経済成長：Economic Growth

Quant Macro (Keio - Mita)

Growth matters

Anything that affects the long-run rate of economic growth will have huge impacts on living standards in the long run.

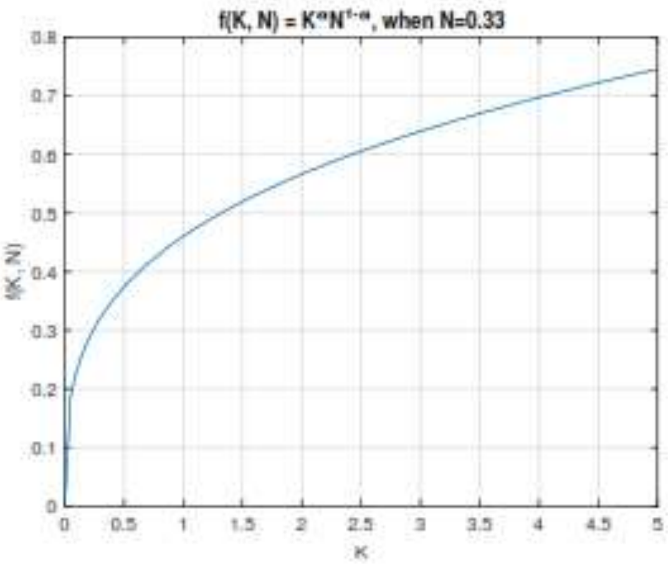
Annual growth rate of income per capita	Increase in standard of living after		
	25 years	50 years	100 years
2.0%	64%	169.2%	624.5%
2.5%	85.4%	243.7%	1081.4%

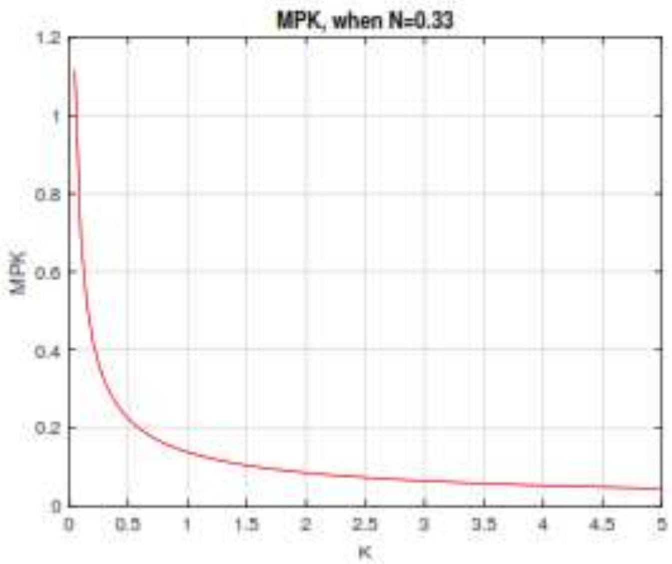
Solow Model

- Production Function
  - $Y_t = A_t F(K_t, N_t)$
  - capital - stock variable (accumulated over time)
  - labour - flow variable (endowment of labour/time)
  - A_t is exogenous productivity
  - $F(\cdot)$ is a function which relates K and N to Y
- Properties
  - $F_K > 0$ and $F_{KK} < 0$
  - $F_N > 0$ and $F_{NN} < 0$
  - Constant Returns to Scale (CRS). See seminar 4
    - $F(\gamma K_t, \gamma N_t) = \gamma F(K_t, N_t)$
  - $F(0, N_t) = 0$
  - $F(K_t, 0) = 0$
  - Example $F(K_t, N_t) = K_t^{\alpha} N_t^{1-\alpha}$, where $0 < \alpha < 1$

Solow Model

- $F_K > 0$ and $F_{KK} < 0$
- $F_N > 0$ and $F_{NN} < 0$
- How does output and MPK change, when capital changes, holding N_t and α fixed





Solow Model

- Firm

$$\max_{K_t, N_t} \Pi_t = AF(K_t, N_t) - \omega_t N_t - R_t K_t$$

  - ω_t is real wage
  - R_t is return on capital
- FOCs
  - $\omega = AF_N(K_t, N_t)$
  - $R_t = AF_K(K_t, N_t)$

Solow Model

- Households
  - Earn income $\omega_t N_t + R_t K_t$
  - Budget constraint:

$$C_t + I_t \leq \omega_t N_t + R_t K_t + \Pi_t$$

    - $\omega_t N_t$ earning wages
    - $R_t K_t$ receiving rent from capital
    - Π_t receiving profit of the firm
  - Firms earn zero profit under CRS (see Seminar 4 for proof)
  - Then

$$C_t + I_t = \omega_t N_t + R_t K_t$$
$$C_t + I_t = Y_t$$

Solow Model

- Investment
  - Capital accumulation equation

$$K_{t+1} = I_t + (1 - \delta)K_t$$

  - δ is a depreciation rate

Solow Model

- Investment
  - Capital accumulation equation

$$K_{t+1} = I_t + (1 - \delta)K_t$$

  - δ is a depreciation rate
  - Assume that investment is a constant fraction of output

$$I_t = sY_t$$

  - Then consumption is a constant fraction of output too
  - Combine $I_t = sY_t$ and $C_t + I_t = Y_t$:

$$C_t = (1 - s)Y_t$$

Solow Model

- The Solow model is characterised by the following equations all holding simultaneously:

$$
\begin{aligned}
Y_t &= A_t F(K_t, N_t) \\
Y_t &= C_t + I_t \\
K_{t+1} &= (1-\delta)K_t + I_t \\
I_t &= sY_t \\
\omega &= AF_N(K_t, N_t) \\
R_t &= AF_K(K_t, N_t)
\end{aligned}
$$

- Endogenous variables - Y_t, C_t, I_t, K_{t+1}, ω_t and R_t
- Exogenous variables - N_t, K_t and A, parameters s and ω

Solow Model

- Combine the following three equations:

$$Y_t = A_t F(K_t, N_t)$$
$$K_{t+1} = (1-\delta)K_t + I_t$$
$$I_t = sY_t$$

- First, substitute the production function into the last expression:

$$I_t = sA_t F(K_t, N_t)$$

- Investment is a constant fraction s of output $A_t F(K_t, N_t)$
- Substitute the result into capital accumulation equation

$$K_{t+1} = (1-\delta)K_t + I_t$$
$$K_{t+1} = (1-\delta)K_t + sA_t F(K_t, N_t)$$

Solow Model

- This equation describes the evolution of K_t

$$K_{t+1} = (1-\delta)K_t + sA_tF(K_t, N_t)$$

- Given K_t, A_t, N_t, s and δ the equation describes how much K_{t+1} the economy will have

Solow Model

- Lets rewrite this equation in terms of capital per work

$$
\begin{align}
K_{t+1} &= (1-\delta)K_t + sA_tF(K_t, N_t) \tag{1}\\
\frac{K_{t+1}}{N_t} &= \frac{(1-\delta)K_t}{N_t} + \frac{sA_tF(K_t, N_t)}{N_t} \tag{2}\\
& \tag{3}
\end{align}
$$

- Define $k_t \equiv \frac{K_t}{N_t}$ as capital per worker
- Then Equation (1) becomes:

$$\frac{K_{t+1}}{N_t} = (1-\delta)k_t + \frac{sA_tF(K_t, N_t)}{N_t}$$

Solow Model

$$\frac{K_{t+1}}{N_t} = (1-\delta)k_t + \frac{sA_tF(K_t, N_t)}{N_t}$$

- Recall the properties of the production function with CRS:

$$\frac{F(K_t, N_t)}{N_t} = \frac{1}{N_t}F(K_t, N_t) = F(\frac{K_t}{N_t}, \frac{N_t}{N_t}) = F(k_t, 1)$$

- Denote $F(k_t, 1) \equiv f(k_t)$ as per worker production function
- then:

$$\frac{K_{t+1}}{N_t} = (1-\delta)k_t + sA_tf(k_t)$$

Solow Model

$$\frac{K_{t+1}}{N_t} = (1-\delta)k_t + sA_t f(k_t)$$

- Finally multiply the RHS by N_{t+1}

$$\frac{K_{t+1}}{N_{t+1}}\frac{N_{t+1}}{N_t} = (1-\delta)k_t + sA_t f(k_t)$$

- Assume that labour is constant $N_{t+1}/N_t = 1$, then

$$k_{t+1} = (1-\delta)k_t + sA_t f(k_t)$$

Solow Model

- Capital accumulation equation in per worker terms:

$$k_{t+1} = (1-\delta)k_t + sA_t f(k_t)$$

- The first derivative of k_{t+1} with respect to k_t

$$\frac{\partial k_{t+1}}{\partial k_t} = (1-\delta) + sA_t f'(k_t)$$

- Since $(1-\delta) > 0$ and $f'(k_t) > 0$ - the slope is positive
- Since $f''(k_t) < 0$, $sA_t f'(k_t)$ gets smaller when capital increases
- Assume **Inada Conditions** hold

$$\lim_{k \to \infty} f'(k) = 0$$
$$\lim_{k \to 0} f'(k) = \infty$$

Solow Model

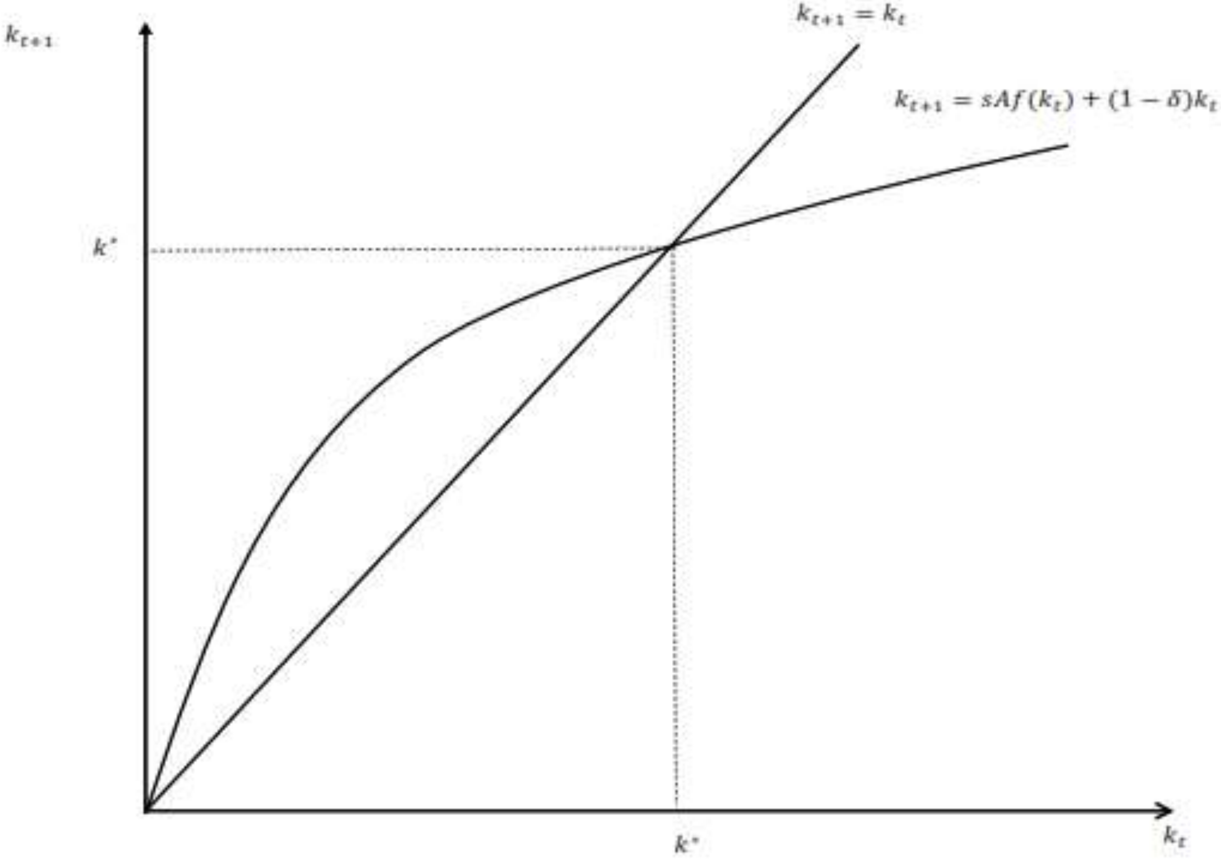


Figure: Capital accumulation

Solow Model

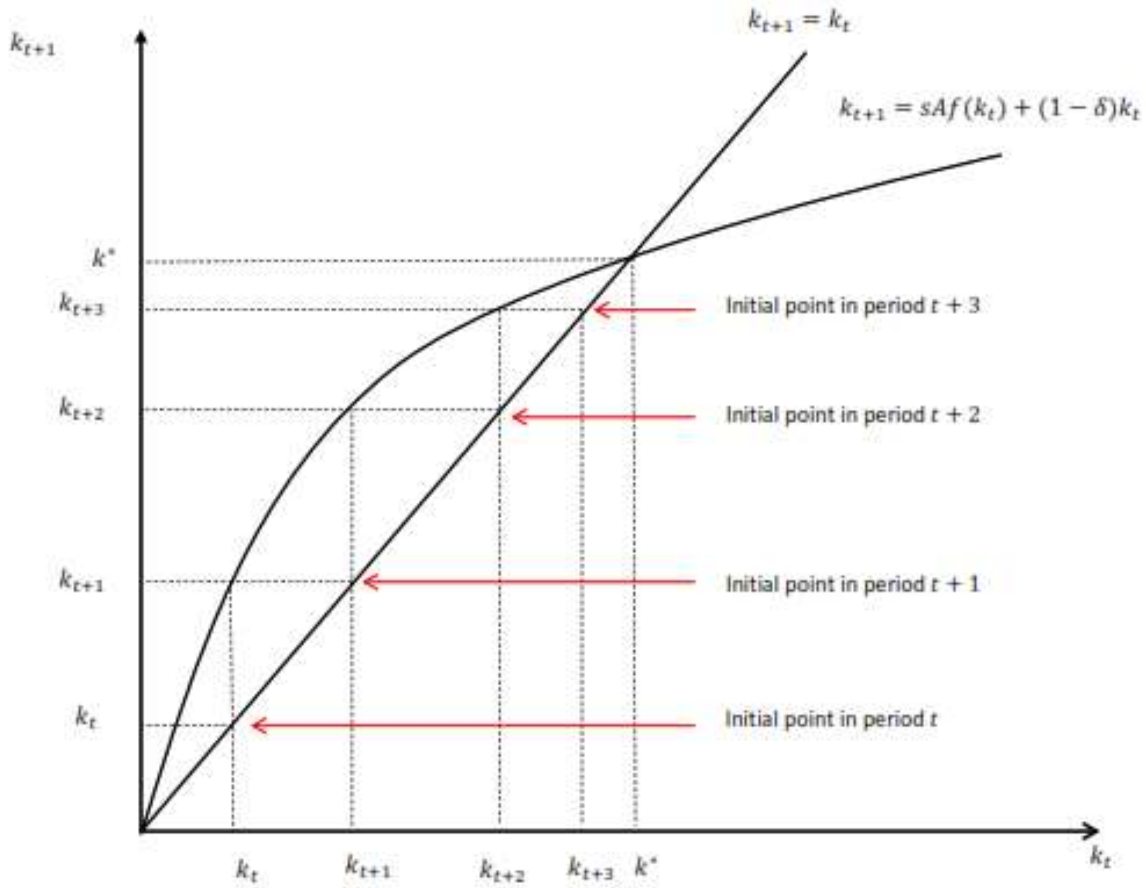


Figure: Capital accumulation

Solow Model

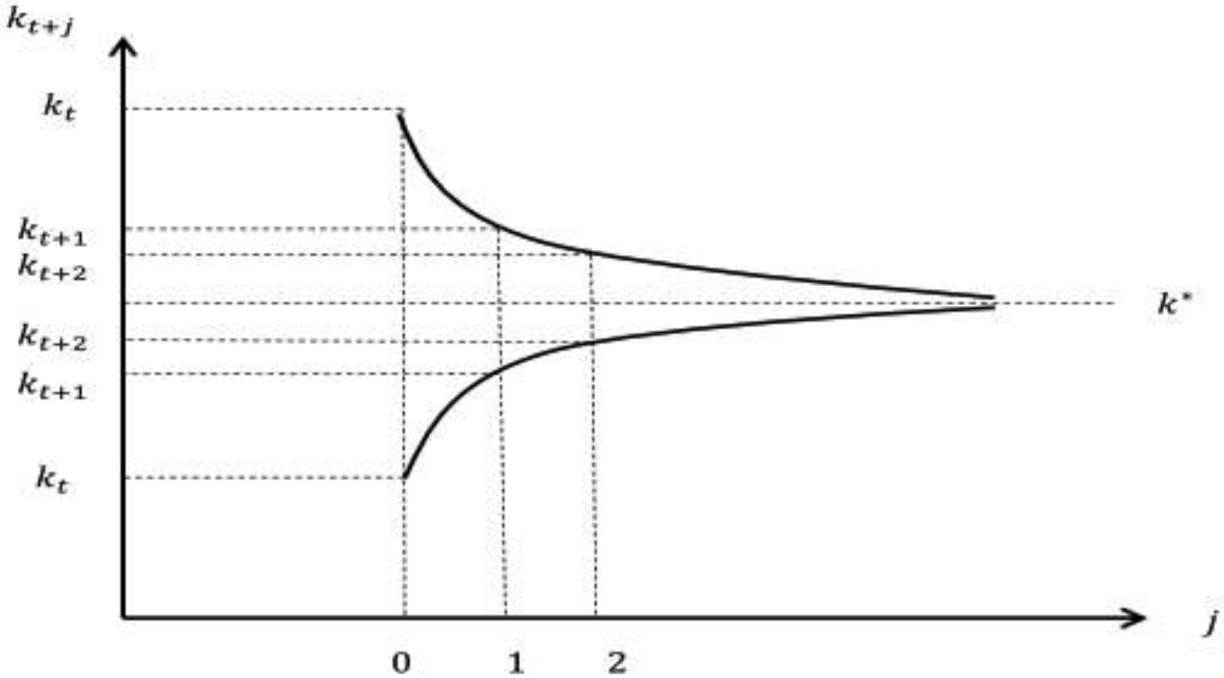


Figure: Convergence to Steady State

Growth Accounting

What do empirical data say about determinants of growth?

- **Production function:** $\boldsymbol{Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}}$

  - Y_t: output
  - A_t : total factor productivity
  - K_t : capital input
  - L_t : labor input
  - $1 - \alpha$: labor share of income

- **Output per worker (working-age pop.) decomposition:** $\boldsymbol{\frac{Y_t}{N_t} = A_t^{\frac{1}{1-\alpha}} (\frac{K_t}{Y_t})^{\frac{\alpha}{1-\alpha}} (\frac{L_t}{N_t})}$

Growth Accounting in Practice

Growth Accounting: $$\frac{Y_t}{N_t} = A_t^{\frac{1}{1-\alpha}} \left(\frac{K_t}{Y_t}\right)^{\frac{\alpha}{1-\alpha}} \left(\frac{L_t}{N_t}\right)$$

- Y_t: real GDP
- N_t : working-age (15-64) population
- K_t : real capital input
- L_t : total annual hours worked
- $1-\alpha$: labor share of income

- We will go over how/where to get the dataset and how to implement this accounting exercise.
  - Using Excel spreadsheets
  - Using R and Penn World Table (PWT)

Growth Accounting for the UK