Quantum Analogs Chapter 1 Student Manual

Standing Sound Waves in a Tube

An Analog to a Quantum Mechanical Particle in a Box

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1. Standing sound waves in a tube – an analog to a quantum mechanical particle in a box

Objective: For a simple tube, use an oscilloscope to compare the sound input by a speaker at one end to the sound received by a microphone at the other end.

Equipment Required:

TeachSpin Quantum Analog System: Controller, V-Channel & Aluminum Cylinders
Sine wave generator capable of producing 1-50 kHz with a peak-to-peak voltage of 0.50 V
Two-Channel Oscilloscope

Setup:

Make a tube using the tube-pieces. Put the end-piece with the speaker on one end and the end-piece with the microphone on the other. Attach a BNC splitter to *SINE WAVE INPUT* on the Controller. Connect the output of your sine wave generator to one side of the splitter. Use a BNC cable to send the sound signal to the Channel 1 input of your oscilloscope. Plug the lead from the speaker end of your experimental tube to *SPEAKER OUTPUT* on the controller. The same sine wave now goes to both the speaker and Channel 1. Connect the microphone output of the tube array to *MICROPHONE INPUT*. Connect *AC MONITOR* on the Controller to Channel 2 of the oscilloscope. Channel 2 will display the sound signal received by the microphone. Trigger the oscilloscope on Channel 1. Use the *ATTENUATOR* dial on the Controller to keep the signal on Channel 2 from going off scale. (Appendix 1 describes the function of each part of the Controller.)

Experiment:

Start at low frequency (100 Hz or less), and slowly increase the frequency.

Question:

What are you observing? How can you tell that you are at a resonance? Did you notice the phase-shift when going through a resonance? (Note that, due to unknown phase shifts in the speaker, microphone, and electronics, the absolute phase between input and output channel can not be interpreted.)

Experiment:

Change the length of the tube and repeat the experiment.

Question:

Do the resonance frequencies change? Are they higher/lower when the tube is longer/shorter?

Take a full set of data for one tube length:

Measure and record the length of the tube. Measure the first 20 resonance frequencies. Assign the lowest resonance frequency the index number n = 1, and plot the resonance frequency f_n as function of its index number, n.

Background:

A resonance occurs when a standing sound wave has developed in the tube. The sound emitted by the speaker is reflected back and forth between the two hard end-walls of the tube. The resonance develops when, after a round trip in the tube, the sound wave is in phase with the wave emitted by the speaker. In this case, the emitted sound interferes with the reflected sound constructively. The condition for resonance is fulfilled when:

$$2L = n\frac{c}{f} = n\lambda$$

with the length of the tube L, the speed of sound c, the frequency f, the wavelength λ and an integer number n=1,2,...∞. Resonances are observed when the tube length is an integer multiple of $\lambda/2$.

Analyze the data:

From the resonance frequencies plotted as function of their index n, you can calculate the speed of sound c. Make a linear fit for your data. Calculate c from the slope and determine the uncertainty of your measurement.

Differential equation for sound and boundary conditions:

The propagation of sound waves in air can be described by differential equations.

On one hand, there is the linearized Euler's equation

$$\frac{\partial \vec{u}}{\partial t} = -\frac{1}{\rho}\,\mathrm{grad}\ p \tag{1.1}$$

with the velocity of the air $\vec{u}$, the mass density of the air ρ and the pressure p. On the other hand, the continuity equation has to be fulfilled.

$$\frac{\partial \rho}{\partial t} = -\rho\ \mathbf{div}\ \vec{u} \tag{1.2}$$

Additionally, representing compressibility as κ; the density and the pressure of the air are connected by

$$\frac{\partial p}{\partial \rho} = \frac{1}{\kappa\rho} \tag{1.3}$$

These equations can be combined to a wave equation for the pressure

$$\frac{\partial^2 p}{\partial t^2} = \frac{1}{\rho\kappa}\Delta p \tag{1.4}$$

with the Laplace operator Δ. In this wave equation, however, the phase relation between velocity and pressure of the wave is lost, since the velocity has been eliminated. We need to refer to the velocity again, since the boundary conditions at the hard wall can be formulated best with the velocity. It is obvious that, at the surface of the wall, the velocity perpendicular to the wall has to be zero. (The air can not move into or out of the wall.) From eqn. (1.1), it also follows that, at the surface of the wall, the derivative of the pressure in the direction perpendicular to the wall is zero. This combination of boundary conditions is called a "Neumann boundary condition".

For frequencies lower than about 16 kHz, the air is not moving perpendicular to the symmetry-axis (x-axis) of the tube. Thus, $u_y(\vec{r}) = 0$, $u_z(\vec{r}) = 0$, $u_x(\vec{r}) = u_x(x)$ and $p(\vec{r}) = p(x)$.

The problem has now been reduced to a quasi one-dimensional problem and we can make a one-dimensional ansatz for the solution in the form:

$$p(x) = p_0 \cos(kx - \omega t + \alpha) \tag{1.5}$$

Here, p_0 represents the amplitude of the wave and must not be confused with the background air pressure of about 1000 mbar. $\omega = 2\pi f$ is the angular frequency and $k = 2\pi/\lambda$ is the wave vector. This function describes a wave propagating in the positive x-direction. In the tube we find a superposition of right and left (positive and negative x-direction) propagating waves, since the waves are reflected at the ends of the tube. The wavefunction is therefore given by

$$p(x) = \tfrac{1}{2} p_0 \cos(kx - \omega t + \alpha) + \tfrac{1}{2} p_0 \cos(-kx - \omega t - \alpha) \tag{1.6}$$

This can be rewritten as

$$p(x) = p_0 \cos(kx + \alpha)\cos(\omega t) \tag{1.7}$$

Solutions of the differential equation are those wave functions $p(x)$ that fulfill the boundary conditions for a certain tube length L at all times. From the boundary conditions $\mathrm{d}p/\mathrm{d}x(0) = 0$ and $\mathrm{d}p/\mathrm{d}x(L) = 0$, we can easily derive the parameters to be $\alpha = 0$ and $k = n\,\pi/L$.

Dispersion of sound waves:
Redraw your graph of frequency as function of resonance-index (f_n *vs.* n) to show angular frequency as function of wave vector $\omega(k)$. This new graph shows the dispersion relation of sound waves.

Analogy to a quantum mechanical particle in a box:

The sound wave in the tube can serve as an analog for a quantum mechanical particle in a one-dimensional square potential well. The differential equation that describes the particle is Schrödinger's equation:

$$i\hbar \frac{\partial}{\partial t}\psi(\vec{r},t) = -\frac{\hbar^2}{2m}\Delta\psi(\vec{r},t) + V(\vec{r})\psi(\vec{r},t) \tag{1.8}$$

with the wave function $\psi(\vec{r},t)$, the particle mass m, and a scalar potential $V(r)$. In the case of a one-dimensional square potential well with infinitely high potential barriers at both ends, and $V = 0$ in the space between the ends, the equation reduces to

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\Delta\psi(x,t) \tag{1.9}$$

This differential equation has as a solution complex waves that are scattered back and forth between the ends of the well. The probability of finding the particle at a certain position x in the well is given by the probability density $|\psi(x,t)|^2$. When multiplied by the elementary charge e, it represents the charge density inside the well.

Most of the solutions of eqn. (1.9) result in time-dependent charge densities. These, however, would emit electromagnetic waves, since charge is moving. On the other hand, there are certain solutions that have a time independent charge density. They can be found by solving the time-independent Schrödinger equation

$$E\psi(\vec{r}) = -\frac{\hbar^2}{2m}\Delta\psi(\vec{r}) + V(\vec{r})\psi(\vec{r}) \qquad (1.10)$$

In our case, for the one-dimensional square potential well, the equation simplifies to

$$E\psi(x) = -\frac{\hbar^2}{2m}\Delta\psi(x) \qquad (1.11)$$

This equation can be solved for certain eigenvalues of energy E. We make an ansatz with standing waves of the form

$$\psi(x) = A\sin(kx+\alpha) \qquad (1.12)$$

At the ends of the box, where the potential is infinitely high, the wave function has to be zero (Dirichlet boundary condition). These boundary conditions, $\psi(0) = 0$ and $\psi(L) = 0$, are fulfilled if $\alpha = 0$ and $k = n\,\pi/L$ where n is an integer. The total probability of finding the particle anywhere in the box has to be one. This determines that the amplitude of the wave function is $A = \sqrt{2/L}$.

The solution of Schrödinger's time-dependent equation (1.9) is obtained from the solution (1.12) by multiplying it with a time dependent phase factor

$$\psi(x,t) = A\sin(kx+\alpha)\,e^{-i\omega t} \qquad (1.13)$$

You can convince yourself that, for this solution, $|\psi(x,t)|^2$ is indeed time-independent. The angular frequency in this expression is given by $\omega = E/\hbar$. Note that in quantum mechanics the energy is in general connected with the frequency by

$$E = h\,f = \hbar\omega \qquad (1.14)$$

We can now calculate the eigenvalues of energy that are given by

$$E(k) = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{2m\,L^2} \qquad (1.15)$$

This is the dispersion relation of the quantum mechanical particle in a box.

What is analogous, what is different?

The classical sound wave in a tube and the quantum mechanical electron in a square potential well are similar in many respects, but some details are different. Both the sound wave and the wave-function of the electron are solutions of a wave equation describing a delocalized object. The particular aspect being described, however, is different. In the classical case, *p(x,t)* is the amplitude of the signal picked up by a microphone located at this position. In the quantum mechanical case, the squared amplitude $|\psi(x,t)|^2$ at a certain position gives the probability of finding the electron at this position.

Both of the differential equations have the Laplace operator on the right side (second derivatives with respect to space). However, with respect to time they are different. In the classical case, we have a second derivative with respect to time that leads to wave-solutions. In the quantum mechanical case, the combination of the complex number i and a first-order derivative with respect to time leads to wave solutions. But these wave-solutions are complex due to this special form. It is also the first-order time-derivative that results in a parabolic dispersion $E(k)$ of the electron. In contrast, the sound wave has a linear dispersion due to the second-order time-derivative. Schrödinger's equation includes, in addition, a potential $V(\vec{r})$ that can not be simulated by the sound wave experiment. However, the reflection at a hard wall can be used to function as an analog to an infinitely high potential barrier. In later experiments, we will use irises as an analog for finite potential barriers with certain reflection and transmission probability.

In both cases, eigenstates are found in a well. For certain wavelengths, standing waves are found, and in both cases the wavevector of these waves is given by $k = n\ \pi/L$. However, the position of the nodes is different, because the boundary conditions are not the same. In the quantum mechanical case, the wave function must be zero at the boundary. In the case of sound waves, we have physical quantities that we use to describe the wave. One is the pressure and the other is the air-velocity. Like the quantum mechanical wave function, the velocity has a node at the boundary, but the velocity is a vector. The pressure has a local maximum at the boundary and is a scalar quantity. As an analog to the *scalar* quantum mechanical wave function, we therefore prefer the *scalar* pressure, even though it has an opposite boundary condition. A scalar "velocity potential" could also be used to describe the wave, but it does not help much, since its nodes are at the same position as those for the pressure. You should be aware of this difference.

To each eigenstate, an eigenfrequency, ω, is assigned. In quantum mechanics, it is found in the time dependent phase factor, $e^{i\omega t}$. In the case of sound waves, the eigenfrequency is simply the frequency of the sound itself, $\omega=2\pi f$. In quantum mechanics, the frequency is directly related to an energy by the equation $E = \eta\omega$. This has no direct analog in the sound experiments. When working with sound, we look at the frequency of the sound and not at an energy. We therefore consider energy-levels in quantum mechanics as being analogous to the "frequency-levels" in the sound experiments that are given by the sharp resonance frequencies. The dispersion $E(k)$, discussed in quantum mechanics, can be compared with $\omega(k)$ in classical mechanics.

Another little difference is related to the absolute phase. The microphone can measure the phase of the sound wave, but in quantum mechanics the absolute phase of a state can not be measured. Relative phases between two wavefunctions can be measured in quantum mechanics and we can measure the phase of an acoustic wave function at different locations and determine the relative phase to compare with a quantum mechanical system. You should be aware that the sound experiments provide an experimentalist with more information about the system than can be extracted from an analogous quantum mechanical system.

1.2 Measure a spectrum in the tube using an oscilloscope

Objective: In this experiment, the independent variable is the frequency provided by the generator, and the dependent variable is the amplitude of the sound wave reaching the microphone. First, we will examine the amplitude of the sound-wave received at the microphone as a function of the frequency of the sound. Then, we will determine how the spectrum (the pattern) observed depends on the length of the tube conducting the sound.

Setup:

With the tube, speaker and microphone arranged as before, connect the output of the sine wave generator to *SINE WAVE INPUT* on the controller and the wire from the speaker to *SPEAKER OUPUT*. Connect the microphone on the experimental tube to *MICROPHONE INPUT*.

Locate the *FREQUENCY-TO-VOLTAGE CONVERTER* module on the controller and set the toggle switch to *ON*. With the oscilloscope in the xy-mode, connect the *DC-OUTPUT* of the converter module to Channel 1, the x-axis. The converter provides a voltage proportional to the instantaneous frequency. The calibration is 1 V per 1 kHz and it can be used for frequencies up to 10 kHz (or, with offsets, up to 20 kHz).

Connect *DETECTOR OUTPUT* to Channel 2, the y-axis of the oscilloscope. The *DETECTOR OUTPUT* connection provides a dc signal that is proportional to the amplitude of the sound wave at the microphone.

You have now set up the oscilloscope to plot the amplitude of the sound at the microphone as a function of the frequency of the sound.

Set the image persistence time on the oscilloscope to infinite.

Now, sweep the frequency by hand. As you change the frequency, the oscilloscope will plot a spectrum with peaks.

You can use the *DC-OFFSET* knob to center the image on the oscilloscope screen.

Experiment:

Take spectra for different tube lengths and compare them with the results you found in section one.

1.3 Measure a spectrum with the computer and compare it to the spectrum found with the oscilloscope.

Objective: This experiment uses a computer sound card both to generate the sound wave and to sweep its frequency. We will use the oscilloscope to observe the actual sine wave signals both going into the speaker and coming from the microphone. Simultaneously, we will use the computer to display a spectrum which shows the amplitude of the signal from the microphone as a function of the frequency of the sound.

Equipment Required:

TeachSpin Quantum Analog System: Controller, V-Channel & Aluminum Cylinders
Two-Channel Oscilloscope
Two adapter cables (BNC - 3.5 mm plug)
Computer with sound card installed and Quantum Analogs “SpectrumSLC.exe” running

WARNING: The BNC-to-3.5-mm adapter cables are provided as a convenient way to couple signals between the controller and sound card. Unfortunately, they could also provide a way for excessive external voltage sources to damage a sound card. Most sound cards are somewhat protected against excessive inputs, but ***it is the user's responsibility to ensure that adapter cable voltages are kept BELOW 5 Volts peak-to-peak***.
The maximum peak-to-peak value for optimum performance of the Quantum Analogs system depends on your sound card and can vary from 500 mV to 2 V.

Setup:

Using the tube-pieces, make a tube with the end-piece containing the speaker on one end and the end-piece with the microphone on the other.

Now, using connectors on the controller, you will send the sound card signal to both the speaker and Channel 1 of the oscilloscope, and the microphone signal to both the microphone input of the computer and to Channel 2 of the oscilloscope.

First, make sure that the *ATTENUATOR* knob on the controller is set at 0.2 (out of 10) turns.

Let’s start with the sound signal. Attach a BNC splitter to *SINE WAVE INPUT* on the controller. Using the adapter cable, connect the output of the sound card to one arm of the splitter. With a BNC cable, convey the sound card signal from the splitter to Channel 1 of your oscilloscope. Plug the lead from the speaker end of your experimental tube to *SPEAKER OUTPUT* on the controller. The sound card signal is now going to both the speaker and Channel 1.

The microphone signal will also be sent two different places. Connect the microphone on your experimental tube to *MICROPHONE INPUT* on the controller. Put a BNC splitter on the controller connector labeled *AC-MONITOR*. From the splitter, use an adapter cable to send the microphone signal to the microphone input on the computer sound card. Use a BNC cable to send the same signal to Channel 2 of the oscilloscope to show the actual signal coming from the microphone.

The computer will plot the instantaneous frequency generated by the sound card on the x-axis and the amplitude of the microphone input signal on the y-axis.

The next job is to start the computer program and adjust the magnitude of both the speaker and microphone signals so that you will have maximum signal while keeping the microphone input to the computer from saturating. Peak-to-peak signals to the microphone input can range from 0.50 to 2.0 volts depending upon your sound card.

Once the program, SpectrumSLC.exe., is running, you can configure the computer. Go to the menu at the top of the screen and choose Configure > Input Channel/Volume At this point, choose *Line In*, if it is available; otherwise choose *Microphone*. On this screen, set the microphone volume slider to the middle of its range.

To set the speaker volume, use the *Amplitude Output Signal* on the lower left of the computer screen. That slider should also be set to middle range.

The microphone signal coming from the apparatus first passes through a built-in amplifier, and then through the *ATTENUATOR*, before reaching the *AC-MONITOR* connector. The ten-turn knob on the attenuator multiplies the incoming signal by a factor ranging from zero to one. For example, a setting of 1.2 turns (out of the 10 turns possible) stands for a transmission of 1.2/10 = 0.12 (or 12%) relative to the maximum possible.

After taking an initial wide range spectrum, choose a section that includes the highest peak and a smaller one next to it. Readjust the scan to cover just this portion. Using the option that allows you to keep successive spectra visible, take Spectrum 1, 2, 3, etc. with the attenuator knob set at 0.1, 0.2, 0.3 . . . turns (out of ten). The changes in the heights of the peaks will tell you whether or not the system is behaving in a linear fashion. Continue to go higher on the 10-turn dial setting until you have visual evidence of saturation.

Once you have reached saturation, drop back into the linear range. Now you can operate with confidence that the signals you see really are proportional to the amplitude of the sound wave you are studying.

Experiment:

Now you can use the computer to collect an overview spectrum from about 100 to 10,000 Hz. You can use coarse steps (~10 Hz) and a short time per step (~50 ms) for this investigation. As the frequency is changing, watch the trace on the oscilloscope. How is the oscilloscope showing the change in frequency? What is happening to the amplitude of the signal? How is this related to the trace being created on the computer?

Compare the spectrum recorded on the computer to the results you found using the oscilloscope in the first experiment.

Linewidth:

Lifetime of quantum mechanical states

In most cases eigenstates do not last forever. In classical physics there is decay due to dissipation of energy by friction. In quantum mechanics only the ground state lasts forever. Excited states with higher energy decay into the ground state, which is the eigenstate of the system with the lowest energy. These effects are not included in the differential equations. However, we can introduce the decay easily into the wave functions by replacing the time dependent factors in the wave function $\cos(\omega t)$ and $e^{i\omega t}$, respectively, with a factor that is oscillating and exponentially damped. With a damping constant λ it results in $e^{-\lambda t}\cos(\omega t)$ and $e^{-\lambda t - i\omega t}$, respectively.

In the case of finite lifetime, the wave function cannot be assigned to a single angular frequency ω_0 but contains a spectrum of angular frequencies that we can determine by Fourier-transformation. Let's write the wave function in a general way as

$$\psi(x,t) = f(x)\, e^{-(\lambda + i\omega_0)t} \tag{1.16}$$

with an arbitrary spatial dependence $f(x)$. For t < 0, the wave function is assumed to be zero. By performing a Fourier-transformation we obtain the so-called spectral function, $A(\omega)$, that describes the amplitude as function of angular frequency in the classical case. In the quantum mechanical case, $\left|A(\omega)\right|^2$ is the probability of measuring the particle to have the energy $E = \eta\omega$. Performing the Fourier-transformation

$$A(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(\lambda + i\omega_0)t}\, e^{i\omega t}\, \mathrm{d}t \tag{1.17}$$

we obtain the spectral function

$$A(\omega) = \frac{\frac{1}{\sqrt{2\pi}}}{\lambda + i(\omega_0 - \omega)} \quad . \tag{1.18}$$

The absolute squared is a so-called Lorentzian peak

$$\left|A(\omega)\right|^2 = \frac{\frac{1}{2\pi}}{(\omega_0 - \omega)^2 + \lambda^2} \,. \tag{1.19}$$

The width of the peak is directly related to the lifetime τ of the eigenstate. The lifetime denotes the time after that the amplitude of the state has been reduced to 1/e. From the half width at half maximum of the peak the damping constant λ can be read directly. In quantum mechanics the width in energy Γ of a metastable state is $\Gamma = \eta\lambda$

$$\Gamma = \frac{\eta}{\tau} \tag{1.20}$$

The spectral function $A(\omega)$ is complex, which can be written as the absolute $\left|A(\omega)\right|$ multiplied by a complex phase factor $A(\omega) = \left|A(\omega)\right| e^{i\varphi}$. Both amplitude and phase depend on the angular frequency.

Linewidth of the resonances in the sound experiment

In the sound experiments the situation is a little bit different, but the result looks almost the same as in quantum mechanics. The sound wave close to an eigenstate can be seen as a damped, driven harmonic oscillator described by the linear differential equation

$$\frac{\mathbf{d}^2 p}{\mathbf{d}t^2} + 2\gamma \frac{\mathbf{d}p}{\mathbf{d}t} + \omega_0^2 p = K\cos(\omega t) \qquad (1.21)$$

This driving force is represented by the speaker that is driving the standing sound wave. The resonance frequency under consideration has the angular frequency ω_0. The solution of this differential equation is a superposition of a transient solution that is a solution of the homogenous differential equation (first part of eqn. 1.22), and a steady-state solution (second part of eqn. 1.22) that is of interest here.

$$p(t) = A_1 e^{-\gamma t}\cos(\omega_1 t + \varphi_1) + A\cos(\omega t + \varphi) \qquad (1.22)$$

For our experiment, we can assume that the transient solution has already damped out, so that we are detecting only the steady state amplitude, A, of the sound wave. This amplitude depends on the frequency ω of the driving force compared to the eigen-frequency ω_0 of the oscillator. It is given by

$$A = \frac{K}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}} \quad . \qquad (1.23)$$

The phase between driving force and oscillating air is given by

$$\varphi = \arctan\frac{2\gamma\omega}{\omega_0^2 - \omega^2}. \qquad (1.24)$$

Using the complex exponential function, the result can be written even more simply.

For this purpose we write the differential equation in the form

$$\frac{\mathbf{d}^2 p}{\mathbf{d}t^2} + 2\gamma \frac{\mathbf{d}p}{\mathbf{d}t} + \omega_0^2 p = Ke^{i\omega t} \qquad (1.25)$$

and the steady-state solution as

$$p_s(t) = Ae^{i(\omega t + \varphi)}. \qquad (1.26)$$

The complex amplitude A as function of angular frequency ω can then be written as

$$A = \frac{Ke^{i\varphi}}{\omega_0^2 - \omega^2 + 2i\gamma\omega}. \qquad (1.27)$$

If only single resonance existed in the tube, the microphone would measure the amplitude

$$|A| = \left|\frac{Ke^{i\varphi}}{\omega_0^2 - \omega^2 + 2i\gamma\omega}\right| = \frac{K}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}}. \tag{1.28}$$

In reality, however, there are a number of resonances, all of which are simultaneously excited. The superposition is coherent because there is a fixed phase-relation between the different resonances.

The entire spectrum is therefore a superposition of all complex amplitudes. That can be written as:

$$|A| = \left|\frac{K_1 e^{i\varphi_1}}{\omega_1^2 - \omega^2 + 2i\gamma_1\omega} + \frac{K_2 e^{i\varphi_2}}{\omega_2^2 - \omega^2 + 2i\gamma_2\omega} + \frac{K_3 e^{i\varphi_3}}{\omega_3^2 - \omega^2 + 2i\gamma_3\omega} + \ldots\right|$$

$$|A(\omega)| = \left|\sum_{i=1}^{n} \frac{K_i e^{i\varphi_i}}{\omega_i^2 - \omega^2 + 2i\gamma_i\omega}\right| \tag{1.29}$$

In this notation, we are using four fitting parameters to model each peak in the spectrum: K_i, $\omega_i, \gamma_i, \varphi_i$. In our simplified theoretical model we describe the resonances in the tube by independent damped, driven oscillators with parameters taken from the experiment. The coupling of the speaker to the standing wave depends on geometry and can be different for different resonances, which results in different K_i's. The friction depends on a different parameter, which results in different γ_i's. Finally, the phase between driving force and oscillating air is also different for different resonances. Therefore, the phase φ_i is also fitted as a parameter.

In a spectrum measured with an oscilloscope or by computer, $|A(\omega)|$ is plotted. The connector marked *DC-OUTPUT* on the Quantum Analogs controller gives a voltage proportional to $|A(\omega)|$. The linewidth of an acoustic resonance is small compared to its frequency; $\gamma << \omega_0$. In this case we can make the approximation

$$\omega_0 + \omega \approx 2\omega \quad \Rightarrow \quad \omega_0^2 - \omega^2 \approx 2\omega(\omega_0 - \omega)$$

and rewrite the absolute value of Amplitude as

$$A(\omega) \approx \frac{Ke^{i\varphi}}{2i\omega[\gamma + i(\omega - \omega_0)]}$$

Since ω can be assumed to be almost constant in the frequency interval across the peak (within the approximation $\gamma << \omega_0$),

$$A(\omega) \propto \frac{1}{\gamma + i(\omega - \omega_0)}$$

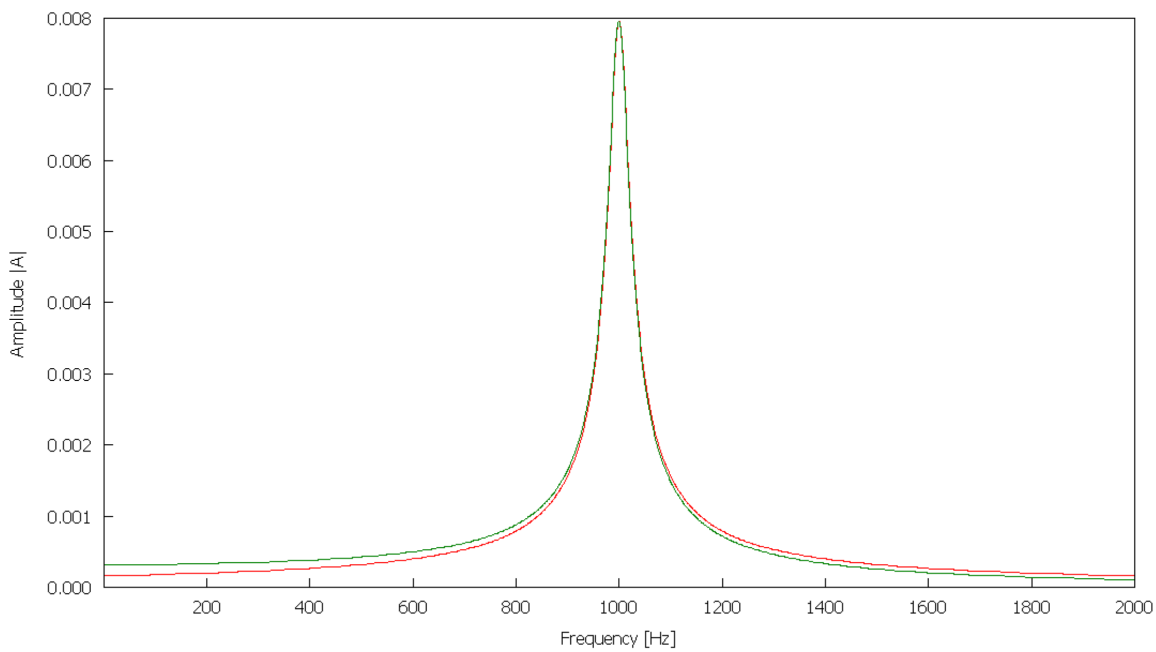
The resonance peak $A(\omega)$, in a classical driven, damped oscillator has the same shape as the spectral function of quantum mechanical eigenstate with finite lifetime (eqn. 1.18).
In the following figure the two line-shapes

$$|A(\omega)| = \frac{2\omega_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}}$$

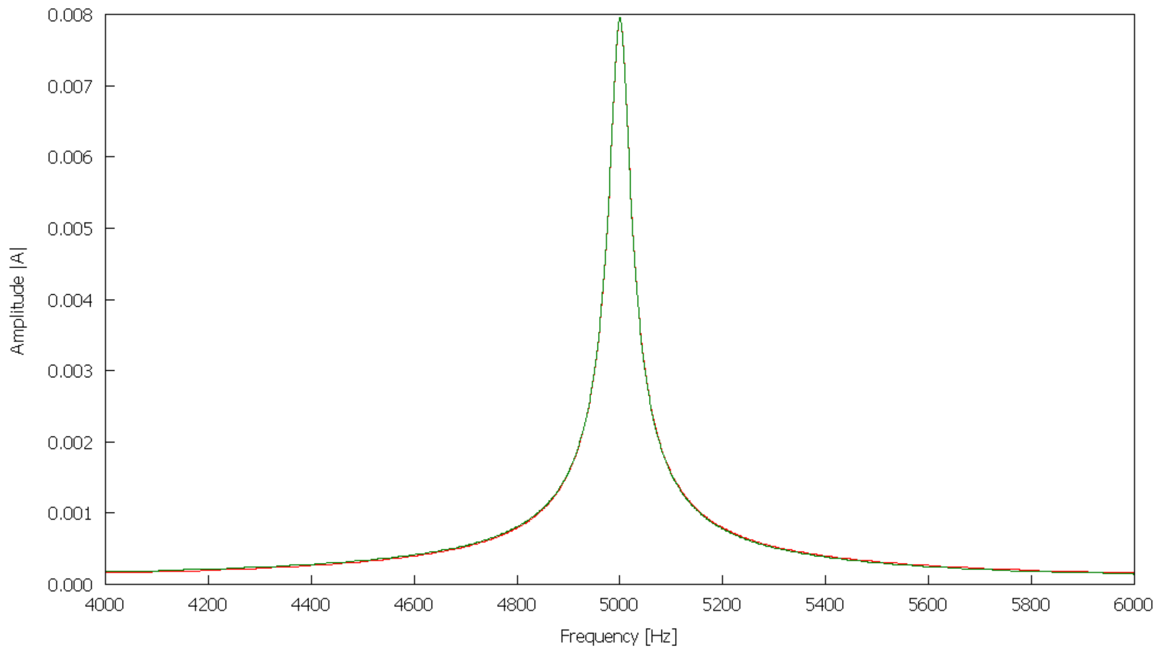
and

$$|A(\omega)| = \frac{1}{\sqrt{(\omega_0 - \omega)^2 + \lambda^2}}$$

are plotted for comparison with the parameter $\omega_0 = 2\pi \cdot 1000\text{Hz}$ and $\gamma = \lambda = 2\pi \cdot 20\text{Hz}$. The full width at half maximum of the peaks is $\Delta\omega = 2\sqrt{3}\lambda$ and $\Delta f = \frac{\sqrt{3}}{\pi}\lambda$, respectively.

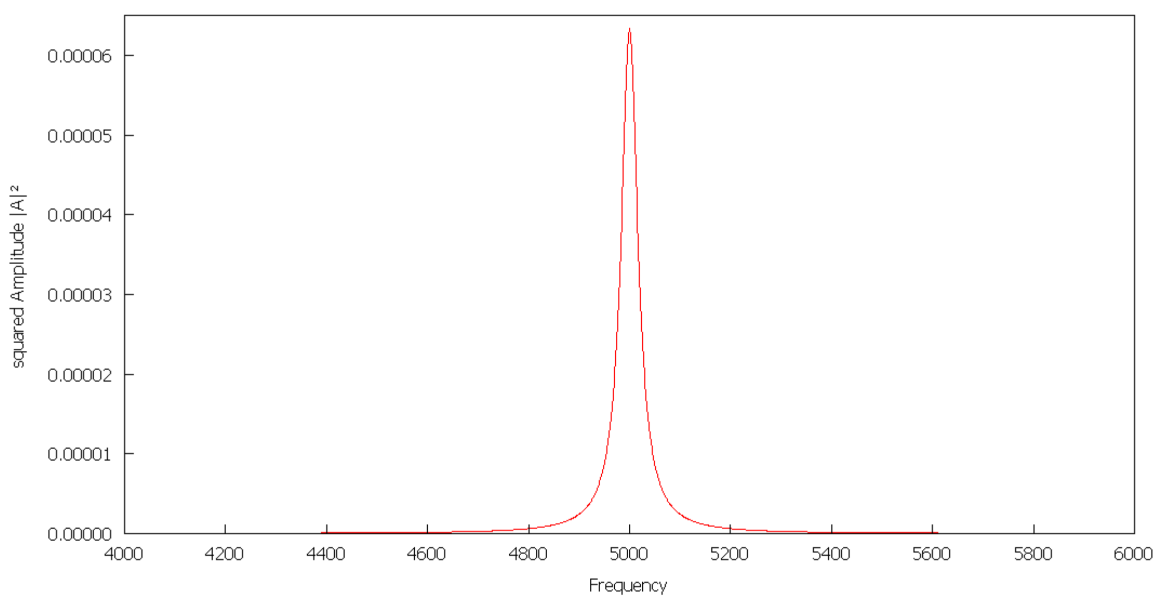


Line-shape of a classical resonance (green line) and for a quantum mechanical eigenstate with finite lifetime (red line) for comparison (parameter see text).



A comparison of the line-shape of a classical resonance (green line) and a quantum mechanical eigenstate with finite lifetime (red line).
Parameters: $\omega_0 = 2\pi \cdot 5000\text{Hz}$ and $\gamma = \lambda = 2\pi \cdot 20\text{Hz}$.

The better known Lorentzian-shape for the same parameter looks as follows



Lorentzian line-shape (squared amplitude) of a quantum mechanical eigenstate with finite lifetime
Parameters: $\omega_0 = 2\pi \cdot 5000\text{Hz}$ and $\gamma = \lambda = 2\pi \cdot 20\text{Hz}$

Objective: In this experiment, we will use the computer to record a spectrum of eight or fewer peaks. We will then use the software program provided to demonstrate that the data generated by Quantum Analogs can be fit to the theoretical models.

Setup:

Create a short tube and set the computer parameters to produce a spectrum with eight or fewer peaks. One possible configuration would be a 150 mm long tube, a sweep from 5000 Hz to 14000 Hz, 5 Hz steps, and 50 ms per step.

Experiment:

Generate a spectrum of eight or fewer peaks. After generating your spectrum, open the fitting window in the software via the sequence: Menu > Windows > Fit. In the fitting window that opens, your first task is to give the software a set of initial estimates for the location and height of up to eight resonances. In the 'Peak Number' menu at the upper left of the window, select Peak 1. Now, point your mouse to the top of the lowest frequency peak, and left-click your mouse. You will see (in blue) the theoretical resonance with the center and height matching the peak you have selected. The blue curve also has a default value for width. If you have a mouse wheel, you may use the wheel to adjust the width estimate to match your data. Perfection is not required in these initial estimates.

When you are done with Peak 1, right-click your mouse and the selection in the Peak Number menu will change to Peak 2. Now locate and left-click the second peak. Repeat this initial-estimate procedure for it and each subsequent peak.

After using the mouse to put in the initial estimates for all of the peaks, you will see a blue curve showing a first approximation of the theoretical model. Now click the button for 'Start Fit', and the software will use your estimates to optimize the match between the data curve (red) and the theoretical model (blue), by adjusting the fitting parameters. If one of the model's peaks 'escapes' from the data of the spectrum during this fitting procedure, you can stop the fit and readjust manually. After you've reset that peak's estimated parameters, just restart the automatic fit.

When the automatic fitting is done, you can use the Peak Number menu (at the window's upper left) to select any peak. The software then shows the values of the parameters for that peak that best-fit your data.

You can now check the repeatability of your data. To do this, first record the parameters for one of your peaks. Next, acquire a fresh set of data. Repeat the fitting procedure, and look again for the center location of your chosen peak. (Prepare to be very impressed!)

You can save the fitting parameters that you generated as an ASCII file. The best-fit theoretical function can be saved either as a data file or an image file.