Ridge regression

STAT 4710

October 13, 2022

Where we are

- **Unit 1:** R for data mining
- **Unit 2:** Prediction fundamentals
- **Unit 3:** Regression-based methods
- **Unit 4:** Tree-based methods
- **Unit 5:** Deep learning

- **Lecture 1:** Linear and logistic regression
- **Lecture 2:** Regression in high dimensions
- $\boxed{\textbf{Lecture 3: } \text{Ridge regression}}$
- **Lecture 4:** Lasso regression
- **Lecture 5:** Unit review and quiz in class

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Ridge regression is defined even if $p > n$, as long as $\lambda > 0$.

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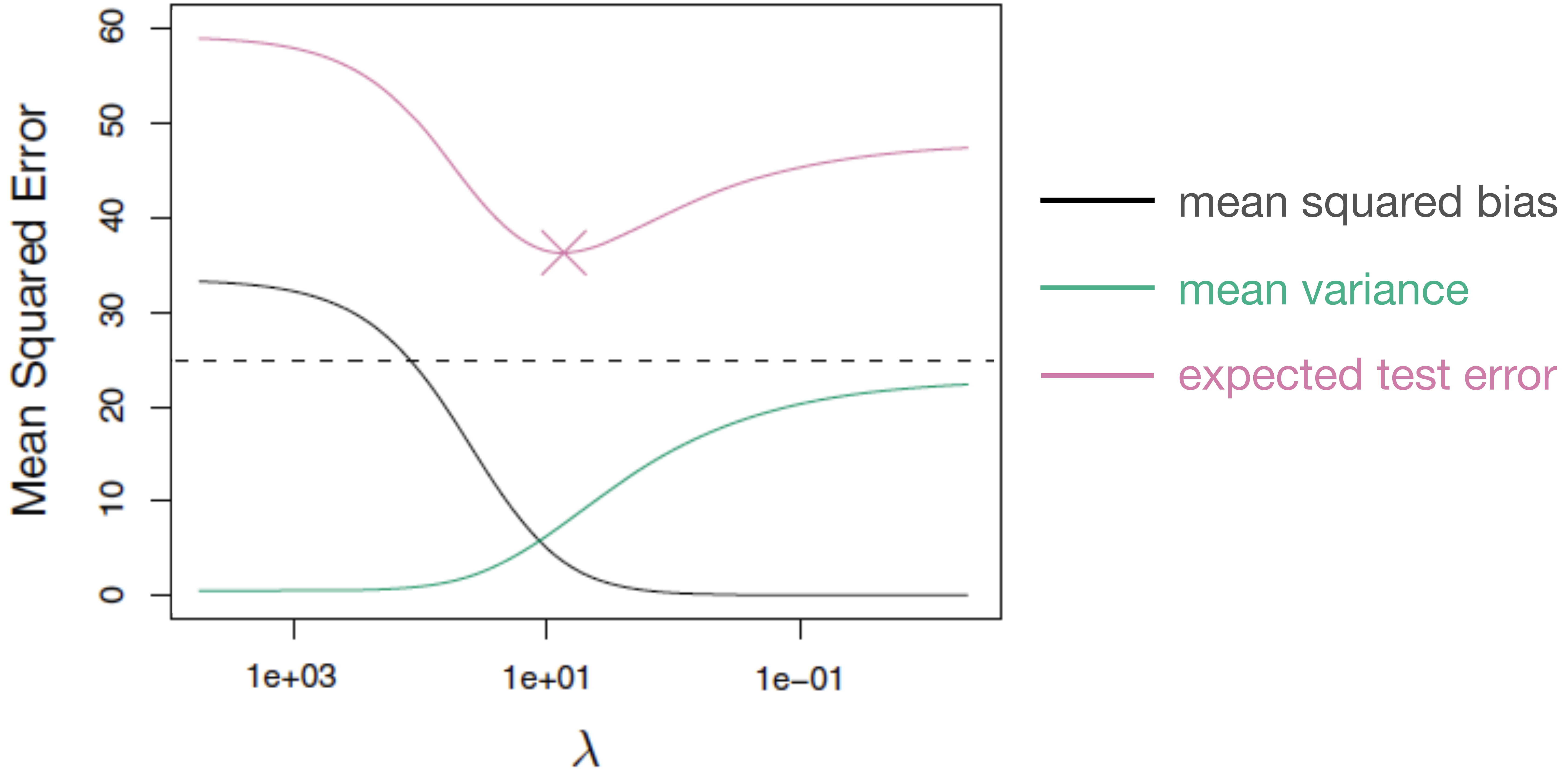
Mathematical expression for the df of ridge regression is complicated; we skip it.

The bias-variance tradeoff for ridge regression



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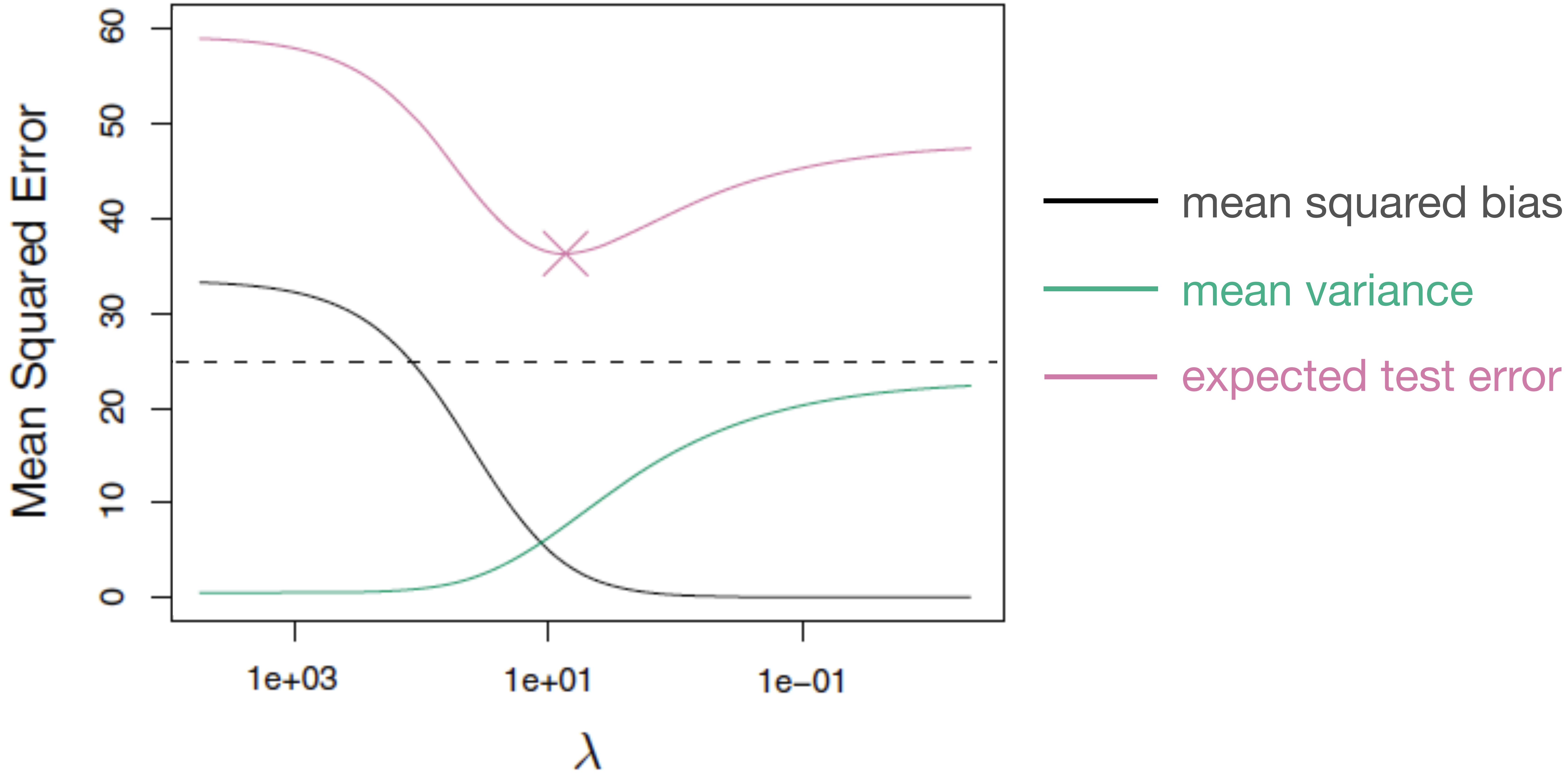

In practice, λ is chosen by cross-validation.

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To put features on the same scale, center each feature and divide by its std. dev.:

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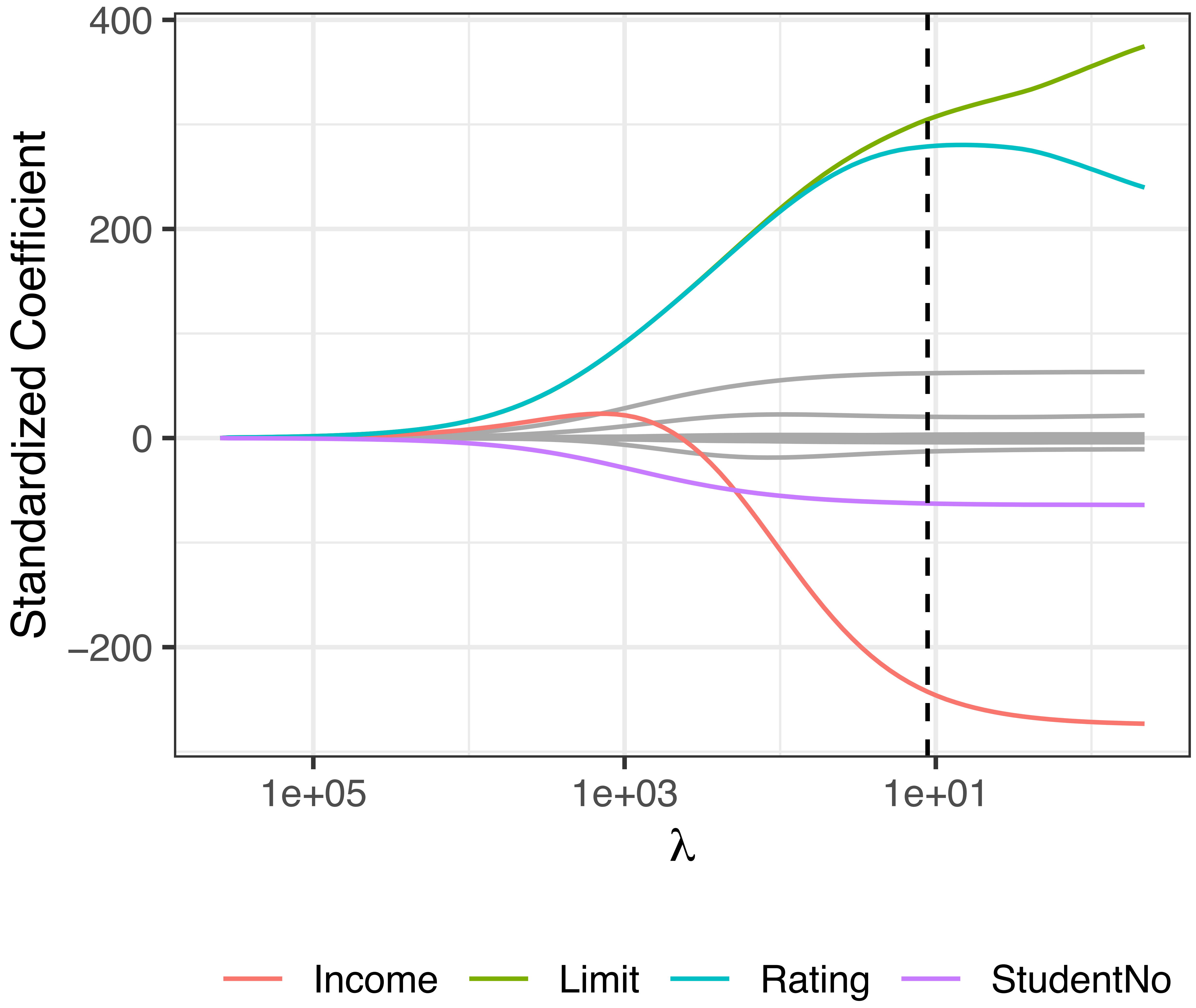
Interpretation: Mean response changes by β_j when X_j is increased by a standard deviation.

Ridge regression trace plot



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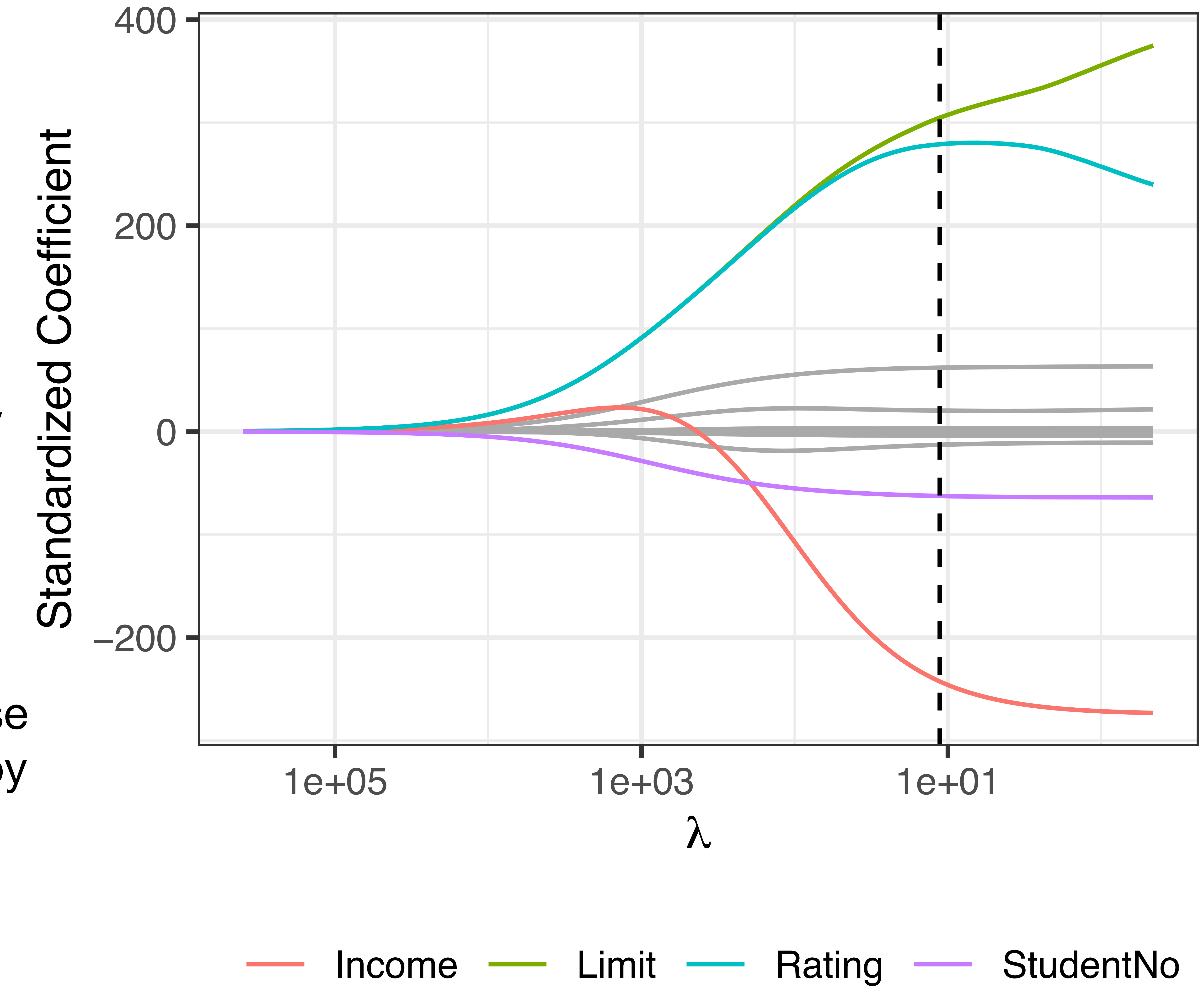

Change in mean response when feature increases by one standard deviation.

Ridge regression in a simple case

Suppose that $n = p$ and $X_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$, i.e. $Y_j = \beta_j + \epsilon_j$.

E.g. $X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

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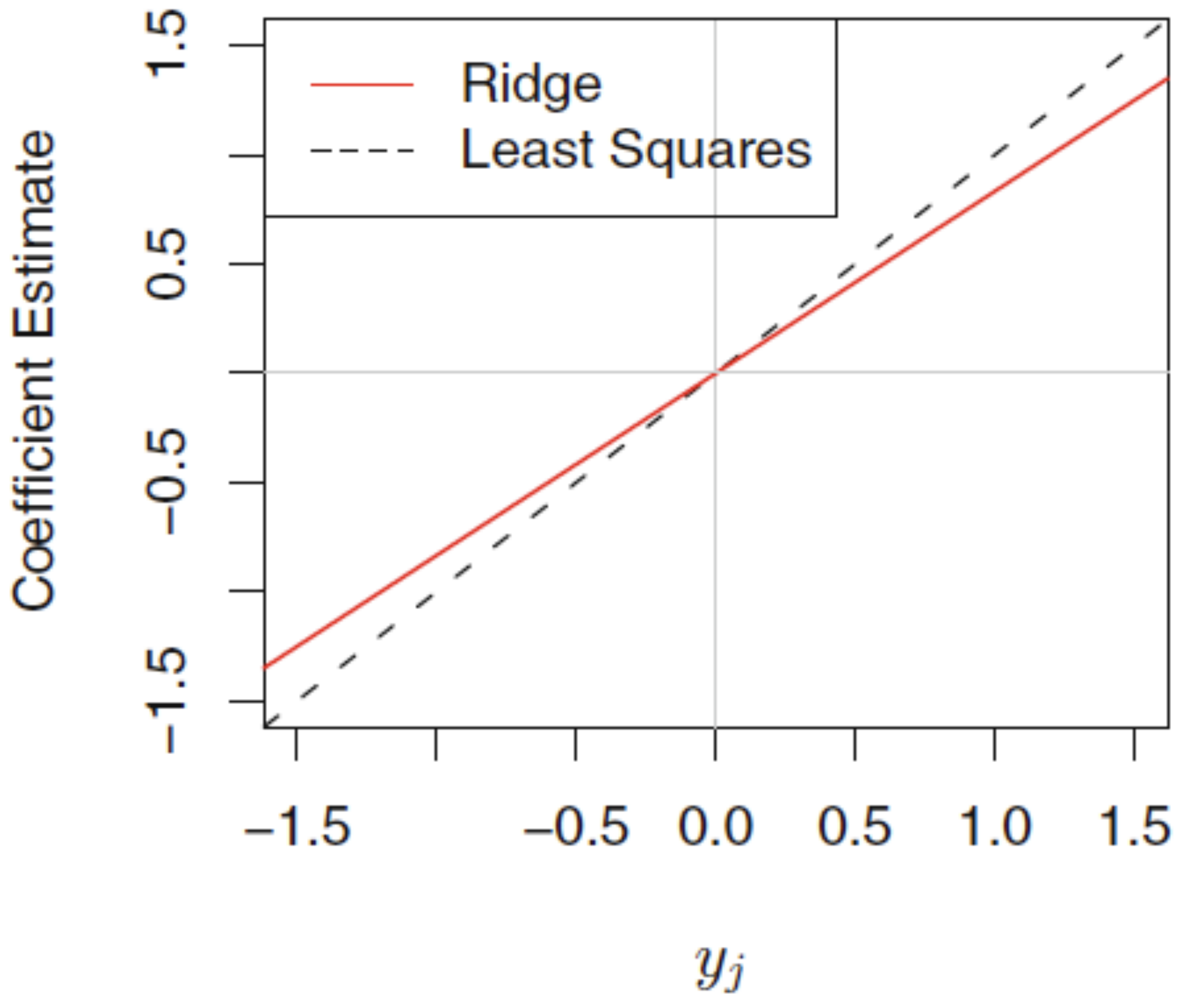

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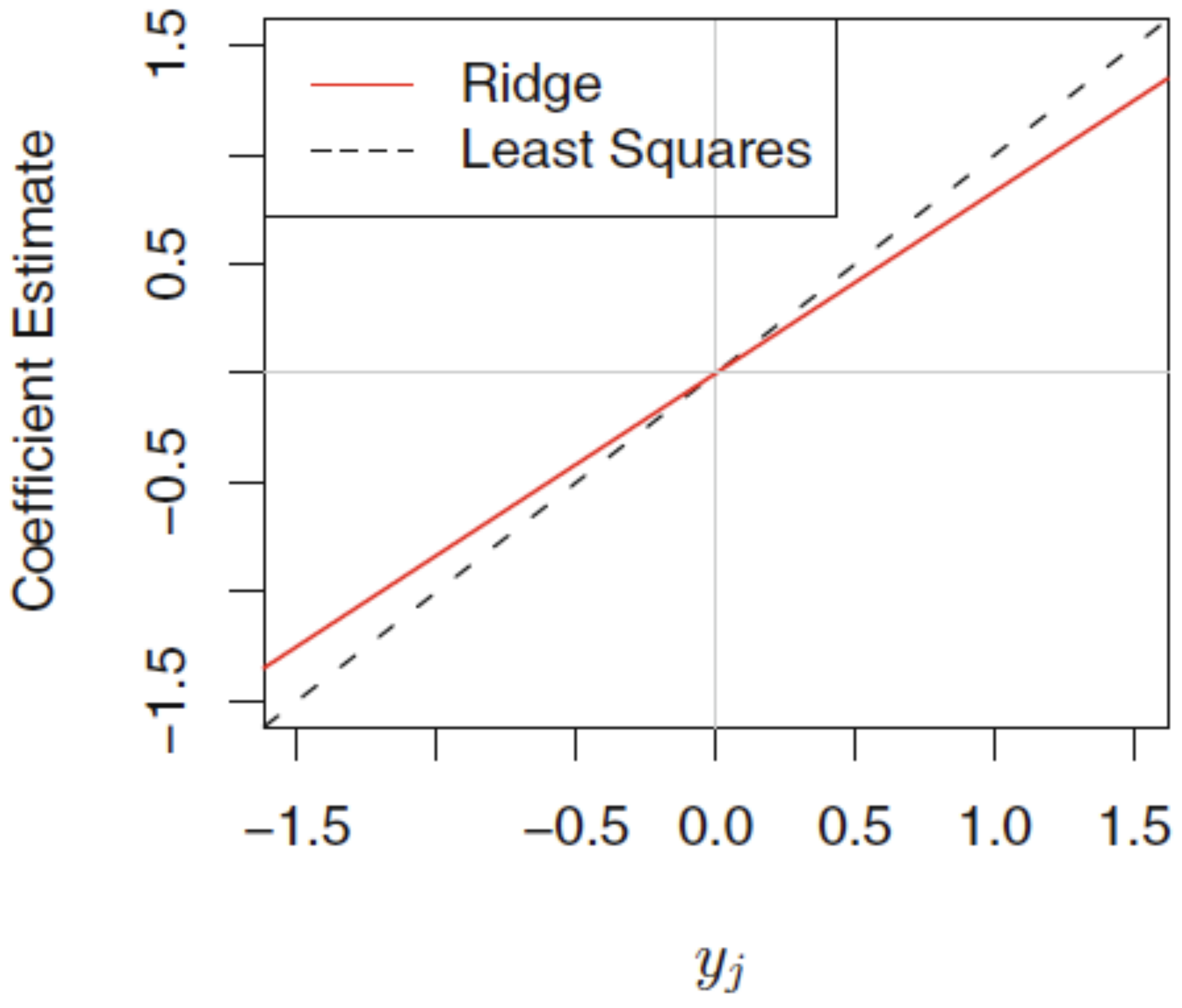

So $\hat{\beta}^{\text{ridge}} = \dfrac{1}{1+\lambda}\hat{\beta}^{\text{OLS}}$, i.e. the ridge estimate is obtained by *shrinking* the OLS estimate by a factor of $1+\lambda$.

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- Ridge regression will obtain $\widehat{\beta}$ from $y = \beta X_1 + \epsilon$, and set $\widehat{\beta}_1 = \widehat{\beta}_2 = \frac{1}{2}\widehat{\beta}$.

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Subtle point: While $\widehat{\beta}^{\text{ridge}}$ is trained based on a (penalized) log-likelihood, during cross-validation we should choose λ based on whatever measure of test error we care about (e.g. weighted misclassification error).

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Quiz practice