Maths course

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Chapter 1

Alternative ways of counting

1.1 Roman numerals

There are a lot of things we have the ancient Romans to thank for, from the invention of concrete to the basics of law and a whole lot inbetween, but the way we count isn't one of them. The numbers we use in our daily life came to Europe about a thousand years ago from Arabic speakers in Spain and North Africa and are therefore called *Arabic numerals*. Little known fact: the numbers were originally invented in India, but Arabic scientists collaborated with Indian scientists and adopted the numeral system way before the Europeans became aware of it. Asia, the Middle East and North Africa were the place to be for scientists in those days, during the Middle Ages Europe didn't have the most civilized culture. If you ever learn about Mesopotamia or Persia in your history lessons, that region of the globe is now called the Middle East. Arabs adopted the new numeral system, translated the Indian symbols into the symbols we all know and use, our digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, and introduced them to Europe from the tenth century onward. Europeans dubbed them "Arabic numerals" and the name stuck.

Europeans were a bit reluctant to say goodbye to the Roman numerals they grew up with and loved. It was a bit hard to acknowledge that the culture you keep going on crusades against actually has a better way of working with numbers than your own system, so the switch from Roman numerals to Arabic numerals was a gradual one. It is only from the fourteenth century onward that the Arabic numerals took over. The use of Roman numerals persists to this day, e.g., in the film industry. No matter how hard it is to decipher in which year a movie or TV show was made, it is always written in Roman numerals in the credits.

In Ancient Rome, writing numbers and calculating with them was a bit different from what we are used to. The main symbols they used were:

Symbol	I	V	X	L	C	D	M
Value	1	5	10	50	100	500	1000

There are two main things about these symbols:

- There is no symbol for 0. This is a bit like discovering a new kind of cookie that is instantly your favourite: before the introduction of Arabic numerals, people didn't know what they were missing.
- The biggest symbol has the value of 1000. Abstract maths weren't invented yet, children weren't forced to do maths with big numbers in schools. In fact, most children never had any formal lessons. Which may sound cool on days where school feels boring or the lesson feels endless, but I bet you still prefer school to long days of doing hard work for barely any money. Education was a privilege for the rich people who were in charge. There are no Roman numerals for numbers like 5000 or 10 000, because there was no need for them. Most people had trouble counting up to 100.

Romans counted by repeating symbols, until they came close to the next symbol. They were fine with repeating a *I, X,* C or M symbol up to three times, but adding a fourth one was simply out of the question. I guess they thought it just looks bad and I have to admit, it does get hard to decipher and it would be easier to make mistakes. They solved that problem by using the symbols V, L and D.

The numbers for 1 to 10 look like this.

Arabic	1	2	3	4	5	6	7	8	9	10
Roman	I	II	III	IV	V	VI	VII	$VIII$	IX	X

The number 4 is written is IV, which you should interpret as I less than V. Similarly, 9 is IX, which means you have to subtract I from X. Roman numerals are written from left to right, starting with the biggest symbol and ending with the smallest. If you see a smaller value symbol before a bigger value symbol, then you have to subtract the smaller value from the bigger value. Unfortunately, there are very strict rules regarding what you can place before a larger symbol.

- You can only use the symbol $I(1)$ to subtract from $V(5)$ or $X(10)$.
- You can use the symbol $X(10)$ to subtract from $L(50)$ or $C(100)$.
- You can use the symbol $C(100)$ to subtract from $D(500)$ or $M(1000)$.
- You can never use the symbols for either $V(5)$, $L(50)$ or $D(500)$ to subtract from a bigger number.

- A number can never contain two V's, nor two L's. Two D's in one number are out of the question as well. This is because $V + V = X$, $L + L = C$, $D + D = M$

It takes some getting used to, but after a while it starts to make sense.

Let's look at some more examples before it's your turn.
$29 = 20 + 9 = X + X + IX = XXIX$
$34 = 30 + 4 = XXX + IV = XXXIV$
$43 = 40 + 3 = XL + III = XLIII$
$51 = 50 + 1 = L + I = LI$
$68 = 60 + 8 = LX + VIII = LXVIII$
$76 = 70 + 6 = LXX + VI = LXXVI$
$87 = 80 + 7 = LXXX + VII = LXXXVII$
$99 = 90 + 9 = XC + IX = XCIX$
$115 = 100 + 10 + 5 = C + X + V = CXV$
$438 = 400 + 30 + 8 = CD + XXX + VIII = CDXXXVIII$
$674 = 600 + 70 + 4 = DC + LXX + IV = DCLXXIV$
$777 = 700 + 70 + 7 = DCC + LXX + VII = DCCLXXVII$
$960 = 900 + 60 = CM + LX = CMLX$
$1,594 = 1000 + 500 + 90 + 4 = M + D + XC + IV = MCXCIV$
$2,023 = 2000 + 20 + 3 = MM + XX + III = MMXXIII$
$3,333 = 3000 + 300 + 30 + 3 = MMM + CCC + XXX + III = MMMCCCXXXIII$
$3,721 = 3000 + 700 + 20 + 1 = MMM + DCC + XX + I = MMMDCCXXI$

1.1.1 Exercises

1. True or false: all seven Roman numerals can repeat in a number, e.g., XX.

2. $LXXXIX$ tiles were made to create a Roman mosaic. IF $LXXII$ of these tiles were stolen how many were left for the mosaic?

3. A Roman man was having a toga made for a party. He took LVI gold coins to the market to buy the material. It cost $XXIV$. How many coins does he have left to pay the seamstress of his toga?

4. Archaeologists expect to find LVI Roman villas in Dewa. They have found $XLII$. How many more are there to find?

5. XI workers are working on painting a fence. If each worker can paint V meters of fence in an hour, how many meters can the workers paint in total over a period of VI hours?

6. Amir is XV years old, his cousin Said is $XXIV$ years old. What is the sum of their ages?

7. There are four Roman numerals that are allowed to repeat. X is one of them. What are the other three?

8. I was born in the year $MCMLXXXIV$. What year is that?

9. Can you write the year you were born in Roman numerals?

10. Rewrite these numbers as Roman numerals.

    (a) $3 =$
    (b) $8 =$
    (c) $16 =$
    (d) $25 =$
    (e) $36 =$
    (f) $81 =$
    (g) $140 =$
    (h) $275 =$
    (i) $567 =$
    (j) $789 =$
    (k) $987 =$
    (l) $1,111 =$
    (m) $1,234 =$
    (n) $2,345 =$
    (o) $3,456 =$

11. Rewrite these numbers in Arabic numerals.

    (a) XV=
    (b) XLIX=
    (c) XXVIII=
    (d) LXXIX=
    (e) MCDXLIV=
    (f) DLIII=
    (g) MDCCCX=
    (h) LXXVII=
    (i) MMMCDXXI=
    (j) MCMLXIV=
    (k) DLV=
    (l) MMCDXLVIII=
    (m) MDCXCIX=
    (n) MMMCDLXXIV=
    (o) MMCMLXXXVII=

12. Jake saw a cornerstone on an old building. It said: *Erected MCMXI.* How old is that building?

13. What is the biggest number you can make with these Roman numerals? What is the value of that number in Arabic numerals?

14. $<$ or $>$

    (a) IX XI
    (b) LVII XLVIII
    (c) XCLXXVIII CXLV
    (d) MMCMXLII MMMCXV
    (e) MMMCDLXXXII MMMDCX
    (f) CDLXXIX CDXXXIV
    (g) D CCCXCVIII
    (h) DCCCLXXXVIII DCCXCLXXX
    (i) MCMCCCXXXIII MMCXCXI
    (j) DCCLXVII DCCLXVI

15. Flora made $XIII$ animals with aquabeads. She wants to give II animals to each of her $VIII$ friends. Has she already made enough animals? If not, how many more does she need to make?

16. Write your answer in Roman numerals.

    (a) $15 + 38 =$
    (b) $101 + 48 =$
    (c) $567 + 725 =$
    (d) $73 + 678 =$
    (e) $7,589 - 5,213 =$
    (f) $1699 + 25 =$
    (g) $123 - 25 =$
    (h) $3450 - 508 =$

17. Write your answer in Arabic numerals.

    (a) XXX+MCMXIV=
    (b) MCDXLIV+DXLIX=
    (c) CDXLIV+DLV=
    (d) M-DCLXVI=
    (e) MMDLXX-MCXVIII=
    (f) CXCIX+LXVI=
    (g) DCCLVI+CMXLVIII=
    (h) DCXXVII-XLIX=

18. You may have felt it coming, and here it is: write your answer in Roman numerals. Can you calculate them without writing down the Arabic numerals?

    (a) XXVIII+XXIII=
    (b) XLVI+XCXIV=
    (c) XCIX+LXXXV=
    (d) CCLV+LXXVII=
    (e) CDIX+DCCCLXXXVIII=
    (f) DLV+MDLV=
    (g) MCMXXIX+MCCXXII=
    (h) MCXI+MMCCXXII=
    (i) CMXCIX+VI=
    (j) DCXXVII+MMIII=
    (k) CXC-LVII=
    (l) D-CCXXII=
    (m) M-II=
    (n) D-C=
    (o) C-V=
    (p) MMXXIII-DLV=
    (q) MDCLVI-CMLXXX=
    (r) MCMXL-DCC=
    (s) MMXXIII-XI=
    (t) MDCLXVI-XCIX=

19. Why should we leave it at additions and subtractions? Let's see if you can do some multiplications and divisions as well. You can use Arabic numerals for the calculations, but write your answer in Roman numerals.

(a) CXI x III =
(b) CXXIII x IV =
(c) LXXVII x VII =
(d) DCXLI x VI =
(e) MDLV x II =
(f) MMDCLXXXIV ÷ II =
(g) MMDCVIII ÷ IV =
(h) MMMDCXCVI ÷ III =
(i) MMCMLXXXVIII ÷ IX =
(j) DLV ÷ V =

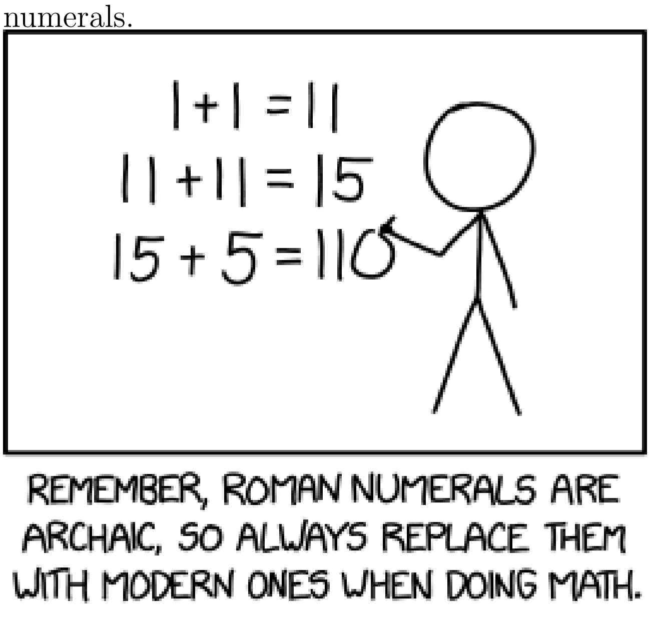
20. Complete the Roman numeral magic squares. In a magic square, each column, row and diagonal adds up to the same sum.

VIII		IV
	V	
VI		

X		
V	VII	
		VI

IV		
	V	
VII		VI

21. Randall Monroe is the creator of *xkcd*, a very cool (and nerdy) webcomic. He makes cartoons about all sorts of scientific subjects. He even made one about Roman numerals.



What are the correct sums that are displayed in this comic? Write them down, either in Roman or Arabic numerals.

22. This cartoon got the following subtext.

100he100k out th1s 1nno5at4e str1ng en100o501ng
1'5e been 500e5e50op1ng! 1t's 6rtua100y perfe100t!
...hang on, what's a *virtuacy*?

(a) Can you translate this message?

(b) What was meant with a *virtuacy*?

(c) Can you use this to encode the message *I am loving this*?

(d) Can you write your own message in this way?

23. I have encrypted a message for you. Can you decipher it?

Letter	*A*	*B*	*C*	*D*	*E*	*F*
Roman	III	MCMIX	MDCL	DCVI	CCLXIX	CMXVI
Arabic						

Letter	*G*	*H*	*I*	*J*	*K*	*L*
Roman	DXCV	MCMXVIII	VII	XLII	XXXVI	MMCCXXII
Arabic						

Letter	*M*	*N*	*O*	*P*	*Q*	*R*
Roman	MDLVII	MMCCCXLII	DCCI	CCCXXVII	MMMDCCXCIX	DXII
Arabic						

Letter	*S*	*T*	*U*	*V*	*W*	*X*
Roman	IX	CDXLVIII	MCDLI	DCCCVI	MMMCCCXL	MMCMLXVI
Arabic						

Letter	*Y*	*Z*
Roman	MDV	MCMXCIX
Arabic		

Now that you managed to complete the encryption key, I am sure that the message won't be a problem for you. Here it goes:

1505/521/1451 3/512/269 595/521/521/606 3/448 448/1918/8/9,

1505/521/1451 3/512/269 606/521/8/2342/595 3/2343

3/3340/269/9/521/1557/269 42/521/1909!

1.2 Binary numbers

You probably learned to count at a young age. You said those magical words *one, two, three* and your parents marveled at how bright their little kid is. A bit later, you managed to count all the way up until ten, without leaving any of the numbers out. By the time you are reading this, I know you can make up an impressive big number. And you know that there is always an even bigger number than whatever your big number is, no matter what it was (even if you used not very specific terms like a million bazillion, there's always a million bazillion and one :P).

Maybe you learned to count in another language at school. Maybe you even learned to count in a different language first. Maybe you have already learned to count in four or five different languages. It's a cool thing to learn to count from one to ten, or a hundred, in another language. But did you know that counting doesn't have to be the way you do it?

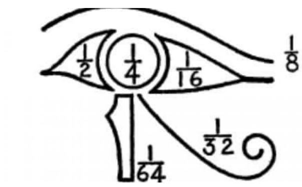
Counting can also be $0, 1, 10, 11, 100...$ and that's just as correct as $0, 1, 2, 3, 4....$ Or at least it is if you are counting in binary. Binary code as we now use it was officially invented by Gottfried Wilhelm Leibniz in the 17th century. The more maths you will learn, the more familiar you will become with the name of this mathematician. He did important work in multiple branches of maths, so his name will keep popping up in your future maths courses.

Leibniz wasn't the very first person who preferred using just 2 digits to calculate. Mathematicians use knowledge of all who came before, explore concepts more deeply, apply ideas in new areas etc. Even Leibniz, the one who invented the framework for binary code as we know and use it now, stands on the shoulders of giants. Maths isn't magic, even though it can definitely be magical.

In some regards, working with binary numbers is a lot easier than working with the decimal numbers we are used to. That is why binary systems were established many times, in many places, long before Leibniz came along. When so many peoples used and formalized a system, independent of each other, it usually means that it is worth the effort. It also helps us to not brush binary off as something that is only useful for computers and modern science, as binary predates computers by thousands of years. So let's take a closer look at some of these older binary systems.

1.2.1 Ancient Egypt

In ancient Egypt they used a form of binary code with Horus Eye fractions about 4400 years ago, to measure grains and liquids. Horus was the sky god in Egyptian mythology. They believed that the Sun was his right eye and the Moon his left and that they traversed the sky when he, a falcon, flew across it. The sun is so much brighter than the moon, because the left eye was smashed to pieces during one of the many battles between Horus and Seth that was meant to determine who got the throne of Egypt. The Eye of Horus was divided into six parts, each of which got their own hieroglyphic sign.

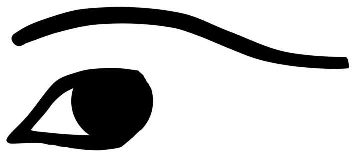


The curious aspect is that these hieroglyphic signs are read as fractions, divided by two from left to right in the eye. This way

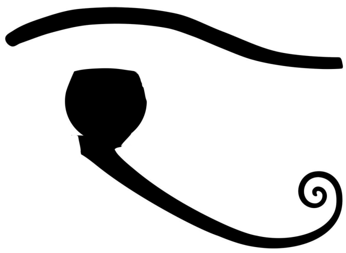
- the left half of the eye is 1/2
- the pupil is 1/4
- the eyebrow is 1/8
- the right half of the eye is 1/16
- the lower parts are 1/32 and 1/64

This means that these hieroglyphs could be used for binary calculations, for the first time in recorded history. For those of you who, like me, prefer a happy ending to a myth: the god of science and wisdom, Thoth, had put the pieces back together and gave them back to Horus. Unfortunately, Thoth was unable to put the entire eye back together. Eyes are complicated body parts, even gods find it hard to put them back together as before. The sum of all those fractions is 63/64, leaving 1/64 for the chunk that Thoth was unable to compose.

By combining parts of the Eye of Horus, you can make a visual representation of fractions, of which the nominator is a power of 2.



$\frac{7}{8} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$



$\frac{13}{32} = \frac{1}{4} + \frac{1}{8} + \frac{1}{32}$

Exercise Can you draw these fractions?

$\frac{5}{16}$ $\frac{55}{64}$

1.2.2 China

The I Ching, or Book of Changes, is an ancient Chinese divination text that is among the oldest of the Chinese classics. The text is around 2800-2900 years old. It reveals at its core the binary number system, that arranges meaningful symbols and uses them to communicate messages. They used hexagrams, what in this context means a figure composed of six horizontal lines, where each line is Yang or Yin.

- Yang is an unbroken, solid line

- Yin is a broken line, an open line with a gap in the center

The hexagram lines are traditionally counted from the bottom up, so the lowest line is considered line one while the top line is line six. A hexagram is formed by combining two trigrams, figures containing three horizontal lines each.

The five elements that you can find in these eight trigrams, Water, Wood, Fire, Earth and Metal, are still used in traditional Chinese medicine and in Feng Shui, a practice that tries to bring harmony between individuals and their surrounding environment.

The binary system explained in the I Ching resembles the binary code we use today, because Leibniz himself studied it thoroughly before formalizing his own version of binary code.

Exercise

- In which two numbers do Water and Flame meet?
- In which numbers is Earth joined by Wind?
- Which number is pure Heaven?
- And which one is pure Earth?

1.2.3 Pingala

Pingala was an Indian scholar who invented a type of binary system around 200 BC, some 2200 years ago. Pingala's binary system had a large number of similarities with the modern binary system. His version of the binary system used short and long syllables to represent 0's and 1's. However, the way you make a Pingala binary number is different from the procedure we follow today.

The procedure of Pingala system is as follows:

- Inspect the number. If it is even, write 1, otherwise write 0. Always add numbers to the right of what you already had.

- If the number was even and bigger than 2, divide by 2. If the number is 2, you are done.
- If the number was odd and bigger than 1, add 1 and then divide by two.
- Repeat these steps.

Using these rules, the numbers from 1 to 16 look as follows

1 = 0	5 = 0 0 1	9 = 0 0 0 1	13 = 0 0 1 1
2 = 1	6 = 1 0 1	10 = 1 0 0 1	14 = 1 0 1 1
3 = 0 1	7 = 0 1 1	11 = 0 1 0 1	15 = 0 1 1 1
4 = 1 1	8 = 1 1 1	12 = 1 1 0 1	16 = 1 1 1 1

Let us look at an example to better understand the Pingala system of Binary Numbers. Let us find the binary equivalent of 122 in Pingala System.

- 122 is an even number, so write 1.
- $122 \div 2 = 61$.
- 61 is an odd number, so write 0. You now have 10.
- $61 + 1 = 62$, $62 \div 2 = 31$
- 31 is an odd number, so write 0. You now have 100.
- $31 + 1 = 32$, $32 \div 2 = 16$.
- 16 is an even number, so write 1. You now have 1001.
- $16 \div 2 = 8$
- 8 is an even number, so write 1, You now have 10011.
- $8 \div 2 = 4$
- 4 is an even number, so write 1. You now have 100111.
- $4 \div 2 = 2$
- 2 is an even number, so write 1. You now have 1001111.
- You got the number 2, so you can stop.

We have found that 122 is equivalent to 1001111 in Pingala binary. We encountered the numbers 122, 61, 31, 16, 8, 4 and 2 along the way. If we stick to the order of these numbers, write 1 for every even number we encounter and 0 for every odd number, we get 1001111. This means that we didn't mix up our 1's and 0's along the way, phew!

We can check that this is the correct representation, but it feels a bit murky.

$$\begin{aligned} 1 \times 1 + 0 \times 2 + 0 \times 4 + 1 \times 8 + 1 \times 16 + 1 \times 32 + 1 \times 64 &= \\ 64 + 32 + 16 + 8 + 1 &= 121 \end{aligned}$$

By adding 1, which we added while dividing 61, to 121 we get 122, which is our desired number.
Don't worry if this verification feels a bit discombobulating, just try to understand the steps that led us to the binary number.

Exercise Can you work out what the binary equivalent is in the Pingala system for the following numbers?

- 25
- 38
- 66
- 2137

1.2.4 Francis Bacon

In 1605, Francis Bacon invented Bacon's Bilateral Cipher, which has similarities with the binary system. The cipher he created was used to encrypt messages so that only the sender and receiver could read the message. He used the letters 'a' and 'b' instead of 0 and 1 to form a binary code for every letter in a message.

aaaaa	aaaab	aaaba	aaabb	aabaa	aabab
A	B	C	D	E	F
aabba	aabbb	abaaa	abaab	ababa	ababb
G	H	I	K	L	M
abbaa	abbab	abbba	abbbb	baaaa	baaab
N	O	P	Q	R	S
baaba	baabb	babaa	babab	babba	babbb
T	V	W	X	Y	Z

In this cipher, the symbol for I can be used for either the letter I or J. The symbol for V can be used for either U or V. It all makes things just a little more exciting, especially if your message contains words like obviously or djinn.

Exercise Can you decipher the following message?
aaabaabbababbaaaabbabaaaaaaaaabaababaabbababaaaaaabaabaabaaaabbababbaabaaab,

babbaabbabbaabb aabbbaaaaabaabbaabaa

aaabbaabaaaaabaabaaaabbbaaabbbaabaabaaaaaabaaaaabb baabaaabbbaabaa

ababbaabaabaaabbaaabaaaaaaabbaaabaa,

babaaaabaaababaababa aaabbabbababbaaaabaa!

There are more types of binary systems, but after deciphering that message, it's time for us to look at the binary code that Leibniz invented and is now used worldwide.

1.2.5 Binary code

The binary code that was formalized by Leibniz is the form of binary code that is now used in pretty much every piece of electronics you own. It is used in smartphones and computers, but also in microwaves, digital pictures, tv's, a birthday card with sound, any clock or watch that is not a family heirloom (it doesn't matter if it is digital or analog, all clocks and watches have been using binary for decades). Digital music is stored by using binary, before that CD's also used it. Everything you can find on the internet is the translation of series of 0's and 1's, whether it's text, pictures, sound of video. Binary code is used so much now, that it is quite impossible to imagine a world without it.

Leibniz's version of binary code is all around us. So much so, that nobody really calls it Leibniz's version of binary code anymore. It's just binary code. So basically, if you invent something and it becomes really really really popular a couple of centuries after you invented it, there is a big chance that almost everybody will use your work and almost nobody will remember your name.

Power of a number

For working with binary numbers, it is useful to know what the power of a number is. So what is that, the power of a number?

Without realising it, you probably have encountered powers before in your maths lessons. This happened every time you had to calculate the square or the cube of a number.
When you are asked to calculate the square of 8, or 8^2, you are asked to work out how much 8×8 is, and your answer would be 64. The number 64 is the square of 8, but you can also say that *8 to the power 2 is 64* or *64 is the 2nd power of 8.*
When you are asked to calculate the cube of 5, or 5^3, you have to work out how much $5 \times 5 \times 5$ is and your answer would 125. The number 125 is the cube of 5, but you can also call it *5 to the power 3 is 125* or *the 3rd power of 5 is 125.*

Definition 1 *The* ***power*** *of a number says how many times to use the number in a multiplication. It is written as a small number to the right and above the base number.*

When you use power 2, you can also call it the square of a number. When you use power 3, you can call it the cube of a number. But there are no special names for higher powers, so after power 3, we just call them power 4, power 5, power 6 and so on.

An other word for the little number that we write, is exponent or index. This word isn't important to you yet, but you will come across it some day in a future maths course. I hope that you'll find working with powers/exponents/indices as much fun as I did when I was in secondary.

There are two special numbers when it comes to powers: 0 and 1.

- The number 0

- No matter how many times you multiply 0 with 0, the result is always 0. This means that $0^2 = 0^3 = 0^{16} = 0^{1,256} = 0$.
- For any number that is not 0, the power 0 of that number is 1. This means that $1^0 = 2^0 = 3^0 = 21^0 = 2,012^0 = 1$
- Even mathematicians don't agree on what the value of 0^0 should be, so in this course we'll pretend that that doesn't exist.

• The number 1

- No matter how many times you multiply 1 with 1, the result is always 1. This means that $1^2 = 1^3 = 1^{25} = 1^{3,721} = 1$.
- For any number, the power 1 of that number is the number itself. This means that $1^1 = 1$, $2^1 = 2$, $3^1 = 3$, $3,981^1 = 3,981$ and so on. It means that you can add the power 1 to whichever number you want and it doesn't change a thing. Though I doubt that your teacher will be pleased if you start answering your homework questions with *Juno had saved* 8^1 *euros and* 2^1 *cents*, it is technically correct.

Exercise Calculate these powers.

1. $6^3 =$
2. $4^3 =$
3. $5^4 =$
4. $11^3 =$
5. $3^5 =$
6. $2^4 =$
7. $2^6 =$
8. $10^2 =$
9. $10^3 =$
10. $10^4 =$
11. $25^1 =$
12. $123,456,789^0 =$
13. $39^2 =$
14. $987,654,321^1 =$
15. $93^0 =$

You may have noticed that the only powers that are very straightforward to calculate are powers of 10: you start by writing down 1, and then add as many zeros as the power. So $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1,000$ and so on.
This works both ways: to know what power of 10 a number is, you count the number of zeros there are at the end and that number is the power you were searching for. So $1,000,000 = 10^6$, $100,000,000,000,000 = 10^{14}$ etc.

Decimal numbers

Before we delve into binary numbers, we'll take a closer look at the numbers we are most used to. Our normal numbers called *decimal numbers*. The name comes from the Latin word for ten, decem, because we form numbers by using ten digits, from 0 to 9. As humans this feels natural to us, as we have ten fingers to count on (and ten toes, but they are usually hidden away in socks, so they're not all that useful for counting).

When you learned how to count, you might have learned that the right-most digit is the "ones' place", the next is the "tens' place", the next is the "hundreds' place", etc. Another way to say that is that the digit in the right-most position is multiplied by 1, the digit one place to its left is multiplied by 10, and the digit two places to its left is multiplied by 100.

Let's visualize the number 234:

2	3	4
hundreds′place	*tens′place*	*ones′place*
100	10	1
10^2	10^1	10^0

When we multiply each digit by its place, we can see that

$$234 = (2 \times 100) + (3 \times 10) + (4 \times 1).$$

When we use the power notation, we get

$$234 = (2 \times 10^2) + (3 \times 10^1) + (4 \times 10^0).$$

We can generalize this for every decimal number we encounter.
For example,

$$\begin{aligned} 321,234,567,890 = &(3 \times 10^{11}) + (2 \times 10^{10}) + (1 \times 10^9) + (2 \times 10^8) + (3 \times 10^7) \\ &+(4 \times 10^6) + (5 \times 10^5) + (6 \times 10^4) + (7 \times 10^3) + (8 \times 10^2) \\ &+(9 \times 10^1) + (0 \times 10^0). \end{aligned}$$

Understanding that we implicitly use powers of 10 to write decimal numbers will help us to understand how binary numbers works.

Exercise Complete the table.

number	10^{11}	10^{10}	10^9	10^8	10^7	10^6	10^5	10^4	10^3	10^2	10^1	10^0
321, 234, 567, 890	3	2	1	2	3	4	5	6	7	8	9	0
25, 268												
20, 102, 012												
198, 438												

Converting decimal numbers to binary numbers

Binary numbers are formed in the same way as decimal numbers. The only differences are that we only use 2 digits, 0 and 1, instead of 10 digits. And instead of multiplying the digit by a power of 10, we multiply it by a power of 2.

Easy peasy, right? Well, to be honest, it probably doesn't feel easy peasy just yet. But I promise that it will all make sense soon.

As mathematicians usually love complicated stuff, but not completely unnecessary complicated stuff, the binary numbers for 0 and 1 are very logical. The binary number for 0 is exactly the same as the decimal number 0, and binary number for 1 is the same as the decimal number 1. It would be completely bonkers if we said that the binary number 0 has a decimal value of 27 and the binary number 1 has a decimal value of 61, or anything other than 0 and 1. I have to admit that it does sound fun, but it would be way too confusing to be useful, so Leibniz decided against it when he made his formal system for binary numbers.

We can only use the digits 0 and 1, so there is no way that the binary number for 2 is the same as the decimal number. But as was mentioned before, the logic behind binary numbers is the same as the logic behind decimal numbers. When we run out of digits in the decimal system, we add an extra digit and start over. Or, easier said, after 9 comes 10, after 99 comes 100 and so on. If we do this when we have only the digits 0 and 1 available to us, we get

- $0 = 0$
- $1 = 1$
- $2 = 10$
- $3 = 11$
- $4 = 100$
- $5 = 101$
- $6 = 110$
- $7 = 111$
- $8 = 1000$
- $9 = 1001$
- $10 = 1010$
- $11 = 1011$

Exercise Can you work out what the next binary numbers will be?

- $12 = 1100$
- $13 =$
- $14 =$
- $15 =$
- $16 =$
- $17 =$
- $18 =$
- $19 =$
- $20 =$
- $21 =$
- $22 =$
- $23 =$
- $24 =$
- $25 =$
- $26 =$
- $27 =$
- $28 =$
- $29 =$
- $30 =$
- $31 =$

Remember, we can write the decimal number 234 as

$$\begin{aligned} 234 &= (2 \times 100) + (3 \times 10) + (4 \times 1) \\ &= (2 \times 10^2) + (3 \times 10^1) + (4 \times 10^0). \end{aligned}$$

Another way to write this is:

2	3	4
100	10	1
10^2	10^1	10^0

We can write the binary number 1011 in the same way, as

$$1011 = (1 \times 2^3) + (0 \times 2^2) + (1 \times 2^1) + (1 \times 2^0),$$

or

1	0	1	1
8	4	2	1
2^3	2^2	2^1	2^0

Writing a binary number in this way makes it easy to check if it is correct. To put your mind at ease, I promise that this check makes a lot more sense than it did with the Pingala binary numbers.

$$\begin{aligned} 1011 &= (1 \times 2^3) + (0 \times 2^2) + (1 \times 2^1) + (1 \times 2^0) \\ &= (1 \times 8) + (0 \times 4) + (1 \times 2) + (1 \times 1) \\ &= 8 + 2 + 1 \\ &= 11 \end{aligned}$$

There's no hocus pocus involved with adding missing 1's or anything. This way of checking works with every binary number that you come across from now on.

It would be a tedious job if you had to make a list of all numbers 0, 1, 10, 11, 100... and keep going like this to convert a random number to it's binary representation, so we won't do that. There are two ways to convert a decimal number to a binary one. The first way is by following a procedure that has some resemblance to how Pingala binary numbers were formed.

The procedure to make convert a number to binary is as follows

1. If the number is 0 or 1, write 0 or 1 and stop.
2. if the number is bigger than 1, there are two options
   - If your number is even, write 0 and divide it by 2.
   - If your number is odd, write 1. Take 1 away from your number to make it even and divide it by 2.

   Always write new digits to the left of what you already had.
3. Repeat as long as is necessary.

Let us use this procedure to find the binary number for 122.

- 122 is an even number, so we write 0 and divide by 2. Our new number is 61 and we have written 0.
- 61 is an odd number, so we write 1 to the left of what we already had, take 1 away so we have 60 and divide that by 2. Our new number is 30 and we have written 10.
- 30 is an even number, so we write 0 and divide by 2. Our new number is 15 and we have written 010.
- 15 is an odd number, so we write 1 to the left of what we already had, take 1 away so we have 14 and divide that by 2. Our new number is 7 and we have written 1010.
- 7 is an odd number, so we write 1 to the left of what we already had, take 1 away so we have 6 and divide that by 2. Our new number is 3 and we have written 11010.
- 3 is an odd number, so we write 1 to the left of what we already had, take 1 away so we have 2 and divide that by 2. Our new number is 1 and we have written 111010.
- 1 is a special case, where we write 1 and stop. We have now written 1111010

The binary number we have found is 1111010. We can check this, by calculating

$$\begin{aligned}
(1\times 2^6)+(1\times 2^5)+(1\times 2^4)+(1\times 2^3)+(0\times 2^2)+(1\times 2^1)+(0\times 2^0) &= \\
(1\times 64)+(1\times 32)+(1\times 16)+(1\times 8)+(0\times 4)+(1\times 2)+(0\times 1) &= \\
64+32+16+8+2 &= 122
\end{aligned}$$

Exercise Can you convert these numbers to binary by using this first method?

- 38
- 66
- 125
- 525
- 2137

For the second way of converting numbers to binary, you need to know the values of powers of 2. So let's find those first.

Exercise Powers of 2

- $2^0 = 1$
- $2^1 =$
- $2^2 =$
- $2^3 =$
- $2^4 =$
- $2^5 = 32$
- $2^6 =$
- $2^7 =$
- $2^8 =$
- $2^9 =$
- $2^{10} = 1024$
- $2^{11} =$
- $2^{12} =$
- $2^{13} =$
- $2^{14} =$
- $2^{15} = 32768$
- $2^{16} =$
- $2^{17} =$

Make sure to check your answers to this exercise before proceeding, as we will be using them a lot from now on.

The second way of converting a decimal number to binary is as follows:

1. Start by making a chart, in which you place powers of 2 up until the greatest power that is smaller than or equal to the number you want to convert. Orden these powers from biggest to smallest. 2^0 is always the last one on the right.
2. Subtract the biggest power of two can fit in your number, and mark it with a 1.
3. Continue until the result of your subtraction is 0.
4. Write 0's in the empty spaces.
5. You now have your binary number.

Let us use this procedure to find the binary number for 122.

- The biggest power of two that is smaller than 122 is $64 = 2^6$, so the table we need to make is as follows.

64	32	16	8	4	2	1
2^6	2^5	2^4	2^3	2^2	2^1	2^0

- 64 fits in 122, so we subtract that. Our new number is $122 - 64 = 58$ and we write a 1 underneath 64.

64	32	16	8	4	2	1
2^6	2^5	2^4	2^3	2^2	2^1	2^0
1						

- 32 fits in 58 so we subtract that. Our new number is $58 - 32 = 26$ and we write 1 underneath 32.

64	32	16	8	4	2	1
2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	1					

- 16 fits in 26 so we subtract that. Our new number is $26 - 16 = 10$ and we write 1 underneath 16.

64	32	16	8	4	2	1
2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	1	1				

- 8 fits in 10 so we subtract that. Our new number is $10 - 8 = 2$ and we write 1 underneath 8.

64	32	16	8	4	2	1
2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	1	1	1			

- 2 fits in 2 so we subtract that. Our new number is $2 - 2 = 0$, so we can stop subtracting and we write 1 underneath 2.

64	32	16	8	4	2	1
2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	1	1	1		1	

- We write 0's in the empty spaces.

64	32	16	8	4	2	1
2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	1	1	1	0	1	0

- We now have the binary number 1111010.

We checked before that 1111010 is the correct binary number for 122, so hurray, this method really does lead to the same result.

Exercise Can you convert these numbers to binary by using this second method? To help you out a bit, you get the tables of powers of two that you need.

- 48 =

32	16	8	4	2	1
2^5	2^4	2^3	2^2	2^1	2^0

- 99 =

64	32	16	8	4	2	1
2^6	2^5	2^4	2^3	2^2	2^1	2^0

- 234 =

128	64	32	16	8	4	2	1
2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

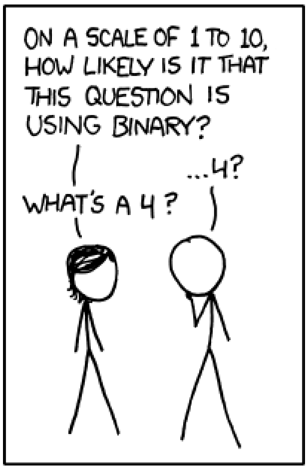
- 579 =

512	256	128	64	32	16	8	4	2	1
2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

- 2173 =

2048	1024	512	256	128	64	32	16	8	4	2	1
2^{11}	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

Converting binary numbers to decimal numbers



xkcd comics cover many maths topics :D

Now that you know how to convert a decimal number to a binary one, we will look at how to convert a binary number to a decimal one. The steps we need to follow are

1. Count the number of digits you have in your binary number. To make sure this works with every possible number, we will call it n. This n can be 1, but it can also be 1067. The steps are always the same.
2. Multiply the left digit with 2^{n-1}, the one next to that with 2^{n-2}, the one next to that with 2^{n-3} and so on. The second to last digit to the right will be multiplied with 2^1, the last one with 2^0. We will have to add everything later, so it is a good idea to already add a +sign between the multiplications.
3. Convert the powers of 2 you used with their decimal values.
4. Add them all up and you get your decimal number.

We actually already did this, when we looked more closely at the binary number 1011 and when checked the binary number we found for 122.

Let us look at another binary number to make sure we really understand the process. We will convert the binary number 10110010101 to a decimal number.

- The number 10110010101 contains 11 digits. So in this case $n = 11$.
- We multiply the left digit with 2^{n-1}, so in our case this means 1×2^{10}.
- We multiply the next digit with 2^{n-2}, which means 0×2^9.

- We keep going like this and get $1 \times 2^{10} + 0 \times 2^9 + 1 \times 2^8 + 1 \times 2^7 + 0 \times 2^6 + 0 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$.
- It may feel a bit pointless to keep writing all these multiplications in the next step and that is a perfectly valid point. When you multiply by 0 it becomes 0, when you multiply a number by 1 you get the number itself, so you can use this to your advantage and reduce the long expression from before to $2^{10} + 2^8 + 2^7 + 2^4 + 2^2 + 2^0$. When you do maths and you don't want to keep writing stuff that doesn't matter anyway, you can call what you're doing *efficient*. Just make sure that you don't leave out too much, otherwise your result won't be efficient anymore, it will just be wrong.
- Anyway, converting those powers to decimals looks like this in our case $1024 + 256 + 128 + 16 + 4 + 1$.
- The sum of these numbers is 1429.

Exercise Convert these binary numbers to decimal numbers. Because this may be the very first time you have to follow steps with a generalized number n, you get a bit of help with the first exercises. And as a friendly reminder, so you don't have to keep turning your pages to find the values you're looking for:

2048	1024	512	256	128	64	32	16	8	4	2	1
2^{11}	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

1. 101 =
   The number 101 has 3 digits, so $n = 3$. You have to multiply the left digit with $2^{n-1} = 2^2$, the one next to that with $2^{n-2} = 2^1$ and the right digit with 2^0.
2. 1001 =
   The number 1001 has 4 digits, so $n = 4$. The left digit is multiplied by $2^{n-1} = 2^3$.
3. 10101 =
   10101 has 5 digits, so the left digit is multiplied by 2^4.
4. 11011 =
5. 101000 =
6. 110011 =
7. 1111111 =
8. 1000000 =
9. 1000011 =
10. 1100011 =
11. 1101011 =
12. 100011000 =
13. 11101010 =
14. 101001101 =
15. 110111100 =
16. 1111101000 =
17. 10001010111 =
18. 10011111100 =
19. 100010101110 =
20. 1101111000 =

Adding binary numbers

Adding binary numbers happens similarly as with decimal numbers. Just as with binary numbers, you add the digits from right to left and carry over a digit when your result exceeds the highest available digit. As we only use 0 and 1 in binary numbers, there are only 4 possibilities when you add 2 numbers:

	carry over	result
$0+0$	0	0
$0+1=1+0$	0	1
$1+1$	1	0
$1+1+1$	1	1

We will examine one addition in more detail to understand how this works. We will add the binary numbers 100011101 and 10001110.

We start by writing them in a way that makes it easy to see which digits you need to add together.

$$\begin{array}{ccccccccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ + & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ \hline \end{array}$$

We start by adding the digits on the right. $1+0=1$, so we get

$$\begin{array}{ccccccccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ + & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ \hline & & & & & & & & 1 \end{array}$$

We move on to the digits on the left of them. $0+1=1$, so we get

$$\begin{array}{ccccccccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ + & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ \hline & & & & & & & 1 & 1 \end{array}$$

We move on to the digits on the left of them. $1+1=10$, so the result is 0 and we carry over 1. We add the 1 that is carried over to the top and get

$$\begin{array}{ccccccccc} & & & & & 1 & & & \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ + & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ \hline & & & & & & 0 & 1 & 1 \end{array}$$

We move on to the digits on the left of them. $1+1+1=11$, so the result is 1 and we carry over 1. We add the 1 that is carried over to the top and get

$$\begin{array}{ccccccccc}
 & & & & 1 & 1 & & & \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
+ & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
 & & & & & 1 & 0 & 1 & 1
\end{array}$$

We move on to the digits on the left of them. $1 + 1 + 0 = 10$, so the result is 0 and we carry over 1. We add the 1 that is carried over to the top and get

$$\begin{array}{ccccccccc}
 & & & 1 & 1 & 1 & & & \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
+ & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
 & & & & 0 & 1 & 0 & 1 & 1
\end{array}$$

We move on to the digits on the left of them. $1+0+0 = 1$, so the result is 1 and we don't carry anything over. We get

$$\begin{array}{ccccccccc}
 & & & 1 & 1 & 1 & & & \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
+ & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
 & & & 1 & 0 & 1 & 0 & 1 & 1
\end{array}$$

We move on to the digits on the left of them. $0 + 0 = 0$, so we get

$$\begin{array}{ccccccccc}
 & & & 1 & 1 & 1 & & & \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
+ & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
 & & 0 & 1 & 0 & 1 & 0 & 1 & 1
\end{array}$$

We move on to the digits on the left of them. $0 + 1 = 1$, so we get

$$\begin{array}{ccccccccc}
 & & & 1 & 1 & 1 & & & \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
+ & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1
\end{array}$$

We don't need to add anything to the digit on the left, so we get

$$\begin{array}{ccccccccc}
 & & & 1 & 1 & 1 & & & \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
+ & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1
\end{array}$$

We have now succesfully added the binary numbers and found that $100011101+10001110 = 110101011$. In decimal numbers, this sum is $285 + 142 = 427$. Adding in binary numbers may look hard, but as you are only adding 0's and 1's, the calculations involved won't give you a headache.

Exercise Adding two binary numbers.

1. $$\begin{array}{r} 1\ 0\ 0\ 1\ 0\ 1 \\ +\quad 1\ 0\ 1\ 0\ 0 \\ \hline \end{array}$$

2. $$\begin{array}{r} 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0 \\ +\quad 1\ 0\ 1\ 0\ 1\ 0\ 1 \\ \hline \end{array}$$

3. $$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0 \\ +\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1 \\ \hline \end{array}$$

4. $$\begin{array}{r} 1\ 0\ 0\ 1\ 0\ 1 \\ +\quad 1\ 1\ 1\ 0\ 0 \\ \hline \end{array}$$

5. $$\begin{array}{r} 1\ 1\ 1\ 0\ 1\ 1\ 0\ 1 \\ +\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 1 \\ \hline \end{array}$$

6. $$\begin{array}{r} 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \\ +\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \\ \hline \end{array}$$

7. $$\begin{array}{r} 1\ 0\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \\ +\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 1 \\ \hline \end{array}$$

8. $$\begin{array}{r} 1\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 1 \\ +\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1 \\ \hline \end{array}$$

9. $$\begin{array}{r} 1\ 1\ 0\ 0\ 0\ 1\ 0\ 1 \\ +\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 1 \\ \hline \end{array}$$

10. $$\begin{array}{r} 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0 \\ +\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0 \\ \hline \end{array}$$

Exercise Can you add two binary numbers when they aren't aligned?

1. $110011 + 100010 =$
2. $1111 + 100111 =$
3. $100000 + 101010 =$
4. $100011 + 101001 =$
5. $101111 + 10100 =$
6. $1111101 + 1010100 =$
7. $11010100 + 11000010 =$
8. $1110001 + 1001101 =$
9. $1010111 + 1100110 =$
10. $1110101 + 1100011 =$
11. $11000110 + 110001001 =$
12. $101101011 + 100100001 =$
13. $101010100 + 10000110 =$
14. $10111011 + 110000010 =$
15. $10110111 + 111011 =$
16. $1111101 + 101101101 =$
17. $11101101 + 10110111 =$
18. $1100011 + 1110111 =$
19. $11011100 + 101110111 =$
20. $101100111 + 1001111 =$

Exercise Can you add three numbers? Be very careful with the numbers you carry over when you have to add $1+1+1+1$ or $1+1+1+1+1$!

1.
$$\begin{array}{ccccccc} & 1 & 1 & 0 & 0 & 0 & 1 \\ & 1 & 0 & 0 & 1 & 0 & 0 \\ + & & 1 & 1 & 0 & 0 & 1 \\ \hline \end{array}$$

2.
$$\begin{array}{ccccccccc} & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ + & & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ \hline \end{array}$$

3.
$$\begin{array}{ccccccc} & 1 & 1 & 1 & 0 & 0 & 0 \\ & 1 & 0 & 1 & 1 & 0 & 1 \\ + & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline \end{array}$$

4.
$$\begin{array}{cccccccccc} & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ + & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ \hline \end{array}$$

5.
$$\begin{array}{ccccccccc} & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ + & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline \end{array}$$

6.
$$\begin{array}{cccccccccc} & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ & & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ + & & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline \end{array}$$

7.
$$\begin{array}{cccccccccc} & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ + & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ \hline \end{array}$$

8.
$$\begin{array}{ccccccccccc} & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ + & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ \hline \end{array}$$

9.
$$\begin{array}{ccccccccccccc} & & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ & & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ + & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ \hline \end{array}$$

10.
$$\begin{array}{ccccccccccc} & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ + & & & & & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline \end{array}$$

Exercise Can you add even more numbers in one go? Keep paying attention to the numbers you carry over.

1.
$$\begin{array}{ccccccccc} & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ + & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ \hline \end{array}$$

2.
$$\begin{array}{ccccccccc} & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ + & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline \end{array}$$

3.

$$\begin{array}{ccccccccccc}
 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
+ & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
\hline
\end{array}$$

4.

$$\begin{array}{cccccccccc}
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
+ & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array}$$

5.

$$\begin{array}{cccccccccc}
 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
+ & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array}$$

6.

$$\begin{array}{ccccccccccc}
 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
+ & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
\hline
\end{array}$$

Subtracting binary numbers

Just like addition, the subtraction of binary numbers happens similarly to what happens with decimal numbers. Just like with decimal numbers, the tricky part is when you have to borrow from the adjecent digit. We will examine one subtraction in more detail to understand how it works. We will subtract 111010010 from 1100101011.

We start by writing them in a way that makes it easy to see which digit you need to be subtract from which.

$$\begin{array}{cccccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
- & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
\hline
\end{array}$$

We start by subtracting the digits on the right. $1 - 0 = 1$, so we get

$$\begin{array}{cccccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
- & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
\hline
 & & & & & & & & & 1
\end{array}$$

We move on to the digits on the left of them. $1 - 1 = 0$, so we get

$$\begin{array}{cccccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
- & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
\hline
 & & & & & & & & 0 & 1
\end{array}$$

We move on to the digits on the left of them. $1 - 1 = 0$, so we get

$$\begin{array}{cccccccccc} 1&1&0&0&1&0&1&0&1&1\\ -&1&1&1&0&1&0&0&1&0\\ \hline &&&&&&&&0&1\end{array}$$

We move on to the digits on the left of them. $0-0=0$, so we get

$$\begin{array}{cccccccccc} 1&1&0&0&1&0&1&0&1&1\\ -&1&1&1&0&1&0&0&1&0\\ \hline &&&&&&&0&0&1\end{array}$$

We move on to the digits on the left of them. $1-0=1$, so we get

$$\begin{array}{cccccccccc} 1&1&0&0&1&0&1&0&1&1\\ -&1&1&1&0&1&0&0&1&0\\ \hline &&&&&&1&0&0&1\end{array}$$

We move on to the digits on the left of them. $0-1$ is impossible, so we have to borrow from the digit on the left. That way, the digit on the left gets reduced by 1 and the original 1 now becomes 10. We now have

$$\begin{array}{cccccccccc} &&&&0&10&&&&\\ 1&1&0&0&\cancel{1}&\cancel{0}&1&0&1&1\\ -&1&1&1&0&1&0&0&1&0\\ \hline &&&&&&1&0&0&1\end{array}$$

We are working in binary, so $10-1=1$, so we get

$$\begin{array}{cccccccccc} &&&&0&10&&&&\\ 1&1&0&0&\cancel{1}&\cancel{0}&1&0&1&1\\ -&1&1&1&0&1&0&0&1&0\\ \hline &&&&&1&1&0&0&1\end{array}$$

We move on to the digits to the left. $0-0=0$, so we get

$$\begin{array}{cccccccccc} &&&&0&10&&&&\\ 1&1&0&0&\cancel{1}&\cancel{0}&1&0&1&1\\ -&1&1&1&0&1&0&0&1&0\\ \hline &&&&0&1&1&0&0&1\end{array}$$

We move on to the digits to the left. We get $0-1$ again, so we have to borrow from the digit on the left. The digit directly on the left is 0 as well, so we have to borrow from even further to the left. We get

$$\begin{array}{cccccccccc} &0&1&10&0&10&&&&\\ 1&\cancel{1}&\cancel{0}&\cancel{0}&\cancel{1}&\cancel{0}&1&0&1&1\\ -&1&1&1&0&1&0&0&1&0\\ \hline &&&&0&1&1&0&0&1\end{array}$$

When we do the actual subtraction, we use $10-1=1$, so we get

$$
\begin{array}{cccccccccc}
 & 0 & 1 & 10 & 0 & 10 & & & & \\
1 & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} & 1 & 0 & 1 & 1 \\
- & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
\hline
 & & & 1 & 0 & 1 & 1 & 0 & 0 & 1
\end{array}
$$

We move on to the digits on the left. $1 - 1 = 0$, so we get

$$
\begin{array}{cccccccccc}
 & 0 & 1 & 10 & 0 & 10 & & & & \\
1 & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} & 1 & 0 & 1 & 1 \\
- & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
\hline
 & & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1
\end{array}
$$

We move to the digits on the left. We encounter $0 - 1$ again, so we have to borrow from the left. We get

$$
\begin{array}{cccccccccc}
 & 10 & \cancel{0} & 1 & 10 & 0 & 10 & & & \\
\cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} & 1 & 0 & 1 & 1 \\
- & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
\hline
 & & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1
\end{array}
$$

We do the actual subtraction $10 - 1 = 1$ and get

$$
\begin{array}{cccccccccc}
 & 10 & \cancel{0} & 1 & 10 & 0 & 10 & & & \\
\cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} & 1 & 0 & 1 & 1 \\
- & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
\hline
 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1
\end{array}
$$

There are no more digits to the left, so we found that $1100101011 - 111010010 = 101011001$. Take another good look at how borrowing worked, take a deep breath in and out and be proud of yourself, because now you know how to subtract binary numbers.

Exercise Subtracting binary numbers.

1. $$\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & 1 \\ - & 1 & 0 & 1 & 0 & 0 \\ \hline \end{array}$$

2. $$\begin{array}{cccccc} 1 & 0 & 1 & 0 & 1 & 1 \\ - & 1 & 0 & 0 & 1 & 1 \\ \hline \end{array}$$

3. $$\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ - & 1 & 1 & 0 & 1 & 1 \\ \hline \end{array}$$

4. $$\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & 1 \\ - & 1 & 1 & 1 & 0 & 0 \\ \hline \end{array}$$

5. $$\begin{array}{ccccccccc} & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ - & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ \hline \end{array}$$

6. $$\begin{array}{cccccccc} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ - & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline \end{array}$$

7.
$$\begin{array}{r} 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0 \\ -\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 0 \\ \hline \end{array}$$

8.
$$\begin{array}{r} 1\ 1\ 0\ 0\ 0\ 1\ 0\ 1 \\ -\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 1 \\ \hline \end{array}$$

9.
$$\begin{array}{r} 1\ 1\ 1\ 0\ 0\ 0\ 1\ 1 \\ -\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 1 \\ \hline \end{array}$$

10.
$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 1 \\ -\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 1 \\ \hline \end{array}$$

Exercise Subtracting binary numbers, where the borrowing part of the calculations gets a bit trickier.

1.
$$\begin{array}{r} 1\ 0\ 0\ 1\ 0\ 0 \\ -\ 1\ 0\ 1\ 1\ 1 \\ \hline \end{array}$$

2.
$$\begin{array}{r} 1\ 1\ 1\ 0\ 0\ 1 \\ -\ 1\ 0\ 1\ 1\ 1\ 0 \\ \hline \end{array}$$

3.
$$\begin{array}{r} 1\ 1\ 1\ 1\ 0\ 1\ 0 \\ -\ 1\ 1\ 0\ 1\ 1\ 1\ 1 \\ \hline \end{array}$$

4.
$$\begin{array}{r} 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0 \\ -\ 1\ 0\ 1\ 1\ 1\ 1\ 0 \\ \hline \end{array}$$

5.
$$\begin{array}{r} 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0 \\ -\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 1 \\ \hline \end{array}$$

6.
$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0 \\ -\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1 \\ \hline \end{array}$$

7.
$$\begin{array}{r} 1\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 0 \\ -\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1 \\ \hline \end{array}$$

8.
$$\begin{array}{r} 1\ 1\ 1\ 0\ 0\ 0\ 1\ 0\ 1 \\ -\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 1 \\ \hline \end{array}$$

9.
$$\begin{array}{r} 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0 \\ -\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 0 \\ \hline \end{array}$$

10.
$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0 \\ -\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 1 \\ \hline \end{array}$$

Exercise Can you subtract two binary numbers when they aren't aligned?

1. $10001 - 10 =$
2. $11111111 - 1001101 =$
3. $10000 - 1011 =$
4. $110011 - 100011 =$
5. $111001 - 101 =$
6. $1001000 - 111 =$

7. $101010 - 11001 =$
8. $11001 - 1111 =$
9. $100110 - 1010 =$
10. $100111 - 1101 =$

Multiplication of binary numbers

You may have suspected it already and indeed, multiplication works the same for binary numbers as it does for decimal ones. The down side: multiplying by 0 turns the whole number into 0 and multiplying by 1 simply gives you back your original number, so all useful binary multiplications are long multiplications. The up side: the multiplications involved are very easy:

- $0 \times 0 = 0$
- $0 \times 1 = 1 \times 0 = 0$
- $1 \times 1 = 1$

And that's it. There are no difficult to remember times tables, like $6 \times 7 = 42$ or whichever multiplication it is that still makes you hesitate just that tiny bit longer before you can give the answer. Everyone has at least one multiplication that their mind refuses to remember as easily as the other ones. If you're lucky, you also have at least one multiplication that is really easy for you to remember, like $3 \times 7 = 21$. But all of those are irrelevant when you multiply binary numbers. Aren't machines that use binary, like calculators and computers, lucky that they never had to learn their times tables? Hmmm, they can't think for themselves and their only way of rebelling against human commands is freezing or shutting down, so I guess I still prefer learning those times tables and being human.

Let's first look at some cool things that happen when we multiply binary numbers.

1. To double a binary number, you multiply it by 2, or the binary number 10, so you add an extra 0 at the right. So multiplying by 2 in binary is like multiplying a decimal number with ten.
2. To quadruple a binary number, you have to multiply it by $4 = 2^2$, or the binary number 100, so you add two zeroes at the right.
3. Multiplying a binary number with $8 = 2^3$, or 1000, gives you 3 added zeroes.
4. The same pattern goes on and on and on.

Conclusion Every time you multiply a binary number with a power of 2, you add as many zeroes as the power of 2 you multiply it with. So if you multiply a binary number with 2^n, you add n zeroes to the binary number.

Exercise Calculate these binary multiplications.

1. $10111 \times 100 =$
2. $1011 \times 10 =$
3. $11011 \times 1000 =$
4. $111 \times 10000 =$
5. $1011 \times 10 =$
6. $11111 \times 1000 =$
7. $10101 \times 100 =$
8. $100111 \times 10 =$
9. $11001 \times 100 =$
10. $1001 \times 1000 =$
11. $10110 \times 100000 =$
12. $100000 \times 100 =$
13. $100101 \times 10 =$
14. $1001001 \times 100 =$
15. $110001 \times 1000 =$
16. $10111 \times 100 =$
17. $11010 \times 10000 =$
18. $10010 \times 10 =$
19. $110101 \times 1000 =$
20. $1101100 \times 100 =$

Let us now refresh our memory of how long multiplication works by looking at an example in more detail. We'll multiply the binary numbers 1111001 and 10110.

We start by writing them in a way that makes it easy to see which digit you need to be multiply with which.

$$\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 0 & 0 & 1 \\
\times & & 1 & 0 & 1 & 1 & 0 \\
\hline
\end{array}$$

We start by multiplying with the most right digit of 10110. This is 0, so we get.

$$\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 0 & 0 & 1 \\
\times & & 1 & 0 & 1 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

We move on to the digit on the left of that. Just like when you work with decimal numbers, you write a 0 at the end. All zeroes that are there by default will be bold so you can see the difference with the zeroes you get as a result of the multiplications. The digit we now multiply with is 1, so we get.

$$\begin{array}{cccccccc}
 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0}
\end{array}$$

We move on to the digit on the left, which is another 1. We now have to write two 0's at the end of our number, so we get

$$\begin{array}{ccccccccc}
 & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0}
\end{array}$$

We move on to the digit on the left, which is 0. We now have to write three 0's at the end of our number, so we get

$$\begin{array}{cccccccccc}
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}$$

We move on to the digit on the left, which is the last digit, another 1. We now have to write four 0's at the end of our number, so we get

$$\begin{array}{ccccccccccc}
 & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
 & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}$$

By writing down every single multiplication we encounter, it is easy to see a system in what we do. Normally, you wouldn't write down the result when you multiply by zero, because when you do maths it is nice to be ~~lazy~~ efficient and adding a bunch of zeroes is not necessary when you already understand how things work. Just make sure that you don't forget any of the extra zeroes at the end!

We have now finished the multiplication part of the calculation, but we aren't done yet. We still have to add all of our numbers. Because there are so many numbers you have to add, I strongly advise you to leave some room at the top, so you have some space to write down digits that are carried over. As we will be adding more than two binary numbers, we will be looking at this addition in more detail as well. I know you already know how to do this, as you have done it in exercises before. This more of a friendly reminder of where to place the carried over digits when your sum exeeds 11 so it doesn't get confusing.

We start by adding some space at the top for carried over digits.

$$
\begin{array}{cccccccccccc}
 & & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & & & & & & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
+ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
\end{array}
$$

No digits are carried over in the digits on the right, or when we add any of the three columns to the left of those, so we get

$$
\begin{array}{cccccccccccc}
 & & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & & & & & & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
+ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
 & & & & & & & & 0 & 1 & 1 & 0
\end{array}
$$

We move to the digits on the left of what we already have. Now we add $0+1+0+0+1 = 10$, so we have to carry over 1 and we get

$$
\begin{array}{cccccccccccc}
 & & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & & & & & 1 & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
+ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
 & & & & & & & 0 & 0 & 1 & 1 & 0
\end{array}
$$

We move to the digits on the left of what we already have. Now we add $1+0+1+1+0+0 = 11$, so we have to carry over 1 and we get

$$
\begin{array}{cccccccccccc}
 & & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & & & & 1 & 1 & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
+ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
 & & & & & & 1 & 0 & 0 & 1 & 1 & 0
\end{array}
$$

We move to the digits on the left of what we already have. Again, we add $1{+}0{+}1{+}1{+}0{+}0 = 11$, so we have to carry over 1 and we get

$$
\begin{array}{cccccccccccc}
 & & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & & & 1 & 1 & 1 & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
+ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
 & & & & & 1 & 1 & 0 & 0 & 1 & 1 & 0
\end{array}
$$

We move to the digits on the left of what we already have. This time we add $1{+}1{+}1{+}0{+}1 = 100$, so we have to carry over 1, but a space further to the left than usually and we get

$$
\begin{array}{cccccccccccc}
 & & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & 1 & & & & & & & & & \\
 & & & & 1 & 1 & 1 & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
+ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
 & & & & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0
\end{array}
$$

We move to the digits on the left of what we already have. We add $1 + 0 + 1 = 10$, so we have to carry over 1 and we get

$$\begin{array}{cccccccccccc}
 & & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & 1 & & & & & & & & & \\
 & & 1 & & 1 & 1 & 1 & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
+ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
 & & & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0
\end{array}$$

We move to the digits on the left of what we already have. We add $1+1+0+1=11$, so we have to carry over 1 and we get

$$\begin{array}{cccccccccccc}
 & & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & 1 & & & & & & & & & \\
 & 1 & 1 & & 1 & 1 & 1 & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
+ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
 & & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0
\end{array}$$

We move to the digits on the left of what we already have. We add $1+1=10$, so we have to carry over 1 and we get

$$\begin{array}{cccccccccccc}
 & & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & 1 & & & & & & & & & \\
1 & 1 & 1 & & 1 & 1 & 1 & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
+ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0
\end{array}$$

We add this last digit to get the complete result and get

$$
\begin{array}{cccccccccccc}
 & & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & \times & & 1 & 0 & 1 & 1 & 0 \\
\hline
 & & 1 & & & & & & & & & \\
1 & 1 & 1 & & 1 & 1 & 1 & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
+ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0
\end{array}
$$

If you convert the binary numbers to decimal numbers, you would find that this multiplication is correct and no mistakes were made in the process. Hurrah! So now it's your turn.

Exercise Multiply these binary numbers. You don't have to write anything down when you multiply by zero, just be mindful of the zeroes at the end. And make sure that you leave enough space at the top to carry over digits. Good luck!

1.
$$
\begin{array}{cccccc}
1 & 0 & 0 & 1 & 0 & 1 \\
\times & 1 & 0 & 1 & 0 & 0 \\
\hline
\end{array}
$$

2.
$$
\begin{array}{cccccc}
1 & 0 & 1 & 0 & 1 & 1 \\
\times & 1 & 0 & 0 & 1 & 1 \\
\hline
\end{array}
$$

3.
$$
\begin{array}{cccccc}
1 & 0 & 1 & 1 & 0 & 1 \\
\times & & & 1 & 1 & 1 \\
\hline
\end{array}
$$

4.
$$
\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 1 & 0 & 1 \\
\times & & & 1 & 0 & 1 & 0 \\
\hline
\end{array}
$$

5.
$$
\begin{array}{cccccc}
1 & 0 & 0 & 1 & 0 & 1 \\
\times & 1 & 1 & 1 & 0 & 0 \\
\hline
\end{array}
$$

6.
$$
\begin{array}{cccccc}
1 & 1 & 0 & 1 & 1 & 1 \\
\times & & 1 & 1 & 0 & 1 \\
\hline
\end{array}
$$

7.

$$\begin{array}{r} 1\,1\,0\,1\,1\,1\,0\,1\,1 \\ \times \quad 1\,1\,0\,0\,1 \\ \hline \\ \hline \end{array}$$

8.

$$\begin{array}{r} 1\,0\,1\,1\,0\,0 \\ \times \; 1\,1\,0\,1\,1 \\ \hline \\ \hline \end{array}$$

9.

$$\begin{array}{r} 1\,1\,1\,0\,1\,1\,1 \\ \times \quad 1\,1\,1\,0\,1\,1 \\ \hline \\ \hline \end{array}$$

10.

$$\begin{array}{r} 1\,0\,1\,1\,0\,1\,0 \\ \times \quad 1\,1\,1\,1\,1 \\ \hline \\ \hline \end{array}$$

11.

$$\begin{array}{r} 1\,0\,0\,1\,1\,0\,1\,1\,1 \\ \times \quad 1\,0\,0\,1\,1 \\ \hline \\ \hline \end{array}$$

12.

$$\begin{array}{r} 1\,1\,0\,1\,1\,1\,0\,1 \\ \times \quad 1\,1\,1\,0\,0\,1 \\ \hline \\ \hline \end{array}$$

13.

$$\begin{array}{r} 1\,0\,1\,0\,1\,0\,1\,1 \\ \times \quad 1\,1\,1\,1\,1 \\ \hline \\ \hline \end{array}$$

14.

$$\begin{array}{r} 1\,1\,0\,1\,0\,1\,1 \\ \times \quad 1\,1\,0\,1\,1\,1 \\ \hline \\ \hline \end{array}$$

Exercise These multiplications are a bit harder, because while adding you might have to carry over more than 1 digit (like when your sum is 111) or the digit you have to carry over is several places away (like when your sum is 1001). But I know you can do it.

1.
$$\begin{array}{r} 1\,0\,1\,1\,1\,1\,0\,1 \\ \times\ 1\,1\,1\,1\,0\,1 \\ \hline \end{array}$$

2.
$$\begin{array}{r} 1\,0\,1\,1\,1\,1\,1\,1 \\ \times\ 1\,1\,0\,1\,1\,1\,1 \\ \hline \end{array}$$

3.
$$\begin{array}{r} 1\,1\,1\,1\,1\,1\,1 \\ \times\ 1\,1\,1\,1\,1\,1 \\ \hline \end{array}$$

4.
$$\begin{array}{r} 1\,1\,1\,1\,1\,1 \\ \times\ 1\,1\,1\,1\,1 \\ \hline \end{array}$$

5.
$$\begin{array}{r} 1\,1\,1\,0\,1\,1\,1 \\ \times\ 1\,1\,1\,1\,1\,1 \\ \hline \end{array}$$

6.
$$\begin{array}{r} 1\,1\,1\,1\,1\,1\,1 \\ \times\ 1\,1\,1\,1\,1\,1\,1 \\ \hline \end{array}$$

Division of binary numbers

Dividing by 0 isn't going to happen any time soon and dividing by 1 isn't very informative... So we're stuck with long divisions when we want to divide binary numbers and gain new information. But before you get discouraged by that prospect, let me remind you that you're only working with 0's and 1's, which makes it a lot easier than when you have to do long divisions of decimal numbers.

When you are asked to divide a binary number by a power of two, you don't even have to do any calculations at all. Binary numbers give us that information very easily, so let us take a look at that first. We will look at the binary numbers from 1 to 32 to get an intuitive feeling of this.

1	1001	10001	11001
10			
11	1011		
100			
101			
110		10110	
111			11111
1000			

Exercise Patterns in binary numbers

1. Complete the table.
2. Underline all even numbers with a blue colour pencil.

3. Underline all multiples of 4 with a red colour pencil.

4. Underline all multiples of 8 with a yellow colour pencil.

5. Underline all multiples of 16 with a green colour pencil.

6. Study the numbers you have underlined.
   (a) What do all multiples of 2 have in common?
   (b) What do all multiples of 4 have in common?
   (c) What do all multiples of 8 have in common?
   (d) What do all multiples of 16 have in common?

By doing the previous exercise, we hope that is has become clear to you that you can determine if a binary number is a multiple of 2, 4, 8 or 16 simply by looking at it.

1. If a binary number ends with 0, it is an even number. If it ends with 1, it is odd. The value of the last digit is crucial to determine whether or not a number is a multiple of 2.

2. A binary number is a multiple of $4 = 2^2$ if the number ends in 00. If it doesn't end in 00, the last two digits give you the remainder you get when you divide the number by 4.

3. If a binary number ends with 000, it is a multiple of $8 = 2^3$. If not, the last three digits give you the remainder you get when you divide the binary number by 8.

Conclusion If you want to determine whether or not a binary number is a multiple of a power of 2, look at as many digits on the right as the power of 2 you are interested in. So if you want to know if it is a multiple of 2^n, you look at the last n digits of the binary number. If they are all zero, then your number is a multiple of 2^n. If not, your last n digits give you the remainder you get when you divide your binary number by 2^n.

Let us look at some more examples, because explaining maths with words sometimes makes it seem more complicated than it actually is. There's a life lesson in there for you: even when it looks hard and sounds even harder, once you start and get the hang of it, you might be surprised at how quickly it became an easy peasy thing to do.

1. $100101 \div 10 = 10010$ r 1
2. $100101 \div 100 = 1001$ r 1
3. $100101 \div 1000 = 100$ r 101
4. $11010 \div 10 = 1101$ r 0
5. $11010 \div 100 = 110$ r 10
6. $11010 \div 1000 = 11$ r 10

7. $111111 \div 10 = 11111$ r 1
8. $111111 \div 100 = 1111$ r 11
9. $111111 \div 1000 = 111$ r 111
10. $111111 \div 10000 = 11$ r 1111
11. $101010100 \div 10 = 10101010$ r 0
12. $101010100 \div 100 = 1010101$ r 0
13. $101010100 \div 1000 = 101010$ r 100
14. $101010100 \div 10000 = 10101$ r 100

Exercise Divide these binary numbers by powers of 2. If there is a remainder, write it down as well. I know you can do it!

1. $110111010 \div 10 =$
2. $110111010 \div 100 =$
3. $110111010 \div 1000 =$
4. $110111010 \div 10000 =$
5. $11001000 \div 10 =$
6. $11001000 \div 100 =$
7. $11001000 \div 1000 =$
8. $11001000 \div 10000 =$
9. $10011001 \div 10 =$
10. $10011001 \div 100 =$
11. $10111011 \div 1000 =$
12. $1101100 \div 100 =$
13. $110110 \div 100 =$
14. $1001100 \div 10 =$
15. $101100 \div 1000 =$
16. $100100 \div 100 =$
17. $10110 \div 10000 =$
18. $11000 \div 100 =$
19. $1101110 \div 1000 =$
20. $11010110 \div 10 =$

Now that you know how to divide a binary number by a power of 2, we will look at what happens when you divide by a random binary number. We will divide 110000101 by 101 as an example for how binary long division works.

We start by writing it down as usual.

$$101 \quad |\overline{110000101}$$

We start by writing 101 underneath it. We subtract it from the number above, and note that our solution starts with 1.

$$\begin{array}{rl} & \underline{\;1\qquad\qquad\;} \\ 101 & |\;\; 110000101 \\ & \underline{-101} \\ & \quad\;\; 1 \end{array}$$

After the subtraction we get 1, which is smaller than 101. So we add another digit and get 10, which still is smaller. This means that we have to add a 0 to our result. We get

$$\begin{array}{ll} & \underline{\;\;10\qquad\qquad} \\ 101 & |\;\; 110000101 \\ & \underline{-101} \\ & \qquad 10 \end{array}$$

We add another digit and get 100, so it stays smaller than 101. We have to add another 0 to our result so we get

$$\begin{array}{ll} & \underline{\;\;100\qquad\quad} \\ 101 & |\;\; 110000101 \\ & \underline{-101} \\ & \qquad 100 \end{array}$$

It is only when we add a fourth digit and get 1000, that we finally have a number that is bigger than 101. So we add 1 to our result, subtract 101 from this number and get.

$$\begin{array}{ll} & \underline{\;\;1001\qquad} \\ 101 & |\;\; 110000101 \\ & \underline{-101} \\ & \qquad 1000 \\ & \qquad \underline{-101} \\ & \qquad\qquad 11 \end{array}$$

We move on to the next digit. We now have 111, which is bigger than 101. So we add 1 to our result, subtract 101 from this number and get.

$$\begin{array}{ll} & \underline{\;\;10011\qquad} \\ 101 & |\;\; 110000101 \\ & \underline{-101} \\ & \qquad 1000 \\ & \qquad \underline{-101} \\ & \qquad\qquad 111 \\ & \qquad\quad \underline{-101} \\ & \qquad\qquad\;\; 10 \end{array}$$

We move on to the next digit. We now have 100, which is smaller than 101. So we add 0 to our result and get.

$$\begin{array}{ll} & \underline{\;\;100110\quad\;\;} \\ 101 & |\;\; 110000101 \\ & \underline{-101} \\ & \qquad 1000 \\ & \qquad \underline{-101} \\ & \qquad\qquad 111 \\ & \qquad\quad \underline{-101} \\ & \qquad\qquad\;\; 100 \end{array}$$

We move on to the last digit. We now have 1001, which is smaller than 101. So we add 1 to our result, subtract 101 from this number and get.

$$\begin{array}{rl}
 & \underline{\ 1001101\ \ } \\
101 & |\ 110000101 \\
 & \underline{-101} \\
 & 1000 \\
 & \underline{-101} \\
 & 111 \\
 & \underline{-101} \\
 & 1001 \\
 & \underline{-101} \\
 & 100
\end{array}$$

We found that $110000101 \div 101 = 1001101$ with remainder 100.

Exercise It is your turn now. Calculate the result of these divisions, don't forget to write down the remainder. You can use the rest of this page and the next one for your calculations.

1. $110111001 \div 101 =$
2. $1000101011 \div 100 =$
3. $11100111101 \div 110 =$
4. $10111001 \div 111 =$
5. $11111111 \div 100 =$
6. $11111111 \div 101 =$
7. $11111111 \div 110 =$
8. $11111111 \div 111 =$
9. $1011011101 \div 1010 =$
10. $110010010001 \div 1100 =$
11. $10010111011 \div 1101 =$
12. $1011100111 \div 1011 =$
13. $110110110110 \div 1111 =$
14. $100101101 \div 11101 =$
15. $110011010100 \div 11010 =$
16. $1101001101001 \div 11011 =$

Binary fractions

We started our adventure in the wonderful world of binary numbers by looking at how the hieroglyphs of the Eye of Horus were used in Ancient Egypt, as a way to count in binary fractions. By now you have learned more about working with binary numbers than most computer programmers know, but we still haven't encountered anything that connects your current knowledge of binary numbers to the Horus Eye fractions. It would be so frustrating to leave you guessing for a link and stop learning about binary right now, so we won't do that.

Let us go above and beyond, and learn about the binary numbers that come after the decimal point. The point that shows you which part of a number is bigger than 1 and which part of a number is smaller is called the **decimal point** and yes, that is a very confusing name when you aren't working with decimal numbers.

I feel like I keep repeating myself, but here we go again: binary numbers work in the same way as decimal numbers. So we will take a closer look at a decimal number first, and then we will look at the equivalent system for binary numbers.

For this, we will take a closer look at the decimal number 12,345.6789. We can write this as

$$\begin{aligned} 12,345.6789 &= (1 \times 10,000) + (2 \times 1,000) + (3 \times 100) + (4 \times 10) + (5 \times 1) \\ &\quad +(6 \times \frac{1}{10}) + (7 \times \frac{1}{100}) + (8 \times \frac{1}{1,000}) + (9 \times \frac{1}{10,000}). \end{aligned}$$

We can also write this in terms of powers of 10, as follows:

$$\begin{aligned} 12,345.6789 &= (1 \times 10^4) + (2 \times 10^3) + (3 \times 10^2) + (4 \times 10^1) + (5 \times 10^0) \\ &\quad +(6 \times \frac{1}{10^1}) + (7 \times \frac{1}{10^2}) + (8 \times \frac{1}{10^3}) + (9 \times \frac{1}{10^4}). \end{aligned}$$

Once you understand this, you might suspect that the binary number 11010.01011 can also be seen as

$$\begin{aligned} 11010.01011 &= (1 \times 2^4) + (1 \times 2^3) + (0 \times 2^2) + (1 \times 2^1) + (0 \times 2^0) \\ &\quad +(0 \times \frac{1}{2^1}) + (1 \times \frac{1}{2^2}) + (0 \times \frac{1}{2^3}) + (1 \times \frac{1}{2^4}) + (1 \times \frac{1}{2^5}). \end{aligned}$$

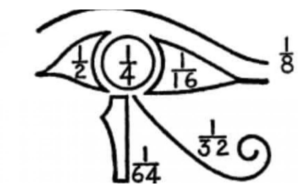
When we calculate the powers of two that we used, we find that

$$\begin{aligned} 11010.01011 &= (1 \times 16) + (1 \times 8) + (0 \times 4) + (1 \times 2) + (0 \times 1) \\ &\quad +(0 \times \frac{1}{2}) + (1 \times \frac{1}{4}) + (0 \times \frac{1}{8}) + (1 \times \frac{1}{16}) + (1 \times \frac{1}{32}). \end{aligned}$$

This means that the decimal number that is equivalent to the binary number 11010.01011 is

$$
\begin{aligned}
11010.01011 &= 16 + 8 + 2 + \frac{1}{4} + \frac{1}{16} + \frac{1}{32} \\
&= 26\frac{11}{32} \\
&= 26 + 0.25 + 0.0625 + 0.03125 \\
&= 26.34375
\end{aligned}
$$

With this understanding of binary fractions, we can convert parts of the Eye of Horus to binary numbers and get the link we were looking for.

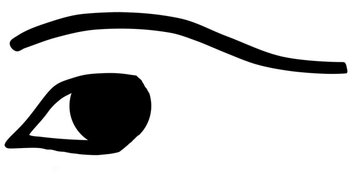


When we convert the parts to binary numbers, we get

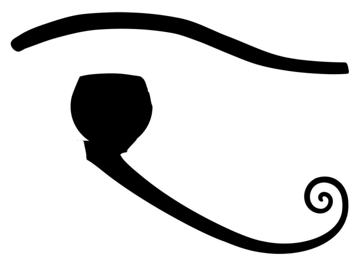
- the left half of the eye is 1/2, or 0.1 in binary numbers
- the pupil is 1/4, or 0.01 in binary.
- the eyebrow is 1/8, or 0.01 in binary.
- the right half of the eye is 1/16, or 0.001 in binary
- the lower parts are 1/32 and 1/64, or 0.0001 in binary.

This means that the hieroglyph that represents the entire Eye of Horus has a binary value of 0.11111. It is less than 1, because even after putting the eye back together, there still was a part that the god Thoth was unable to compose.

We can now convert the pictures that contain only parts of the Eye of Horus to binary numbers.

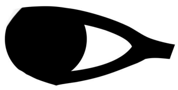


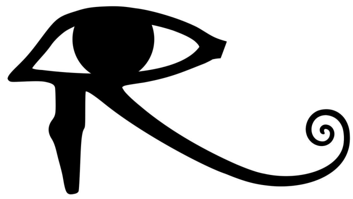
$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$
0.111



$\frac{1}{4} + \frac{1}{8} + \frac{1}{32} = \frac{13}{32}$
0.01101

Exercise Write the binary value of these hieroglyphs.





Exercise Can you draw the hieroglyphs that correspond to these binary values?

0.10011

0.1101

Final words

You can now give yourself a pat on the back, followed by a thumbs up, because you deserve it!

It took you years and years to get to know the decimal system and to be able to do operations on the numbers. Counting, adding, subtracting, multiplying and dividing numbers, none of that was knowledge you just woke up with one day. You learned to count and learned those operations one by one. Even if you learned them at an incredible speed, it still took you years to go from $1,\ 2,\ 3...$ to $16,042 \div 13 = 1,234$.

But now, in just a short period of time, you learned to do all those things with binary numbers and that is simply amazing! You should be proud of yourself!

To end this part, I give you two binairos to solve. They don't really have anything to do with binary numbers, but they are logical puzzles so you use the maths part of your brain to solve them, they only use 0's and 1's and I think they are fun, so I hope you enjoy solving them.

The rules:

1. Each box should contain either a zero or a one.

2. It is not allowed to have more than two equal numbers immediately next to or below each other. So every 00 has to be preceded and followed by a 1, and every 11 has to be preceded and followed by a 0. When you have 0?0, the ? can not be 0 because that would give three zeroes in a row.

3. Each row and each column should contain an equal amount of zeroes and ones.

4. Each row is unique and each column is unique. Thus, any row cannot be exactly equal to another row, and any column cannot be exactly equal to another column.

The first binairo can be completed without using the fourth rule, but you probably need it for the second one. If you have some columns or rows with just a few missing numbers and you have no idea what else you can do, check if you can find another column or row that is already complete and that has all the same numbers as the one that is almost completed.

		1		1							
				0		1					1
0										0	1
		1		1							
0		1					1			1	
				0			1				
1					1			0			1
		0			1				0		
		0							0		
							0				
		0		1			0			0	
					0	0				0	

			1	0		1	1			1	1
	0							1			1
	1				1			1	1		
							1			0	1
0		0									
	0	0				1		0			
	1					0	0				
		1								0	0
					1		0				0
0			0				0				
1	1			1	0				1		
		0		1		1	0				

In conclusion:

Here is our list of top 10 binary numbers:

- 1
- 0

1.3 Solutions to the exercises

1.3.1 Roman numerals

1. This is false: V, L and D are not allowed to repeat.
2. $89 - 72 = 15$, so there are only XV tiles left for the mosaic.
3. $56 - 24 = 32$, so he can pay the seamstress $XXXII$ gold coins.
4. $56 - 42 = 14$, so there are XIV Roman villas left to be found.
5. $11 \times 5 \times 6 = 330$, so they can paint $CCCXXX$ meters in that time.
6. $15 + 24 = 39$, so the sum of their ages is $XXXIX$ years.
7. Aside from X, the numerals I, C and M are also allowed to repeat.
8. 1984
9. MMX for 2010, $MMXI$ for 2011, $MMXII$ for 2012, $MMXIII$ for 2013, $MMXIV$ for 2014, $MMXV$ for 2015 and so on.
10. (a) 3 =III (b) 8 =VIII (c) 16 =XVI (d) 25 =XXV (e) 36 =XXXVI (f) 81 =LXXXI (g) 140 =CXL (h) 275 =CCLXXV (i) 567 =DLXVII (j) 789 =DCCLXXXIX (k) 987 =CMLXXXVII (l) $1,111$ =MCXI (m) $1,234$ =MCCXXXIV (n) $2,345$ =MMCCCXLV (o) $3,456$ =MMMCDLVI
11. (a) XV=15 (b) XLIX=49 (c) XXVIII=28 (d) LXXIX=79 (e) MCDXLIV=1,444 (f) DLIII=553 (g) MDCCCX=1,810 (h) LXXVII=77 (i) MMMCDXXI=3,421 (j) MCMLXIV=1,964 (k) DLV=555 (l) MMCDXLVIII=2,448 (m) MDCXCIX=1,699 (n) MMMCDLXXIV=3,474 (o) MMCMLXXXVII=2,987
12. The building was built in 1911, so in 2023 it is 112 years old.
13. $MMMCMXCIX$, also known as 3999. As there is no symbol for $5,000$, it is also impossible to write $4,000$, as this would be written in the same way as $IV = 4$, $XL = 40$ and $CD = 400$.
14.

(a) IX < XI
(b) LVII > XLVIII
(c) XCLXXVIII < CXLV
(d) MMCMXLII < MMMCXV
(e) MMMCDLXXXII < MMMDCX
(f) CDLXXIX > CDXXXIV
(g) D > CCCXCVIII
(h) DCCCLXXXVIII > DCCXCLXXX
(i) MCMCCCXXXIII < MMCXCXI
(j) DCCLXVII > DCCLXVI

15. Flora needs III more animals to be able to give II to each of her friends.

16. (a) $53 = LIII$
(b) $149 = CXLIX$
(c) $1,292 = MCCXCII$
(d) $751 = DCCLI$
(e) $2,376 = MMCCCLXXVI$
(f) $1,724 = MDCCXXIV$
(g) $98 = XCVIII$
(h) $2942 = MMCMXLII$

17. (a) $MCMXLIV = 1944$
(b) $MCMXCIII = 1993$
(c) $CMXCIX = 999$
(d) $CCCXLIV = 344$
(e) $MCDLII = 1452$
(f) $CLXV = 165$
(g) $MDCCIV = 1704$
(h) $DLXXVIII = 578$

18. (a) LI
(b) CXL
(c) CLXXIV
(d) CCCXXXII
(e) MCCXCVII
(f) MMCX
(g) MMMCLI
(h) MMMCCCXXXIII
(i) MV
(j) MMDCXXX
(k) CXXXIII
(l) CCLXXVIII
(m) CMXCVIII
(n) CD
(o) XCV
(p) MCDLXVIII
(q) DCCXXVI
(r) MCCXL
(s) MMXII
(t) MDLXVII

19. (a) CCCXXXIII
(b) CDXCII
(c) DXXXIX
(d) MMMDCCCXLVI
(e) MMCX
(f) MCCCXLII
(g) DCLII
(h) MCCXXXII
(i) CCCXXXII
(j) CXI

20.

VIII	III	IV
I	V	IX
VI	VII	II

X	III	VIII
V	VII	IX
VI	XI	VI

IV	VIII	III
IV	V	VI
VII	II	VI

21. (a) $1 + 1 = 2$ or $I + I = II$

(b) $2 + 2 = 4$ or $II + II = IV$

(c) $4 + 5 = 9$ or $IV + V = IX$

22. (a) Check out this innovative string encoding I've been developing! It's virtuacy perfect! ... Hang on, what's a *virtuacy*?

(b) It was meant to be the word virtually, but $L + L = C$.

(c) 1 a1000 50o6ng th1s.

23.

Letter	*A*	*B*	*C*	*D*	*E*	*F*
Roman	III	MCMIX	MDCL	DCVI	CCLXIX	CMXVI
Arabic	3	1909	1650	606	269	916

Letter	*G*	*H*	*I*	*J*	*K*	*L*
Roman	DXCV	MCMXVIII	VII	XLII	XXXVI	MMCCXXII
Arabic	595	1918	8	42	36	2222

Letter	*M*	*N*	*O*	*P*	*Q*	*R*
Roman	MDLVII	MMCCCXLII	DCCI	CCCXXVII	MMMDCCXCIX	DXII
Arabic	1557	2342	521	327	3799	512

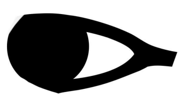
Letter	*S*	*T*	*U*	*V*	*W*	*X*
Roman	IX	CDXLVIII	MCDLI	DCCCVI	MMMCCCXL	MMCMLXVI
Arabic	9	448	1451	806	3340	2966

Letter	*Y*	*Z*
Roman	MDV	MCMXCIX
Arabic	1505	1999

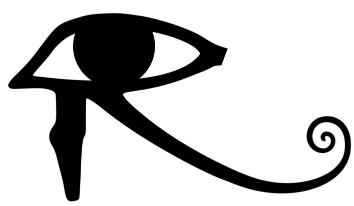
It would ruin the surprise if I put the decoded message here, but I know you can figure it out with the deciphering code!

1.3.2 Binary numbers

Ancient Egypt



$\frac{5}{16} = \frac{1}{4} + \frac{1}{16}$



$\frac{55}{64} = \frac{1}{2} + \frac{1}{4} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64}$

China

- Water and Flame are joined to make the numbers 63 and 64.
- Earth and Wind make up the numbers 20 and 46.
- The number 1 is pure Heaven.
- The number 2 is pure Earth.

Pingala

- If you start from 25, by following the steps you should encounter the numbers 25, 13, 7, 4 and 2, which means that the binary equivalent is 00011.
- Starting with 38, you encounter 38, 19, 10, 5, 3 and 2, which leads to the Pingala binary number of 101001.
- You encounter 66, 33, 17, 9, 5, 3 and 2, so you get 1000001.
- The numbers you encounter are 2137, 1069, 535, 268, 134, 67, 34, 17, 9, 5, 3 and 2, so you get 000110100001.

Francis Bacon

You should be proud of yourself if you managed to decipher the message. If you peeked here in the hope to get some help because that amount of a's and b's is overwhelming, here's a tip for you: every letter is encoded in a string of exactly 5 letters, so it is more manageable if you place / symbols after every 5 letters. Ideally, you do this before you start searching for what those blobs of a's and b's stand for. Good luck!

Leibniz's binary code

Exercise Calculate these powers.

1. $6^3 = 216$
2. $4^3 = 64$
3. $5^4 = 625$
4. $11^3 = 1,331$
5. $3^5 = 243$
6. $2^4 = 14$
7. $2^6 = 64$
8. $10^2 = 100$
9. $10^3 = 1,000$
10. $10^4 = 10,000$
11. $25^1 = 25$
12. $123,456,789^0 = 1$

13. $39^2 = 1,521$

14. $987,654,321^1 = 987,654,321$

15. $93^0 = 1$

Exercise Complete the table - decimal numbers.

number	10^{11}	10^{10}	10^9	10^8	10^7	10^6	10^5	10^4	10^3	10^2	10^1	10^0
321,234,567,890	3	2	1	2	3	4	5	6	7	8	9	0
25,268								2	5	2	6	8
20,102,012					2	0	1	0	2	0	1	2
198,438							1	9	8	4	3	8

Exercise Can you work out what the next binary numbers will be?

- 12 = 1100
- 13 = 1101
- 14 = 1110
- 15 = 1111
- 16 = 10000
- 17 = 10001
- 18 = 10010
- 19 = 10011
- 20 = 10100
- 21 = 10101
- 22 = 10110
- 23 = 10111
- 24 = 11000
- 25 = 11001
- 26 = 11010
- 27 = 11011
- 28 = 11100
- 29 = 11101
- 30 = 11110
- 31 = 11111

Exercise Can you convert these numbers to binary by using this first method?

- If you follow the steps, you encounter 38, 19, 9, 4, 2 and 1, so your binary number is 100110.
- You start with 66, then get 33, 16, 8, 4, 2 and 1, so you get 1000010.
- You encounter 125, 62, 31, 15, 7, 3 and 1, so you get 1111101.
- You start with 525, then get 262, 131, 65, 32, 16, 8, 4, 2 and 1. Your binary number is 1000001101.
- You get 2137, 1068, 534, 267, 133, 66, 33, 16, 8, 4, 2 and 1, so you get 100001011001.

Exercise Powers of 2

- $2^0 = 1$
- $2^1 = 2$
- $2^2 = 4$
- $2^3 = 8$
- $2^4 = 16$
- $2^5 = 32$
- $2^6 = 64$
- $2^7 = 128$
- $2^8 = 256$

- $2^9 = 512$
- $2^{10} = 1,024$
- $2^{11} = 2,048$
- $2^{12} = 4,096$
- $2^{13} = 8,192$
- $2^{14} = 16,384$
- $2^{15} = 32,768$
- $2^{16} = 65,536$
- $2^{17} = 131,072$

Exercise Can you convert these numbers to binary by using this second method? To help you out a bit, you get the tables of powers of two that you need.

- $48 = 110000$

32	16	8	4	2	1
2^5	2^4	2^3	2^2	2^1	2^0
1	1	0	0	0	0

- $99 = 1100011$

64	32	16	8	4	2	1
2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	1	0	0	0	1	1

- $234 = 11101010$

128	64	32	16	8	4	2	1
2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	1	1	0	1	0	1	0

- $579 = 1001000011$

512	256	128	64	32	16	8	4	2	1
2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	0	0	1	0	0	0	0	1	1

- $2173 = 100001111101$

2048	1024	512	256	128	64	32	16	8	4	2	1
2^{11}	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	0	0	0	0	1	1	1	1	1	0	1

Exercise Convert these binary numbers to decimal numbers.

1. $101 = 5$
2. $1001 = 9$
3. $10101 = 21$
4. $11011 = 27$
5. $101000 = 40$
6. $110011 = 51$
7. $1111111 = 127$
8. $1000000 = 64$
9. $1000011 = 67$
10. $1100011 = 99$

11. 1101011 = 107
12. 100011000 = 280
13. 11101010 = 234
14. 101001101 = 333
15. 110111100 = 444
16. 1111101000 = 1000
17. 10001010111 = 1111
18. 10011111100 = 1276
19. 100010101110 = 2222
20. 1101111000 = 888

Exercise Adding two binary numbers.

1.
$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & 1 & 0 & 0 & 1 & 0 & 1 \\
+ & & 1 & 0 & 1 & 0 & 0 \\
\hline
 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}$$

2.
$$\begin{array}{ccccccc}
1 & 1 & 1 & 1 & & & \\
 & 1 & 0 & 0 & 1 & 0 & 1 \\
+ & & 1 & 1 & 1 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 1 & 0 & 1
\end{array}$$

3.
$$\begin{array}{cccccccccc}
1 & 1 & & 1 & 1 & & & & & \\
 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
+ & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
\hline
1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1
\end{array}$$

4.
$$\begin{array}{ccccccc}
1 & 1 & 1 & 1 & & & \\
 & 1 & 0 & 0 & 1 & 0 & 1 \\
+ & & 1 & 1 & 1 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}$$

5.
$$\begin{array}{ccccccccc}
1 & 1 & & & 1 & 1 & & 1 & \\
 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
+ & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
\hline
1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0
\end{array}$$

6.
$$\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
+ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{array}$$

7.
$$\begin{array}{cccccccccc}
1 & & 1 & 1 & 1 & 1 & & 1 & 1 & \\
 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
+ & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
\hline
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0
\end{array}$$

8.
$$\begin{array}{cccccccccc}
1 & & & & 1 & 1 & 1 & 1 & 1 & \\
 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
+ & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
\hline
1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0
\end{array}$$

9.
$$\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & & 1 & \\
 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
+ & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}$$

10.
$$\begin{array}{ccccccccc}
1 & 1 & 1 & & & & & & \\
 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
+ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\hline
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0
\end{array}$$

Exercise Can you add two binary numbers when they aren't aligned?

1. $110011 + 100010 = 1010101$
2. $1111 + 100111 = 110110$
3. $100000 + 101010 = 1001010$
4. $100011 + 101001 = 1001100$
5. $101111 + 10100 = 1000011$
6. $1111101 + 1010100 = 11010001$
7. $11010100 + 11000010 = 110010110$
8. $1110001 + 1001101 = 10111110$
9. $1010111 + 1100110 = 10111101$
10. $1110101 + 1100011 = 11011000$
11. $11000110 + 110001001 = 1001001111$
12. $101101011 + 100100001 = 1010001100$
13. $101010100 + 10000110 = 111011010$
14. $10111011 + 110000010 = 1000111101$
15. $10110111 + 111011 = 11110010$
16. $1111101 + 101101101 = 111101010$
17. $11101101 + 10110111 = 110100100$
18. $1100011 + 1110111 = 11011010$
19. $11011100 + 101110111 = 1001010011$
20. $101100111 + 1001111 = 110110110$

Exercise Can you add three numbers? As we can get 100 and 101 as the result of adding 4 or 5 digits, and the purpose of these corrections is to understand any mistake you might have made, there are two lines for digits that are carried over where this happens. The top line is for digits that come from 100, the bottom line is for digits that come from 10 or 11.

1.

$$\begin{array}{ccccccc}
1 & 1 & & & & 1 & \\
 & 1 & 1 & 0 & 0 & 0 & 1 \\
 & 1 & 0 & 0 & 1 & 0 & 0 \\
+ & & 1 & 1 & 0 & 0 & 1 \\
\hline
1 & 1 & 0 & 1 & 1 & 1 & 0
\end{array}$$

2.

$$\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
+ & & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1
\end{array}$$

3.

$$\begin{array}{cccccccc}
1 & & & & & & & \\
 & & 1 & 1 & 1 & & & \\
 & & 1 & 1 & 1 & 0 & 0 & 0 \\
 & & 1 & 0 & 1 & 1 & 0 & 1 \\
+ & & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1
\end{array}$$

4.

$$\begin{array}{ccccccccccc}
1 & & & & & & & & & & \\
 & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\
 & & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
 & & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
+ & & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0
\end{array}$$

5.

$$\begin{array}{ccccccccccc}
 & 1 & & & & 1 & & & & & \\
 & & & 1 & 1 & 1 & 1 & 1 & 1 & & \\
 & & & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
 & & & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
+ & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1
\end{array}$$

6.

$$\begin{array}{ccccccccccc}
 & 1 & 1 & 1 & 1 & & & & & & \\
 & & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
 & & & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
+ & & & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1
\end{array}$$

7.

$$\begin{array}{cccccccccccc}
 & 1 & & 1 & & 1 & & 1 & & & & \\
 & & & & & 1 & & & & 1 & 1 & \\
 & & & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
 & & & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
+ & & & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
\hline
 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}$$

8.

$$\begin{array}{ccccccccccccc}
 & 1 & & & 1 & & 1 & & & & & & \\
 & & & 1 & 1 & & & & 1 & 1 & 1 & 1 & \\
 & & & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
 & & & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
+ & & & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
\hline
 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0
\end{array}$$

9.

$$\begin{array}{cccccccccccccc}
 & 1 & & & & & 1 & & 1 & & & & & \\
 & & & 1 & 1 & 1 & 1 & & 1 & & 1 & 1 & & \\
 & & & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
+ & & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
\hline
 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1
\end{array}$$

10.

$$\begin{array}{cccccccccccc}
 & 1 & 1 & 1 & 1 & 1 & & & 1 & & & \\
 & & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
+ & & & & & & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1
\end{array}$$

Exercise Can you add even more numbers in one go? Where adding digits leads to a sum of 1000 or higher, I have added a third line for carrying digits over.

1.

$$\begin{array}{ccccccccccc}
 & 1 & & 1 & & 1 & & 1 & & & \\
 & & 1 & 1 & & 1 & 1 & 1 & & 1 & \\
 & & & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
 & & & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
 & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
+ & & & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
\hline
 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}$$

2.

$$\begin{array}{ccccccccccc}
 & 1 & & 1 & & 1 & & 1 & & & \\
 & & 1 & 1 & & & & 1 & & 1 & \\
 & & & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
 & & & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & & & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
 & & & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
+ & & & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{array}$$

3.

$$\begin{array}{cccccccccccc}
1 & & 1 & & 1 & 1 & & 1 & & & & \\
 & & 1 & 1 & 1 & & & 1 & & 1 & & \\
 & & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
 & & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
 & & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
+ & & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
\hline
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1
\end{array}$$

4.

$$\begin{array}{cccccccccccc}
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & \\
 & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
 & & & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
 & & & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
 & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
+ & & & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0
\end{array}$$

5.

$$\begin{array}{cccccccccccc}
1 & & & & & & & & & & & \\
 & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
 & & & 1 & 1 & & & & 1 & & & \\
 & & & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
 & & & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
 & & & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
 & & & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
 & & & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
+ & & & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1
\end{array}$$

6.

$$\begin{array}{ccccccccccccc}
1 & & & 1 & & & & & & & & & \\
 & & & 1 & & 1 & 1 & 1 & 1 & 1 & 1 & & \\
 & & & 1 & & & 1 & 1 & & & & & \\
 & & & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
 & & & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
 & & & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
 & & & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
 & & & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
+ & & & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
\hline
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0
\end{array}$$

Exercise Subtracting binary numbers.

1.

$$\begin{array}{cccccc}
0 & 10 & & & & \\
\cancel{1} & \cancel{0} & 0 & 1 & 0 & 1 \\
- & 1 & 0 & 1 & 0 & 0 \\
\hline
 & 1 & 0 & 0 & 0 & 1
\end{array}$$

2.

$$\begin{array}{cccccc}
0 & 10 & & & & \\
\cancel{1} & \cancel{0} & 1 & 0 & 1 & 1 \\
- & 1 & 0 & 0 & 1 & 1 \\
\hline
 & 1 & 1 & 0 & 0 & 0
\end{array}$$

3.

$$\begin{array}{cccccc}
0 & 10 & & 1 & 1 & 10 \\
\cancel{1} & \cancel{0} & 1 & \cancel{1} & \cancel{0} & \cancel{0} \\
- & 1 & 1 & 0 & 1 & 1 \\
\hline
 & 1 & 0 & 0 & 0 & 1
\end{array}$$

4.

$$\begin{array}{ccccccc}
 & 0 & 1 & 10 & & & \\
 & \cancel{1} & \cancel{0} & \cancel{0} & 1 & 0 & 1 \\
- & & 1 & 1 & 1 & 0 & 0 \\
\hline
 & & & 1 & 0 & 0 & 1
\end{array}$$

5.

$$\begin{array}{ccccccccc}
 & & 0 & 10 & 0 & 10 & & & \\
 & 1 & \cancel{1} & \cancel{0} & \cancel{1} & \cancel{0} & 1 & 0 & 1 \\
- & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & & & 1 & 0 & 1 & 1 & 0 & 1
\end{array}$$

6.

$$\begin{array}{cccccccc}
0 & 10 & 0 & 10 & 0 & 10 & & \\
\cancel{1} & \cancel{0} & \cancel{1} & \cancel{0} & \cancel{1} & \cancel{0} & 1 & 0 \\
- & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
\hline
 & 1 & 0 & 1 & 0 & 1 & 1 & 1
\end{array}$$

7.

$$\begin{array}{ccccccccc}
0 & 10 & 0 & 10 & & & & & \\
\cancel{1} & \cancel{0} & \cancel{1} & \cancel{0} & 1 & 0 & 1 & 0 & 0 \\
- & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
\hline
 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0
\end{array}$$

8.

$$\begin{array}{ccccccccc}
 & & 0 & 1 & 1 & 10 & & & \\
 & 1 & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} & 1 & 0 & 1 \\
- & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
\hline
 & & & 1 & 1 & 1 & 0 & 0 & 0
\end{array}$$

9.

$$\begin{array}{ccccccccc}
 & & & 0 & 1 & 1 & 10 & & \\
 & 1 & 1 & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} & 1 & 1 \\
- & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
\hline
 & & 1 & 0 & 0 & 0 & 1 & 1 & 0
\end{array}$$

10.

$$\begin{array}{cccccccccc}
 & & 0 & 10 & & 0 & 10 & & & \\
 & 1 & \cancel{1} & \cancel{0} & 1 & \cancel{1} & \cancel{0} & 0 & 1 & 1 \\
- & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
\hline
 & & & 1 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}$$

Exercise Subtracting binary numbers, where the borrowing part of the calculations gets a bit trickier.

1.

$$\begin{array}{cccccc}
 & & & 10 & & \\
0 & 1 & 1 & \cancel{0} & 1 & 10 \\
\cancel{1} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} & \cancel{0} \\
- & 1 & 0 & 1 & 1 & 1 \\
\hline
 & & 1 & 1 & 0 & 1
\end{array}$$

2.

$$\begin{array}{ccccccc}
 & & & 10 & & & \\
 & & 0 & \cancel{0} & 1 & 10 & \\
 & 1 & \cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} & 1 \\
- & 1 & 0 & 1 & 1 & 1 & 0 \\
\hline
 & & & 1 & 0 & 1 & 1
\end{array}$$

3.

$$\begin{array}{cccccccc}
 & & & & 10 & & 10 & \\
 & & & 0 & \cancel{0} & 1 & \cancel{0} & 10 \\
 & 1 & 1 & \cancel{1} & \cancel{1} & \cancel{0} & \cancel{1} & \cancel{0} \\
- & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
\hline
 & & & & 1 & 0 & 1 & 1
\end{array}$$

4.

$$\begin{array}{cccccccc}
 & 10 & & & 10 & & & \\
0 & \cancel{0} & 1 & 1 & \cancel{0} & 1 & 10 & \\
\cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} & \cancel{0} & 0 \\
- & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
\hline
 & 1 & 1 & 0 & 1 & 0 & 1 & 0
\end{array}$$

5.

$$\begin{array}{ccccccccc}
 & & & 10 & & & & & \\
0 & 1 & 1 & \cancel{0} & 10 & & 0 & 1 & 10 \\
\cancel{1} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} & 1 & \cancel{1} & \cancel{0} & \cancel{0} \\
- & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
\hline
 & & 1 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}$$

6.

$$\begin{array}{cccccccccc}
 & & & & & 10 & & & & \\
 & & & & 0 & \cancel{0} & 10 & & 0 & 10 \\
 & 1 & 1 & 0 & \cancel{1} & \cancel{1} & \cancel{0} & 1 & \cancel{1} & \cancel{0} \\
- & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
\hline
 & & & & & 1 & 1 & 1 & 0 & 1
\end{array}$$

7.

$$\begin{array}{cccccccccc}
 & & & & & & & & 10 & \\
 & & 0 & 1 & 1 & 1 & 1 & 1 & \cancel{0} & 10 \\
 & 1 & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} \\
- & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
\hline
 & & & & 1 & 0 & 0 & 1 & 1 & 1
\end{array}$$

8.

$$\begin{array}{ccccccccc}
 & 10 & 10 & & & & & & \\
0 & \cancel{0} & \cancel{0} & 1 & 1 & 10 & & & \\
\cancel{1} & \cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} & 1 & 0 & 1 \\
- & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
\hline
 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0
\end{array}$$

9.

$$\begin{array}{cccccccccc}
 & 10 & 10 & 10 & & & & & & \\
0 & \cancel{0} & \cancel{0} & \cancel{0} & 1 & 1 & 1 & 1 & 1 & 10 \\
\cancel{1} & \cancel{1} & \cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\
- & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
\hline
 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{array}$$

10.

$$\begin{array}{cccccccccc}
 & & & & & & 10 & & 10 & \\
 & & 0 & 10 & & 0 & \cancel{0} & 1 & \cancel{0} & 10 \\
 & 1 & \cancel{1} & \cancel{0} & 1 & \cancel{1} & \cancel{1} & \cancel{0} & \cancel{1} & \cancel{0} \\
- & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
\hline
 & & & 1 & 0 & 0 & 1 & 1 & 1 & 1
\end{array}$$

Exercise Can you subtract two binary numbers when they aren't aligned?

1. $10001 - 10 = 1111$
2. $11111111 - 1001101 = 10110010$
3. $10000 - 1011 = 101$
4. $110011 - 100011 = 10000$
5. $111001 - 101 = 110100$
6. $1001000 - 111 = 1000001$
7. $101010 - 11001 = 10001$
8. $11001 - 1111 = 1010$
9. $100110 - 1010 = 11010$
10. $100111 - 1101 = 11010$

Exercise Calculate these binary multiplications.

1. $10111 \times 100 = 1011100$
2. $1011 \times 10 = 10110$
3. $11011 \times 1000 = 1101100$
4. $111 \times 10000 = 1110000$
5. $1011 \times 10 = 10110$
6. $11111 \times 1000 = 11111000$
7. $10101 \times 100 = 1010100$
8. $100111 \times 10 = 1001110$
9. $11001 \times 100 = 1100100$
10. $1001 \times 1000 = 1001000$
11. $10110 \times 100000 = 1011000000$
12. $100000 \times 100 = 10000000$
13. $100101 \times 10 = 1001010$
14. $1001001 \times 100 = 100100100$
15. $110001 \times 1000 = 110001000$
16. $10111 \times 100 = 1011100$
17. $11010 \times 10000 = 110100000$
18. $10010 \times 10 = 100100$
19. $110101 \times 1000 = 110101000$
20. $1101100 \times 100 = 110110000$

Exercise Multiply these binary numbers.

1.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & 0 & 0 & 1 & 0 & 1 \\
 & & & & & \times & 1 & 0 & 1 & 0 & 0 \\
\hline
 & & & & & 1 & & & & & \\
 & & & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
+ & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0
\end{array}$$

2.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & 0 & 1 & 0 & 1 & 1 \\
 & & & & & \times & 1 & 0 & 0 & 1 & 1 \\
\hline
 & & & 1 & 1 & 1 & 1 & 1 & 1 & & \\
 & & & & & 1 & 0 & 1 & 0 & 1 & 1 \\
 & & & & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
+ & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
\hline
 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1
\end{array}$$

3.

$$\begin{array}{ccccccccc}
 & & & 1 & 0 & 1 & 1 & 0 & 1 \\
 & & & \times & & & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & & & \\
 & & & 1 & 0 & 1 & 1 & 0 & 1 \\
 & & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
+ & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}$$

4.

$$\begin{array}{ccccccccccc}
 & & & & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
 & & & & \times & & & 1 & 0 & 1 & 0 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\
 & & & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
+ & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0
\end{array}$$

5.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & 0 & 0 & 1 & 0 & 1 \\
 & & & & & \times & 1 & 1 & 1 & 0 & 0 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\
 & & & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
 & & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
+ & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{array}$$

6.

$$\begin{array}{cccccccccc}
 & & & & 1 & 1 & 0 & 1 & 1 & 1 \\
 & & & & \times & & 1 & 1 & 0 & 1 \\
\hline
 & & & 1 & & & & & & \\
1 & 1 & 1 & 1 & & 1 & 1 & & & \\
 & & & & 1 & 1 & 0 & 1 & 1 & 1 \\
 & & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
+ & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
\hline
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1
\end{array}$$

7.

$$
\begin{array}{cccccccccccccc}
&&&&&1&1&0&1&1&1&0&1&1\\
&&&&&\times&&&&1&1&0&0&1\\
\hline
&&&&1&&&1&&&&&&\\
1&1&1&1&1&&1&1&&1&&&&\\
&&&&&1&1&0&1&1&1&0&1&1\\
&&1&1&0&1&1&1&0&1&1&0&0&0\\
+&1&1&0&1&1&1&0&1&1&0&0&0&0\\
\hline
1&0&1&0&1&1&0&1&0&0&0&0&1&1
\end{array}
$$

8.

$$
\begin{array}{ccccccccccc}
&&&&&1&0&1&1&0&0\\
&&&&&\times&1&1&0&1&1\\
\hline
&&1&&&&&&&&\\
1&1&&&1&1&1&&&&\\
&&&&&1&0&1&1&0&0\\
&&&&1&0&1&1&0&0&0\\
&&1&0&1&1&0&0&0&0&0\\
+&1&0&1&1&0&0&0&0&0&0\\
\hline
1&0&0&1&0&1&0&0&1&0&0
\end{array}
$$

9.

$$
\begin{array}{ccccccccccccc}
&&&&&&1&1&1&0&1&1&1\\
&&&&&\times&&1&1&1&0&1&1\\
\hline
&1&&1&1&1&1&&&&&&\\
1&1&&1&&&&&1&1&1&&\\
&&&&&&1&1&1&0&1&1&1\\
&&&&&1&1&1&0&1&1&1&0\\
&&&1&1&1&0&1&1&1&0&0&0\\
&&1&1&1&0&1&1&1&0&0&0&0\\
+&1&1&1&0&1&1&1&0&0&0&0&0\\
\hline
1&1&0&1&1&0&1&1&0&1&1&0&1
\end{array}
$$

10.

$$
\begin{array}{cccccccccccc}
&&&&&1&0&1&1&0&1&0\\
&&&&\times&&&1&1&1&1&1\\
\hline
&1&&1&&1&&&&&&\\
1&&&1&&1&&1&&&&\\
&&&&&1&0&1&1&0&1&0\\
&&&&1&0&1&1&0&1&0&0\\
&&&1&0&1&1&0&1&0&0&0\\
&&1&0&1&1&0&1&0&0&0&0\\
+&1&0&1&1&0&1&0&0&0&0&0\\
\hline
1&0&1&0&1&1&1&0&0&1&1&0
\end{array}
$$

11.

$$
\begin{array}{cccccccccccccc}
&&&&&1&0&0&1&1&0&1&1&1\\
&&&&&\times&&&&1&0&0&1&1\\
\hline
&&&&&&1&&&&&&&\\
&&&1&1&1&1&&1&1&1&1&&\\
&&&&&1&0&0&1&1&0&1&1&1\\
&&&&1&0&0&1&1&0&1&1&1&0\\
+&1&0&0&1&1&0&1&1&1&0&0&0&0\\
\hline
&1&0&1&1&1&0&0&0&1&0&1&0&1
\end{array}
$$

12.

$$
\begin{array}{cccccccccccccc}
&&&&&&1&1&0&1&1&1&0&1\\
&&&&&&\times&&1&1&1&0&0&1\\
\hline
1&&1&&1&1&&&&&&&&\\
&&&1&&1&&&1&1&1&&&\\
&&&&&&1&1&0&1&1&1&0&1\\
&&&1&1&0&1&1&1&0&1&0&0&0\\
&&1&1&0&1&1&1&0&1&0&0&0&0\\
+&1&1&0&1&1&1&0&1&0&0&0&0&0\\
\hline
1&1&0&0&0&1&0&0&1&1&0&1&0&1
\end{array}
$$

13.

$$
\begin{array}{ccccccccccccc}
&&&&&1&0&1&0&1&0&1&1\\
&&&&&\times&&&1&1&1&1&1\\
\hline
&1&&1&&1&&1&&&&&\\
1&&&1&&1&&1&&1&1&&\\
&&&&&1&0&1&0&1&0&1&1\\
&&&&1&0&1&0&1&0&1&1&0\\
&&&1&0&1&0&1&0&1&1&0&0\\
&&1&0&1&0&1&0&1&1&0&0&0\\
+&1&0&1&0&1&0&1&1&0&0&0&0\\
\hline
1&0&1&0&0&1&0&1&1&0&1&0&1
\end{array}
$$

14.

$$
\begin{array}{ccccccccccccc}
&&&&&&1&1&0&1&0&1&1\\
&&&&&\times&&1&1&0&1&1&1\\
\hline
&&&1&&1&&&&&&&\\
1&1&1&1&&1&&1&1&1&1&&\\
&&&&&&1&1&0&1&0&1&1\\
&&&&&1&1&0&1&0&1&1&0\\
&&&&1&1&0&1&0&1&1&0&0\\
&&1&1&0&1&0&1&1&0&0&0&0\\
+&1&1&0&1&0&1&1&0&0&0&0&0\\
\hline
1&0&1&1&0&1&1&1&1&1&1&0&1
\end{array}
$$

Exercise These multiplications are a bit harder, but I know you can do it.

1.

$$
\begin{array}{cccccccccccccc}
 & & & & & & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
 & & & & & & & \times & 1 & 1 & 1 & 1 & 0 & 1 \\
\hline
 & & 1 & 1 & 1 & 1 & 1 & 1 & & & & & & \\
1 & 1 & 1 & & & 1 & & & & 1 & 1 & & & \\
 & & & & & & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
 & & & & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
 & & & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
 & & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
+ & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1
\end{array}
$$

2.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & & & & \times & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
\hline
 & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\
1 & 1 & 1 & & & 1 & 1 & 1 & 1 & & & & 1 & & \\
 & & & & & & & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & & & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
 & & & & & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
 & & & & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
 & & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
+ & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}
$$

3.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & & & \times & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
 & & & & 1 & & & & & & & & \\
 & 1 & 1 & 1 & 1 & & 1 & 1 & 1 & & & & \\
1 & 1 & & 1 & 1 & 1 & & 1 & & & 1 & & \\
 & & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
 & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
 & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
+ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

4.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & & \times & 1 & 1 & 1 & 1 & 1 \\
\hline
 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\
1 & 1 & & 1 & 1 & 1 & & & 1 & & \\
 & & & & & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
 & & & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
 & & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
+ & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

5.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
 & & & & & & \times & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
 & 1 & 1 & 1 & 1 & 1 & 1 & & 1 & & & & \\
1 & 1 & & & 1 & 1 & 1 & 1 & 1 & & 1 & & \\
 & & & & & & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
 & & & & & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
 & & & & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
 & & & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
 & & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
+ & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{array}
$$

6.

$$
\begin{array}{cccccccccccccc}
 & & & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & & & \times & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
 & & & & 1 & 1 & & & & & & & & \\
 & 1 & 1 & 1 & 1 & & & 1 & 1 & 1 & & & & \\
1 & 1 & & 1 & 1 & 1 & & & 1 & & & 1 & & \\
 & & & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
 & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
 & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
 & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
+ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

Exercise Patterns in binary numbers

1. Complete the table.

2. Underline all even numbers with a blue colour pencil. Even numbers are printed bold (which means thicker)
3. Underline all multiples of 4 with a red colour pencil. Multiples of 4 are printed italic (which means not quite straight, but leaning to the right)
4. Underline all multiples of 8 with a yellow colour pencil. Multiples of 8 are underlined.
5. Underline all multiples of 16 with a green colour pencil. Multiples of 16 are a bit bigger than the rest.
6. Study the numbers you have underlined.
   (a) What do all multiples of 2 have in common? They end with 0.
   (b) What do all multiples of 4 have in common? They end with 00.
   (c) What do all multiples of 8 have in common? They end with 000.
   (d) What do all multiples of 16 have in common? They end with 0000.

1	1001	10001	11001
10	**1010**	**10010**	**11010**
11	1011	10011	11011
100	***1100***	***10100***	***11100***
101	1101	10101	11101
110	**1110**	**10110**	**11110**
111	1111	10111	11111
1000	***10000***	***11000***	***100000***

Exercise Divide these binary numbers by powers of 2. If there is a remainder, write it down as well.

1. $110111010 \div 10 = 11011101$
2. $110111010 \div 100 = 1101110$ r 10
3. $110111010 \div 1000 = 110111$ r 10
4. $110111010 \div 10000 = 11011$ r 1010
5. $11001000 \div 10 = 1100100$
6. $11001000 \div 100 = 110010$
7. $11001000 \div 1000 = 11001$
8. $11001000 \div 10000 = 1100$ r 1000
9. $10011001 \div 10 = 1001100$ r 1
10. $10011001 \div 100 = 100110$ r 1
11. $10111011 \div 1000 = 10111$ r 11
12. $1101100 \div 100 = 11011$
13. $110110 \div 100 = 1101$ r 10
14. $1001100 \div 10 = 100110$
15. $101100 \div 1000 = 101$ r 100
16. $100100 \div 100 = 1001$
17. $10110 \div 10000 = 1$ r 110
18. $11000 \div 100 = 110$
19. $1101110 \div 1000 = 1101$ r 110
20. $11010110 \div 10 = 1101011$

Exercise Calculate the result of these divisions, don't forget to write down the remainder.

1. $110111001 \div 101 = 1011000$ r 1
2. $1000101011 \div 100 = 10001010$ r 11
3. $11100111101 \div 110 = 100110100$ r 101
4. $10111001 \div 111 = 11010$ r 11
5. $11111111 \div 100 = 111111$ r 11
6. $11111111 \div 101 = 110011$
7. $11111111 \div 110 = 101010$ r 11
8. $11111111 \div 111 = 100100$ r 11
9. $1011011101 \div 1010 = 1001001$ r 11
10. $110010010001 \div 1100 = 1000011$ r 1
11. $10010111011 \div 1101 = 10110101$ r 10
12. $1011100111 \div 1011 = 100001$ r 110
13. $110110110110 \div 1111 = 11101010$
14. $100101101 \div 11101 = 1010$ r 1011
15. $110011010100 \div 11010 = 1111110$ r 1000
16. $1101001101001 \div 11011 = 11111010$ r 1011

To check where you made a mistake, you can compare your work to the long divisions below.

1.

```
         1011000
101  | 110111001
       -101
          111
        -101
           101
         -101
            0001
```

2.

```
         10001010
100  | 1000101011
       -100
          00101
          -100
              101
            -100
                11
```

3.

```
         100110100
110  | 11100111101
       -110
          1001
          -110
             111
           -110
              111
            -110
                101
```

4.

```
         11010
111  | 10111001
        -111
         1001
        -111
           1000
          -111
              11
```

5.

```
         111111
100  | 11111111
       -100
          111
        -100
           111
         -100
            111
          -100
             111
           -100
              111
            -100
                11
```

6.

```
         110011
101  | 11111111
       -101
          101
        -101
           0111
           -101
              101
             -101
                0
```

7.

```
         101010
110  | 11111111
       -110
          111
        -110
           111
          -110
              11
```

8.

```
         110011
111  | 11111111
       -111
          0111
          -111
              011
```

9.

```
          1001001
1010  | 1011011101
        -1010
            1011
          -1010
              1101
            -1010
                 11
```

10.

$$\begin{array}{l|l}
 & \overline{1000011} \\
1100 & 110010010001 \\
 & \underline{-1100} \\
 & \quad 010010 \\
 & \quad \underline{-1100} \\
 & \qquad 1100 \\
 & \qquad \underline{-1100} \\
 & \qquad\quad 001
\end{array}$$

11.

$$\begin{array}{l|l}
 & \overline{10110101} \\
1101 & 10010111011 \\
 & \underline{-1101} \\
 & \quad 10111 \\
 & \quad \underline{-1101} \\
 & \quad\; 10101 \\
 & \quad\; \underline{-1101} \\
 & \qquad 10000 \\
 & \qquad \underline{-1101} \\
 & \qquad\quad 1111 \\
 & \qquad\quad \underline{-1101} \\
 & \qquad\qquad 10
\end{array}$$

12.

$$\begin{array}{l|l}
 & \overline{1000011} \\
1011 & 1011100111 \\
 & \underline{-1011} \\
 & \quad 010011 \\
 & \quad \underline{-1011} \\
 & \qquad 10001 \\
 & \qquad \underline{-1011} \\
 & \qquad\quad 110
\end{array}$$

13.

$$\begin{array}{l|l}
 & \overline{11101010} \\
1111 & 110110110110 \\
 & \underline{-1111} \\
 & \; 11000 \\
 & \; \underline{-1111} \\
 & \quad 10011 \\
 & \quad \underline{-1111} \\
 & \qquad 10010 \\
 & \qquad \underline{-1111} \\
 & \qquad\quad 1111 \\
 & \qquad\quad \underline{-1111} \\
 & \qquad\qquad 00
\end{array}$$

14.

$$\begin{array}{l|l}
 & \overline{1010} \\
11101 & 100101101 \\
 & \underline{-11101} \\
 & \quad 100010 \\
 & \quad \underline{-11101} \\
 & \qquad 1011
\end{array}$$

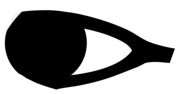
15.

$$\begin{array}{l|l}
 & \overline{1111110} \\
11010 & 110011010100 \\
 & \underline{-11010} \\
 & \; 110010 \\
 & \; \underline{-11010} \\
 & \quad 110001 \\
 & \quad \underline{-11010} \\
 & \quad\; 101110 \\
 & \quad\; \underline{-11010} \\
 & \qquad 101001 \\
 & \qquad \underline{-11010} \\
 & \qquad\quad 11110 \\
 & \qquad\quad \underline{-11010} \\
 & \qquad\qquad 1000
\end{array}$$

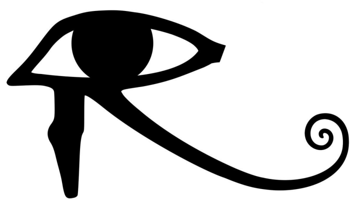
16.

```
               11111010
11011  |  1101001101001
          -11011
            110011
           -11011
              110001
             -11011
                101100
               -11011
                  100011
                 -11011
                      100000
                     -11011
                          1011
```

Exercise Write the binary value of these hieroglyphs.



$\frac{5}{16} = \frac{1}{4} + \frac{1}{16}$
0.0101

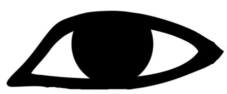


$\frac{55}{64} = \frac{1}{2} + \frac{1}{4} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64}$
0.110111

Exercise Can you draw the hieroglyphs that correspond to these binary values?

0.10011

0.1101



Binairo

1	0	1	0	1	1	0	0	1	0	1	0
0	0	1	1	0	0	1	0	1	0	1	1
0	1	0	1	0	0	1	1	0	1	0	1
1	0	1	0	1	1	0	0	1	1	0	0
0	0	1	0	1	0	0	1	1	0	1	1
0	1	0	1	0	0	1	1	0	1	1	0
1	0	1	0	0	1	1	0	0	1	0	1
0	1	0	0	1	1	0	1	1	0	0	1
1	1	0	1	0	0	1	1	0	0	1	0
0	0	1	1	0	1	1	0	0	1	1	0
1	1	0	0	1	1	0	0	1	0	0	1
1	1	0	1	1	0	0	1	0	1	0	0

0	0	1	1	0	0	1	1	0	0	1	1
0	0	1	0	0	1	0	1	1	0	1	1
1	1	0	0	1	1	0	0	1	1	0	0
0	0	1	1	0	0	1	1	0	1	0	1
0	1	0	0	1	0	0	1	1	0	1	1
1	0	0	1	0	1	1	0	0	1	1	0
0	1	1	0	1	1	0	0	1	0	0	1
1	0	1	0	1	0	1	1	0	1	0	0
1	1	0	1	0	1	0	0	1	0	1	0
0	0	1	0	0	1	1	0	1	0	1	1
1	1	0	1	1	0	0	1	0	1	0	0
1	1	0	1	1	0	1	0	0	1	0	0