $C1A551C + F00D =$
7. $D06 - AB =$
8. $DADD1E5 - BAB135 =$
9. $54 \times 7 =$
10. $FEE \times 3 =$
11. $$\begin{array}{r} 6\ 1\ A\ D \\ \times \quad\ D\ 3 \\ \hline \\ \\ + \qquad\qquad \\ \hline \end{array}$$
12. $BADD1E \div 4 =$
13. $C1A55 \div 3 =$

6.4 Octal numbers

What do Halloween and Christmas have in common?

$$31_{oct} = 25_{dec}$$

1. What is the octal representation of 39_{dec}?
2. What is the octal equivalent of the binary number 1101110011?
3. What is the octal value of $FA57_{hex}$?
4. What is the decimal value of 75_{oct}?
5. What is the binary value of 326_{oct}?
6. What is the hexadecimal value of 7420_{oct}?
7. $65 + 76 =$
8. $6453 + 7035 =$
9. $1004 - 5 =$
10. $650032 - 274375 =$
11. $56 \times 4 =$
12. $325 \times 6 =$
13. $$\begin{array}{rrrrrrrr} & & & 6 & 3 & 5 & 4 & 2 \\ & & & & \times & 5 & 3 & 7 \\ \hline \\ \\ \\ + & & & & & & & \\ \hline \end{array}$$
14. $547 \div 2 =$
15. $3636 \div 5 =$
16. $75310642 \div 6 =$

Chapter 7

Solutions to the exercises

7.1 Roman numerals

1. This is false: V, L and D are not allowed to repeat.
2. $89 - 72 = 15$, so there are only XV tiles left for the mosaic.
3. $56 - 24 = 32$, so he can pay the seamstress $XXXII$ gold coins.
4. $56 - 42 = 14$, so there are XIV Roman villas left to be found.
5. $11 \times 5 \times 6 = 330$, so they can paint $CCCXXX$ meters in that time.
6. $15 + 24 = 39$, so the sum of their ages is $XXXIX$ years.
7. Aside from X, the numerals I, C and M are also allowed to repeat.
8. 1984
9. MMX for 2010, $MMXI$ for 2011, $MMXII$ for 2012, $MMXIII$ for 2013, $MMXIV$ for 2014, $MMXV$ for 2015 and so on.
10. (a) 3 =III
    (b) 8 =VIII
    (c) 16 =XVI
    (d) 25 =XXV
    (e) 36 =XXXVI
    (f) 81 =LXXXI
    (g) 140 =CXL
    (h) 275 =CCLXXV
    (i) 567 =DLXVII
    (j) 789 =DCCLXXXIX
    (k) 987 =CMLXXXVII
    (l) 1,111 =MCXI
    (m) 1,234 =MCCXXXIV
    (n) 2,345 =MMCCCXLV
    (o) 3,456 =MMMCDLVI
11.

(a) XV=15
(b) XLIX=49
(c) XXVIII=28
(d) LXXIX=79
(e) MCDXLIV=1,444
(f) DLIII=553
(g) MDCCCX=1,810
(h) LXXVII=77
(i) MMMCDXXI=3,421
(j) MCMLXIV=1,964
(k) DLV=555
(l) MMCDXLVIII=2,448
(m) MDCXCIX=1,699
(n) MMMCDLXXIV=3,474
(o) MMCMLXXXVII=2,987

12. The building was built in 1911, so in 2023 it is 112 years old.

13. $MMMCMXCIX$, also known as 3999. As there is no symbol for $5,000$, it is also impossible to write $4,000$, as this would be written in the same way as $IV = 4$, $XL = 40$ and $CD = 400$.

14.

(a) IX < XI
(b) LVII > XLVIII
(c) XCLXXVIII < CXLV
(d) MMCMXLII < MMMCXV
(e) MMMCDLXXXII < MMMDCX
(f) CDLXXIX > CDXXXIV
(g) D > CCCXCVIII
(h) DCCCLXXXVIII > DCCXCLXXX
(i) MCMCCCXXXIII < MMCXCXI
(j) DCCLXVII > DCCLXVI

15. Flora needs III more animals to be able to give II to each of her friends.

16. (a) $53 = LIII$
(b) $149 = CXLIX$
(c) $1,292 = MCCXCII$
(d) $751 = DCCLI$
(e) $2,376 = MMCCCLXXVI$
(f) $1,724 = MDCCXXIV$
(g) $98 = XCVIII$
(h) $2942 = MMCMXLII$

17. (a) $MCMXLIV = 1944$
(b) $MCMXCIII = 1993$
(c) $CMXCIX = 999$
(d) $CCCXLIV = 344$
(e) $MCDLII = 1452$
(f) $CLXV = 165$
(g) $MDCCIV = 1704$
(h) $DLXXVIII = 578$

18. (a) LI
(b) CXL
(c) CLXXIV
(d) CCCXXXII
(e) MCCXCVII
(f) MMCX
(g) MMMCLI
(h) MMMCCCXXXIII
(i) MV
(j) MMDCXXX

(k) CXXXIII
(l) CCLXXVIII
(m) CMXCVIII
(n) CD
(o) XCV
(p) MCDLXVIII
(q) DCCXXVI
(r) MCCXL
(s) MMXII
(t) MDLXVII

19. (a) CCCXXXIII
(b) CDXCII
(c) DXXXIX
(d) MMMDCCCXLVI
(e) MMCX
(f) MCCCXLII
(g) DCLII
(h) MCCXXXII
(i) CCCXXXII
(j) CXI

20.

VIII	III	IV
I	V	IX
VI	VII	II

X	III	VIII
V	VII	IX
VI	XI	IV

IV	VIII	III
IV	V	VI
VII	II	VI

21. (a) $1 + 1 = 2$ or $I + I = II$

(b) $2 + 2 = 4$ or $II + II = IV$

(c) $4 + 5 = 9$ or $IV + V = IX$

22. (a) Check out this innovative string encoding I've been developing! It's virtuacy perfect! ... Hang on, what's a *virtuacy*?

(b) It was meant to be the word virtually, but $L + L = C$.

(c) 1 a1000 50o6ng th1s.

23.

Letter	*A*	*B*	*C*	*D*	*E*	*F*
Roman	III	MCMIX	MDCL	DCVI	CCLXIX	CMXVI
Arabic	3	1909	1650	606	269	916

Letter	*G*	*H*	*I*	*J*	*K*	*L*
Roman	DXCV	MCMXVIII	VIII	XLII	XXXVI	MMCCXXII
Arabic	595	1918	8	42	36	2222

Letter	*M*	*N*	*O*	*P*	*Q*	*R*
Roman	MDLVII	MMCCCXLII	DXXI	CCCXXVII	MMMDCCXCIX	DXII
Arabic	1557	2342	521	327	3799	512

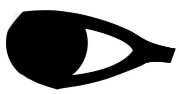
Letter	*S*	*T*	*U*	*V*	*W*	*X*
Roman	IX	CDXLVIII	MCDLI	DCCCVI	MMMCCCXL	MMCMLXVI
Arabic	9	448	1451	806	3340	2966

Letter	*Y*	*Z*
Roman	MDV	MCMXCIX
Arabic	1505	1999

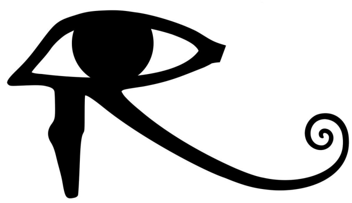
It would ruin the surprise if I put the decoded message here, but I know you can figure it out with the deciphering code!

7.2 Binary numbers

7.2.1 Ancient Egypt



$\frac{5}{16} = \frac{1}{4} + \frac{1}{16}$



$\frac{55}{64} = \frac{1}{2} + \frac{1}{4} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64}$

7.2.2 China

- Water and Flame are joined to make the numbers 63 and 64.
- Earth and Wind make up the numbers 20 and 46.
- The number 1 is pure Heaven.
- The number 2 is pure Earth.

7.2.3 Pingala

- If you start from 25, by following the steps you should encounter the numbers 25, 13, 7, 4 and 2, which means that the binary equivalent is 00011.
- Starting with 38, you encounter 38, 19, 10, 5, 3 and 2, which leads to the Pingala binary number of 101001.
- You encounter 66, 33, 17, 9, 5, 3 and 2, so you get 1000001.
- The numbers you encounter are 2137, 1069, 535, 268, 134, 67, 34, 17, 9, 5, 3 and 2, so you get 000110100001.

7.2.4 Francis Bacon

You should be proud of yourself if you managed to decipher the message. If you peeked here in the hope to get some help because that amount of a's and b's is overwhelming, here's a tip for you: every letter is encoded in a string of exactly 5 letters, so it is more manageable if you place / symbols after every 5 letters. Ideally, you do this before you start searching for what those blobs of a's and b's stand for. Good luck!

7.2.5 Leibniz's binary code

Power of a number

Exercise Calculate these powers.

1. $6^3 = 216$
2. $4^3 = 64$
3. $5^4 = 625$
4. $11^3 = 1,331$
5. $3^5 = 243$
6. $2^4 = 16xxxxxxxxx$

7. $2^6 = 64$
8. $10^2 = 100$
9. $10^3 = 1,000$
10. $10^4 = 10,000$
11. $25^1 = 25$
12. $123,456,789^0 = 1$
13. $39^2 = 1,521$
14. $987,654,321^1 = 987,654,321$
15. $93^0 = 1$

Decimal numbers

Exercise Complete the table - decimal numbers.

number	10^{11}	10^{10}	10^9	10^8	10^7	10^6	10^5	10^4	10^3	10^2	10^1	10^0
321, 234, 567, 890	3	2	1	2	3	4	5	6	7	8	9	0
25, 268								2	5	2	6	8
20, 102, 012					2	0	1	0	2	0	1	2
198, 438							1	9	8	4	3	8

Converting decimal numbers to binary numbers

Exercise Can you work out what the next binary numbers will be?

- 12 = 1100
- 13 = 1101
- 14 = 1110
- 15 = 1111
- 16 = 10000
- 17 = 10001
- 18 = 10010
- 19 = 10011
- 20 = 10100
- 21 = 10101
- 22 = 10110
- 23 = 10111
- 24 = 11000
- 25 = 11001
- 26 = 11010
- 27 = 11011
- 28 = 11100
- 29 = 11101
- 30 = 11110
- 31 = 11111

Exercise Can you convert these numbers to binary by using this first method?

- If you follow the steps, you encounter 38, 19, 9, 4, 2 and 1, so your binary number is 100110.
- You start with 66, then get 33, 16, 8, 4, 2 and 1, so you get 1000010.
- You encounter 125, 62, 31, 15, 7, 3 and 1, so you get 1111101.
- You start with 525, then get 262, 131, 65, 32, 16, 8, 4, 2 and 1. Your binary number is 1000001101.

- You get 2137, 1068, 534, 267, 133, 66, 33, 16, 8, 4, 2 and 1, so you get 100001011001.

Exercise Powers of 2

- $2^0 = 1$
- $2^1 = 2$
- $2^2 = 4$
- $2^3 = 8$
- $2^4 = 16$
- $2^5 = 32$
- $2^6 = 64$
- $2^7 = 128$
- $2^8 = 256$
- $2^9 = 512$
- $2^{10} = 1,024$
- $2^{11} = 2,048$
- $2^{12} = 4,096$
- $2^{13} = 8,192$
- $2^{14} = 16,384$
- $2^{15} = 32,768$
- $2^{16} = 65,536$
- $2^{17} = 131,072$

Exercise Can you convert these numbers to binary by using this second method? To help you out a bit, you get the tables of powers of two that you need.

- 48 = 110000

32	16	8	4	2	1
2^5	2^4	2^3	2^2	2^1	2^0
1	1	0	0	0	0

- 99 = 1100011

64	32	16	8	4	2	1
2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	1	0	0	0	1	1

- 234 = 11101010

128	64	32	16	8	4	2	1
2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	1	1	0	1	0	1	0

- 579 = 1001000011

512	256	128	64	32	16	8	4	2	1
2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	0	0	1	0	0	0	0	1	1

- 2173 = 100001111101

2048	1024	512	256	128	64	32	16	8	4	2	1
2^{11}	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	0	0	0	0	1	1	1	1	1	0	1

Converting binary numbers to decimal

Exercise Convert these binary numbers to decimal numbers.

1. $101 = 5$
2. $1001 = 9$
3. $10101 = 21$
4. $11011 = 27$
5. $101000 = 40$
6. $110011 = 51$
7. $1111111 = 127$
8. $1000000 = 64$
9. $1000011 = 67$
10. $1100011 = 99$
11. $1101011 = 107$
12. $100011000 = 280$
13. $11101010 = 234$
14. $101001101 = 333$
15. $110111100 = 444$
16. $1111101000 = 1000$
17. $10001010111 = 1111$
18. $10011111100 = 1276$
19. $100010101110 = 2222$
20. $1101111000 = 888$

Adding binary numbers

Exercise Adding two binary numbers.

1.
$$\begin{array}{ccccccc} & & & 1 & & & \\ & 1 & 0 & 0 & 1 & 0 & 1 \\ + & & 1 & 0 & 1 & 0 & 0 \\ \hline & 1 & 1 & 1 & 0 & 0 & 1 \end{array}$$

2.
$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & & & \\ & 1 & 0 & 0 & 1 & 0 & 1 \\ + & & 1 & 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

3.
$$\begin{array}{cccccccccc} 1 & 1 & & 1 & 1 & & & & & \\ & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ + & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}$$

4.
$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & & & \\ & 1 & 0 & 0 & 1 & 0 & 1 \\ + & & 1 & 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

5.
$$\begin{array}{ccccccccc} 1 & 1 & & & 1 & 1 & & 1 & \\ & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ + & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ \hline 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \end{array}$$

6.
$$\begin{array}{ccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ + & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}$$

7.

$$\begin{array}{cccccccccc}
1 & & 1 & 1 & 1 & 1 & & 1 & 1 & \\
 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
+ & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
\hline
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0
\end{array}$$

8.

$$\begin{array}{cccccccccc}
1 & & & & 1 & 1 & 1 & 1 & 1 & \\
 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
+ & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
\hline
1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0
\end{array}$$

9.

$$\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & & 1 & \\
 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
+ & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}$$

10.

$$\begin{array}{ccccccccc}
1 & 1 & 1 & & & & & & \\
 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
+ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\hline
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0
\end{array}$$

Exercise Can you add two binary numbers when they aren't aligned?

1. $110011 + 100010 = 1010101$
2. $1111 + 100111 = 110110$
3. $100000 + 101010 = 1001010$
4. $100011 + 101001 = 1001100$
5. $101111 + 10100 = 1000011$
6. $1111101 + 1010100 = 11010001$
7. $11010100 + 11000010 = 110010110$
8. $1110001 + 1001101 = 10111110$
9. $1010111 + 1100110 = 10111101$
10. $1110101 + 1100011 = 11011000$
11. $11000110 + 110001001 = 1001001111$
12. $101101011 + 100100001 = 1010001100$
13. $101010100 + 10000110 = 111011010$
14. $10111011 + 110000010 = 1000111101$
15. $10110111 + 111011 = 11110010$
16. $1111101 + 101101101 = 111101010$
17. $11101101 + 10110111 = 110100100$
18. $1100011 + 1110111 = 11011010$
19. $11011100 + 101110111 = 1001010011$
20. $101100111 + 1001111 = 110110110$

Exercise Can you add three numbers? As we can get 100 and 101 as the result of adding 4 or 5 digits, and the purpose of these corrections is to understand any mistake you might have made, there are two lines for digits that are carried over where this happens. The top line is for digits that come from 100, the bottom line is for digits that come from 10 or 11.

1.

$$\begin{array}{ccccccc}
1 & 1 & & & & 1 & \\
 & 1 & 1 & 0 & 0 & 0 & 1 \\
 & 1 & 0 & 0 & 1 & 0 & 0 \\
+ & & 1 & 1 & 0 & 0 & 1 \\
\hline
1 & 1 & 0 & 1 & 1 & 1 & 0
\end{array}$$

2.

$$\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
+ & & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1
\end{array}$$

3.

$$\begin{array}{cccccccc}
1 & & & & & & & \\
 & & 1 & 1 & 1 & & & \\
 & & 1 & 1 & 1 & 0 & 0 & 0 \\
 & & 1 & 0 & 1 & 1 & 0 & 1 \\
+ & & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1
\end{array}$$

4.

$$\begin{array}{ccccccccccc}
1 & & & & & & & & & & \\
 & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\
 & & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
 & & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
+ & & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0
\end{array}$$

5.

$$\begin{array}{cccccccccc}
1 & & & & 1 & & & & & \\
 & & 1 & 1 & 1 & 1 & 1 & 1 & & \\
 & & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
 & & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
+ & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1
\end{array}$$

6.

$$\begin{array}{cccccccccc}
1 & 1 & 1 & 1 & & & & & & \\
 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
 & & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
+ & & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1
\end{array}$$

7.

$$\begin{array}{ccccccccccc}
1 & & 1 & & 1 & & 1 & & & & \\
 & & & & 1 & & & & 1 & 1 & \\
 & & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
 & & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
+ & & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
\hline
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}$$

8.

$$\begin{array}{cccccccccccc}
1 & & & 1 & & 1 & & & & & & \\
 & & 1 & 1 & & & & 1 & 1 & 1 & 1 & \\
 & & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
 & & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
+ & & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0
\end{array}$$

9.

$$\begin{array}{ccccccccccccc}
1 & & & & & 1 & & 1 & & & & & \\
 & & 1 & 1 & 1 & 1 & & 1 & & 1 & 1 & & \\
 & & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
+ & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
\hline
1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1
\end{array}$$

10.

$$\begin{array}{ccccccccccc}
1 & 1 & 1 & 1 & 1 & & & 1 & & & \\
 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
+ & & & & & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1
\end{array}$$

Exercise Can you add even more numbers in one go? Where adding digits leads to a sum of 1000 or higher, I have added a third line for carrying digits over.

1.

$$\begin{array}{cccccccccc}
1 & & 1 & & 1 & & 1 & & & \\
 & 1 & 1 & & 1 & 1 & 1 & & 1 & \\
 & & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
 & & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
 & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
+ & & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
\hline
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}$$

2.

$$\begin{array}{cccccccccc}
1 & & 1 & & 1 & & 1 & & & \\
 & 1 & 1 & & & & 1 & & 1 & \\
 & & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
 & & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
 & & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
+ & & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{array}$$

3.

$$\begin{array}{cccccccccccc}
1 & & 1 & & 1 & 1 & & 1 & & & & \\
 & & 1 & 1 & 1 & & & 1 & & 1 & & \\
 & & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
 & & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
 & & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
+ & & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
\hline
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1
\end{array}$$

4.

$$\begin{array}{cccccccccccc}
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & \\
 & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
 & & & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
 & & & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
 & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
+ & & & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0
\end{array}$$

5.

$$\begin{array}{cccccccccccc}
1 & & & & & & & & & & & \\
 & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
 & & & 1 & 1 & & & & 1 & & & \\
 & & & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
 & & & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
 & & & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
 & & & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
 & & & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
+ & & & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1
\end{array}$$

6.

$$\begin{array}{ccccccccccccc}
 & & & & & & & & & & & & \\
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
1 & 1 & 1 & & 1 & 1 & & 1 & & & & & \\
 & & & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
 & & & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
 & & & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
 & & & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
 & & & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
+ & & & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0
\end{array}$$

Subtracting binary numbers

Exercise Subtracting binary numbers.

1.
$$\begin{array}{rrrrrr} 0 & 10 & & & & \\ \cancel{1} & \cancel{0} & 0 & 1 & 0 & 1 \\ - & 1 & 0 & 1 & 0 & 0 \\ \hline & 1 & 0 & 0 & 0 & 1 \end{array}$$

2.
$$\begin{array}{rrrrrr} 0 & 10 & & & & \\ \cancel{1} & \cancel{0} & 1 & 0 & 1 & 1 \\ - & 1 & 0 & 0 & 1 & 1 \\ \hline & 1 & 1 & 0 & 0 & 0 \end{array}$$

3.
$$\begin{array}{rrrrrr} 0 & 10 & & 1 & 1 & 10 \\ \cancel{1} & \cancel{0} & 1 & \cancel{1} & \cancel{0} & \cancel{0} \\ - & 1 & 1 & 0 & 1 & 1 \\ \hline & 1 & 0 & 0 & 0 & 1 \end{array}$$

4.
$$\begin{array}{rrrrrrr} & 0 & 1 & 10 & & & \\ & \cancel{1} & \cancel{0} & \cancel{0} & 1 & 0 & 1 \\ - & & 1 & 1 & 1 & 0 & 0 \\ \hline & & & 1 & 0 & 0 & 1 \end{array}$$

5.
$$\begin{array}{rrrrrrrrr} & & 0 & 10 & 0 & 10 & & & \\ & 1 & \cancel{1} & \cancel{0} & \cancel{1} & \cancel{0} & 1 & 0 & 1 \\ - & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ \hline & & & 1 & 0 & 1 & 1 & 0 & 1 \end{array}$$

6.
$$\begin{array}{rrrrrrrr} 0 & 10 & 0 & 10 & 0 & 10 & 0 & 10 \\ \cancel{1} & \cancel{0} & \cancel{1} & \cancel{0} & \cancel{1} & \cancel{0} & \cancel{1} & \cancel{0} \\ - & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}$$

7.
$$\begin{array}{rrrrrrrrr} 0 & 10 & 0 & 10 & & & & & \\ \cancel{1} & \cancel{0} & \cancel{1} & \cancel{0} & 1 & 0 & 1 & 0 & 0 \\ - & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{array}$$

8.
$$\begin{array}{rrrrrrrrr} & & 0 & 1 & 1 & 10 & & & \\ & 1 & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} & 1 & 0 & 1 \\ - & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ \hline & & & 1 & 1 & 1 & 0 & 0 & 0 \end{array}$$

9.
$$\begin{array}{rrrrrrrrr} & & & 0 & 1 & 1 & 10 & & \\ & 1 & 1 & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} & 1 & 1 \\ - & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ \hline & & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

10.
$$\begin{array}{rrrrrrrrrr} & & 0 & 10 & & 0 & 10 & & & \\ & 1 & \cancel{1} & \cancel{0} & 1 & \cancel{1} & \cancel{0} & 0 & 1 & 1 \\ - & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ \hline & & & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}$$

Exercise Subtracting binary numbers, where the borrowing part of the calculations gets a bit trickier.

1.
$$\begin{array}{rrrrrr} & & & 10 & & \\ 0 & 1 & 1 & \cancel{0} & 1 & 10 \\ \cancel{1} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} & \cancel{0} \\ - & 1 & 0 & 1 & 1 & 1 \\ \hline & & 1 & 1 & 0 & 1 \end{array}$$

2.
$$\begin{array}{rrrrrrr} & & & 10 & & & \\ & & 0 & \cancel{0} & 1 & 10 & \\ & 1 & \cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} & 1 \\ - & 1 & 0 & 1 & 1 & 1 & 0 \\ \hline & & & 1 & 0 & 1 & 1 \end{array}$$

3.

$$\begin{array}{cccccccc}
 & & & & 10 & & 10 & \\
 & & & 0 & \cancel{0} & 1 & \cancel{0} & 10 \\
 & 1 & 1 & \cancel{1} & \cancel{1} & \cancel{0} & \cancel{1} & \cancel{0} \\
- & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
\hline
 & & & & 1 & 0 & 1 & 1
\end{array}$$

4.

$$\begin{array}{cccccccc}
 & 10 & & & 10 & & & \\
0 & \cancel{0} & 1 & 1 & \cancel{0} & 1 & 10 & \\
\cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} & \cancel{0} & 0 \\
- & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
\hline
 & 1 & 1 & 0 & 1 & 0 & 1 & 0
\end{array}$$

5.

$$\begin{array}{ccccccccc}
 & & & 10 & & & & & \\
0 & 1 & 1 & \cancel{0} & 10 & & 0 & 1 & 10 \\
\cancel{1} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} & 1 & \cancel{1} & \cancel{0} & \cancel{0} \\
- & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
\hline
 & & 1 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}$$

6.

$$\begin{array}{cccccccccc}
 & & & & & 10 & & & & \\
 & & & & 0 & \cancel{0} & 10 & & 0 & 10 \\
 & 1 & 1 & 0 & \cancel{1} & \cancel{1} & \cancel{0} & 1 & \cancel{1} & \cancel{0} \\
- & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
\hline
 & & & & & 1 & 1 & 1 & 0 & 1
\end{array}$$

7.

$$\begin{array}{cccccccccc}
 & & & & & & & & 10 & \\
 & & 0 & 1 & 1 & 1 & 1 & 1 & \cancel{0} & 10 \\
 & 1 & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} \\
- & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
\hline
 & & & & 1 & 0 & 0 & 1 & 1 & 1
\end{array}$$

8.

$$\begin{array}{ccccccccc}
 & 10 & 10 & & & & & & \\
0 & \cancel{0} & \cancel{0} & 1 & 1 & 10 & & & \\
\cancel{1} & \cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} & 1 & 0 & 1 \\
- & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
\hline
 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0
\end{array}$$

9.

$$\begin{array}{cccccccccc}
 & 10 & 10 & 10 & & & & & & \\
0 & \cancel{0} & \cancel{0} & \cancel{0} & 1 & 1 & 1 & 1 & 10 & \\
\cancel{1} & \cancel{1} & \cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & 0 \\
- & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
\hline
 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0
\end{array}$$

10.

$$\begin{array}{cccccccccc}
 & & & & & & 10 & & 10 & \\
 & & 0 & 10 & & 0 & \cancel{0} & 1 & \cancel{0} & 10 \\
 & 1 & \cancel{1} & \cancel{0} & 1 & \cancel{1} & \cancel{1} & \cancel{0} & \cancel{1} & \cancel{0} \\
- & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
\hline
 & & & 1 & 0 & 0 & 1 & 1 & 1 & 1
\end{array}$$

Exercise Can you subtract two binary numbers when they aren't aligned?

1. $10001 - 10 = 1111$
2. $11111111 - 1001101 = 10110010$
3. $10000 - 1011 = 101$
4. $110011 - 100011 = 10000$
5. $111001 - 101 = 110100$
6. $1001000 - 111 = 1000001$
7. $101010 - 11001 = 10001$
8. $11001 - 1111 = 1010$
9. $100110 - 1010 = 11100$
10. $100111 - 1101 = 11010$

Multiplication of binary numbers

Exercise Calculate these binary multiplications.

1. $10111 \times 100 = 1011100$
2. $1011 \times 10 = 10110$
3. $11011 \times 1000 = 1101100$
4. $111 \times 10000 = 1110000$
5. $1011 \times 10 = 10110$
6. $11111 \times 1000 = 11111000$
7. $10101 \times 100 = 1010100$
8. $100111 \times 10 = 1001110$
9. $11001 \times 100 = 1100100$
10. $1001 \times 1000 = 1001000$
11. $10110 \times 100000 = 1011000000$
12. $100000 \times 100 = 10000000$
13. $100101 \times 10 = 1001010$
14. $1001001 \times 100 = 100100100$
15. $110001 \times 1000 = 110001000$
16. $10111 \times 100 = 1011100$
17. $11010 \times 10000 = 110100000$
18. $10010 \times 10 = 100100$
19. $110101 \times 1000 = 110101000$
20. $1101100 \times 100 = 110110000$

Exercise Multiply these binary numbers.

1.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & 0 & 0 & 1 & 0 & 1 \\
 & & & & & \times & 1 & 0 & 1 & 0 & 0 \\
\hline
 & & & & & 1 & & & & & \\
 & & & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
+ & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0
\end{array}$$

2.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & 0 & 1 & 0 & 1 & 1 \\
 & & & & & \times & 1 & 0 & 0 & 1 & 1 \\
\hline
 & & & 1 & 1 & 1 & 1 & 1 & 1 & & \\
 & & & & & 1 & 0 & 1 & 0 & 1 & 1 \\
 & & & & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
+ & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
\hline
 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1
\end{array}$$

3.

$$\begin{array}{ccccccccc}
 & & & 1 & 0 & 1 & 1 & 0 & 1 \\
 & & & \times & & & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & & & \\
 & & & 1 & 0 & 1 & 1 & 0 & 1 \\
 & & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
+ & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}$$

4.

$$\begin{array}{ccccccccccc}
 & & & & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
 & & & & \times & & & 1 & 0 & 1 & 0 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\
 & & & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
+ & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0
\end{array}$$

5.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & 0 & 0 & 1 & 0 & 1 \\
 & & & & & \times & 1 & 1 & 1 & 0 & 0 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\
 & & & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
 & & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
+ & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{array}$$

6.

$$\begin{array}{cccccccccc}
 & & & & 1 & 1 & 0 & 1 & 1 & 1 \\
 & & & & \times & & 1 & 1 & 0 & 1 \\
\hline
 & & & 1 & & & & & & \\
1 & 1 & 1 & 1 & & 1 & 1 & & & \\
 & & & & 1 & 1 & 0 & 1 & 1 & 1 \\
 & & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
+ & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
\hline
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1
\end{array}$$

7.

$$\begin{array}{cccccccccccccc}
 & & & & & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
 & & & & & \times & & & & 1 & 1 & 0 & 0 & 1 \\
\hline
 & & & & 1 & & & 1 & & & & & & \\
1 & 1 & 1 & 1 & 1 & & 1 & 1 & & 1 & & & & \\
 & & & & & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
 & & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
+ & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}$$

8.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & 0 & 1 & 1 & 0 & 0 \\
 & & & & & \times & 1 & 1 & 0 & 1 & 1 \\
\hline
 & & 1 & & & & & & & & \\
1 & 1 & & & 1 & 1 & 1 & & & & \\
 & & & & & 1 & 0 & 1 & 1 & 0 & 0 \\
 & & & & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
 & & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
+ & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0
\end{array}$$

9.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
 & & & & & \times & & 1 & 1 & 1 & 0 & 1 & 1 \\
\hline
 & 1 & & 1 & 1 & 1 & 1 & & & & & & \\
1 & 1 & & 1 & & & & & 1 & 1 & 1 & & \\
 & & & & & & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
 & & & & & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
 & & & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
 & & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
+ & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1
\end{array}$$

10.

$$\begin{array}{cccccccccccc}
 & & & & & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
 & & & & \times & & & 1 & 1 & 1 & 1 & 1 \\
\hline
 & 1 & & 1 & & 1 & & & & & & \\
1 & & & 1 & & 1 & & 1 & & & & \\
 & & & & & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
 & & & & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
 & & & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
 & & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
+ & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0
\end{array}$$

11.

$$\begin{array}{cccccccccccccc}
 & & & & & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
 & & & & & \times & & & & 1 & 0 & 0 & 1 & 1 \\
\hline
 & & & & & & 1 & & & & & & & \\
 & & & 1 & 1 & 1 & 1 & & 1 & 1 & 1 & 1 & & \\
 & & & & & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
 & & & & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
+ & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\hline
 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1
\end{array}$$

12.

$$\begin{array}{cccccccccccccc}
 & & & & & & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
 & & & & & & \times & & 1 & 1 & 1 & 0 & 0 & 1 \\
\hline
1 & & 1 & & 1 & 1 & & & & & & & & \\
 & & & 1 & & 1 & & & 1 & 1 & 1 & & & \\
 & & & & & & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
 & & & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
 & & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
+ & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1
\end{array}$$

13.

$$
\begin{array}{ccccccccccccc}
 & & & & & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
 & & & & & \times & & & 1 & 1 & 1 & 1 & 1 \\
\hline
 & 1 & & 1 & & 1 & & 1 & & & & & \\
1 & & & 1 & & 1 & & 1 & & 1 & 1 & & \\
 & & & & & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
 & & & & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
 & & & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
 & & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
+ & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1
\end{array}
$$

14.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
 & & & & & \times & & 1 & 1 & 0 & 1 & 1 & 1 \\
\hline
 & & & 1 & & 1 & & & & & & & \\
1 & 1 & 1 & 1 & & 1 & & 1 & 1 & 1 & 1 & & \\
 & & & & & & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
 & & & & & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
 & & & & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
 & & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
+ & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1
\end{array}
$$

Exercise These multiplications are a bit harder, but I know you can do it.

1.

$$
\begin{array}{cccccccccccccc}
 & & & & & & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
 & & & & & & & \times & 1 & 1 & 1 & 1 & 0 & 1 \\
\hline
 & & 1 & 1 & 1 & 1 & 1 & 1 & & & & & & \\
1 & 1 & 1 & & & 1 & & & & 1 & 1 & & & \\
 & & & & & & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
 & & & & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
 & & & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
 & & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
+ & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1
\end{array}
$$

2.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & & & & \times & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
\hline
 & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\
1 & 1 & 1 & & & 1 & 1 & 1 & 1 & & & & 1 & & \\
 & & & & & & & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & & & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
 & & & & & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
 & & & & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
 & & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
+ & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}
$$

3.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & & & \times & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
 & & & & 1 & & & & & & & & \\
 & 1 & 1 & 1 & 1 & & 1 & 1 & 1 & & & & \\
1 & 1 & & 1 & 1 & 1 & & 1 & & & 1 & & \\
 & & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
 & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
 & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
+ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

4.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & & \times & 1 & 1 & 1 & 1 & 1 \\
\hline
 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\
1 & 1 & & 1 & 1 & 1 & & & 1 & & \\
 & & & & & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
 & & & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
 & & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
+ & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

5.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
 & & & & & & \times & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
 & 1 & 1 & 1 & 1 & 1 & 1 & & 1 & & & & \\
1 & 1 & & & 1 & 1 & 1 & 1 & 1 & & 1 & & \\
 & & & & & & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
 & & & & & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
 & & & & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
 & & & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
 & & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
+ & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{array}
$$

6.

$$
\begin{array}{cccccccccccccc}
 & & & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & & & \times & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
 & & & & 1 & 1 & & & & & & & & \\
 & 1 & 1 & 1 & 1 & & & 1 & 1 & 1 & & & & \\
1 & 1 & & 1 & 1 & 1 & & & 1 & & & 1 & & \\
 & & & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
 & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
 & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
 & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
+ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

Division of binary numbers

Exercise Patterns in binary numbers

1. Complete the table.

2. Underline all even numbers with a blue colour pencil. Even numbers are printed bold (which means thicker)

3. Underline all multiples of 4 with a red colour pencil. Multiples of 4 are printed italic (which means not quite straight, but leaning to the right)

4. Underline all multiples of 8 with a yellow colour pencil. Multiples of 8 are underlined.

5. Underline all multiples of 16 with a green colour pencil. Multiples of 16 are a bit bigger than the rest.

6. Study the numbers you have underlined.

   (a) What do all multiples of 2 have in common? They end with 0.

   (b) What do all multiples of 4 have in common? They end with 00.

   (c) What do all multiples of 8 have in common? They end with 000.

   (d) What do all multiples of 16 have in common? They end with 0000.

1	1001	10001	11001
10	**1010**	**10010**	**11010**
11	1011	10011	11011
100	***1100***	***10100***	***11100***
101	1101	10101	11101
110	**1110**	**10110**	**11110**
111	1111	10111	11111
1000	***10000***	***11000***	***100000***

Exercise Divide these binary numbers by powers of 2. If there is a remainder, write it down as well.

1. $110111010 \div 10 = 11011101$
2. $110111010 \div 100 = 1101110$ r 10
3. $110111010 \div 1000 = 110111$ r 10
4. $110111010 \div 10000 = 11011$ r 1010
5. $11001000 \div 10 = 1100100$
6. $11001000 \div 100 = 110010$
7. $11001000 \div 1000 = 11001$
8. $11001000 \div 10000 = 1100$ r 1000
9. $10011001 \div 10 = 1001100$ r 1
10. $10011001 \div 100 = 100110$ r 1
11. $10111011 \div 1000 = 10111$ r 11
12. $1101100 \div 100 = 11011$
13. $110110 \div 100 = 1101$ r 10
14. $1001100 \div 10 = 100110$
15. $101100 \div 1000 = 101$ r 100
16. $100100 \div 100 = 1001$
17. $10110 \div 10000 = 1$ r 110
18. $11000 \div 100 = 110$
19. $1101110 \div 1000 = 1101$ r 110
20. $11010110 \div 10 = 1101011$

Exercise Calculate the result of these divisions, don't forget to write down the remainder.

1. $110111001 \div 101 = 1011000$ r 1
2. $1000101011 \div 100 = 10001010$ r 11
3. $11100111101 \div 110 = 100110100$ r 101
4. $10111001 \div 111 = 11010$ r 11
5. $11111111 \div 100 = 111111$ r 11
6. $11111111 \div 101 = 110011$
7. $11111111 \div 110 = 101010$ r 11
8. $11111111 \div 111 = 100100$ r 11
9. $1011011101 \div 1010 = 1001001$ r 11
10. $110010010001 \div 1100 = 100001100$ r 1
11. $10010111011 \div 1101 = 1011101$ r 10
12. $1011100111 \div 1011 = 1000011$ r 110
13. $110110110110 \div 1111 = 11101010$
14. $100101101 \div 11101 = 1010$ r 1011
15. $110011010100 \div 11010 = 1111110$ r 1000
16. $1101001101001 \div 11011 = 11111010$ r 1011

To check where you made a mistake, you can compare your work to the long divisions below.

1.
```
       1011000
101 ) 110111001
      -101
         111
        -101
          101
         -101
           0001
```

2.
```
       10001010
100 ) 1000101011
      -100
         00101
          -100
             101
            -100
              11
```

3.
```
       100110100
110 ) 11100111101
      -110
         1001
         -110
            111
           -110
              111
             -110
                101
```

4.
```
       11010
111 ) 10111001
       -111
        1001
        -111
          1000
          -111
             11
```

5.
```
       111111
100 ) 11111111
      -100
         111
        -100
           111
          -100
             111
            -100
               111
              -100
                 111
                -100
                   11
```

6.

```
        110011
101 ) 11111111
      -101
         101
       -101
          0111
          -101
             101
            -101
               0
```

7.

```
        101010
110 ) 11111111
      -110
         111
        -110
           111
          -110
             11
```

8.

```
        110011
111 ) 11111111
      -111
        0111
        -111
           011
```

9.

```
         1001001
1010 ) 1011011101
       -1010
          1011
         -1010
            1101
           -1010
              11
```

10.

```
         100001100
1100 ) 110010010001
       -1100
           010010
           -1100
              1100
             -1100
                 001
```

11.

```
         1011101
1101 ) 10010111011
        -1101
          10111
          -1101
           10101
           -1101
            10000
            -1101
               1111
              -1101
                 10
```

12.

```
         1000011
1011 ) 1011100111
       -1011
          010011
           -1011
            10001
            -1011
              110
```

13.

```
          11101010
1111  )  110110110110
         -1111
           11000
          -1111
            10011
           -1111
              10010
             -1111
                 1111
                -1111
                    00
```

14.

```
          1010
11101  )  100101101
          -11101
            100010
           -11101
               1011
```

15.

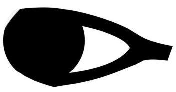
```
           1111110
11010  )  110011010100
          -11010
            110010
           -11010
             110001
            -11010
              101110
             -11010
                101001
               -11010
                   11110
                  -11010
                      1000
```

16.

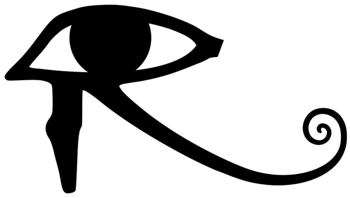
```
           11111010
11011  )  1101001101001
          -11011
            110011
           -11011
             110001
            -11011
              101100
             -11011
                100011
               -11011
                   100000
                  -11011
                      1011
```

Binary fractions

Exercise Write the binary value of these hieroglyphs.



$\frac{5}{16} = \frac{1}{4} + \frac{1}{16}$
0.0101

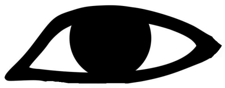


$\frac{55}{64} = \frac{1}{2} + \frac{1}{4} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64}$
0.110111

Exercise Can you draw the hieroglyphs that correspond to these binary values?

0.10011

0.1101



Binairo

1	0	1	0	1	1	0	0	1	0	1	0
0	0	1	1	0	0	1	0	1	0	1	1
0	1	0	1	0	0	1	1	0	1	0	1
1	0	1	0	1	1	0	0	1	1	0	0
0	0	1	0	1	0	0	1	1	0	1	1
0	1	0	1	0	0	1	1	0	1	1	0
1	0	1	0	0	1	1	0	0	1	0	1
0	1	0	0	1	1	0	1	1	0	0	1
1	1	0	1	0	0	1	1	0	0	1	0
0	0	1	1	0	1	1	0	0	1	1	0
1	1	0	0	1	1	0	0	1	0	0	1
1	1	0	1	1	0	0	1	0	1	0	0

0	0	1	1	0	0	1	1	0	0	1	1
0	0	1	0	0	1	0	1	1	0	1	1
1	1	0	0	1	1	0	0	1	1	0	0
0	0	1	1	0	0	1	1	0	1	0	1
0	1	0	0	1	0	0	1	1	0	1	1
1	0	0	1	0	1	1	0	0	1	1	0
0	1	1	0	1	1	0	0	1	0	0	1
1	0	1	0	1	0	1	1	0	1	0	0
1	1	0	1	0	1	0	0	1	0	1	0
0	0	1	0	0	1	1	0	1	0	1	1
1	1	0	1	1	0	0	1	0	1	0	0
1	1	0	1	1	0	1	0	0	1	0	0

7.3 Hexadecimal numbers

7.3.1 Converting decimal numbers to hexadecimal and vice versa

Exercise 1 Practice using these notations by converting the first 40 decimal numbers to hexadecimal ones.

- $0_{dec} = 0_{hex}$
- $1_{dec} = 1_{hex}$
- $2_{dec} = 2_{hex}$
- $3_{dec} = 3_{hex}$
- $4_{dec} = 4_{hex}$
- $5_{dec} = 5_{hex}$
- $6_{dec} = 6_{hex}$
- $7_{dec} = 7_{hex}$
- $8_{dec} = 8_{hex}$
- $9_{dec} = 9_{hex}$
- $10_{dec} = A_{hex}$
- $11_{dec} = B_{hex}$
- $12_{dec} = C_{hex}$
- $13_{dec} = D_{hex}$
- $14_{dec} = E_{hex}$
- $15_{dec} = F_{hex}$
- $16_{dec} = 10_{hex}$
- $17_{dec} = 11_{hex}$
- $18_{dec} = 12_{hex}$
- $19_{dec} = 13_{hex}$
- $20_{dec} = 14_{hex}$
- $21_{dec} = 15_{hex}$
- $22_{dec} = 16_{hex}$
- $23_{dec} = 17_{hex}$
- $24_{dec} = 18_{hex}$
- $25_{dec} = 19_{hex}$
- $26_{dec} = 1A_{hex}$
- $27_{dec} = 1B_{hex}$
- $28_{dec} = 1C_{hex}$
- $29_{dec} = 1D_{hex}$
- $30_{dec} = 1E_{hex}$
- $31_{dec} = 1F_{hex}$
- $32_{dec} = 20_{hex}$
- $33_{dec} = 21_{hex}$
- $34_{dec} = 22_{hex}$
- $35_{dec} = 23_{hex}$
- $36_{dec} = 24_{hex}$
- $37_{dec} = 25_{hex}$
- $38_{dec} = 26_{hex}$
- $39_{dec} = 27_{hex}$

Exercise 2 Fill in the 16 times table

1. $1 \times 16 = 16$
2. $2 \times 16 = 32$
3. $3 \times 16 = 48$
4. $4 \times 16 = 64$
5. $5 \times 16 = 80$
6. $6 \times 16 = 96$
7. $7 \times 16 = 112$
8. $8 \times 16 = 128$
9. $9 \times 16 = 144$
10. $10 \times 16 = 160$
11. $11 \times 16 = 176$
12. $12 \times 16 = 192$
13. $13 \times 16 = 208$
14. $14 \times 16 = 224$
15. $15 \times 16 = 240$

Exercises

3. Convert these decimal numbers to hexadecimal numbers.

(a) $21_{dec} = 15_{hex}$
(b) $50_{dec} = 32_{hex}$
(c) $80_{dec} = 50_{hex}$
(d) $99_{dec} = 63_{hex}$
(e) $123_{dec} = 7B_{hex}$
(f) $165_{dec} = A5_{hex}$
(g) $250_{dec} = FA_{hex}$
(h) $255_{dec} = FF_{hex}$
(i) $345_{dec} = 159_{hex}$
(j) $505_{dec} = 1F9_{hex}$
(k) $627_{dec} = 273_{hex}$
(l) $750_{dec} = 2EE_{hex}$
(m) $806_{dec} = 326_{hex}$
(n) $918_{dec} = 396_{hex}$
(o) $1000_{dec} = 3E8_{hex}$
(p) $1111_{dec} = 457_{hex}$
(q) $1,506_{dec} = 5E2_{hex}$
(r) $2,433_{dec} = 981_{hex}$
(s) $2,137_{dec} = 859_{hex}$
(t) $3,721_{dec} = E89_{hex}$
(u) $7,213_{dec} = 1C2D_{hex}$
(v) $4,000_{dec} = FA0_{hex}$
(w) $16,884_{dec} = 41F4_{hex}$
(x) $123,456_{dec} = 1E240_{hex}$
(y) $345,678_{dec} = 5464E_{hex}$
(z) $567,890_{dec} = 8AA52_{hex}$

4. Convert these hexadecimal numbers to decimal ones.

(a) $16_{hex} = 22$
(b) $21_{hex} = 33$
(c) $2C_{hex} = 44$
(d) $39_{hex} = 57$
(e) $5D_{hex} = 93$
(f) $7B_{hex} = 123$
(g) $8E_{hex} = 142$
(h) $99_{hex} = 153$
(i) $AB_{hex} = 171$
(j) $AF_{hex} = 175$
(k) $BA_{hex} = 186$
(l) $BF_{hex} = 191$
(m) $EE_{hex} = 238$
(n) $FF_{hex} = 255$
(o) $ABC_{hex} = 2,748$
(p) $123_{hex} = 291$
(q) $BFF_{hex} = 3,071$
(r) $CAB_{hex} = 3,243$
(s) $FAB_{hex} = 4,011$
(t) $FEE_{hex} = 4,078$
(u) $2A7_{hex} = 679$
(v) $5B7_{hex} = 1,463$
(w) $234_{hex} = 564$
(x) $ABCDE_{hex} = 703,710$
(y) $111111_{hex} = 1,118,481$
(z) $1010011_{hex} = 16,842,769$

7.3.2 Converting binary numbers to hexadecimal and vice versa

Exercises

1. Match the hexadecimal numbers with their corresponding binary representations.

(a) $ABBA = 1010101110111010$
(b) $3E4 = 1111100100$
(c) $7C4 = 11111000100$
(d) $AB96 = 1010101110010110$
(e) $6E9 = 11011101001$

(f) $3F5 = 1111110101$

(g) $BEAD = 1011111010101101$

(h) $B75D = 1011011101011101$

2. Convert these binary numbers to hexadecimal ones.

(a) $1001\,0110\,1011\,1010\,0101 = 96BA5$

(b) $1\,1110\,0100\,0101\,0101\,0111 = 1E4557$

(c) $1\,0101\,0011\,1110\,0011\,1001 = 153E39$

(d) $1\,1110\,0101\,1100\,1110\,0101 = 1E5CE5$

(e) $1\,1100\,1011\,1010\,0011\,0011 = 1CBA33$

(f) $10\,0001\,0101\,0010\,1000\,1010 = 21528A$

(g) $11\,1101\,0111\,1110\,1011\,1110 = 3D7EBE$

(h) $1\,1110\,0111\,1101\,0101\,1110 = 1E7D5E$

(i) $10\,0011\,0110\,0111\,1101\,0110 = 2367D6$

(j) $101\,0101\,0111\,0110\,0101\,0101 = 557655$

(k) $1\,0100\,1010\,1010\,1010\,0101\,0101\,0001\,0101\,0101\,0100\,1110 = 14AAA551554E$

(l) $1101\,0111\,1111\,0101\,1110\,1101\,1111\,0111\,0111\,0111\,1111 = D7F5EDF777F$

(m) $1\,0001\,0101\,1000\,0011\,0101\,0110\,1000\,0010\,0101\,0100\,0010 = 115835682542$

(n) $100\,1010\,1010\,0100\,1001\,0010\,1100\,1000\,1001\,0100\,1001 = 4AA492C894$

(o) $1\,0100\,1010\,1001\,0101\,0010\,0010\,1001\,1101\,0010\,1001\,0110 = 14A95229D296$

(p) $1111\,1111\,1000\,1000\,0010\,0000\,1000\,0010\,1010\,0100\,1000 = FF882082A48$

(q) $1\,0100\,1001\,0100\,1001\,0010\,0010\,0001\,0000\,0111\,1101\,0011 = 1494922107D3$

(r) $1\,0100\,1100\,1010\,0101\,0010\,1001\,0010\,1010\,0101\,0101\,0001 = 14CA5292A551$

(s) $1\,0100\,1010\,0101\,0100\,0101\,1010\,1010\,1011\,0010\,0101\,0110 = 14A545AAB256$

(t) $1\,0100\,1010\,0100\,1010\,1010\,0101\,0100\,1010\,1010\,1010\,0101 = 14A4AA54AAA5$

3. Convert these hexadecimal numbers to binary ones.

(a) $60 = 1100\,000$

(b) $BEEF = 1011\,1110\,1110\,1111$

(c) $C0DE = 1100\,0000\,1101\,1110$

(d) $FACE = 1111\,1010\,1100\,1110$

(e) $DECAF = 1101\,1110\,1100\,1010\,1111$

(f) $CAFE = 1100\,1010\,1111\,1110$

(g) $98765 = 1001\,1000\,0111\,0110\,0101$

(h) $2137 = 10\,0001\,0011\,0111$

(i) $F37EA = 1111\,0011\,0111\,1110\,1010$

(j) $FE715 = 1111\,1110\,0111\,0001\,0101$

(k) $F104A = 1111\,0001\,0000\,0100\,1010$

(l) $B0D1CE = 1011\,0000\,1101\,0001\,1100\,1110$

(m) $D06FACE = 1101\,0000\,0110\,1111\,1010\,1100\,1110$

(n) $ABCDEF = 1010\,1011\,1100\,1101\,1110\,1111$

(o) $57160 = 101\,0111\,0001\,0110\,0000$

(p) $14E7A8D = 1\,0100\,1110\,0111\,1010\,1000\,1101$

(q) $5A7AD = 101\,1010\,0111\,1010\,1101$

(r) $7E771CE = 111\,1110\,0111\,0111\,0001\,1100\,1110$

(s) $1234567 = 1\,0010\,0011\,0100\,0101\,0110\,0111$

(t) $C705E4 = 1100\,0111\,0000\,0101\,1110\,0100$

7.3.3 Adding hexadecimal numbers

1. Add these hexadecimal numbers.

(a) $5 + 7 = C$
(b) $4 + 6 = A$
(c) $6 + 8 = E$
(d) $4 + 4 = 8$
(e) $7 + 8 = F$
(f) $7 + 7 = E$
(g) $9 + 4 = D$
(h) $8 + 4 = C$
(i) $3 + 4 = 7$
(j) $4 + 5 = 9$
(k) $5 + 6 = B$
(l) $6 + 7 = D$
(m) $9 + 5 = E$
(n) $2 + 9 = B$
(o) $3 + 8 = B$
(p) $7 + 4 = B$
(q) $3 + 9 = C$
(r) $5 + 5 = A$
(s) $6 + 9 = F$
(t) $6 + 6 = C$
(u) $8 + 3 = B$
(v) $5 + 8 = D$
(w) $9 + 1 = A$
(x) $7 + 3 = A$

2. Add these hexadecimal numbers.

(a) $1 + A = B$
(b) $A + 5 = F$
(c) $2 + B = D$
(d) $B + 3 = E$
(e) $3 + C = F$
(f) $C + 1 = D$
(g) $2 + D = F$
(h) $A + 4 = E$
(i) $D + 1 = E$
(j) $2 + A = C$
(k) $1 + B = C$
(l) $B + 4 = F$
(m) $E + 1 = F$
(n) $3 + A = D$
(o) $C + 2 = E$
(p) $1 + D = E$

3. Add these hexadecimal numbers. It keeps getting slightly harder.

(a) $F + 1 = 10$
(b) $4 + E = 12$
(c) $E + 2 = 10$
(d) $5 + B = 10$
(e) $A + 6 = 10$
(f) $6 + E = 14$
(g) $C + 5 = 11$
(h) $3 + D = 10$
(i) $A + 4 = E$
(j) $2 + C = E$
(k) $F + 6 = 15$
(l) $5 + E = 13$
(m) $F + 2 = 11$
(n) $6 + D = 13$
(o) $C + 4 = 10$
(p) $3 + B = E$
(q) $F + 5 = 14$
(r) $3 + E = 11$
(s) $B + 6 = 11$
(t) $5 + D = 12$
(u) $F + 4 = 13$
(v) $6 + C = 12$
(w) $D + 4 = 11$
(x) $3 + F = 12$

4. Add these hexadecimal numbers.

(a) $F + 9 =$
(b) $8 + E =$
(c) $D + 7 =$
(d) $9 + C =$
(e) $B + 8 =$
(f) $7 + A =$
(g) $9 + 9 =$
(h) $F + 8 =$
(i) $8 + 7 =$
(j) $7 + E =$
(k) $A + 9 =$
(l) $8 + 8 =$
(m) $B + 7 =$
(n) $9 + 7 =$
(o) $C + 8 =$
(p) $9 + E =$
(q) $F + 7 =$
(r) $8 + 9 =$
(s) $B + 9 =$
(t) $8 + E =$
(u) $D + 9 =$
(v) $7 + 7 =$
(w) $B + 8 =$
(x) $7 + C =$

5. Add these letters to form new words. Nope, just kidding. Add these hexadecimal numbers.

(a) $A + A = 14$
(b) $B + B = 16$
(c) $C + C = 18$
(d) $D + D = 1A$
(e) $E + E = 1C$
(f) $F + F = 1E$
(g) $A + B = 15$
(h) $B + C = 17$
(i) $C + D = 19$
(j) $D + E = 1B$
(k) $E + F = 1D$
(l) $F + A = 19$
(m) $A + E = 18$
(n) $B + D = 18$
(o) $C + F = 1B$
(p) $D + A = 17$
(q) $E + B = 19$
(r) $F + B = 1A$
(s) $A + C = 16$
(t) $B + F = 1A$
(u) $C + E = 1A$
(v) $D + F = 1C$
(w) $E + A = 18$
(x) $F + C = 1B$

6. Can you add 3 hexadecimal digits?

(a) $4 + 4 + 4 = C$
(b) $5 + 5 + 5 = F$
(c) $6 + 6 + 6 = 12$
(d) $7 + 7 + 7 = 15$
(e) $8 + 8 + 8 = 18$
(f) $9 + 9 + 9 = 1B$
(g) $A + A + A = 1E$
(h) $B + B + B = 21$
(i) $C + C + C = 24$
(j) $D + D + D = 27$
(k) $E + E + E = 2A$
(l) $F + F + F = 2D$
(m) $B + F + F = 29$
(n) $B + E + D = 26$
(o) $A + D + D = 24$
(p) $B + A + D = 22$
(q) $F + A + B = 24$
(r) $D + A + D = 24$
(s) $9 + A + 8 = 1B$
(t) $8 + 6 + F = 1D$
(u) $5 + E + C = 1F$
(v) $4 + F + C = 1F$
(w) $6 + B + 8 = 19$
(x) $3 + F + 7 = 19$

7. Can you add even more?

(a) $D + A + D + A = 2E$
(b) $F + A + 9 + 5 = 27$
(c) $D + E + A + D = 32$
(d) $2 + 7 + 6 + 8 = 17$
(e) $F + 4 + A + 9 = 26$
(f) $3 + C + A + 5 = 1E$
(g) $C + 4 + B + B = 26$
(h) $C + 8 + 6 + E = 28$
(i) $A + B + C + D = 2E$
(j) $E + F + 9 + 8 = 2E$
(k) $5 + F + A + 6 = 24$
(l) $F + A + C + E = 33$

(m) $D{+}A{+}B{+}B{+}E$ =3B (o) $F{+}F{+}F{+}F{+}F$ =4B (q) B+A+B+B+E+1=3A

(n) $9{+}C{+}8{+}F{+}7 = 33$ (p) E+F+F+E+C+7=4D (r) $5{+}7{+}A{+}E{+}D = 31$

8. Find the answers to these additions. Can you find a pattern in what happens?

(a) $50 + F = 5F$ (e) $C4 + F = D3$ (i) $A8 + F = B7$ (m) $CC{+}F = DB$

(b) $41 + F = 50$ (f) $E5 + F = F4$ (j) $39 + F = 48$ (n) $8D + F = 9C$

(c) $B2 + F = C1$ (g) $76 + F = 85$ (k) $9A + F = A9$ (o) $DE{+}F = ED$

(d) $23 + F = 32$ (h) $17 + F = 26$ (l) $3B + F = 4A$ (p) $2F + F = 3E$

Pattern:Adding F to a hexadecimal number works in the same way as adding 9 to a decimal number: if your last digit is 0, it becomes F. If your last digit is anything else, your last digit is decreased by one and your second to last digit is increased by one.

9. Find the answers to these additions. Is there a clear pattern here as well?

(a) $50 + A = 5A$ (e) $C4 + A = CE$ (i) $A8 + A = B2$ (m) $CC + A = D6$

(b) $41 + A = 4B$ (f) $E5 + A = EF$ (j) $39 + A = 43$ (n) $8D + A = 97$

(c) $B2 + A = BC$ (g) $76 + A = 80$ (k) $9A + A = A4$ (o) $DE + A = E8$

(d) $23 + A = 2D$ (h) $17 + A = 21$ (l) $3B + A = 45$ (p) $2F + A = 39$

Pattern: There is a logic to it, but it is not as clear cut as it was while adding F. Adding 10 to a decimal number is a lot easier than adding A to a hexadecimal one.

10. Add these hexadecimal numbers of 2 digits each, without the need to carry over digits.

(a) $47 + 85 = CC$ (g) $25 + 52 = 77$ (m) $C0{+}3F = FF$ (s) $73 + 37 = AA$

(b) $AB + 42$ =ED (h) $55 + 56 = AB$ (n) $21 + DE$ =FF (t) $BA{+}35 = EF$

(c) $56 + 67 = BD$ (i) $69 + 56 = BF$ (o) $67 + 67 = CE$ (u) $38 + 87 = BF$

(d) $2C{+}A0 = CC$ (j) $75 + 75 = EA$ (p) $88 + 55 = DD$ (v) $CC + 21$ =ED

(e) $58 + 96 = EE$ (k) $48 + A4 = EC$ (q) $66 + 99 = FF$ (w) $EE{+}10 = FE$

(f) $7A + 44 = BE$ (l) $C4{+}2B = EF$ (r) $21 + 37 = 58$ (x) $97 + 68 = FF$

11. Add these hexadecimal numbers of 2 digits each, where sometimes you do need to carry over digits.

(a) $23+9E=C1$
(b) $38+39=71$
(c) $79+5F=D8$
(d) $2C+76=A2$
(e) $48+AF=F7$
(f) $5A+9E=F8$
(g) $AE+44=F2$
(h) $BA+21=$DB
(i) $6E+5F=$CD
(j) $CA+27=F1$
(k) $25+FE=123$
(l) $63+7D=E0$
(m) $68+DA=142$
(n) DE+AF=18D
(o) CA+FE=1C8
(p) $B9+87=140$
(q) C1+AA=16B
(r) FE+E1=1DF
(s) AB+BA=165
(t) $BA+79=133$
(u) $99+57=F0$
(v) $87+CD=154$
(w) AB+CD=178
(x) FF+FF=1FE

12. Can you add 3 hexadecimal numbers containing 2 digits each?

(a) $AA+BB+CC=231$
(b) DD+EE+FF=2CA
(c) $38+AB+E0=1C3$
(d) $21+98+54=10D$
(e) $75+BC+25=156$
(f) $CA+88+53=1A5$
(g) $98+6A+3E=140$
(h) $FA+BC+56=20C$
(i) $A9+8B+46=17A$
(j) $37+CD+2A=12E$
(k) $61+58+74=12D$
(l) $1A+2B+3C=81$
(m) $4D+5E+6F=11A$
(n) $A4+A5+A6=1EF$
(o) $BA+BB+E1=256$

13. Add these hexadecimal numbers.

(a)
$$\begin{array}{ccccccc}
 & 1 & 1 & 1 & 1 & 1 & \\
 & 2 & A & 3 & B & 7 & F \\
+ & A & B & C & D & E & F \\
\hline
 & D & 6 & 0 & 9 & 6 & E
\end{array}$$

(b)
$$\begin{array}{cccccccc}
 & 1 & & & 1 & & 1 & \\
 & 5 & C & 8 & 9 & 3 & 5 & B \\
+ & 9 & A & 2 & 4 & D & 7 & 9 \\
\hline
 & F & 6 & A & E & 0 & D & 4
\end{array}$$

(c)
$$\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\
 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 \\
+ & 8 & 9 & A & B & C & D & E & F \\
\hline
1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1
\end{array}$$

(d)
$$\begin{array}{ccccccccc}
1 & 1 & 1 & & & 1 & 1 & 1 & \\
 & 7 & 8 & 9 & 7 & 8 & 9 & 7 & 8 \\
+ & A & B & C & 5 & 6 & A & B & C \\
\hline
1 & 2 & 4 & 5 & C & F & 4 & 3 & 4
\end{array}$$

(e)

$$\begin{array}{ccccccc}
 & 1 & 1 & & 1 & 1 & \\
 & 2 & 8 & D & 5 & 1 & E \\
+ & A & 9 & 5 & 7 & E & 6 \\
\hline
 & D & 2 & 2 & D & 0 & 4
\end{array}$$

(f)
$$\begin{array}{ccccccccc}
 & 1 & 3 & 5 & 7 & 9 & B & D & F \\
+ & E & C & A & 8 & 6 & 4 & 2 & 0 \\
\hline
 & F & F & F & F & F & F & F & F
\end{array}$$

(g)
$$\begin{array}{ccccccccc}
1 & 1 & & & & & 1 & 1 & \\
 & 9 & 9 & A & A & 5 & 5 & E & E \\
+ & B & B & 5 & 5 & 9 & 9 & 5 & 5 \\
\hline
1 & 5 & 4 & F & F & E & F & 4 & 3
\end{array}$$

(h)
$$\begin{array}{ccccccccc}
 & 1 & & 1 & 1 & 1 & & & \\
 & 3 & 7 & 2 & 1 & 5 & 5 & 2 & 5 \\
+ & 9 & 9 & 0 & F & E & E & 9 & 5 \\
\hline
 & D & 0 & 3 & 1 & 4 & 3 & B & A
\end{array}$$

(i)

$$\begin{array}{ccccccc}
1 & & & & 1 & 1 & \\
 & B & 0 & D & 1 & E & 5 \\
+ & F & 0 & 0 & D & 1 & E \\
\hline
1 & A & 0 & D & F & 0 & 3
\end{array}$$

(j)

$$\begin{array}{cccccccc}
 & & 1 & 1 & & 1 & 1 & \\
 & D & 0 & 6 & F & A & C & E \\
+ & 2 & B & A & B & 1 & E & 5 \\
\hline
 & F & C & 1 & A & C & B & 3
\end{array}$$

14. I feel like I am repeating myself here, but I promise that this is the very last time: add these hexadecimal numbers. Any resemblance between the hexadecimal numbers and real words is purely coincidental. Or isn't it?

(a)

$$\begin{array}{cccccccc}
2 & 1 & 1 & 1 & 2 & 2 & 1 & \\
 & E & 1 & E & C & 7 & 3 & D \\
 & F & A & C & A & D & E & 5 \\
+ & D & E & F & 1 & E & C & 7 \\
\hline
2 & B & B & A & 9 & 4 & 9 & 9
\end{array}$$

(b)

$$\begin{array}{ccccccccc}
1 & 1 & 1 & 2 & 1 & 1 & & 1 & \\
 & & C & A & B & 0 & 0 & 5 & E \\
 & B & A & 5 & E & B & A & 1 & 1 \\
+ & F & 0 & 1 & D & A & B & 1 & E \\
\hline
1 & B & 7 & 2 & 7 & 6 & 5 & 8 & D
\end{array}$$

(c)

$$\begin{array}{ccccccccc}
2 & 1 & & 1 & 2 & 1 & 2 & 2 & \\
 & B & 0 & 1 & D & F & A & C & E \\
 & B & E & 5 & 0 & 7 & 7 & E & D \\
+ & C & A & 5 & 5 & E & 7 & 7 & E \\
\hline
2 & 3 & 8 & C & 4 & 5 & A & 3 & 9
\end{array}$$

(e)

$$\begin{array}{cccccccccc}
2 & & & 1 & 2 & 2 & & & 1 & \\
 & E & 5 & 7 & A & F & E & 7 & 7 & E \\
 & F & 0 & 0 & 7 & B & A & 1 & 1 & 5 \\
+ & D & A & 7 & A & B & A & 5 & 3 & 5 \\
\hline
2 & A & F & F & D & 7 & 2 & D & C & 8
\end{array}$$

(d)

$$\begin{array}{ccccccccc}
1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & \\
 & B & E & D & 5 & 7 & E & A & D \\
 & D & E & 7 & E & C & 7 & E & D \\
 & + & D & E & C & 0 & D & E & D \\
\hline
1 & A & B & 4 & 0 & 5 & 4 & 8 & 7
\end{array}$$

(f)

$$\begin{array}{cccccccccc}
1 & 1 & & 1 & 2 & 1 & 2 & 1 & 1 & \\
 & 5 & 7 & 3 & A & D & F & A & 5 & 7 \\
 & 7 & A & 5 & 7 & E & 1 & E & 5 & 5 \\
+ & 7 & E & 5 & 5 & E & 1 & A & 7 & E \\
\hline
1 & 4 & F & E & 8 & A & 3 & 3 & 2 & A
\end{array}$$

(g)

$$\begin{array}{ccccccc}
3 & 2 & 2 & 1 & 2 & 1 & \\
 & A & C & C & E & 5 & 5 \\
 & B & A & 1 & 1 & A & D \\
 & C & A & B & 1 & E & 5 \\
+ & F & A & B & 1 & E & 5 \\
\hline
3 & 2 & C & 4 & 3 & C & C
\end{array}$$

$$\begin{array}{ccccccc}
2 & 1 & 2 & 1 & 1 & 2 & \\
 & F & 1 & 3 & 3 & C & 3 \\
 & 5 & A & F & 3 & 5 & 7 \\
 & 7 & 3 & A & 5 & 3 & 5 \\
 & C & 0 & A & 7 & 3 & D \\
+ & 1 & A & 5 & 5 & 0 & 5 \\
\hline
2 & 9 & A & C & 8 & 9 & 1
\end{array}$$

(h)

(i)

(j)
$$\begin{array}{cccccccccc}
3 & 1 & 1 & 1 & & 2 & 2 & 2 & 2 & \\
 & E & 5 & C & A & 1 & A & 7 & E & D \\
 & F & 1 & 0 & A & 7 & A & B & 1 & E \\
 & A & C & C & 0 & 1 & A & D & E & 5 \\
+ & F & 0 & 0 & 7 & 1 & 0 & 0 & 5 & E \\
\hline
3 & 7 & 3 & 9 & B & C & 0 & 1 & 4 & E
\end{array}
\qquad
\begin{array}{cccccc}
2 & 2 & 3 & 1 & 1 & \\
 & 1 & 0 & A & D & 5 \\
 & 5 & 7 & A & 1 & 3 \\
 & 7 & A & B & 1 & 3 \\
 & B & A & 5 & 3 & D \\
+ & C & A & B & 1 & 3 \\
\hline
2 & 6 & 8 & 0 & 4 & B
\end{array}$$

7.3.4 Subtracting hexadecimal numbers

Exercises

1. Subtract these hexadecimal numbers.

(a) $F - 9 = 6$
(b) $B - 5 = 6$
(c) $C - 1 = B$
(d) $E - 6 = 8$
(e) $A - 7 = 3$
(f) $D - 8 = 5$
(g) $A - 3 = 7$
(h) $B - 9 = 2$
(i) $C - 7 = 5$
(j) $D - 4 = 9$
(k) $E - 2 = C$
(l) $F - 1 = E$
(m) $B - 1 = A$
(n) $D - 6 = 7$
(o) $C - 9 = 3$
(p) $F - 5 = A$
(q) $A - 9 = 1$
(r) $E - 5 = 9$
(s) $F - 2 = D$
(t) $E - 9 = 5$
(u) $D - 3 = A$
(v) $C - 5 = 7$
(w) $B - 4 = 7$
(x) $A - 6 = 4$

2. Subtract these hexadecimal numbers. You can consider this a part two, because there still aren't enough letters in the alphabet to keep going.

(a) $E - 8 = 6$
(b) $A - 2 = 8$
(c) $B - 8 = 3$
(d) $F - 8 = 7$
(e) $F - 3 = C$
(f) $C - 2 = A$
(g) $A - 8 = 2$
(h) $B - 2 = 9$
(i) $C - 3 = 9$
(j) $D - 1 = C$
(k) $E - 4 = A$
(l) $F - 6 = 9$
(m) $D - 5 = 8$
(n) $F - 7 = 8$
(o) $A - 5 = 5$
(p) $E - 1 = D$
(q) $B - 6 = 5$
(r) $C - 8 = 4$
(s) $A - 1 = 9$
(t) $B - 3 = 8$
(u) $C - 6 = 6$
(v) $D - 2 = B$
(w) $E - 3 = B$
(x) $F - 4 = B$

3. Subtract these hexadecimal numbers. It's full on letter time!

(a) $F - A = 5$
(b) $E - C = 2$
(c) $D - B = 2$
(d) $C - B = 1$
(e) $B - A = 1$
(f) $A - A = 0$
(g) $F - C = 3$
(h) $E - A = 4$
(i) $D - B = 2$
(j) $C - A = 2$
(k) $F - F = 0$
(l) $D - A = 3$

(m) $B - B = 0$ (o) $F - B = 4$ (q) $E - A = 4$ (s) $F - D = 2$
(n) $E - D = 1$ (p) $F - E = 1$ (r) $D - C = 1$ (t) $E - B = 2$

4. Work out these subtractions.

(a) $AB - 9 = A2$ (g) $CA - 89 = 41$ (m) $AA - 32 = 78$ (s) $2C - 7 = 25$
(b) $27 - 16 = 11$ (h) $FF - CD = 32$ (n) $BB - 56 = 65$ (t) $37 - 21 = 16$
(c) $FF - 35 = CA$ (i) $9A - 78 = 22$ (o) $CC - 2A = A2$ (u) $CD - BA = 13$
(d) $CA - B9 = 11$ (j) $8B - 7A = 11$ (p) $DD - 79 = 64$ (v) $FE - 9C = 62$
(e) $EA - 99 = 51$ (k) $FA - 82 = 78$ (q) $EE - 40 = AE$ (w) $9B - 69 = 32$
(f) $F6 - A4 = 52$ (l) $1E - 13 = A$ (r) $FF - 1A = E5$ (x) $DC - 45 = 97$

5. Find the answers to these subtractions. Remember that they're all hexadecimal numbers, so be careful when you have to borrow a digit. Can you find a pattern in what happens?

(a) $50 - F = 41$ (e) $C4 - F = B5$ (i) $A8 - F = 99$ (m) $CC - F = BD$
(b) $41 - F = 32$ (f) $E5 - F = D6$ (j) $F9 - F = EA$ (n) $8D - F = 7E$
(c) $B2 - F = A3$ (g) $76 - F = 67$ (k) $9A - F = 8B$ (o) $DE - F = CF$
(d) $23 - F = 14$ (h) $17 - F = 8$ (l) $3B - F = 2C$ (p) $2F - F = 20$

Pattern: Subtracting F from a hexadecimal number leads to a similar results as subtracting 9 from a decimal number: if your last digit is not F, then your last digit increases by one and the second to last digit decreases by one. If your last digit is F, it gets reduced to 0.

6. Find the answers to these subtractions. Is there a clear pattern here as well?

(a) $50 - A = 46$ (e) $C4 - A = BA$ (i) $A8 - A = 9E$ (m) $CC - A = C2$
(b) $41 - A = 37$ (f) $E5 - A = DB$ (j) $F9 - A = EF$ (n) $8D - A = 83$
(c) $B2 - A = A8$ (g) $76 - A = 6C$ (k) $9A - A = 90$ (o) $DE - A = D4$
(d) $23 - A = 19$ (h) $17 - A = D$ (l) $3B - A = 31$ (p) $2F - A = 25$

Pattern: Despite the fact that subtracting 10 from a decimal number is quite easy, subtracting A from a hexadecimal number isn't that straightforward. We're stuck with converting the number to decimal, subtracting 10 from that and converting it back to hexadecimal.

7. Work out these subtractions.

(a) $11 - 9 = 8$
(b) $14 - B = 9$
(c) $22 - F = 13$
(d) $56 - 3A = 1C$
(e) $78 - 4B = 2D$
(f) $9A - 7C = 1E$
(g) $15 - 8 = D$
(h) $23 - D = 16$
(i) $4F - 19 = 36$
(j) $DC - CD = F$
(k) $56 - 4E = 8$
(l) $21 - F = 12$
(m) $37 - 2C = B$
(n) $50 - 3E = 12$
(o) $DA - AD =$2D
(p) $CD - BE = F$
(q) $7E - 5F = 1F$
(r) $5A - 4F = B$
(s) $A1 - 93 = E$
(t) $BA - AB = F$
(u) $10 - 1 = F$
(v) $F1 - 34 = BD$
(w) $D3 - A5 = 2E$
(x) $16 - F = 7$

8. Let us try some subtractions with 3 digits.

(a) $123 - 45 = DE$
(b) $456 - AB = 3AB$
(c) $789 - 5A = 72F$
(d) $ABC - 96 = A26$
(e) $DEF - F5 = CFA$
(f) $DAD - FF = CAE$
(g) $D06 - CA7 = 5F$
(h) $CA7 - DA = BCD$
(i) $709 - EF = 61A$
(j) $FAB - 9D = F0E$
(k) $AC7 - 60 = A67$
(l) $6E1 - B1 = 630$
(m) $370 - 21 = 34F$
(n) $C1A - B0 = B6A$
(o) $555 - 1E = 537$

9. Subtractions with hexadecimal numbers don't get any harder than this.

(a)
$$\begin{array}{cccccccc}
 & 8 & 1A & 4 & 1B & B & 14 \\
 & \cancel{9} & \cancel{A} & \cancel{5} & \cancel{B} & \cancel{C} & \cancel{4} \\
- & 6 & F & 2 & E & A & 9 \\
\hline
 & 2 & B & 2 & D & 1 & B
\end{array}$$

(d)
$$\begin{array}{ccccc}
 & C & F & 16 & \\
 & \cancel{D} & \cancel{0} & \cancel{6} & 5 \\
- & C & A & 7 & 5 \\
\hline
 & & 5 & F & 0
\end{array}$$

(b)
$$\begin{array}{ccccccc}
 & E & F & F & 11 & & \\
 & \cancel{F} & \cancel{0} & \cancel{0} & \cancel{1} & E & D \\
- & 7 & 6 & 9 & 8 & 5 & A \\
\hline
 & 7 & 9 & 6 & 9 & 9 & 3
\end{array}$$

(e)
$$\begin{array}{ccccccccc}
 & & 8 & 10 & & & & & \\
 & 9 & \cancel{9} & \cancel{0} & F & E & E & 9 & 5 \\
- & 3 & 7 & 2 & 1 & 5 & 5 & 2 & 5 \\
\hline
 & 6 & 1 & E & E & 9 & 9 & 7 & 0
\end{array}$$

(c)
$$\begin{array}{ccccccc}
 & & 18 & 14 & & & \\
 & A & \cancel{8} & \cancel{4} & F & 18 & \\
 & \cancel{B} & \cancel{9} & \cancel{5} & \cancel{0} & \cancel{8} & A \\
- & 9 & A & B & E & A & 3 \\
\hline
 & 1 & E & 9 & 1 & E & 7
\end{array}$$

(f)
$$\begin{array}{cccccc}
 & & & 16 & 15 & \\
 & & 7 & \cancel{6} & \cancel{5} & 15 \\
 & 9 & \cancel{8} & \cancel{7} & \cancel{6} & \cancel{5} \\
- & 5 & 7 & 9 & B & D \\
\hline
 & 4 & 0 & D & A & 8
\end{array}$$

(g)
$$\begin{array}{ccccccc}
 & & & & 1B & 1A & \\
 & & & C & \cancel{B} & \cancel{A} & 1A \\
 & F & E & \cancel{D} & \cancel{C} & \cancel{B} & \cancel{A} \\
- & A & B & C & D & E & F \\
\hline
 & 5 & 3 & 0 & E & C & B
\end{array}$$

(h)

$$\begin{array}{cccccccc}
 & & 16 & & & & \\
 & B & \cancel{6} & 15 & D & 19 & \\
 & \cancel{C} & \cancel{7} & \cancel{5} & \cancel{E} & \cancel{9} & F \\
- & B & A & 8 & 0 & E & 1 \\
\hline
 & & C & D & D & B & E
\end{array}$$

(i)

$$\begin{array}{ccccccccc}
 & & & & & 15 & 13 & 11 & \\
 & & & & 7 & \cancel{5} & \cancel{3} & \cancel{1} & 10 \\
 & E & C & A & \cancel{8} & \cancel{6} & \cancel{4} & \cancel{2} & \cancel{0} \\
- & 1 & 3 & 5 & 7 & 9 & B & D & F \\
\hline
 & D & 9 & 5 & 0 & C & 8 & 4 & 1
\end{array}$$

(j)

$$\begin{array}{ccccccccc}
 & & & 14 & & & & 14 & \\
 & & A & \cancel{4} & 15 & & 8 & \cancel{4} & 15 \\
 & B & \cancel{B} & \cancel{5} & \cancel{5} & 9 & \cancel{9} & \cancel{5} & \cancel{5} \\
- & 9 & 9 & A & A & 5 & 5 & E & E \\
\hline
 & 2 & 1 & A & B & 4 & 3 & 6 & 7
\end{array}$$

10. Solve these subtractions. Again, I claim that any resemblance to existing words is purely coincidental. But I will admit that making these exercises was fun!

(a)

$$\begin{array}{cccccccc}
 & & & & 15 & 10 & & \\
 & & & 5 & \cancel{5} & \cancel{0} & 1E & \\
 & D & 0 & \cancel{6} & \cancel{6} & \cancel{1} & \cancel{E} & 5 \\
- & & & & F & 1 & F & 1 \\
\hline
 & D & 0 & 5 & 6 & F & F & 4
\end{array}$$

(b)

$$\begin{array}{ccccccc}
 & E & F & 10 & C & 11 & \\
 & \cancel{F} & \cancel{0} & \cancel{0} & \cancel{D} & \cancel{1} & E \\
- & B & 0 & D & 1 & E & 5 \\
\hline
 & 3 & F & 3 & B & 3 & 9
\end{array}$$

(c)

$$\begin{array}{cccccccc}
 & C & F & 16 & & 9 & 1C & \\
 & \cancel{D} & \cancel{0} & \cancel{6} & F & \cancel{A} & \cancel{C} & E \\
- & 2 & B & A & B & 1 & E & 5 \\
\hline
 & A & 4 & C & 4 & 8 & E & 9
\end{array}$$

(d)

$$\begin{array}{cccccc}
A & F & 15 & & & \\
\cancel{B} & \cancel{0} & \cancel{5} & 5 & E & 5 \\
- & & B & 0 & 5 & 5 \\
\hline
A & F & A & 5 & 9 & 0
\end{array}$$

(e)

$$\begin{array}{cccccc}
 & & 14 & & & \\
 & 9 & \cancel{4} & 17 & & \\
C & \cancel{A} & \cancel{5} & \cancel{7} & 1 & E \\
- & & C & E & 1 & 1 \\
\hline
C & 9 & 8 & 9 & 0 & D
\end{array}$$

(f)

$$\begin{array}{ccccc}
 & B & 11 & & \\
 & \cancel{C} & \cancel{1} & E & F \\
- & B & A & 5 & E \\
\hline
 & & 7 & 9 & 1
\end{array}$$

(g)

$$\begin{array}{ccccccccc}
 & E & F & F & 17 & & & & \\
 & \cancel{F} & \cancel{0} & \cancel{0} & \cancel{7} & B & A & 1 & 1 \\
- & B & A & 5 & E & B & A & 1 & 1 \\
\hline
 & 3 & 5 & A & 9 & 0 & 0 & 0 & 0
\end{array}$$

(h)

$$\begin{array}{cccccc}
D & E & F & E & A & 7 \\
- & & 6 & 0 & A & 1 \\
\hline
D & E & 9 & E & 0 & 6
\end{array}$$

(i)

$$\begin{array}{ccccc}
E & 11 & & 4 & 15 \\
\cancel{F} & \cancel{1} & 0 & \cancel{5} & \cancel{5} \\
- & 6 & 0 & 0 & D \\
\hline
E & B & 0 & 4 & 8
\end{array}$$

(j)

$$\begin{array}{ccccccc}
 & & & & 9 & 17 & \\
 & D & E & B & \cancel{A} & \cancel{7} & E \\
- & C & 1 & 0 & 5 & E & D \\
\hline
 & 1 & D & B & 4 & 9 & 1
\end{array}$$

7.3.5 Multiplication of hexadecimal numbers

1. The 16 times table in decimal numbers. Convert the results to hexadecimal numbers as well.

   (a) $1 \times 16 = 16_{dec} = 10_{hex}$
   (b) $2 \times 16 = 32_{dec} = 20_{hex}$
   (c) $3 \times 16 = 48_{dec} = 30_{hex}$
   (d) $4 \times 16 = 64_{dec} = 40_{hex}$
   (e) $5 \times 16 = 80_{dec} = 50_{hex}$
   (f) $6 \times 16 = 96_{dec} = 60_{hex}$
   (g) $7 \times 16 = 112_{dec} = 70_{hex}$
   (h) $8 \times 16 = 128_{dec} = 80_{hex}$
   (i) $9 \times 16 = 144_{dec} = 90_{hex}$
   (j) $10 \times 16 = 160_{dec} = A0_{hex}$
   (k) $11 \times 16 = 176_{dec} = B0_{hex}$
   (l) $12 \times 16 = 192_{dec} = C0_{hex}$
   (m) $13 \times 16 = 208_{dec} = D0_{hex}$
   (n) $14 \times 16 = 224_{dec} = E0_{hex}$
   (o) $15 \times 16 = 240_{dec} = F0_{hex}$
   (p) $16 \times 16 = 256_{dec} = 100_{hex}$

2. The 2 times table

   (a) $1 \times 2 = 2_{dec} = 2_{hex}$
   (b) $2 \times 2 = 4_{dec} = 4_{hex}$
   (c) $3 \times 2 = 6_{dec} = 6_{hex}$
   (d) $4 \times 2 = 8_{dec} = 8_{hex}$
   (e) $5 \times 2 = 10_{dec} = A_{hex}$
   (f) $6 \times 2 = 12_{dec} = C_{hex}$
   (g) $7 \times 2 = 14_{dec} = E_{hex}$
   (h) $8 \times 2 = 16_{dec} = 10_{hex}$
   (i) $9 \times 2 = 18_{dec} = 12_{hex}$
   (j) $10 \times 2 = 20_{dec} = 14_{hex}$
   (k) $11 \times 2 = 22_{dec} = 16_{hex}$
   (l) $12 \times 2 = 24_{dec} = 18_{hex}$
   (m) $13 \times 2 = 26_{dec} = 1A_{hex}$
   (n) $14 \times 2 = 28_{dec} = 1C_{hex}$
   (o) $15 \times 2 = 30_{dec} = 1E_{hex}$
   (p) $16 \times 2 = 32_{dec} = 20_{hex}$

3. The 3 times table is next.

   (a) $1 \times 3 = 3_{dec} = 3_{hex}$
   (b) $2 \times 3 = 6_{dec} = 6_{hex}$
   (c) $3 \times 3 = 9_{dec} = 9_{hex}$
   (d) $4 \times 3 = 12_{dec} = C_{hex}$
   (e) $5 \times 3 = 15_{dec} = F_{hex}$
   (f) $6 \times 3 = 18_{dec} = 12_{hex}$
   (g) $7 \times 3 = 21_{dec} = 15_{hex}$
   (h) $8 \times 3 = 24_{dec} = 18_{hex}$
   (i) $9 \times 3 = 27_{dec} = 1B_{hex}$
   (j) $10 \times 3 = 30_{dec} = 1E_{hex}$
   (k) $11 \times 3 = 33_{dec} = 21_{hex}$
   (l) $12 \times 3 = 36_{dec} = 24_{hex}$
   (m) $13 \times 3 = 39_{dec} = 27_{hex}$
   (n) $14 \times 3 = 42_{dec} = 2A_{hex}$
   (o) $15 \times 3 = 45_{dec} = 2D_{hex}$
   (p) $16 \times 3 = 48_{dec} = 30_{hex}$

4. Do the 4 times table now.

   (a) $1 \times 4 = 4_{dec} = 4_{hex}$
   (b) $2 \times 4 = 8_{dec} = 8_{hex}$
   (c) $3 \times 4 = 12_{dec} = C_{hex}$
   (d) $4 \times 4 = 16_{dec} = 10_{hex}$
   (e) $5 \times 4 = 20_{dec} = 14_{hex}$
   (f) $6 \times 4 = 24_{dec} = 18_{hex}$
   (g) $7 \times 4 = 28_{dec} = 1C_{hex}$
   (h) $8 \times 4 = 32_{dec} = 20_{hex}$
   (i) $9 \times 4 = 36_{dec} = 24_{hex}$
   (j) $10 \times 4 = 40_{dec} = 28_{hex}$
   (k) $11 \times 4 = 44_{dec} = 2C_{hex}$
   (l) $12 \times 4 = 48_{dec} = 30_{hex}$
   (m) $13 \times 4 = 52_{dec} = 34_{hex}$
   (n) $14 \times 4 = 56_{dec} = 38_{hex}$
   (o) $15 \times 4 = 60_{dec} = 3C_{hex}$
   (p) $16 \times 4 = 64_{dec} = 40_{hex}$

5. Next up is the 5 times table.

   (a) $1 \times 5 = 5_{dec} = 5_{hex}$
   (b) $2 \times 5 = 10_{dec} = A_{hex}$
   (c) $3 \times 5 = 15_{dec} = F_{hex}$
   (d) $4 \times 5 = 20_{dec} = 14_{hex}$
   (e) $5 \times 5 = 25_{dec} = 19_{hex}$
   (f) $6 \times 5 = 30_{dec} = 1E_{hex}$
   (g) $7 \times 5 = 35_{dec} = 23_{hex}$
   (h) $8 \times 5 = 40_{dec} = 28_{hex}$
   (i) $9 \times 5 = 45_{dec} = 2D_{hex}$
   (j) $10 \times 5 = 50_{dec} = 32_{hex}$
   (k) $11 \times 5 = 55_{dec} = 37_{hex}$
   (l) $12 \times 5 = 60_{dec} = 3C_{hex}$
   (m) $13 \times 5 = 65_{dec} = 41_{hex}$
   (n) $14 \times 5 = 70_{dec} = 46_{hex}$
   (o) $15 \times 5 = 75_{dec} = 4B_{hex}$
   (p) $16 \times 5 = 80_{dec} = 50_{hex}$

6. The 6 times table is next. Can you find the decimal 6 times tables up to $10 \times 6 = 60$ in your hexadecimal results?

   (a) $1 \times 6 = 6_{dec} = 6_{hex}$
   (b) $2 \times 6 = 12_{dec} = C_{hex}$
   (c) $3 \times 6 = 18_{dec} = 12_{hex}$
   (d) $4 \times 6 = 24_{dec} = 18_{hex}$
   (e) $5 \times 6 = 30_{dec} = 1E_{hex}$
   (f) $6 \times 6 = 36_{dec} = 24_{hex}$
   (g) $7 \times 6 = 42_{dec} = 2A_{hex}$
   (h) $8 \times 6 = 48_{dec} = 30_{hex}$
   (i) $9 \times 6 = 54_{dec} = 36_{hex}$
   (j) $10 \times 6 = 60_{dec} = 3C_{hex}$
   (k) $11 \times 6 = 66_{dec} = 42_{hex}$
   (l) $12 \times 6 = 72_{dec} = 48_{hex}$
   (m) $13 \times 6 = 78_{dec} = 4E_{hex}$
   (n) $14 \times 6 = 84_{dec} = 54_{hex}$
   (o) $15 \times 6 = 90_{dec} = 5A_{hex}$
   (p) $16 \times 6 = 96_{dec} = 60_{hex}$

7. Let's do the 7 times table.

(a) $1 \times 7 = 7_{dec} = 7_{hex}$
(b) $2 \times 7 = 14_{dec} = E_{hex}$
(c) $3 \times 7 = 21_{dec} = 15_{hex}$
(d) $4 \times 7 = 28_{dec} = 1C_{hex}$
(e) $5 \times 7 = 35_{dec} = 23_{hex}$
(f) $6 \times 7 = 42_{dec} = 2A_{hex}$
(g) $7 \times 7 = 49_{dec} = 31_{hex}$
(h) $8 \times 7 = 56_{dec} = 38_{hex}$
(i) $9 \times 7 = 63_{dec} = 3F_{hex}$
(j) $10 \times 7 = 70_{dec} = 46_{hex}$
(k) $11 \times 7 = 77_{dec} = 4D_{hex}$
(l) $12 \times 7 = 84_{dec} = 54_{hex}$
(m) $13 \times 7 = 91_{dec} = 5B_{hex}$
(n) $14 \times 7 = 98_{dec} = 62_{hex}$
(o) $15 \times 7 = 105_{dec} = 69_{hex}$
(p) $16 \times 7 = 112_{dec} = 70_{hex}$

8. At this point you are over halfway with the times tables. Next up is the 8 times table, which has a pretty easy conversion from decimal to hexadecimal.

(a) $1 \times 8 = 8_{dec} = 8_{hex}$
(b) $2 \times 8 = 16_{dec} = 10_{hex}$
(c) $3 \times 8 = 24_{dec} = 18_{hex}$
(d) $4 \times 8 = 32_{dec} = 20_{hex}$
(e) $5 \times 8 = 40_{dec} = 28_{hex}$
(f) $6 \times 8 = 48_{dec} = 30_{hex}$
(g) $7 \times 8 = 56_{dec} = 38_{hex}$
(h) $8 \times 8 = 64_{dec} = 40_{hex}$
(i) $9 \times 8 = 72_{dec} = 48_{hex}$
(j) $10 \times 8 = 80_{dec} = 50_{hex}$
(k) $11 \times 8 = 88_{dec} = 58_{hex}$
(l) $12 \times 8 = 96_{dec} = 60_{hex}$
(m) $13 \times 8 = 104_{dec} = 68_{hex}$
(n) $14 \times 8 = 112_{dec} = 70_{hex}$
(o) $15 \times 8 = 120_{dec} = 78_{hex}$
(p) $16 \times 8 = 128_{dec} = 80_{hex}$

9. The 9 times table is next.

(a) $1 \times 9 = 9_{dec} = 9_{hex}$
(b) $2 \times 9 = 18_{dec} = 12_{hex}$
(c) $3 \times 9 = 27_{dec} = 1B_{hex}$
(d) $4 \times 9 = 36_{dec} = 24_{hex}$
(e) $5 \times 9 = 45_{dec} = 2D_{hex}$
(f) $6 \times 9 = 54_{dec} = 36_{hex}$
(g) $7 \times 9 = 63_{dec} = 3F_{hex}$
(h) $8 \times 9 = 72_{dec} = 48_{hex}$
(i) $9 \times 9 = 81_{dec} = 51_{hex}$
(j) $10 \times 9 = 90_{dec} = 5A_{hex}$
(k) $11 \times 9 = 99_{dec} = 63_{hex}$
(l) $12 \times 9 = 108_{dec} = 6C_{hex}$
(m) $13 \times 9 = 117_{dec} = 75_{hex}$
(n) $14 \times 9 = 126_{dec} = 7E_{hex}$
(o) $15 \times 9 = 135_{dec} = 87_{hex}$
(p) $16 \times 9 = 144_{dec} = 90_{hex}$

10. The 10 times table is next. At least finding you decimal answers is easy!

(a) $1 \times 10 = 10_{dec} = A_{hex}$
(b) $2 \times 10 = 20_{dec} = 14_{hex}$
(c) $3 \times 10 = 30_{dec} = 1E_{hex}$
(d) $4 \times 10 = 40_{dec} = 28_{hex}$
(e) $5 \times 10 = 50_{dec} = 32_{hex}$
(f) $6 \times 10 = 60_{dec} = 3C_{hex}$
(g) $7 \times 10 = 70_{dec} = 46_{hex}$
(h) $8 \times 10 = 80_{dec} = 50_{hex}$
(i) $9 \times 10 = 90_{dec} = 5A_{hex}$
(j) $10 \times 10 = 100_{dec} = 64_{hex}$
(k) $11 \times 10 = 110_{dec} = 6E_{hex}$
(l) $12 \times 10 = 120_{dec} = 78_{hex}$
(m) $13 \times 10 = 130_{dec} = 82_{hex}$
(n) $14 \times 10 = 140_{dec} = 8C_{hex}$
(o) $15 \times 10 = 150_{dec} = 96_{hex}$
(p) $16 \times 10 = 160_{dec} = A0_{hex}$

11. Next up is the 11 times table. Do you notice any patterns in your decimal results?

(a) $1 \times 11 = 11_{dec} = B_{hex}$
(b) $2 \times 11 = 22_{dec} = 16_{hex}$
(c) $3 \times 11 = 33_{dec} = 21_{hex}$
(d) $4 \times 11 = 44_{dec} = 2C_{hex}$
(e) $5 \times 11 = 55_{dec} = 37_{hex}$
(f) $6 \times 11 = 66_{dec} = 42_{hex}$
(g) $7 \times 11 = 77_{dec} = 4D_{hex}$
(h) $8 \times 11 = 88_{dec} = 58_{hex}$
(i) $9 \times 11 = 99_{dec} = 63_{hex}$
(j) $10 \times 11 = 110_{dec} = 6E_{hex}$
(k) $11 \times 11 = 121_{dec} = 79_{hex}$
(l) $12 \times 11 = 132_{dec} = 84_{hex}$
(m) $13 \times 11 = 143_{dec} = 8F_{hex}$
(n) $14 \times 11 = 154_{dec} = 9A_{hex}$
(o) $15 \times 11 = 165_{dec} = A5_{hex}$
(p) $16 \times 11 = 176_{dec} = B0_{hex}$

12. The 12 times table is next.

(a) $1 \times 12 = 12_{dec} = C_{hex}$
(b) $2 \times 12 = 24_{dec} = 18_{hex}$
(c) $3 \times 12 = 36_{dec} = 24_{hex}$
(d) $4 \times 12 = 48_{dec} = 30_{hex}$
(e) $5 \times 12 = 60_{dec} = 3C_{hex}$
(f) $6 \times 12 = 72_{dec} = 48_{hex}$
(g) $7 \times 12 = 84_{dec} = 54_{hex}$
(h) $8 \times 12 = 96_{dec} = 60_{hex}$
(i) $9 \times 12 = 108_{dec} = 6C_{hex}$
(j) $10 \times 12 = 120_{dec} = 78_{hex}$
(k) $11 \times 12 = 132_{dec} = 84_{hex}$
(l) $12 \times 12 = 144_{dec} = 90_{hex}$
(m) $13 \times 12 = 156_{dec} = 9C_{hex}$
(n) $14 \times 12 = 168_{dec} = A8_{hex}$
(o) $15 \times 12 = 180_{dec} = B4_{hex}$
(p) $16 \times 12 = 192_{dec} = C0_{hex}$

13. The 13 times table is next.

(a) $1 \times 13 = 13_{dec} = D_{hex}$
(b) $2 \times 13 = 26_{dec} = 1A_{hex}$
(c) $3 \times 13 = 39_{dec} = 27_{hex}$
(d) $4 \times 13 = 52_{dec} = 34_{hex}$

(e) $5 \times 13 = 65_{dec} = 41_{hex}$
(f) $6 \times 13 = 78_{dec} = 4E_{hex}$
(g) $7 \times 13 = 91_{dec} = 5B_{hex}$
(h) $8 \times 13 = 104_{dec} = 68_{hex}$
(i) $9 \times 13 = 117_{dec} = 75_{hex}$
(j) $10 \times 13 = 130_{dec} = 82_{hex}$
(k) $11 \times 13 = 143_{dec} = 8F_{hex}$
(l) $12 \times 13 = 156_{dec} = 9C_{hex}$
(m) $13 \times 13 = 169_{dec} = A9_{hex}$
(n) $14 \times 13 = 182_{dec} = B6_{hex}$
(o) $15 \times 13 = 195_{dec} = C3_{hex}$
(p) $16 \times 13 = 208_{dec} = D0_{hex}$

14. The second to last times table you need is the 14 times table.

(a) $1 \times 14 = 14_{dec} = E_{hex}$
(b) $2 \times 14 = 28_{dec} = 1C_{hex}$
(c) $3 \times 14 = 42_{dec} = 2A_{hex}$
(d) $4 \times 14 = 56_{dec} = 38_{hex}$
(e) $5 \times 14 = 70_{dec} = 46_{hex}$
(f) $6 \times 14 = 84_{dec} = 54_{hex}$
(g) $7 \times 14 = 98_{dec} = 62_{hex}$
(h) $8 \times 14 = 112_{dec} = 70_{hex}$
(i) $9 \times 14 = 126_{dec} = 7E_{hex}$
(j) $10 \times 14 = 140_{dec} = 8C_{hex}$
(k) $11 \times 14 = 154_{dec} = 9A_{hex}$
(l) $12 \times 14 = 168_{dec} = A8_{hex}$
(m) $13 \times 14 = 182_{dec} = B6_{hex}$
(n) $14 \times 14 = 196_{dec} = C4_{hex}$
(o) $15 \times 14 = 210_{dec} = D2_{hex}$
(p) $16 \times 14 = 224_{dec} = E0_{hex}$

15. The last times table we'll do is the 15 times table.

(a) $1 \times 15 = 15_{dec} = F_{hex}$
(b) $2 \times 15 = 30_{dec} = 1E_{hex}$
(c) $3 \times 15 = 45_{dec} = 2D_{hex}$
(d) $4 \times 15 = 60_{dec} = 3C_{hex}$
(e) $5 \times 15 = 75_{dec} = 4B_{hex}$
(f) $6 \times 15 = 90_{dec} = 5A_{hex}$
(g) $7 \times 15 = 105_{dec} = 69_{hex}$
(h) $8 \times 15 = 120_{dec} = 78_{hex}$
(i) $9 \times 15 = 135_{dec} = 87_{hex}$
(j) $10 \times 15 = 150_{dec} = 96_{hex}$
(k) $11 \times 15 = 165_{dec} = A5_{hex}$
(l) $12 \times 15 = 180_{dec} = B4_{hex}$
(m) $13 \times 15 = 195_{dec} = C3_{hex}$
(n) $14 \times 15 = 210_{dec} = D2_{hex}$
(o) $15 \times 15 = 225_{dec} = E1_{hex}$
(p) $16 \times 15 = 240_{dec} = F0_{hex}$

16. Now that you have reviewed all your times tables, we'll start multiplying hexadecimal numbers. Be very careful when you convert numbers from decimal to hexadecimal and the other way around.

(a) $8 \times 8 = 40$
(b) $9 \times 9 = 51$
(c) $A \times A = 64$
(d) $B \times B = 79$
(e) $C \times C = 90$
(f) $D \times D = A9$
(g) $E \times E = C4$
(h) $F \times F = E1$
(i) $A \times F = 96$

(j) $B \times C = 84$
(k) $C \times A = 78$
(l) $D \times E = B6$
(m) $E \times B = 9A$
(n) $F \times C = B4$
(o) $A \times D = 82$
(p) $B \times F = A5$
(q) $C \times E = A8$
(r) $D \times B = 8F$
(s) $E \times A = 8C$
(t) $F \times D = C3$
(u) $7 \times F = 69$
(v) $A \times B = 6E$
(w) $C \times D = 9C$
(x) $E \times F = D2$

17. Now it's time to multiply numbers with more than one digit. Be mindful of conversions and of the digits you carry over!

(a) $17 \times 5 = 73$
(b) $23 \times 8 = 118$
(c) $46 \times A = 2BC$
(d) $67 \times C = 4D4$
(e) $9B \times D = 7DF$
(f) $AB \times F = A05$
(g) $C4 \times E = AB8$
(h) $8E \times 9 = 4FE$
(i) $FF \times 2 = 1FE$
(j) $5D \times B = 3FF$
(k) $123 \times F = 110D$
(l) $456 \times 6 = 1A04$
(m) $789 \times 7 = 34BF$
(n) $ABC \times A = 6B58$
(o) $DEF \times 4 = 37BC$

18. And now, finally, the thing that we've all been waiting for... drumroll... long multiplications with hexadecimal numbers!

(a)

$$
\begin{array}{ccccccc}
 & & & 1 & 1 & 1 & \\
 & & 1 & 2 & 3 & 4 & \\
 & & 2 & 4 & 6 & 8 & A \\
 & & \times & & & 3 & 7 \\
\hline
 & 1 & 1 & 1 & 1 & & \\
 & & F & E & D & C & 6 \\
+ & 6 & D & 3 & 9 & E & 0 \\
\hline
 & 7 & D & 2 & 7 & A & 6
\end{array}
$$

(b)

$$
\begin{array}{cccccccc}
 & & & & 3 & 2 & 4 & \\
 & & & & 1 & & 1 & \\
 & & & 9 & 1 & 5 & 3 & 7 \\
 & & & \times & & & A & 4 \\
\hline
 & & 1 & & & 1 & & \\
 & & 2 & 4 & 5 & 4 & D & C \\
+ & 5 & A & D & 4 & 2 & 6 & 0 \\
\hline
 & 5 & D & 1 & 9 & 7 & 3 & C
\end{array}
$$

(c)

$$
\begin{array}{cccccccc}
 & & & & 1 & 8 & 2 & \\
 & & & & 1 & A & 3 & \\
 & & & A & 1 & 1 & E & 5 \\
 & & & \times & & & 9 & C \\
\hline
 & 1 & 1 & & & 1 & & \\
 & & 7 & 8 & D & 6 & B & C \\
+ & 5 & A & A & 1 & 0 & D & 0 \\
\hline
 & 6 & 2 & 2 & E & 7 & 8 & C
\end{array}
$$

(d)

$$
\begin{array}{cccccccc}
 & & & & & 8 & 5 & \\
 & & & & & 5 & 3 & \\
 & & & F & 1 & 0 & A & 7 \\
 & & & \times & & & D & 8 \\
\hline
 & & 1 & 1 & & & & \\
 & & 7 & 8 & 8 & 5 & 3 & 8 \\
+ & C & 3 & D & 8 & 7 & B & 0 \\
\hline
 & C & B & 6 & 0 & C & E & 8
\end{array}
$$

(e)

$$
\begin{array}{cccccccc}
 & & & & & & 1 & \\
 & & & & & 1 & 3 & \\
 & & & F & 1 & 0 & 4 & A \\
 & & & \times & & & 2 & 5 \\
\hline
 & 1 & & & & & & \\
 & & 4 & B & 5 & 1 & 7 & 2 \\
+ & 1 & E & 2 & 0 & 9 & 4 & 0 \\
\hline
 & 2 & 2 & D & 5 & A & B & 2
\end{array}
$$

(f)

$$
\begin{array}{cccccccc}
 & & & 1 & 3 & 8 & & \\
 & & & 1 & 5 & 2 & 5 & \\
 & & & B & 2 & E & 5 & F \\
 & & & \times & & & 9 & 6 \\
\hline
 & & 4 & 3 & 1 & 6 & 3 & A \\
+ & 6 & 4 & A & 1 & 5 & 7 & 0 \\
\hline
 & 6 & 8 & D & 2 & B & A & A
\end{array}
$$

(g)

(h)

$$
\begin{array}{ccccccc}
 & & 8 & 8 & 1 & A & \\
 & & F & A & B & 1 & E \\
 & & \times & & & 1 & C \\
\hline
 & 1 & & & 1 & & \\
 & B & C & 0 & 5 & 6 & 8 \\
+ & F & A & B & 1 & E & 0 \\
\hline
1 & B & 6 & B & 7 & 4 & 8
\end{array}
$$

$$\begin{array}{cccccccc}
 & & & 2 & 4 & 4 & 2 & \\
 & & & & 1 & 1 & & \\
 & & & 6 & 4 & 8 & 8 & 5 \\
 & & & \times & & & 9 & 3 \\
\hline
 & & & 1 & 1 & 1 & & \\
 & & 1 & 2 & D & 9 & 8 & F \\
+ & 3 & 8 & 8 & C & A & D & 0 \\
\hline
 & 3 & 9 & B & A & 4 & 5 & F
\end{array}$$

(i)

$$\begin{array}{cccccccc}
 & & & & 7 & 3 & 0 & 2 & 1 \\
 & & & & \times & & & 7 & 2 \\
\hline
 & & & 1 & & & & & \\
 & & & & E & 6 & 0 & 4 & 2 \\
+ & 3 & 2 & 5 & 0 & E & 7 & 0 \\
\hline
 & 3 & 3 & 3 & 6 & E & B & 2
\end{array}$$

(j)

$$\begin{array}{cccccccc}
 & & & 4 & 5 & 2 & 1 & \\
 & & & 9 & B & 5 & 2 & \\
 & & & B & A & D & 6 & 3 \\
 & & & \times & & & 6 & E \\
\hline
 & 1 & & & 1 & & & \\
 & & A & 3 & 7 & B & 6 & A \\
+ & 4 & 6 & 1 & 0 & 5 & 2 & 0 \\
\hline
 & 5 & 0 & 4 & 8 & 0 & 8 & A
\end{array}$$

(k)

$$\begin{array}{cccccccc}
 & & & 4 & 1 & 6 & 9 & \\
 & & & 4 & 1 & 6 & 9 & \\
 & & & 2 & 5 & 1 & 6 & 4 \\
 & & & \times & & & F & F \\
\hline
 & & & 1 & 1 & 1 & & \\
 & & 2 & 2 & C & 5 & 3 & 6 \\
+ & 2 & 2 & C & 5 & 3 & 6 & 0 \\
\hline
 & 2 & 4 & F & 1 & 8 & 9 & 6
\end{array}$$

(l)

$$\begin{array}{cccccccc}
 & & & & 2 & & & \\
 & & & 1 & 8 & 2 & 2 & \\
 & & & B & 1 & E & 5 & 5 \\
 & & & \times & & & 3 & 9 \\
\hline
 & & & & 1 & 1 & & \\
 & & 6 & 4 & 1 & 0 & F & D \\
+ & 2 & 1 & 5 & A & F & F & 0 \\
\hline
 & 2 & 7 & 9 & C & 0 & E & D
\end{array}$$

(m)

$$\begin{array}{ccccccccc}
 & & & 1 & 1 & 1 & 1 & & \\
 & & C & 0 & F & F & E & E & \\
 & & \times & & & & 2 & 1 & \\
\hline
 & & & 1 & 1 & 1 & 1 & & \\
 & & & C & 0 & F & F & E & E \\
+ & 1 & 8 & 1 & F & F & D & C & 0 \\
\hline
 & 1 & 8 & E & 0 & F & D & A & E
\end{array}$$

(n)

$$\begin{array}{ccccccccc}
 & & & 4 & 3 & 3 & 2 & 4 & \\
 & & & 3 & 2 & 2 & 1 & 3 & \\
 & & & D & E & B & A & 7 & E \\
 & & & \times & & & & 5 & 4 \\
\hline
 & & 1 & 1 & 1 & 1 & 1 & & \\
 & & 3 & 7 & A & E & 9 & F & 8 \\
+ & 4 & 5 & 9 & A & 4 & 7 & 6 & 0 \\
\hline
 & 4 & 9 & 1 & 5 & 3 & 1 & 5 & 8
\end{array}$$

(o)

$$\begin{array}{cccccccccc}
 & & & & & 1 & D & 5 & 6 & \\
 & & & & & 1 & 8 & 3 & 4 & \\
 & & & C & 0 & 0 & 1 & E & 5 & 7 \\
 & & & \times & & & & & F & A \\
\hline
 & & & & & & 1 & & & \\
 & & 7 & 8 & 0 & 1 & 2 & F & 6 & 6 \\
+ & B & 4 & 0 & 1 & C & 7 & 1 & 9 & 0 \\
\hline
 & B & B & 8 & 1 & D & A & 0 & F & 6
\end{array}$$

(p)

$$\begin{array}{ccccccccc}
 & & & 2 & 2 & 1 & & 2 & \\
 & & & 1 & 1 & & & 1 & \\
 & & & F & E & E & 7 & 1 & E \\
 & & & \times & & & & 3 & 2 \\
\hline
 & 1 & 1 & 1 & 1 & 1 & & & \\
 & & 1 & F & D & C & E & 3 & C \\
+ & 2 & F & C & B & 5 & 5 & A & 0 \\
\hline
 & 3 & 1 & C & 9 & 2 & 3 & D & C
\end{array}$$

(q)

$$\begin{array}{cccccccccc}
 & & & & 2 & 2 & & 2 & & \\
 & & & & 5 & 5 & & 7 & 2 & \\
 & & C & 0 & B & B & 1 & E & 5 & \\
 & & \times & & & & & 3 & 8 & \\
\hline
 & & & & & & 1 & 1 & & \\
 & & 6 & 0 & 5 & D & 8 & F & 2 & 8 \\
+ & 2 & 4 & 2 & 3 & 1 & 5 & A & F & 0 \\
\hline
 & 2 & A & 2 & 8 & E & E & A & 1 & 8
\end{array}$$

(r)

$$\begin{array}{cccccccccc}
 & & & 1 & 7 & 4 & 4 & A & 3 & \\
 & & & & 1 & 1 & 1 & 2 & & \\
 & & & 6 & 1 & A & 5 & 5 & E & 5 \\
 & & & \times & & & & & C & 3 \\
\hline
 & & & 1 & & & & 1 & & \\
 & & 1 & 2 & 4 & F & 0 & 1 & A & F \\
+ & 4 & 9 & 3 & C & 0 & 6 & B & C & 0 \\
\hline
 & 4 & A & 6 & 0 & F & 6 & D & 6 & F
\end{array}$$

(s)

$$\begin{array}{cccccccccc}
 & & 1 & 2 & 2 & 3 & 3 & 4 & 4 & \\
 & & & & & 1 & 1 & 1 & 1 & \\
 & & 2 & 3 & 5 & 6 & 8 & 9 & B & C \\
 & & \times & & & & & & 6 & 2 \\
\hline
 & & & 1 & 1 & & & & & \\
 & & 4 & 6 & A & D & 1 & 3 & 7 & 8 \\
+ & D & 4 & 0 & 7 & 3 & A & 6 & 8 & 0 \\
\hline
 & D & 8 & 7 & 2 & 0 & B & 9 & F & 8
\end{array}$$

(t)

$$\begin{array}{cccccccccc}
 & & & 3 & 6 & 1 & & 2 & 6 & \\
 & & & 3 & 6 & 1 & & 2 & 6 & \\
 & & & F & 5 & B & 2 & 0 & 3 & A \\
 & & & \times & & & & & A & A \\
\hline
 & 1 & 1 & 1 & 1 & & & & & \\
 & & 9 & 9 & 8 & F & 4 & 2 & 4 & 4 \\
+ & 9 & 9 & 8 & F & 4 & 2 & 4 & 4 & 0 \\
\hline
 & A & 3 & 2 & 8 & 3 & 6 & 6 & 8 & 4
\end{array}$$

19. And now some really long long multiplications to end this part with.

(a)

$$\begin{array}{cccccccccc}
 & & & & & 9 & 5 & 7 & D & \\
 & & & & & 8 & 4 & 6 & B & \\
 & & & & & 7 & A & 5 & 7 & E \\
 & & & & & \times & F & 0 & 0 & D \\
\hline
 & & & & & & & & & \\
 & & & & 6 & 3 & 6 & 7 & 6 & 6 \\
+ & 7 & 2 & B & 2 & 6 & 2 & 0 & 0 & 0 \\
\hline
 & 7 & 2 & B & 8 & 9 & 8 & 7 & 6 & 6
\end{array}$$

(b)

$$\begin{array}{ccccccccccccc}
 & & & & & 6 & & E & E & 1 & 6 & D & \\
 & & & & & 3 & & 6 & 6 & & 3 & 6 & \\
 & & & & & 5 & 7 & 0 & F & F & 1 & 6 & E \\
 & & & & & \times & & & & F & 0 & 7 & 0 \\
\hline
 & & & 1 & 1 & & 1 & 1 & & & & & \\
 & & & 2 & 6 & 1 & 6 & F & 9 & A & 0 & 2 & 0 \\
+ & 5 & 1 & 9 & E & F & 2 & 5 & 7 & 2 & 0 & 0 & 0 \\
\hline
 & 5 & 1 & C & 5 & 0 & 9 & 5 & 0 & C & 0 & 2 & 0
\end{array}$$

(c)

$$\begin{array}{cccccccccc}
 & & & & 7 & 7 & 1 & 9 & 3 & \\
 & & & & 9 & 9 & 1 & C & 4 & \\
 & & & & 8 & 9 & 1 & B & 4 & \\
 & & & & B & A & B & 1 & E & 5 \\
 & & & & \times & & & B & E & D \\
\hline
 & & 1 & 1 & 1 & 1 & 1 & 1 & & \\
 & & & 9 & 7 & B & 0 & 8 & A & 1 \\
 & & A & 3 & 5 & B & A & 8 & 6 & 0 \\
+ & 8 & 0 & 5 & A & 4 & D & 7 & 0 & 0 \\
\hline
 & 8 & B & 2 & 7 & B & 8 & 8 & 0 & 1
\end{array}$$

(d)

$$\begin{array}{cccccccccc}
 & & & & & 3 & 4 & 4 & 4 & \\
 & & & & & & & & & \\
 & & & & 1 & 6 & 8 & 8 & 8 & \\
 & & & & 6 & 1 & A & D & D & E \\
 & & & & \times & & & 5 & 1 & A \\
\hline
 & & 1 & 1 & 1 & 1 & 1 & 1 & & \\
 & & & 3 & D & 0 & C & A & A & C \\
 & & & 6 & 1 & A & D & D & E & 0 \\
+ & 1 & E & 8 & 6 & 5 & 5 & 6 & 0 & 0 \\
\hline
 & 1 & F & 2 & 5 & 0 & F & E & 8 & C
\end{array}$$

(e)

$$
\begin{array}{ccccccccc}
 & & & & & 8 & 5 & 4 & \\
 & & & & & 3 & 2 & 1 & \\
 & & & & & 3 & 2 & 1 & \\
 & & & & & C & A & 7 & 5 \\
 & & & & \times & D & 0 & 6 & 5 \\
\hline
 & & & 1 & 1 & 1 & 1 & & \\
 & & & & 3 & F & 4 & 4 & 9 \\
 & & & 4 & B & E & B & E & 0 \\
+ & A & 2 & 7 & F & 1 & 0 & 0 & 0 \\
\hline
 & A & 2 & C & E & F & 0 & 2 & 9
\end{array}
$$

(f)

$$
\begin{array}{ccccccccccc}
 & & & & & 6 & 6 & 3 & & 6 & \\
 & & & & & 9 & 9 & 4 & 1 & 8 & \\
 & & & & & C & B & 5 & 1 & B & \\
 & & & & & B & E & E & 7 & 1 & E \\
 & & & & & \times & & 7 & 0 & A & D \\
\hline
 & & 1 & 1 & 1 & & 1 & & 1 & & \\
 & & & & 9 & B & 1 & B & C & 8 & 6 \\
 & & & 7 & 7 & 5 & 0 & 7 & 2 & C & 0 \\
+ & 5 & 3 & 8 & 5 & 1 & D & 2 & 0 & 0 & 0 \\
\hline
 & 5 & 4 & 0 & 6 & 1 & F & 4 & F & 4 & 6
\end{array}
$$

(g)

$$
\begin{array}{ccccccccc}
 & & & & 8 & 1 & 9 & B & \\
 & & & & 6 & 1 & 6 & 8 & \\
 & & & & 9 & 1 & A & C & \\
 & & & & 5 & A & 1 & A & D \\
 & & & & \times & 1 & E & A & F \\
\hline
 & 1 & 1 & 1 & 1 & 1 & & & \\
 & & & 5 & 4 & 7 & 9 & 2 & 3 \\
 & & 3 & 8 & 5 & 0 & C & 2 & 0 \\
 & 4 & E & D & 7 & 7 & 6 & 0 & 0 \\
+ & 5 & A & 1 & A & D & 0 & 0 & 0 \\
\hline
 & A & C & C & B & C & B & 4 & 3
\end{array}
$$

(h)

$$
\begin{array}{ccccccccc}
 & & & & & 7 & 4 & 9 & \\
 & & & & & 6 & 3 & 8 & \\
 & & & & & & & & \\
 & & & & & B & A & 5 & E \\
 & & & & \times & B & A & 1 & 1 \\
\hline
 & & & 1 & 2 & 1 & 1 & & \\
 & & & & & B & A & 5 & E \\
 & & & & B & A & 5 & E & 0 \\
 & & 7 & 4 & 7 & A & C & 0 & 0 \\
+ & 8 & 0 & 2 & 0 & A & 0 & 0 & 0 \\
\hline
 & 8 & 7 & 7 & 4 & A & C & 3 & E
\end{array}
$$

(i)

$$
\begin{array}{cccccccccccccc}
 & & & & & & & & & 6 & 6 & 2 & 3 & \\
 & & & & & & & & 1 & 8 & 8 & 3 & 4 & \\
 & & & & & & & & 1 & 9 & 9 & 3 & 4 & \\
 & & & & & & & & & & & & & \\
 & & & & & & & & 1 & C & C & 4 & 6 & \\
 & & & & & & & & & 6 & 6 & 2 & 3 & \\
 & & & & & & & C & 0 & 1 & D & E & 5 & 7 \\
 & & & & & & & \times & 7 & A & B & 1 & E & 7 \\
\hline
 & & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & & & \\
 & & & & & & 5 & 4 & 0 & D & 1 & 4 & 6 & 1 \\
 & & & & & A & 8 & 1 & A & 2 & 8 & C & 2 & 0 \\
 & & & & & C & 0 & 1 & D & E & 5 & 7 & 0 & 0 \\
 & & & 8 & 4 & 1 & 4 & 8 & D & B & D & 0 & 0 & 0 \\
 & & 7 & 8 & 1 & 2 & A & F & 6 & 6 & 0 & 0 & 0 & 0 \\
+ & 5 & 4 & 0 & D & 1 & 4 & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
 & 5 & C & 1 & 3 & C & 1 & 5 & D & F & C & 7 & 8 & 1
\end{array}
$$

(j)

$$
\begin{array}{ccccccccccccc}
 & & & & & & 7 & 4 & 5 & 1 & A & & \\
 & & & & & & 4 & 2 & 3 & & 6 & & \\
 & & & & & & 9 & 4 & 6 & 1 & C & & \\
 & & & & & & 3 & 1 & 2 & & 4 & & \\
 & & & & & & C & A & 5 & 7 & 1 & E & \\
 & & & & & \times & C & 1 & 7 & 1 & E & 5 & \\
\hline
 & & 1 & 1 & 2 & 2 & 2 & 3 & 1 & 1 & & & \\
 & & & & & & 3 & F & 3 & B & 3 & 9 & 6 \\
 & & & & & B & 1 & 0 & C & 3 & A & 4 & 0 \\
 & & & & & C & A & 5 & 7 & 1 & E & 0 & 0 \\
 & & & 5 & 8 & 9 & 6 & 1 & D & 2 & 0 & 0 & 0 \\
 & & & C & A & 5 & 7 & 1 & E & 0 & 0 & 0 & 0 \\
+ & 9 & 7 & C & 1 & 5 & 6 & 8 & 0 & 0 & 0 & 0 & 0 \\
\hline
 & 9 & 8 & E & 5 & B & 3 & 1 & 2 & 2 & B & D & 6
\end{array}
$$

7.3.6 Division of hexadecimal numbers

1. Work out these divisions. Write down the remainder if there is any.

(a) $89356 \div 2 = 449AB$
$2\ \underline{)\,89^13^15^16}$
$44\,9\ A\ B$

(b) $F48E4 \div 3 = 5184C$
$3\ \underline{)\,F4^18E^24}$
$51\,84\ C$

(c) $B10B \div 4 = 2C42\ R3$
$4\ \underline{)\,B^31^10B}$
$2\ C\ 42\ R3$

(d) $FEEA \div 5 = 32$FB R3
$5\ \underline{)\,FE^4E^3A}$
$32\ F\ B\ R3$

(e) 60EDE÷6 =1027A R2
$6\ \underline{)\,60E^2D^3E}$
$102\ 7\ A\ R2$

(f) $87654 \div 7 = 13579\ R5$
$7\ \underline{)\,8^17^26^35^44}$
$1\,3\,5\,7\,9\ R5$

(g) $51AA \div 8 = A35\ R2$
$8\ \underline{)\,51^1A^2A}$
$A\ 3\ 5\ R2$

(h) $7839 \div 9 = D5B\ R6$
$9\ \underline{)\,78^33^69}$
$D\ 5\ B\ R6$

(i) FEDCBA ÷2 =7F6E5D
$2\ \underline{)\,F^1ED^1CB^1A}$
$7\ F6\ E5\ D$

(j) $7002 \div 3 = 2556$
$3\ \underline{)\,7^10^10^12}$
$2\,5\,5\,6$

(k) $B1AB \div 4 = 2C8A\ R3$
$4\ \underline{)\,B^31^1A^2B}$
$2\ C\ 8\ A\ R3$

(l) $7826 \div 5 = 1807\ R3$
$5\ \underline{)\,7^2826}$
$1\,807\ R3$

(m) $9043 \div 6 = 180B\ R1$
$6\ \underline{)\,9^3043}$
$1\,80B\ R1$

(n) $7DB1 \div 7 = 11F4\ R5$
$7\ \underline{)\,7D^6B^21}$
$11\ F\ 4\ R5$

(o) $2BB8 \div 8 = 577$
$8\ \underline{)\,2B^3B^38}$
$5\ 7\ 7$

(p) $C1A3 \div 9 = 1583\ R8$
$9\ \underline{)\,C^31^4A^23}$
$1\,5\ 8\ 3\ R8$

(q) $DCCD \div 5 = 2C29$
$5\ \underline{)\,D^3CC^2D}$
$2\ C2\ 9$

(r) $B105 \div 3 = 3B01\ R2$
$3\ \underline{)\,B^2105}$
$3\ B01\ R2$

2. Work out these divisions.

(a) $1234 \div A = 1D2$
$A\ )\ \underline{12^{8}3^{1}4}$
$1\ D\ 2$

(b) $5678 \div B = 7DC\ R4$
$B\ )\ \underline{56^{9}7^{8}8}$
$7\ D\ C\ R4$

(c) $5107 \div C = 6C0\ R7$
$C\ )\ \underline{51^{9}07}$
$6\ C0\ R7$

(d) $2348 \div D = 2B6\ RA$
$D\ )\ \underline{23^{9}4^{5}8}$
$2\ B\ 6\ RB$

(e) $9102 \div E = A5B\ R8$
$E\ )\ \underline{91^{5}0^{A}2}$
$A\ 5\ B\ R8$

(f) $8067 \div F = 88F\ R6$
$F\ )\ \underline{80^{8}6^{E}7}$
$8\ 8\ F\ R6$

(g) $ABBA \div A =$ 112C R2
$A\ )\ \underline{AB^{1}B^{7}A}$
$11\ 2\ C\ R2$

(h) $2137 \div B = 305$
$B\ )\ \underline{2137}$
305

(i) $C1FE \div C = 102A\ R6$
$C\ )\ \underline{C1F^{7}E}$
$102\ A\ R6$

(j) $8372 \div D = A1C\ R6$
$D\ )\ \underline{83^{1}7^{A}2}$
$A\ 1\ C\ R6$

(k) $C1A6 \div E = DD5$
$E\ )\ \underline{C1^{B}A^{4}6}$
$D\ D\ 5$

(l) $F1E5 \div F = 1020\ R5$
$F\ )\ \underline{F1E5}$
$10\,20\ R5$

(m) BABE50$\div A =$12ACA1 R6:
$A\ )\ \underline{B^{1}A^{6}B^{7}E^{6}5^{1}0}$
$1\ 2\ A\ C\ A\ 1\ R6$

(n) DEBA7E$\div B =$143F7F R9:
$B\ )\ \underline{D^{2}E^{2}B^{A}A^{5}7^{A}E}$
$1\ 4\ 3\ F\ 7\ F\ R9$

(o) B1E55ED$\div C =$ED31D3 R9:
$C\ )\ \underline{B1^{9}E^{2}5^{1}5^{9}E^{2}D}$
$E\ D\ 3\ 1\ D\ 3\ R9$

(p) 383940$\div D$ =4532C R4
$D\ )\ \underline{38^{4}3^{2}9^{2}4^{A}0}$
$4\ 5\ 3\ 2\ C\ R4$

(q) CADE75$\div E =$E7D9A R9:
$E\ )\ \underline{CA^{6}D^{B}E^{8}7^{9}5}$
$E\ 7\ D\ 9\ A\ R9$

(r) F104A$\div F$ =10116
$F\ )\ \underline{F10^{1}4^{5}A}$
$101\ 1\ 6$

3. Can you divide by powers of the decimal number 16, just for the fun of it? Everything is written in hexadecimal numbers, so $16_{dec} = 10_{hex}$, $16^2_{dec} = 256_{dec} = 100_{hex}$ etc. Write down the remainder if there is one.

(a) $ABC \div 10 = AB\ RC$

(b) $BEE7 \div 100 = BE\ RE7$

(c) $87F4 \div 10 = 87F\ R4$

(d) $57AB1E5 \div 100 = 57AB1\ RE5$

(e) $C0C0A0 \div 10 = C0C0A$

(f) $B1AB1AB1A \div 1000 = B1AB1A\ RB1A$

(g) $ABBA \div 10 = ABB\ RA$

(h) $B269017 \div 1000 = B269\ R17$

(i) $BABBE1 \div 100 = BABB\ RE1$

(j) $F10FE \div 100 = F10\ RFE$

(k) $F10EF1E \div 10 = F10EF1\ RE$

(l) 3489632DBC÷10000=348963 r2DBC

4. Can you figure out how to do these long divisions?

(a) $8EDF \div 23 = 415$

$$\begin{array}{r} 415 \\ 23\ \overline{)\ 8EDF} \\ \underline{-8\text{C}} \\ 2D \\ \underline{-23} \\ AF \\ \underline{-\text{AF}} \\ 0 \end{array}$$

(b) $2B8F \div 15 = 213$

$$\begin{array}{r} 213 \\ 15\ \overline{)\ 2B8F} \\ \underline{-2\text{A}} \\ 18 \\ \underline{-15} \\ 3F \\ \underline{-3\text{F}} \\ 0 \end{array}$$

(c) $1F46A \div 21 = F2A$

$$\begin{array}{r} F2A \\ 21\ \overline{)\ 1F46A} \\ \underline{-1\text{EF}} \\ 56 \\ \underline{-42} \\ 14A \\ \underline{-14\text{A}} \\ 0 \end{array}$$

7.4 Octal numbers

7.4.1 Converting decimal numbers to octal and vice versa

Exercise 1 Practice using these notations by converting the first 40 decimal numbers to octal ones.

- $0_{dec} = 0_{oct}$
- $1_{dec} = 1_{oct}$
- $2_{dec} = 2_{oct}$
- $3_{dec} = 3_{oct}$
- $4_{dec} = 4_{oct}$
- $5_{dec} = 5_{oct}$
- $6_{dec} = 6_{oct}$
- $7_{dec} = 7_{oct}$
- $8_{dec} = 10_{oct}$
- $9_{dec} = 11_{oct}$
- $10_{dec} = 12_{oct}$
- $11_{dec} = 13_{oct}$
- $12_{dec} = 14_{oct}$
- $13_{dec} = 15_{oct}$
- $14_{dec} = 16_{oct}$
- $15_{dec} = 17_{oct}$
- $16_{dec} = 20_{oct}$
- $17_{dec} = 21_{oct}$
- $18_{dec} = 22_{oct}$
- $19_{dec} = 23_{oct}$
- $20_{dec} = 24_{oct}$
- $21_{dec} = 25_{oct}$
- $22_{dec} = 26_{oct}$
- $23_{dec} = 27_{oct}$
- $24_{dec} = 30_{oct}$
- $25_{dec} = 31_{oct}$
- $26_{dec} = 32_{oct}$
- $27_{dec} = 33_{oct}$
- $28_{dec} = 34_{oct}$
- $29_{dec} = 35_{oct}$
- $30_{dec} = 36_{oct}$
- $31_{dec} = 37_{oct}$
- $32_{dec} = 40_{oct}$
- $33_{dec} = 41_{oct}$
- $34_{dec} = 42_{oct}$
- $35_{dec} = 43_{oct}$
- $36_{dec} = 44_{oct}$
- $37_{dec} = 45_{oct}$
- $38_{dec} = 46_{oct}$
- $39_{dec} = 47_{oct}$

Exercises

2. Convert these decimal numbers to their octal counterparts.

(a) $21_{dec} = 25_{oct}$
(b) $54_{dec} = 66_{oct}$
(c) $65_{dec} = 101_{oct}$
(d) $77_{dec} = 115_{oct}$
(e) $80_{dec} = 120_{oct}$
(f) $99_{dec} = 143_{oct}$
(g) $100_{dec} = 144_{oct}$
(h) $187_{dec} = 273_{oct}$
(i) $218_{dec} = 332_{oct}$
(j) $356_{dec} = 544_{oct}$
(k) $437_{dec} = 665_{oct}$
(l) $529_{dec} = 1021_{oct}$
(m) $678_{dec} = 1246_{oct}$
(n) $769_{dec} = 1401_{oct}$
(o) $801_{dec} = 1441_{oct}$
(p) $955_{dec} = 1673_{oct}$
(q) $1,000_{dec} = 1750_{oct}$
(r) $1,111_{dec} = 2127_{oct}$
(s) $1,234_{dec} = 2322_{oct}$
(t) $2,849_{dec} = 5441_{oct}$
(u) $3,566_{dec} = 6756_{oct}$
(v) $7,890_{dec} = 17322_{oct}$
(w) $32,768_{dec} = 100000_{oct}$
(x) $80,888_{dec} = 235770_{oct}$
(y) $100,000_{dec} = 303240_{oct}$
(z) $1,234,567_{dec} = 4553207_{oct}$

3. Convert these octal numbers to decimal ones.

(a) $7_{oct} = 7_{dec}$
(b) $17_{oct} = 15_{dec}$
(c) $27_{oct} = 23_{dec}$
(d) $37_{oct} = 31_{dec}$
(e) $47_{oct} = 39_{dec}$
(f) $57_{oct} = 47_{dec}$
(g) $67_{oct} = 55_{dec}$
(h) $77_{oct} = 63_{dec}$
(i) $100_{oct} = 64_{dec}$
(j) $102_{oct} = 66_{dec}$
(k) $213_{oct} = 139_{dec}$
(l) $324_{oct} = 212_{dec}$
(m) $435_{oct} = 285_{dec}$
(n) $546_{oct} = 358_{dec}$
(o) $657_{oct} = 431_{dec}$
(p) $777_{oct} = 511_{dec}$
(q) $1111_{oct} = 585_{dec}$
(r) $2222_{oct} = 1,170_{dec}$
(s) $3333_{oct} = 1,755_{dec}$
(t) $4444_{oct} = 2,340_{dec}$
(u) $5555_{oct} = 2,925_{dec}$
(v) $6666_{oct} = 3,510_{dec}$
(w) $7777_{oct} = 4,095_{dec}$
(x) $12345_{oct} = 5,349_{dec}$
(y) $10770_{oct} = 4,600_{dec}$
(z) $102030_{oct} = 33,816_{dec}$

7.4.2 Converting binary numbers to octal and vice versa

Exercises

1. Match the octal numbers with their corresponding binary representations

(a) 1744 = 1111100100
(b) 3704 = 11111000100
(c) 1765 = 1111110101
(d) 133535 = 1011011101011101
(e) 125626 = 1010101110010110
(f) 125672 = 1010101110111010

(g) 137255 = 1011111010101101
(h) 3351 = 11011101001

2. Convert these binary numbers to octal ones.
   (a) 1 001 011 010 110 100 101 = 1132645
   (b) 111 100 100 010 101 010 111 = 7442527
   (c) 101 010 011 111 000 111 001 = 5237071
   (d) 111 100 101 110 011 100 101 = 7456345
   (e) 111 001 011 101 000 110 011 = 7135063
   (f) 1 000 010 101 001 010 001 010 = 10251212
   (g) 1 111 010 111 111 010 111 110 = 17277276
   (h) 111 100 111 110 101 011 110 = 7476536
   (i) 1 000 110 110 011 111 010 110 = 10663726
   (j) 10 101 010 111 011 001 010 101 = 25273125
   (k) 101 001 010 101 010 100 101 010 100 010 101 010 101 001 110 = 512524524252516
   (l) 11 010 111 111 101 011 110 110 111 110 111 011 101 111 111 = 327753667673577
   (m) 100 010 101 100 000 110 101 011 010 000 010 010 101 000 010 = 425406532022502
   (n) 1 001 010 101 001 001 001 001 011 001 000 100 101 001 001 = 112511113104511
   (o) 101 001 010 100 101 010 010 001 010 011 101 001 010 010 110 = 512452212351226
   (p) 11 111 111 100 010 000 010 000 010 000 010 101 001 001 000 = 377420202025110
   (q) 101 001 001 010 010 010 010 001 000 010 000 011 111 010 011 = 511222210203723
   (r) 101 001 100 101 001 010 010 100 100 101 010 010 101 010 001 = 514512244522521
   (s) 101 001 010 010 101 000 101 101 010 101 011 001 001 010 110 = 512250552531126
   (t) 101 001 010 010 010 101 010 010 101 001 010 101 010 100 101 = 512225225125245

3. Can you convert these octal numbers to binary?
   (a) 1357 = 1 011 101 111
   (b) 2460 = 10 100 110 000
   (c) 2137 = 10 001 011 111
   (d) 1625 = 1 110 010 101
   (e) 3664 = 11 110 110 100
   (f) 75623 = 111 101 110 010 011
   (g) 52365 = 101 010 011 110 101
   (h) 23311 = 10 011 011 001 001
   (i) 12365 = 1 010 011 110 101
   (j) 34567 = 11 100 101 110 111
   (k) 1234567 = 1 010 011 100 101 110 111
   (l) 76543210 = 111 110 101 100 011 010 001 000

(m) $40176037 = 100\,000\,001\,111\,110\,000\,011\,111$

(n) $65130027123 = 110\,101\,001\,011\,000\,000\,010\,111\,001\,010\,011$

(o) $123562041 = 1\,010\,011\,101\,110\,010\,000\,100\,001$

(p) $5473021465 = 101\,100\,111\,011\,000\,010\,001\,100\,110\,101$

(q) $42705123352 = 100\,010\,111\,000\,101\,001\,010\,011\,011\,101\,010$

(r) $120005265321 = 1\,010\,000\,000\,000\,101\,010\,110\,101\,011\,010\,001$

(s) $21065540 = 10\,001\,000\,110\,101\,101\,100\,000$

(t) $755010013152 = 111\,101\,101\,000\,001\,000\,000\,001\,011\,001\,101\,010$

7.4.3 Converting hexadecimal numbers to octal and vice versa

Exercises

1. Convert these hexadecimal numbers to octal.

   (a) $D06 = 1101\,0000\,0110 = 110\,100\,000\,110 = 6406$

   (b) $CA7 = 1100\,1010\,0111 = 110\,010\,100\,111 = 6247$

   (c) $57A7E = 101\,0111\,1010\,0111\,1110 = 1\,010\,111\,101\,001\,111\,110 = 1275176$

   (d) $C01055A1 = 1100\,0000\,0001\,0000\,0101\,0101\,1010\,0001$
   $= 11\,000\,000\,000\,100\,000\,101\,010\,110\,100\,001 = 30004052641$

   (e) $C1A551F1ED = 1100\,0001\,1010\,0101\,0101\,0001\,1111\,0001\,1110\,1101$
   $= 1\,100\,000\,110\,100\,101\,010\,100\,011\,111\,000\,111\,101\,101 = 14064524370755$

   (f) $50C1A1 = 101\,0000\,1100\,0001\,1010\,0001 = 10\,100\,001\,100\,000\,110\,100\,001 = 24140641$

   (g) $FACED = 1111\,1010\,1100\,1110\,1101 = 11\,111\,010\,110\,011\,101\,101 = 3726355$

   (h) $BAB1E5 = 1011\,1010\,1011\,0001\,1110\,0101 = 101\,110\,101\,011\,000\,111\,100\,101 = 56530745$

   (i) $C0A57 = 1100\,0000\,1010\,0101\,0111 = 11\,000\,000\,101\,001\,010\,111 = 3005127$

   (j) $5A1AD = 101\,1010\,0001\,1010\,1101 = 1\,011\,010\,000\,110\,101\,101 = 1320655$

   (k) $F0CA1 = 1111\,0000\,1100\,1010\,0001 = 11\,110\,000\,110\,010\,100\,001 = 3606241$

   (l) $F1EECE = 1111\,0001\,1110\,1110\,1100\,1110\quad 111\,100\,011\,110\,111\,011\,001\,110 = 74367316$

   (m) $DEBA7E = 1101\,1110\,1011\,1010\,0111\,1110 = 110\,111\,101\,011\,101\,001\,111\,110 = 67535176$

   (n) $C1A551C = 1100\,0001\,1010\,0101\,0101\,0001\,1100 = 1\,100\,000\,110\,100\,101\,010\,100\,011\,100 = 1406452434$

2. Convert these octal numbers to hexadecimal.

   (a) $4321 = 100\,011\,010\,001 = 1000\,1101\,0001 = 8D1$

   (b) $623104 = 110\,010\,011\,001\,000\,100 = 11\,0010\,0110\,0100\,0100 = 32644$

   (c) $12760453 = 1\,010\,111\,110\,000\,100\,101\,011 = 10\,1011\,1110\,0001\,0010\,1011 = 2BE12B$

   (d) $27403262 = 10\,111\,100\,000\,011\,010\,110\,010 = 101\,1110\,0000\,0110\,1011\,0010 = 5E06B2$

   (e) $31362034 = 11\,001\,011\,110\,010\,000\,011\,100 = 110\,0101\,1110\,0100\,0001\,1100 = 65E41C$

   (f) $40123762 = 100\,000\,001\,010\,011\,111\,110\,010 = 1000\,0000\,1010\,0111\,1111\,0010 = 80A7F2$

   (g) $50213407 = 101\,000\,010\,001\,011\,100\,000\,111 = 1010\,0001\,0001\,0111\,0000\,0111 = A11707$

   (h) $65730204 = 110\,101\,111\,011\,000\,010\,000\,100 = 1101\,0111\,1011\,0000\,1000\,0100 = D7B084$

   (i) $12367210 = 1\,010\,011\,110\,111\,010\,001\,000 = 10\,1001\,1110\,1110\,1000\,1000 = 29EE88$

   (j) $72030124 = 111\,010\,000\,011\,000\,001\,010\,100 = 1110\,1000\,0011\,0000\,0101\,0100 = E83054$

   (k) $3567212355 = 11\,101\,110\,111\,010\,001\,010\,011\,101\,101$
   $= 1\,1101\,1101\,1101\,0001\,0100\,1110\,1101 = 1DDD14ED$

   (l) $1123663323 = 1\,001\,010\,011\,110\,110\,011\,011\,010\,011$
   $= 1001\,0100\,1111\,0110\,0110\,1101\,0011 = 94F66D3$

   (m) $7766554433221100 = 111\,111\,110\,110\,101\,101\,100\,100\,011\,011\,010\,010\,001\,001\,000\,000 = 1111\,1111\,0110\,1011\,0110\,0100\,0110\,1101\,0010\,0010\,0100\,0000 = FF6B646D2240$

   (n) $1234567654321 = 1\,010\,011\,100\,101\,110\,111\,110\,101\,100\,011\,010\,001$
   $= 1\,0100\,1110\,0101\,1101\,1111\,0101\,1000\,1101\,0001 = 14E5DF58D1$

   (o) $765432101234567 = 111\,110\,101\,100\,011\,010\,001\,000\,001\,010\,011\,100\,101\,110\,111$
   $= 1\,1111\,0101\,1000\,1101\,0001\,0000\,0101\,0011\,1001\,0111\,0111 = 1F58D1053977$

7.4.4 Operations on octal numbers

Exercises

1. We'll start with the good old 8 times table.

(a) $1 \times 8 = 8_{dec} = 10_{oct}$
(b) $2 \times 8 = 16_{dec} = 20_{oct}$
(c) $3 \times 8 = 24_{dec} = 30_{oct}$
(d) $4 \times 8 = 32_{dec} = 40_{oct}$
(e) $5 \times 8 = 40_{dec} = 50_{oct}$
(f) $6 \times 8 = 48_{dec} = 60_{oct}$
(g) $7 \times 8 = 56_{dec} = 70_{oct}$
(h) $8 \times 8 = 64_{dec} = 100_{oct}$
(i) $9 \times 8 = 72_{dec} = 110_{oct}$

2. Work out these sums.

(a) $3 + 3 = 6$
(b) $4 + 4 = 10$
(c) $5 + 5 = 12$
(d) $6 + 6 = 14$
(e) $7 + 7 = 16$
(f) $3 + 7 = 12$
(g) $6 + 5 = 13$
(h) $12 + 6 = 20$
(i) $17 + 5 = 24$
(j) $23 + 7 = 32$
(k) $25 + 25 = 52$
(l) $14 + 35 = 51$
(m) $23 + 54 = 77$
(n) $36 + 63 = 121$
(o) $72 + 46 = 140$
(p) $45 + 66 = 133$
(q) $51 + 34 = 105$
(r) $65 + 26 = 113$
(s) $123 + 456 = 601$
(t) $234 + 567 = 1023$
(u) $306 + 444 = 752$
(v) $562 + 370 = 1152$
(w) $434 + 217 = 653$
(x) $3721 + 2516 = 6437$
(y) $2604 + 1107 = 3713$
(z) $1234567 + 7654321 = 11111110$

3. Can you work out sums with more than two numbers as well? I bet you can!

(a)
$$\begin{array}{cccccc} 1&2&1&1&1& \\ &3&1&2&4&6 \\ &4&7&5&2&3 \\ +&3&7&2&1&5 \\ \hline 1&4&0&2&0&6 \end{array}$$

(b)
$$\begin{array}{cccccc} 1&1&2&1&1& \\ &3&5&7&0&4 \\ &2&6&7&5&0 \\ +&6&0&7&5&4 \\ \hline 1&4&5&6&3&0 \end{array}$$

(c)
$$\begin{array}{ccccccc} 1&1&1&1&1&1& \\ &3&6&1&1&1&4 \\ &5&2&7&7&6&5 \\ +&6&0&3&6&5&4 \\ \hline 1&7&1&4&7&5&5 \end{array}$$

(d)
$$\begin{array}{cccccc} &1&&2&1& \\ &2&5&0&3&1 \\ &1&1&0&7&2 \\ &2&6&0&4&3 \\ +&1&0&0&2&4 \\ \hline &7&4&2&1&2 \end{array}$$

(e)
$$\begin{array}{ccccccc} 1&1&1&1&1&2& \\ &1&6&2&5&3&6 \\ &6&4&1&0&0&4 \\ &5&1&2&1&4&4 \\ +&2&2&5&3&2&4 \\ \hline 1&7&6&3&2&3&2 \end{array}$$

(f)
$$\begin{array}{cccccc} 3&2&1&1&1& \\ &7&5&3&0&2 \\ &6&2&0&4&1 \\ &7&6&4&1&3 \\ +&4&6&0&3&6 \\ \hline 3&2&4&0&1&4 \end{array}$$

(g)
$$\begin{array}{ccccc} 3&2&2&3& \\ &2&3&4&5 \\ &6&7&4&5 \\ &7&4&2&5 \\ &5&1&7&5 \\ +&3&0&2&5 \\ \hline 3&1&1&6&1 \end{array}$$

(h)
$$\begin{array}{cccccc} 3&2&2&1&2& \\ &7&3&5&0&3 \\ &6&0&3&7&2 \\ &2&5&1&0&4 \\ &4&5&3&2&7 \\ +&7&5&6&3&4 \\ \hline 3&4&4&3&6&4 \end{array}$$

(i)
$$\begin{array}{cccccc} 2&2&3&1&2& \\ &1&0&2&3&4 \\ &2&3&6&4&5 \\ &7&3&5&0&4 \\ &2&7&6&1&3 \\ +&5&0&4&2&7 \\ \hline 2&3&0&0&4&7 \end{array}$$

4. Work out these subtractions.

(a) $12 - 7 = 3$
(b) $15 - 6 = 7$
(c) $23 - 7 = 14$
(d) $32 - 5 = 25$
(e) $61 - 4 = 55$
(f) $52 - 3 = 47$
(g) $60 - 2 = 56$
(h) $35 - 16 = 17$
(i) $44 - 27 = 15$
(j) $56 - 34 = 22$
(k) $31 - 27 = 2$
(l) $52 - 33 = 17$
(m) $43 - 34 = 7$
(n) $70 - 51 = 17$
(o) $205 - 106 = 77$
(p) $370 - 21 = 347$
(q) $105 - 27 = 56$
(r) $463 - 172 = 271$
(s) $326 - 175 = 131$
(t) $432 - 234 = 176$
(u) $753 - 624 = 127$

5. These subtractions are about as hard as they get.

(a)
$$\begin{array}{rrrrrrr} & & 4 & 13 & 0 & 12 & \\ & 7 & \cancel{5} & \cancel{3} & \cancel{1} & \cancel{2} & 4 \\ - & 4 & 3 & 7 & 0 & 6 & 3 \\ \hline & 3 & 1 & 4 & 0 & 4 & 1 \end{array}$$

(b)
$$\begin{array}{rrrrrrr} & & 6 & 7 & 12 & 4 & 13 \\ & 5 & \cancel{7} & \cancel{0} & \cancel{2} & \cancel{5} & \cancel{3} \\ - & 5 & 5 & 3 & 5 & 2 & 4 \\ \hline & & 1 & 4 & 5 & 2 & 7 \end{array}$$

(c)
$$\begin{array}{rrrrrrr} & & 2 & 7 & 15 & 6 & 12 \\ & 5 & \cancel{3} & \cancel{0} & \cancel{5} & \cancel{7} & \cancel{2} \\ - & 4 & 1 & 7 & 6 & 5 & 4 \\ \hline & 1 & 1 & 0 & 7 & 1 & 6 \end{array}$$

(d)
$$\begin{array}{rrrrrrr} & & 12 & & & & \\ & 1 & \cancel{2} & 15 & 6 & 7 & 11 \\ & \cancel{2} & \cancel{3} & \cancel{5} & \cancel{7} & \cancel{0} & \cancel{1} \\ - & 1 & 7 & 6 & 6 & 3 & 4 \\ \hline & & 3 & 7 & 0 & 4 & 5 \end{array}$$

(e)
$$\begin{array}{rrrrrrr} & & 13 & 15 & & & \\ & 4 & \cancel{3} & \cancel{5} & 7 & 7 & 12 \\ & \cancel{5} & \cancel{4} & \cancel{6} & \cancel{0} & \cancel{0} & \cancel{2} \\ - & 2 & 4 & 6 & 5 & 4 & 3 \\ \hline & 2 & 7 & 7 & 2 & 3 & 7 \end{array}$$

(f)
$$\begin{array}{rrrrrrr} & & 11 & & 12 & & \\ & 4 & \cancel{1} & 7 & \cancel{2} & 11 & \\ & \cancel{5} & \cancel{2} & \cancel{0} & \cancel{3} & \cancel{1} & 4 \\ - & 2 & 4 & 7 & 5 & 6 & 1 \\ \hline & 2 & 5 & 0 & 5 & 3 & 3 \end{array}$$

(g)
$$\begin{array}{rrrrrrr} & & & & 11 & & \\ & & 2 & 7 & \cancel{1} & 14 & \\ & 6 & \cancel{3} & \cancel{0} & \cancel{2} & \cancel{4} & 5 \\ - & 4 & 1 & 3 & 6 & 7 & 2 \\ \hline & 2 & 1 & 4 & 3 & 5 & 3 \end{array}$$

(h)
$$\begin{array}{rrrrrrr} & & & 10 & 11 & 12 & \\ & 4 & 7 & \cancel{0} & \cancel{1} & \cancel{2} & 14 \\ & \cancel{5} & \cancel{0} & \cancel{1} & \cancel{2} & \cancel{3} & \cancel{4} \\ - & 4 & 7 & 6 & 5 & 6 & 7 \\ \hline & & & 2 & 4 & 4 & 5 \end{array}$$

(i)
$$\begin{array}{rrrrrrr} & & 11 & 16 & & 13 & \\ & 4 & \cancel{1} & \cancel{6} & 7 & \cancel{3} & 11 \\ & \cancel{5} & \cancel{2} & \cancel{7} & \cancel{0} & \cancel{4} & \cancel{1} \\ - & 1 & 3 & 7 & 7 & 5 & 2 \\ \hline & 3 & 6 & 7 & 0 & 6 & 7 \end{array}$$

6. It's time for multiplications!

(a) $5 \times 2 = 12$
(b) $4 \times 3 = 14$
(c) $2 \times 6 = 14$
(d) $6 \times 6 = 44$
(e) $7 \times 4 = 34$
(f) $3 \times 5 = 17$
(g) $7 \times 7 = 61$
(h) $13 \times 5 = 67$
(i) $21 \times 4 = 104$
(j) $35 \times 6 = 256$
(k) $47 \times 3 = 165$
(l) $52 \times 7 = 446$
(m) $66 \times 2 = 154$
(n) $74 \times 4 = 360$
(o) $456 \times 3 = 1612$
(p) $765 \times 2 = 1752$
(q) $666 \times 6 = 5104$
(r) $525 \times 7 = 4523$
(s) $304 \times 5 = 1724$
(t) $236 \times 4 = 1170$
(u) $123456 \times 7 = 1111102$

7. Can you figure out how to do long multiplications as well?

(a)
$$\begin{array}{cccccccc}
 & & & 3 & & 4 & 2 & \\
 & & & 1 & & 2 & 1 & \\
 & & & 3 & 5 & 0 & 7 & 3 \\
 & & & \times & & & 6 & 3 \\
\hline
 & & 1 & 1 & 1 & 1 & & \\
 & & 1 & 2 & 7 & 2 & 6 & 1 \\
+ & 2 & 5 & 6 & 5 & 4 & 2 & 0 \\
\hline
 & 2 & 7 & 1 & 4 & 7 & 0 & 1
\end{array}$$

(b)
$$\begin{array}{ccccccccc}
 & & & 3 & 1 & & 2 & 1 & \\
 & & & 3 & 1 & & 2 & 1 & \\
 & & & 4 & 6 & 2 & 1 & 4 & 3 \\
 & & & \times & & & & 5 & 4 \\
\hline
 & 1 & 1 & & 1 & 1 & 1 & & \\
 & & 2 & 3 & 1 & 0 & 6 & 1 & 4 \\
+ & 2 & 7 & 7 & 2 & 7 & 5 & 7 & 0 \\
\hline
 & 3 & 2 & 2 & 4 & 0 & 4 & 0 & 4
\end{array}$$

(c)
$$\begin{array}{cccccccccc}
 & & & 5 & 4 & 3 & 2 & 1 & & \\
 & & & 1 & 1 & 1 & & & & \\
 & & & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
 & & & \times & & & & & 7 & 2 \\
\hline
 & 1 & 1 & 1 & 1 & & 1 & 1 & & \\
 & & 1 & 7 & 5 & 3 & 0 & 6 & 4 & 2 \\
+ & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 7 & 0 \\
\hline
 & 7 & 0 & 6 & 4 & 1 & 7 & 5 & 3 & 2
\end{array}$$

(d)
$$\begin{array}{ccccccccc}
 & & & & 1 & & 2 & & \\
 & & & & & & & & \\
 & & & & 2 & & 3 & & \\
 & & & & 7 & 3 & 0 & 4 & 1 \\
 & & & & \times & & 5 & 1 & 6 \\
\hline
 & & 2 & 1 & & 1 & & & \\
 & & & 5 & 4 & 2 & 3 & 0 & 6 \\
 & & & 7 & 3 & 0 & 4 & 1 & 0 \\
+ & 4 & 4 & 7 & 2 & 4 & 5 & 0 & 0 \\
\hline
 & 4 & 6 & 4 & 1 & 7 & 4 & 1 & 6
\end{array}$$

(e)
$$\begin{array}{ccccccccc}
 & & & & 2 & 1 & & 2 & \\
 & & & & 1 & & 1 & & \\
 & & & & 4 & 2 & 1 & 3 & \\
 & & & & 2 & 5 & 3 & 1 & 4 \\
 & & & & \times & & 4 & 2 & 7 \\
\hline
 & & 1 & 1 & 2 & 1 & & & \\
 & & & 2 & 2 & 5 & 6 & 2 & 4 \\
 & & & 5 & 2 & 6 & 3 & 0 & 0 \\
+ & 1 & 2 & 5 & 4 & 6 & 0 & 0 & 0 \\
\hline
 & 1 & 3 & 5 & 2 & 2 & 1 & 2 & 4
\end{array}$$

(f)
$$\begin{array}{ccccccccc}
 & & & & & 1 & & & \\
 & & & & & 2 & & 1 & \\
 & & & & & & & & \\
 & & & & 3 & 0 & 4 & 1 & 3 \\
 & & & & \times & & 2 & 5 & 1 \\
\hline
 & 1 & 1 & & 3 & 1 & & & \\
 & & & & 3 & 0 & 4 & 1 & 3 \\
 & & 1 & 7 & 2 & 4 & 6 & 7 & 0 \\
+ & & 6 & 1 & 0 & 2 & 6 & 0 & 0 \\
\hline
 & 1 & 0 & 0 & 6 & 0 & 1 & 0 & 3
\end{array}$$

(g)
$$\begin{array}{ccccccccc}
 & & & & & 1 & 2 & 3 & \\
 & & & & & 1 & 2 & 2 & \\
 & & & & & 2 & 4 & 5 & \\
 & & & & & & 1 & 1 & \\
 & & & & & 6 & 2 & 5 & 7 \\
 & & & & \times & 4 & 3 & 6 & 2 \\
\hline
 & & 1 & 1 & 1 & 1 & & & \\
 & & & & 1 & 4 & 5 & 3 & 6 \\
 & & & 4 & 6 & 0 & 3 & 2 & 0 \\
 & & 2 & 3 & 0 & 1 & 5 & 0 & 0 \\
+ & 3 & 1 & 2 & 7 & 4 & 0 & 0 & 0 \\
\hline
 & 3 & 4 & 2 & 7 & 2 & 5 & 5 & 6
\end{array}$$

(h)
$$\begin{array}{ccccccccc}
 & & & & & & 1 & 1 & \\
 & & & & & 1 & 2 & 1 & \\
 & & & & & & 2 & 1 & \\
 & & & & & & 1 & 1 & \\
 & & & & & 7 & 1 & 3 & 2 \\
 & & & & \times & 4 & 6 & 5 & 4 \\
\hline
 & 1 & 1 & 1 & 1 & 1 & & & \\
 & & & & 3 & 4 & 5 & 5 & 0 \\
 & & & 4 & 3 & 7 & 0 & 2 & 0 \\
 & & 5 & 3 & 0 & 3 & 4 & 0 & 0 \\
+ & 3 & 4 & 5 & 5 & 0 & 0 & 0 & 0 \\
\hline
 & 4 & 2 & 5 & 4 & 7 & 1 & 7 & 0
\end{array}$$

(i)
$$\begin{array}{ccccccccc}
 & & & & & 3 & 2 & 2 & \\
 & & & & & 1 & 1 & 1 & \\
 & & & & & 5 & 4 & 3 & \\
 & & & & & & & & \\
 & & & & & 3 & 6 & 5 & 4 \\
 & & & & \times & 4 & 2 & 7 & 1 \\
\hline
 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
 & & & & & 3 & 6 & 5 & 4 \\
 & & & 3 & 2 & 6 & 6 & 4 & 0 \\
 & & & 7 & 5 & 3 & 0 & 0 & 0 \\
+ & 1 & 7 & 2 & 6 & 0 & 0 & 0 & 0 \\
\hline
 & 2 & 0 & 5 & 6 & 5 & 5 & 1 & 4
\end{array}$$

8. Solve these divisions and write down the remainder if there is one.

(a) $34 \div 2 = 16$
$$\begin{array}{l} 2\ \overline{)\,3^{1}4} \\ \quad\ \ 1\ 6 \end{array}$$

(b) $56 \div 3 = 17\ R1$
$$\begin{array}{l} 3\ \overline{)\,5^{2}6} \\ \quad\ \ 1\ 7\ R1 \end{array}$$

(c) $21 \div 4 = 4\ R1$
$$\begin{array}{l} 4\ \overline{)\,21} \\ \quad\ \ \ 4\ R1 \end{array}$$

(d) $66 \div 5 = 12\ R4$
$5\ \overline{)\ 6^{1}6}$
$1\ 2\ R4$

(e) $53 \div 6 = 7\ R1$
$6\ \overline{)\ 53}$
$7\ R1$

(f) $47 \div 7 = 5\ R4$
$7\ \overline{)\ 47}$
$5\ R4$

(g) $531 \div 2 = 254\ R1$
$2\ \overline{)\ 5^{1}3^{1}1}$
$2\ 5\ 4\ R1$

(h) $672 \div 3 = 223\ R1$
$3\ \overline{)\ 67^{1}2}$
$22\ 3\ R1$

(i) $326 \div 4 = 65\ R2$
$4\ \overline{)\ 32^{2}6}$
$6\ 5\ R2$

(j) $275 \div 5 = 45\ R4$
$5\ \overline{)\ 27^{3}5}$
$4\ 5\ R4$

(k) $704 \div 6 = 113\ R2$
$6\ \overline{)\ 7^{1}0^{2}4}$
$1\ 1\ 3\ R2$

(l) $654 \div 7 = 75\ R1$
$7\ \overline{)\ 65^{4}4}$
$7\ 5\ R1$

(m) $1234 \div 2 = 516$
$2\ \overline{)\ 123^{1}4}$
$51\ 6$

(n) $2345 \div 3 = 641\ R2$
$3\ \overline{)\ 23^{1}45}$
$6\ 41\ R2$

(o) $3456 \div 4 = 713\ R2$
$4\ \overline{)\ 345^{1}6}$
$7\ 13\ R2$

(p) $4567 \div 5 = 744\ R3$
$5\ \overline{)\ 45^{2}6^{2}7}$
$7\ 4\ 4\ R3$

(q) $5670 \div 6 = 764$
$6\ \overline{)\ 56^{4}7^{3}0}$
$7\ 6\ 4$

(r) $6701 \div 7 = 767$
$7\ \overline{)\ 67^{6}0^{6}1}$
$7\ 6\ 7$

9. Can you figure out these long divisions?

(a) $734 \div 21 = 34$

$$\begin{array}{r} 34 \\ 21\ \overline{)\ 734} \\ \underline{-62} \\ 74 \\ \underline{-74} \\ 0 \end{array}$$

(b) $3264 \div 12 = 312$

$$\begin{array}{r} 312 \\ 12\ \overline{)\ 3744} \\ \underline{-36} \\ 14 \\ \underline{-12} \\ 24 \\ \underline{-24} \\ 0 \end{array}$$

(c) $3264 \div 13 = 234$

$$\begin{array}{r} 234 \\ 13\ \overline{)\ 3264} \\ \underline{-26} \\ 46 \\ \underline{-41} \\ 54 \\ \underline{-54} \\ 0 \end{array}$$

7.5 Unicode

I am dreadfully sorry, but as the answers to the first two questions are different for everyone, you will have to double check them yourself. And as I don't want to spoil the fun of decoding the messages from the other exercises by simply typing out the answers for you, I can only wish you good luck with them! I hope you like the quotes. The older I get, the more I know them to be true.

7.6 Revision

7.6.1 Roman numerals

1. Right now, the youngest is V and the eldest is X years old. In IX years time, they will be XIV and XIX years old.

2. The centurion is very wrong, winning the battle won't be easy at all. The other army has CXXII soldiers more than they have. The centurion might have been misled because the length of the number CCCLXXVIII makes it seem more impressive than the number D.

3. The girl will be XLIV years old in the year MMLVI

4. MMCCCXLV

5. $DCX + LXVIII = DCLXVIII$

6. $MMCDXLIX + CCLXVI = MMDCCXV$

7. The original number is $MDCLXIX$

8. $MCDLIV - DCCCXXV = DCXXIX$

9. $LXII \times V = CCCX$

10. $VI \times XXI = CXXI$

11. $DCXXVII \div III = CCIX$

12. $MMCXXXV \div VII = CCCV$

7.6.2 Binary numbers

1. What is binary code useful for? Binary code is used in computers, from more simple things like a digital watch or the clock on a microwave oven to big supercomputers used for scientific experiments or, you know, the internet.

2. What is the binary representation of the decimal number 39? 100111

3. What is the decimal value of the binary number 110110? 54

4. $11011 + 101 = 100000$

5. $11010011 + 1010101011 = 1001111110$

6. $1101 - 111 = 110$

7. $110100110 - 11011001 = 11001101$

8. $1011001 \times 1000 = 1011001000$

9.
$$\begin{array}{cccccccc}
 & & & 1 & 0 & 1 & 1 & 0 \\
 & & & \times & & 1 & 0 & 1 \\
\hline
 & & 1 & & & & & \\
 & & & 1 & 0 & 1 & 1 & 0 \\
+ & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
\hline
 & 1 & 1 & 0 & 1 & 1 & 1 & 0
\end{array}$$

10.
$$\begin{array}{cccccccc}
 & & & 1 & 1 & 1 & 1 & 1 \\
 & & & \times & & & 1 & 1 \\
\hline
 & 1 & 1 & 1 & 1 & 1 & & \\
 & & & 1 & 1 & 1 & 1 & 1 \\
+ & & 1 & 1 & 1 & 1 & 1 & 0 \\
\hline
 & 1 & 0 & 1 & 1 & 1 & 0 & 1
\end{array}$$

11.
$$\begin{array}{ccccccccccc}
 & & & & & 1 & 1 & 0 & 1 & 1 & 1 \\
 & & & & & \times & 1 & 0 & 1 & 1 & 1 \\
\hline
 & & 1 & & 1 & & 1 & & & & \\
1 & 1 & & & 1 & & 1 & & 1 & & \\
 & & & & & 1 & 1 & 0 & 1 & 1 & 1 \\
 & & & & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
 & & & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
+ \; 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \\
\hline
1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1
\end{array}$$

12. $1100111 \div 101 = 10100\ R11$
$$\begin{array}{rl}
 & \underline{\quad 10100} \\
101 &)\ 1100111 \\
 & \underline{-101} \\
 & 101 \\
 & \underline{-101} \\
 & 011
\end{array}$$

13. $10010111001001 \div 110011 = 10111101\ R100010$
$$\begin{array}{rl}
 & \underline{\quad 10111101} \\
110011 &)\ 10010111001001 \\
 & \underline{-110011} \\
 & 1100010 \\
 & \underline{-110011} \\
 & 1011110 \\
 & \underline{-110011} \\
 & 1010111 \\
 & \underline{-110011} \\
 & 1001000 \\
 & \underline{-110011} \\
 & 1010101 \\
 & \underline{-110011} \\
 & 100010
\end{array}$$

14. What is the decimal value of the binary number 0.11?

$$\frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 0.75$$

7.6.3 Hexadecimal numbers

1. What is the hexadecimal equivalent of 39_{dec}? 27_{hex}

2. What is the decimal value of the $5A_{hex}$? 90_{dec}

3. What is the binary representation of CD_{hex}? 11001101

4. What is the hexadecimal value of the binary number 1100101001110101? $CA75_{hex}$
5. $106 + FEE = 1014$
6. $C1A551C + F00D = C1B4529$
7. $D06 - AB = C5B$
8. $DADD1E5 - BAB135 = CF320B0$
9. $54 \times 7 = 24C$
10. $FEE \times 3 = 2FCA$
11.
$$\begin{array}{ccccccc}
 & & & 1 & 8 & A & \\
 & & & & 2 & 2 & \\
 & & & 6 & 1 & A & D \\
 & & & \times & & D & 3 \\
\hline
 & 1 & & 1 & & & \\
 & & 1 & 2 & 5 & 0 & 7 \\
+ & 4 & F & 5 & C & 9 & 0 \\
\hline
 & 5 & 0 & 8 & 1 & 9 & 7
\end{array}$$
12. $BADD1E \div 4 = 2EB747\ R2$
$$\begin{array}{r}
4\ \underline{)\ B^3A^2D^1D^11^1E} \\
2\ E\ B\ \ 7\ 4\ 7\ R2
\end{array}$$
13. $C1A55 \div 3 = 408C7$
$$\begin{array}{r}
3\ \underline{)\ C1^1A^25^15} \\
4\ 0\ 8\ C\ 7
\end{array}$$

7.6.4 Octal numbers

1. What is the octal representation of 39_{dec}? 47_{oct}
2. What is the octal equivalent of the binary number 1101110011? 1563_{oct}
3. What is the octal value of $FA57_{hex}$? 175127_{oct}
4. What is the decimal value of 75_{oct}?61_{dec}
5. What is the binary value of 326_{oct}?11010110
6. What is the hexadecimal value of 7420_{oct}? $F10_{hex}$
7. $65 + 76 = 163$

8. $6453 + 7035 = 15510$

9. $1004 - 5 = 777$

10. $650032 - 274375 = 353435$

11. $56 \times 4 = 270$

12. $325 \times 6 = 2376$

13.

$$
\begin{array}{cccccccc}
 & & & 3 & 2 & 1 & & \\
 & & & & 2 & 1 & & \\
 & & & 4 & 3 & 1 & & \\
 & & & 6 & 3 & 5 & 4 & 2 \\
 & & & & \times & 5 & 3 & 7 \\
\hline
 & & 1 & & 1 & 1 & & \\
 & & & 3 & 1 & 6 & 5 & 6 \\
 & & 1 & 3 & 0 & 4 & 6 & 0 \\
+ & 2 & 2 & 3 & 5 & 2 & 0 & 0 \\
\hline
 & 1 & 4 & 1 & 7 & 5 & 3 & 6
\end{array}
$$

14. $547 \div 2 = 263\ R1$

$$
\begin{array}{ll}
2 & \big) \underline{\ 5^1 47} \\
 & \quad 2\ 63\ R1
\end{array}
$$

15. $3636 \div 5 = 606$

$$
\begin{array}{ll}
4 & \big) \underline{\ 3636} \\
 & \quad\ 606
\end{array}
$$

16. $75310642 \div 6 = 12166633$

$$
\begin{array}{ll}
4 & \big) \underline{\ 7^1 5^1 3^5 1^5 0^4 6^2 4^2 2} \\
 & \quad 1\ 2\ 1\ 6\ 6\ 6\ 3\ 3
\end{array}
$$