Determination of Young's modulus and Poisson's ratio of a material using Cornu's apparatus

Gayatri P
NISER, Bhubaneswar
2nd year, Integrated M.Sc. Physics
Roll No.: 2211185

(Dated: August 22, 2023)

In this experiment, try to determine the Young's Modulus and Poisson's Ratio of a given glass using Cornu's apparatus. We study the deformations obtained on the surface of the glass when subjected to a uniform bending moment using the phenomena of interference of light to perform the observations.

I. THEORY

A. Modulus of Elasticity

Young's Modulus or Modulus of Elasticity of a material is defined as the ratio of longitudinal stress and the corresponding longitudinal strain.

$$Y = \frac{FL_0}{A\Delta L} \tag{1}$$

Here, F/A is the force applied per unit area, and $\Delta L/L$ is the extension cause by the applied force with respect to the initial length L.

Usually when a sample material is stretched in one direction, it tends to get thinner in the other direction. Poisson's ratio is the measure of this tendency and is defined as the ratio of the strain in the direction of applied load to the strain in the transverse direction. A perfectly in-compressible material has Poisson's ratio $\sigma = 0.5$.

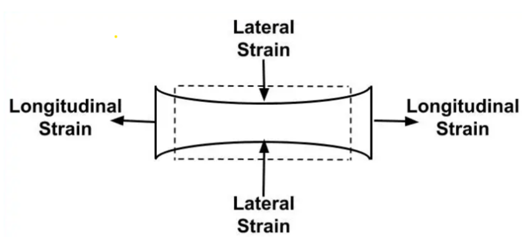

FIG. 1: Schematic diagram of the deformation of a material under longitudinal stress

B. Experimental Principle

Cornu proposed a method to measure the deformation of a solid under load using the interference phenomenon of light. According to his method, a glass plate is kept on top of a glass beam and load is applied on both sides of the glass beam. Therefore, the glass beam will be deformed due to strain in the longitudinal direction (x-axis), and since Poisson's ration $\sigma \neq 0$, it will also be deformed in the transverse direction (y-axis).Thus the beam deforms into the shape of horse saddle forming a thin film of air between them. When the film is illuminated by monochromatic light, interference occurs between the light reflected from the bottom of the glass plate and the top of the beam as shown in Fig. 1.

Let x and y be the coordinates along longitudinal and transverse direction centering around O. Also, let R_x and R_y be the radius of curvature of the glass beam in longitudinal and transverse directions respectively. In order to obtain the shape of the interference fringes, consider that the thickness of air film between the glass plate and the beam to be $t(x, y)$ at a point (x,y) in the XY-plane.

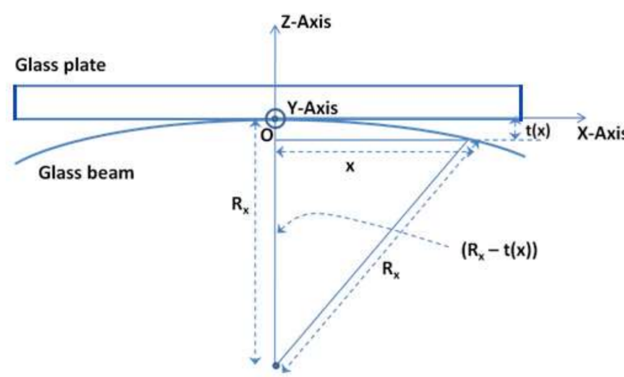

FIG. 2: Schematic drawing of the geometry to obtain the fringes

Firstly, consider the x dependence of $t(x, y)$ i.e. $t(x)$. As evident in Fig 1, we can obtain $t(x)$ as:

$$(R_x - t(x))^2 = R_x^2 - x^2 \tag{2}$$

Assuming $t(x)$ to be very small, we get

$$t(x) = \frac{x^2}{2R_x} \tag{3}$$

Similarly, $t(y)$ can be obtained from

$$t(y) = -\frac{y^2}{2R_y} \tag{4}$$

Here, there appears a negative sign because the glass beam bends upwards along the transverse direction (y-axis). Thus, $t(x, y)$ can be expressed as

$$t(x,y) = \frac{x^2}{2R_x} - \frac{y^2}{2R_y} \tag{5}$$

The fringe shapes are determined by the locus of all points with identical path difference, in the case, constant thickness, i.e. $t(x, y)$ is constant.

$$\frac{x^2}{2R_x} - \frac{y^2}{2R_y} = a^2 \tag{6}$$

which represents the equation of a hyperbola, where a is a constant. So, the fringes will be hyperbolic in nature (Fig. 2).

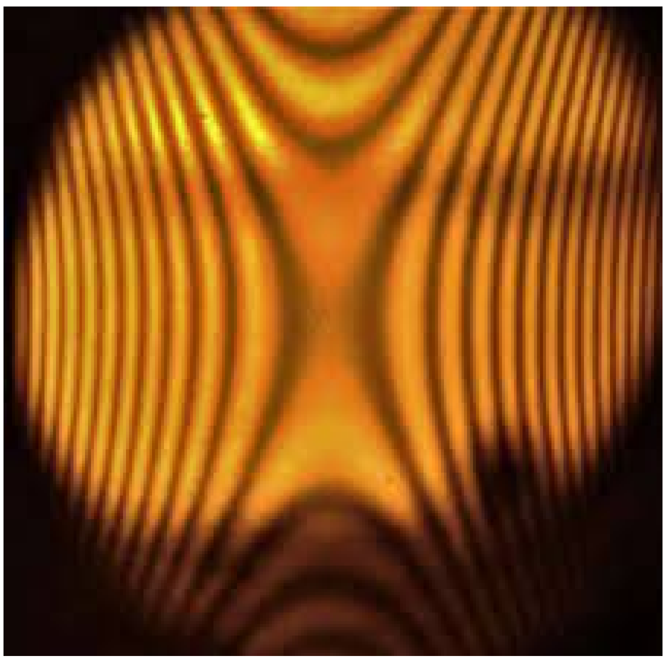

FIG. 3: Hyperbolic fringes observed through the travelling microscope

The light waves passing through glass plate will be divided into two parts — one from the reflection from the bottom of the glass plate-air interface and the second one from the top of air film-glass beam interface (which would under go a phase change of π due to reflection at the air film-glass beam interface and would traverse the width of the air film twice). These two components would interfere and produce the fringe pattern. Therefore, the optical phase difference between these two waves (for almost normal incidence) is given by,

$$\Delta\phi = \frac{2\pi}{\lambda}(2\mu t(x,y)) + \pi \tag{7}$$

Where μ is the refractive index of the air film and λ is the wavelength of light. Let's assume $\mu = 1$ for air. If the N^{th} dark fringe along the x-axis is at a distance x_N from the origin, then the interfering waves are out of phase by,

$$\Delta\phi = (2N+1)\pi \tag{8}$$

$$\Rightarrow 2t_N(x) = \frac{x_N^2}{R_x} = N\lambda \tag{9}$$

$$\text{and, } 2t_{N+s}(x) = \frac{x_{N+s}^2}{R_x} = (N+s)\lambda \tag{10}$$

where x_{N+s} is the $(N+s)^{th}$ dark fringe along x-axis. Subtracting eqn. (9) from (10) we get,

$$R_x = \frac{x_{N+s}^2 - x_N^2}{s\lambda} = \frac{\rho_x(s)}{s\lambda} \tag{11}$$

where we define $\rho_x(s) = x_{N+s}^2 - x_N^2$ for convenience.

Since it is difficult to find the origin, we measure the diameter (D) of the fringe, which is the distance between the N-th dark fringe on the right and left sides correspondingly. Hence $D_{N_x} = 2x_N$.

Once the radius of curvature R_x is obtained, we relate it to the internal bending moment of the glass slab, given by

$$G_x = \frac{Ybd^3}{12}\frac{1}{R_x} = \frac{Ybd^3}{12}\frac{s\lambda}{\rho_x(s)} \tag{12}$$

where b and d and Y are the thickness, depth and Young's Modulus of the glass beam respectively. This internal bending moment will be balanced by the external bending moment applied by the loads hanging from the beam.

If l is the distance between the knife-edge and the suspension point of the load (of mass m each), hanging on both sides, then we can write,

$$(mg)l = \frac{Ybd^3}{12}\frac{s\lambda}{\rho_x(s)} \tag{13}$$

If we carry out the measurements for two different loads, we can obtain this expression to calculate the **Young's modulus** of the glass beam as,

$$(m_1 - m_2)gl = \frac{Ybd^3}{12} s\lambda \left(\frac{1}{\rho_x^2} - \frac{1}{\rho_x^3}\right) \tag{14}$$

To Calculate Poisson's Ratio, we need the ratio of radius of curvature of the fringes in the longitudinal direction to that in the transverse direction. Similar to the process of finding R_x in eqn. (11), we can obtain R_y using,

$$R_y = \frac{y_{N+s}^2 - y_N^2}{s\lambda} = \frac{\rho_y(s)}{s\lambda} \tag{15}$$

Thus, **Poisson's ratio** can be calculated by,

$$\sigma = \frac{R_x}{R_y} = \frac{\rho_x(s)}{\rho_y(s)} \tag{16}$$

II. OBSERVATIONS AND CALCULATIONS

A. Observational Data

- Least count of the micrometer = 0.01mm
- Breadth of the glass slab (b) = 49.4mm
- Depth of the glass slab (d) = 2.07mm
- Distance from the knife-edge to the load (l) = 115mm
- Wavelength of the monochromatic light used (λ) = 589.3nm

1. *For m_1 = 200g*

Fringe Order	Fringes on the left (mm)	Fringes on the right (mm)	D_x (mm)	ρ_x (mm^2)	R_x (mm)
1	9.24	5.40	3.84		
2	10.38	3.89	6.49	6.84	11.62
3	11.18	3.02	8.16	12.96	11.00
4	12.41	2.23	10.18	22.22	12.58
5	13.05	1.48	11.57	29.78	12.64
6	13.55	0.82	12.73	36.83	12.50
7	14.10	0.18	13.92	44.76	12.66

TABLE I: Observed fringe pattern on along the longitudinal direction

Fringe Order	Fringes on the bottom (mm)	Fringes on the top (mm)	D_y (mm)	ρ_y (mm^2)	R_y (mm)
1	17.5	8.01	9.49		
2	20.02	6.55	13.47	22.85	38.79
3	22.24	5.01	17.23	51.70	43.89
4	24.07	4.52	19.55	73.04	41.33
5	25.99	3.53	22.46	103.60	43.97

TABLE II: Observed fringe pattern on along the transverse direction

2. *For m_2 = 250g*

Fringe Order	Fringes on the left (mm)	Fringes on the right (mm)	D_x (mm)	ρ_x (mm^2)	R_x (mm)
1	13.58	11.65	1.93		
2	15.00	10.72	4.28	3.65	6.19
3	15.86	9.40	6.46	9.50	8.07
4	16.53	8.13	8.40	16.71	9.46
5	17.13	7.50	9.63	22.25	9.45
6	17.64	6.92	10.72	27.80	9.44
7	18.10	6.38	11.72	33.41	9.45
8	18.55	5.89	12.66	39.14	9.49

TABLE III: Observed fringe pattern on along the longitudinal direction

Fringe Order	Fringes on the bottom (mm)	Fringes on the top (mm)	D_y (mm)	ρ_y (mm^2)	R_y (mm)
1	15.21	9.6	5.61		
2	18.00	6.97	11.03	22.55	38.28
3	19.99	5.66	14.33	43.47	36.90
4	21.66	4.72	16.94	63.87	36.15
5	23.36	3.84	19.52	87.39	37.09
6	25.62	3.40	22.22	115.56	39.24

TABLE IV: Observed fringe pattern on along the transverse direction

3. *For m_3 = 300g*

Fringe Order	Fringes on the left (mm)	Fringes on the right (mm)	D_x (mm)	ρ_x (mm^2)	R_x (mm)
1	13.03	9.94	3.09		
2	14.24	9.04	5.20	4.37	7.42
3	15.00	8.37	6.63	8.60	7.30
4	16.17	7.82	8.35	15.04	8.51
5	16.69	7.38	9.31	19.28	8.18
6	17.10	6.91	10.19	23.57	8.00
7	17.56	6.51	11.05	28.14	7.96
8	17.98	6.12	11.86	32.78	7.95

TABLE V: Observed fringe pattern on along the longitudinal direction

Fringe Order	Fringes on the bottom (mm)	Fringes on the top (mm)	D_y (mm)	ρ_y (mm^2)	R_y (mm)
1	15.48	10.64	4.84		
2	17.95	7.84	10.11	19.70	33.44
3	19.76	5.68	14.08	43.71	37.10
4	21.29	4.26	17.03	66.65	37.72
5	21.85	3.07	18.78	82.32	34.94
6	22.69	1.99	20.70	101.27	34.39
7	23.47	0.91	22.56	121.38	34.35

TABLE VI: Observed fringe pattern on along the transverse direction

B. Calculations

1. *Calculation of Young's Modulus (Y)*

We can rearrange equation (14) to form

$$\left(\frac{1}{\rho_x^1(s)} - \frac{1}{\rho_x^2(s)}\right)^{-1} = \left(\frac{Ybd^3\lambda}{12gl(m_1 - m_2)}\right) s \qquad (17)$$

which is of form $y = mx + c$, where m = slope and c = y-intercept of a straight line. Thus we can plot the LHS of this eqn. vs s to find out Y using,

$$Y = \frac{12gl(m_1 - m_2)}{bd^3\lambda} \cdot m \qquad (18)$$

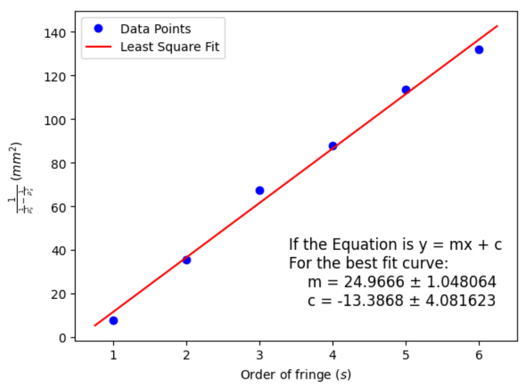

FIG. 4: $(\frac{1}{\rho_x^1} - \frac{1}{\rho_x^2})^{-1}$ vs Order of fringe (s)

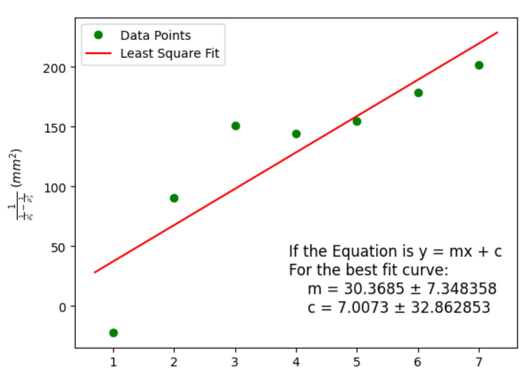

FIG. 5: $(\frac{1}{\rho_x^2} - \frac{1}{\rho_x^3})^{-1}$ vs Order of fringe (s)

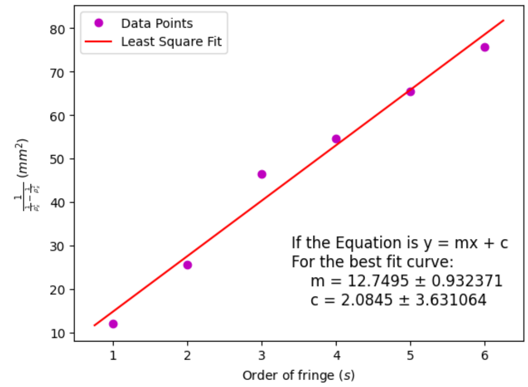

FIG. 6: $(\frac{1}{\rho_x^1} - \frac{1}{\rho_x^3})^{-1}$ vs Order of fringe (s)

Using eq. (18),

- For m_1 and m_2, Y_{12} = 66.80 GPa
- For m_2 and m_3, Y_{23} = 75.34 GPa
- For m_1 and m_3, Y_{13} = 66.88 GPa

Thus, average value of Y = 69.67 GPa

2. *Calculation of Poisson's Ratio (σ)*

We can rearrange equation (16) to form

$$\rho_x(s) = \sigma \rho_y(s) \qquad (19)$$

which is of form $y = mx + c$, where m =slope and c = y-intercept of a straight line. Thus we can plot this $rho_x(s)$ vs $\rho_y(s)$ to find out σ which is equal to the slope (m).

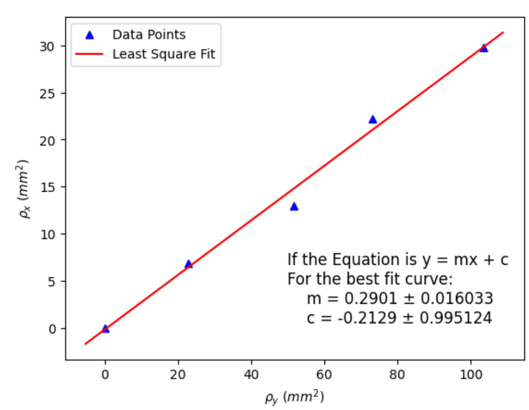

FIG. 7: ρ_x vs ρ_y for m = 200g

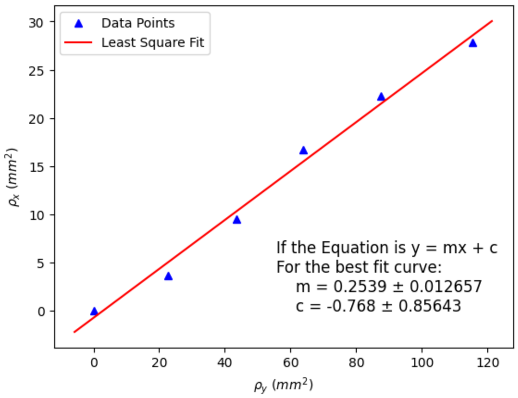

FIG. 8: ρ_x vs ρ_y for m = 250g

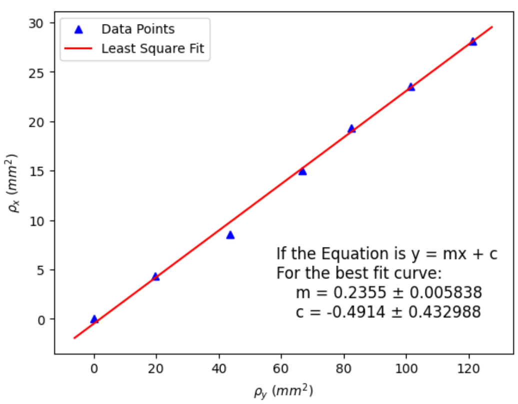

FIG. 9: ρ_x vs ρ_y for m = 300g

Thus, the average value of σ is observed to be 0.261.

III. ERROR ANALYSIS

From eqn. (18), we can see that the error in Young's modulus (Y) will be propagated from the error values of measured quantities b, d, l, and slope (m) as follows:

$$\frac{\Delta Y_{ij}}{Y_{ij}} = \sqrt{\left(\frac{\Delta l}{l}\right)^2 + \left(\frac{\Delta b}{b}\right)^2 + \left(\frac{3\Delta d}{d}\right)^2 + \left(\frac{\Delta m_{ij}}{m_{ij}}\right)^2}$$

Using $\Delta l = 1$ mm, $\Delta b = 0.1$ mm and $\Delta d = 0.01$ mm, the calculated errors are $\Delta Y_{12} = 3.47$ GPa, $\Delta Y_{23} = 17.02$ GPa and $\Delta Y_{13} = 5.02$ GPa. Therefore, the average error will be measured by,

$$\Delta Y = \frac{1}{3}\sqrt{(\Delta Y_{12})^2 + (\Delta Y_{23})^2 + (\Delta Y_{13})^2}$$
$$= 6.03\text{GPa}$$

Similarly, from eqn (19), error in Poisson's ratio ($\Delta\sigma$) is the same as the error in slope of the ρ_x vs ρ_y plot. Thus the average error value in Poisson's ratio can be written as,

$$\Delta\sigma = \frac{1}{3}\sqrt{(\Delta\sigma_{200})^2 + (\Delta\sigma_{250})^2 + (\Delta\sigma_{300})^2}$$
$$= 0.007$$

IV. RESULTS

From the experiment, I have measured the value of Young's modulus (Y) and Poisson's ratio (σ) of the given glass slab to be,

$$Y = 69.67 \pm 6.03 \text{ GPa}$$
$$\sigma = 0.261 \pm 0.007$$

V. CONCLUSION

Our measured values of Young's modulus and Poisson's ratio is within the the literature values of Y = 50 to 90 GPa and $\sigma = 0.2$ to 0.7 respectively.

Glass is a brittle material and thus shows very little strain subject to a considerable amount of stress upto the elastic limit. Thus, the high value of Young's modulus observed is justified.

Using this experiment, we can determine the refractive index of an unknown liquid by substituting it with the air-film.

VI. PRECAUTIONS & SOURCES OF ERROR

1. The beam should be placed on the knife–edges symmetrically.
2. All the glass plates must be plane (optically) and clean.
3. The travelling microscope must be handled carefully in order to avoid backlash error.

[1] SPS. Lab manual. *Website*, 2023. `https://www.niser.ac.in/sps/sites/default/files/2_Young's%20modulus%20by%20Cornu's%20method.pdf`.