Regression in high dimensions

STAT 4710

October 11, 2022

Where we are

- **Unit 1:** R for data mining
- **Unit 2:** Prediction fundamentals
- **Unit 3:** Regression-based methods
- **Unit 4:** Tree-based methods
- **Unit 5:** Deep learning

- **Lecture 1:** Linear and logistic regression
- $\boxed{\textbf{Lecture 2: } \text{Regression in high dimensions}}$
- **Lecture 3:** Ridge regression
- **Lecture 4:** Lasso regression
- **Lecture 5:** Unit review and quiz in class

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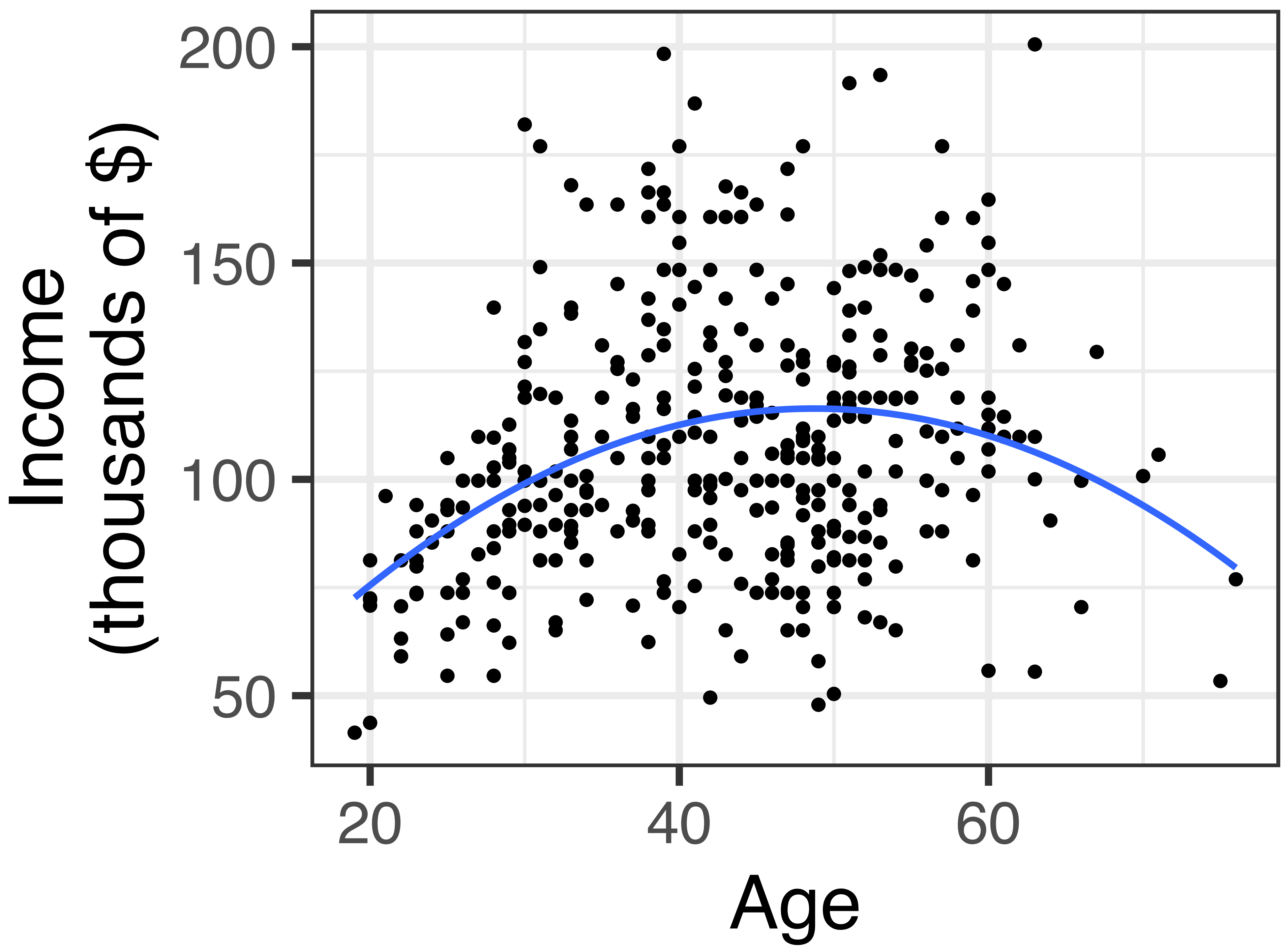
Income $\approx \beta_0 + \beta_1$**Age** $+ \beta_2$**Age**^{2}



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In modern applications, can collect very many features for each observation, e.g.:

- Natural language processing
- Image processing
- Genetics/Genomics
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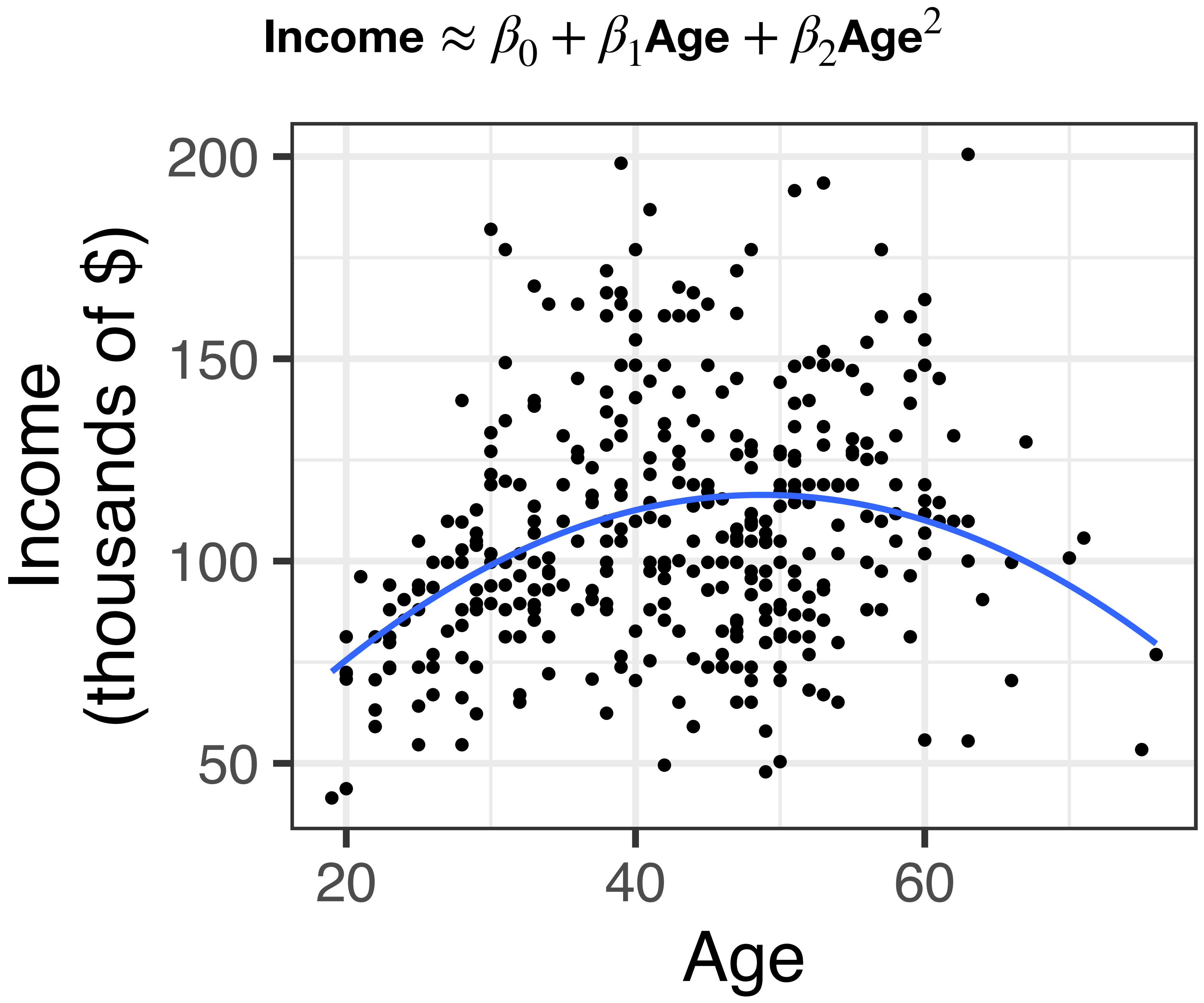

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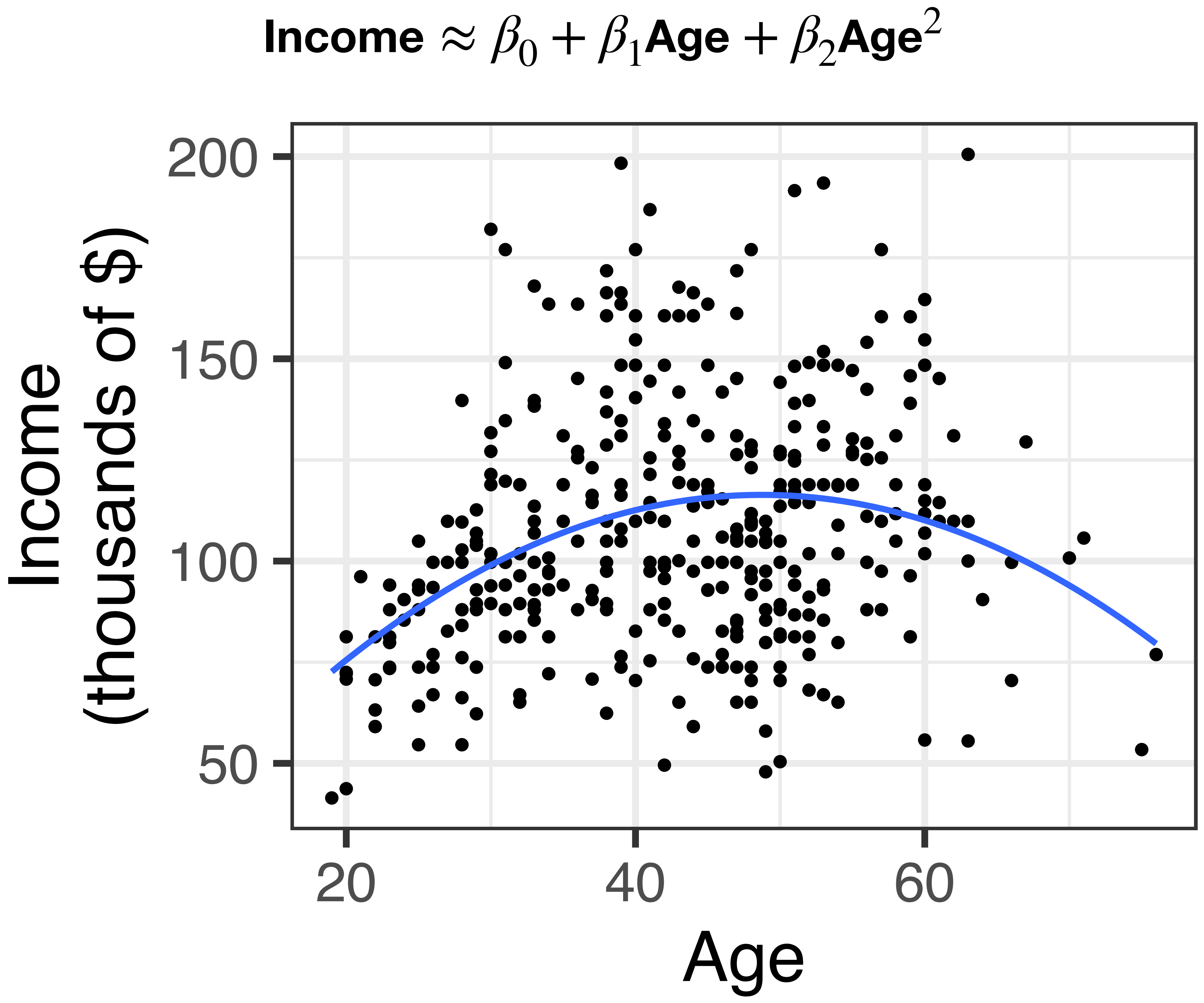

High-dimensional data: Data with $p > n$ or $p \approx n$

Challenges in high dimensions

Let's consider fitting a linear regression with n observations and p features.

If $p > n$, the columns of the feature matrix X guaranteed to be multi-collinear, so the least squares linear regression estimate is not even defined.

If $p = n$, linear regression will perfectly fit training set, even with "junk" features.

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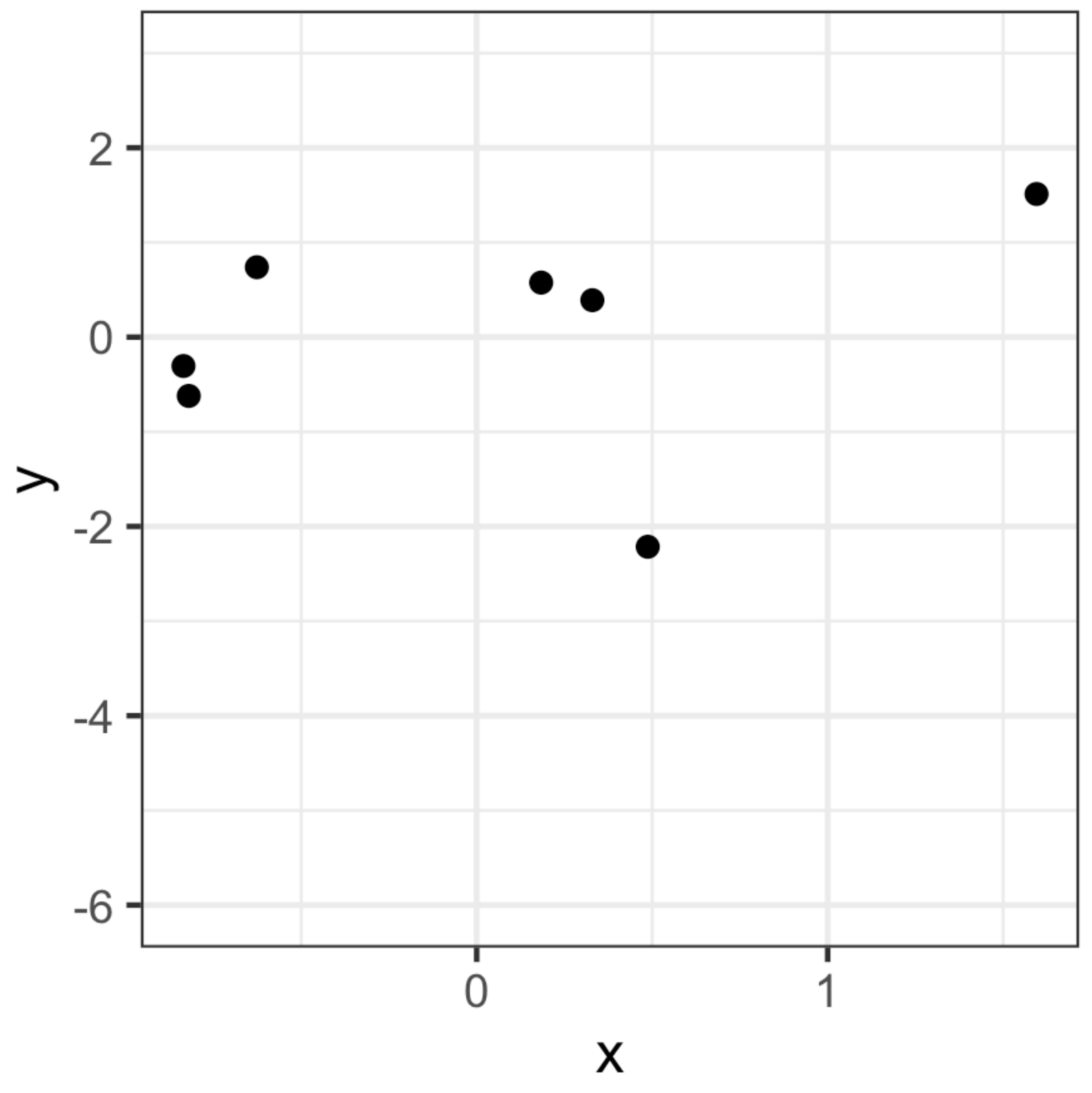

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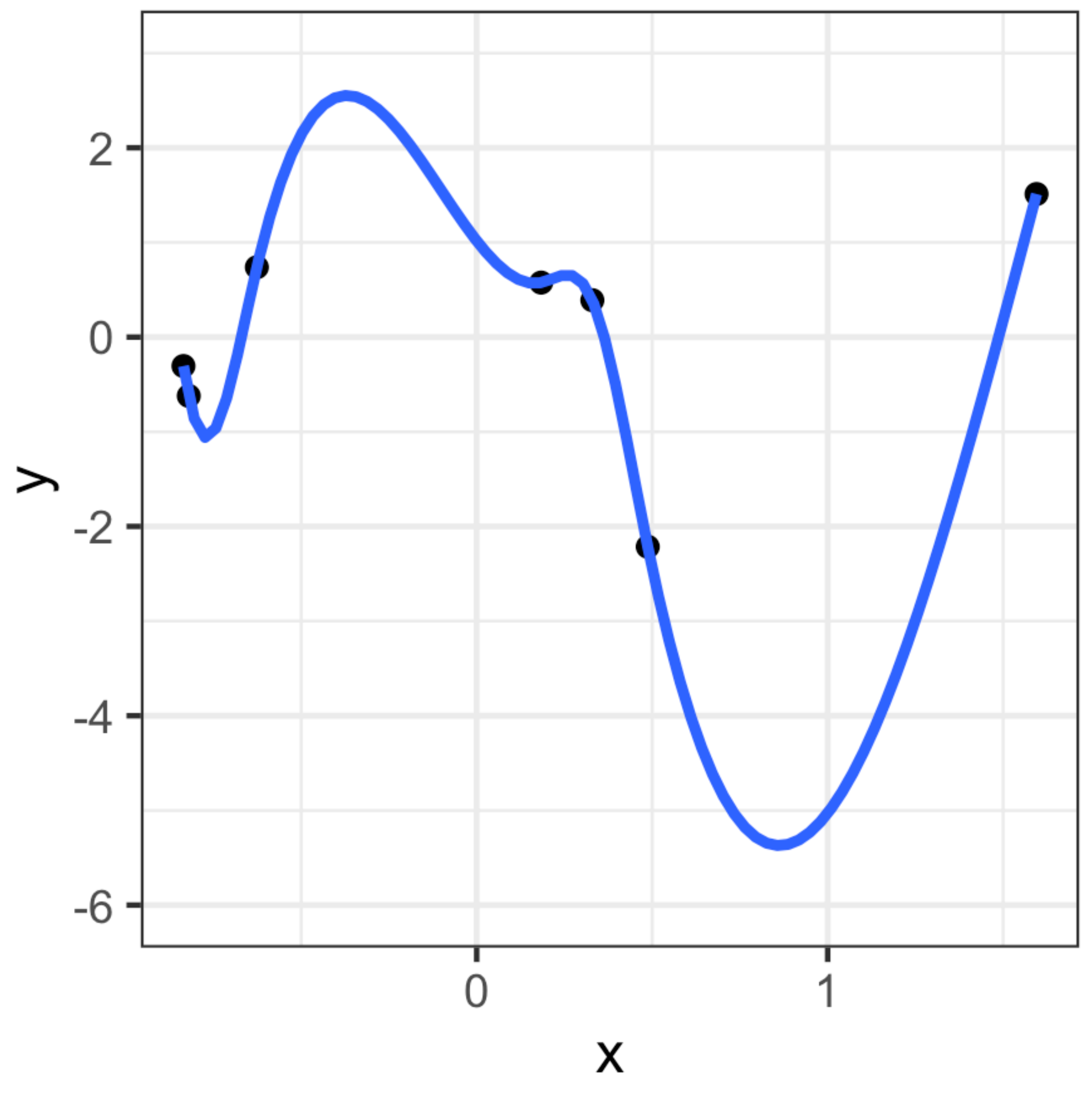

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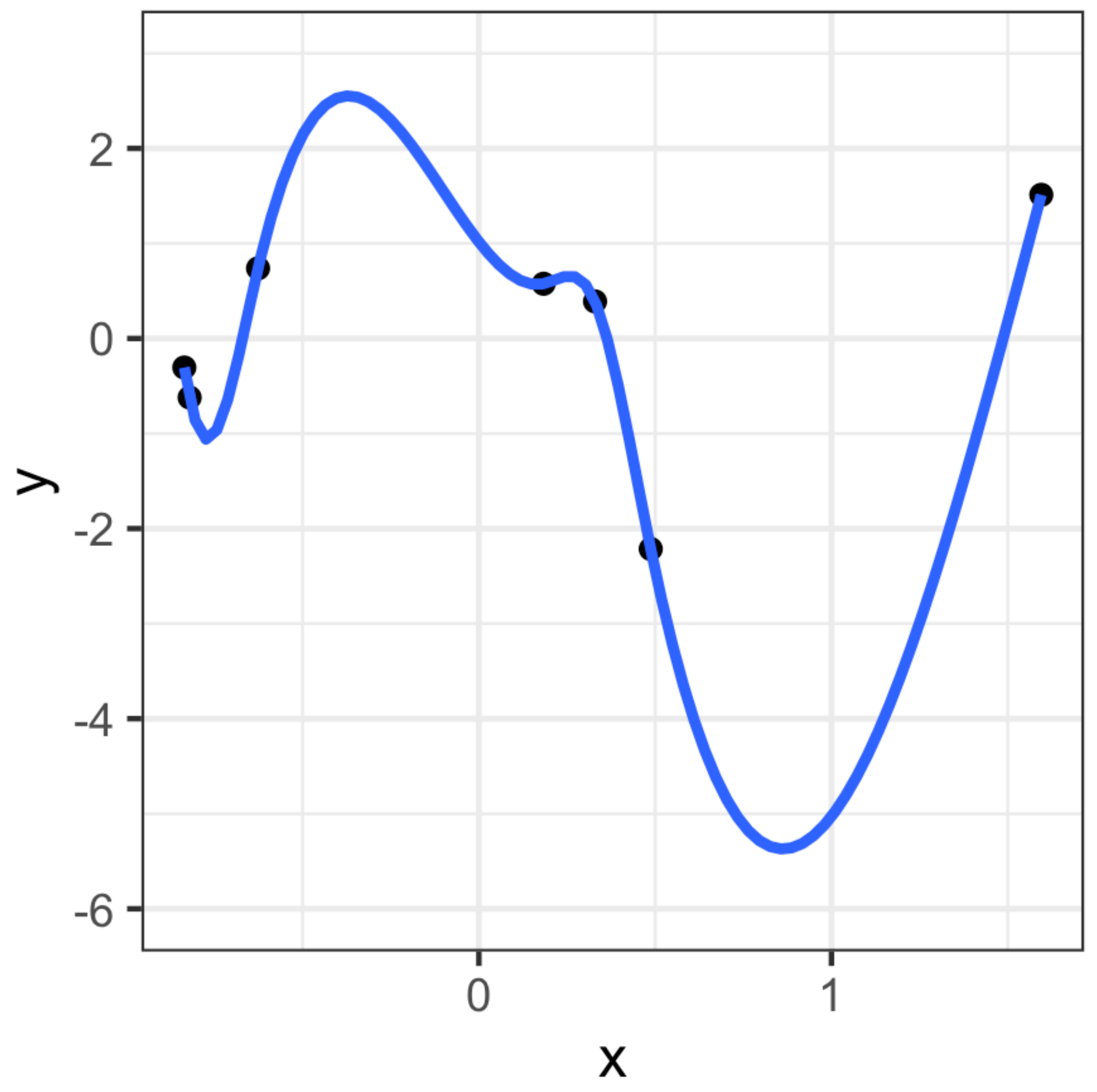
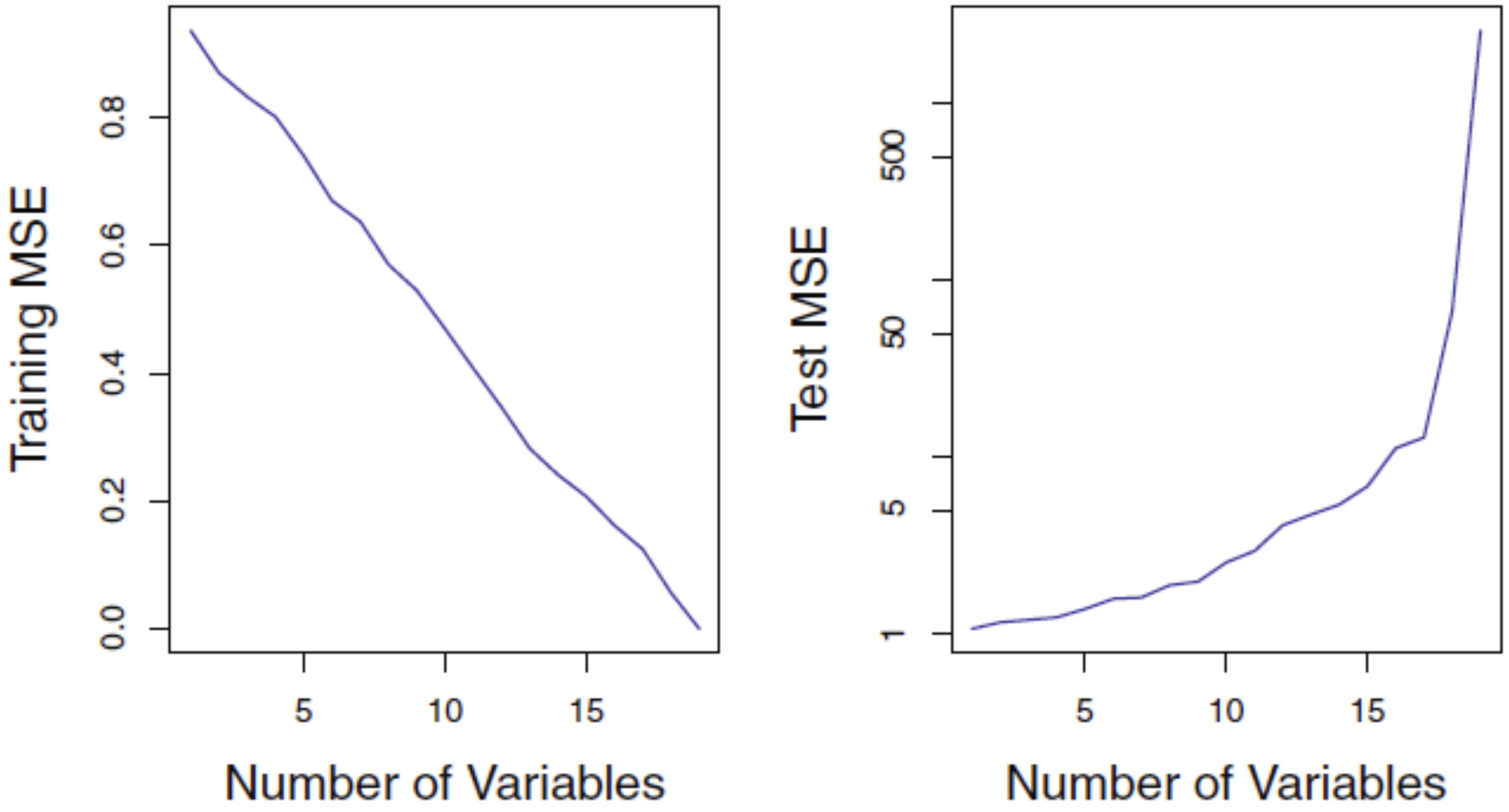
If $p < n$, recall that linear regression variance is $\sigma^2 p/n$. Therefore, if $p \approx n$ then variance will be very high.



Linear models fit using too many features (i.e. too many degrees of freedom) perform poorly due to high variance.

Challenges in high dimensions (illustration)

Linear regression for $n = 20$; p features unrelated to response



The solution

The solution is to constrain the fitted coefficients in some way, e.g.:

1. Make sure fitted coefficients are not too large (ridge regression).

2. Make sure fitted coefficients are mostly equal to zero (lasso regression).

These constraints reduce the degrees of freedom of the fit, reducing variance.

We are still fitting p coefficients, but using fewer than p degrees of freedom.

Penalization: A way of constraining the fit

Recall least squares solution:

$$\widehat{\beta} = \underset{\beta_0,\beta_1,\ldots,\beta_{p-1}}{\arg\min} \sum_{i=1}^{n} (y_i - (\beta_0 + \beta_1 X_{i1} + \cdots + \beta_{p-1} X_{i,p-1}))^2.$$

Here we let $\widehat{\beta}$ fit the data as close as possible, putting no constraints.

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how well β fits the data $\longleftarrow$ compromise $\longrightarrow$ how wild β is

Example: L0-penalized regression

Consider the penalized regression

$$\widehat{\beta}' = \underset{\beta_0,\beta_1,\ldots,\beta_{p-1}}{\arg\min} \sum_{i=1}^{n} (y_i - (\beta_0 + \beta_1 X_{i1} + \cdots + \beta_{p-1} X_{i,p-1}))^2 + \lambda \cdot P(\beta),$$

with $P(\beta) = |\{j : \beta_j \neq 0\}|$.

The L0 penalty P counts the number of nonzero entries in β, and creates sparse solutions $\widehat{\beta}$.

The optimization above is computationally infeasible, so in practice we use a different penalty (called the lasso) to achieve sparsity (stay tuned for Lecture 4).

How and when penalization works

Penalization reduces the variance, but increases the bias of the predictions.

$\Rightarrow$ Reduces test error when reduction in variance outweighs increase in bias.

The bias is a function of the complexity of the underlying model, and in high dimensions, we can have some very complex underlying models.

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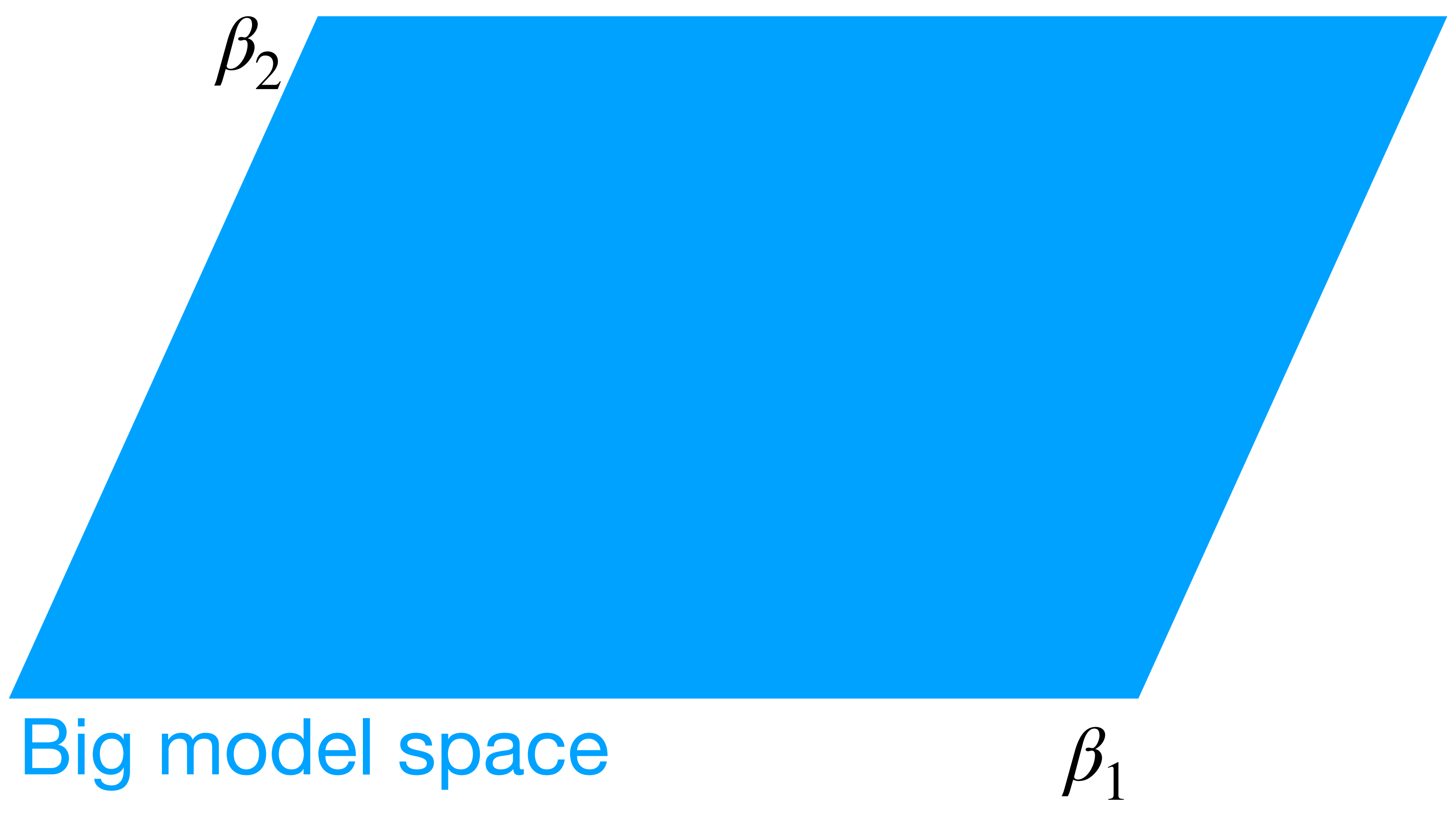

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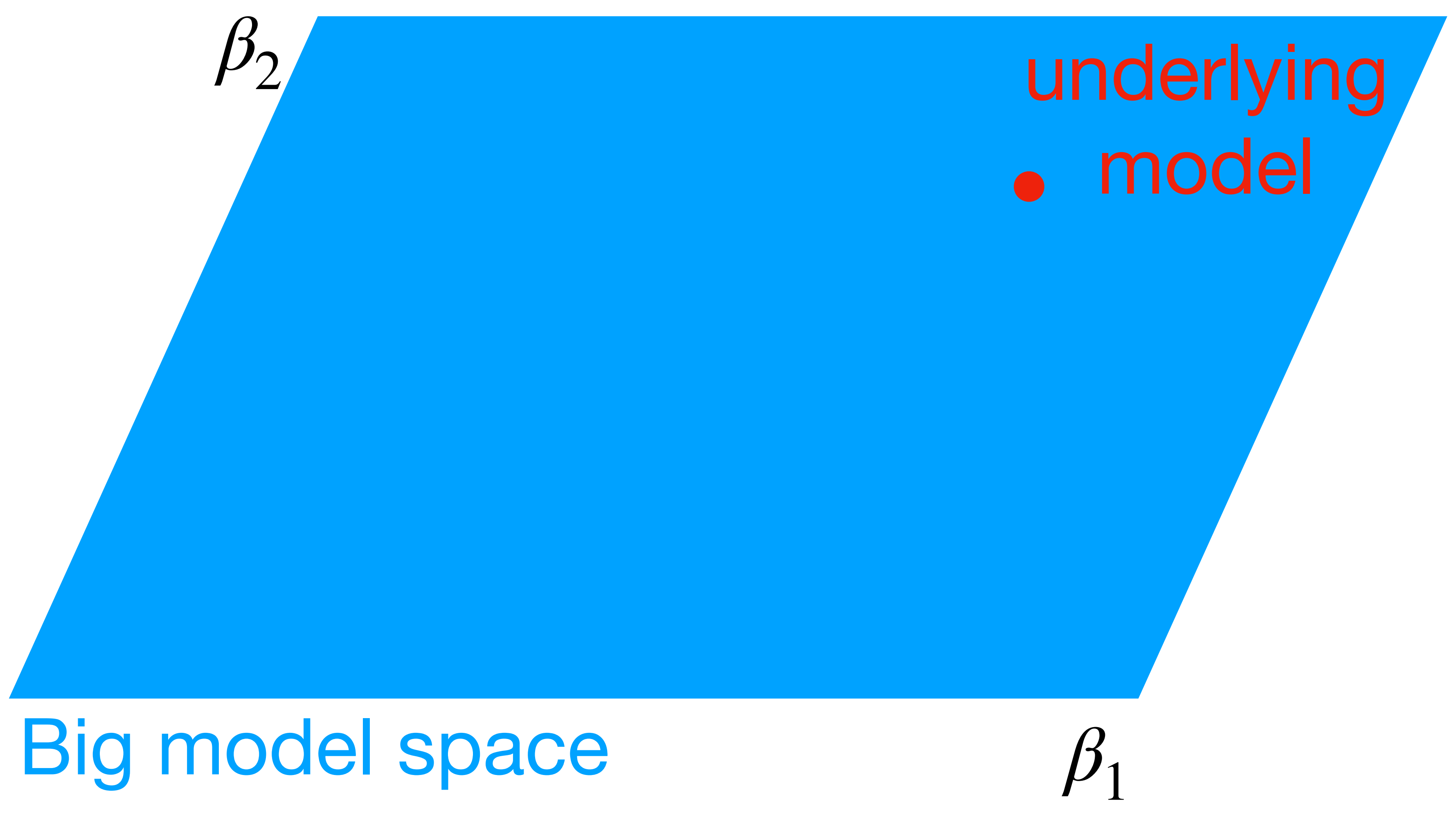

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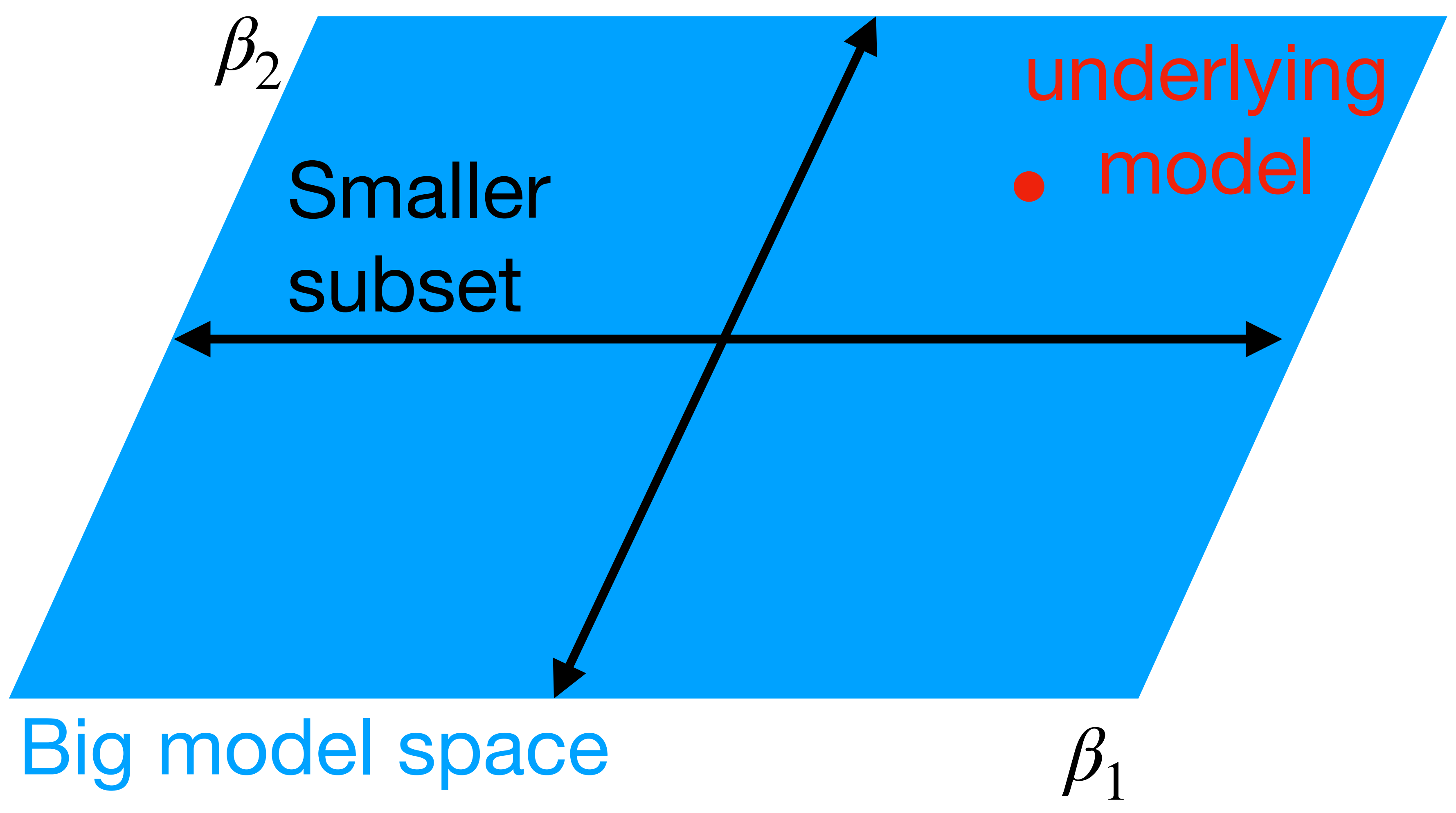

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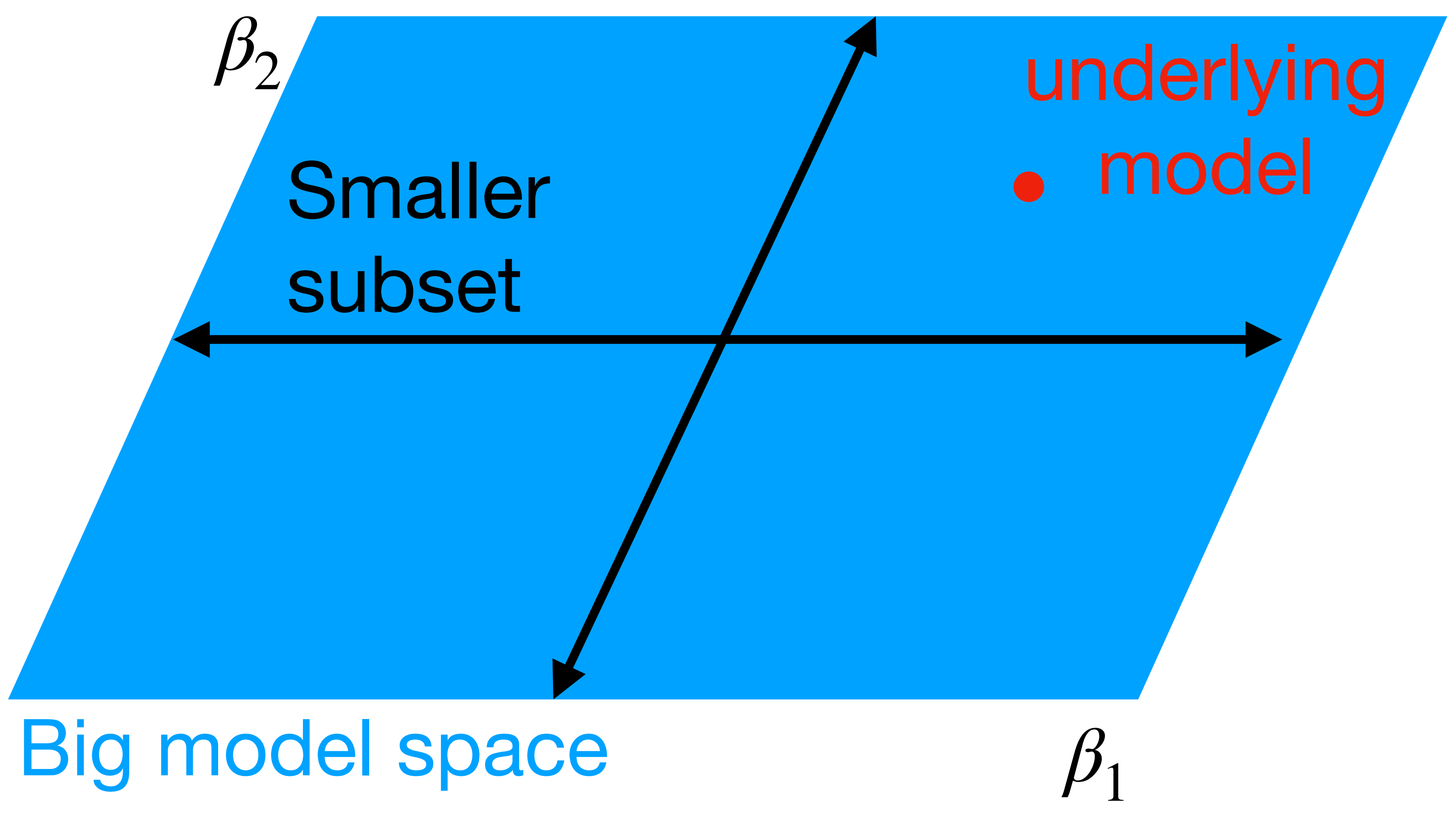

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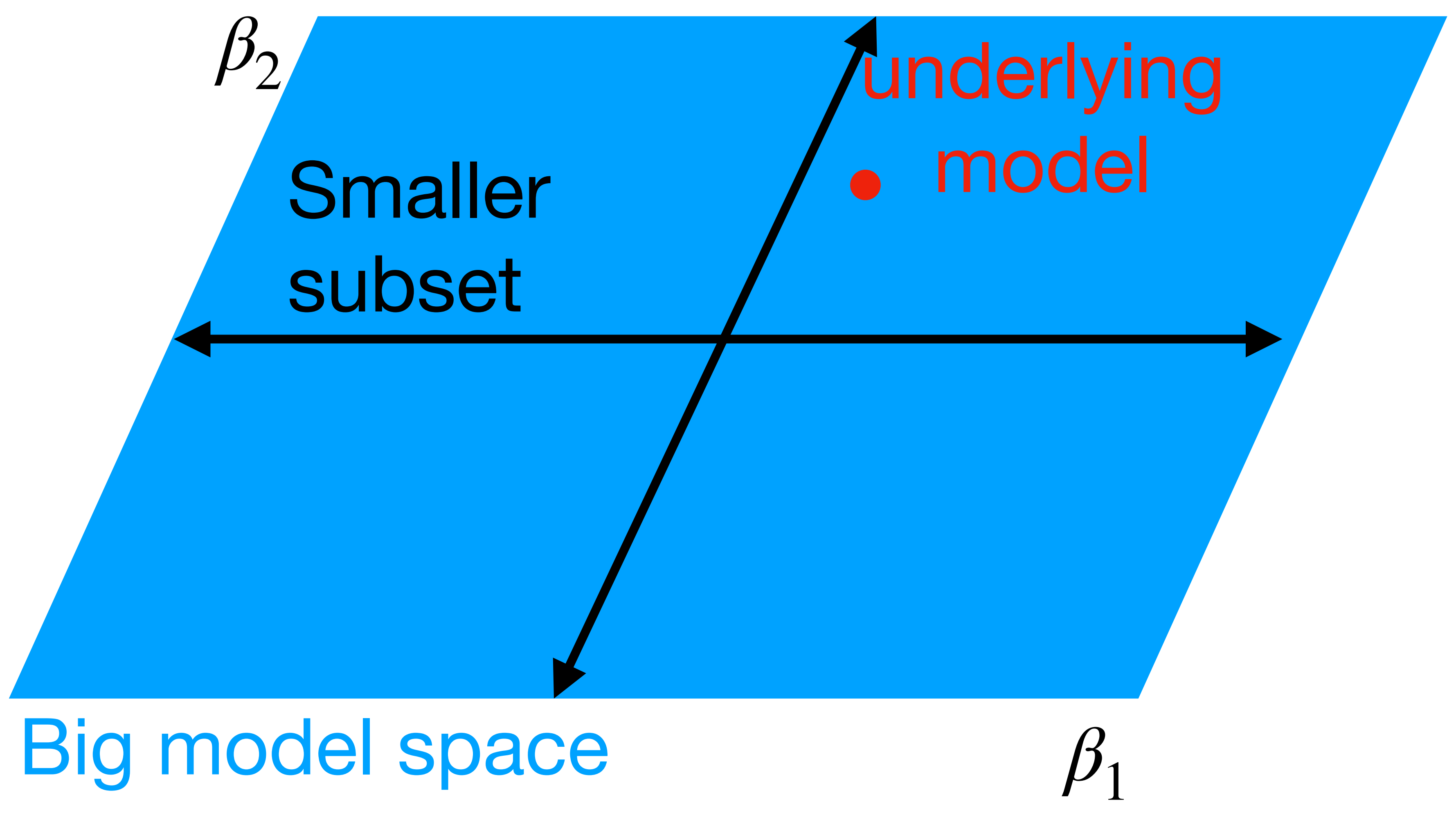

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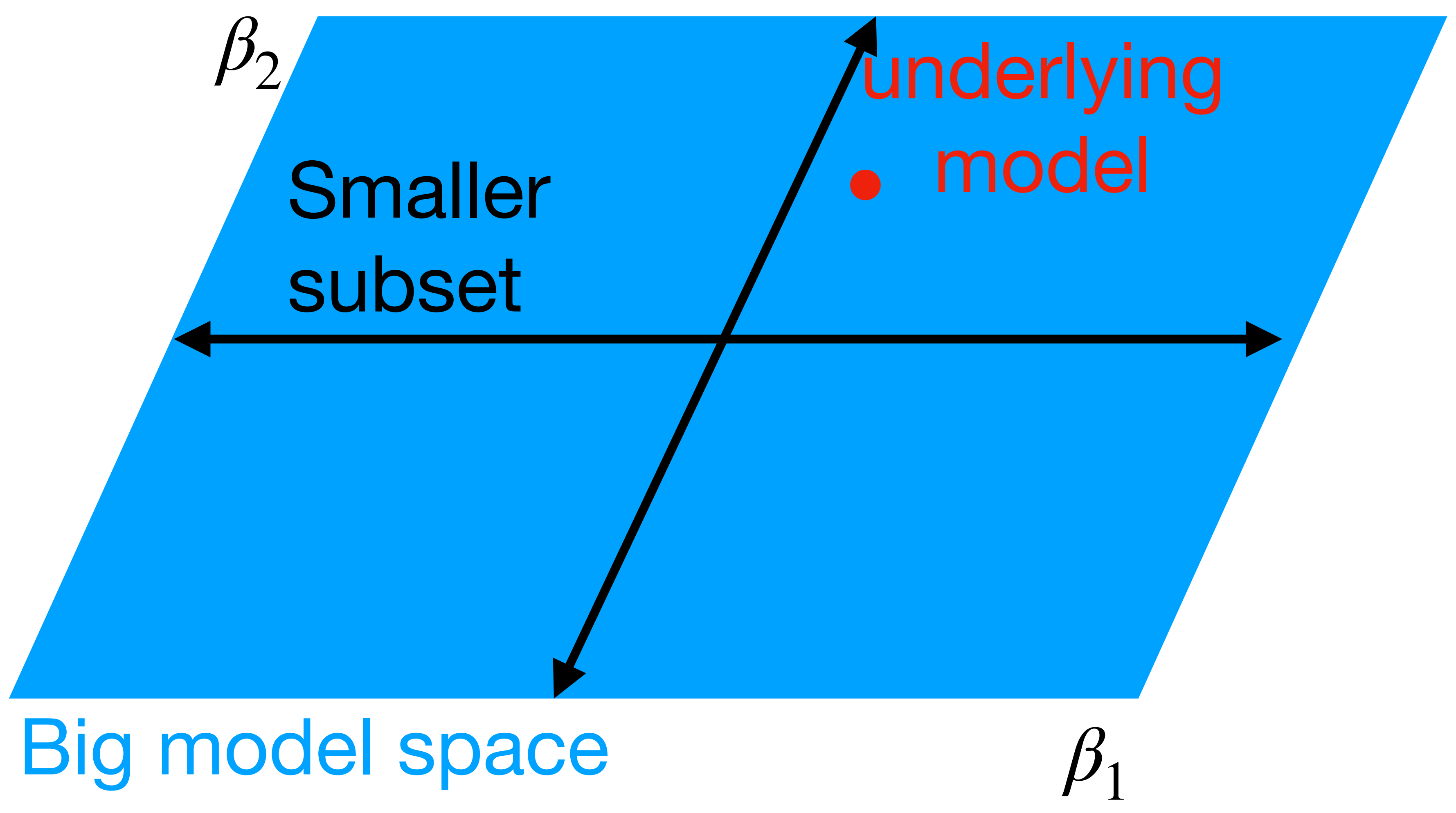

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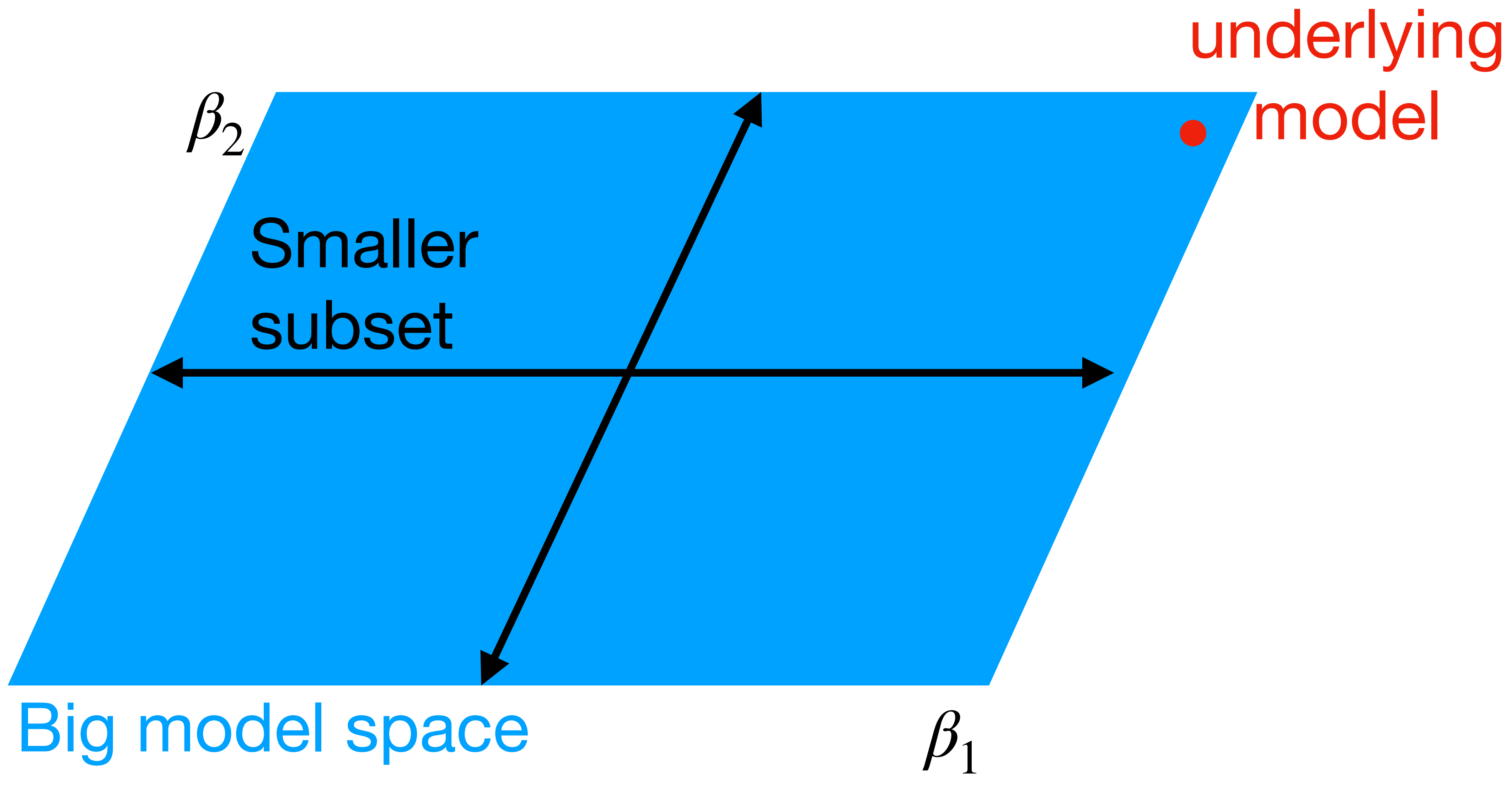

Looking ahead to lectures 3 and 4

Lecture 3: Ridge regression (constraining coefficients not to be too large)

Lecture 4: Lasso regression (constraining coefficients to be sparse)

We'll learn about the theory and practice of these penalized regression methods.

Quiz practice

Link to Canvas