Classification

STAT 4710

September 26, 2023

Where we are

Unit 1: R for data mining

Unit 2: Prediction fundamentals

Unit 3: Regression-based methods

Unit 4: Tree-based methods

Unit 5: Deep learning

Lecture 1: Model complexity

Lecture 2: Bias-variance trade-off

Lecture 3: Cross-validation

Lecture 4: Classification

Lecture 5: Unit review and quiz in class

Recall: Clinical decision support

A patient comes into the emergency room with stroke symptoms. Based on her CT scan, is the stroke ischemic or hemorrhagic?

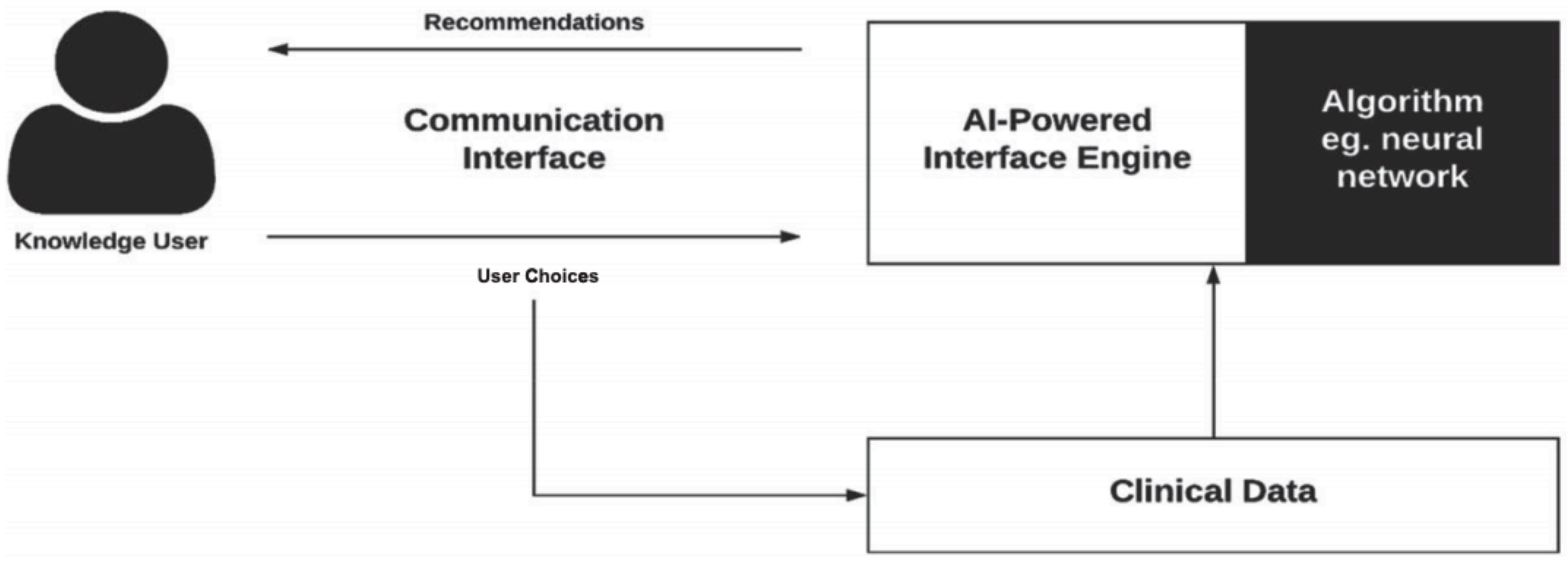

Image source: Sutton et al. 2020 (npj Digit. Med.)

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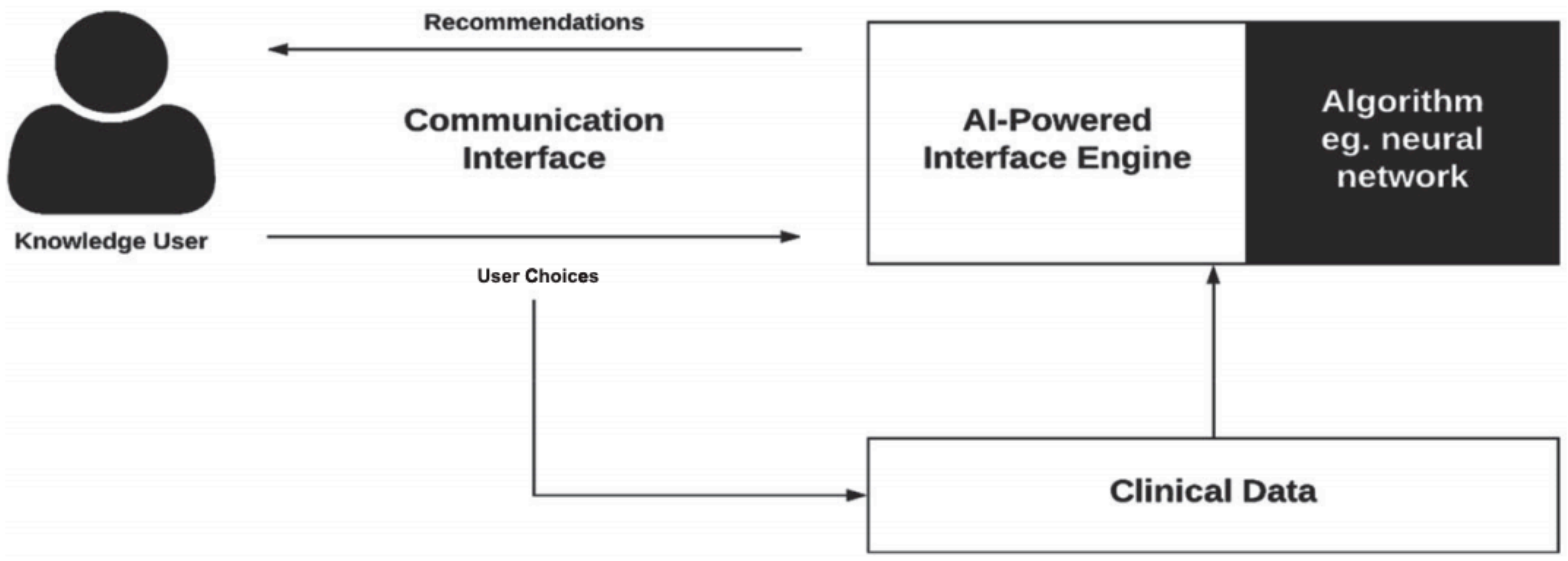

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This is a binary classification problem: $Y \in \{0,1\}$.

Given features $X = (X_1, \ldots, X_p)$, the goal is to guess a response $\widehat{Y} = \hat{f}(X)$ that is close to the true response, i.e. $\widehat{Y} \approx Y$. Measure of success is usually the

$$\text{test misclassification error} = \frac{1}{N}\sum_{i=1}^{N} I(Y_i^{\text{test}} \neq \hat{f}(X_i^{\text{test}})).$$

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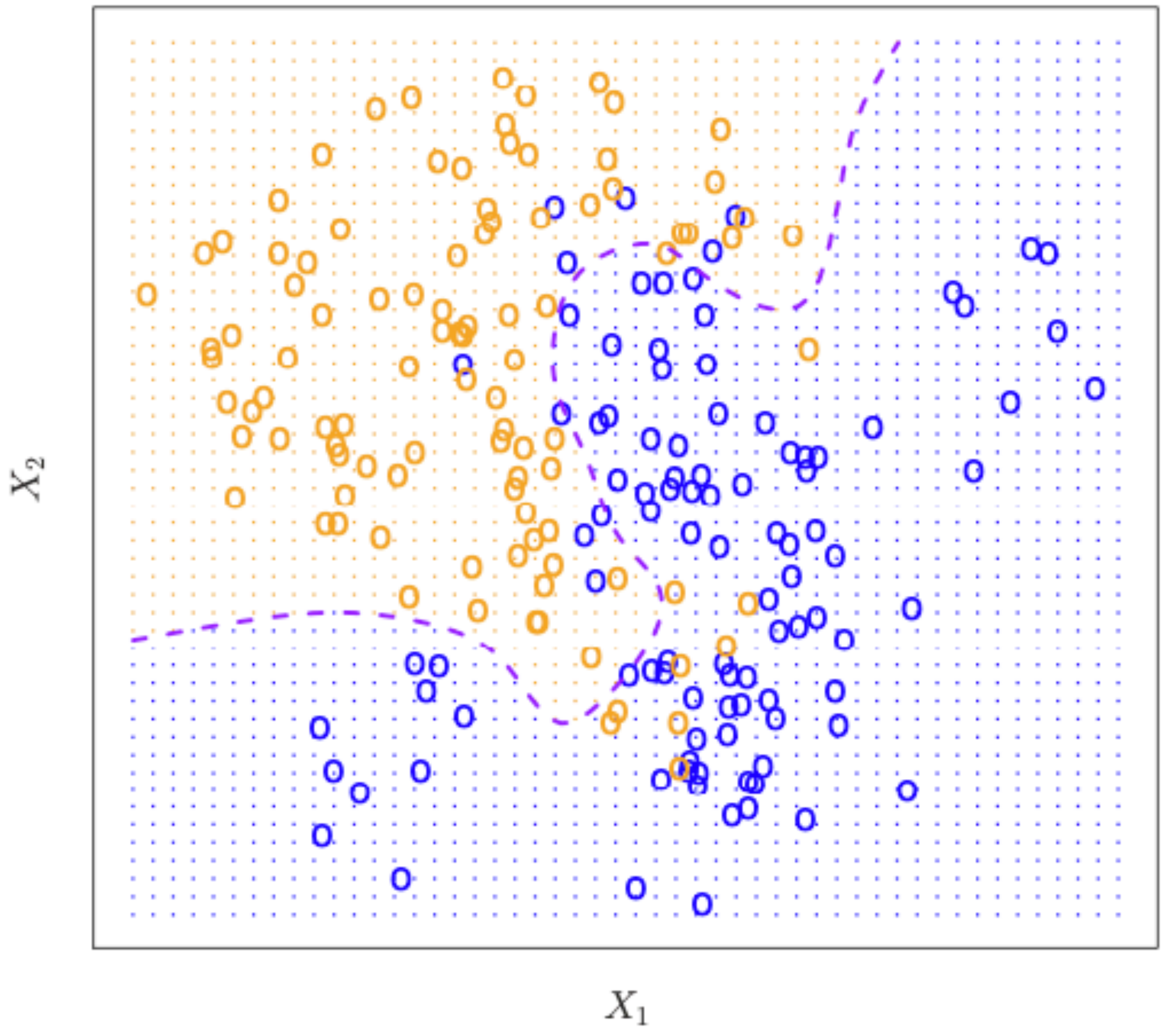
Classifiers usually build an approximation $\widehat{p}(X) \approx \mathbb{P}[Y = 1 \mid X]$, and define

$$\hat{f}(X) = \begin{cases} 1, & \text{if } \widehat{p}(X) \geq 0.5; \\ 0 & \text{if } \widehat{p}(X) < 0.5. \end{cases}$$

Example: K-nearest neighbors



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Bayes classifier in purple.
E.g., color = stroke type, (X_1, X_2) = CT image.

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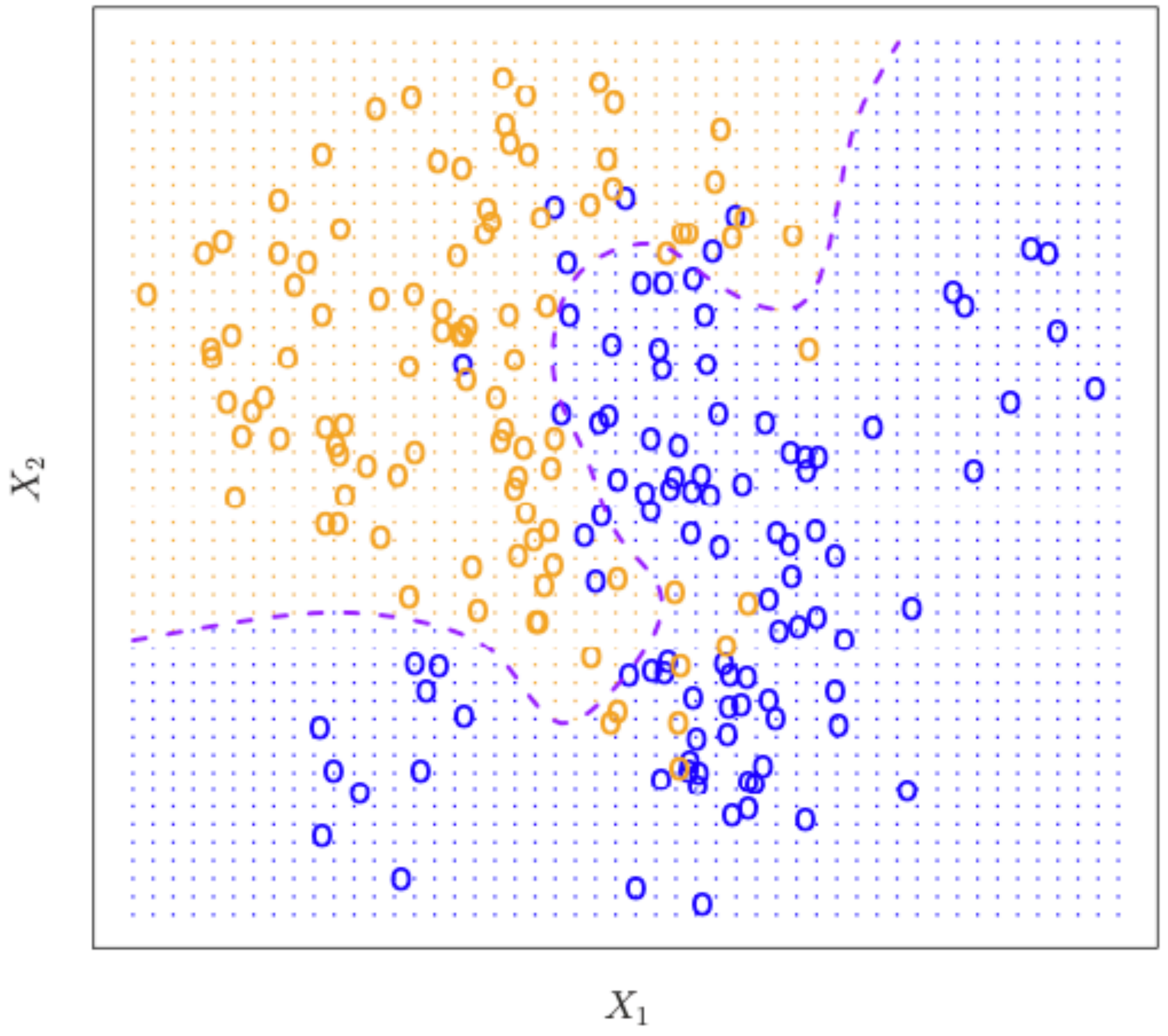

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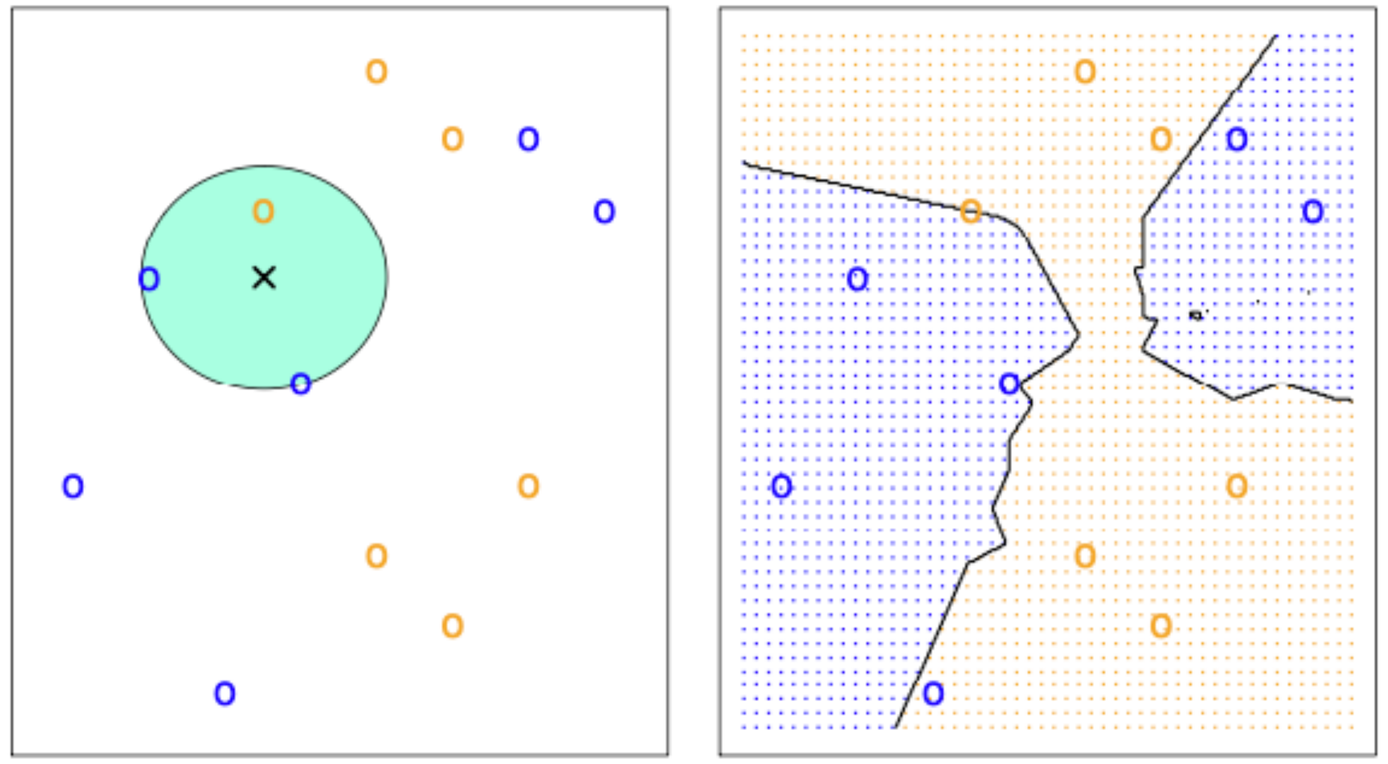

KNN illustration: Classify a test point based on majority vote among 3 nearest neighbors.

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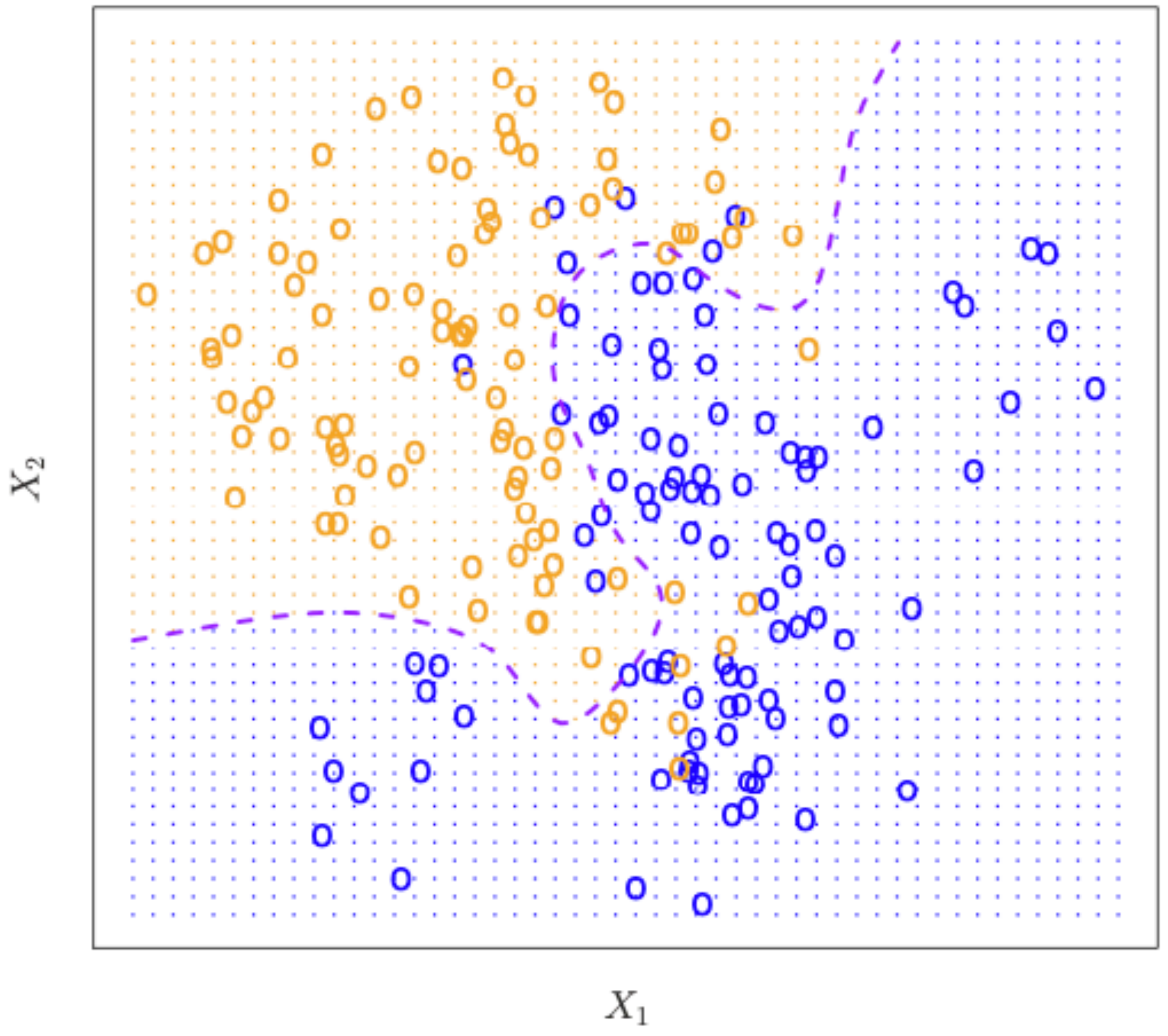

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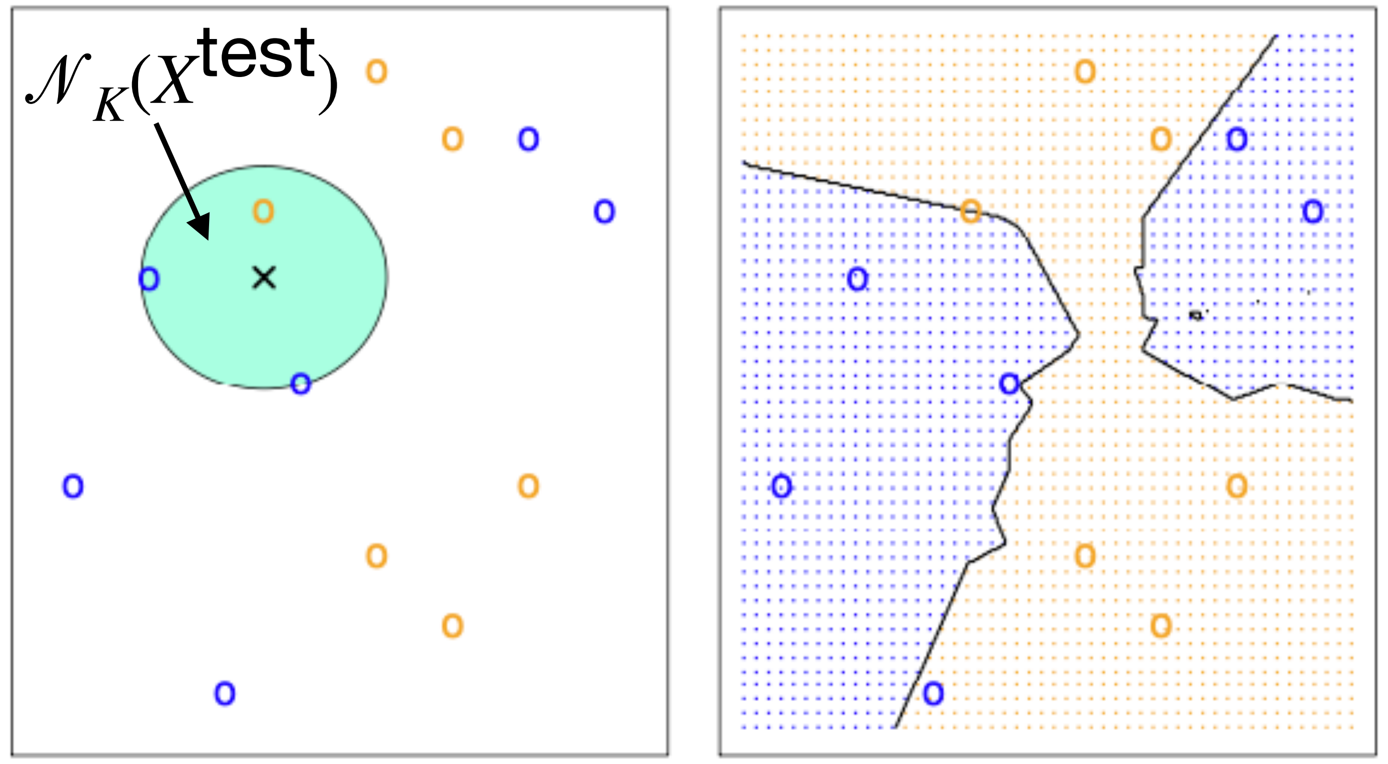

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$$\hat{p}(X^{\text{test}}) = \frac{1}{K} \sum_{i \in \mathcal{N}_K} I(Y_i^{\text{train}} = 1)\,.$$

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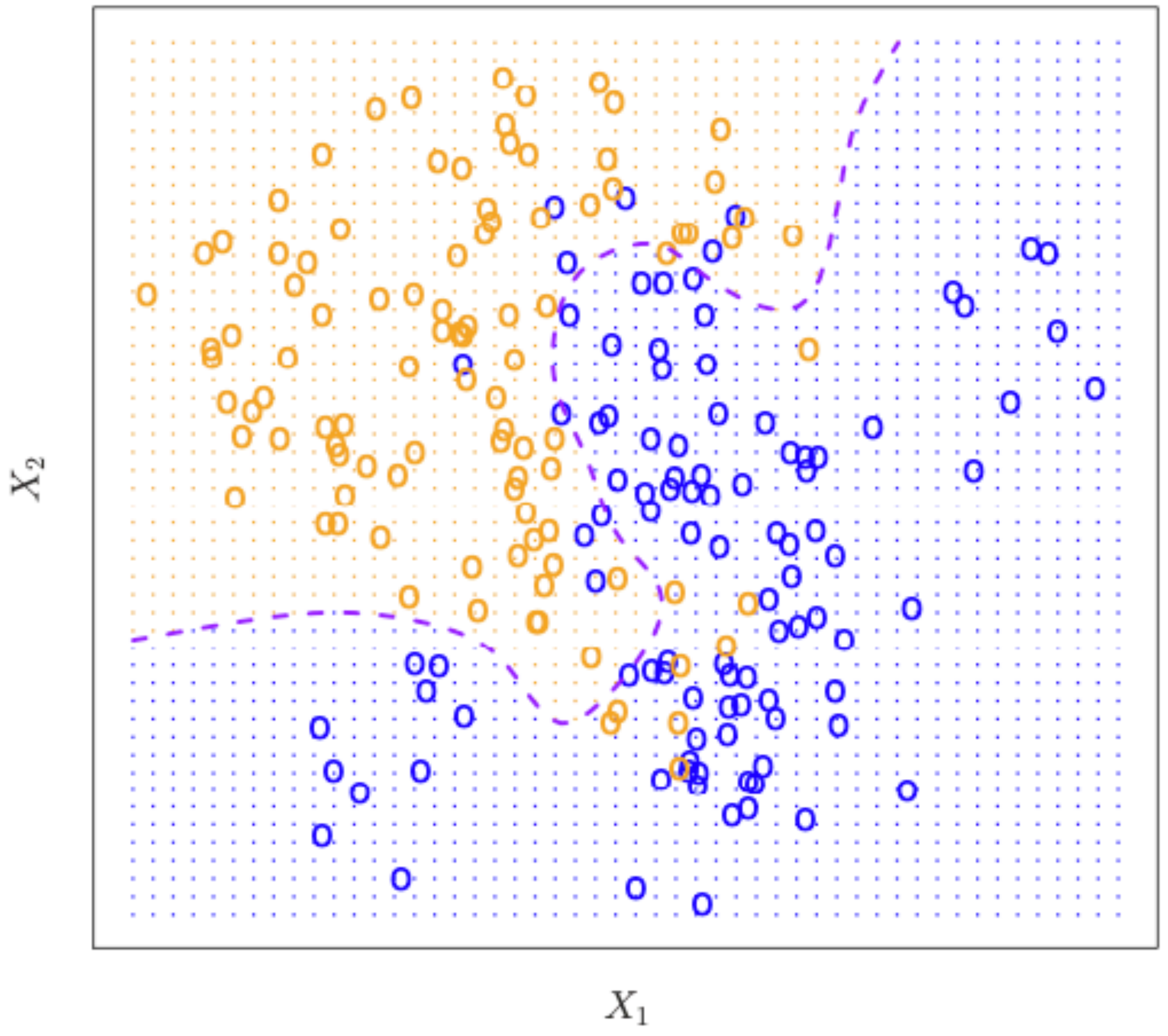

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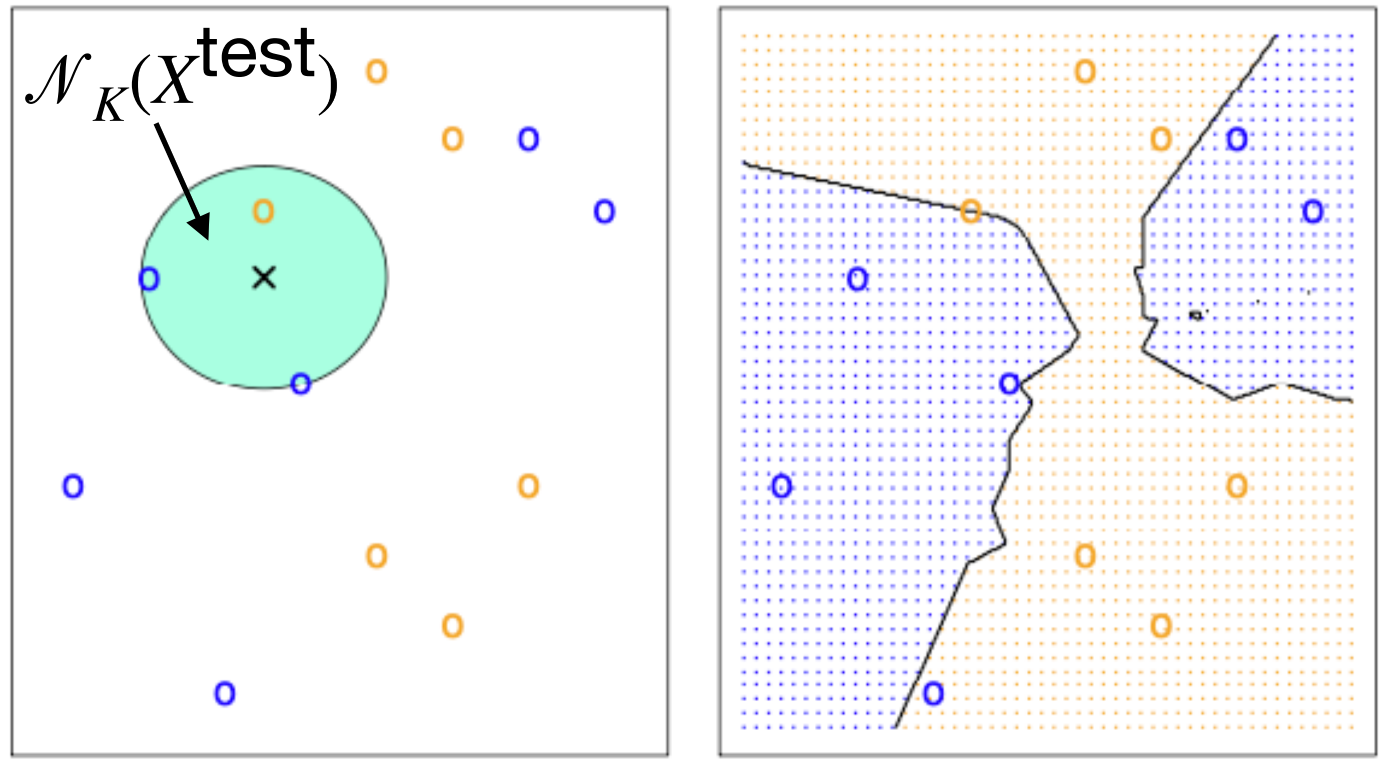

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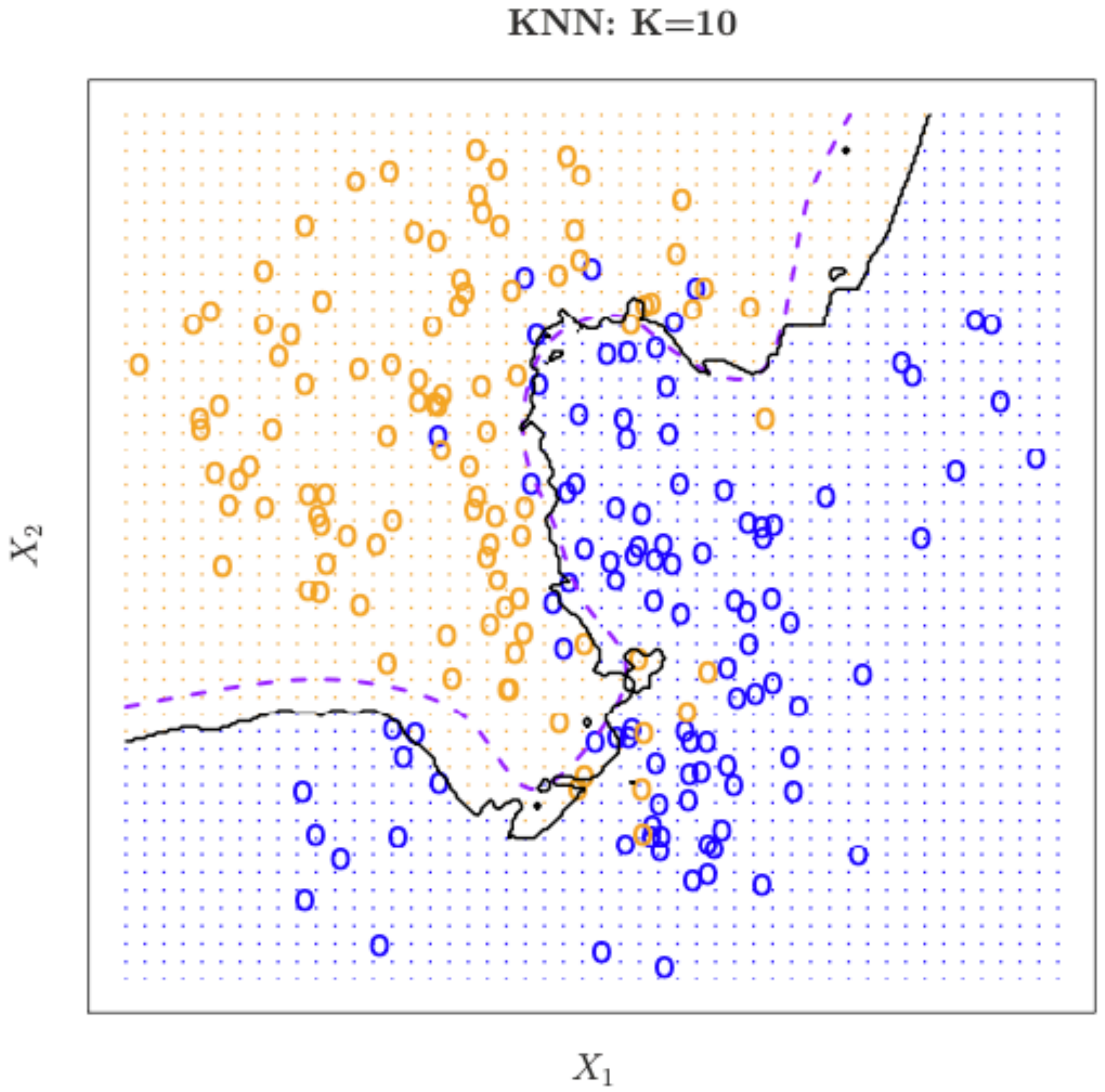
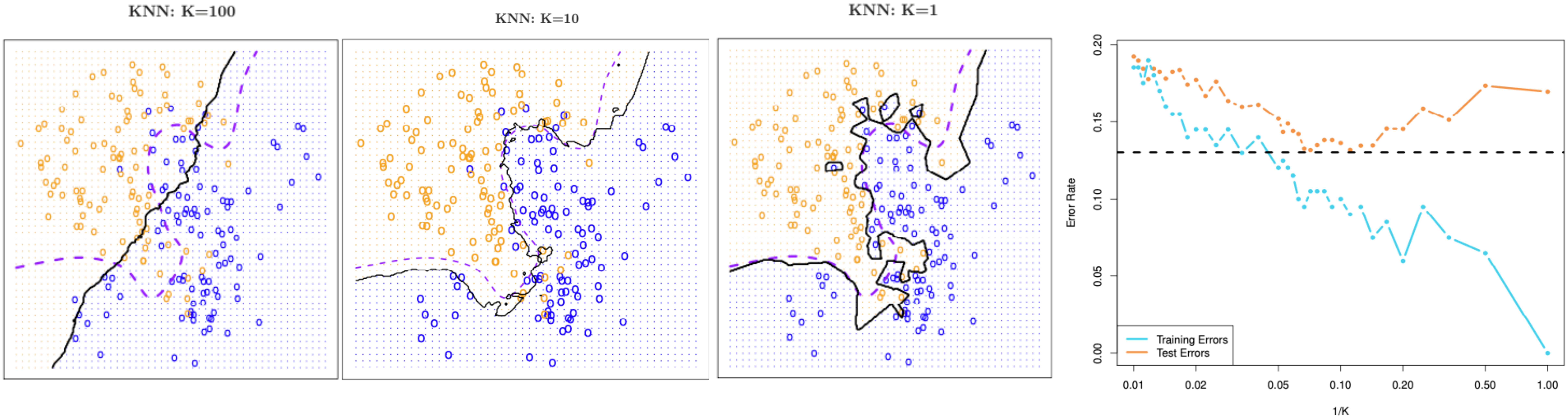


Applying KNN with K = 10 to simulated data.

Model complexity and misclassification error



Same Goldilocks principle as in regression case:

- Too little complexity: Can't capture the true trend in the data.
- Too much complexity: Too sensitive to noise in the training data (overfitting).

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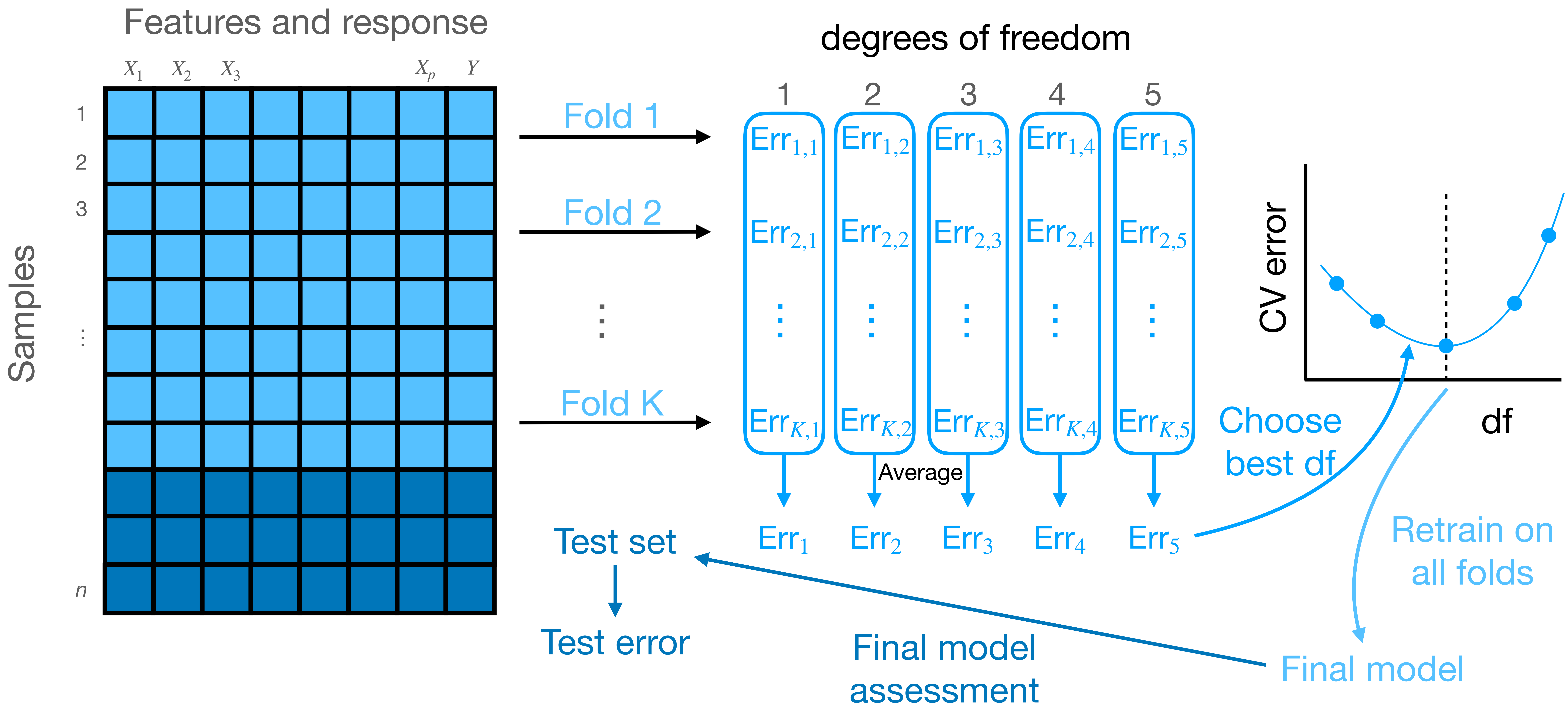
- Variance: Prediction varies with training set, to the extent $\widehat{p}$ fluctuates above or below 0.5
- Irreducible error (AKA Bayes error): Error incurred by Bayes classifier because $0 < \mathbb{P}[Y = 1 \,|\, X] < 1$.

Cross-validation based on misclassification error
(otherwise same as before)



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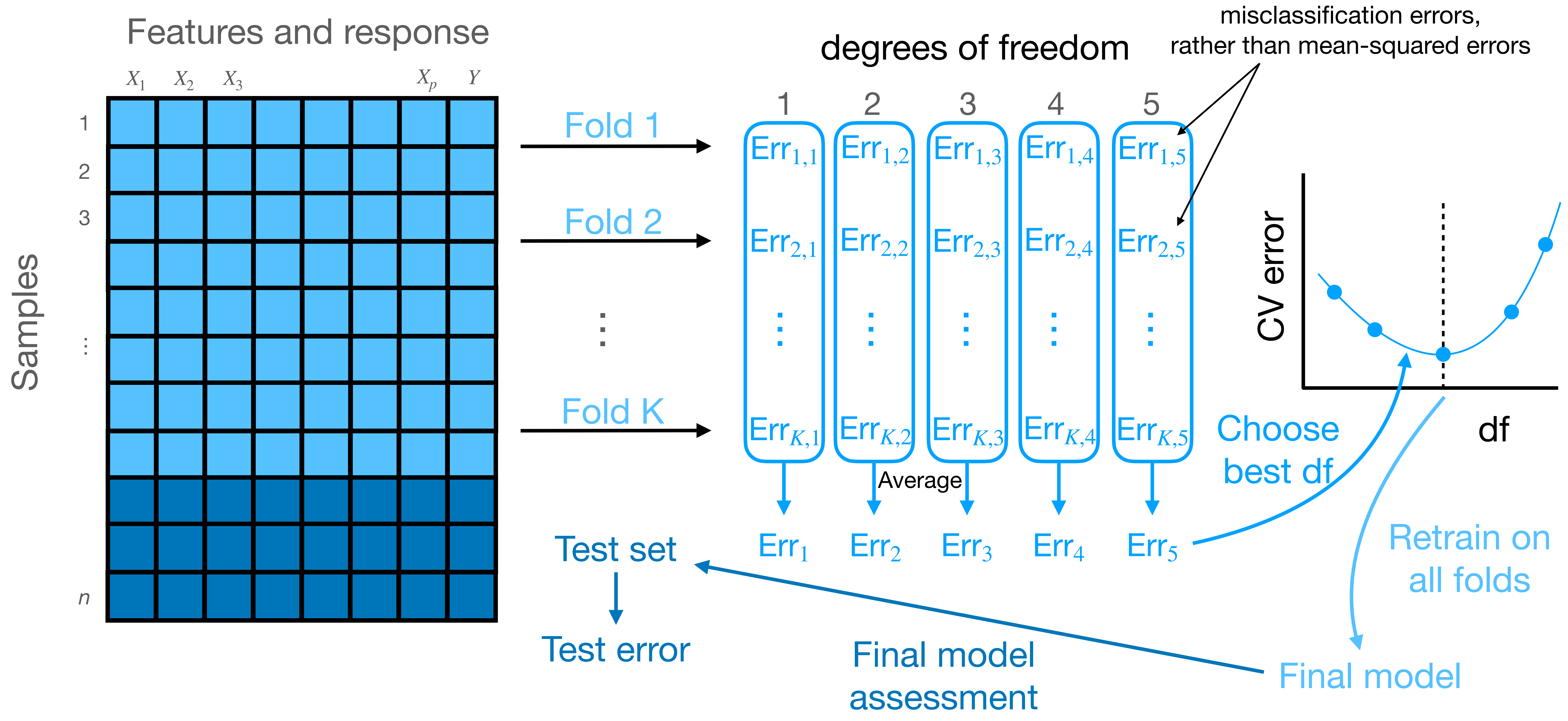

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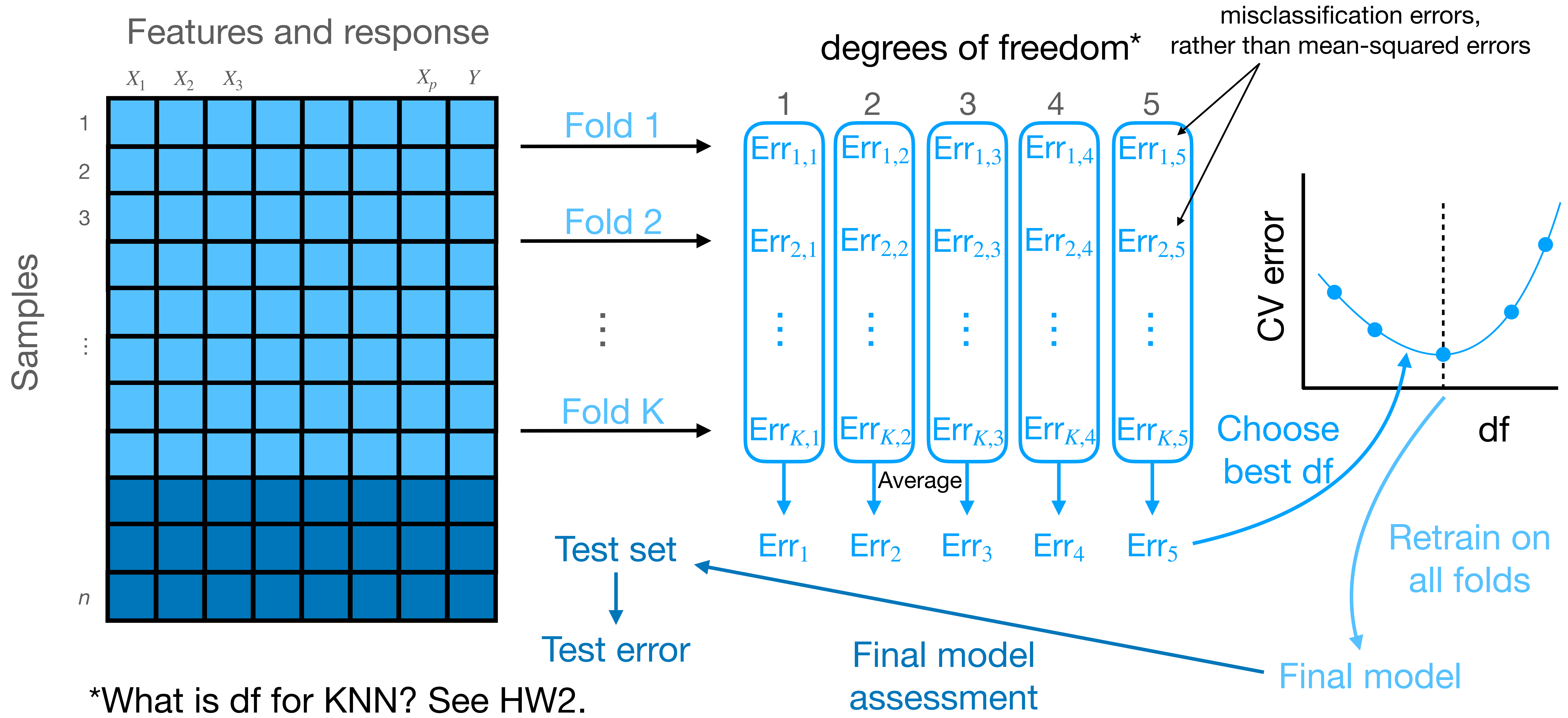
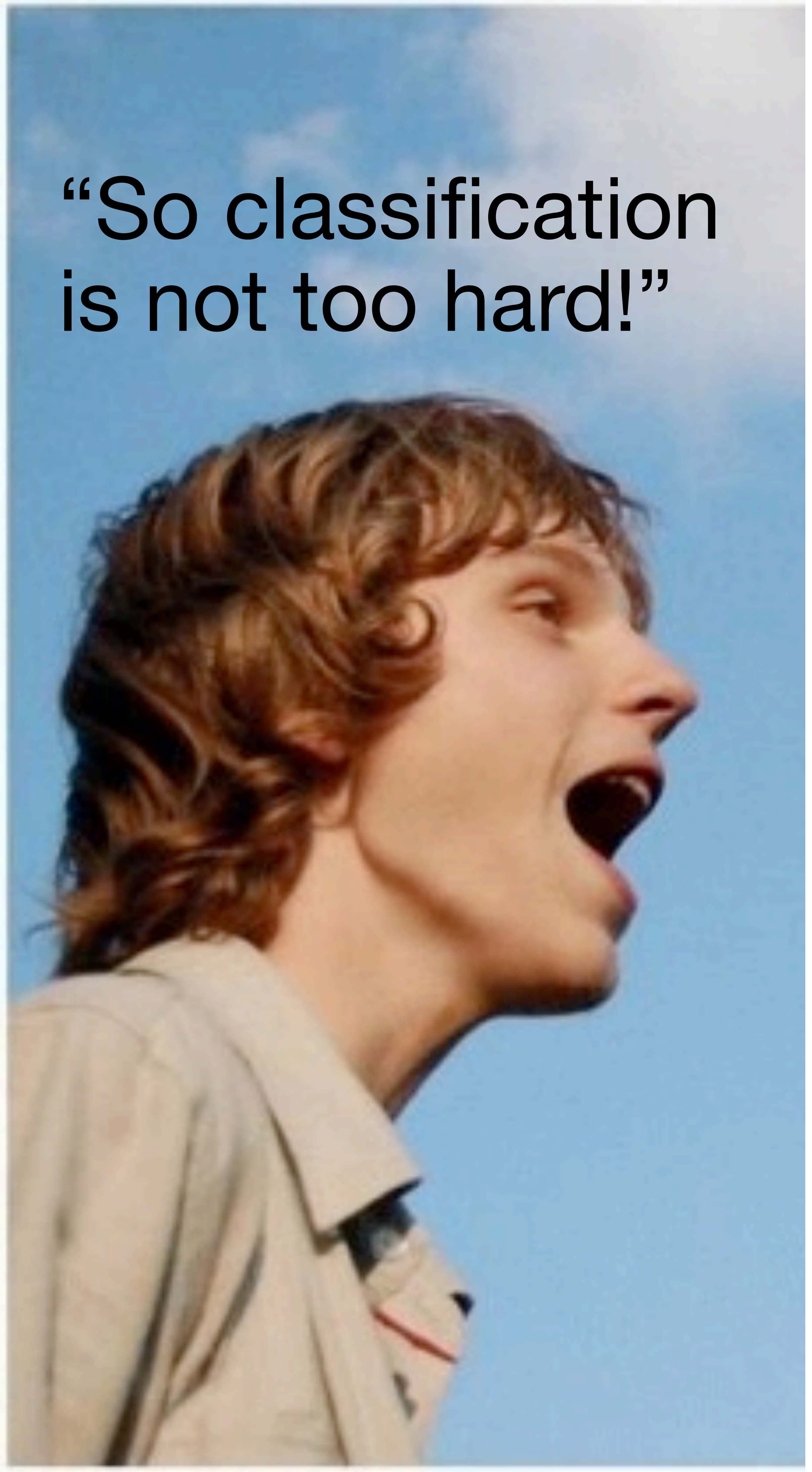


*What is df for KNN? See HW2.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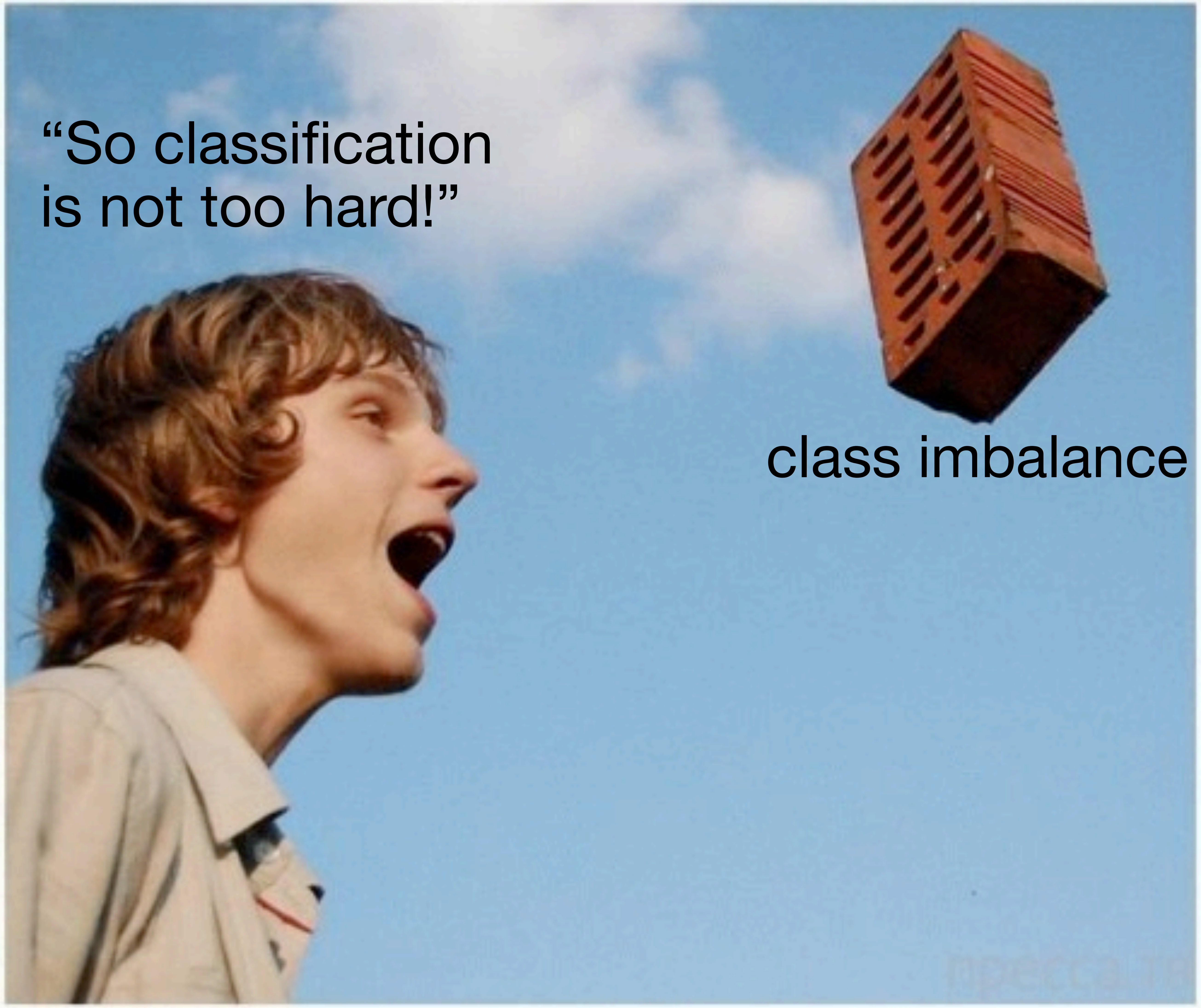
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Cross-validation based on misclassification error leads to overly simple models that ignore the minority class.

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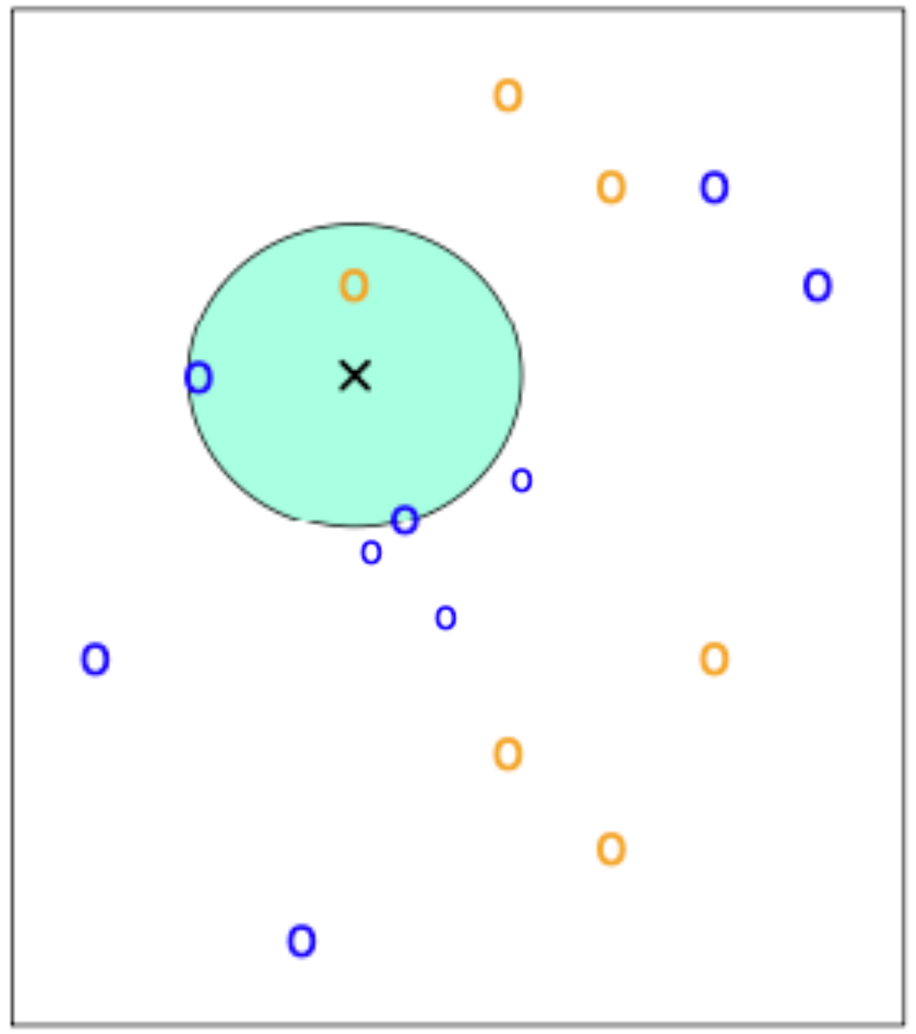

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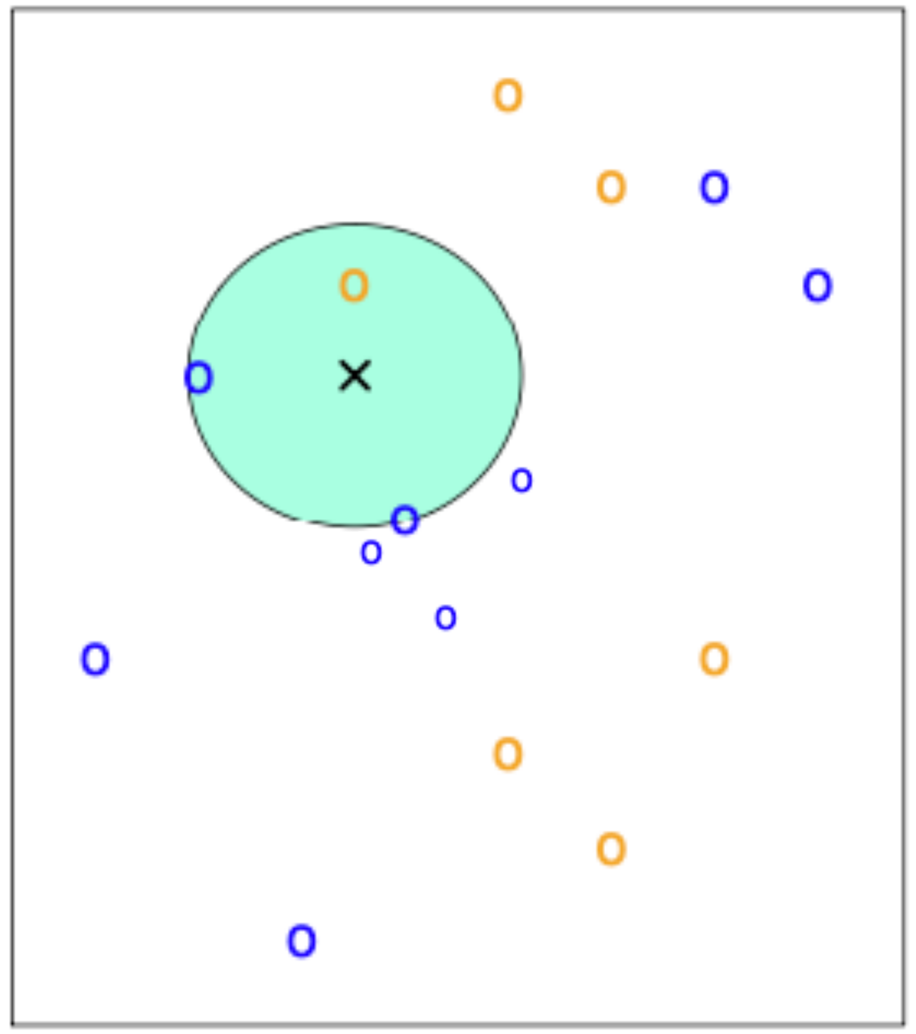

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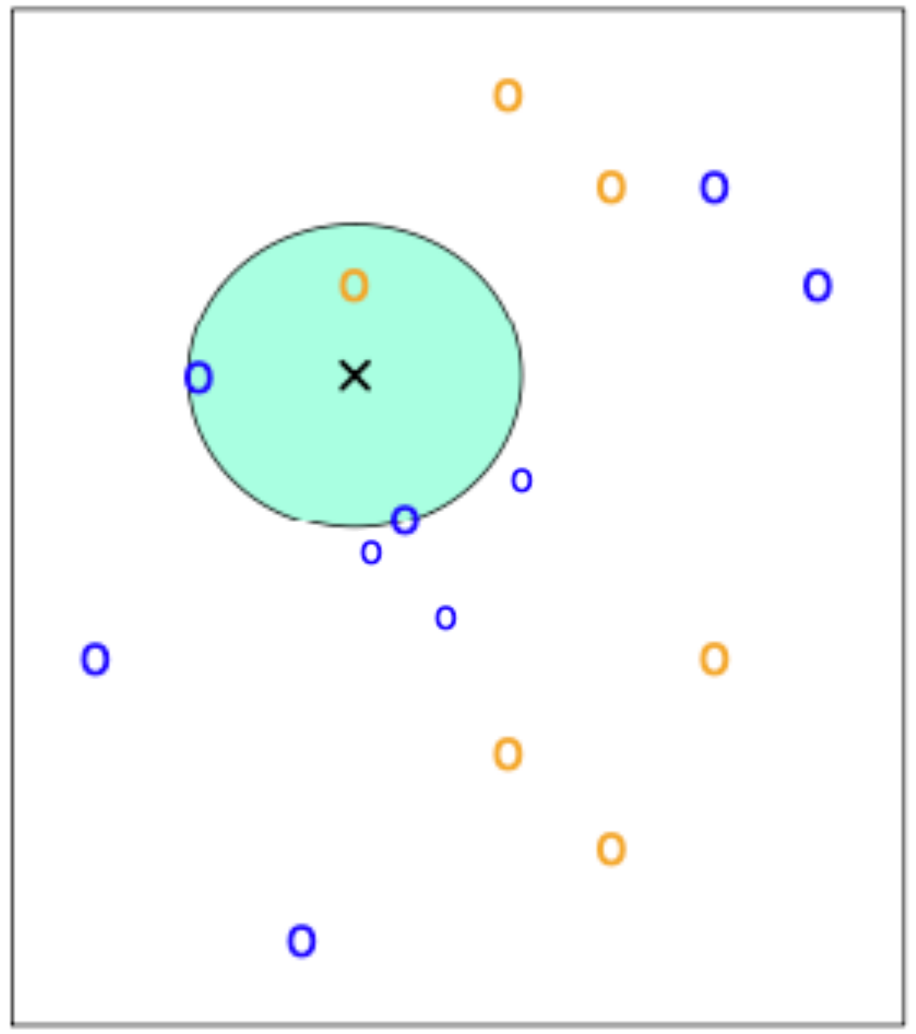

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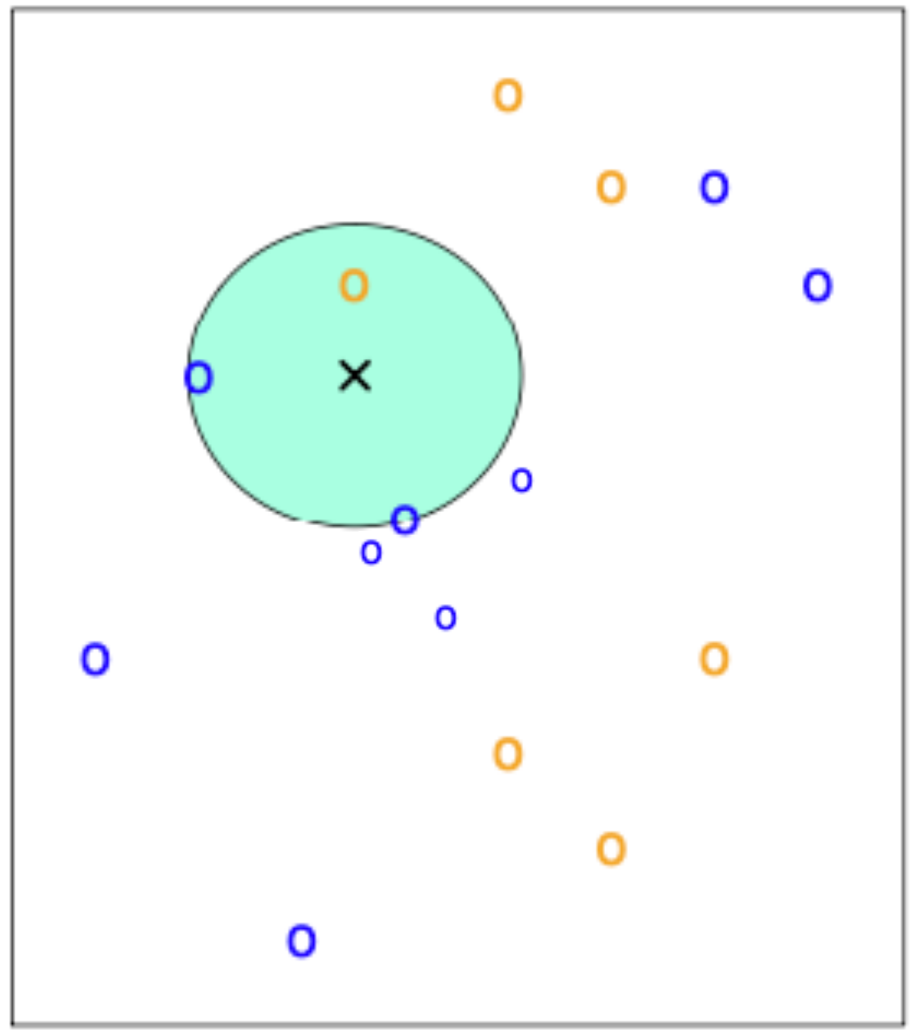

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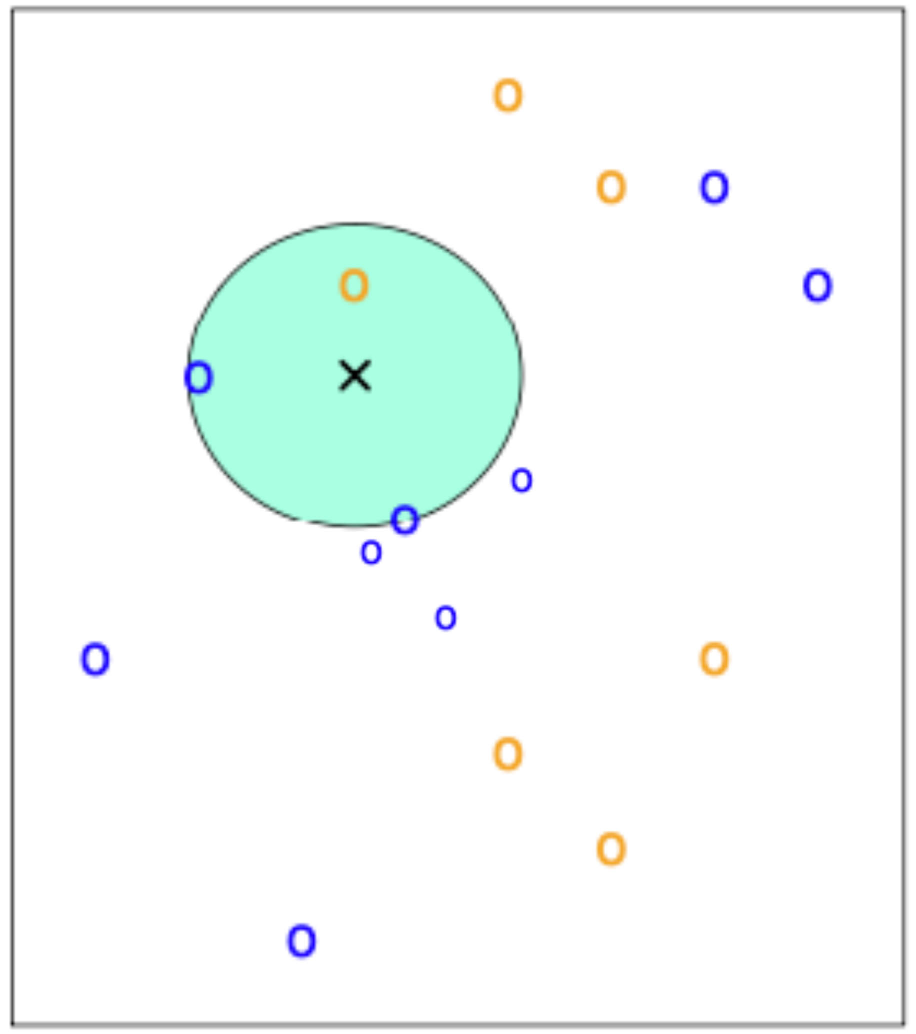

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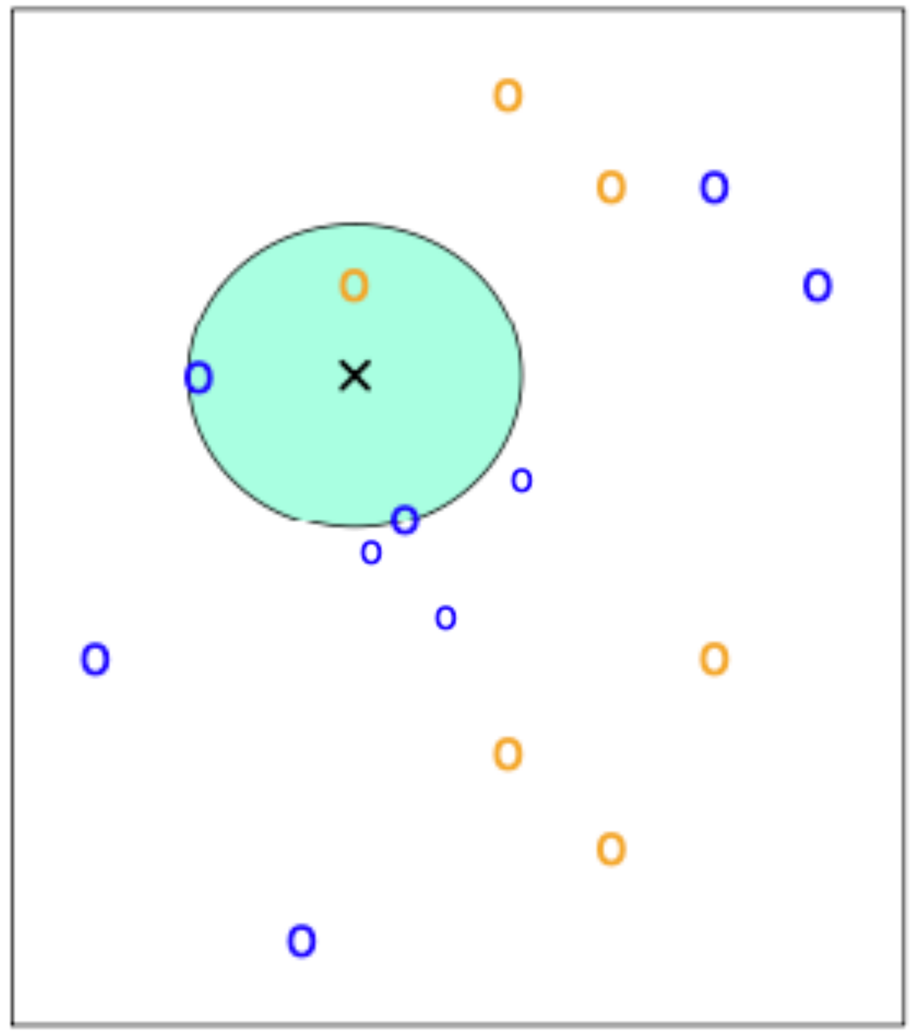

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- Use weighted misclassification error when assessing models on in-fold data.

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Another way of assessing classification performance—without quantifying costs—is the confusion matrix and associated metrics (e.g. precision and recall).

The confusion matrix and associated metrics

Confusion matrix

	Actually Positive	Actually Negative
Predicted Positive	10 True Positives (TP) (E.g. Sick person testing positive)	20 False Positives (FP) (E.g. Healthy person testing positive)
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$$\text{F-score} = \left(\frac{1}{2}\left(\frac{1}{\text{prec.}} + \frac{1}{\text{rec.}}\right)\right)^{-1} = \frac{1}{4}$$

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Metrics based on confusion matrix

$$\text{Precision} = \frac{\text{TP}}{\text{TP} + \text{FP}} = \frac{10}{30}$$

$$\text{Recall} = \frac{\text{TP}}{\text{TP} + \text{FN}} = \frac{10}{50}$$

$$\text{F-score} = \left(\frac{1}{2}\left(\frac{1}{\text{prec.}} + \frac{1}{\text{rec.}}\right)\right)^{-1} = \frac{1}{4}$$

One number summarizing performance of the classifier.

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- Performance metrics for classifiers include the weighted misclassification error and confusion matrix based metrics like precision and recall.