Dutch Books

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1 Introduction

One argument that agents ought to apply Bayesian reasoning is the so called dutch book argument. The argument purport to show that, given that an agent acts rationally, they should also bet rationally on beliefs so as to make sure to avoid guaranteed losses. The dutch book theorem shows that if the agent's betting behavior violates the standard (Kolmogorov) axioms of probability theory, a sure loss can be created. The converse dutch book theorem shows that if the agent's betting behavior follows the axioms of probability, then they cannot be subject to a sure loss. Similar theorems also hold for why agents ought to conditionalize their beliefs on new evidence via Bayes's formula. Using these theorems, one can use them as arguments for normative rational belief. We end the discussion by discussing some problems with Dutch book arguments, some alternatives, and in the end argue for a pragmatic and risk-averse approach.

2 Introductory definitions

The general idea of the dutch book theorem will be to construct a set of dutch books, one for each axiom of probability theory. We construct them using gambles, that the bookie agent B will set up for agent A to take. Before we can set up dutch books, we must first set forth some elementary definitions.

Two agents enter a gamble on belief b with odds $p : q$ whenever agent A bets $\$p$ that b is true, agent B bets $\$q$ that b is false, and if b is true agent A wins the bet and gets the entire pot i.e $\$p + \$q$ and if b is false then agent B gets the pot instead.

A credence is the degree of belief that an agent has in some belief, which in the context of dutch books will be proportional to the amount of money the agent is willing to risk on a bet that their belief is true. Formally, for belief b the credence is $P(b)$.

Let's assume that the credence for agent A on b being true is p. this means odds up to $p : 1 - p$ will give positive expected payout. So the more strongly you hold your belief, the more you're willing to pay. For example if you are

90% sure it will rain tomorrow, then taking a $9:1$ gamble will give you about 0 expected utility, that is to say, you'd be atleast a bit willing to spend $9 to win $10 because you're very certain it's going to rain tomorrow, but you'd be willing to pay less aswell. For any k such that you take an $\$k:\$1$ gamble for $k<9$ will give positive expected payout.

The axioms of standard (Kolmogorov) probability theory when applied to P, for all beliefs b and c:

$$\text{(Non-zero probability) } P(b) \geq 0.$$

$$\text{(Tautologies } \top \text{ are certain) } P(\top) = 1.$$

$$\text{(when } b \text{ and } c \text{ disjoint, their probabilities sum) } P(b \vee c) = P(b) + P(c).$$

3 Setting up dutch books

The idea is that if agent A's credence's P_A violates one of the axioms above, then a gamble on b can be set up by some agent B (sometimes called "the bookie") such that B always wins and A always looses, no matter if b is true or false.

So a dutch book can be setup for the first axiom. Assume that $P_A(b) < 0$, then Agent B can buy a bet that pays out $1 is b is true and $0 if b is false, for a price of $P_A(b)$ (which is negative). A is betting against b being true. As such we have odds of $P_A(b) : 1 - P_A(b)$. So now, if b turns out true, then A has to pay B $1, thus ending up loosing money. However, if b turns out false, then A looses $\$P_A(b)$. In either case, A ends up loosing. [5]

Now for the second axiom. Assume $P(q) = 0.51$ and $P(\neg q) = 0.51$. Now we can set up a bet such that the agent will always loose. Create a bet such that the agent wins back $1. The agent would be upwards of $0.51 on q being true and $0.51 on q being false. Whether q is true or false, the agent pays $1.02 in total, and will win back at most 1, so the agent will loose $0.02 no matter what.[5]

For the third axiom there can be other dutch books setup. Let's say p and q are mutually exclusive. Then $P(p \vee q) \neq P(p) + P(q)$, in which case it's either $P(p \vee q) < P(p) + P(q)$ or $P(p \vee q) > P(p) + P(q)$. We won't go over it here, but in both cases a gamble can be constructed so as to guarantee a loss.[5]

The dutch book theorem then follows, in that one can always create a fair (expectation-value 0) bet that guarantees a loss to an agent whose credence fails the probability axioms. Also, Lehman and Kemeny independently proved the converse: For probability credences that follow the probability axioms, there can be no guaranteed loss or win. [5]

Both of these results are very powerful, and in section 5 will be used to justify that rational agent action should adjust their credence's to the laws of probability, known as the dutch book arguments. But first let's discuss similar results on conditionalization on credence's as new data comes in.

4 Conditionalization

The dutch book arguments state that conditional probability, and therefore Bayes's theorem doesn't need its own dutch book arguments, since the 3 axioms is enough to derive Bayes's theorem. However what of conditionalization? We still need new credence functions as they update over time.

Conditionalization is a central point of Bayesian epistemology. It states that whenever an agent has a prior credence $P_1(b)$ at time $t = 1$, and some evidence e is gathered, it ought to be updated by the rule $P_t(b) = P_{t-1}(b|e)$. This essay will not contain a lengthy dutch book example for conditionalization. However a presentation off Lewis's proof, and a revised proof, will be presented. The ideas are credited to Lewis, but a revised version is sourced.

Let E be a set of mutually different propositions. At time $t = 2$ agent A learns which partition proposition e is true i.e $P_2(e) = 1$. That is to say, some evidence is gathered and believed in with certainty. We define the credence update as $P_2(b) = P_1(b|e)$. Assume P_1 abides by the standard probability axioms. Agent B ("the bookie") then creates a gamble G such that if agent A accepts G if it yields positive expected value. If $P_2(b) \neq P_1(b|e)$ then there exists a G such that it is a dutch book. The converse is also true, so if $P_2(b) = P_1(b|e)$, then there does not exist a dutch book.[3]

There is a similar theorem in [3] that proves this without assuming that $P_2(e) = 1$, that is to say, one can be uncertain whether or not the evidence is the case. The idea is that evidence can be forgotten, and that all possibilities should have a non-zero probability. [5]

5 The Dutch Book Argument

Now with the theorem's out the way, the main argument[2] is quite straightforward and simple. Let A be an agent. Then

1. It is irrational for A to leave themselves open to a dutch book
2. A's credence's are open to a dutch book iff they violate the standard axioms of probability theory
3. Conclusion: It is irrational for A to have credence's that violate the standard axioms of probability theory

If you assume rationality has normative force, that is to say, you ought to be rational, then it is also the case that you ought, or are in some way obliged, to conform your credence's to the probability axioms.

6 Problems with the Dutch Book Argument

While the dutch book theorems are uncontroversial mathematical theorems[3], a range of attacks have been laid forth against dutch book argument itself. That we ought to hold our actual credence's to the laws of probability theory is perhaps, not as clear cut as the theorems make it seem. This section will lay out some of those problems, and propose some potential solutions.

For instance, the connection between betting behavior and credence's is questionable. Do people really bet like that, and more importantly for the normative force of the DBA, ought we to bet like what the DBA compels us to do? For example, do, or ought, poor people who don't have enough money, spend on bets they cannot afford?[4] One response to such criticism is to state that we have real psychological credence's, that are separate from betting behavior, as evidenced by empirical research in psychology.[4]

Another problem with the dutch book argument that has been pointed out is the possibility for guaranteed wins if one violates the axioms of probability. An bet can be created by swapping the odds around by B to create a guaranteed winning scenario. For instance, by create a bet in a way such that only incoherent agents can receive a guaranteed win.[5] This is related to the so called Czech Book Arguments[2], which can be seen as a kind of inversion of the dutch book arguments. The CBA include a set of gambles where the violator of the axioms of probability have a guaranteed win. This has been purported to show that it cancels out the DBA. However, Hajek emphasizes [2] that this can be solved somewhat by adding in the assumption that agents don't take bets that are deemed unfair.

Yet another problem is that the second axiom is unrealistic for any agents to uphold. It might require logical omniscience[4], which means that if an agent believes p and $p \to q$, then they must believe q. They're even compelled to know every tautology in every logical system, or are atleast able to prove it on the spot before betting. This is clearly unreasonable, if not straight up impossible for some logical systems that are incomplete (for instance, what should you bet on the continuum hypothesis?).

And since the second axiom states that credence's on inevitability's should be 1, one might risk betting everything they have on a gamble. For instance, so long as one believes bivalence applies to meteorology, one should have a credence of 1 on the proposition "It is either raining or not raining". So the bets would have to be up to ratio $1 : 0$. So should one take a one million dollar gamble? One billion? According to axiom 2, you should still bet.

There's also the problem of risk aversion. However it is worth noting that the dutch book arguments need not be exactly as the way people actually bet, just about expected payout and the rational behavior associated with trying to get positive expected payouts. However, in the non-factative paper, The argument itself doesn't even use expected utility maximization as a necessary premise, but that it's just one reasonable decision making procedure that agents can follow. This generalizes the argument.[3]

The last problem for the dutch books is that dutch books are rare in the sense that they would rarely occur in reality. However it has been argued [4] that even under the assumption that it is rare, it is not a bad idea to guard against potential rare large losses. In fact, it's what we do when it comes to insurance all the time. Even if the examples given above have small amounts of money involved, one can multiple the losses by an arbitrary constant. It can be rational to have strategies in place that prevent this.

7 Alternatives to Dutch Books

Since the dutch books seem to have a lot of problems to deal with, how about we take a look at some alternatives to justifying Bayesian epistemology?

One such alternative is Van Fraasen's[1] where he argues uses symmetry properties of credences. He points out that it is trivial to model any agent as a conditionalizer, but for it to be Bayesian conditionalization this is because of some isomorphism between two credence functions P_A and P'_A such that the logical connectives between them.

One can also associate beliefs with preference relations, and show that breaking axioms of preference relations (such as transitivity) might also lead to violations of belief that follows the probability axioms. In fact, certain representation theorems show that preference relations that satisfy certain axioms specify beliefs that must uphold the probability axioms. [5]

8 Pragmatic and Risk-averse approach towards Bayesian Epistemology

Instead of justifying Bayesian epistemology a priori with the Dutch book argument or any of the other alternatives, one can do a weakening of the argument and instead conclude with a more pragmatic approach. In such an approach, violators are not guaranteed to accept dutch books, but rather have a tendency, all things being equal, in very specific circumstances, to accept dutch books.[4]

So my own case is the following: I will argue that the pragmatic approach offers a case for Bayesian epistemology in the sense that it avoids the problems stated above, though at the cost of Bayesian epistemology no longer being universally applicable. It allows us to instead use Bayesian epistemology, and probabilistic credence's, in a way that doesn't require us to be committed to logical omniscience, certain properties on rational action, and that our better behavior could be different from our credence's.

The important part seems to be to take fair bets, and to take bets that guarantee a win, even if it means you'd have to disconnect your credence's from your betting behavior. If we generalize the gamble to any option in life, then it is preferable to simply take actions that avoid Dutch books, guarantee Czech books, and generally avoid risky gambles. Even if this would generate a preference which avoids risks even at the cost of less expected value (A so called Allai's preference). This is still preferable since it allows one to avoid risky losses while still optimizing for one's preferences and values.

In a way, what I'm proposing is a similar strategy employed by a causal decision theorist who is still a one-boxer in Newcomb's problem. An agent can employ causal decision theory for general decision theory, but avoiding risking loosing a million dollars by setting up exceptions to the general theory. That exception being a predisposed behavior towards one-boxing, since you risk a lot of potential money otherwise. Especially if you set up an opposite-Newcomb's problem where you'd risk accruing debts inside of the boxes. The predictor in the problem have set up two boxes and says they contain some debt amount, and states that if they predicted you will be taking both boxes then you only accrue \$1000 debt but taking only one gives \$1001000 debt. So you should take both, and have a disposition towards taking both, even if there's a risk of more debt since you're taking on potentially two debts.

In a similar fashion, one can change one's behavior so as to avoid Dutch Books, and accepting Czech books, and to not gamble one's life savings on tautologies or supposed certainties. Even if such scenarios are potentially rare, just like Newcomb's problem, it is still preferable to have some kind of behavior where you avoid risks in general, while value maximizing at the same time.

Radical risk reduction might not be optimal, we take actions all the time where we risk our lives with some usually insignificant probability. But looking both ways before crossing the street is one way to minimize this risk.

9 Conclusion

We have seen how the dutch book theorems can be used to derive the dutch book arguments for rational agent belief. While it seems clear that dutch book arguments have enough problems to warrant being careful to applying Bayesian epistemology universally, it can still be generally applicable in a large range of situations one find in life. As they fail to take into account certain risks, such as telling you to not take Czech books, one can make exceptions for these cases.

References

[1] Fraasen, B. V. *Notes on Bayesian Confirmation Theory.* 1989.

[2] Hajek. *Dutch Book Arguments.* 2008.

[3] Rescorla, M. An improved dutch book theorem for conditionalization.

[4] STREVENS, M. *Notes on Bayesian Confirmation Theory*. 2012.

[5] VINEBERG, S. Dutch book arguments. *Stanford Encyclopedia of philosophy* (2022).