Linear and logistic regression

STAT 4710

October 4, 2022

Rolling into Unit 3

- ✓ **Unit 1:** R for data mining
- ✓ **Unit 2:** Prediction fundamentals
- **Unit 3:** Regression-based methods
- **Unit 4:** Tree-based methods
- **Unit 5:** Deep learning

Lecture 1: Linear and logistic regression

Lecture 2: Regression in high dimensions

Lecture 3: Ridge regression

Lecture 4: Lasso regression

Lecture 5: Unit review and quiz in class

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Let's review:

- Continuous and categorical features in linear models
- Interpretation of linear regression coefficients
- How to fit a linear regression model

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Example 3 (categorical feature): $X_3 =$ education. It does not make sense to write $\beta_3 X_3$. Instead, map education onto multiple dummy variables: $X_3 = I(\text{education} = \text{high school})$, $X_4 = I(\text{education} = \text{"college"})$, etc.

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To avoid redundancy, use dummy variables for all levels except one baseline.

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Note: Linear regression coefficients do not necessarily imply causation.

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The least squares optimization problem can be solved in closed form.

What if the response is binary?

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   default student balance income
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 1 No      No         730. 44362.
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$$\mathbb{P}[\text{default} = \text{Yes} \,|\, \text{student}, \text{balance}, \text{income}]$$

and then predict

$$\text{default} = \begin{cases} \text{Yes}, & \text{if } \widehat{\mathbb{P}}[\text{default}] \geq 0.5; \\ \text{No}, & \text{if } \widehat{\mathbb{P}}[\text{default}] < 0.5. \end{cases}$$

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How do we model probability of default?

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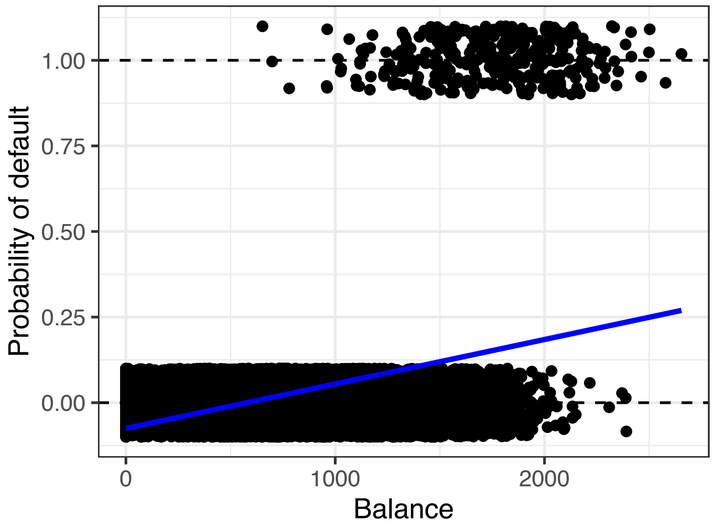

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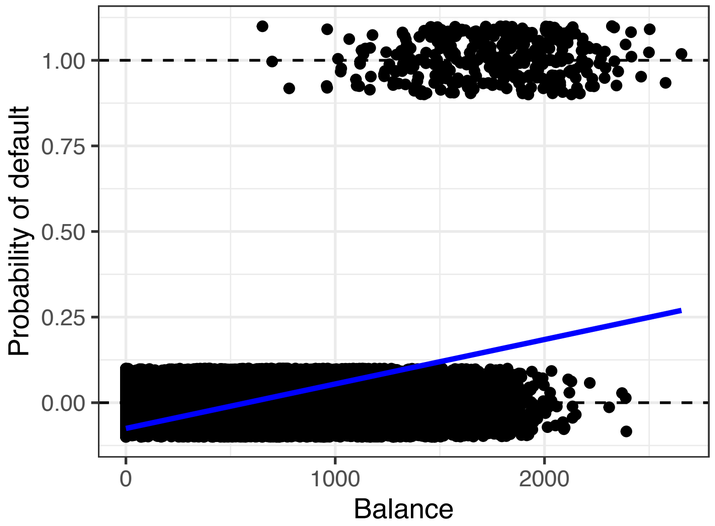

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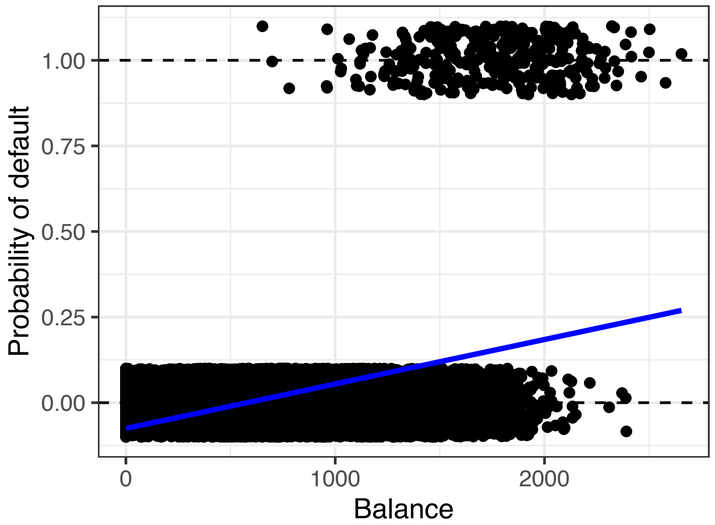

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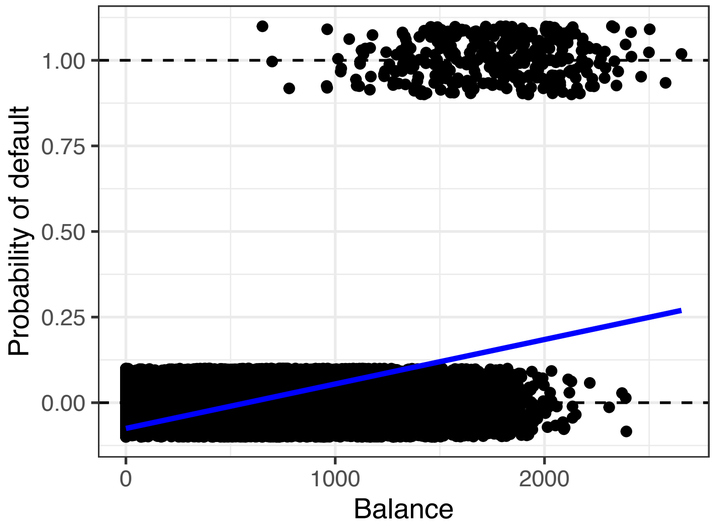

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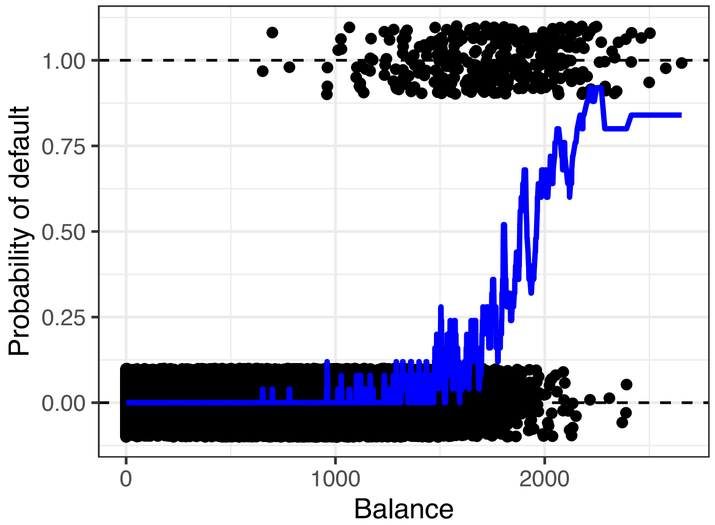

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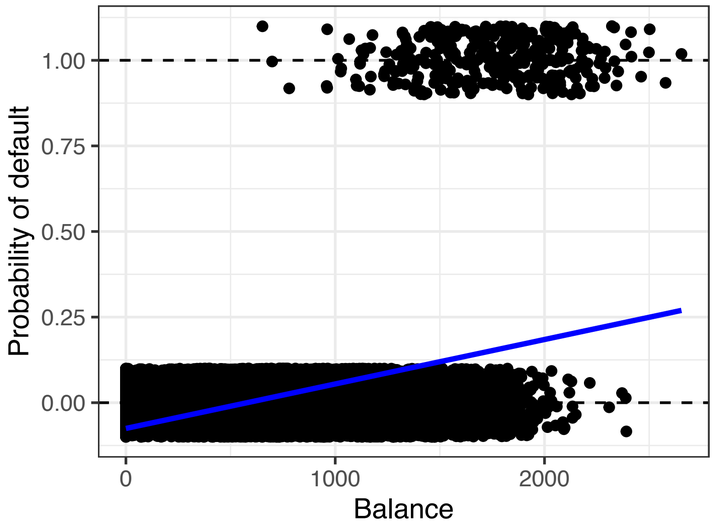

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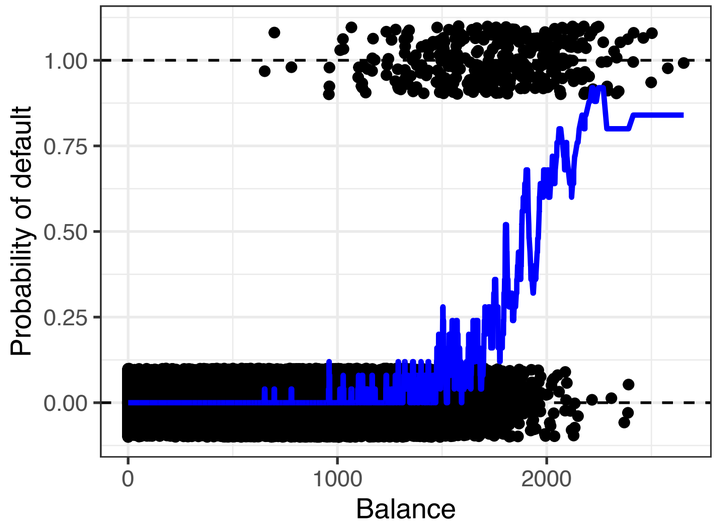

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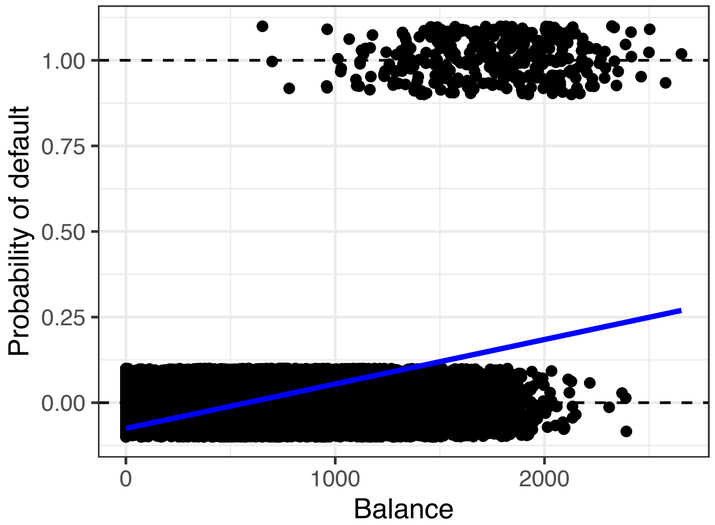

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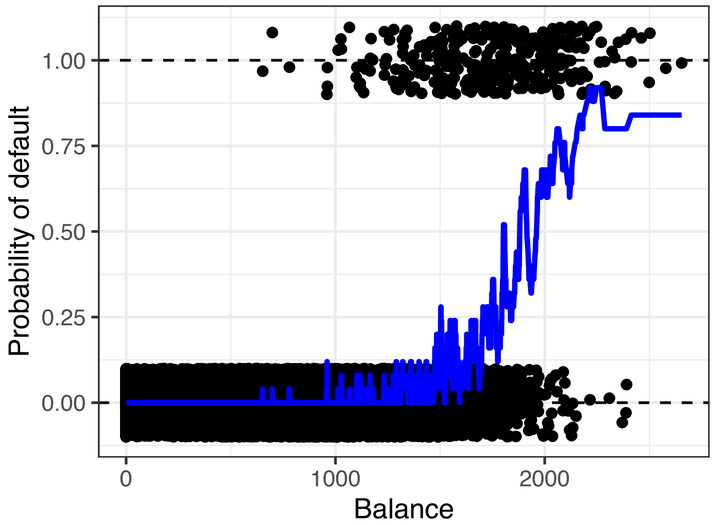

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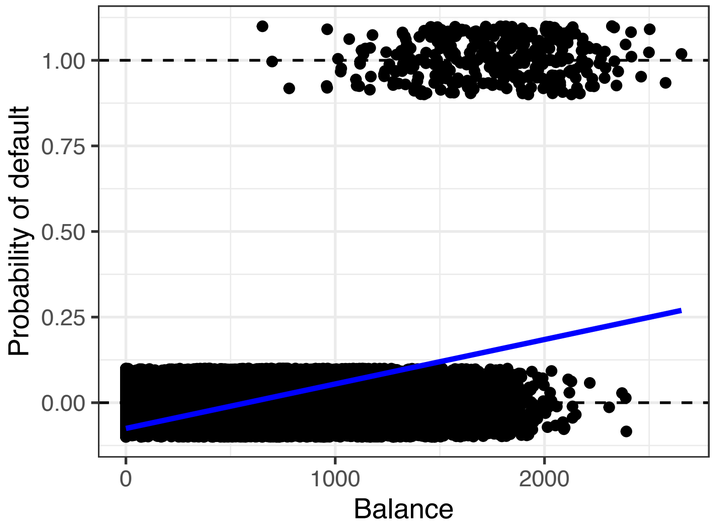

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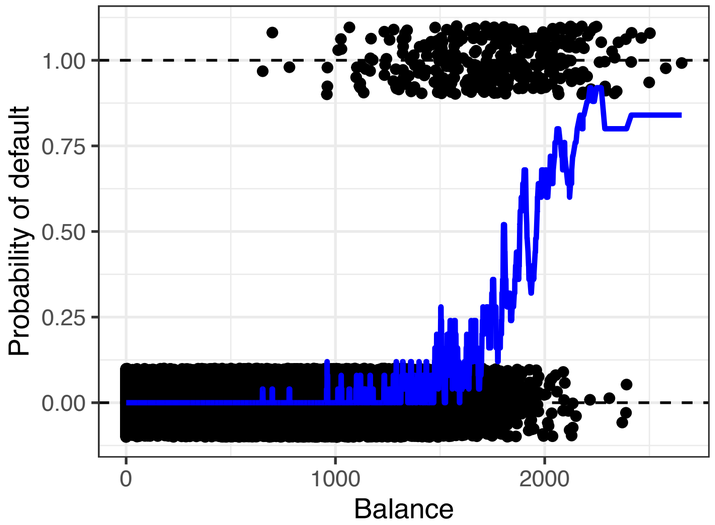

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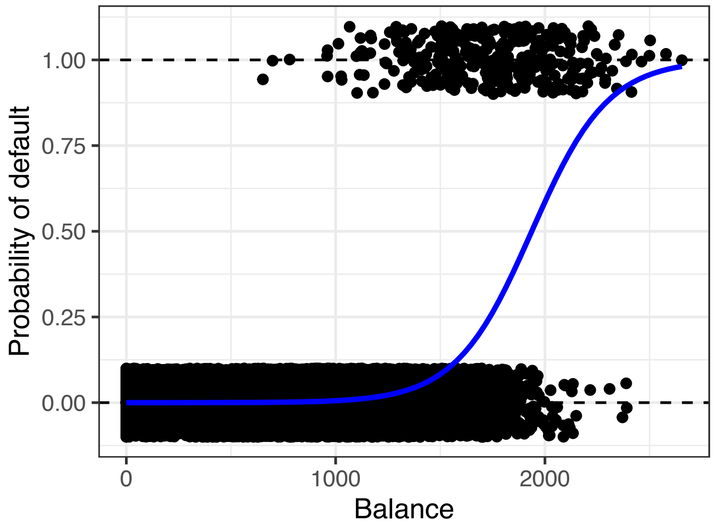

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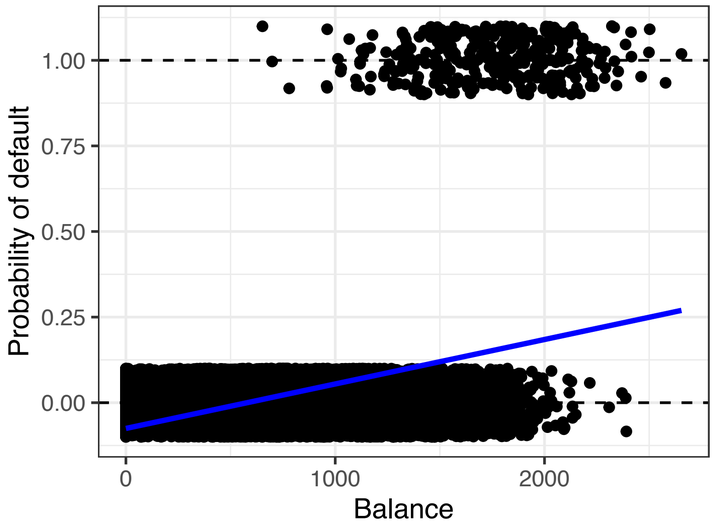

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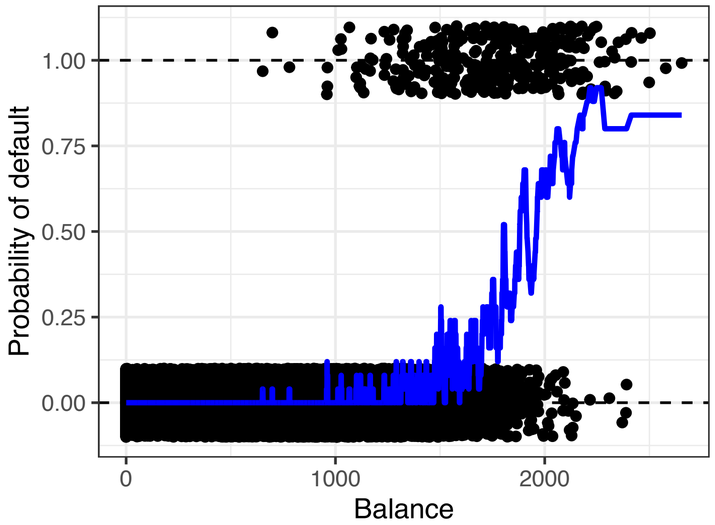

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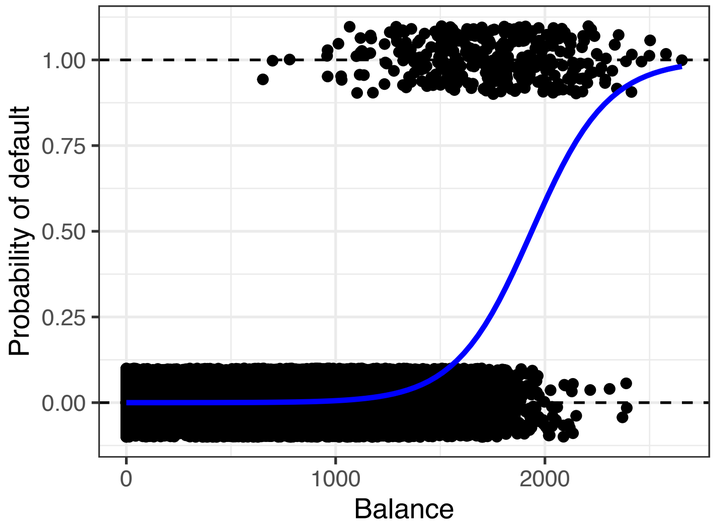

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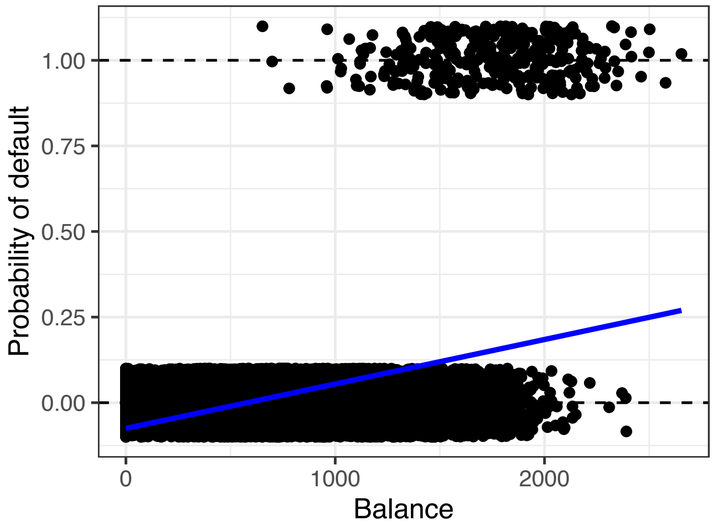

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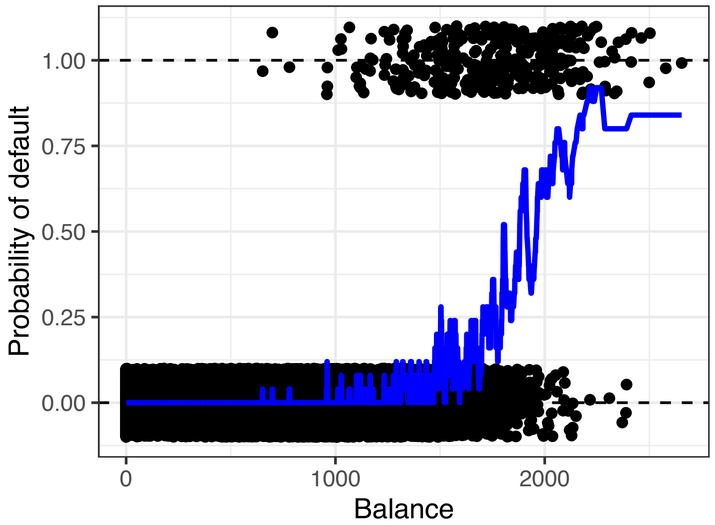

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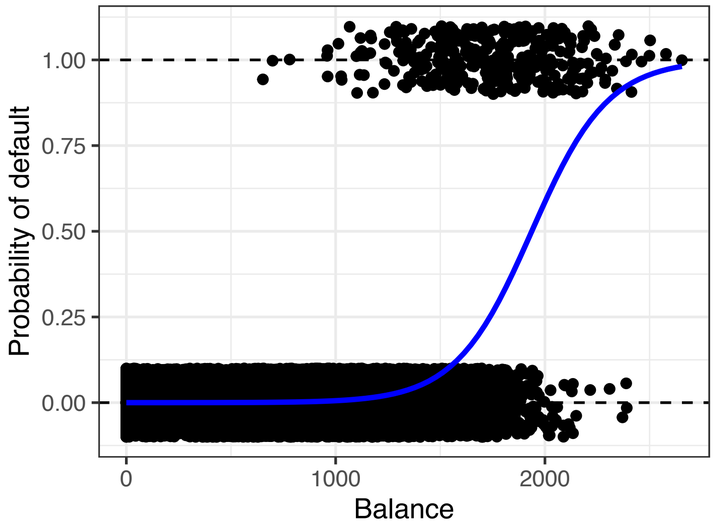

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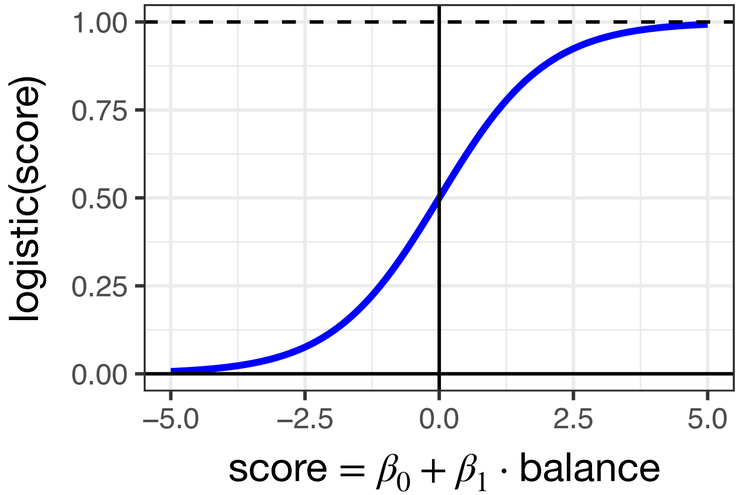

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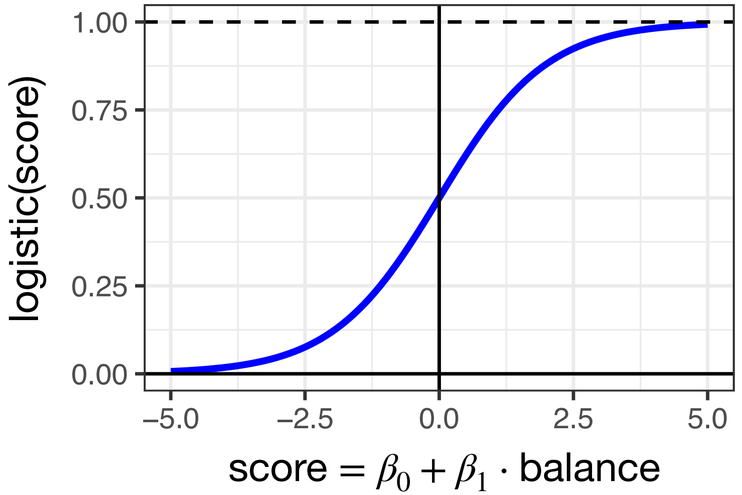

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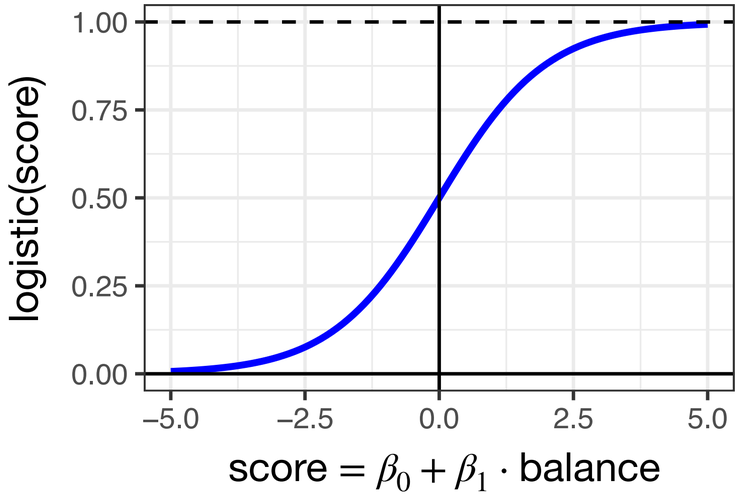
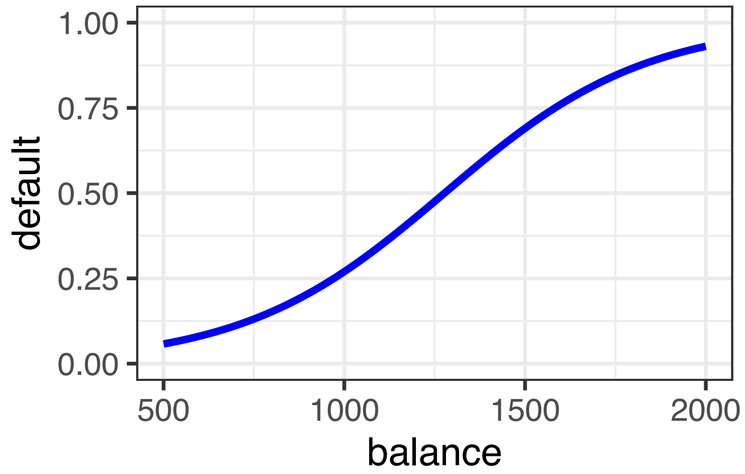
Use $\beta_0 + \beta_1 \cdot \text{balance}$ as a "score", then map the score onto [0,1] using logistic transformation:

$$\text{logistic}(\text{score}) = \frac{e^{\text{score}}}{1 + e^{\text{score}}}$$

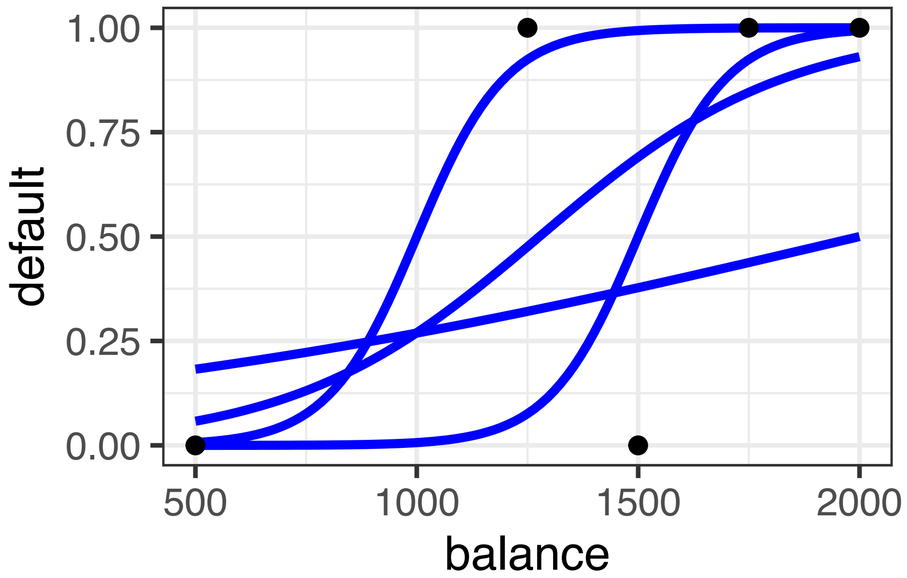


Logistic regression model:

$$\mathbb{P}[\text{default} \mid \text{balance}] = \text{logistic}(\beta_0 + \beta_1 \cdot \text{balance})$$



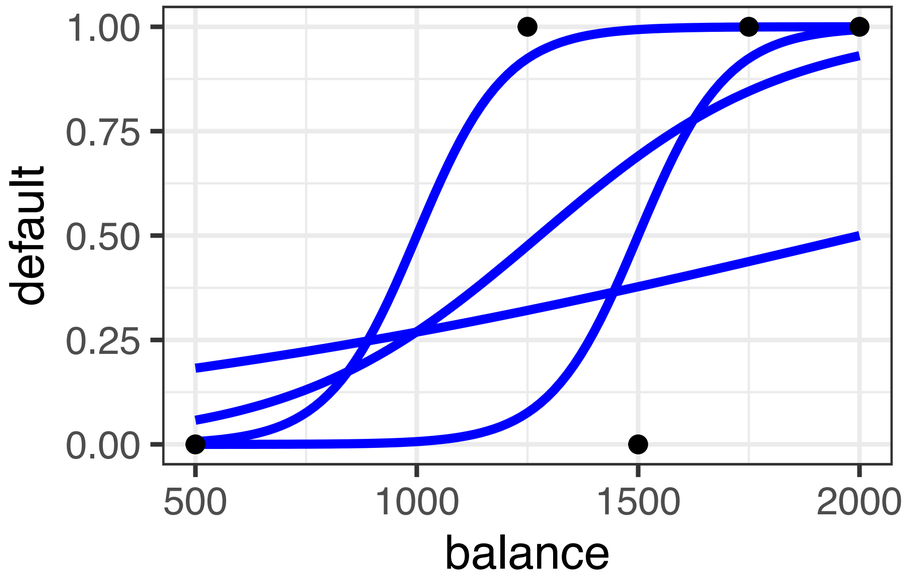
Fitting logistic regression models



Fitting logistic regression models

Each choice of (β_0, β_1) traces out a different logistic regression curve fit

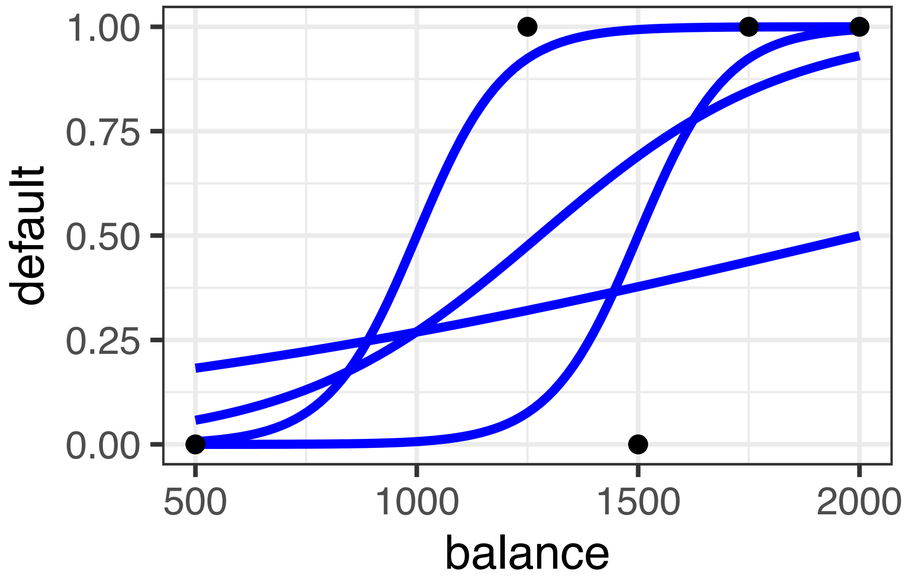
$$\mathbb{P}[\text{default} \mid \text{balance}] = \text{logistic}(\beta_0 + \beta_1 \cdot \text{balance}).$$



Fitting logistic regression models

Each choice of (β_0, β_1) traces out a different logistic regression curve fit

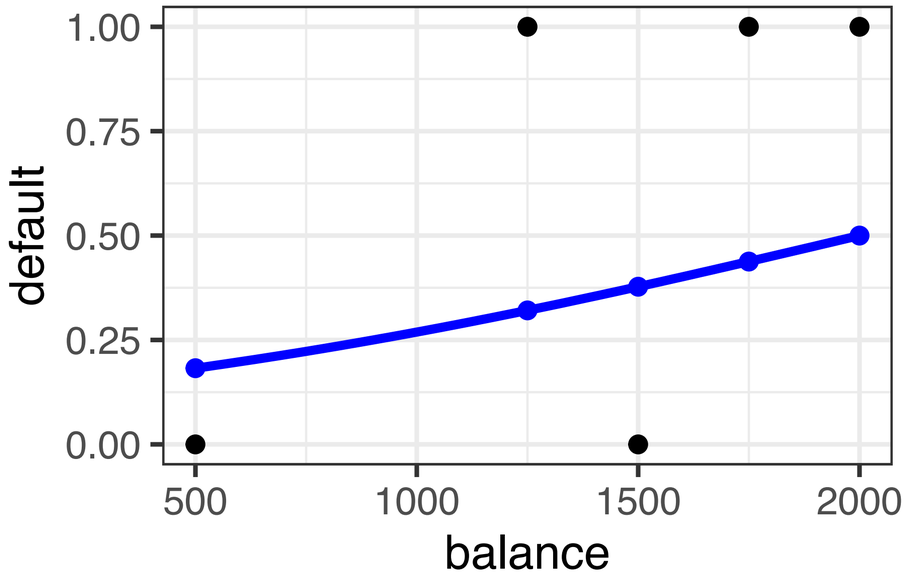
$$\mathbb{P}[\text{default} \mid \text{balance}] = \text{logistic}(\beta_0 + \beta_1 \cdot \text{balance}).$$



Which logistic regression curve fits the data the best?

Maximum likelihood estimation

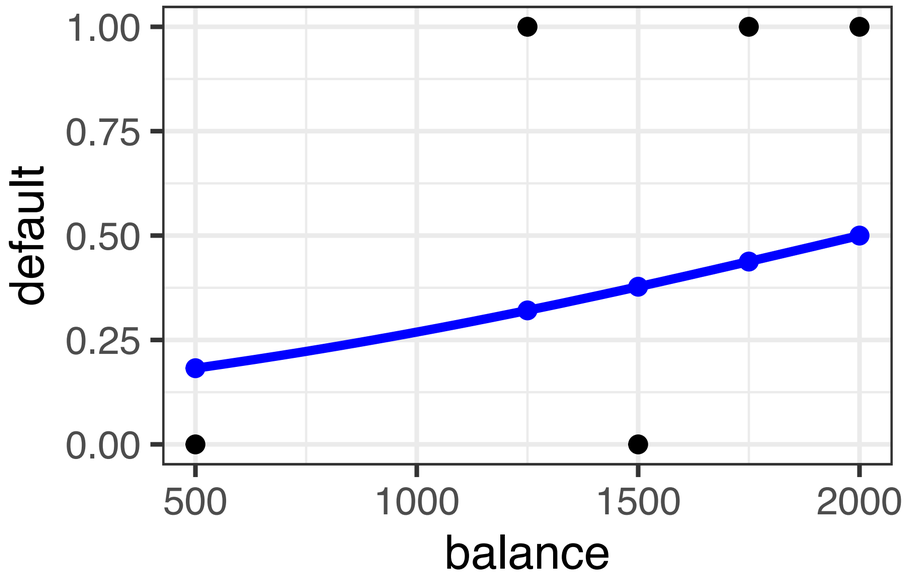
Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:



β_0	β_1
-2.0	0.001

Maximum likelihood estimation

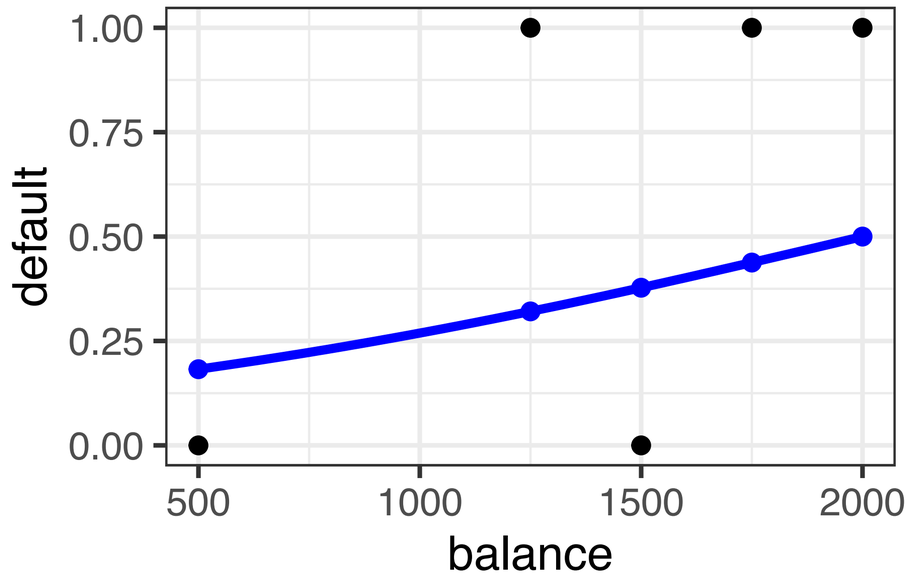
Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:



β_0	β_1	Predicted probabilities
-2.0	0.001	

Maximum likelihood estimation

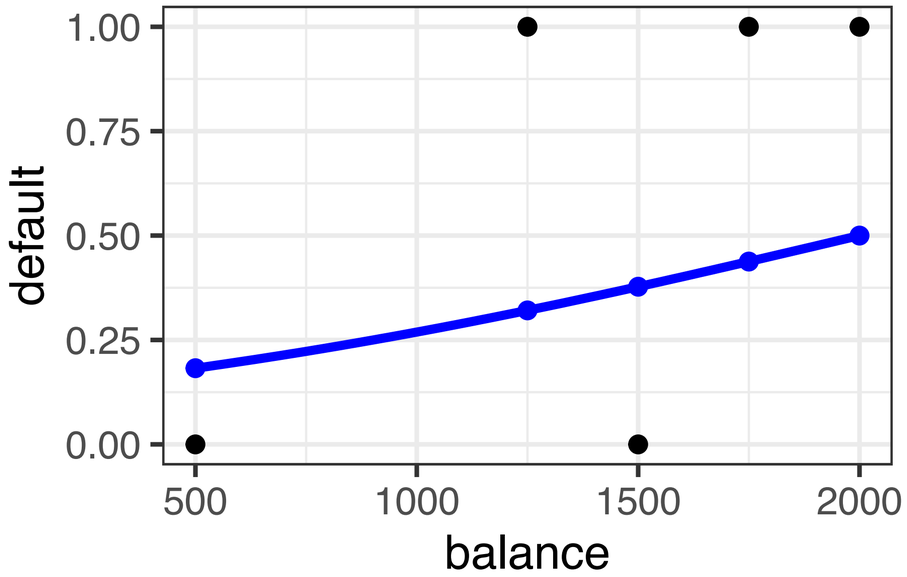
Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:



β_0	β_1	Predicted probabilities
-2.0	0.001	0.8

Maximum likelihood estimation

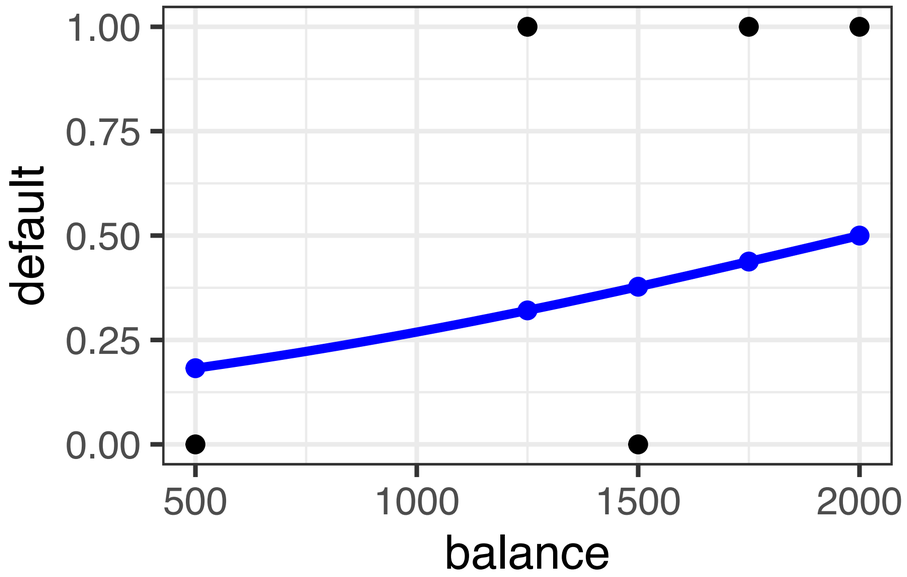
Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:



β_0	β_1	Predicted probabilities	
-2.0	0.001	0.8	0.3

Maximum likelihood estimation

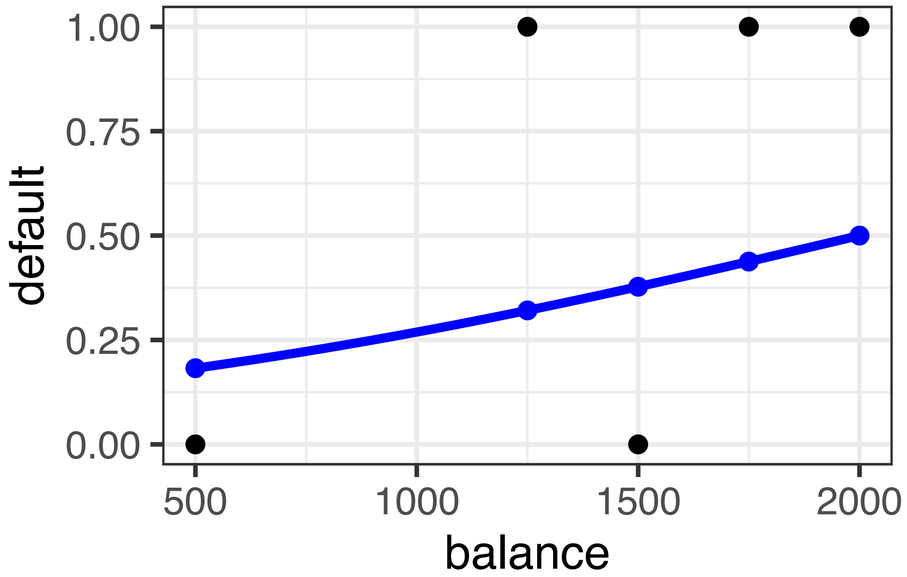
Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:



β_0	β_1	Predicted probabilities		
-2.0	0.001	0.8	0.3	0.6

Maximum likelihood estimation

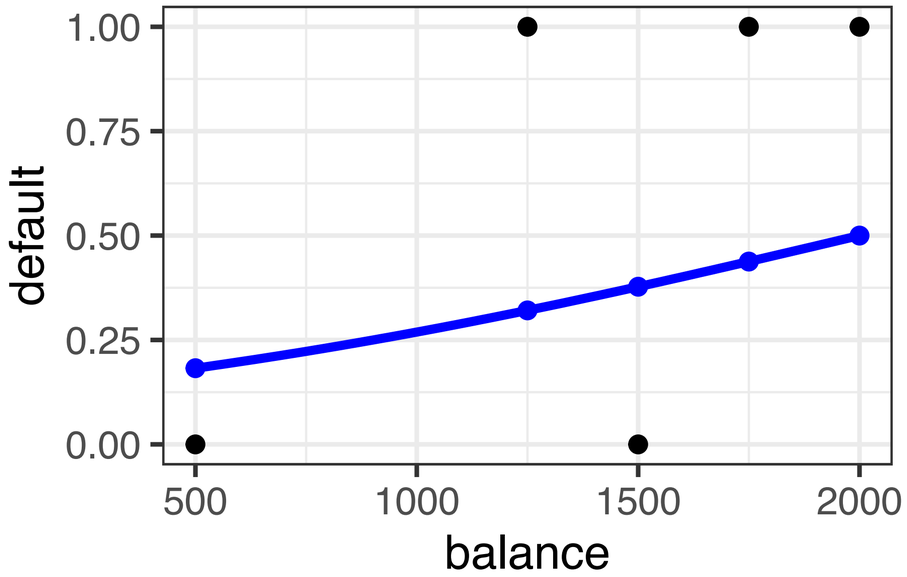
Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:



β_0	β_1	Predicted probabilities				
-2.0	0.001	0.8	0.3	0.6	0.4	0.5

Maximum likelihood estimation

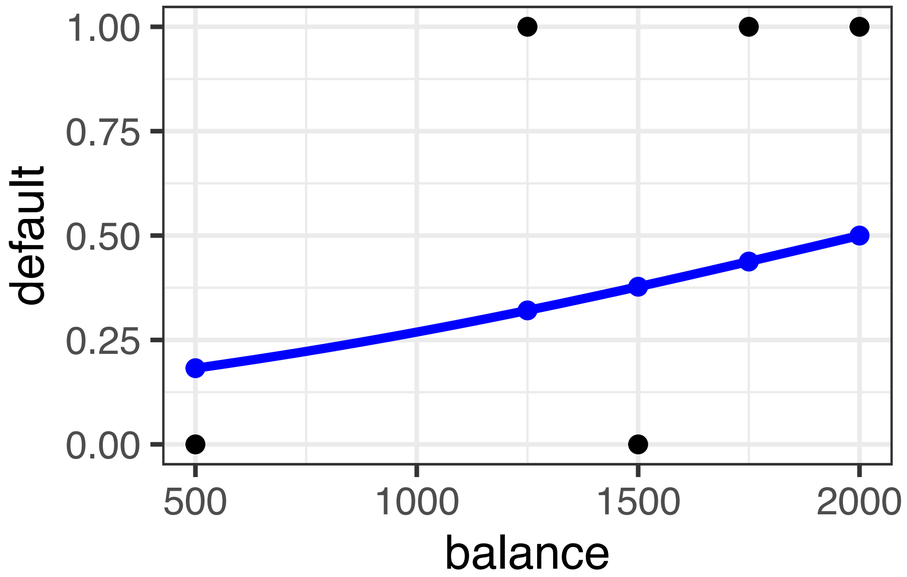
Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:



β_0	β_1	Predicted probabilities					$\mathscr{L}(\beta_0, \beta_1)$
-2.0	0.001	0.8	0.3	0.6	0.4	0.5	

Maximum likelihood estimation

Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:



β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
-2.0	0.001	0.8	×	0.3×0.6×0.4×0.5	

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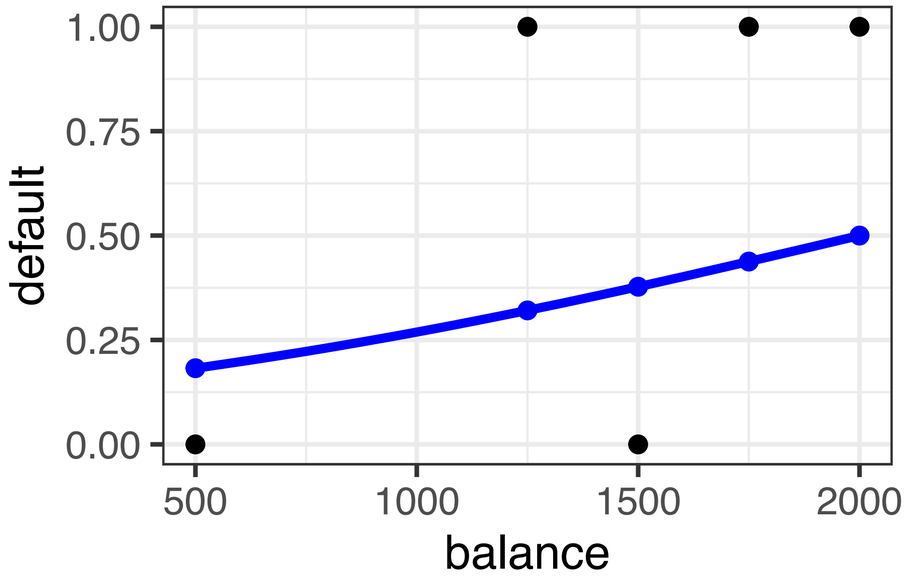
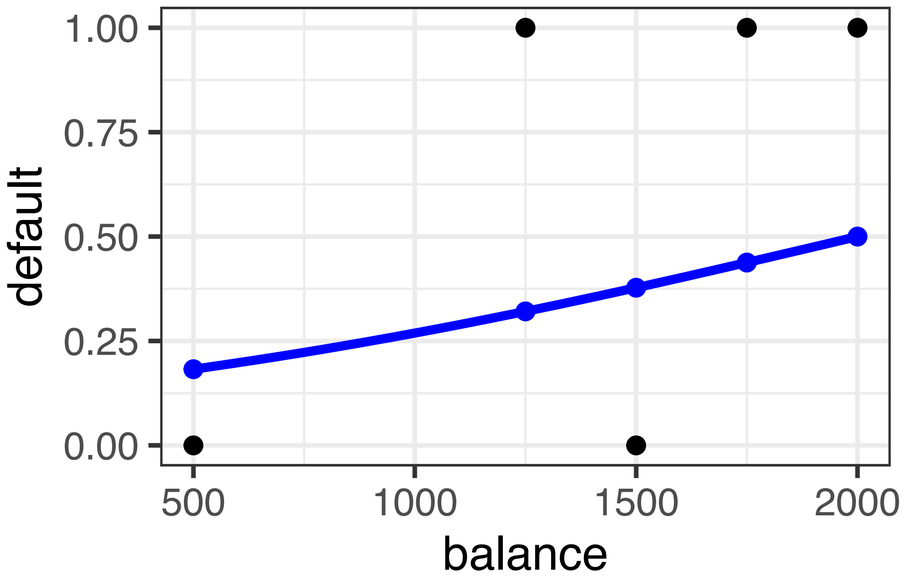


β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
-2.0	0.001	0.8	×	0.3×0.6×0.4×0.5	= 0.03

Maximum likelihood estimation

Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:

The maximum likelihood estimate (MLE) $(\hat{\beta}_0, \hat{\beta}_1)$ is defined as the maximizer of $\mathscr{L}(\beta_0, \beta_1)$.



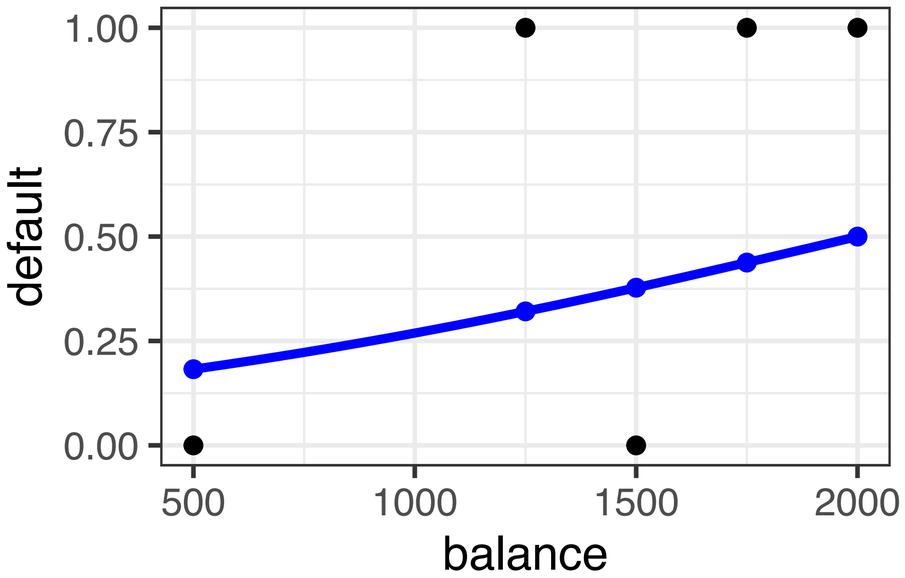
β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
-2.0	0.001	0.8	×	0.3×0.6×0.4×0.5	= 0.03

Maximum likelihood estimation

Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:

The maximum likelihood estimate (MLE) $(\hat{\beta}_0, \hat{\beta}_1)$ is defined as the maximizer of $\mathscr{L}(\beta_0, \beta_1)$.

It cannot be written in closed form; it is found via iterative algorithm.



β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
-2.0	0.001	0.8	×	0.3×0.6×0.4×0.5	= 0.03

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Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:

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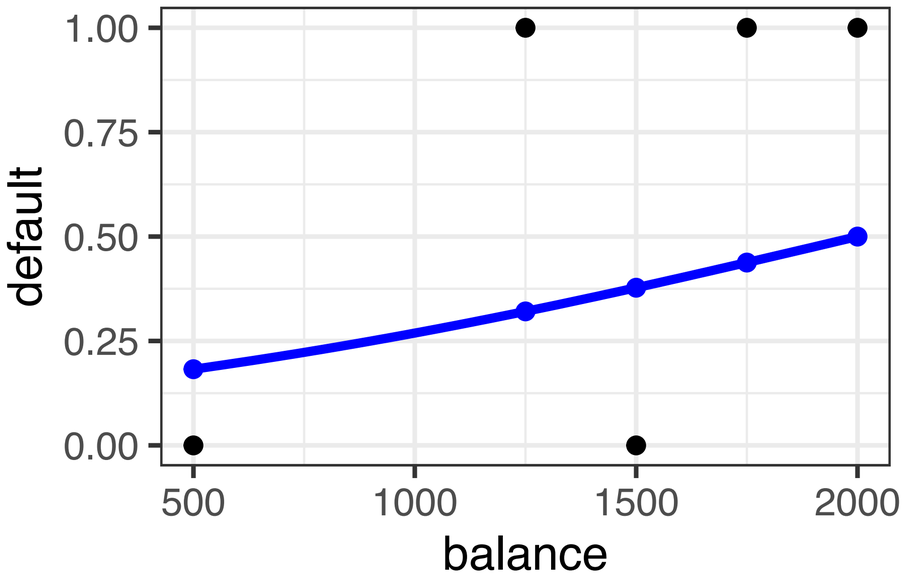

β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
-2.0	0.001	0.8	×	0.3×0.6×0.4×0.5	= 0.03
-4.6	0.004				

Maximum likelihood estimation

Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:

The maximum likelihood estimate (MLE) $(\hat{\beta}_0, \hat{\beta}_1)$ is defined as the maximizer of $\mathscr{L}(\beta_0, \beta_1)$.

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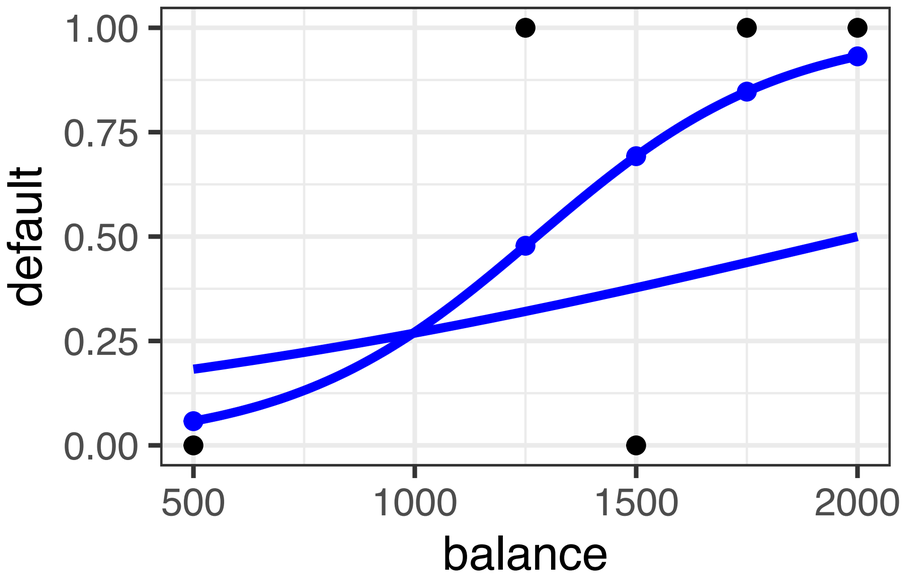

β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
-2.0	0.001	0.8	×	0.3×0.6×0.4×0.5	= 0.03
-4.6	0.004				

Maximum likelihood estimation

Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:

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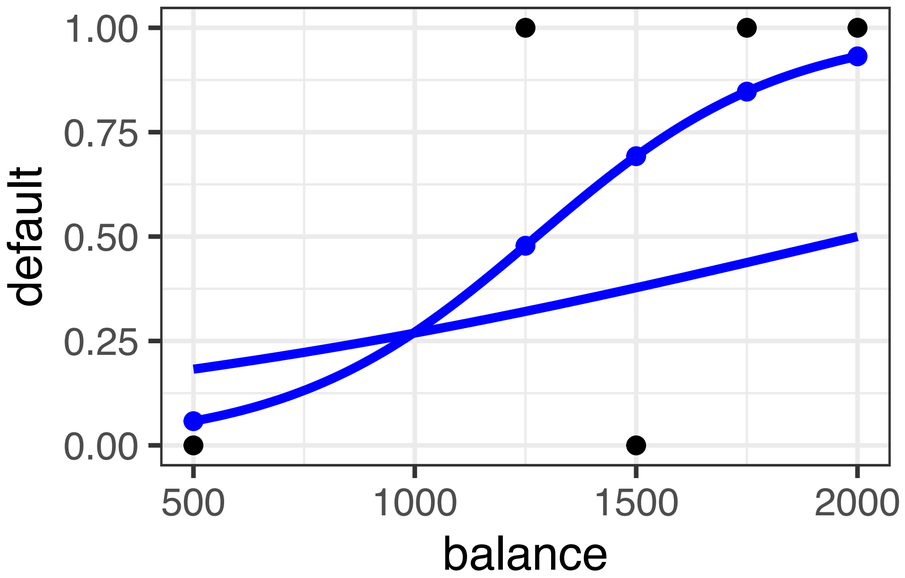

β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
-2.0	0.001	0.8	×	0.3×0.6×0.4×0.5	= 0.03
-4.6	0.004	0.9		0.5 0.3 0.8 0.9	

Maximum likelihood estimation

Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:

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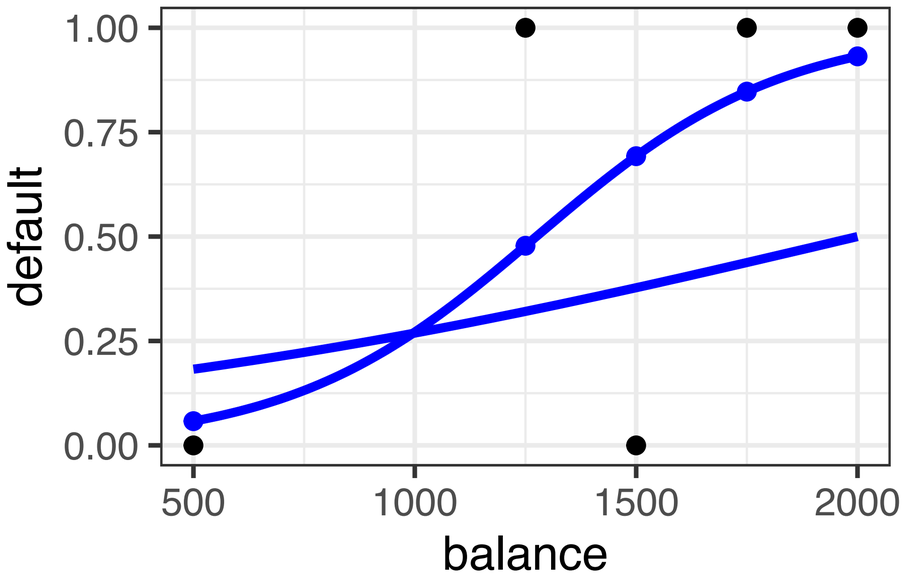

β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
-2.0	0.001	0.8	×	0.3×0.6×0.4×0.5	= 0.03
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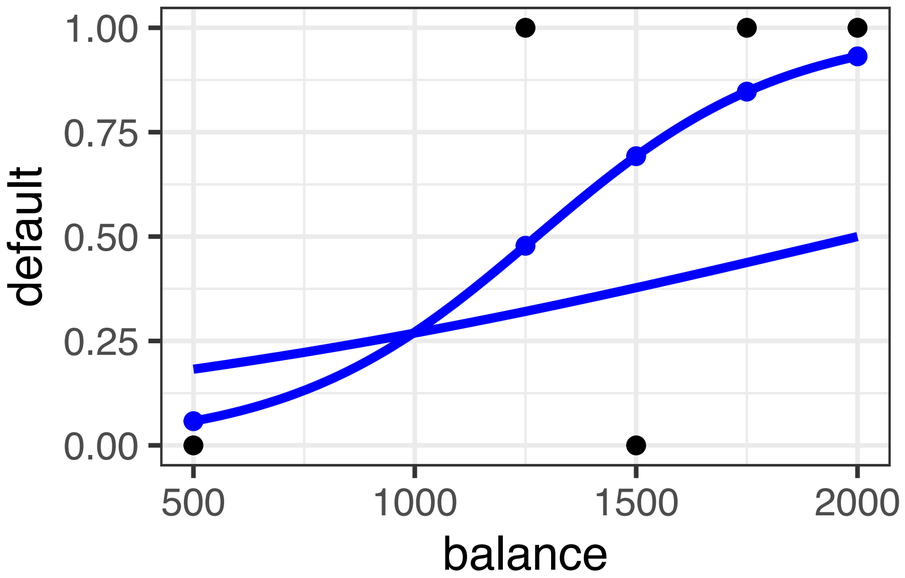

β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
-2.0	0.001	0.8	×	0.3×0.6×0.4×0.5	= 0.03
-4.6	0.004	0.9	×	0.5×0.3×0.8×0.9	= 0.1

Maximum likelihood estimation

Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:

The maximum likelihood estimate (MLE) $(\hat{\beta}_0, \hat{\beta}_1)$ is defined as the maximizer of $\mathscr{L}(\beta_0, \beta_1)$.

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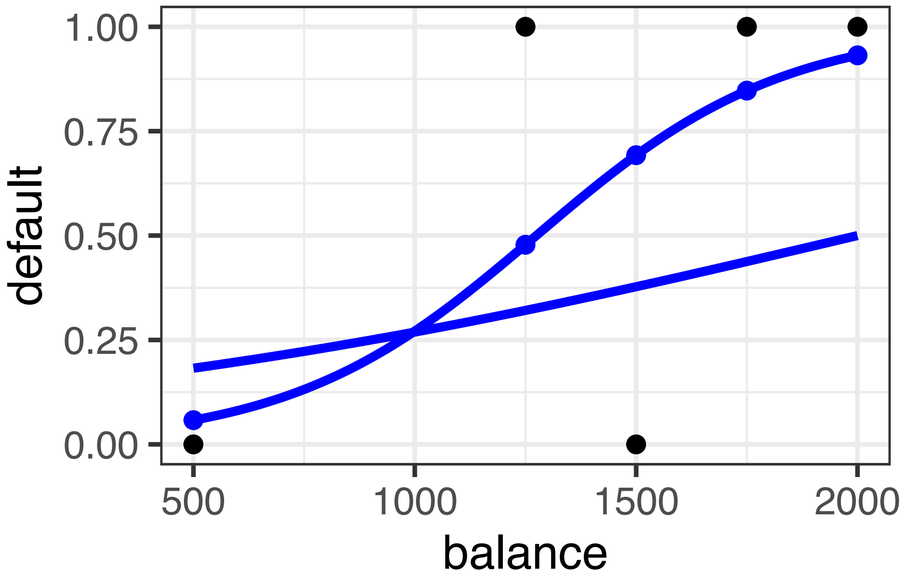

β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
-2.0	0.001	0.8	×	0.3×0.6×0.4×0.5	= 0.03
-4.6	0.004	0.9	×	0.5×0.3×0.8×0.9	= 0.1
-15.0	0.01				

Maximum likelihood estimation

Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:

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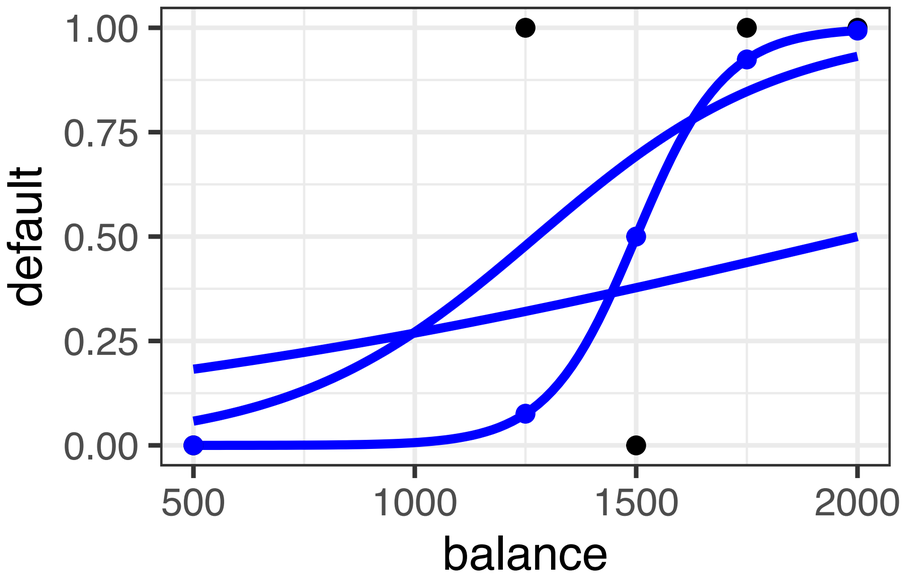

β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
-2.0	0.001	0.8	×	0.3×0.6×0.4×0.5	= 0.03
-4.6	0.004	0.9	×	0.5×0.3×0.8×0.9	= 0.1
-15.0	0.01				

Maximum likelihood estimation

Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:

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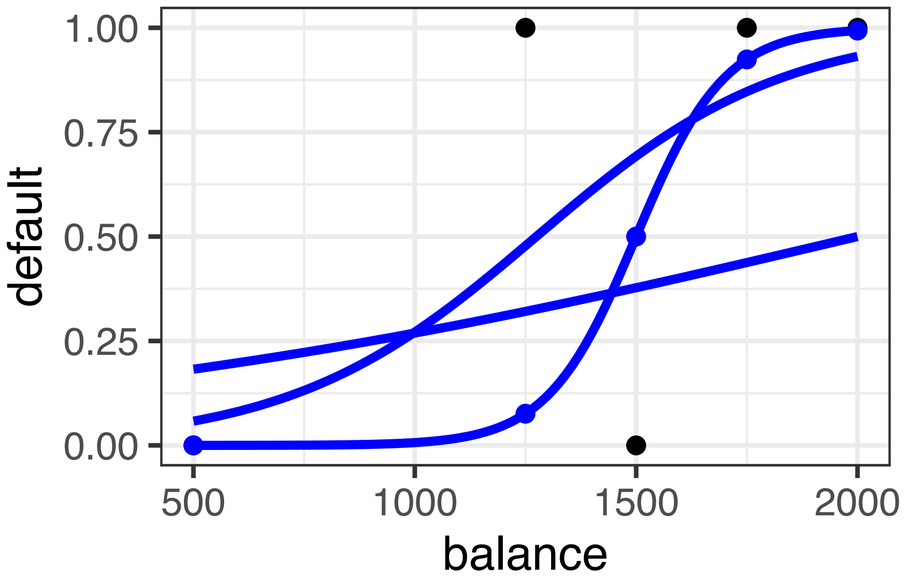

β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
-2.0	0.001	0.8	×	0.3×0.6×0.4×0.5	= 0.03
-4.6	0.004	0.9	×	0.5×0.3×0.8×0.9	= 0.1
-15.0	0.01	1.0		0.1 0.5 0.9 1.0	

Maximum likelihood estimation

Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:

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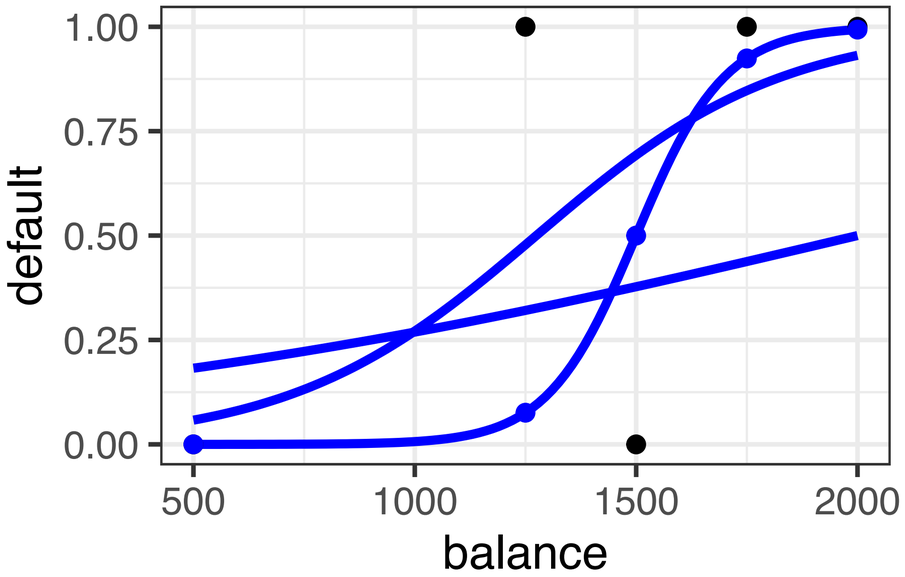

β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
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Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:

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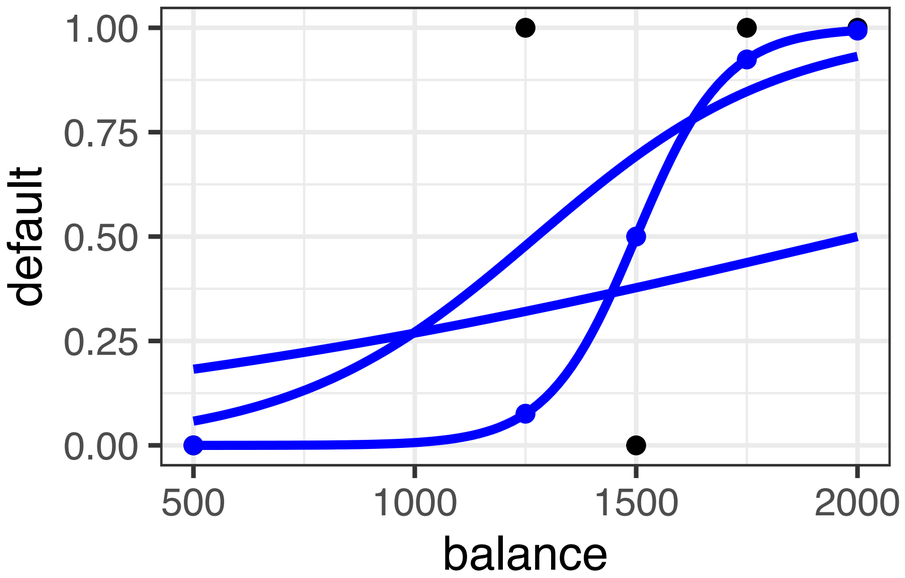

β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
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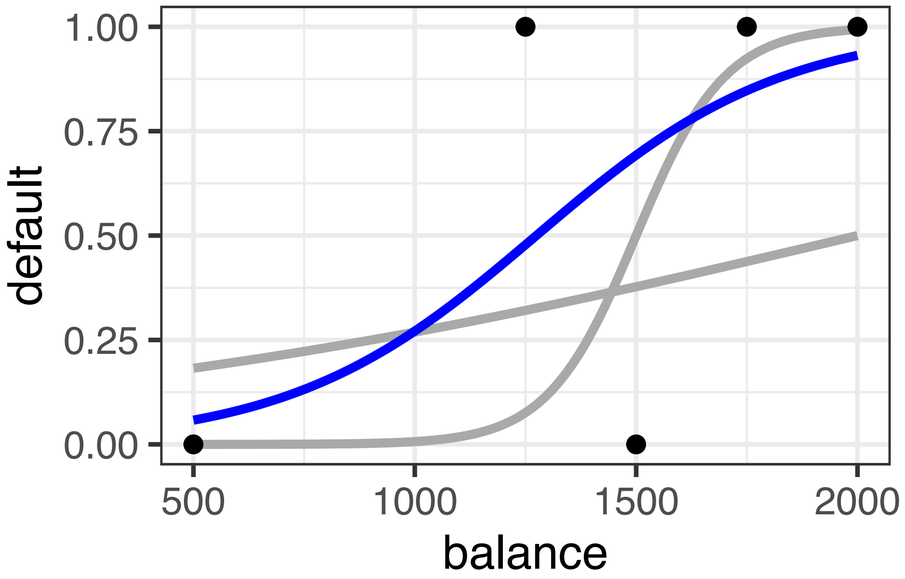

β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
-2.0	0.001	0.8	×	0.3×0.6×0.4×0.5	= 0.03
-4.6	0.004	0.9	×	0.5×0.3×0.8×0.9	= 0.1
-15.0	0.01	1.0	×	0.1×0.5×0.9×1.0	= 0.05

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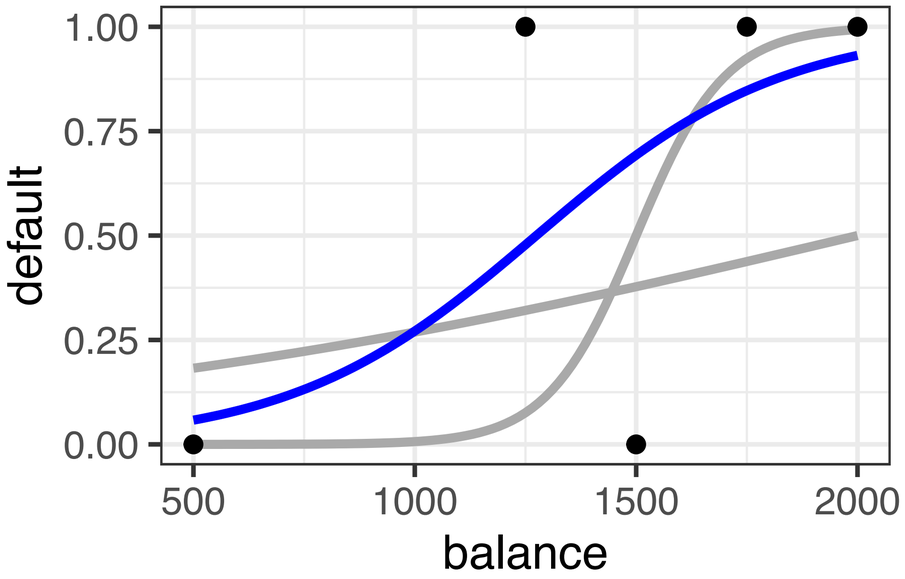

	β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
	-2.0	0.001	0.8	×	0.3×0.6×0.4×0.5	= 0.03
$(\hat{\beta}_0, \hat{\beta}_1) =$	(-4.6 ,	0.004)	0.9	×	0.5×0.3×0.8×0.9	= 0.1
	-15.0	0.01	1.0	×	0.1×0.5×0.9×1.0	= 0.05

Multiple logistic regression

Like with linear regression, can include multiple features, e.g.

$$\begin{aligned}\mathbb{P}[\text{default} \mid \text{student}, \text{balance}, \text{income}] \\ = \text{logistic}(\beta_0 + \beta_1 \cdot \text{student} + \beta_2 \cdot \text{balance} + \beta_3 \cdot \text{income})\end{aligned}$$

The logistic regression likelihood, as well as the maximum likelihood estimates $(\widehat{\beta}_0, \widehat{\beta}_1, \widehat{\beta}_2, \widehat{\beta}_3)$ are defined analogously.

Interpreting logistic regression coefficients

$$\mathbb{P}[\text{default}] = \text{logistic}(\beta_0 + \beta_1 \cdot \text{student} + \beta_2 \cdot \text{balance} + \beta_3 \cdot \text{income})$$

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For given (student, balance, income), suppose $\mathbb{P}[\text{default}] = 1/4$.

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Then, odds = 1:3 = 1/3 and log-odds = log(1/3) $\approx$ -1.

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Increasing balance by 500 while controlling for the other features tends to (additively) increase the log-odds of default by $500 \cdot \beta_2$.

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New odds are $e^1 \approx 2.7 = 2.7 : 1$, so new prob is 2.7/3.7 ≈ 0.7.
Odds went from e^{-1} (1/3) to e^1 (2.7), increase by factor of $e^2 \approx 7.5$.

Classification via logistic regression

$$\text{default} = \begin{cases} \text{Yes}, & \text{if } \widehat{\mathbb{P}}[\text{default}] \geq 0.5; \\ \text{No}, & \text{if } \widehat{\mathbb{P}}[\text{default}] < 0.5. \end{cases}$$

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Logistic regression has a linear decision boundary.

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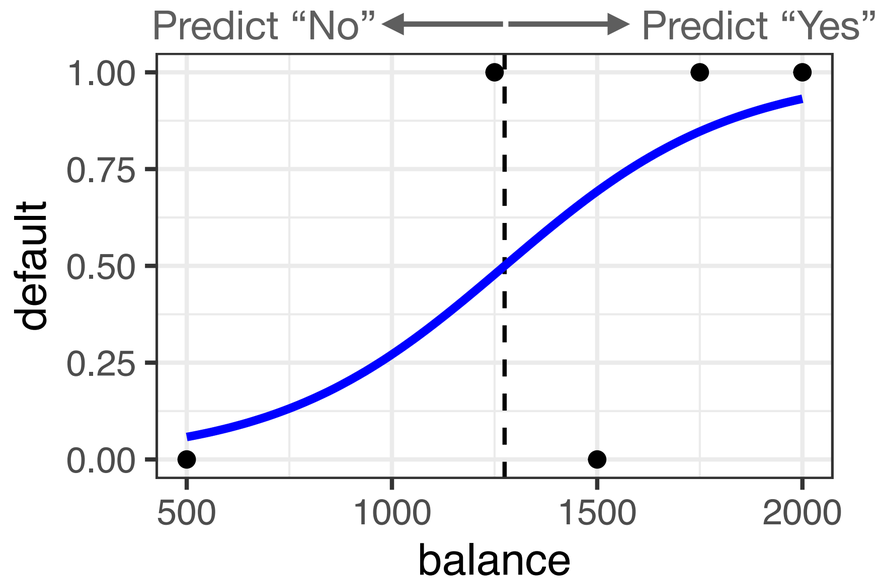

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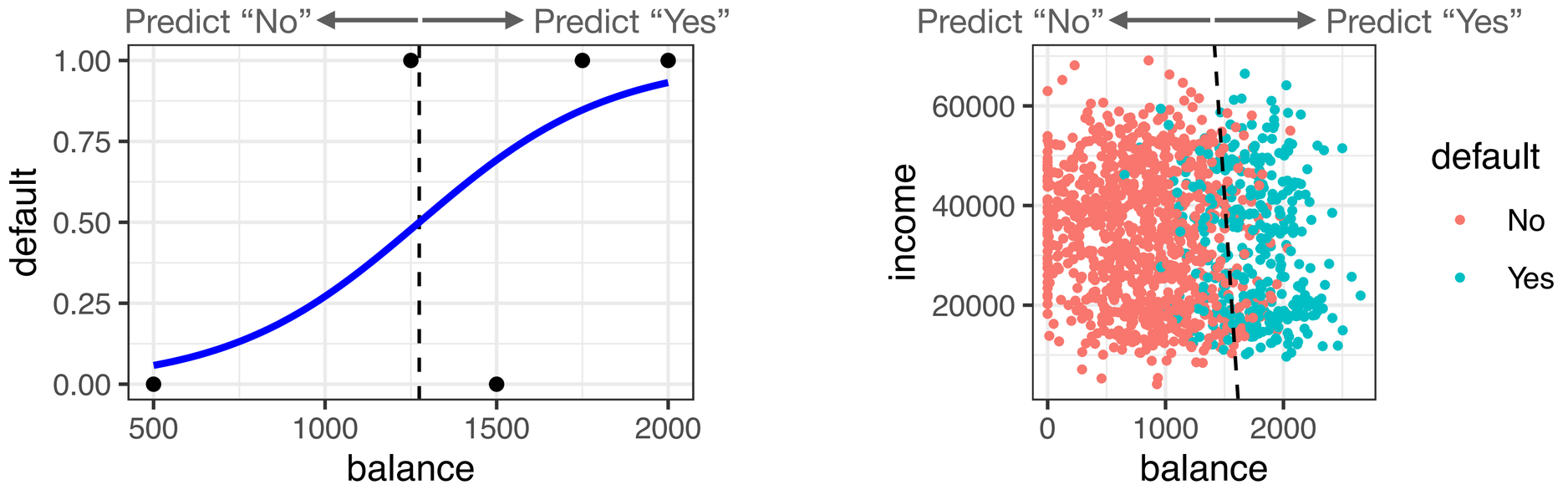
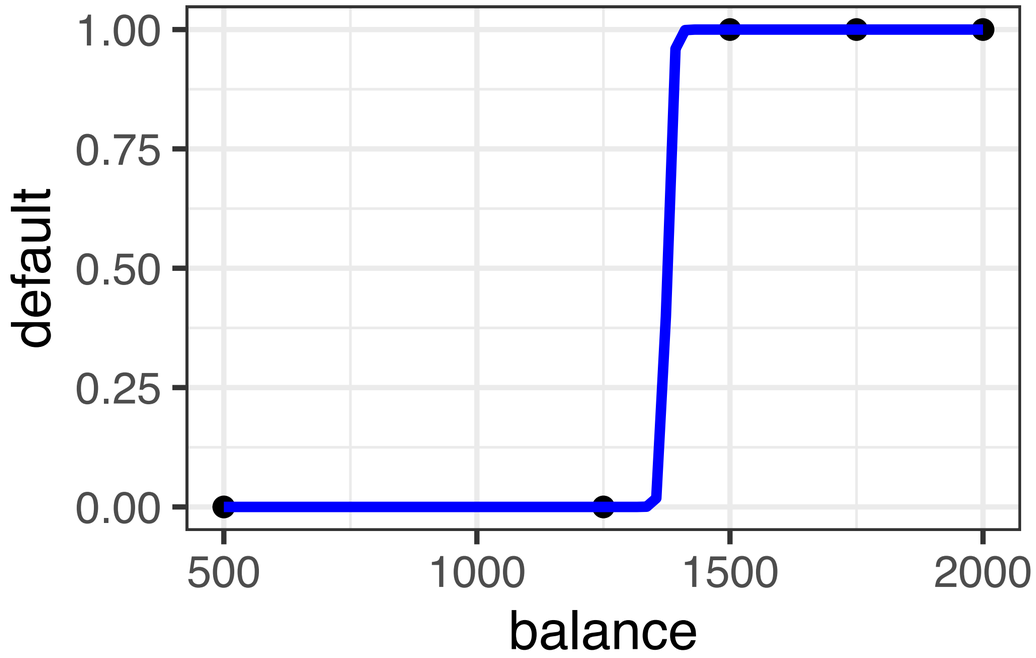


Caution: Separable data

When the two classes of response variable can be perfectly separated in feature space, logistic regression solution undefined, though perfect predictions possible.



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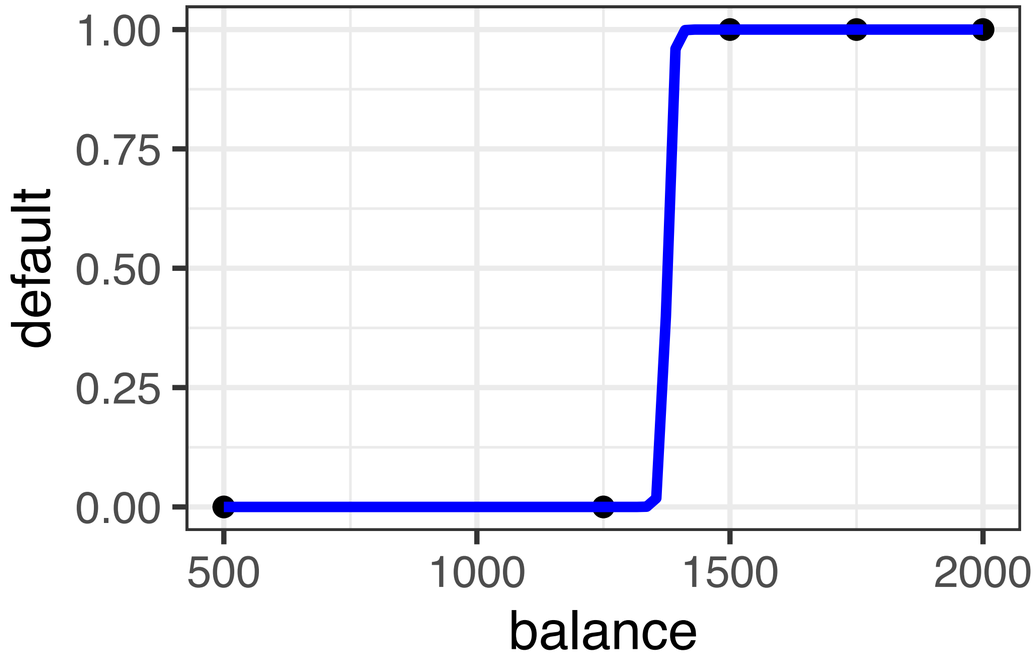

A similar phenomenon occurs in linear regression under perfect multicollinearity: The coefficient estimates are undefined but good prediction still possible.

Summary

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