Integers

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2023

Contents

Chapter 1

Numbers

A good place to start this topic is a short review what the numbers you are familiar with are called.

- $1, 2, 3, 4, \ldots$ are **natural numbers**, or **positive integers**. Sometimes 0 is included in this set, sometimes it isn't, because 0 is such a special case. The set of all these natural numbers is noted as $\mathbb{N}$, and if we ignore the number 0, we get

$$\mathbb{N} = \{1, 2, 3, \ldots, \infty\}$$

- Numbers like $-1, -2, -3, \ldots$ are called **negative integers**. If we gather all these negative integers and combine them with all the positive ones, we get the set of integers, $\mathbb{Z}$. Regardless of whether you think of 0 as a positive number, a negative number or neither of those, we get

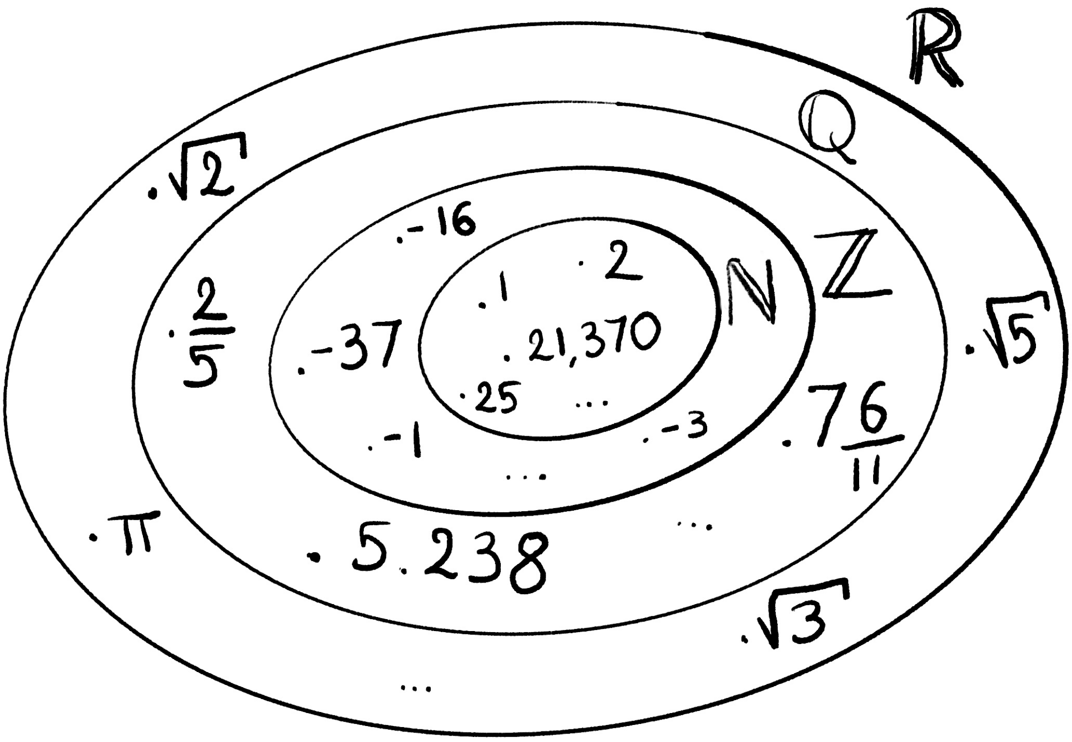
$$\mathbb{Z} = \{-\infty, \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots, \infty\}$$

- Numbers that can be written as fractions (or as mixed numbers) are called **rational numbers**. The set of all the rational numbers is called $\mathbb{Q}$. As all integers can be written as a fraction (the easiest way to do this is to give them the denominator 1), all the numbers above are also a part of the set $\mathbb{Q}$.

- Numbers that can not be written as a fraction, because the part after the decimal point never repeats itself in any sort of patter, are called **irrational numbers**. They are the numbers that are hard to wrap your head around, like π or $\sqrt{2}$. The set of all rational and irrational numbers is also called the set of all real numbers, noted as $\mathbb{R}$.

- All numbers above are called real numbers, so you might conclude that imaginary numbers must also exist. Why else would mathematicians have felt the need to call these real? You are correct, imaginary numbers do exist, the letter i will pop up in your maths courses in a couple of years to form complex numbers and if you are lucky, the letters j and k will join i to make quaternions.

When we visualise the sets of real numbers, it looks as follows:



It is very tempting to teach you about imaginary numbers and the cool properties they have, but that would mean giving you a head start on topics that will be covered in school and possible boredom during those lessons, which is not at all the aim of this course. Instead, you get to learn stuff about integers (everything inside the set $\mathbb{Z}$) that is mostly left out of the normal maths curriculum.

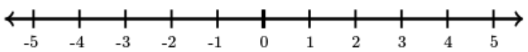
For now, it's enough to know that numbers will keep being fascinating and there is always so much left to learn, whoohoow!

Chapter 2

Negative integers

Definition 1 *A **negative number** is a number whose value is less than zero and has a minus (-) sign before it.*

On a number line, negative numbers are represented on the left of zero.



We automatically use negative numbers in situations such as

- when it's freezing outside and the temperature drops below zero.
- when you borrow money. The money you own someone can be viewed as a negative amount of money you have. If you have €5, but you want to buy something that costs €10, then you can ask a parent to help you out temporarily. After your purchase, you have €-5, because you will have to pay back the remaining €5.

You can also work with negative numbers just for the fun of it, because the operations you perform on them feel counter intuitive and you are training your brain into being flexible by doing so. What makes working with negative numbers feel so counter intuitive? I'm glad you asked.

Addition

When you add a positive number, you move to the right on the number line. But when you add a negative number, you move to the left. This means that adding a negative number will give you a result that is smaller than the number you started from. This means that $+(-) = -$.

Example: $5 + (-3) = 2$

Exercise 1 Solve these additions.

1. $7 + (-5) =$
2. $-8 + 3 =$
3. $-5 + (-6) =$
4. $4 + (-4) =$
5. $-7 + 12 =$
6. $3 + 4 =$
7. $3 + (-4) =$
8. $-3 + 4 =$
9. $-3 + (-4) =$
10. $8 + (-11) =$
11. $-15 + 18 + (-9) =$
12. $7 + (-98) + 2 =$
13. $25 + (-16) + (-8) =$
14. $-16 + (-17) + 5 =$
15. $11 + (-38) + (-9) =$

Subtraction

When you subtract a positive number, you move to the left on the number line. But when you subtract a negative number, you move to the right. This means that subtracting a negative number will give you a result that is greater than the number you started from. It also means that $-(-) = +$.

Example: $5 - (-3) = 5 + 3 = 8$

Exercise 2 Solve these subtractions.

1. $5 - (-6) =$
2. $5 - 6 =$
3. $-5 - 6 =$
4. $-5 - (-6) =$
5. $6 - 5 =$
6. $6 - (-5) =$
7. $7 - (-11) =$
8. $7 - 11 =$
9. $-15 - (-19) =$
10. $254 - 546 =$
11. $25 - (-36) - 3 =$
12. $-16 - (-25) - (-4) =$
13. $-12 - (-14) - 16 =$
14. $10 - (-20) - 15 =$
15. $7 - (-61) - 41 =$

Multiplication

When you multiply any number with a negative number, the sign switches. This means that multiplying a positive number with a negative one results in a negative number. And if you multiply a negative number with another negative number, your result will be a positive number. This means that you basically have to count the amount of negative numbers in your multiplication. If you have an odd amount of negative numbers, your result will be negative. If you have an even amount of negative numbers, your result will be positive. You can remember this as $(-) \times (+) = (+) \times (-) = (-)$ and $(-) \times (-) = (+) \times (+) = (+)$.

Example: $5 \times (-3) = -15$, $(-5) \times 3 = -15$ and $(-5) \times (-3) = 15$.

Exercise 3 Solve these multiplications.

1. $3 \times 7 =$
2. $(-3) \times 7 =$
3. $3 \times (-7) =$
4. $(-3) \times (-7) =$
5. $5 \times (-5) =$
6. $(-4) \times (-4) =$
7. $7 \times (-8) =$
8. $38 \times (-16) =$
9. $(-25) \times (-11) =$
10. $(-81) \times (27) =$
11. $2 \times (-3) \times (-4) =$
12. $(-3) \times (-5) \times (-2) =$
13. $(-28) \times 0 \times (-16) =$
14. $15 \times (-9) \times (-2) =$
15. $7 \times (-8) \times 4 =$

Division

Similar to what happens with multiplication, the sign of a number switches when you divide it by a negative number. The summary of what happens is $(-) \div (+) = (+) \div (-) = (-)$ and $(-) \div (-) = (+) \div (+) = (+)$.

Example: $21 \div (-3) = -7$, $(-21) \div 3 = -7$ and $(-21) \div (-3) = 7$.
You can think of that last one $(-21) \div (-3) = 7$ as follows: you were broke and needed €21 urgently. None of your friends could lend you more than €3, so you had to keep asking until you found 7 friends who each loaned you €3. You have to pay €21 back, so you actually have €-21. You made loans of €-3 each, 7 in total. The money balances are all negative, but for your 7 friends it will be a positive thing if you pay them back.

Exercise 4 Solve these divisions.

1. $35 \div (-5) =$
2. $(-121) \div 11 =$
3. $(-12,639) \div (-11) =$
4. $456 \div (-12) =$
5. $(-81) \div 9 =$
6. $72 \div (-3) =$
7. $(-72) \div 3 =$
8. $(-72) \div (-3) =$
9. $66 \div (-11) =$
10. $(-25) \div (-5) =$
11. $7,345 \div (-5) =$
12. $(-1,043) \div 7 =$
13. $(-851) \div (-23) =$
14. $195 \div (-13) =$
15. $(-3,038) \div (-98) =$

When you have finished the following exercises, you can proudly state that negative numbers have lost their mysterious aura because you have learned all their secrets.
Exercises

5. Now that you know everything that happens when you work with negative numbers, we can combine it all.

   (a) $245 + (-38) - (-87) + 21 =$
   (b) $(36 \div (-6)) \times (-4) + (-50) =$
   (c) $((-144) \div 6) \div (39 \div (-13)) =$
   (d) $(24 - (-16)) \div ((-2) \times (-2)) =$
   (e) $((-85) + 99) - ((-3) \times (-7)) =$
   (f) $14 + ((-36) \times (-2) \div (-9)) =$
   (g) $(-24) \times (-9) \div (-3) \div 6 =$
   (h) $((-3) \times 4) - (28 \div (-4)) =$
   (i) $((3 - (-7)) \div (-2)) \times (-8) =$
   (j) $((-6) \times (-3)) + ((-231) \div 11) =$
   (k) $4 + ((-9) \times 3) + (-37) =$
   (l) $(-11) \times (-38) + (108 \div (-9)) =$
   (m) $(225 \div (-15)) - ((-364) \div 4) =$
   (n) $(16 \div (-4)) \times (-17 - (-15)) =$
   (o) $((-17) + (-8)) \div (-17 + 22) =$
   (p) $((-63) \div 3) \div ((-21) \div (-7)) =$

6. The Campidoglio is one of the seven hills on which the city Rome was built. It was seen as the center of the entire Roman empire and housed the most important temples in the city. One of those is the temple of Juno Moneta, one of the most loved and important Roman gods. It was built in 345 BC. The temple was destroyed during the great fire of Rome in 64 BC. Many centuries later, the Basilica di Santa Maria in Ara Coeli al Campidoglio was built on the same location. This building was completed in the 12th century. In 1797, the basilica was deconsecrated (meaning the place underwent a ritual through which it officially lost its holy properties as a Catholic church, after which it can be used for purposes that have nothing to do with religion) and it was turned into a stable. It has since resumed its role as a Catholic church, with daily masses.

   (a) How long did the temple of Juno Moneta last?
   (b) How many years have passed between the great fire of Rome that destroyed the temple of Juno and the deconsecration of the Basilica that was built on the same site?

7. Lauren's bank account currently has €150.87. She buys a new dress for €119.95 and a backpack that costs €49.95. What is the balance of her bank account after she went shopping?

8. On World Book Day 2023, it was 6° C in Dublin, Ireland; -18° C in Winnipeg, Canada; 18° C in Hong Kong and 30° C in São Paulo, Brazil.

   (a) What was the difference in temperature between Dublin and Winnipeg?
   (b) And between Dublin and Hong Kong?
   (c) And between Dublin and São Paulo?
   (d) Between which two cities is the temperature difference the greatest?
   How much is this difference?

Chapter 3

Power of a number

3.1 Definition

Later on in this section, it will be useful to know what the power of a number is. The goal of this course is to teach you things that you wouldn't normally learn, but in this case you need a bit of knowledge of secondary school in advance. To prevent boredom later on, we'll stick to the basics here. So what is that, the power of a number?

Without realising it, you probably have encountered powers before in your maths lessons. This happened every time you had to calculate the square or the cube of a number.
When you are asked to calculate the square of 8, or 8^2, you are asked to work out how much 8×8 is, and your answer would be 64. The number 64 is the square of 8, but you can also say that *8 to the power 2 is 64* or *64 is the 2nd power of 8.*
When you are asked to calculate the cube of 5, or 5^3, you have to work out how much $5 \times 5 \times 5$ is and your answer would 125. The number 125 is the cube of 5, but you can also call it *5 to the power 3 is 125* or *the 3rd power of 5 is 125.*

Definition 2 *The* ***power*** *of a number tells you how many times this number is multiplied by itself. It is written as a small number to the right and above the base number.*

If we use the letter a as a substitution for any number you want, and n for any natural number, then

$$a^n = \underbrace{a \times a \times a \cdots \times a}_{n \text{ times}}$$

An other word for the little number that we write, is **exponent** or **index**. This word isn't important to you yet, but you may come across it some day in a future maths course. I hope that you'll find working with powers/exponents/indices as much fun as I did when I was in secondary.

As you may have guessed, the power 2 is also called the square of a number. The power 3 is also known as the cube of a number. But there are no special names for higher powers, so after power 3, we just call them power 4, power 5, power 6 and so on.

The words may seem a bit unimaginative, but at least they are a lot easier to learn than, for example, the names of regular geometrical figures (pentagon, hexagon, septagon, octagon and so on). This means that you can focus all your attention on what they mean. Instead of trying to remember how many sides a decagon has, you have to make sure you don't miscount the power, so that when you're calculating 3^5, you do $3 \times 3 \times 3 \times 3 \times 3$ and not add an extra multiplication or forget one of them. Maths is one of those things that is quite hard to combine with daydreaming, even when the words themselves are pretty straightforward. Luckily there is joy and beauty that can be found in maths, so you may not even feel the need to daydream.

There are two special numbers when it comes to powers: 0 and 1.

- The number 0
  - No matter how many times we multiply 0 with 0, the result is always 0. This means that $0^2 = 0^3 = 0^{16} = 0^{1,256} = 0$.
  - For any number that is not 0, the power 0 of that number is 1. This means that $1^0 = 2^0 = 3^0 = 21^0 = 2,012^0 = 1$.
  - Even mathematicians don't all agree on what the value of 0^0 should be, so in this course we'll pretend that that doesn't exist.
- The number 1
  - No matter how many times we multiply 1 with 1, the result is always 1. This means that $1^2 = 1^3 = 1^{25} = 1^{3,721} = 1$.
  - For any number, the power 1 of that number is the number itself. This means that $1^1 = 1$, $2^1 = 2$, $3^1 = 3$, $3,981^1 = 3,981$ and so on. It means that you can add the power 1 to whichever number you want and it doesn't change a thing. Though I doubt that your teacher will be pleased if you start answering your homework questions with *Juno had saved* 8^1 *euros and* 2^1 *cents*, it is technically correct.

Exercise 1 Calculate these powers.

1. $3^3 =$
2. $4^4 =$
3. $5^2 =$
4. $7^3 =$
5. $2^5 =$
6. $3^4 =$
7. $1^{166} =$
8. $10^5 =$
9. $11^3 =$
10. $10^8 =$
11. $26,987^1 =$
12. $123,456,789^1 =$
13. $37^2 =$
14. $987,654,321^0 =$
15. $93^0 =$

You may have noticed that the only powers that are very straightforward to calculate are powers of 10: you start by writing down 1, and then add as many zeros as the power. So $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1,000$ and so on.
This works both ways: to know what power of 10 a number is, you count the number of zeros there are at the end and that number is the power you were searching for. So $1,000,000 = 10^6$, $100,000,000,000,000 = 10^{14}$ etc.

3.2 Relationship between different powers of the same number

3.2.1 Multiplying powers of the same number

When we take a closer look at the powers of 2, we get

2^0	2^1	2^2	2^3	2^4	2^5	2^6
1	2	4	8	16	32	64

We see that, in order to get from one power of 2 to the next one, we multiply the number by 2. This means that we can also write

2^0	2^1	2^2	2^3	2^4	2^5	2^6	
1	2	4	8	16	32	64	
2^0	$2^0 \times 2$	$2^1 \times 2$	$2^2 \times 2$	$2^{\times}2$	$2^3 \times 2$	$2^4 \times 2$	$2^5 \times 2$

This always works, no matter what the power is or what number. This means that, if we use the letter a once more to mean any number you want, and n for any positive whole number that is larger than 1, we can say that

$$a^n \quad = \quad a \times a^{n-1},$$

or the equivalent statement

$$a^{n+1} = a \times a^n$$

If we repeat this, we get

$$\begin{aligned} a^{n+2} &= a \times a^{n+1} \\ &= \underbrace{a \times a}_{a^2} \times a^n \\ &= a^2 \times a^n \end{aligned}$$

Or the equivalent

$$a^n = a^2 \times a^{n-2}$$

If we keep going like this, we will find that

$$a^{n+3} = a^3 \times a^n$$

or the equivalent

$$a^n = a^3 \times a^{n-3}$$

We can keep going like this. If the letters k and l represent any two numbers so that $k + l = n$, then it is true that

$$a^n = a^k \times a^l$$

If k and l represent any two numbers, no matter what their sum is, it is still true that

$$a^k \times a^l = a^{k+l}$$

Exercise 2 Work out these multiplications. Use the power notation a^n in your results.

1. $2^2 \times 2 =$
2. $3 \times 3^3 =$
3. $2^3 \times 3^2 =$
4. $2^5 \times 2^3 =$
5. $5^3 \times 5^7 =$
6. $4^6 \times 4^1 =$
7. $17^5 \times 17^2 =$
8. $1^{25} \times 1^{36} =$
9. $4^8 \times 4^7 =$
10. $0^{567} \times 7^8 =$
11. $6^3 \times 6^{323} =$
12. $11^0 \times 11^{11} =$

3.2.2 Dividing powers of the same number

2^0	2^1	2^2	2^3	2^4	2^5	2^6
1	2	4	8	16	32	64

If we take another look at our table with the powers of 2, we can see that in order to go from one power of 2 to the previous one, we divide that number by 2. This means that we can also write.

2^0	2^1	2^2	2^3	2^4	2^5	2^6
1	2	4	8	16	32	64
$2^1 \div 2$	$2^2 \div 2$	$2^3 \div 2$	$2^4 \div 2$	$2^5 \div 2$	$2^6 \div 2$	$2^7 \div 2$

As before, this is true no matter the power or the number. If we use a again to represent any number and n for any positive whole number, we get

$$a^n = a^{n+1} \div a$$

and the equivalent

$$a^{n-1} = a^n \div a$$

Repeating this gives us

$$\begin{aligned} a^{n-2} &= a^{n-1} \div a \\ &= a^n \div a \div a \\ &= a^n \div (a \times a) \\ &= a^n \div a^2 \end{aligned}$$

or equivalently

$$a^n = a^{n+2} \div a^2$$

Just as before, we can keep going. If the letters k and l represent any two numbers so that $k - l = n$, then it is true that

$$a^n = a^k \div a^l$$

For any two numbers k and l, it is true that

$$a^k \div a^l = a^{k-l}$$

This property of powers is also a neat explanation of why the power 0 has to lead to the number 1:

$$\begin{aligned} 1 &= \frac{a}{a} \\ &= \frac{a^1}{a^1} \\ &= a^{1-1} \\ &= a^0 \end{aligned}$$

The lesson you can take away from this is that, in maths, even if it isn't clear at the start *why* we do what we do (like proclaim that for any number $a \neq 0$ it is true that $a^0 = 1$), there usually is a good reason for it. And the more maths you learn, the more often the mysterious things you simply had to accept before turn out to have a very logical reason to be the way they are.

Exercise 3 Work out these divisions. Write your answers in the power notation a^n.

1. $10^7 \div 10^3 =$
2. $3^5 \div 5^2 =$
3. $3^5 \div 3^2 =$
4. $8^{15} \div 8^0 =$
5. $11^{11} \div 11^7 =$
6. $3^9 \div 3 =$
7. $9^3 \div 9^3 =$
8. $2^{16} \div 2^9 =$
9. $1^{89} \div 1^{99} =$

Exercise 4 Combine everything you learned this far. Use the power notation a^n to write down your answers.

1. $2^4 \times 2^8 \div 2^5 =$
2. $3^8 \div 3^4 \div 3^2 =$
3. $5^4 \times 5^3 \div 5^7 =$
4. $4^4 \div 4^5 \times 4^3 =$
5. $2^8 \times 3 \div 2^6 =$
6. $3^7 \div 3^2 \times 3^0 =$
7. $7 \times 7^7 \times 7^2 =$
8. $1^{97} \div 1^{17} \times 1^0 =$
9. $2^4 \div 3^3 \times 3^5 =$
10. $7^7 \times 7^2 \times 1^7 =$
11. $2^4 \times 8^0 \div 2^2 =$
12. $9^3 \times 9^7 \div 9^8 =$
13. $2^7 \div 2^2 \div 2^5 =$
14. $8^5 \times 8^8 \times 8^3 =$
15. $5 \times 5^2 \times 5^5 \div 5^3 =$
16. $7^0 \times 7^1 \times 7^2 \times 7^3 \times 7^4 =$
17. $2^9 \times 3^4 \div 3^3 \times 2^3 =$
18. $3^5 \div 3^4 \times 4^4 \div 4^2 \div 3 =$
19. $2^3 \times 5^5 \div 2^2 \times 5^3 \times 5 =$
20. $3^3 \times 7^9 \div 7^3 \div 7^4 \div 3^2 =$
21. $9^7 \times 4^3 \times 5^4 \div 9^5 \div 5^2 =$

In secondary school, you will come across powers again. You may encounter negative powers, powers that aren't whole numbers, powers of powers and other stuff that will turn out to be useful in both maths and other science courses. This also means that we'll leave this topic for now, so you will still have new things to learn about powers in the years to come.

Chapter 4

Divisibility

In this chapter, we'll look at natural numbers and learn to recognize characteristics of these numbers that tell us whether or not they are divisible by 2, 3, 4, 5, 8, 9, 11 or powers of 10, without actually working out the division.

A number is divisible by 3 if and only if it is a multiple of 3. This means that we could also say that we are learning to recognize multiples of 3 (and 2, 4, 5 and all the others), but you have to admit that saying that you can check any number for divisibility by 3 sounds even more impressing.

This is one of the many ways in which the words in maths can sound really fancy and impressing, and sometimes so overwhelming that is seems like you are talking about a foreign concept that you have to be a genius to be able to understand them, but it sounds more difficult than it actually is. Language and maths don't always get along very well, it can take a lot of words to explain a simple thing. If you ever encounter something in maths that sounds too difficult to wrap you head around it, always remember that it's very hard to put maths into words and that difficult words don't mean it's actually about a difficult subject.

Once you have the lightbulb moment of *Oooh, that's what they mean by it!*, doing the work usually becomes a lot easier. Just remember to breathe and don't panic when it seems impossibly hard at first glance. Things are very rarely actually impossible to understand.

4.1 Divisibility by 2

A number is even if and only if it ends on 0, 2, 4, 6 or 8.
A number is divisible by 2 if and only if it is an even number.

This means that you only have to look at the very last digit to know whether or not a number is divisible by 2 or not. Because 10 is a multiple of 2 it doesn't matter how big your number is, determining if it is divisible by 2 is something you can figure out in a matter of mere seconds. It usually takes you longer to read out a number than it is to determine its divisibility by 2. Even saying "I have to check divisibility by 2" takes more time than actually doing it. Do you see why you were warned that long words don't have to mean difficult things?

Exercise Is the number divisible by 2? Write your answer as ✓or ×.

1. 36
2. 89
3. 123
4. 438
5. 903
6. 1,231
7. 91,234
8. 983,476
9. 304,897,734
10. 42,109,341,087
11. 42,179,803,410
12. 2,314,898,413,092
13. 12,347,120,798,325
14. 2,902,138,436,712,309
15. 9,834,012,573,584,912

4.2 Divisibility by 3

A number is divisible by 3 if and only if the sum of its digits is divisible by 3.

We'll use this technique on the number 21,369. The sum of its digits is $2+1+3+6+9 = 21$, which is a multiple of 3 as $21 = 7 \times 3$.

Divisibility by 3 may feel a bit disappointing compared to divisibility by 2, because you have to add digits and you still have to divide them. But fear not. Because this rule applies to all numbers, it also applies to the sum of your digits. This means that you can keep adding up the digits until you get a one digit number. In our example, we could have said that the sum of the digits of 21 is $2 + 1 = 3$. If the one digit number you obtain is 3, 6 or 9, your number is divisible by 3.

Another nice feature of the sum of the digits is that it immediately gives you the remainder when a number is not divisible by 3.

- If your one digit number is 1, 4, or 7, then your remainder is 1.
- If your one digit number is 2, 5 or 8, then your remainder is 2.

If we look at the number 456,728 and we keep adding the digits to get the one digit number, we get

$$\begin{aligned} 456,728 &\Rightarrow 4+5+6+7+2+8=32 \\ &\Rightarrow 3+2=5 \end{aligned}$$

The fact that this number is 5 tells us that 456,728 is not divisible by 3, but has a remainder of 2. If we actually do the division, we get $456,728 \div 3 = 152,242\ R2$ and this confirms what we found earlier. If you don't like adding all the digits of a number, you can immediately skip all multiples of 3 as these won't change the outcome.

You may even just keep track of the remainders you encounter and still get the same result. For 456,728, this would mean the following: 4 has remainder 1, so keep that number 1 in your mind. Next is 5, which has remainder 2. You already had remainder 1, so when you add 2 to it, you get 3. This is a multiple of 3 so you can reset the remainder to 0. Next up is 6, which is a multiple of 3 so your remainder stays 0. The next digit is 7, so your remainder becomes 1. 2 is next, so you add 2 to the remainder you already had. Once more, you get $1 + 2 = 3$, so your remainder resets to 0. The last digit is 8, so your final remainder is 2, which is the same thing you found earlier.

You could also try to match the digits in your number to form multiples of 3, because each time you encounter a multiple of 3 you can discard it in your sum. For 456,728, you can discard the digit 6 as this is a multiple of 3. You could add $4 + 5 = 9$. As 9 is a multiple of 3, you can now discard it as it won't change the outcome. You could add $7+2 = 9$ and discard them as well, as this is once again a multiple of 3. The only digit that is left is 8, so your number isn't divisible by 3 and your remainder is 2. Alternatively, you could have added $7+8 = 15$ as this is another multiple of 3. That way, the only digit left would have been 2. Whichever digits you add together to make multiples of 3, your conclusion about divisibility by 3 and the remainder will always stay the same.

While this other method can be easier when you are dealing with large numbers, you should always pick the method that works best for you.

Exercise Is the number divisible by 3? Write your answer as ✓or ×. If your number isn't divisible by 3, write down the remainder as well.

1. 36
2. 89
3. 123
4. 438
5. 903
6. 1,231
7. 91,234
8. 983,476
9. 304,897,734
10. 42,109,341,087
11. 42,179,803,410
12. 2,314,898,413,092
13. 12,347,120,798,325
14. 2,902,138,436,712,309
15. 9,834,012,573,584,912

4.3 Divisibility by 4

A number is divisible by 4 if and only if the 2 lowest digits are divisible by 4.

Because 100 is a multiple of 4 ($25 \times 4 = 100$), none of the higher digits play a role for divisibility by 4. You still have to divide the last 2 digits by 4 to know whether or not the number is divisible by 4, but I am confident you can do this in no time.

If your 2 lowest digits are divisible by 4, then your number is divisible by 4. If your 2 lowest digits aren't divisible by 4, then you learn what your remainder is.

For example, if we want to know whether or not the number 742,383,246 is divisible by 4, we will look at the 2 lowest digits, which are 46. Because $46 \div 4 = (44 + 2) \div 4 = 11\ R2$, the number 742,383,246 is not divisible by 4, but has a remainder of 2.

Exercise Is the number divisible by 4? Write your answer as ✓or ×. If your number isn't divisible by 4, write down the remainder as well.

1. 36
2. 89
3. 123
4. 438
5. 903
6. 1,231
7. 91,234
8. 983,476
9. 304,897,734
10. 42,109,341,087
11. 42,179,803,410
12. 2,314,898,413,092
13. 12,347,120,798,325
14. 2,902,138,436,712,309
15. 9,834,012,573,584,912

4.4 Divisibility by 5

A number is divisible by 5 if and only if it ends with 5 or 0.

Because 10 is a multiple of 5, the lowest digit determines whether or not a number is divisible by 5. If a number isn't divisible by 5, the remainder you get when you divide the lowest digit by 5 is the same as the remainder you get when you divide your entire number by 5.

Exercise Is the number divisible by 5? Write your answer as ✓or ×. If your number isn't divisible by 5, write down the remainder as well.

1. 36
2. 89
3. 123
4. 438
5. 903
6. 1,231

7. 91,234
8. 983,476
9. 304,897,734
10. 42,109,341,087
11. 42,179,803,410
12. 2,314,898,413,092
13. 12,347,120,798,325
14. 2,902,138,436,712,309
15. 9,834,012,573,584,912

4.5 Divisibility by 6

As $6 = 2 \times 3$, a number is divisible by 6 if and only if it is an even number and the sum of its digits is divisible by 3.

It takes a bit more work to determine the remainder if a number is not divisible by 6. Because you had to check divisibility by 3, you know the remainder after dividing by 3. You can use this information to calculate your remainder after dividing by 6. If subtracting the remainder from your number gives you an even number, then the remainder you get after dividing by 6 is the same as the remainder after dividing by 3. If subtracting the remainder from your number gives you an odd number, the remainder you get after dividing by 6 is the three more than the remainder after dividing by 3. Don't worry if this sounds confusing, when it comes to maths, this will not be the only time that words make a concept sound harder than it actually is. Let's look at some examples to understand this whole remainder thing a bit better.

- 782,904. This number is even and divisible by 3, so it is a multiple of 6.
- 56,731. This number is odd and not divisible by 3. The remainder after division by 3 is 1. To determine the remainder after dividing by 6, we subtract 1 from the number and get 56,730. This number is even and divisible by 3, so it is a multiple of 6. This means that the remainder we were looking for is 1.
- 24,734. This number is even, but not divisible by 3. The remainder after division by 3 is 2. To determine the remainder after dividing by 6, we subtract 2 from the number and get 24,272. This number is even and divisible by 3, so it is a multiple of 6. This means that the remainder after division by 6 is 2.
- 123,456,789. This number is odd, but it is divisible by 3. This means that the remainder after dividing by 6 is 3.
- 634,216. This number is even, but not divisible by 3. The remainder after division by 3 is 1. To determine the remainder after dividing by 6, we subtract 2 from the number and get 634,215. This is a multiple of 3, but not a multiple of 6. This means that we have to add 3 to the 1 we found and the remainder after division by 6 is 4.

- 98,705. This number is odd and not divisible by 3. The remainder after division by 3 is 2. To find the remainder we are looking for, we subtract 2 from our number and we get 98,703. This is a multiple of 3 but not of 6, so we have to add 3 to it. The remainder after dividing this number by 6 is 5.

Exercise Is the number divisible by 6? Write your answer as ✓or ×. If your number isn't divisible by 6, write down the remainder as well.

1. 36
2. 89
3. 123
4. 438
5. 903
6. 1,231
7. 91,234
8. 983,476
9. 304,897,734
10. 42,109,341,087
11. 42,179,803,410
12. 2,314,898,413,092
13. 12,347,120,798,325
14. 2,902,138,436,712,309
15. 9,834,012,573,584,912

4.6 Divisibility by 7

Bad news: there is no easy way to check whether or not a number is divisible by 7. It is the only 1 digit number for which you are stuck with doing the actual division to determine if a number is a multiple of 7 or not. Urgh.

But hey, at least it is the only one digit number that gives you so much work!

4.7 Divisibility by 8

A number is divisible by 8 if and only if the 3 lowest digits are divisible by 8.

After the bad news about divisibility by 7, the news that you still have to divide a 3 digit number by 8 feels a bit less bad, doesn't it? It is still far from ideal, but at least you're not stuck with dividing the entire number by 8. And that's all thanks to the fact that $8 \times 125 = 1,000$, so the higher digits don't play a part in divisibility by 8. Similarly to divisibility by 4, the remainder you get when you divide the lowest 3 digits by 8 is the same remainder as when you divide your entire number by 8.

As an example. we will look at the number 65,123. To check divisibility by 8, we look at 123. Because $123 \div 8 = (120 + 3) \div 8 = 15\ R3$, the number 65,123 is not divisible by 8 and has a remainder of 3.

Exercise Is the number divisible by 8? Write your answer as ✓or ×. If your number isn't divisible by 8, write down the remainder as well.

1. 36
2. 89
3. 123
4. 438
5. 903
6. 1,231
7. 91,234
8. 983,476
9. 304,897,734
10. 42,109,341,087
11. 42,179,803,410
12. 2,314,898,413,092
13. 12,347,120,798,325
14. 2,902,138,436,712,309
15. 9,834,012,573,584,912

4.8 Divisibility by 9

A number is divisible by 9 if and only if the sum of its digits is divisible by 9.

Just like with divisibility by 3, we can keep adding the digits until we get a 1 digit number. If this is 9, then the number is divisible by 9, otherwise the remainder after division is the same as your 1 digit number.

Similar to what we were allowed to do while determining whether or not a number is divisible by 3, we may discard multiples of 9 while we are adding up digits.

We will try this out with the number 985,310,128. To determine divisibility by 9, we do

$$\begin{aligned} 985,310,389,127 &\Rightarrow 9+8+5+3+1+0+1+2+8 \\ \text{discard } 9'\text{s} &\Rightarrow 8+5+3+1+1+2+8=28 \\ &\Rightarrow 2+8=10 \\ &\Rightarrow 1+0=1 \end{aligned}$$

Because we end up with the 1 digit number 1, we know that the number 985,310,128 is not divisible by 9 and has a remainder of 1 after division by 9.

Exercise Is the number divisible by 9? Write your answer as ✓or ×. If your number isn't divisible by 9, write down the remainder as well.

1. 36
2. 89
3. 123
4. 438
5. 903
6. 1,231
7. 91,234
8. 983,476
9. 304,897,734
10. 42,109,341,087
11. 42,179,803,410
12. 2,314,898,413,092

13. 12,347,120,798,325 14. 2,902,138,436,712,309 15. 9,834,012,573,584,912

4.9 Divisibility by 11

There are two methods you can use to determine whether or not a number is divisible by 11. Both lead to the same result, so you can choose whichever one works best for you. I would just advise you to try out both of them a couple of times before you decide which one you'll keep using.

As there is no straightforward method to know the remainder after division by 11, we'll leave figuring out the remainder for those times where you are asked to do the entire division.

First method

1. Calculate the sum of the digits in even places.
2. Calculate the sum of the digits in odd places.
3. If the sums are the same, or the difference between them is a multiple of 11, then the number is divisible by 11. Otherwise it is not.

Let us look at some numbers to understand this method a little better.

1. Let's look at the number 52,138,729,214.
   (a) The digits in even places are marked in bold in 5**2**,1**3**8,**7**2**9**,2**1**4, so their sum is $2 + 3 + 7 + 9 + 1 = 22$.
   (b) The digits in odd places are marked in bold in **5**2,**1**3**8**,7**2**9,**2**1**4**, so their sum is $5 + 1 + 8 + 2 + 2 + 4 = 22$.
   (c) These sums are the same, so the number 52,138,729,214 is divisible by 11.
2. Let's look at the number 506,396.
   (a) The digits in even places are marked in bold in **5**0**6**,3**9**6, so their sum is $5+6+9 = 20$
   (b) The digits in even places are marked in bold in 5**0**6,**3**9**6**, so their sum is $0+3+6 = 9$
   (c) The difference between these two sums is $20 - 9 = 11$, so a multiple of 11. This means that the number 506,396 is divisible by 11.

3. Let's look at the number 213,480,345.
   (a) The digits in even places are marked in bold in 2**1**3,**4**8**0**,3**4**5, so their sum is $1 + 4 + 0 + 4 = 9$
   (b) The digits in even places are marked in bold in **2**1**3**,4**8**0,**3**4**5**, so their sum is $2 + 3 + 8 + 3 + 5 = 21$
   (c) The difference between these two sums is $21 - 9 = 12$, which is not a multiple of 11. This means that 213,480,345 is not a multiple of 11.

Second method

1. Place alternating + and - signs between all digits of the number.
2. Work out the result.
3. If your result is 0 or a multiple of 11, then your number is divisible by 11. If it is not, then it is not.

We will look at the same examples to fully understand this second method.

1. Let's look at the number 52,138,729,214.
   (a) $+5 - 2 + 1 - 3 + 8 - 7 + 2 - 9 + 2 - 1 + 4$
   (b) The results you encounter while calculating this are 3, 4, 1, 9, 2, 4, -5, -3, -4, 0. So as you can see, it is likely that you have to use negative numbers to get your result.
   (c) The result is 0, so the number 52,138,729,214 is divisible by 11.
2. Let's look at the number 506,396.
   (a) $+5 - 0 + 6 - 3 + 9 - 6$
   (b) The results you encounter while calculating are 5, 11, 8, 17, 11.
   (c) The result is 11, which is a multiple of 11. This means that the number 506,396 is divisible by 11.
3. Let's look at the number 213,480,345.
   (a) $+2 - 1 + 3 - 4 + 8 - 0 + 3 - 4 + 5$
   (b) The results you encounter while calculating this are 1, 4, 0, 8, 8, 11, 7, 12.
   (c) The result is 12, so the number 213,480,345 is not divisible by 11.

Exercise Is the number divisible by 11? Write your answer as ✓or ×.

1. 36
2. 89
3. 123
4. 438
5. 903
6. 1,231
7. 91,234
8. 983,476
9. 304,897,734
10. 42,109,341,087
11. 42,179,803,410
12. 2,314,898,413,092
13. 12,347,120,798,325
14. 2,902,138,436,712,309
15. 9,834,012,573,584,912

4.10 Divisibility by powers of 10

A number is divisible by 10 if and only if it ends with 0.
A number is divisible by 100 if and only if it ends with 00.
A number is divisible by 1,000 if and only if it ends with 000.
A number is divisible by 10,000 if and only if it ends with 0000.

You can keep going like that. You can even summarize all of the above in one sentence, if you use the power notation.

A number is divisible by 10^n if and only if the last n digits of the number are 0. You replace n by any positive whole number and the rule stays true.

Exercise Is ... divisible by ...? Write your answer as ✓or ×.

1. 12,300 by 10^1?
2. 12,300 by 10^2?
3. 12,300 by 10^3?
4. 872,234,000 by 10^1?
5. 872,234,000 by 10^2?
6. 872,234,000 by 10^3?
7. 872,234,000 by 10^4?
8. 872,234,000 by 10^5?
9. 872,234,000 by 10^6?
10. 5,234,900,000 by 10^1?
11. 5,234,900,000 by 10^2?
12. 5,234,900,000 by 10^3?
13. 5,234,900,000 by 10^4?
14. 5,234,900,000 by 10^5?
15. 5,234,900,000 by 10^6?

4.11 Divisibility by other numbers

To check whether or not a number is divisible by 6, we combined the rules of divisibility by 2 and 3, because $6 = 2 \times 3$. In a similar way, we can deduce rules for divisibility by different numbers.

Be mindful that you group your powers of 2 and of 3 while deducing the rules. The rule for divisibility by 4 ($=2^2$) is a lot stricter than the rule for divisibility by 2 and the rule for 8 ($= 2^3$) is even stricter than that one.

When we look at divisibility by 12, we can see 12 as 2×6 or 3×4. Because combining the rules for 2 and 6 is the same as only using the rule for 6, this will not give you the correct results. But if you combine the rules for 3 and 4, you get the strictest possible rules and it will lead you to the correct results.

A number is divisible by $12 = 3 \times 4$ if and only if it is divisible by 3 and by 4.
Another way of saying this is
A number is divisible by 12 if and only if the sum of its digits is a multiple of 3 and the lowest 2 digits are a multiple of 4.

A number is divisible by $15 = 3 \times 5$ if and only if it is divisible by 3 and by 5, so it ends with either 5 or 0 and the sum of its digits is divisible by 3.

Exercises

1. What are the rules for divisibility by 18? Be careful, will you use $18 = 2 \times 9$ or $18 = 3 \times 6$?

2. What are the rules for divisibility by 22? You can write out the rule for divisibility by 11, but as this is the longest rule we've encountered you don't have to do this.

3. What are the rules for divisibility by 24?

4. What are the rules for divisibility by 30?

5. What are the rules for divisibility by 33?

6. What are the rules for divisibility by 40?

7. What are the rules for divisibility by 44?

8. What are the rules for divisibility by 45?

9. What are the rules for divisibility by 50?

10. What are the rules for divisibility by 55?

11. What are the rules for divisibility by 66?

12. What are the rules for divisibility by 72?

13. What are the rules for divisibility by 88?

14. What are the rules for divisibility by 99?

4.12 Revision

Fill in the table. Write ✓ if the number is divisible, × if it is not. Write down the remainder if the number is not divisible by 2, 3, 4, 5, 6, 8 or 9.

number	is divisible by										
	2	3	4	5	6	8	9	10	11	33	45
924											
1,345,976											
3,960											
22,275											
89,042,157											
40,095											
5,402,529											
7,095,424											
6,587											
1,234,567,890											

Chapter 5

Prime numbers

> A **prime number** is a natural number greater than 1 that is divisible only by itself and 1.

This means that a prime number is not the product of two smaller natural numbers.

> A **composite number** is a natural number that can be formed by multiplying two smaller natural numbers.

This means that a composite number has at least one divisor other than 1 and itself.

When we look at the numbers up to twelve, we find that

- 2 is a prime number, as its only divisors are 1 and 2.
- 3 is a prime number, as its only divisors are 1 and 3.
- 4 is a composite number, as it can be written as $2 \times 2 = 2^2$. The divisors of 4 are 1, 2 and 4.
- 5 is a prime number, as its only divisors are 1 and 5.
- 6 is a composite number, as it can be written as 2×3. The divisors of 6 are 1, 2, 3 and 6.
- 7 is a prime number, as its only divisors are 1 and 7.
- 8 is a composite number, as it can be written as 2×4 or as $2 \times 2 \times 2 = 2^3$. The divisors of 8 are 1, 2, 4 and 8.

- 9 is a composite number, as it can be written as $3 \times 3 = 3^2$. The divisors of 9 are 1, 3 and 9.
- 10 is a composite number, as it can be written as 2×5. The divisors of 10 are 1, 2, 5 and 10.
- 11 is a prime number, as its only divisors are 1 and 11.
- 12 is a composite number, as it can be written as 3×4 or as 2×6 or as $2\times2\times3 = 2^2\times3$. The divisors of 10 are 1, 2, 3, 4, 6 and 12.

The quickest way to find all prime numbers up to any given limit is called the **sieve of Eratosthenes**.

Side note: Eratosthenes lived over 2,200 years ago in the ancient Greek empire. He is best known for being the first person known to calculate the circumference of the earth and having a remarkably accurate result. He was also the first to calculate the Earth's axial tilt. This tilt is responsible for our seasons, as it is the reason why our days are longer or shorter depending on where the Earth is on its orbit around the Sun.

Eratosthenes' method for finding prime numbers up until any number n is as follows:

1. Write down a list of all natural numbers from 2 to n: 2, 3, ..., n.
2. Start with the smallest prime number you have, which is 2. Call this number p, so at this point $p = 2$.
3. Cross out all multiples of p, Do not cross out the number p itself.
4. Find the smallest number in your list, that is bigger than the value of p. If there is no such number, stop. Otherwise, this is the next prime number you encounter, so this number gets the role of p. Repeat from step 3.
5. When you stopped, the uncrossed numbers remaining in your list are all prime numbers up until the number n.

Let's illustrate this by looking for all prime numbers up till 25.

- Our list is (2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25).
- $p = 2$ at the start, so we cross out all multiples of 2. Our list becomes (2, 3, $\cancel{4}$, 5, $\cancel{6}$, 7, $\cancel{8}$, 9, $\cancel{10}$, 11, $\cancel{12}$, 13, $\cancel{14}$, 15, $\cancel{16}$, 17, $\cancel{18}$, 19, $\cancel{20}$, 21, $\cancel{22}$, 23, $\cancel{24}$, 25).

- 3 is the first number we come across that is bigger than 2. So now $p = 3$ and we cross out the multiples of 3. Our list becomes (2, 3, $\cancel{4}$, 5, $\cancel{6}$, 7, $\cancel{8}$, $\cancel{9}$, $\cancel{10}$, 11, $\cancel{12}$, 13, $\cancel{14}$, $\cancel{15}$, $\cancel{16}$, 17, $\cancel{18}$, 19, $\cancel{20}$, $\cancel{21}$, $\cancel{22}$, 23, $\cancel{24}$, 25). Notice that you don't have to cross out all multiples of 3 anymore, as all the even multiples of 3 have already been crossed out (i.e. 6, 12, 18 and 24).

- 5 is the first number we come across that is bigger than 3. So now $p = 5$ and we cross out the multiples of 5. Our list becomes (2, 3, $\cancel{4}$, 5, $\cancel{6}$, 7, $\cancel{8}$, $\cancel{9}$, $\cancel{10}$, 11, $\cancel{12}$, 13, $\cancel{14}$, $\cancel{15}$, $\cancel{16}$, 17, $\cancel{18}$, 19, $\cancel{20}$, $\cancel{21}$, $\cancel{22}$, 23, $\cancel{24}$, $\cancel{25}$). The only extra number that got crossed out at this point is the number 25.

- 7 is the first number we come across that is bigger than 5. So now $p = 7$ and we cross out the multiples of 7. Our list is (2, 3, $\cancel{4}$, 5, $\cancel{6}$, 7, $\cancel{8}$, $\cancel{9}$, $\cancel{10}$, 11, $\cancel{12}$, 13, $\cancel{14}$, $\cancel{15}$, $\cancel{16}$, 17, $\cancel{18}$, 19, $\cancel{20}$, $\cancel{21}$, $\cancel{22}$, 23, $\cancel{24}$, $\cancel{25}$). No new numbers were crossed out.

- 11 is the first number we come across that is bigger than 7. So now $p = 11$ and we cross out the multiples of 11. Our list is (2, 3, $\cancel{4}$, 5, $\cancel{6}$, 7, $\cancel{8}$, $\cancel{9}$, $\cancel{10}$, 11, $\cancel{12}$, 13, $\cancel{14}$, $\cancel{15}$, $\cancel{16}$, 17, $\cancel{18}$, 19, $\cancel{20}$, $\cancel{21}$, $\cancel{22}$, 23, $\cancel{24}$, $\cancel{25}$). No new numbers were crossed out.

- 13 is the first number we come across that is bigger than 11. So now $p = 13$ and we cross out the multiples of 13. Our list is (2, 3, $\cancel{4}$, 5, $\cancel{6}$, 7, $\cancel{8}$, $\cancel{9}$, $\cancel{10}$, 11, $\cancel{12}$, 13, $\cancel{14}$, $\cancel{15}$, $\cancel{16}$, 17, $\cancel{18}$, 19, $\cancel{20}$, $\cancel{21}$, $\cancel{22}$, 23, $\cancel{24}$, $\cancel{25}$). No new numbers were crossed out.

- 17 is the first number we come across that is bigger than 13. So now $p = 17$ and we cross out the multiples of 17. Our list is (2, 3, $\cancel{4}$, 5, $\cancel{6}$, 7, $\cancel{8}$, $\cancel{9}$, $\cancel{10}$, 11, $\cancel{12}$, 13, $\cancel{14}$, $\cancel{15}$, $\cancel{16}$, 17, $\cancel{18}$, 19, $\cancel{20}$, $\cancel{21}$, $\cancel{22}$, 23, $\cancel{24}$, $\cancel{25}$). No new numbers were crossed out.

- 19 is the first number we come across that is bigger than 17. So now $p = 19$ and we cross out the multiples of 19. Our list is (2, 3, $\cancel{4}$, 5, $\cancel{6}$, 7, $\cancel{8}$, $\cancel{9}$, $\cancel{10}$, 11, $\cancel{12}$, 13, $\cancel{14}$, $\cancel{15}$, $\cancel{16}$, 17, $\cancel{18}$, 19, $\cancel{20}$, $\cancel{21}$, $\cancel{22}$, 23, $\cancel{24}$, $\cancel{25}$). No new numbers were crossed out.

- 23 is the first number we come across that is bigger than 19. So now $p = 23$ and we cross out the multiples of 23. Our list is (2, 3, $\cancel{4}$, 5, $\cancel{6}$, 7, $\cancel{8}$, $\cancel{9}$, $\cancel{10}$, 11, $\cancel{12}$, 13, $\cancel{14}$, $\cancel{15}$, $\cancel{16}$, 17, $\cancel{18}$, 19, $\cancel{20}$, $\cancel{21}$, $\cancel{22}$, 23, $\cancel{24}$, $\cancel{25}$). No new numbers were crossed out.

- There are no uncrossed numbers left in the list, so we can stop. The prime numbers up to 25 are 2, 3, 5, 7, 11, 13, 17, 19 and 23.

You may have noticed that you did not cross out any new numbers after $p = 5$ and there is a reason for this. As $25 = 5 \times 5 = 5^2$, this means that every composite number that is less than 25 would have at least one divisor that is smaller than 5. E.g., you can write 24 as 2×12, 3×8 or 4×6, but whichever product you look at that proves that 24 is a composite number, one of the two factors is a number smaller than 5.

It is impossible for a number that is smaller than 5×5 to be written as the product of two numbers that are both bigger than 5. One or both of them have to be smaller.

This means that if you have to find all prime numbers up to a certain number n, you can stop searching once you encounter a prime number p for which $p \times p > n$.

In our example we were looking for prime numbers up to 25. We had to keep going when $p = 5$, because $5 \times 5 = 25$. But we could have stopped when we encountered $p = 7$, because $7 \times 7 = 49 > 25$ so there would be no new numbers crossed out.

Understanding this reasoning requires a high level of abstract reasoning that is not easy to grasp. If you don't understand it fully, you can keep writing any composite number as a product of two numbers to understand it better. If you do understand it completely, congratulations. You have just grasped a concept that a lot of engineering students I tutored had a hard time understanding.

Exercise
Find all prime numbers up to 150.

	2	3	4	5	6	7	8	9	10	11	12	13	14	15
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70	71	72	73	74	75
76	77	78	79	80	81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100	101	102	103	104	105
106	107	108	109	110	111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130	131	132	133	134	135
136	137	138	139	140	141	142	143	144	145	146	147	148	149	150

What was the biggest prime number of which you had to cross out the multiples?

Chapter 6

Prime factor decomposition

6.1 What is prime factor decomposition?

Every natural number can be written as a product of prime numbers. Each of these prime numbers is called a prime factor of a number.

Writing a number as a product of prime factors is called **prime factor decomposition**.

The strategy to do this is:

1. Divide the number by the smallest possible prime number it is divisible by. Write down that prime number and the quotient.

2. Divide the resulting number by the smallest possible prime number it is divisible by. Write down that prime number and the quotient.

3. Repeat this until the quotient is 1.
   The prime factor decomposition of your number is the product of all the prime numbers you have written down.

We will apply this process to two examples.

First, we'll look at the number 210.

- 210 is an even number, so it is divisible by 2, the smallest prime number we can pick. $210 \div 2 = 105$. We write down 2 and continue with the quotient 105.

- 105 is an odd number, so we check divisibility by the next prime number, which is 3. The sum of the digits of 105 is 6, so it is divisible by 3. $105 \div 3 = 35$. We write down 3 and proceed with the quotient 35.
- The sum of the digits of 35 is 8, so it is not divisible by 3. We proceed to the next prime number, which is 5. As 35 ends with 5, it is divisible by 5. $35 \div 5 = 7$, so we write down 5 and proceed with 7.
- 7 is itself a prime number, so we do $7 \div 7 = 1$. We write down 7 and, as the quotient is 1, we know that we can stop.
- The prime factor decomposition of 210 is $2 \times 3 \times 5 \times 7$.

A shorter way to write this down is as follows, with the prime numbers on the right and the quotients you encounter on the left.

$$\begin{array}{r|l} 210 & 2 \\ 105 & 3 \\ 35 & 5 \\ 7 & 7 \\ 1 & \end{array}$$

Multiplying all numbers that you have written down on the right hand side give you the decomposition: $210 = 2 \times 3 \times 5 \times 7$.

For the number 25,164 the process looks as follows.

- 25,164 is an even number, so it is divisible by 2, which is the smallest prime number we can pick. $25,164 \div 2 = 12,582$. We write down the number 2 and continue with the quotient 12,582.
- 12,582 is also an even number, so we divide it by 2. $12,582 \div 2 = 6,291$. We write down the number 2 and continue with 6,291.
- The number 6,291 is odd, so it is not divisible by 2. We move on to check divisibility by the next prime number, which is 3. The sum of the digits of $6,291$ is 18, so it is divisible by 3. $6,291 \div 3 = 2,097$. We write down the number 3 and continue with 2,097.
- The sum of the digits of $2,097$ is 18, so it is divisible by 3. $2,097 \div 3 = 699$. We write down the number 3 and continue with 699.
- The sum of the digits of 699 is 24, so it is divisible by 3. $699 \div 3 = 233$. We write down 3 and continue with 233.

- The sum of the digits of 233 is 8, so it is not divisible by 3. We check divisibility of the next prime numbers.
  - 233 doesn't end with 0 or 5, so it is not divisible by 5.
  - 233=210+23, so it is not divisible by 7.
  - We can check whether or not 233 is divisible by 11 by using the rule, or we can see it as 233=220+13. Either way, it is not divisible by 11.
  - Next, we have to check if it is divisible by 13. $233 = 130 + 103 = 130 + (130 - 27) = 130 + (130 - 2 \times 13 - 1)$, so it is not divisible by 13.
  - Next prime number to check is 17. $233 = 170 + 63 = 170 + 51 + 12 = 170 + 3 \times 17 + 12$, so it is not divisible by 17. $17 \times 17 = 10 \times 17 + 7 \times 10 + 7 \times 7 = 170 + 70 + 49 = 289$, so this is bigger than the number 233 we are looking at. This means that it was the last prime number we had to check, we will not find any other number that 233 is divisible by. This means that 233 itself is a prime number.

  $233 \div 233 = 1$, we write down 233.
- The quotient is 1, so we stop.
  The prime factor decomposition of $25,164 = 2 \times 2 \times 3 \times 3 \times 3 \times 233 = 2^2 \times 3^3 \times 233$

Written down, this becomes

$$\begin{array}{r|l} 25{,}164 & 2 \\ 12{,}582 & 2 \\ 6{,}291 & 3 \\ 2{,}097 & 3 \\ 699 & 3 \\ 233 & 233 \\ 1 & \end{array}$$

We have found that we can write 25,164 as $2^2 \times 3^3 \times 233$.

And now it's your turn.

Exercise Find the prime decompostion of the following numbers.

1. 60=
   $$\begin{array}{r|l} 60 & \\ & \\ & \\ & \end{array}$$
2. 61=
   $$\begin{array}{r|l} 61 & \\ & \\ & \\ & \end{array}$$
3. 62=
   $$\begin{array}{r|l} 62 & \\ & \\ & \\ & \end{array}$$
4. 63=
   $$\begin{array}{r|l} 63 & \\ & \\ & \\ & \end{array}$$

5. 64=
64

6. 65=
65

7. 66=
66

8. 67=
67

9. 68=
68

10. 69=
69

11. 70=
70

12. 71=
71

13. 98=
98

14. 163=
163

15. 189=
189

16. 222=
222

17. 144=
144

18. 234=
234

19. 267=
267

20. 324=
324

17. 721=
721

18. 681
681

19. 873=
873

20. 999=
999

21. $1,584 =$
1,584

22. $2,057 =$
2,057

23. $5,083 =$
5,087

24. $7,267 =$
7,267

6.2 Uses of prime factor decomposition

6.2.1 Finding all factors of a number

Once you have found the prime factor decomposition of a number, you can use it to list all its factors. To do this, you start your list with the number 1, then proceed by listing prime numbers you encountered, and all possible combinations of them.

Even while writing this down, I wonder how much sense it makes without an example to clarify what is meant by the previous sentence. So let's look at the number 12. The prime factor decomposition is $12 = 2^2 \times 3$

12	2
6	2
3	3
1	

We use this to make a list of all the factors of 12.

- We start with the number 1.
- We list all the prime numbers we encountered, so 2, 2 and 3. We encountered 2 twice, but it's enough to write it down just once.
- We list all possible combinations of them
  - We can combine 2 with 2, so we get $2 \times 2 = 4$
  - We can combine 2 with 3, so we get $2 \times 3 = 6$
  - We can combine all of them, so we get $2 \times 2 \times 3 = 12$

We have found that the factors of 12 are 1, 2, 3, 4, 6 and 12. To check if our list is complete, we can multiply the first factor with the last ($1 \times 12 = 12$), the second with second last ($2 \times 6 = 12$) and so on ($3 \times 4 = 12$). If all these multiplications result in our number, than we have a complete list.

The second number we'll look at together is 600.

600	2
300	2
150	2
75	3
25	5
5	5
1	

The prime factor decomposition of 600 is $2^3 \times 3 \times 5^2$. Once again, we will use this information to list all the factors of 600.

- We start with the number 1.
- We list all the prime numbers we encountered: 2, 3 and 5. This time we immediately wrote down each of them just once, even though we encountered 2 and 5 multiple times.
- We list all possible combinations. As we needed more steps than with the number 12 to complete the prime factor decomposition, we can expect to encounter more possible combination than before.
  - $2 \times 2 = 4$
  - $2 \times 3 = 6$
  - $2 \times 5 = 10$
  - $3 \times 5 = 15$
  - $5 \times 5 = 25$
  - $2 \times 3 \times 5 = 30$
  - $2^2 \times 2 = 8$
  - $2^2 \times 3 = 12$
  - $2^2 \times 5 = 20$
  - $2^2 \times 5^2 = 100$
  - $5^2 \times 2 = 50$
  - $5^2 \times 3 = 75$
  - $2^2 \times 3 \times 5 = 60$
  - $2^2 \times 3 \times 5^2 = 300$
  - $2 \times 3 \times 5^2 = 150$
  - $2^3 \times 3 = 24$
  - $2^3 \times 5 = 40$
  - $2^3 \times 5^2 = 200$
  - $2^3 \times 3 \times 5 = 120$
  - $2^3 \times 3 \times 5^2 = 600$

We have found that the factors of 600 are, when we order them from smallest to greatest: 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 25, 30, 40, 50, 60, 75, 100, 120, 150, 200, 300, 600. Our completeness check is $1 \times 600 = 2 \times 300 = 3 \times 200 = 4 \times 150 = 5 \times 120 = 6 \times 100 =$

$8 \times 75 = 10 \times 60 = 12 \times 50 = 15 \times 40 = 20 \times 30 = 24 \times 25$. I had almost forgotten to include the combination $2 \times 3 \times 5^2 = 150$ in the list, this check showed me that something was wrong and I knew I needed another factor with a value between 120 and 200. Checking your results may feel tedious, but that doesn't automatically mean that it's a waste of time.

Exercise List all the factors of the following numbers, by using prime factor decomposition.

1. $1,398 =$

   $$\begin{array}{r|l} 1{,}398 & \\ & \\ & \\ & \\ \end{array}$$

   The factors of 1,398 are

2. $1,575 =$

   $$\begin{array}{r|l} 1{,}575 & \\ & \\ & \\ & \\ \end{array}$$

   The factors of 1,575 are

3. $2,023 =$

   $$\begin{array}{r|l} 2{,}023 & \\ & \\ & \\ & \\ \end{array}$$

   The factors of 2,023 are

4. $2,024 =$

   $$\begin{array}{r|l} 2{,}024 & \\ & \\ & \\ & \\ \end{array}$$

   The factors of 2,024 are

6.2.2 Finding the highest common factor

Prime factor decomposition can be used to find all factors of a number, and it can be used to find the highest common factor of numbers as well. You already learned how to find the highest common factor by listing all factors, now you will learn a method that gives you the answer without needing to know all factors first.

We'll find out how to do this by searching for the highest common factor of 132 and 168. Firstly, we'll use prime factor decomposition on both numbers. We get

$$\begin{array}{r|l} 132 & 2 \\ 66 & 2 \\ 33 & 3 \\ 11 & 11 \\ 1 & \end{array} \qquad \begin{array}{r|l} 168 & 2 \\ 84 & 2 \\ 42 & 2 \\ 21 & 3 \\ 7 & 7 \\ 1 & \end{array}$$

When we have the prime factor decomposition of each number, we'll search for all the prime factors that both numbers have in common. The highest common factor is the multiplication of all these factors. In our case, we can see 2, 2 and 3 in both decompositions, so the highest common factor is $2 \times 2 \times 3 = 12$.

It's even easier to find the highest common factor when we write the numbers as follows:
$132 = 2^2 \times 3 \times 11$
$168 = 2^3 \times 3 \times 7$
Because the prime factors are ordered from smallest to largest and the power notation is used, it is easy to see that both numbers have $2^2 \times 3$ in common. It's the same thing as before, the notation is just a bit more structured.

Things to consider:

- If two numbers don't have any prime factors in common, the highest common factor is 1.
- You can use the same method to find the highest common factor of three or more numbers.

Exercises

1. Find the highest common factor of the following numbers by using prime factor decomposition.

   (a) $hcf(267, 198) =$

   267 |

   198 |

   (c) $hcf(183, 186) =$

   183 |

   186 |

   (b) $hcf(242, 286) =$

   267 |

   198 |

   (d) $hcf(231, 363) =$

   231 |

   363 |

(e) $hcf(333;\ 2,775) =$

333	2,775

(f) $hcf(258;\ 1,104) =$

258	1,104

(g) $hcf(51, 68, 85) =$

51	68	85

(h) $hcf(60, 75, 105) =$

60	75	105

2. (a) Find the prime factor decomposition of the following numbers.

135	144	150	162	200

(b) Find the following highest common factors.

i. $hcf(135, 144) =$
ii. $hcf(135, 150) =$
iii. $hcf(135, 162) =$
iv. $hcf(135, 200) =$
v. $hcf(144, 150) =$
vi. $hcf(144, 162) =$
vii. $hcf(144, 200) =$
viii. $hcf(150, 162) =$
ix. $hcf(150, 200) =$
x. $hcf(162, 200) =$
xi. $hcf(135, 144, 150) =$
xii. $hcf(135, 144, 162) =$
xiii. $hcf(135, 144, 200) =$
xiv. $hcf(135, 150, 162) =$
xv. $hcf(135, 150, 200) =$
xvi. $hcf(135, 162, 200) =$
xvii. $hcf(144, 150, 162) =$
xviii. $hcf(144, 150, 200) =$

xix. $hcf(144, 162, 200) =$

xx. $hcf(150, 162, 200) =$

xxi. $hcf(135, 144, 150, 162) =$

xxii. $hcf(135, 144, 150, 200) =$

xxiii. $hcf(135, 144, 162, 200) =$

xxiv. $hcf(135, 150, 162, 200) =$

xxv. $hcf(144, 150, 162, 200) =$

xxvi. $hcf(135, 144, 150, 162, 200) =$

Euclid's algorithm to find the highest common factor of two numbers

I feel like I would be consciously withholding information from you if I were to end a part about finding the highest common factor without mentioning Euclid's method for it. And as I don't like withholding you information, I won't do that. Especially not about rather fun maths stuff. I only learned about this method to find the highest common factor after graduating as a mathematician, so it's not like you are missing a crucial bit of information. But it is a nice thing to know and it felt like a bit of a revelation when I finally learned about it. It gives you a way to find the highest common factor of two numbers without searching for any factors at all. The only thing you need are simple subtractions!

So what is this method that everybody has been keeping from you?

Euclid's algorithm to find the highest common factor of two numbers works as follows:

1. Given two whole numbers, subtract the smaller number from the larger number and note the result.
2. Repeat the process of subtracting the smaller number from the result until the result is equal to or smaller than the original small number.
3. If the result is equal, this number is the highest common factor you were searching for.
4. If the result is smaller, use the original small number as the new larger number. Subtract the result from Step 2 from the new larger number.
5. Repeat the process for every new larger number and smaller number until both numbers are equal.

Let's look at some examples. First, let's find the highest common factor of 18 and 27. The steps we encounter when using the algorithm are:

- $27 - 18 = 9$
- $18 - 9 = 9$

- At this point, both numbers are 9, so $hcf(18, 27) = 9$.

Next, we'll search for the highest common factor of 120 and 50. The steps we encounter are:

- $120 - 50 = 70$
- $70 - 50 = 20$
- $50 - 20 = 30$
- $30 - 20 = 10$
- $20 - 10 = 10$
- At this point, both numbers are 10, so $hcf(120, 50) = 10$.

You can lower the amount of seperate steps by combining the subtractions of the same number. In that case, your steps would be

- $120 - 50 - 50 = 20$
- $50 - 20 - 20 = 10$
- $20 - 10 = 10$

Please note that, as usual, if you combine steps, then you have to pay a bit more attention to them. If you don't stop the algorithm in time, you may get 0 as your smallest number. In this case that would happen if you do $20 - 10 - 10 = 0$, in order to get the bigger number 20 to a number that is smaller than 10. When this happens, you have to go back one calculation. Your highest common factor is always the number you found just before getting a zero result.

The biggest advantage of Euclid's algorithm is that it can be used on big numbers in a pretty straightforward way. Where it would take you quite a lot of time to find all the factors, or even to use prime factor decomposition, for the numbers 182,664 and 154,875, using the algorithm would lead to the following steps:

- $182,664 - 154,875 = 27,789$
- $154,875 - (27,789 \times 5) = 154,875 - 138,945 = 15,930$
- $27,789 - 15,930 = 11,859$

- $15,930 - 11,859 = 4,071$
- $11,859 - (4,071 \times 2) = 11,859 - 8,142 = 3,717$
- $4,071 - 3,717 = 354$
- $3,717 - (354 \times 10) = 3,717 - 3,540 = 177$
- $354 - 177 = 177$ OR $354 - (177 \times 2) = 0$

We found that $hcf(182,664; 154,875) = 177$. You may object and say that this method isn't all that quick and easy either, and you would be absolutely right about that. But if you try to find the answer by listing all factors, or by using prime factor decomposition, it may take you even longer to find the solution.

As a rule, the highest common factor of three numbers is the same as the highest common factor of one of those numbers and the highest common factor of the other two numbers. Or, if $x, /y$ and z are numbers, $hcf(x, y, z) = hcf(x, hcf(y, z)) = hcf(y, hcf(x, z)) = hcf(z, hcf(x, y))$. You can use this to calculate the highest common factor of any amount of numbers you like, by using Euclid's algorithm or another method.

Exercises

3. Find the highest common factor by using Euclid's algorithm.
   (a) $hcf(267, 198) =$
   (b) $hcf(242, 286) =$
   (c) $hcf(183, 186) =$
   (d) $hcf(231, 363) =$
   (e) $hcf(333;\ 2,775) =$
   (f) $hcf(258;\ 1,104) =$
   (g) $hcf(51, 68, 85) =$
   (h) $hcf(60, 75, 105) =$
4. (a) Did you find your answers quicker or slower than when you searched for the same highest common factors by using prime factor decomposition in Exercise 1?
   (b) Do you think the algorithm would be a smart choice time-wise to solve part b) of Exercise 2 that came before? Why or why not?
5. Find the highest common factor, by whichever method you like.

(a) $hcf(153,217;\ 58,883) =$
(b) $hcf(101,69) =$
(c) $hcf(62,68) =$
(d) $hcf(62,64) =$
(e) $hcf(99;\ 1,683) =$
(f) $hcf(3,375;\ 8,000) =$
(g) $hcf(35,49) =$
(h) $hcf(1,836;\ 1,287) =$
(i) $hcf(440,671) =$
(j) $hcf(234,432) =$

6.2.3 Finding the lowest common multiple

We can not only deduce the highest common factor from the prime factor decomposition of numbers, we can also find their lowest common multiple. Where before we had to find all factors that the numbers have in common, now we have to make sure that no factors are left out.

We'll search for the lowest common multiple of 132 and 168 to show how it's done. Remember, the prime factor decomposition of these numbers is

$132 = 2^2 \times 3 \times 11$

$$\begin{array}{r|l} 132 & 2 \\ 66 & 2 \\ 33 & 3 \\ 11 & 11 \\ 1 & \end{array}$$

$168 = 2^3 \times 3 \times 7$

$$\begin{array}{r|l} 168 & 2 \\ 84 & 2 \\ 42 & 2 \\ 21 & 3 \\ 7 & 7 \\ 1 & \end{array}$$

The lowest common multiple of two numbers must include all prime factors of both numbers. When we look at the prime factor decompositions, we see that both numbers share the factor 2. It appears twice in 132 and three times in 168. To make sure that we include all prime factors of both numbers, the lowest common multiple must contain 2^3 as a factor. The factor 3 appears once in both numbers, so the lowest common multiple must contain the factor 3 as well. The factor 11 appears once in 132, and not at all in 168. This means that the lowest common multiple still must contain the factor 11. The factor 7 does not appear in 132, but it does appear in 168 and hence it must appear as a factor of the lowest common multiple as well.

The combination of all this gives you the factors of the lowest common multiple: it must be $2^3 \times 3 \times 7 \times 11$. When you work out the multiplications, you find that $lcm(132, 168) = 2^3 \times 3 \times 7 \times 11 = 132 \times 2 \times 7 = 168 \times 11 = 1,848$.

It is again easier to find what we are looking for when we write the numbers as follows:
$132 = 2^2 \times 3 \times 11$
$168 = 2^3 \times 3 \times 7$
Because the prime factors are ordered from smallest to largest and the power notation is used, it is easy to see which powers are the highest and to make sure that all factors are included. In this example, we get $2^3 \times 3 \times 7 \times 11 = 1,848$.

Let's look at another example and search for the lowest common multiple of 42 and 20. Prime factor decomposition gives us:

$42 = 2 \times 3 \times 7$

$$\begin{array}{r|l} 42 & 2 \\ 21 & 3 \\ 7 & 7 \\ 1 & \end{array}$$

$20 = 2^2 \times 5$

$$\begin{array}{r|l} 20 & 2 \\ 10 & 2 \\ 5 & 5 \\ 1 & \end{array}$$

The number that contains all factors of both 42 and 20 is $2^2 \times 3 \times 5 \times 7 = 420$.

Things to consider:

- If two numbers don't have any prime factors in common, you get the the lowest common multiple is by multiplying both numbers.
- You can use the same method to find the lowest common multiple of three or more numbers.

Exercises

1. Find the lowest common multiple of the following numbers by using prime factor decompostion. Write you answer as a multiplication of prime factors and, if you feel up to it, work out the multiplications to find the actual number.

(a) $lcm(36, 21) =$ $\begin{array}{r|l} 36 & \\ & \\ & \end{array}$ $\begin{array}{r|l} 21 & \\ & \\ & \end{array}$

(b) $lcm(32, 18) =$

32 | 18 |

(c) $lcm(34, 85) =$

34 | 85 |

(d) $lcm(52, 46) =$

52 | 46 |

(e) $lcm(144, 147) =$

144 | 147 |

(f) $lcm(236, 252) =$

236 | 252 |

(g) $lcm(549, 488) =$

549 | 488 |

(h) $lcm(666, 848) =$

666 | 848 |

(i) $lcm(24, 39, 54) =$

24 | 39 | 54 |

(j) $lcm(63, 15, 33) =$

63 | 15 | 33 |

2. (a) Find the prime factor decomposition of the following numbers.

35 | 44 | 50 |

72		90	

(b) Find the following lowest common multiples. Write down (at least) the prime factor decomposition of your answers.

i. $lcm(35, 44) =$
ii. $lcm(35, 50) =$
iii. $lcm(35, 72) =$
iv. $lcm(35, 90) =$
v. $lcm(44, 50) =$
vi. $lcm(44, 72) =$
vii. $lcm(44, 90) =$
viii. $lcm(50, 72) =$
ix. $lcm(50, 90) =$
x. $lcm(72, 90) =$
xi. $lcm(35, 44, 50) =$
xii. $lcm(35, 44, 72) =$
xiii. $lcm(35, 44, 90) =$
xiv. $lcm(35, 50, 72) =$
xv. $lcm(35, 50, 90) =$
xvi. $lcm(35, 72, 90) =$
xvii. $lcm(44, 50, 72) =$
xviii. $lcm(44, 50, 90) =$
xix. $lcm(44, 72, 90) =$
xx. $lcm(50, 72, 90) =$
xxi. $lcm(35, 44, 50, 72) =$
xxii. $lcm(35, 44, 50, 90) =$
xxiii. $lcm(35, 44, 72, 90) =$
xxiv. $lcm(35, 50, 72, 90) =$
xxv. $lcm(44, 50, 72, 90) =$
xxvi. $lcm(35, 44, 50, 72, 90) =$

6.2.4 Relationship between the highest common factor and the lowest common multiple

By using the prime factor decomposition, we found that

- The highest common factor of two numbers is the product of all the factors that both of them have in common.
- The lowest common multiple of two numbers is the product of all the factors that occur in either of the two numbers.

This means that if we multiply the highest common factor with the lowest common multiple, the result will be the same as the product of the original numbers.

Or, if you want to formulate that in a more mathematical way (because in maths language can be more confusing than symbols):

If a and b are whole numbers, then

$$hcf(a, b) \times lcm(1, b) = a \times b$$

This is a very neat connection between what we previously saw. It binds the highest common factor and the lowest common multiple together in a simple and elegant way, which is something that gives joy to many mathematicians. I hope you feel good about yourself every single time you mastered a difficult skill or completed a long calculation without making any mistakes. I also hope that you can see beauty when maths provides you with simple connections between concepts that feel like they are supposed to be linked together somehow.

We'll explore what this means for the numbers 48 and 54. Once again, we'll start with the prime factor decomposition of the numbers.

$$\begin{array}{r|l} 48 & 2 \\ 24 & 2 \\ 12 & 2 \\ 6 & 2 \\ 3 & 3 \\ 1 & \end{array} \qquad\qquad \begin{array}{r|l} 54 & 2 \\ 27 & 3 \\ 9 & 3 \\ 3 & 3 \\ 1 & \end{array}$$

We found that $48 = 2^4 \times 3$ and $54 = 2 \times 3^3$. We can use this to determine the highest common factor and the lowest common multiple. We find:

- $hcf(48, 54) = 2 \times 3 = 6$
- $lcm(48, 54) = 2^4 \times 3^3 = 432$

Multiplying them gives us

$$\begin{aligned} hcf(48, 54) \times lcm(48, 54) &= 6 \times 432 = 2,592 \\ &= (2 \times 3) \times (2^4 \times 3^3) \\ &= 2^5 \times 3^4 \\ &= (2^4 \times 3) \times (2 \times 3^3) \\ &= 48 \times 54 \end{aligned}$$

So we get exactly what was predicted. This means that, if you have to find both the highest common factor and the lowest common multiple, if you calculate one of them then you can find the other one.

Because, if a and b are whole numbers, then

$$hcf(a,b) \times lcm(1,b) = a \times b$$

This also means that

$$\begin{aligned} hcf(a,b) &= a \times b \div lcm(a,b) = \frac{a \times b}{lcm(a,b)} \\ lcm(a,b) &= a \times b \div hcf(a,b) = \frac{a \times b}{hcf(a,b)} \end{aligned}$$

This is the perfect time to let out a little cheer of excitement, as you just came across, and hopefully understood, formulas that look like very impressive. Welcome to *maths with letters*!

You can put them to use immediately.

Exercise
Find the lowest common multiple and the highest common factor of the following numbers.

1. 12 and 16
2. 32 and 50
3. 36 and 48
4. 25 and 35

Chapter 7

Ratio and proportion

We'll start this chapter in a very serious way, because we are very serious people doing very serious things. Or maybe we are not such serious people all of the time and we enjoy laughter a lot more, but in any case, right now we'll start off with formal definitions.

Definition 3 *A* ***ratio*** *is a comparison of two quantities. The ratio of a to b is most commonly expressed as $a : b$, though the notation $\frac{a}{b}$ can also be used.*

Definition 4 *A* ***proportion*** *is an equality of two ratios.*

Two common types of ratios that are used are

- part to part
- part to whole

The best way to explain these types is by using an example, such as making cordial. Cordial is a non alcoholic fruity drink where you add a syrup-like substance to water.

- We use a part to part ratio when we say that we use one part syrup for six parts water. This compares the amount of two ingredients. In our case, the ratio of syrup to water is $1 : 6$.
- We use a part to whole ratio when we say how much of the cordial consists of syrup. This compares the amount of one ingredient to the sum of all ingredients. In our case, the ratio of syrup to the cordial is $1 : 7$.

What is always true is that

$$\text{part} : \text{whole} = \text{part} : \text{sum of all parts}$$

Since you can write a ratio as a fraction, equivalent ratios exist.

For example, the ratio 2 : 3 is equivalent to 4 : 6, 6 : 9, 212 : 318 and so on, in the same way that the corresponding fractions are equivalent: $\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{212}{318}$.

This means that you can simplify a ratio by writing it as a fraction, simplify this fraction to its lowest terms and rewrite this fraction as an $a : b$ ratio.

Another way to simplify a ratio is by using what you learned before:

1. Search for the highest common factor of the numbers that appear in the ratio. If your highest common factor is 1, then you can skip step 2.
2. Divide both numbers in the ratio by their highest common fraction.
3. You have found the simplest version of your ratio!

Similar to loving fractions in their lowest terms, people are a lot fonder of simple ratios that cannot be simplified any further. However much we may like complexity, simple ratios are preferable.

If a recipe for making cordial would say that you have to add 7 parts of syrup to 42 parts water, then it might make you start doubting whether or not you really want to make cordial at all. The ratio is the same as in the example before, but where 1 part syrup to 6 parts water sounds like something you can easily do, the 7 : 42 ratio would scare off many people.

Exercise 1 Simplify these ratios when possible.

1. 15 : 75
2. 21 : 49
3. 28 : 16
4. 3 : 17
5. 45 : 9
6. 12 : 16
7. 50 : 75
8. 84 : 14
9. 98 : 66

It will not happen often, but at some point you may come across a ratio that consists of fractions, like $\frac{1}{2} : \frac{2}{3}$. This can either scare you into thinking that you have no idea what to do with such information or annoy you a lot because people are making it unnecessarily hard for you. Or maybe you feel a combination of both or have no idea why anybody would think that fractions in a ratio would be any harder than any other ratio. Whatever your feelings are, we will now see a way to convert a ratio that contains fractions into an equivalent ratio without any trace of a fraction. To do this you have to:

1. Search for the lowest common multiple of the denominators of your fractions.
2. Multiply both numbers with this lowest common multiple. You should now have an equivalent ratio that doesn't contain any fractions.
3. Simplify your ratio if possible.

By using this procedure for the ratio $\frac{1}{2} : \frac{2}{3}$, we get:

1. The denominators are 2 and 3, so $lcm(2, 3) = 6$
2. $\frac{1}{2} \times 6 = 3$ and $\frac{2}{3} \times 6 = 4$, so our equivalent ratio is $3 : 4$.
3. Since $hcf(3, 4) = 1$, we cannot simplify the ratio any further.

We have found that the intimidating looking ratio $\frac{1}{2} : \frac{2}{3}$ is equivalent to the ratio $3 : 4$.

We will use the procedure a second time, this time for the ratio $4 : \frac{8}{9}$.

1. There is only one denominator written, so we can either say that $4 = \frac{4}{1}$ and proceed as before, i.e. $lcm(1, 9) = 9$, or we can immediately say that the number we will have to multiply our numbers with is 9.
2. $4 \times 9 = 36$ and $\frac{8}{9} \times 9 = 8$, so our equivalent ratio is $36 : 8$.
3. As $hcf(36, 8) = 4$, we have not yet found the simplest version of this ratio. We divide both numbers by their highest common factor: $36 \div 4 = 9$ and $8 \div 4 = 2$. The simplest equivalent ratio is $9 : 2$.

Exercise 2 Find the simplest possible equivalent ratios that do not contain a fraction.

1. $\frac{1}{2} : \frac{1}{3}$
2. $\frac{3}{4} : \frac{2}{5}$
3. $\frac{4}{7} : \frac{7}{8}$
4. $\frac{1}{4} : \frac{2}{9}$
5. $\frac{2}{7} : \frac{3}{5}$
6. $\frac{6}{9} : \frac{6}{7}$
7. $\frac{2}{3} : \frac{4}{7}$
8. $\frac{2}{5} : 3$
9. $2 : \frac{1}{3}$
10. $8 : \frac{2}{7}$

The procedure to find a ratio is as follows:

1. Determine whether the ratio is part to part or part to whole.
2. Calculate the parts, and the whole if needed.
3. Plug your values into the ratio.
4. Simplify your ratio if possible.

Let us look at an example to understand the way we can find a ratio.
Example There are 15 senior class students and 10 junior class students playing hockey. What is the ratio of senior class students to the total number of students playing hockey?

1. The ratio we are looking for is that of senior class students to the total number of students, so it is a part to whole ratio.
2. There are 15 senior class students. The total number of students includes both students from the senior and junior classes, so there are $15 + 10 = 25$ students.
3. The ratio of senior class students to the total number of students is 15 to 25, or $15 : 25$.
4. To see if we can simplify this ratio, we write it as a fraction:
$$\frac{15}{25} = \frac{3 \times 5}{5 \times 5} = \frac{3}{5}$$
The simplified ratio of senior class students to the total number of students playing hockey is $3 : 5$.

Once you start paying attention to it, you may notice that ratios are used more often than you realized, especially in maths problems. But they don't always show up in the notation that, from now on, you will now immediately recognize as a ratio. The ratio $3 : 5$ may be formulated in other ways, such as

- $\frac{3}{5}$ of the students playing hockey are in senior class.
- 3 in 5 students playing hockey are in senior class.

Exercises

3. A pancake recipe uses $\frac{1}{4}$ cup of all-purpose flour and $\frac{1}{4}$ cup of rice flour. What is the ratio of all-purpose flour to rice flour in the recipe?

(a) 1 : 4 (b) 1 : 2 (c) 1 : 1 (d) 2 : 1 (e) 4 : 1

4. Luna owns 3 cats, 2 owls and a pigmy puff as pets. What is the ratio of the number of cats to the total number of pets Luna owns?

   (a) 1 : 6 (b) 1 : 3 (c) 2 : 5 (d) 1 : 2 (e) 2 : 3

5. The Land of Stories is a books series that contains 6 books. A Tale of Magic is a trilogy (meaning 3 books) that take place in the same setting. For the sake of this exercise, we will assume that these are the only books the author Chris Colfer has written about that magical fairy tale land.

   (a) What is the ratio of the number of books of The Land of Stories to the total number of books written about the magical fairy tale land?
   (b) What is the ratio of books in the The Land of Stories series to the books in the A Tale of Magic series?
   (c) What is the ratio of the number of books of A Tale of Magic to the total number of books written about the magical fairy tale land?
   (d) What is the ratio of books in the A Tale of Magic series to the books in the The Land of Stories series?

6. Mom makes a smoothie for the whole family. She uses 4 pears, 3 apples, 4 bananas and 5 oranges to make it.

   (a) What is the ratio of bananas to the total amount of pieces of fruit?
   (b) What is the ratio of apples to the total amount of pieces of fruit?
   (c) What is the ratio of oranges to the total amount of pieces of fruit?
   (d) What is the ratio of apples to oranges?
   (e) What is the ratio of pears to apples?
   (f) What is the ratio of bananas to pears?

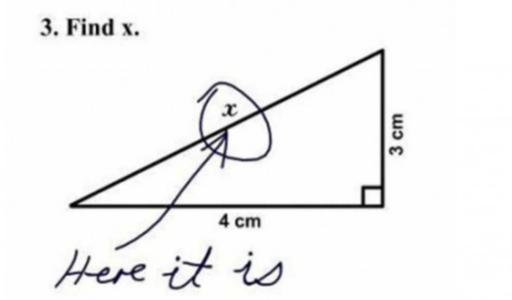
When you have a recipe that gives you all ingredients to make 10 cookies, but you want 25 cookies to treat all your friends, you can use ratios and proportions to calculate the ingredients you need.

If we know a ratio and want to apply it to a different quantity, we can use **proportional relationships**, or equations of equivalent ratios, to calculate any unknown quantities.

The procedure to use a proportional relationship to find an unknown quantity is as follows:

1. Write an equation using equivalent ratios. Write the ratios in the shape of fractions.

2. Plug in the known values and use a variable to represent the unknown quantity. Mathematicians love using the letter x for this, but you can use any letter or symbol you like.



3. If part of one ratio is a multiple of the corresponding part of the other ratio, we can calculate the unknown quantity by multiplying the other part of the given ratio by the same number.
4. If the relationship between the two ratios is not obvious and you can not use the previous step, you have to solve the equation for the unknown quantity by isolating the variable representing it.

Don't panic if these steps feel a bit as if I just threw a bunch of long words together in order to overwhelm you and make you feel like this is too hard for you. I promise you that it isn't too hard and you will be able to do it. It's just another one of those moments where trying to explain what you have to do is just as hard, or even harder, than actually doing it. Let's look at some examples so you can see it for yourself.

Example There are 60 students at Sharavogue. If the student-to-teacher ratio is 15:2, how many teachers are there?

1. The only ratio we got is the student-to-teacher ratio, so we will use this to make a proportional relationship. We get
$$\frac{\text{students}}{\text{teachers}} = \frac{\text{students}}{\text{teachers}}$$
2. We use x for the unknown number of teachers that work at Sharavogue. We plug in the known values and get:
$$\frac{15 \text{ students}}{2 \text{ teachers}} = \frac{60 \text{ students}}{x \text{ teachers}}$$
At this point, it is easiest to drop all the words and continue working with only the numbers and the unknown variable x in the fractions. We get
$$\frac{15}{2} = \frac{60}{x}$$

3. If you are not only a smart cookie, but a smart cookie that has (part of) the 15 times table memorized, you may notice that $60 = 4 \times 15$. If you noticed that, you can continue in the way step 3 suggests:

$$\frac{15}{2} = \frac{60}{x} = \frac{4 \times 15}{x}$$

Two fractions are equivalent / have the same value if multiplying the top and bottom of one fraction with the same number results in the other fraction.
Here, we found that the numerator 60 is 4 times the numerator 15. This means that the denominator x has to be 4 times the denominator 2, so $x = 4 \times 2 = 8$.

4. If elements of the 15 times table are not shouting *look at me, I am a multiple of 15* to you, because they rarely do that and usually disguise themselves so they appear to be numbers that aren't multiples of 15, then you can fall back on step 4 from the procedure. This means that you get to solve equations to find out what your unknown value is (shhh, at this point we pretend that we never saw that statement about x being a number between 7 and 9. Or maybe we do keep that knowledge in mind and we want to find out if this other way of finding the solution leads to the same result). We have found

$$\frac{15}{2} = \frac{60}{x}$$

Working with equations can be quite fun, because you can multiply and divide them by any number you want. As long as you do everything on both sides, the new equations you get will give you the same result as the original ones. *We do this all the time when we calculate prices. If one apple costs 40 cents, then 2 apples cost* $2 \times 40 = 80$ *cents. Equations work similarly.* This means you can get rid of those pesky denominators!
We start with getting rid of the denominator 2, by multiplying both sides by 2. We get

$$\frac{15}{2} \times 2 = \frac{60}{x} \times 2, \text{ or } 15 = \frac{120}{x}$$

To get rid of the denominator x, we do the same as before. We multiply both sides by x (I know that this is weird because we don't know what number x is, but as long as we multiply both sides it's all grand). We get

$$15 \times x = \frac{120}{x} \times x, \text{ or } 15 \times x = 120$$

We know how to solve this equation: we have to find out how many times 15 fits in 120.

$$x = 120 \div 15 = 8$$

Whether you used step 3 or step 4 of the procedure, we found that the unknown value x, the number of teachers that work at Sharavogue, is 8.

Example You were very lucky and you received money for your birthday. You are not sure yet what you want to use the money for, because you love books and lego sets equally. If the ratio of the price of books you want to buy to the price of lego sets you would like is 5 : 2, and you could buy 6 lego sets with your birthday money, how many books would you be able to buy?

1. The relevant ratio we got is the books-to-lego-sets ratio, so we will use this to make a proportional relationship. We get

$$\frac{\text{books}}{\text{lego sets}} = \frac{\text{books}}{\text{lego sets}}$$

2. We use f for the unknown amount of books you would be able to buy (because who knows, maybe always using x in equations makes the other letters of the alphabet feel a bit unappreciated). We plug in the known values and we get:

$$\frac{5 \text{ books}}{2 \text{ lego sets}} = \frac{f \text{ books}}{6 \text{ lego sets}}$$

   Once more, we drop the words and we get

$$\frac{5}{2} = \frac{f}{6}$$

3. You may notice that the denominator 6 on the right hand side is 3 times the denominator 2 on the left hand side. If we use this knowledge for step 3 of the procedure to calculate the unknown value of f, we rewrite the fractions as

$$\frac{5}{2} = \frac{f}{6} = \frac{f}{3 \times 2}$$

   As the denominator 6 is 3 times the denominator 2, this means that the numerator f has to be 3 times the numerator 5, so $f = 3 \times 5 = 15$.

4. If we ignore step 3 and try to solve the equation, we can breathe a sigh of relief because having an unknown value in the numerator is always a bit easier to do than when it appears in the denominator. Our denominators are 2 and 6. You can eliminate them one by one, or you can eliminate both at once by multiplying both sides by the lowest common multiple of the two denominators (there is that lowest common multiple again, it really seems to like popping up as often as possible). In our case, $lcm(2, 6) = 6$, so we multiply both sides by 6. We get

$$\frac{5}{2} \times 6 = \frac{f}{6} \times 6, \text{ or } 15 = f$$

   That was solved even quicker than I expected!

Whichever way you used to solve the equation, you found that you can buy 15 books with your birthday money. Now that's a lot of stories to disappear into!

Exercises

7. Santa drinks 20 cl of milk for every 5 cookies he eats. If he eats 20 cookies, how much milk does he drink?
8. The ratio of fiction books to non-fiction books in the school library is 7 to 4. If there are 182 fiction books, how many non-fiction books does the school library have?
9. Saoirse wants to paint the walls of her bedroom in a light blue colour. To get the right shade, she has to mix white and bright blue paint with a ratio of 3:1. If she needs 20l of the light blue paint to give all her walls 2 coats of paint, how much white and bright blue paint does she have to buy?
10. Mal uses the practice of rotating crops to keep the plants and the soil of her fields as healthy as possible and prevent pests and diseases without using chemicals. This means that instead of using the same field to grow the same crop over and over again, the fields are used to grow a different crop every year. If this is done with crops from different plant families, it is possible to set up a cycle of 4 or more years to make sure that the soil never gets depleted.
    Mal has 4 fields, and she wants to grow crops from 4 different plant families: salads, carrots, beans and tomatoes. She has set up everything she needs to organise her fields as efficiently as possible. She knows that she has enough room to plant 76 tomato plants. She can plant twice as many bean stalks as salads, as many tomato plants as bean stalks and the ratio of carrots to salads is 3:1. How many of each crop can she plant?
    (a) Bean stalks
    (b) Salads
    (c) Carrots
11. Jantje has 240 animals at his farm. The ratio of sheep to cattle is 5:3.
    (a) How many sheep does he have?
    (b) And how many cattle?
12. The Helix theatre has 433 seats. During the performance of a popular play, there was only one empty seat in the entire room. For every adult that was there, there were two children attending. For every 7 boys, there were 9 girls and for every 5 men, there were 4 women.
    (a) How many girls saw that performance of the play?

(b) How many boys were there?

(c) How many men attended?

(d) How many women saw it?

13. Your sibling is being annoying and tells you that "Anything you can do, they can do twice as well". You decide to put them to the test and see if that is actually true or just a huge exaggeration.

    (a) You let them time you while you run 100m. You finish in 18 seconds. What does your sibling have to accomplish now to show they weren't lying?

    (b) You build a tower with Jenga blocks and reach a height of 110cm. How high does your siblings tower need to become before toppling over?

    (c) You take a skipping rope and manage to jump 50 times in 2 minutes. What does your sibling have to do?

    (d) You stretch out your arms as wide as you can and let your sibling measure the distance between the fingertips of your left hand and those of your right hand. You managed to stretch them 155cm apart. How far does your sibling have to stretch their arms?

    (e) Do you think the sibling would be able to prove their statement every single time?

14. August 6th was a busy day at Harbour Splash Aqua Park, they received many visitors. Most of the visitors were children, only two in seven visitors were adults. But despite adults being the minority of the visitors, 156 of them visited.

    (a) How many visitors enjoyed the Aqua Park on August 6th?

    (b) How many of them were children?

15. To make a bonsai tree out of lego, you need 432 pieces that are either brown for the stem or green for the leaves. There are twice as many green leaf pieces as there are brown stem ones. If you want to replace the summery green look of your bonsai tree to a pink blooming spring look, you have to replace each green lego leaf piece by 2 pink lego frogs (I also don't really understand why the frogs look so much as flowers in the end result, but despite it being very surprising, they actually do). How many pink frogs do you need to give the bonsai tree the appearance of having an abundance of blossoms?

16. A cake recipe called quatre quart (which means four quarters in French) can be used to make 1 big cake or about 16 cupcakes. It is as follows:
    Ingredients: 4 eggs, 4ml of vanilla essence, 200g self raising flour, 200g sugar, 200g of butter - the butter can be replaced by 200ml of sunflower oil.
    Instructions: Mix all ingredients together. Pour them into your cake or muffin tin.

Put them in the oven (preferably preheated) at 180° C for 45 minutes for a big cake, 30 minutes for cupcakes. To test if your cake or cupcakes are ready, prick them with a fork. If the fork comes away clean, your cake is done. If not, leave it to bake for another 5 minutes and do the prick-it-with-a-fork-test again. Repeat this step until the fork comes away clean.

(a) If an average egg weighs about 50 grams, what is the ratio of the following ingredients?

  i. Eggs to self raising flour
  ii. Sugar to butter
  iii. Eggs to sugar
  iv. Self raising flour to sugar
  v. Butter to eggs
  vi. Butter to self raising flour

(b) i. If you only have one egg, but really really want to make some cupcakes (maybe because you want a taste of the delicious cake batter), how much of the other ingredients would you need to do this?
  ii. How many cupcakes can you make with this?

(c) i. If you want to make 36 cupcakes instead of 16, how much eggs, vanilla essence, self raising flour, sugar and butter do you need?
  ii. Does the amount of cupcakes you place in the oven affect the baking time of them?

Chapter 8

Proportionality

Is there a better way to introduce a new topic, proportionality in this case, than with an explanation of what it is? I can not think of one, so let's start with what a dictionary has to say about the word proportion.

Proportion

1. A part or amount considered in relation to a whole.
2. A relationship between things or parts of things with respect to comparative quantity, magnitude or degree.
3. A relationship between quantities such that if one varies then another varies in a manner dependent on the first.

It is the third meaning of the word proportion that we will be using here. You have actually already encountered many examples of this during your maths lessons and during homework, but now we'll add fancy names to what you did. As you've already used this type of thinking on fractions and decimal numbers, we'll ignore that everything in this course is supposed to be about integers and include rational numbers as well in our examples. It's a bit sneaky of me, I know, but hey, now you can tell your teacher that you've reviewed working with decimal numbers as well... Let's dive in.

8.1 Direct proportionality

Definition 5 *Two things are called* ***proportional*** *or* ***directly proportional*** *if their corresponding elements have a constant ratio.*

Let's look at an example to see what this means and how we are able to use it.

A plumber earns €50 for every hour they work. In this case, the ratio between the plumber's work hours and their earnings is 1:50. This ratio is always the same, so it's constant. This means that the work hours and earnings are directly proportional. Once we know the ratio between the different quantities, we can deduce other things from it.

- If the plumber works for 2 hours, they will have earned €100.
- If the plumber wants to earn €350, they will have to work for 7 hours, because $350 \div 50 = 7$.

We can generalise this as follows:

- For any positive number n, we can say that the plumber will have earned €$n \times 50$ after working n hours.
- For any positive number m, we can say that the plumber will have earned €m after working for $m \div 50$ hours

When we get a table of proportional values, such as

3.72	6.48	2.6	4.44
3.1	5.4	2	3.7

We can calculate the ratio that is used by dividing any of the top numbers by the corresponding bottom number and simplifying this as much as possible. To use the proportionality, it may be easiest to work with a ratio that contains fractions or decimal numbers.

With this table, The easiest numbers to start from are 2.6 and 2. As you see, we start off with decimal numbers. To get rid of the decimal point, we multiply both numbers with 10, or 100, or 1000, whichever it takes. We get

$$\frac{2.6}{2} = \frac{26}{20}$$

Next, we will simplify this fraction as much as possible.

$$\frac{26}{20} = \frac{2 \times 13}{2 \times 10} \quad = \quad \frac{13}{10}$$

The ratio we found is $13 : 10$, or $1.3 : 1$. Using that last ratio we can check the other elements of the table. We get $3.1 \times 1.3 = 3.72$, so we would have gotten the same ratio by using the numbers 3.72 and 3.1. Similarly, we can check that $6.48 = 5.4 \times 1.3$ and

$4.44 = 3.7 \times 1.3$, so the ratio holds up. Even though the ratio 13 : 10 is the most elegant one, the ratio 1.3 : 1 is easier to use to check the other elements of the table.

Exercises

1. Complete these tables by using direct proportionality, and calculate the ratio that is being used.

(a)

6		14.4		62.4
1.5	7.5		25	

Ratio:

(b)

2	7	11	23
50			

Ratio:

(c)

1	2	3	7	16	25
		4			

Ratio:

(d)

11			55	1	
3	1	4			$1\frac{1}{11}$

Ratio:

(e)

		5	1	$\frac{1}{7}$
3	4.2	7		

Ratio:

2. Are these tables using direct proportionality? If they are, give the ratio used. If not, explain why.

(a) Price of rice

Weight of rice (in kg)	2.2	5.3	1.7	512
Price (in €)	2.86	6.89	2.21	66.56

(b) Price of fabric sold per meter

Length of fabric	5	2	10	12
Price (in €)	45.50	18.20	91	109.20

(c) Average height of a child

Age (in years)	2	5	10	14
Height (in cm)	92	105	125	162

(d) Price of stamps sold by An Post

Number of stamps	2	4	5	10	100
Price (in €)	2.70	5.40	6.75	13.00	125.00

(e) Prices of a magazine subscription

Number of months	2	6	9	12
Price (in €)	8	20	26	36

(f) Distance traveled by two people in the same amount of time, one of them cycling, the other one walking.

Cycling (in km)	7.7	19.25	38.5	2.625
Walking (in km)	2.2	5.5	11	0.75

(g) Distance it takes a car to come to a full stop in function of its speed

Speed (in km/h)	40	60	80	90
Distance (in m)	20	36	57.5	67.5

3. Maggie gets a weekly allowance of €2. She saw an awesome toy in the shop that costs €17.95. How many weeks does she have to save her allowance money to be able to buy it?

4. To warm up for football practice, Sammy has to run 5 laps around the football pitch that is 120m wide and 70m long.

   (a) What is the length of one lap around the field?

   (b) How far does she have to run during warm up?

   (c) Which ratio did you use to calculate your result?

5. Arne made running his new year's resolution. He used to be able to run 10km in 1 hour and he wants to get back to that level of fitness. He currently runs 1 km in 8 minutes. How much faster will he have to run to reach his goal?

6. David wants to make a phone call to his family in Denmark. He still has €4 phone credit, and the website of the phone company gives him the following table with costs for international phone calls:

Minutes	1	15	2.5	
Cost (in €)	0.40			4

   (a) Complete the table.

   (b) How long can David's phone call last?

7. Oscar wants to make strawberry jam. He knows that when he has 800 g of fruits, he can make 1.2 kg of jam by adding sugar to them and boiling everything.

(a) What is the ratio of sugar to fruit used to make jam?

(b) How much jam can he make with 1 kg of strawberries? And with 1.5 kg?

(c) How much fruit does he need to fill 5 jars, that have a capacity of 540 g, with homemade strawberry jam?

8. Ciara and her brother Dermot want to surprise their parents by making a cake. They found a delicious looking recipe for a cake for 6 people. The ingredients for that recipe are

   - $\frac{3}{4}$ l milk
   - 6 eggs
   - 270 g of sugar
   - 300 g of self raising flour
   - 180 g of butter

   (a) How do they have to adjust the list of ingredients if they want to make a cake for 4 people?

   (b) How much do they need of each ingredient?

9. Katrijn likes to crochet. She finds it very hard to resist SpecialBuys of colourful wool, and she justifies these impulsive purchases by turning the wool into scarves that are donated in winter. A scarf is usually 30 stitches wide and crocheting 8 rows gives about 10 cm of length. The ideal length of a scarf is about 1m 60cm.

   - How many rows does Katrijn have to crochet to make a scarf?
   - How many stitches is this?
   - If you know that it takes approximately one minute and a half to crochet one row, how long does it take to make a scarf?
   - Do you think this is a good trade off compared to the joy of buying nice wool?

10. Arian wants to send a gift to her former au pair in Spain. The items she wants to send weigh 3.75 kg altogether and they can fit in either 2 packets or 1 big parcel. The An Post website gives the following information:

Zone 3

	Standard (USO)				**Registered** (USO)				**Express**◊	**Int Courier** (EU)[1]#	**Int Courier** (Non EU)#
Weight not over	**Letter/ postcard**	**Large envelope**	**Packet**	**‡Parcel**	**Letter/ postcard**	**Large envelope**	**Packet**	**‡Parcel**	**Letter/ large envelope/ packet**	**Letter/large envelope/ packet/ parcel**	**Letter/large envelope/ packet/ parcel**
100g	€2.20	€3.50	€8.00	€30.00	€8.70	€10.50	€13.00	€35.00	€14.00	€43.00	€49.00
250g	•	€5.50	€9.00	€30.00	•	€12.50	€14.00	€35.00	€15.00	€43.00	€49.00
500g	•	€7.00	€11.00	€30.00	•	€14.00	€16.00	€35.00	€17.00	€43.00	€49.00
1kg	•	•	€14.50	€30.00	•	•	€19.50	€35.00	€20.50	€59.50	€64.00
1.5kg	•	•	€17.50	€30.00	•	•	€22.50	€35.00	€23.50	€63.50	€68.50
2kg	•	•	€17.50	€30.00	•	•	€22.50	€35.00	€23.50	€67.50	€73.00
2.5kg	•	•	•	€45.00	•	•	•	€50.00	Δ	€71.50	€77.50
3kg	•	•	•	€45.00	•	•	•	€50.00	Δ	€75.50	€82.00
3.5kg	•	•	•	€45.00	•	•	•	€50.00	Δ	€79.50	€86.50
4kg	•	•	•	€45.00	•	•	•	€50.00	Δ	€83.50	€91.00
4.5kg	•	•	•	€45.00	•	•	•	€50.00	Δ	€87.50	€95.50
5kg	•	•	•	€45.00	•	•	•	€50.00	Δ	€91.50	€100.00
10kg	•	•	•	€60.00	•	•	•	€65.00	Δ	#	#
Each additional 1kg max 20kg	•	•	•	€3.00	•	•	•	€3.00	Δ	€8.00	€9.00

• This weight exceeds the current category and automatically moves to a higher category.
Δ This weight exceeds the current category size, see International Courier.
International Courier Service, in conjunction with DHL, is available from selected Post Offices. See **anpost.com/internationalcourier** for more details.
◊ Delivery to selected destinations only, please see **anpost.com** for more details or ask at your local Post Office.
‡ Domestic and International Parcels up to 10kg are covered by the USO.
1 VAT @ the Standard Rate is included on International Courier rates to the EU.

(a) Are these rates proportional or not? Why?

(b) If Arian wants to send her gift through standard post (meaning untracked),

i. what is the least expensive way to do this, 2 packets or 1 parcel?

ii. How much would it cost?

iii. What is the price difference with the other way she could send her gift?

(c) How much more would she have to pay to send her gift through registered post (meaning tracked) when she sends it as

i. a parcel?

ii. packets?

Which option is most feasible for registered post?

8.2 Inverse proportionality

Definition 6 *Two things are called inversely proportional if their corresponding elements have a constant product.*

We will once more again look at an example to see what this means.

It takes 20 workers 8 days to harvest all beans on a farm. In this case, if there are more workers, the work would be done in less time. With twice as many workers, it would only take half as long to harvest all the beans. But if there are less people, the harvest would take longer. If we halve the amount of workers, the work would take twice as long. The product of those 2 factors, the number of workers and the amount of days the harvest lasts, is a constant: $20 \times 8 = 160$. Now that we know that constant, we can deduce other things from it.

- If one person would have to do all the work on their own, it would take them 160 days (which is over 5 months, most beans wouldn't be edible anymore by the time they're finally harvested).
- If the work needs to be done in just 2 days (this is $8 \div 4$ days), then it would take $20 \times 4 = 80$ people to make that happen.

We can generalise this as follows:

- For any amount n of workers, the work will get done in $160 \div n$ days.
- For any amount m of days, to get the harvest over with there need to be $160 \div m$ workers.

When we get a table of inverse proportional values, such as

3	2	8	15
12	18	4.5	2.4

We can calculate the constant product by multiplying any of the top numbers with the corresponding bottom number. We can then use this to check if it holds up with the other values.

In our table, the constant product is $3 \times 12 = 36$. This holds up for all values of the table, as $2 \times 18 = 36$, $8 \times 4.5 = 36$ and $15 \times 2.4 = 36$.

Exercises

1. Complete these tables by using inverse proportionality, and calculate the constant product that is being used.

   (a)

25	5		20	
16		32		50

   Constant product:

(b)

22		42		13.2
	5	33	14	

Constant product:

(c)

24	3		10	
12		36		18

Constant product:

(d)

12.5		8	12	
	3.2		40	9

Constant product:

(e)

24	0.4		2.5	38
		64		16

Constant product:

2. Are these tables using inverse proportionality? If they are, give the product they used. If not, explain why.

(a) Cost of a circular table

Radius of the table (in cm)	40	60	75	100
Price (in euro)	90	135	168.75	225

(b) Waiting time in a restaurant

Number of waiters	6	9	10	15
Average waiting time (in minutes)	90	60	54	36

(c) Records of trips from Dublin to Galway (180 km)

Average speed (in km/h)	60	72	90	100	120
Time (in hours)	3	2.5	2	1.8	1.5

(d) Waiting time for check-in

Number of staff	3	4	5	6	7
Average waiting time (in minutes)	83	47	30	21	15

(e) The force exerted by a magnet on a metal object at a distance d

Force (in Newtons)	60	$26\frac{2}{3}$	15	9.6	$6\frac{2}{3}$
Distance (in cm)	2	3	4	5	6

(f) The force exerted by a magnet on a metal object at a distance d

Force (in Newtons)	60	$26\frac{2}{3}$	15	9.6	$6\frac{2}{3}$
Square of the distance (in cm^2)	4	9	16	25	36

3. It was an exceptionally warm summer's day (as in, it was 20° C, in Ireland!) and the weather forecast predicted that this warm spell would last for another four days. If there ever was a good time to set up the inflatable pool in the garden, it was definitely now! When it came to filling the pool with water, everyone was eager to help. Mom calculated that it would take 4 hours to fill the pool if they could attach hoses to 3 taps.

   (a) How long would it take to fill the pool is they could use 5 hoses with the same water flow?

   (b) How long would it take if they could only use 2 hoses?

4. It takes 35 workers 8 days to harvest the apples from half an orchard. But then, a nasty flu did the rounds and 15 of them got sick. How many days will it take the remaining 20 workers to harvest the other half of the orchard?

5. Two trucks have just picked up the apple harvest and they arrived at the factory where the apples will be used to make apple juice and applesauce. It took 3 people 2 hours to unload the first truck. When that job is done, 5 people are ready to start emptying the second truck. How long will it take them to unload it?

6. The research base in Antarctica had enough supplies for 300 researchers for 90 days. After 20 days, 50 researchers left the base. How long will the food now last?

7. Back in the middle ages, before the invention of the press, books were incredibly valuable and expensive. Each one was written by hand and being able to read and write was a skill not many people possessed. People who were employed to write were calles scribes.

   It took 6 scribes, working 5 hours a day, 16 days to copy a particular book. How long would it have taken 4 scribes to copy the same book, if they all were working 6 hours a day?

8. In Santa's toy shop, the elves don't have to do all the work themselves as they are assisted by machines. Last year, there were 36 machines in use to make puzzles. They could make the required amount of puzzles in 54 days. Unfortunately it isn't all that easy to get a technician all the way up to the North Pole for the maintenance of the machines, so some are temporarily out of order. The elves in charge are freaking out a bit, as they need to produce the exact same amount of puzzles as last year and they have less machines available to do it. Luckily, they still have 81 days left to finish

Figure 8.1: The Book of Kells is a famous copy of the bible. In this book the scribes added many decorations to the text, with such fine details that it is very impressive. Especially when you realise that magnifying lenses and other equipment to easily enlarge an image wasn't available when it was made

the job. What is the least amount of machines they need to be fully functioning in order to reach their goal?

9. In junior infants, a box of pencils is distributed among 25 children and each child receives 3 pencils. In a parallel class, another box is distributed among the children. If there are 10 children less in the second class, how many pencils will each child receive?

10. A big pharmaceutical project takes 125 days to complete if 16 scientists work on it. How many scientists are needed when the boss wants to see results in only 40 days?

11. An animal shelter has bought enough food to feed the 80 dogs they currently take care of for 60 days. After 15 days, they are trusted to care for another 20 dogs. How long will the food they bought last once there are 20 more dogs to feed?

Chapter 9

Maths riddles

1. Farmer Nelis went to the market to sell carrots. His first client bought half of his stock, and half a kg extra. His second client bought half of what was left, plus half a kg. The third client did the same: buy half of what is left, plus half a kilo. The fourth and fifth clients followed the same pattern: they bought half of the stock that was left, and half a kg more than that. After the fifth client, farmer Nelis had sold out.
   How many kg of carrots did farmer Nelis bring with him to the market place?

2. There are 20 yellow socks in a box, together with 24 red socks and 27 blue ones. The 28^{th} blue sock is used to blindfold you.
   (a) What is the least amount of socks that you have to take out of the box to ensure that you have two socks of the same colour?
   (b) How many socks do you need to take out to be sure that you have one pair of yellow socks?
   (c) How many socks do you need to take to ensure that you have at least two socks of a different colour?

3. In this addition, some digits were replaced by other symbols. The value of the symbols stays the same. Crack the code!

$$\begin{array}{ccccc} & \% & 8 & 3 & \& \\ & 2 & \# & 8 & 3 \\ + & @ & 3 & \% & \& \\ \hline & 8 & 8 & \# & \& \end{array}$$

4. A grandfather clock chimes on the hour, as many times as the hour. For example, at 10 o'clock it chimes 10 times. At half past, it chimes exactly once.
   Since midnight, the clock has chimed 54 times. What time will it be the next time the clock chimes?

Chapter 10

Solutions

10.1 Negative numbers

Exercise 1 Solve these additions.

1. $7 + (-5) = 2$
2. $-8 + 3 = -5$
3. $-5 + (-6) = -11$
4. $4 + (-4) = 0$
5. $-7 + 12 = 5$
6. $3 + 4 = 7$
7. $3 + (-4) = -1$
8. $-3 + 4 = 1$
9. $-3 + (-4) = -7$
10. $8 + (-11) = -3$
11. $-15 + 18 + (-9) = -6$
12. $7 + (-98) + 2 = -89$
13. $25 + (-16) + (-8) = 1$
14. $-16 + (-17) + 5 = -28$
15. $11 + (-38) + (-9) = -36$

Exercise 2 Solve these subtractions.

1. $5 - (-6) = 11$
2. $5 - 6 = -1$
3. $-5 - 6 = -11$
4. $-5 - (-6) = 1$
5. $6 - 5 = 1$
6. $6 - (-5) = 11$
7. $7 - (-11) = 18$
8. $7 - 11 = -4$
9. $-15 - (-19) = 4$
10. $254 - 546 = -292$
11. $25 - (-36) - 3 = 58$
12. $-16 - (-25) - (-4) = 13$
13. $-12 - (-14) - 16 = -18$
14. $10 - (-20) - 15 = 15$
15. $7 - (-61) - 41 = 27$

Exercise 3 Solve these multiplications.

1. $3 \times 7 = 21$
2. $(-3) \times 7 = -21$
3. $3 \times (-7) = -21$
4. $(-3) \times (-7) = 21$
5. $5 \times (-5) = -25$
6. $(-4) \times (-4) = 16$
7. $7 \times (-8) = -56$
8. $38 \times (-16) = -608$
9. $(-25) \times (-11) = 275$
10. $(-81) \times (27) = -2187$
11. $2 \times (-3) \times (-4) = 24$
12. $(-3) \times (-5) \times (-2) = -30$
13. $(-28) \times 0 \times (-16) = 0$
14. $15 \times (-9) \times (-2) = 270$
15. $7 \times (-8) \times 4 = -224$

Exercise 4 Solve these divisions.

1. $35 \div (-5) = -7$
2. $(-121) \div 11 = -11$
3. $(-12,639) \div (-11) = 1,149$
4. $456 \div (-12) = -38$
5. $(-81) \div 9 = -9$
6. $72 \div (-3) = -24$
7. $(-72) \div 3 = -24$
8. $(-72) \div (-3) = 24$
9. $66 \div (-11) = -6$
10. $(-25) \div (-5) = 5$
11. $7,345 \div (-5) = 1,469$
12. $(-1,043) \div 7 = -149$
13. $(-851) \div (-23) = 37$
14. $195 \div (-13) = -15$
15. $(-3,038) \div (-98) = 31$

Exercises

5. (a) $245 + (-38) - (-87) + 21 = 315$
   (b) $(36 \div (-6)) \times (-4) + (-50) = -26$
   (c) $((-144) \div 6) \div (39 \div (-13)) = 8$
   (d) $(24 - (-16)) \div ((-2) \times (-2)) = 10$
   (e) $((-85) + 99) - ((-3) \times (-7)) = -7$
   (f) $14 + ((-36) \times (-2) \div (-9)) = 6$
   (g) $(-24) \times (-9) \div (-3) \div 6 = -12$
   (h) $((-3) \times 4) - (28 \div (-4)) = -5$
   (i) $((3 - (-7)) \div (-2)) \times (-8) = 40$
   (j) $((-6) \times (-3)) + ((-231) \div 11) = -3$
   (k) $4 + ((-9) \times 3) + (-37) = -60$
   (l) $(-11) \times (-38) + (108 \div (-9)) = 406$
   (m) $(225 \div (-15)) - ((-364) \div 4) = 76$
   (n) $(16 \div (-4)) \times (-17 - (-15)) = 8$
   (o) $((-17) + (-8)) \div (-17 + 22) = -5$
   (p) $((-63) \div 3) \div ((-21) \div (-7)) = -7$

6. (a) How long did the temple of Juno Moneta last? 281 years
   (b) How many years have passed between the great fire of Rome that destroyed the temple of Juno and the deconsecration of the Basilica that was built on the same site? 1,861 years

7. The balance on her bank account is €-19.03.

8. (a) What was the difference in temperature between Dublin and Winnipeg? 24° C

(b) And between Dublin and Hong Kong? 12° C

(c) And between Dublin and São Paulo? 24° C

(d) Between which two cities is the temperature difference the greatest? Winnipeg and São Paulo How much is this difference? 48° C

10.2 Power of a number

Exercise 1 Calculate these powers.

1. $3^3 = 27$
2. $4^4 = 256$
3. $5^2 = 25$
4. $7^3 = 343$
5. $2^5 = 32$
6. $3^4 = 81$
7. $1^166 = 1$
8. $10^5 = 100,000$
9. $11^3 = 1331$
10. $10^8 = 100,000,000$
11. $26,987^1 = 26,987$
12. $123,456,789^1 = 123,456,789$
13. $37^2 = 1,369$
14. $987,654,321^0 = 1$
15. $93^0 = 1$

Exercise 2 Work out these multiplications. Use the power notation a^n in your results.

1. $2^2 \times 2 = 2^3$
2. $3 \times 3^3 = 3^4$
3. $2^3 \times 3^2 = 2^3 \times 3^2$
4. $2^5 \times 2^3 = 2^8$
5. $5^3 \times 5^7 = 5^{10}$
6. $4^6 \times 4^1 = 4^7$
7. $17^5 \times 17^2 = 17^7$
8. $1^{25} \times 1^{36} = 1$
9. $4^8 \times 4^7 = 4^{15}$
10. $0^{567} \times 7^8 = 0$
11. $6^3 \times 6^{323} = 6^{326}$
12. $11^0 \times 11^{11} = 11^{11}$

Important to remember: You can not add the powers in $2^3 \times 3^2$ because you are working with powers of different numbers.

Exercise 3 Work out these divisions. Write your answers in the power notation a^n.

1. $10^7 \div 10^3 = 10^4$
2. $3^5 \div 5^2 = 3^5 \div 5^2$
3. $3^5 \div 3^2 = 3^3$
4. $8^{15} \div 8^0 = 8^{15}$
5. $11^{11} \div 11^7 = 11^4$
6. $3^9 \div 3 = 3^8$
7. $9^3 \div 9^3 = 1$
8. $2^{16} \div 2^9 = 2^7$
9. $1^{89} \div 1^{99} = 1$

Exercise 4 Combine everything you learned this far. Use the power notation a^n to write down your answers.

1. $2^4 \times 2^8 \div 2^5 = 2^7$
2. $3^8 \div 3^4 \div 3^2 = 3^2$
3. $5^4 \times 5^3 \div 5^7 = 1$
4. $4^4 \div 4^5 \times 4^3 = 4^2$
5. $2^8 \times 3 \div 2^6 = 2^2 \times 3$
6. $3^7 \div 3^2 \times 3^0 = 3^5$
7. $7 \times 7^7 \times 7^2 = 7^{10}$
8. $1^{97} \div 1^{17} \times 1^0 = 1$
9. $2^4 \div 3^3 \times 3^5 = 2^4 \times 3^2$
10. $7^7 \times 7^2 \times 1^7 = 7^9$
11. $2^4 \times 8^0 \div 2^2 = 2^2$
12. $9^3 \times 9^7 \div 9^8 = 9^2$
13. $2^7 \div 2^2 \div 2^5 = 1$
14. $8^5 \times 8^8 \times 8^3 = 8^{16}$
15. $5 \times 5^2 \times 5^5 \div 5^3 = 5^5$
16. $7^0 \times 7^1 \times 7^2 \times 7^3 \times 7^4 = 7^{10}$
17. $2^9 \times 3^4 \div 3^3 \times 2^3 = 2^{12} \times 3$
18. $3^5 \div 3^4 \times 4^4 \div 4^2 \div 3 = 4^2$
19. $2^3 \times 5^5 \div 2^2 \times 5^3 \times 5 = 2 \times 5^9$
20. $3^3 \times 7^9 \div 7^3 \div 7^4 \div 3^2 = 3 \times 7^2$
21. $9^7 \times 4^3 \times 5^4 \div 9^5 \div 5^2 = 4^3 \times 5^6 \times 9^2$

10.3 Divisibility

10.3.1 Divisibility by 2

Exercise Is the number divisible by 2? Write your anwer as ✓or ×.

1. 36 ✓
2. 89 ×
3. 123 ×
4. 438 ✓
5. 903 ×
6. 1,231 ×
7. 91,234 ✓
8. 983,476 ✓
9. 304,897,734 ✓
10. 42,109,341,087 ×
11. 42,179,803,410 ✓
12. 2,314,898,413,092 ✓
13. 12,347,120,798,325 ×
14. 2,902,138,436,712,309 ×
15. 9,834,012,573,584,912 ✓

10.3.2 Divisibility by 3

Exercise Is the number divisible by 3? Write your anwer as ✓or ×. If your number isn't divisible by 3, write down the remainder as well.

1. 36 ✓
2. 89 × R2
3. 123 ✓
4. 438 ✓
5. 903 ✓
6. 1,231 × R1

7. 91,234 × R1
8. 983,476 × R1
9. 304,897,734 ✓
10. 42,109,341,087 ✓
11. 42,179,803,410 ✓
12. 2,314,898,413,092 ✓
13. 12,347,120,798,325 ✓
14. 2,902,138,436,712,309 ✓
15. 9,834,012,573,584,912 × R2

10.3.3 Divisibility by 4

Exercise Is the number divisible by 4? Write your answer as ✓or ×. If your number isn't divisible by 4, write down the remainder as well.

1. 36 ✓
2. 89 × R1
3. 123 × R3
4. 438 × R2
5. 903 × R3
6. 1,231 × R3
7. 91,234 × R2
8. 983,476 ✓
9. 304,897,734 × R2
10. 42,109,341,087 × R3
11. 42,179,803,410 × R2
12. 2,314,898,413,092 ✓
13. 12,347,120,798,325 × R1
14. 2,902,138,436,712,309 × R1
15. 9,834,012,573,584,912 ✓

10.3.4 Divisibility by 5

Exercise Is the number divisible by 5? Write your answer as ✓or ×. If your number isn't divisible by 5, write down the remainder as well.

1. 36 × R1
2. 89 × R4
3. 123 × R3
4. 438 × R3
5. 903 × R3
6. 1,231 × R1
7. 91,234 × R4
8. 983,476 × R1
9. 304,897,734 × R4
10. 42,109,341,087 × R2
11. 42,179,803,410 ✓
12. 2,314,898,413,092 × R2
13. 12,347,120,798,325 ✓
14. 2,902,138,436,712,309 × R4
15. 9,834,012,573,584,912 × R2

10.3.5 Divisibility by 6

Exercise Is the number divisible by 6? Write your answer as ✓or ×. If your number isn't divisible by 6, write down the remainder as well.

1. 36 ✓
2. 89 × R5
3. 123 × R3
4. 438 ✓
5. 903 × R3
6. 1,231 × R1
7. 91,234 × R4
8. 983,476 × R4
9. 304,897,734 ✓
10. 42,109,341,087 × R3
11. 42,179,803,410 ✓
12. 2,314,898,413,092 ✓
13. 12,347,120,798,325 × R3
14. 2,902,138,436,712,309 × R3
15. 9,834,012,573,584,912 × R2

10.3.6 Divisibility by 8

Exercise Is the number divisible by 8? Write your answer as ✓or ×. If your number isn't divisible by 8, write down the remainder as well.

1. 36 × R4
2. 89 × R1
3. 123 × R3
4. 438 × R6
5. 903 × R7
6. 1,231 × R7
7. 91,234 × R2
8. 983,476 × R4
9. 304,897,734 × R6
10. 42,109,341,087 × R7
11. 42,179,803,410 × R2
12. 2,314,898,413,092 × R6
13. 12,347,120,798,325 × R5
14. 2,902,138,436,712,309 × R5
15. 9,834,012,573,584,912 ✓

10.3.7 Divisibility by 9

Exercise Is the number divisible by 9? Write your answer as ✓or ×. If your number isn't divisible by 9, write down the remainder as well.

1. 36 ✓
2. 89 × R8
3. 123 × R6
4. 438 × R6
5. 903 × R3
6. 1,231 × R7
7. 91,234 × R1
8. 983,476 × R1
9. 304,897,734 ✓

10. 42,109,341,087 × R3
11. 42,179,803,410 × R3
12. 2,314,898,413,092 ✓
13. 12,347,120,798,325 ✓
14. 2,902,138,436,712,309 × R6
15. 9,834,012,573,584,912 × R8

10.3.8 Divisibility by 11

Exercise Is the number divisible by 11? Write your answer as ✓or ×.

1. 36 ×
2. 89 ×
3. 123 ×
4. 438 ×
5. 903 ×
6. 1,231 ×
7. 91,234 ✓
8. 983,476 ×
9. 304,897,734 ×
10. 42,109,341,087 ✓
11. 42,179,803,410 ×
12. 2,314,898,413,092 ×
13. 12,347,120,798,325 ×
14. 2,902,138,436,712,309 ×
15. 9,834,012,573,584,912 ×

10.3.9 Divisibility by powers of 10

Exercise Is ... divisible by ...? Write your answer as ✓or ×.

1. 12,300 by 10^1? ✓
2. 12,300 by 10^2? ✓
3. 12,300 by 10^3? ×
4. 872,234,000 by 10^1? ✓
5. 872,234,000 by 10^2? ✓
6. 872,234,000 by 10^3? ✓
7. 872,234,000 by 10^4? ×
8. 872,234,000 by 10^5? ×
9. 872,234,000 by 10^6? ×
10. 5,234,900,000 by 10^1? ✓
11. 5,234,900,000 by 10^2? ✓
12. 5,234,900,000 by 10^3? ✓
13. 5,234,900,000 by 10^4? ✓
14. 5,234,900,000 by 10^5? ✓
15. 5,234,900,000 by 10^6? ×

10.3.10 Divisibility by other numbers

Exercises

1. What are the rules for divisibility by 18? Be careful, will you use $18 = 2 \times 9$ or $18 = 3 \times 6$? A number is divisible by $18 = 2 \times 9$ if and only if it is divisible by 2 and by 9, so it is an even number and the sum of its digits is divisible by 9.
2. What are the rules for divisibility by 22? A number is divisible by $22 = 2 \times 11$ if and only if it is divisible by 2 and 11, so it has to be an even multiple of 11.
3. What are the rules for divisibility by 24? A number is divisible by $24 = 3 \times 8$ if and only if it is divisible by 3 and by 8, so the sum of its digits has to be divisible by 3 and the 3 lowest digits have to be divisible by 8.
4. What are the rules for divisibility by 30? A number is divisible by $30 = 3 \times 10 = 2 \times 3 \times 5$ if and only if it is divisible by 3 and by 10, so the sum of its digits has to be divisible by 3 and the last digit has to be 0 (this is actually the combination of the rules for divisibility by 2 and by 5).
5. What are the rules for divisibility by 33? A number is divisible by $33 = 3 \times 11$ if and only if it is divisible by 3 and by 11, so the sum of the digits has to be a multiple of 3 and the number has to be a multiple of 11.
6. What are the rules for divisibility by 40? A number is divisible by $40 = 4 \times 10$ if and only if it is divisible by 4 and by 10, so its last digit has to be 0 and its last two digits have to be a multiple of 4. This basically means that the number has to end with 00, 20, 40, 60 or 80.
7. What are the rules for divisibility by 44? A number is divisible by $44 = 4 \times 11$ if and only if it is divisible by 4 and by 11, so its last two digits have to be a multiple of 4 and the number has to be divisible by 11.
8. What are the rules for divisibility by 45? A number is divisible by $45 = 5 \times 9$ if and only if it is divisible by 5 and by 9, so the sum of its digits has to be divisible by 9 and the last digit has to be 0 or 5.
9. What are the rules for divisibility by 50? A number is divisible by 50 if and only if it ends with 00 or 50. This rule is a bit tricky, I hope you noticed that divisibility by 5 and 10 wasn't enough.
10. What are the rules for divisibility by 55? A number is divisible by $55 = 5 \times 11$ if and only if it is divisible by 5 and 11, so it is a multiple of 11 that ends in 0 or 5.
11. What are the rules for divisibility by 66? A number is divisible by $66 = 2 \times 3 \times 11$ if and only if it is divisible by 2, 3 and 11, so it is an even number, the sum of the digits is divisible by 3 and it is a multiple of 11.
12. What are the rules for divisibility by 72? A number is divisible by $72 = 8 \times 9$ if and only if it is divisible by 8 and by 9, so its last three digits are a multiple of 8 and the sum of its digits is a multiple of 9.

13. What are the rules for divisibility by 88? A number is divisible by $88 = 8 \times 11$ if and only if it is divisible by 8 and by 11, so its 3 lowest digits are divisible by 8 and the number is divisible by 11.

14. What are the rules for divisibility by 99? A number is divisible by $99 = 9 \times 11$ if and only if it is divisible by 9 and 11, so the sum of its digits is a multiple of 9 and the number is divisible by 11.

10.3.11 Revision

Fill in the table. Write ✓if the number is divisible, $\times$ if it is not. Write down the remainder if the number is not divisible by 2, 3, 4, 5, 6, 8, 9 or 10.

number	is divisible by										
	2	3	4	5	6	8	9	10	11	33	45
924	✓	✓	✓	$\times$ R4	✓	$\times$ R4	$\times$ R6	$\times$ R4	✓	✓	$\times$
1,345,976	✓	$\times$ R2	✓	$\times$ R1	$\times$ R2	✓	$\times$ R8	$\times$ R6	$\times$	$\times$	$\times$
3,960	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
22,275	$\times$ R1	✓	$\times$ R3	✓	$\times$ R3	$\times$ R3	✓	$\times$ R5	✓	✓	✓
89,042,157	$\times$ R1	✓	$\times$ R1	$\times$ R2	$\times$ R3	$\times$ R5	✓	$\times$ R7	$\times$	$\times$	$\times$
40,095	$\times$ R1	✓	$\times$ R3	✓	$\times$ R3	$\times$ R7	✓	$\times$ R5	✓	✓	✓
5,402,529	$\times$ R1	✓	$\times$ R1	$\times$ R4	$\times$ R3	$\times$ R1	✓	$\times$ R9	✓	✓	$\times$
7,095,424	✓	$\times$ R1	✓	$\times$ R4	$\times$ R4	$\times$ R4	$\times$ R4	$\times$ R4	$\times$	$\times$	$\times$
6,587	$\times$ R1	$\times$ R2	$\times$ R3	$\times$ R2	$\times$ R5	$\times$ R3	$\times$ R8	$\times$ R7	$\times$	$\times$	$\times$
1,234,567,890	✓	✓	$\times$ R2	✓	✓	$\times$ R2	✓	✓	$\times$	$\times$	✓

10.4 Prime numbers

The prime numbers smaller than 150 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29,31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139 and 149. The biggest prime number for which you had to cross out the multiples is $p = 13$.

10.5 Prime factor decomposition

10.5.1 What is prime factor decomposition

Exercise Find the prime decompostion of the following numbers.

1. $60 = 2^2 \times 3 \times 5$

$$\begin{array}{r|l} 60 & 2 \\ 30 & 2 \\ 15 & 3 \\ 5 & 5 \\ 1 & \end{array}$$

2. 61=61

61 is a prime number

$$\begin{array}{r|l} 61 & 61 \\ 1 & \end{array}$$

3. $62 = 2 \times 31$

$$\begin{array}{r|l} 62 & 2 \\ 31 & 31 \\ 1 & \end{array}$$

4. $63 = 3^2 \times 7$

$$\begin{array}{r|l} 63 & 3 \\ 21 & 3 \\ 7 & 7 \\ 1 & \end{array}$$

5. $64 = 2^6$

$$\begin{array}{r|l} 64 & 2 \\ 32 & 2 \\ 16 & 2 \\ 8 & 2 \\ 4 & 2 \\ 2 & 2 \\ 1 & \end{array}$$

6. $65 = 5 \times 13$

$$\begin{array}{r|l} 65 & 5 \\ 13 & 13 \\ 1 & \end{array}$$

7. $66 = 2 \times 3 \times 11$

$$\begin{array}{r|l} 66 & 2 \\ 33 & 3 \\ 11 & 11 \\ 1 & \end{array}$$

8. $67 = 67$

$$\begin{array}{r|l} 67 & 67 \\ 1 & \end{array}$$

9. $68 = 2^2 \times 17$

$$\begin{array}{r|l} 68 & 2 \\ 34 & 2 \\ 17 & 17 \\ 1 & \end{array}$$

10. $69 = 3 \times 23$

$$\begin{array}{r|l} 69 & 3 \\ 23 & 23 \\ 1 & \end{array}$$

11. $70 = 2 \times 5 \times 7$

$$\begin{array}{r|l} 70 & 2 \\ 35 & 5 \\ 7 & 7 \\ 1 & \end{array}$$

12. $71 = 71$

$$\begin{array}{r|l} 71 & 71 \\ 1 & \end{array}$$

13. $98 = 2 \times 7^2$

$$\begin{array}{r|l} 98 & 2 \\ 49 & 7 \\ 7 & 7 \\ 1 & \end{array}$$

14. $163 = 163$

$$\begin{array}{r|l} 163 & 163 \\ 1 & \end{array}$$

15. $189 = 3^3 \times 7$

$$\begin{array}{r|l} 189 & 3 \\ 63 & 3 \\ 21 & 3 \\ 7 & 7 \\ 1 & \end{array}$$

16. $222 = 2 \times 3 \times 37$

$$\begin{array}{r|l} 222 & 2 \\ 111 & 3 \\ 37 & 37 \\ 1 & \end{array}$$

17. $144 = 2^4 \times 3^2$

$$\begin{array}{r|l} 144 & 2 \\ 72 & 2 \\ 36 & 2 \\ 18 & 2 \\ 9 & 3 \\ 3 & 3 \\ 1 & \end{array}$$

18. $234 = 2 \times 3^2 \times 13$

$$\begin{array}{r|l} 234 & 2 \\ 117 & 3 \\ 39 & 3 \\ 13 & 13 \\ 1 & \end{array}$$

19. $267 = 3 \times 89$

$$\begin{array}{r|l} 267 & 3 \\ 89 & 89 \\ 1 & \end{array}$$

20. $324 = 2^2 \times 3^4$

324	2
162	2
81	3
27	3
9	3
3	3
1	

17. $721 = 7 \times 103$

721	7
103	103
1	

18. $681 = 3 \times 227$

681	3
227	227
1	

19. $873 = 3^2 \times 97$

873	3
291	3
97	97
1	

20. $999 = 3^3 \times 37$

999	3
333	3
111	3
37	37
1	

21. $1,584 = 2^4 \times 3^2 \times 11$

1,584	
792	2
396	2
198	2
99	3
33	3
11	11
1	

22. $2,057 = 11^2 \times 17$

2,057	11
187	11
17	17
1	

23. $5,083 = 13 \times 17 \times 23$

5,087	13
391	17
23	23
1	

24. $7,267 = 13^2 \times 43$

7,267	13
559	13
43	43
1	

10.5.2 Uses of prime factor decomposition

Finding all factors of a number

Exercise List all the factors of the following numbers, by using prime factor decomposition.

1. $1,398 = 2 \times 3 \times 233$

1,398	2
699	3
233	

The factors of 1,398 are 1, 2, 3, 6, 233, 466, 699, 1,398.

2. $1,575 = 3^2 \times 5^2 \times 7$

1,575	3
525	3
175	5
35	5
7	7
1	

The factors of 1,575 are 1, 3, 5, 7, 9, 15, 21, 25, 35, 45, 63, 75, 105, 175,225,315, 525, 1,575.

3. $2,023 = 7 \times 17^2$

2,023	7
289	17
17	17
1	

The factors of 2,023 are 1, 7, 17, 119, 289, 2,023.

4. $2,024 = 2^3 \times 11 \times 23$

$$\begin{array}{r|l} 2{,}024 & 2 \\ 1{,}012 & 2 \\ 506 & 2 \\ 253 & 11 \\ 23 & 23 \\ 1 & \end{array}$$

The factors of 2,024 are 1, 2, 4, 8, 11, 22, 23, 44, 46, 88, 92, 184, 253, 506, 1,012, 2,024.

Finding the highest common factor

Exercises

1. Find the highest common factor of the following numbers by using prime factor decomposition.

(a) $hcf(267, 198) = 3$

$$\begin{array}{r|l} 267 & 3 \\ 89 & 89 \\ 1 & \end{array} \qquad \begin{array}{r|l} 198 & 2 \\ 99 & 3 \\ 33 & 3 \\ 11 & 11 \\ 1 & \end{array}$$

(b) $hcf(242, 286) = 22$

$$\begin{array}{r|l} 242 & 2 \\ 121 & 11 \\ 11 & 11 \\ 1 & \end{array} \qquad \begin{array}{r|l} 286 & 2 \\ 143 & 11 \\ 13 & 13 \\ 1 & \end{array}$$

(c) $hcf(183, 186) = 3$

$$\begin{array}{r|l} 183 & 3 \\ 61 & 61 \\ 1 & \end{array} \qquad \begin{array}{r|l} 186 & 2 \\ 93 & 3 \\ 31 & 31 \\ 1 & \end{array}$$

(d) $hcf(231, 363) = 33$

$$\begin{array}{r|l} 231 & 3 \\ 77 & 7 \\ 11 & 11 \\ 1 & \end{array} \qquad \begin{array}{r|l} 363 & 3 \\ 121 & 11 \\ 11 & 11 \\ 1 & \end{array}$$

(e) $hcf(333;\ 2,775) = 111$

$$\begin{array}{r|l} 333 & 3 \\ 111 & 3 \\ 37 & 37 \\ 1 & \end{array} \qquad \begin{array}{r|l} 2775 & 3 \\ 925 & 5 \\ 185 & 5 \\ 37 & 37 \\ 1 & \end{array}$$

(f) $hcf(258;\ 1,104) = 6$

$$\begin{array}{r|l} 258 & 2 \\ 129 & 3 \\ 43 & 43 \\ 1 & \end{array} \qquad \begin{array}{r|l} 1104 & 2 \\ 552 & 2 \\ 276 & 2 \\ 138 & 2 \\ 69 & 3 \\ 23 & 23 \\ 1 & \end{array}$$

(g) $hcf(51, 68, 85) = 17$

(h) $hcf(60, 75, 105) = 15$

$$\begin{array}{r|l} 51 & 3 \\ 17 & 17 \\ 1 & \end{array} \qquad \begin{array}{r|l} 68 & 2 \\ 34 & 2 \\ 17 & 17 \\ 1 & \end{array} \qquad \begin{array}{r|l} 85 & 5 \\ 17 & 17 \\ 1 & \end{array} \qquad \begin{array}{r|l} 60 & 2 \\ 30 & 2 \\ 15 & 3 \\ 5 & 5 \\ 1 & \end{array} \qquad \begin{array}{r|l} 75 & 3 \\ 25 & 5 \\ 5 & 5 \\ 1 & \end{array} \qquad \begin{array}{r|l} 105 & 3 \\ 35 & 5 \\ 7 & 7 \\ 1 & \end{array}$$

2. (a) Find the prime factor decomposition of the following numbers.

$$\begin{array}{r|l} 135 & 3 \\ 45 & 3 \\ 15 & 3 \\ 5 & 5 \\ 1 & \end{array} \qquad \begin{array}{r|l} 144 & 2 \\ 72 & 2 \\ 36 & 2 \\ 18 & 2 \\ 9 & 3 \\ 3 & 3 \\ 1 & \end{array} \qquad \begin{array}{r|l} 150 & 2 \\ 75 & 3 \\ 25 & 5 \\ 5 & 5 \\ 1 & \end{array} \qquad \begin{array}{r|l} 162 & 2 \\ 81 & 3 \\ 27 & 3 \\ 9 & 3 \\ 3 & 3 \\ 1 & \end{array} \qquad \begin{array}{r|l} 200 & 2 \\ 100 & 2 \\ 50 & 2 \\ 25 & 5 \\ 5 & 5 \\ 1 & \end{array}$$

(b) Find the following highest common factors.

i. $hcf(135, 144) = 9$
ii. $hcf(135, 150) = 15$
iii. $hcf(135, 162) = 27$
iv. $hcf(135, 200) = 5$
v. $hcf(144, 150) = 6$
vi. $hcf(144, 162) = 18$
vii. $hcf(144, 200) = 8$
viii. $hcf(150, 162) = 6$
ix. $hcf(150, 200) = 50$
x. $hcf(162, 200) = 2$
xi. $hcf(135, 144, 150) = 3$
xii. $hcf(135, 144, 162) = 9$
xiii. $hcf(135, 144, 200) = 1$
xiv. $hcf(135, 150, 162) = 3$
xv. $hcf(135, 150, 200) = 5$
xvi. $hcf(135, 162, 200) = 1$
xvii. $hcf(144, 150, 162) = 6$
xviii. $hcf(144, 150, 200) = 2$
xix. $hcf(144, 162, 200) = 2$
xx. $hcf(150, 162, 200) = 2$

xxi. $hcf(135, 144, 150, 162) = 3$
xxii. $hcf(135, 144, 150, 200) = 1$
xxiii. $hcf(135, 144, 162, 200) = 1$
xxiv. $hcf(135, 150, 162, 200) = 1$
xxv. $hcf(144, 150, 162, 200) = 2$
xxvi. $hcf(135, 144, 150, 162, 200) = 1$

3. Find the highest common factor by using Euclid's algorithm.

   (a) $hcf(267, 198) = hcf(267 - 198 = 63, 198) = hcf(63, 198 - 3 \times 63 = 9) = hcf(63 - 7 \times 9 = 0 \text{ oops, went too far, so } 63 - 6 \times 9 = 9, 9) = 9$

   (b) $hcf(242, 286) = hcf(242, 286 - 242 = 44) = hcf(242 - 5 \times 44 = 242 - 220 = 22, 44) = hcf(22, 44 - 22 = 22) = 22$

   (c) $hcf(183, 186) = hcf(183, 186 - 183 = 3) = hcf(183 - 60 \times 3 = 3, 3) = 3$

   (d) $hcf(231, 363) = hcf(231, 363 - 231 = 132) = hcf(231 - 132 = 99, 132) = hcf(99, 132 - 99 = 33) = hcf(99 - 2 \times 33 = 33, 33) = 33$

   (e) $hcf(333;\ 2,775) = hcf(333;\ 2,775-8\times 333 = 2,775-2,664 = 111) = hcf(333-2 \times 111, 111) = 111$

   (f) $hcf(258;\ 1,104) = hcf(258;\ 1,104-4\times 258 = 1,104-1,032 = 72) = hcf(258-3\times 72 = 258-216 = 42, 72) = hcf(42, 72-42 = 30) = hcf(42-30 = 12, 30) = hcf(12, 30 - 2 \times 12 = 6) = hcf(12 - 6 = 6, 6) = 6$

   (g) $hcf(51, 68, 85) = hcf(hcf(51, 68), 85) = hcf(hcf(51, 68 - 51 = 17), 85) = hcf(hcf(51 - 2 \times 17 = 17, 17), 85) = hcf(17, 85) = hcf(17, 85 - 5 \times 17 = 0 \text{ oops, went too far, so } 85 - 4 \times 85 = 17) = 17$

   (h) $hcf(60, 75, 105) = hcf(hcf(60, 75), 105) = hcf(hcf(60, 75 - 60 = 15), 105) = hcf(hcf(60 - 3 \times 15 = 15, 15), 105) = hcf(15, 105) = hcf(15, 105 - 6 \times 15 = 15) = 15$

4. (a) Did you find your answers quicker or slower than when you searched for the same highest common factors by using prime factor decomposition in Exercise 1? It's definitely quicker if the numbers are close, but otherwise it depends on the numbers.

   (b) Do you think the algorithm would be a smart choice time-wise to solve part b) of Exercise 2 that came before? Why or why not? Porbably not, because you only have to do the prime factor decompositions once, but you would have to use the algorithm over and over again.

5. Find the highest common factor, by whichever method you like.

   (a) $hcf(153, 217;\ 58, 883) = 101$

$153,217 = 37 \times 41 \times 101$ $58,883 = 11 \times 53 \times 101$

(b) $hcf(101, 69) = 1$

101 is a prime number. $69 = 3 \times 23$

(c) $hcf(62, 68) = 2$

$62 = 2 \times 31$ $68 = 2^2 \times 17$

(d) $hcf(62, 64) = 2$

$62 = 2 \times 31$ $64 = 2^6$

(e) $hcf(99;\ 1,683) = 99$

$99 = 3^2 \times 11$ $1,683 = 3^2 \times 11 \times 17$

(f) $hcf(3,375;\ 8,000) = 125$

$3,375 = 3^3 \times 5^3$ $8,000 = 2^6 \times 5^3$

(g) $hcf(35, 49) = 7$

$35 = 5 \times 7$ $49 = 7^2$

(h) $hcf(1,836;\ 1,287) = 9$

$1,836 = 2^2 \times 3^3 \times 17$ $1,287 = 3^3 \times 11 \times 13$

(i) $hcf(440, 671) = 11$

$440 = 2^3 \times 5 \times 11$ $671 = 11 \times 61$

(j) $hcf(234, 432) = 18$

$234 = 2 \times 3^2 \times 13$ $432 = 2^3 \times 3^3$

Finding the lowest common multiple

Exercises

1. Find the lowest common multiple of the following numbers by using prime factor decompostion. Write you answer as a multiplication of prime factors and, if you feel up to it, work out the multiplications to find the actual number.

(a) $lcm(36, 21) = 2^2 \times 3^2 \times 7 = 252$

$$\begin{array}{r|l r|l}
36 & 2 & 21 & 3 \\
18 & 2 & 7 & 7 \\
9 & 3 & 1 & \\
3 & 3 & & \\
1 & & &
\end{array}$$

(b) $lcm(32, 18) = 2^5 \times 3^2 = 288$

$$\begin{array}{r|l r|l}
32 & 2 & 18 & 2 \\
16 & 2 & 9 & 3 \\
8 & 2 & 3 & 3 \\
4 & 2 & 1 & \\
2 & 2 & & \\
1 & & &
\end{array}$$

(c) $lcm(34, 85) = 2 \times 5 \times 17 = 170$

$$\begin{array}{r|l r|l}
34 & 2 & 85 & 5 \\
17 & 17 & 17 & 17 \\
1 & & 1 &
\end{array}$$

(d) $lcm(52, 46) = 2^2 \times 13 \times 23 = 1,196$

$$\begin{array}{r|l r|l}
52 & 2 & 46 & 2 \\
26 & 2 & 23 & 23 \\
13 & 13 & 1 & \\
1 & & &
\end{array}$$

(e) $lcm(144, 147) = 2^4 \times 3^2 \times 7^2 = 7,056$

$$\begin{array}{r|l r|l}
144 & 2 & 147 & 3 \\
72 & 2 & 49 & 7 \\
36 & 2 & 7 & 7 \\
18 & 2 & 1 & \\
9 & 3 & & \\
3 & 3 & & \\
1 & & &
\end{array}$$

(f) $lcm(236, 252) = 2^2 \times 3^2 \times 7 \times 59 = 14,868$

$$\begin{array}{r|l r|l}
236 & 2 & 252 & 2 \\
118 & 2 & 126 & 2 \\
59 & 59 & 63 & 3 \\
1 & & 21 & 3 \\
 & & 7 & 7 \\
 & & 1 &
\end{array}$$

(g) $lcm(549, 488) = 2^3 \times 3^2 \times 61 = 4,392$

$$\begin{array}{r|l r|l}
549 & 3 & 488 & 2 \\
183 & 3 & 244 & 2 \\
61 & 61 & 122 & 2 \\
1 & & 61 & 61 \\
 & & 1 &
\end{array}$$

(h) $lcm(666, 848) = 2^4 \times 3^2 \times 37 \times 53 = 282,384$

$$\begin{array}{r|l r|l}
666 & 2 & 848 & 2 \\
333 & 3 & 424 & 2 \\
111 & 3 & 212 & 2 \\
37 & 37 & 106 & 2 \\
1 & & 53 & 53 \\
 & & 1 &
\end{array}$$

(i) $lcm(24, 39, 54) = 2^3 \times 3^3 \times 13 = 2,808$

$$\begin{array}{r|l r|l r|l}
24 & 2 & 39 & 3 & 54 & 2 \\
12 & 2 & 13 & 13 & 27 & 3 \\
6 & 2 & 1 & & 9 & 3 \\
3 & 3 & & & 3 & 3 \\
1 & & & & 1 &
\end{array}$$

(j) $lcm(63, 15, 33) = 3^2 \times 5 \times 7 \times 11 = 3,465$

$$\begin{array}{r|l r|l r|l}
63 & 3 & 15 & 3 & 33 & 3 \\
21 & 3 & 5 & 5 & 11 & 11 \\
7 & 7 & 1 & & 1 & \\
1 & & & & &
\end{array}$$

2. (a) Find the prime factor decomposition of the following numbers.

$$\begin{array}{r|l} 35 & 5 \\ 7 & 7 \\ 1 & \end{array} \qquad \begin{array}{r|l} 44 & 2 \\ 22 & 2 \\ 11 & 11 \\ 1 & \end{array} \qquad \begin{array}{r|l} 50 & 2 \\ 25 & 5 \\ 5 & 5 \\ 1 & \end{array} \qquad \begin{array}{r|l} 72 & 2 \\ 36 & 2 \\ 18 & 2 \\ 9 & 3 \\ 3 & 3 \\ 1 & \end{array} \qquad \begin{array}{r|l} 90 & 2 \\ 45 & 3 \\ 15 & 3 \\ 5 & 5 \\ 1 & \end{array}$$

(b) Find the following lowest common multiples. Write down (at least) the prime factor decomposition of your answers.

i. $lcm(35, 44) = 2^2 \times 5 \times 7 \times 11 = 1,540$
ii. $lcm(35, 50) = 2 \times 5^2 \times 7 = 350$
iii. $lcm(35, 72) = 2^3 \times 3^2 \times 5 \times 7 = 2,520$
iv. $lcm(35, 90) = 2 \times 3^2 \times 5 \times 7 = 630$
v. $lcm(44, 50) = 2^2 \times 5^2 \times 11 = 1,100$
vi. $lcm(44, 72) = 2^3 \times 3^2 \times 11 = 792$
vii. $lcm(44, 90) = 2^2 \times 3^2 \times 5 \times 11 = 1,980$
viii. $lcm(50, 72) = 2^3 \times 3^2 \times 5^2 = 1,800$
ix. $lcm(50, 90) = 2 \times 3^2 \times 5^2 = 450$
x. $lcm(72, 90) = 2^3 \times 3^2 \times 5 = 360$
xi. $lcm(35, 44, 50) = 2^2 \times 5^2 \times 7 \times 11 = 7,700$
xii. $lcm(35, 44, 72) = 2^3 \times 3^2 \times 5 \times 7 \times 11 = 27,720$
xiii. $lcm(35, 44, 90) = 2^2 \times 3^2 \times 5 \times 7 \times 11 = 13,860$
xiv. $lcm(35, 50, 72) = 2^3 \times 3^2 \times 5^2 \times 7 = 12,600$
xv. $lcm(35, 50, 90) = 2 \times 3^2 \times 5^2 \times 7 = 3,150$
xvi. $lcm(35, 72, 90) = 2^3 \times 3^2 \times 5 \times 7 = 2,520$
xvii. $lcm(44, 50, 72) = 2^3 \times 3^2 \times 5^2 \times 11 = 19,800$
xviii. $lcm(44, 50, 90) = 2^2 \times 3^2 \times 5^2 \times 11 = 9,900$
xix. $lcm(44, 72, 90) = 2^3 \times 3^2 \times 5 \times 11 = 3,960$
xx. $lcm(50, 72, 90) = 2^3 \times 3^2 \times 5^2 = 1,800$
xxi. $lcm(35, 44, 50, 72) = 2^3 \times 3^2 \times 5^2 \times 7 \times 11 = 138,600$
xxii. $lcm(35, 44, 50, 90) = 2^2 \times 3^2 \times 5^2 \times 7 \times 11 = 69,300$
xxiii. $lcm(35, 44, 72, 90) = 2^3 \times 3^2 \times 5 \times 7 \times 11 = 27,720$
xxiv. $lcm(35, 50, 72, 90) = 2^3 \times 3^2 \times 5^2 \times 7 = 12,600$
xxv. $lcm(44, 50, 72, 90) = 2^3 \times 3^2 \times 5^2 \times 11 = 19,800$
xxvi. $lcm(35, 44, 50, 72, 90) = 2^3 \times 3^2 \times 5^2 \times 7 \times 11 = 138,600$

Relationship between the highest common factor and the lowest common multiple

Exercise
Find the lowest common multiple and the highest common factor of the following numbers.

1. 12 and 16: $hcf(12, 16) = 4$, $lcm(12, 16) = 48$
2. 32 and 50: $hcf(32, 50) = 2$, $lcm(32, 50) = 800$
3. 36 and 48: $hcf(36, 48) = 12$, $lcm(36, 48) = 144$
4. 25 and 35: $hcf(25, 35) = 5$, $lcm(25, 35) = 175$

10.6 Ratio and proportion

Exercise 1 Simplify these ratios when possible.

1. $15 : 75 \overset{\div 15}{=} 1 : 5$
2. $21 : 49 \overset{\div 7}{=} 3 : 7$
3. $28 : 16 \overset{\div 4}{=} 7 : 4$
4. $3 : 17$
5. $45 : 9 \overset{\div 9}{=} 5 : 1$
6. $12 : 16 \overset{\div 4}{=} 3 : 4$
7. $50 : 75 \overset{\div 25}{=} 2 : 3$
8. $84 : 14 \overset{\div 14}{=} 6 : 1$
9. $98 : 66 \overset{\div 2}{=} 49 : 33$

Exercise 2 Find the simplest possible equivalent ratios that do not contain a fraction.

1. $\frac{1}{2} : \frac{1}{3} \overset{\times 6}{=} \frac{6}{2} : \frac{6}{3} = 3 : 2$
2. $\frac{3}{4} : \frac{2}{5} \overset{\times 20}{=} \frac{60}{4} : \frac{40}{5} = 15 : 8$
3. $\frac{4}{7} : \frac{7}{8} \overset{\times 56}{=} \frac{224}{7} : \frac{392}{8} = 32 : 49$
4. $\frac{1}{4} : \frac{2}{9} \overset{\times 36}{=} \frac{36}{4} : \frac{72}{9} = 9 : 8$
5. $\frac{2}{7} : \frac{3}{5} \overset{\times 35}{=} \frac{70}{7} : \frac{105}{5} = 10 : 21$
6. $\frac{6}{9} : \frac{6}{7} \overset{\times 63}{=} \frac{378}{9} : \frac{378}{7} = 42 : 54 \overset{\div 6}{=} 7 : 9$
7. $\frac{2}{3} : \frac{4}{7} \overset{\times 21}{=} \frac{42}{3} : \frac{84}{7} = 13 : 12$
8. $\frac{2}{5} : 3 \overset{\times 5}{=} \frac{10}{5} : 15 = 2 : 15$
9. $2 : \frac{1}{3} \overset{\times 3}{=} 6 : \frac{3}{3} = 6 : 1$
10. $8 : \frac{2}{7} \overset{\times 7}{=} 56 : \frac{14}{7} = 56 : 2 \overset{\div 2}{=} 28 : 1$

3. option (c)
4. option (d)
5. (a) What is the ratio of the number of books of The Land of Stories to the total number of books written about the magical fairy tale land? $2 : 3$

(b) What is the ratio of books in the The Land of Stories series to the books in the A Tale of Magic series? 2 : 1

(c) What is the ratio of the number of books of A Tale of Magic to the total number of books written about the magical fairy tale land? 1 : 3

(d) What is the ratio of books in the A Tale of Magic series to the books in the The Land of Stories series? 1 : 2

6. (a) What is the ratio of bananas to the total amount of pieces of fruit? 1 : 4
   (b) What is the ratio of apples to the total amount of pieces of fruit? 3 : 16
   (c) What is the ratio of oranges to the total amount of pieces of fruit? 5 : 16
   (d) What is the ratio of apples to oranges? 3 : 5
   (e) What is the ratio of pears to apples? 4 : 3
   (f) What is the ratio of bananas to pears? 1 : 1

7. Santa drinks 80cl of milk.

8. There are 104 non-fiction books in the school library.

9. Saoirse needs 15l of white paint and 5 liters of bright blue paint.

10. (a) Bean stalks: 76
    (b) Salads: 38
    (c) Carrots: 114

11. Jantje has 240 animals at his farm. The ratio of sheep to cattle is 5:3.

    (a) How many sheep does he have? 150
    (b) And how many cattle? 90

    ?

12. (a) How many girls saw that performance of the play? 162 girls
    (b) How many boys were there? 126 boys
    (c) How many men attended? 80 men
    (d) How many women saw it? 64 women

13. Your sibling is being annoying and tells you that "Anything you can do, they can do twice as well". You decide to put them to the test and see if that is actually true or just a huge exaggeration.

    (a) The sibling has to run either 200m in 18 seconds, or 100m in 9 seconds (just know that 9.58 seconds is the current world record for the 100m sprint).

(b) Their tower has to be 2m20cm high.

(c) They have to jump either 50 times in 1 minute or 100 times in 2 minutes.

(d) They have to stretch their arms 310cm apart. As the maximum distance you can stretch out in this way roughly correspronds to your heigth, you sibling may have a problem to do this as the tallest person in recorded history was Robert Wadlow who was 272cm tall.

(e) i and iv would be as good as impossible, but you can make them try over and over again to accomplish ii and iii.

14. (a) How many visitors enjoyed the Aqua Park on August 6th? 702 visitors

    (b) How many of them were children? 546 children

15. You need 576 frogs.

16. (a) All of the ratios are 1 : 1.

    (b) i. You need 1 ml of vanilla essence and 50g of each of the other ingredients

        ii. You can make 4 cupcakes with it.

    (c) i. You need 9 eggs, 9ml vanilla essence, 450g self raising flour, 450g sugar and 450 g butter to make 36 cupcakes.

        ii. It doesn't really affect the baking time. If you fill up your oven completely with cupcakes, they may be done a bit slower than if you bake only 4 of them, but the difference will never be very big. The reason for this slightly longer baking time is because if you place many items together, they shield each other a bit from the environment. You may have noticed that giving someone a big hug is warmer than staying apart. During the hug, you shield each other from the cold that surrounds you and gives you extra warmth. Unless there is a dangerous heat wave going on, your surroundings will never be warmer than your body temperature, so shielding from the environment means that you give each other extra warmth. In the oven, the opposite happens when you fill it with a lot of cupcakes. The cupcakes are room temperature when they go in an oven that is over a hundred degrees warmer than they are. The closer they are placed together, the more successful they are in shielding each other from the heat that surrounds them. By doing so, they may increase the required baking time by a few minutes if you filled your oven with more than two trays of cupcakes. And your oven also simply has to work harder to bake so many cupcakes in one go.

10.7 Proportionality

10.7.1 Direct proportionality

Exercises

1. Complete these tables by using direct proportionality, and calculate the ratio that is being used.

   (a)

6	30	14.4	100	62.4
1.5	7.5	3.6	25	13.1

   Ratio: 4 : 1

   (b)

2	7	11	23
50	175	275	575

   Ratio: 1 : 25

   (c)

1	2	3	7	16	25
$1\frac{1}{3}$	$2\frac{2}{3}$	4	$9\frac{1}{3}$	$21\frac{1}{3}$	$31\frac{1}{3}$

   Ratio: $3 : 4 = 1 : \frac{4}{3} = \frac{3}{4} : 1$

   (d)

11	$3\frac{2}{3}$	$14\frac{2}{3}$	55	1	4
3	1	4	15	$\frac{3}{11}$	$1\frac{1}{11}$

   Ratio: $11 : 3 = 1 : \frac{3}{11} = \frac{11}{3} : 1$

   (e)

$\frac{15}{7} = 2\frac{1}{7}$	3	5	1	$\frac{1}{7}$
3	4.2	7	$\frac{7}{5} = 1\frac{2}{5}$	$\frac{1}{5}$

   Ratio:$5 : 7 = 1 : \frac{7}{5} = \frac{5}{7} : 1$

2. Are these tables using direct proportionality? If they are, give the ratio used. If not, explain why.

   (a) Price of rice

Weight of rice (in kg)	2.2	5.3	1.7	512
Price (in €)	2.86	6.89	2.21	66.56

   Yes, ratio 1 : 1.3

   (b) Price of fabric sold per meter

Length of fabric	5	2	10	12
Price (in €)	45.50	18.20	91	109.20

   Yes, ratio 1 : 9.1

(c) Average height of a child

Age (in years)	2	5	10	14
Height (in cm)	92	105	125	162

No, at 2 years the ratio is 1 : 46, but at 10 years old, it is 1 : 1.25. The ratio is different for each age.

(d) Price of stamps sold by An Post

Number of stamps	2	4	5	10	100
Price (in €)	2.70	5.40	6.75	13.00	125.00

The first part is directly proportional with ratio 1 : 1.35, but the prices for 10 and 100 stamps don't follow this ratio and are cheaper.

(e) Prices of a magazine subscription

Number of months	2	6	9	12
Price (in €)	8	20	26	36

No, the subscription prices vary depending on the duration. The ratio for 2 months is 1 : 4, that for a year is 1 : 3. The cheapest formula is that for 9 months, as that ratio is $1 : 2\frac{8}{9}$.

(f) Distance traveled by two people in the same amount of time, one of them cycling, the other one walking.

Cycling (in km)	7.7	19.25	38.5	2.625
Walking (in km)	2.2	5.5	11	0.75

Yes, the ratio is 3.5 : 1.

(g) Distance it takes a car to come to a full stop in function of its speed

Speed (in km/h)	40	60	80	90
Distance (in m)	20	36	57.5	67.5

No, the distance necessary for a full stop increases significantly with the speed. The ratio at 40 km/h is 2 : 1, but at 90 km/h it is $1\frac{1}{3} : 1$.

3. Maggie has to save her allowance for 9 weeks.

4. To warm up for football practice, Sammy has to run 5 laps around the football pitch that is 120m wide and 70m long.

   (a) One lap equals 380 m.

   (b) 1,900 m or 1.9 km.

   (c) Ratio: 1 : 380.

5. He needs to increase his speed to 1 km every 6 minutes, because at 8 minutes per kilometer it would take him 1 hr 20 mins to run 10 km.

6.

Minutes	1	15	2.5	10
Cost (in €)	0.40	6	1	4

(b) His call can last for 10 minutes.

7. (a) 1 : 2

   (b) 1 kg of strawberries is good for 1.5 kg jam, 1.5 kg of strawberries gives 2.25 kg of jam.

   (c) Oscar needs 1.8 kg of strawberries to fill all five jars.

8. (a) They have to multiply the ingredients with the factor $\frac{4}{6} = \frac{2}{3}$.

   (b) Ciara and Dermot need 0.5 l milk, 4 eggs, 180 g sugar, 200 g self raising flour and 120 g butter.

9.
   - Katrijn has to crochet about 128 rows.
   - That's 3,840 stitches.
   - Crocheting 128 rows takes 192 minutes, or 3 hr 12 mins.
   - It is totally worth it!

10. (a) Are these rates proportional or not? Why? No, because there are only 3 rates for parcels under 10 kg and the rates for packets don't have a constant ratio.

    (b) i. Sending the gift as 2 packets is cheaper than sending it as 1 parcel.
       ii. It would cost € 35.
       iii. Sending it as a parcel would cost € 10 more.

    (c) i. The price difference for registered post is € 5 for one parcel.
       ii. The difference is € 10 for 2 packets
       Even when using registered post, sending the gift as 2 packets is still the cheapest option.

10.7.2 Inverse proportionality

Exercises

1. (a)

25	5	12.5	20	8
16	80	32	20	50

Constant product: 400

(b)

22	277.2	42	99	13.2
63	5	33	14	105

Constant product: 1386

(c)

24	3	8	10	16
12	96	36	28.8	18

Constant product: 288

(d)

12.5	150	8	12	$53\frac{1}{3}$
38.4	3.2	60	40	9

Constant product: 480

(e)

24	0.4	9	2.5	38
24	1440	64	230.4	16

Constant product: 576

2. Are these tables using inverse proportionality? If they are, give the product they used. If not, explain why.

   (a) Cost of a circular table

Radius of the table (in cm)	40	60	75	100
Price (in euro)	90	135	168.75	225

   The radius is directly proportional to the price, with ratio 1 : 2.25, not inversely proportional.

   (b) Waiting time in a restaurant

Number of waiters	6	9	10	15
Average waiting time (in minutes)	90	60	54	36

   The waiting time is inversely proportional, with constant product 540.

   (c) Records of trips from Dublin to Galway (180 km)

Average speed (in km/h)	60	72	90	100	120
Time (in hours)	3	2.5	2	1.8	1.5

   The speed is inversely proportional to the time, with constant product 180. This makes sense, as speed is calculated as distance divided by time.

   (d) Waiting time for check-in

Number of staff	3	4	5	6	7
Average waiting time (in minutes)	83	47	30	21	15

   No inverse proportionality, the products aren't constant. The ratio isn't constant either.

(e) The force exerted by a magnet on a metal object at a distance d

Force (in Newtons)	60	$26\frac{2}{3}$	15	9.6	$6\frac{2}{3}$
Distance (in cm)	2	3	4	5	6

Force and distance are not inversely proportional, nor are they directly proportional.

(f) The force exerted by a magnet on a metal object at a distance d

Force (in Newtons)	60	$26\frac{2}{3}$	15	9.6	$6\frac{2}{3}$
Square of the distance (in cm^2)	4	9	16	25	36

Force and the square of the distance are inversely proportional, with constant product 240.

3. (a) The constant product is $4 \times 3 = 12$, so with 5 hoses it would take $12 \div 5 = 2.4$ hours, so 2 hours and 24 minutes.

   (b) As $12 \div 2 = 6$, it would take 6 hours to fill the pool with only 2 hoses.

4. The constant product is $35 \times 8 = 280$, so it would take 20 workers $280 \div 20 = 14$ days to complete the harvest.

5. The constant product is $3 \times 2 = 6$, so it takes 5 people $6 \div 5 = 1.2$ hours, or 1 hour and 12 minutes to unload the second truck.

6. The constant product is $300 \times 90 = 27,000$. In 20 days, the 300 people have already eaten $20 \times 300 = 6,000$ day supplies of food, so there are $27,000 - 6,000 = 21,000$ left for the remaining 250 researchers. This will last them $21,000 \div 250 = 84$ days.

7. The total number of working hours to copy the book is $6 \times 5 \times 16 = 480$, so with 4 scribes working 6 hours a day, it would take $480 \div (4 \times 6) = 480 \div 24 = 20$ days to copy the book.

8. Last year's constant product was $36 \times 54 = 1,944$, so the elves need at least $1,944 \div 81 = 24$ machines to make the same amount of puzzles in 81 days.

9. A box contains $25 \times 3 = 75$ pencils, so each child from the second class will receive $75 \div 15 = 5$ pencils.

10. The constant product is $125 \times 16 = 2,000$, so they need $2,000 \div 40 = 50$ scientist to see results in 40 days. And we won't ponder about the delays caused by working together with such a large team or the stress they are under by having such an impatient boss, because luckily, that is something outside the scope of this course.

11. There are $80 \times 60 = 4,800$ day portions of dog food. After 15 days with 80 dogs, $15 \times 80 = 1,200$ day portions have been eaten and there are $4,800 - 1,200 = 3,600$ portions left. Adding 20 dogs brings the total to 100 dogs, so the food will last them for another $3,600 \div 100 = 36$ days.

10.8 Maths riddles

1. The fifth client buys half of what is left, plus half a kilo. After that, farmer Nelis has nothing left. This means that when the fifth client arrived, there was only one kg of carrots left. The fourth client bought half of the stock that was left, plus half a kg and afterwards there was one kg left. This means that, at that point, half the stock was equal to $1\frac{1}{2}$ kg, so there were 3 kg left when the fourth client arrived. Using the same reasoning leads you to 7 kg when the third client arrived, 15kg when the second client arrived and **31kg of carrots** when the farmer arrived at the market.

2. (a) You need to take out at least 4 socks to be sure you have two socks of the same colour.
   (b) To be absolutely sure you have 2 yellow socks, you need to take out 53 socks (27 of those could be blue, 24 of them could be red, so at least 2 of them are bound to be yellow).
   (c) You need to take out 28 socks to be sure that at least two of them have a different colour (you might have picked all 27 blue ones).

3. $\& = 7$
   $\% = 4$
   $\# = 6$
   $@ = 1$

4. The next time the clock chimes, it will be 9:30.