Basic Seminar in Mathematics
An Introduction to Computational Stochastic PDEs

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Contents

1 An Introduction to Computational Stochastic PDEs

1.1 Variational formulation of elliptic PDEs

Let us develop the Galerkin approximation and the finite element method for two-dimensional elliptic PDEs,such as

$$-\nabla \cdot (a(\mathbf{x})\nabla u(\mathbf{x})) = -\sum_{j=1}^{2} \frac{\partial}{\partial x_j} a(\mathbf{x}) \frac{\partial u(\mathbf{x})}{\partial x_j} = f(\mathbf{x}), \quad \mathbf{x} \in D, \tag{1.1.1}$$

$$u(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \partial D \tag{1.1.2}$$

where $D \subset \mathbb{R}^2$ is a bounded domain with piecewise smooth boundary ∂D and
$u : D \to \mathbb{R}$ is an unknown function , and $g : \partial D \to \mathbb{R}$,$a, f : D \to \mathbb{R}$ is given suitable functions.

Def 1.1.1. A function $u \in C^2(D) \cap C(D \cup \partial D)$ that satisfies these PDE's conditions is called **classical solution**

In many practical cases,a is discontinuous and we do not have classical solutions and we are content with weak solutions. In this section, we often assume that the diffusion coefficient $a(x)$ satisfies following regularity.

Def 1.1.2. The diffusion coefficient $a(x)$ is regular if $a(x)$ satisfies

$$0 < a_{min} \leq a(x) \leq a_{max} < \infty \text{ for almost all } x \in D$$

for some $a_{min}, a_{max} \in \mathbb{R}$.

To set up a variational formulation, We will check the definitions of function spaces below.

Def 1.1.3. Let D be a domain and Y be a Banach space. And we use α as multi-index.
For $p \geq 1$, the Sobolev space $W^{r,p}(D, Y)$ is the set of functions whose weak derivatives up to order $r \in \mathbb{N}$ are $L^p(D, Y)$.That is

$$W^{r,p}(D, Y) = \{u : \mathcal{D}^\alpha u \in L^p(D, Y) \text{ if} |\alpha| \leq r\}$$

and Sobolev space $W^{r,p}(D, Y)$ is a Banach space with norm

$$\|u\|_{W^{r,p}(D,Y)} := \left(\sum_{0 \leq \alpha \leq r} \|\mathcal{D}^\alpha u\|^p_{L^P(D,Y)} \right)^{1/p}$$

If $p = 2$ and Y is Hilbert space , $H^r(D, Y)$ is used to denote $W^{r,2}(D, Y)$.
$H^r(D, Y)$ is a Hilbert space with inner product

$$\langle u, v \rangle_{H^r(D,Y)} := \sum_{0 \leq \alpha \leq r} \langle \mathcal{D}^\alpha u, \mathcal{D}^\alpha v \rangle_{L^2(D,H)}$$

Especially, $H^r(D, \mathbb{R})$ is abbreviated to $H^r(D)$.
$H_0^1(D)$ is the completion of $C_c^\infty(D)$ with respect to the $H^1(D)$ norm and is a Hilbert space with the $H^1(D)$ inner product
The correct solution space is written as,

$$H_g^1(D) = \{w \in H^1(D) : \ w|_{\partial D} = g\}$$

which needs precise definition later because $w \in H^1(D)$ can be non-continuous function on $\bar{D}$,and we can assign any value to w on ∂D because of this and the fact that measure of ∂D is zero.

Now, we are ready to set up a variational formulation of elliptic PDE1.1.1.

Def 1.1.4. A **weak solution** to the BVP (1.1.1) – (1.1.2) is a function $u \in W = H_g^1(D)$ that satisfies

$$a(u, v) = l(v) \ \forall v \in V = H_0^1(D)$$

where

$$a(u, v) := \int_D a\nabla u \cdot \nabla v d\mathbf{x}$$
$$l(v) = \langle f, v\rangle_{L^2(D)}$$

Prop 1.1.5. The classical solution to the BVP (1.1.1) – (1.1.2) is a weak solution

Proof. First, we multiply both sides of (1.1.1) by a test function $\phi \in C_c^\infty(D)$ and integrate over D, and obtain

$$\int_D -\nabla \cdot (a(\mathbf{x})\nabla u(\mathbf{x}))\phi(\mathbf{x})d\mathbf{x} = \int_D f(\mathbf{x})\phi(\mathbf{x})d\mathbf{x}$$

and by the product rule for ∇

$$\nabla \cdot (\phi a \nabla u) = \nabla \cdot (a\nabla u)\phi + \nabla\phi \cdot a\nabla u$$

we can split the intergal of the left-hand side and get

$$\int_D a\nabla u \cdot \nabla\phi d\mathbf{x} - \int_D \nabla \cdot (\phi a\nabla u)d\mathbf{x} = \int_D f(\mathbf{x})\phi(\mathbf{x})d\mathbf{x}$$

And the divergence theorem in $\mathbb{R}^2$ gives

$$\int_D \nabla \cdot (\phi a\nabla u)d\mathbf{x} = \int_{\partial D} (\phi a\nabla u) \cdot \mathbf{n}ds$$

Here $\phi \in C_c^\infty(D)$, we have $\phi(x) = 0$ on ∂D and the left-hand side is 0. Hence

$$\int_D a\nabla u \cdot \nabla\phi d\mathbf{x} = \int_D f(\mathbf{x})\phi(\mathbf{x})d\mathbf{x}$$

□

We cannot use the Lax-Milgram lemma because $W \neq V$ when $g\neg 0$.So we need some modifications to prove the existence and uniqueness of the weak solutions.

Def 1.1.6. Let $D \subset \mathbb{R}^2$ be a bounded domain.$L^2(\partial D)$ is the Hilbert space $L^2(\partial, \mathbb{R})$ equipped with the norm

$$\|g\|_{L^2(\partial D)} := \left(\int_{\partial D} g(x)^2 dV(x) \right)^{1/2}$$

Lem 1.1.7. Let $D \subset \mathbb{R}^2$ be a bounded domain with a sufficiently smooth boundary ∂D. Then there exists a bounded linear operator $\gamma : H^1(D) \to L^2(\partial D)$such that

$$\gamma w = w|_{\partial D}, \ \forall w \in C^1(\bar{D})$$

Proof. $w \in H^1(D)$ can be approximated by a sequence in $C(\bar{D})$, $\{w_n\}$ and the restrictions of these functions are a Cauchy sequence so we can define γ by

$$\gamma(w) := \lim_{n\to\infty} w_n|_{\partial D}$$

as a linear operator $H^1(D) \to L^2(\partial D)$ □

Def 1.1.8. Let $D \subset \mathbb{R}^2$ be a bounded domain.The trace space $H^{1/2}(\partial D)$ is defined as

$$H^{1/2}(\partial D) := \gamma(H^1(D)) = \{\gamma w | w \in H^1(D)\}$$

$H^{1/2}(\partial D)$ is a Hilbert space equipped with the norm

$$\|g\|_{H^{1/2}(\partial D)} := \inf\{\|w\|_{H^1(D)} | \gamma w = g, \ w \in H^1(D)\}$$

Prop 1.1.9. There exists $K_\gamma > 0$ such that , for all $g \in H^{1/2}(\partial D)$, there exists $u_g \in H^1(D)$ such that

$$\|u_g\|_{H^1(D)} \leq K_\gamma \|g\|_{H^{1/2}(\partial D)}$$

and

$$\gamma(u_g) = g$$

Proof. See Hackbusch Theorem6.2.28 □

Thm 1.1.10. *Let a be a regular diffusion coefficient , $f \in L^2(D), \ \ g \in H^{1/2}(\partial D)$. Then BVP (1.1.1) – (1.1.2) has a unique solution $u \in H^1_g(D)$*

Proof. Let $g \in H^{1/2}(\partial D)$.
$u_g \in H^1(D)$ such that $\gamma(u_g) = g$
and solve the variational problem to find $u_0 \in V$

$$a(u_0, v) = \hat{l}(v) := l(v) - a(u_g, v)$$

Solving this problem is equivalent to finding the weak solution of the BVP.
So We can use the Lax-Milgram lemma to new variational problem by the lemma below and can prove this theorem.

□

Lem 1.1.11. Let a be a regular diffusion coefficient. Then the bilinear form $a(,)$is bounded form.And the seminorm $|\cdot|_E$ defined by

$$|u|_E := a(u, u)^{1/2}$$

is equivalent to the semi-norm $|\cdot|_{H^1(D)}$ on $H^1(D)$

Proof. $\square$

Thm 1.1.12. *Assume the same conditions of the theorem above and let $u \in W$ be a weak solution of the BVP. Then*

$$|u|_{H^1(D)} \leq K(\|f\|_{L^2(D)} + \|g\|_{H^{1/2}(\partial D)})$$

This variational formulation gives upper bound for the errors of approximations.

Thm 1.1.13. *Consider a weak problem to find $\tilde{u} \in W$ such that*

$$\tilde{a}(\tilde{u}, v) = \tilde{l}(v) \ \forall v \in V$$

where $\tilde{a} : W \times V \to \mathbb{R}$,$\tilde{l} : V \to \mathbb{R}$ are defined as

$$\tilde{a}(u, v) := \int_D \tilde{a} \nabla u \cdot \nabla v d\mathbf{x}$$

$$\tilde{l}(v) = \langle \tilde{f}, v \rangle_{L^2(D)}$$

Now let $\tilde{a}$ be a regular diffusion coefficient , $\tilde{f} \in L^2(D)$ $g \in H^{1/2}(\partial D)$.Then this weak problem has a unique solution $\tilde{u} \in W$. And let $u \in W$ be the weak solution of the original BVP. Then,

$$|u - \tilde{u}|_{H^1(D)} \leq \frac{K_p}{\tilde{a}_{min}} \|f - \tilde{f}\|_{L^2(D)} + \frac{1}{\tilde{a}_{min}} \|a - \tilde{a}\|_{L^\infty(D)} |u|_{H^1(D)}$$

1.2 Galerkin approximation

We return to the approximation of the original BVP.

Def 1.2.1. Let $V^h \subset H_0^1(D)$,$W^h \subset H_g^1(D)$be the finite dimensional subspaces of test solution space and solution space such that

$$v - w \in V^h, \ \forall v, w \in W^h$$

Then the Galerkin approximation for the $(1.1.1) - (1.1.2)$ is the function $u_h \in W^h$ satisfying

$$a(u_h, v) = l(v) \ \forall v \in V^h \tag{1.2.1}$$

Thm 1.2.2. *Let a be a regular diffusion coefficient , $f \in L^2(D)$, $g \in H^{1/2}(\partial D)$.*
Then Galerkin approximation (1.2.1) *will be defined uniquely and will be the best approximation. i.e.*

$$|u - u_h|_E = \inf_{w \in W^h} |u - w|_E$$

Finally we consider the accuracy of the Galerkin approximation $\tilde{u}_h$ when a and f are approximated.
I have run out of my energy.