Zeros of Eisenstein Series of $\Gamma_0(p)$

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Fundamental Domain

A **fundamental domain** for a congruence subgroup Γ is a region $\mathcal{F} \subset \mathbb{H}$ such that for every $z \in \mathbb{H}$, there is a unique $\gamma \in \Gamma$ such that $\gamma z \in \mathcal{F}$. Note that there are many fundamental domains for a given Γ.
We look at a fundamental domain of $\Gamma_0(p)$, $\mathcal{F}_p$ that looks like the following

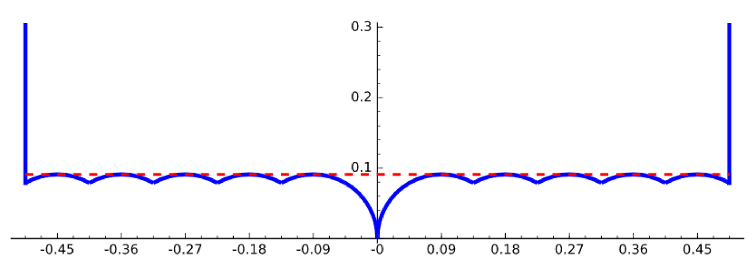


Figure: Fundamental domain for $p = 11$

Generally, the boundary of this fundamental domain is given by the arcs $\{e^{i\theta}/p \pm q/p\}$ for $-(p-1)/2 \le q \le (p-1)/2$ and $q \neq 0$. In addition, we have that for all $z = x + iy \in \mathcal{F}_p$, $-1/2 \le x \le 1/2$.

Rankin and Swinnerton-Dyer Method

F.K. Rankin and Swinnerton-Dyer proved that all of the zeros of the level 1 Eisenstein series lie on the arc $\{e^{i\theta} \mid \theta \in (\pi/3, 2\pi/3)\}$ using the function $F_k(\theta) = e^{i\theta/2}E_k(\theta)$. Then along this arc, F_k takes on real values so we can apply the intermediate value theorem to find zeros of F_k. Then there is a bijection between the zeros of F_k and E_k, so we are finding zeros of E_k.
Shigezumi generalized the Rankin and Swinnterton-Dyer Method to show that there are zeros on the arcs $\{\frac{e^{i\theta}}{p} - \frac{1}{p} \mid (0, 2\pi/3)\}$ for $p = 2, 3, 5$ and bounded k. We aimed to generalize his results with our result.

Valence Formulas

Another aspect of Rankin and Swinnerton-Dyer's proof was the presence of a valence formula: a formula that counts the number of zeros of a modular form in the fundamental domain. We derived the following valence formula for $\Gamma_0(p)$ with complex analysis and integration:

Let $v_x(f)$ denote the order of the zero of f at a point x. Let f be a modular form of weight k for $\Gamma_0(p)$ where k is even. Let $\rho_{p,1}, \rho_{p,2}$ be the two ρ-elliptic points, and $i_{p,1}, i_{p,2}$ be the two i-elliptic points, if any of them exist. Then the following relation holds:

$$v_\infty(f) + v_0(f) + \sum_{x \in \mathcal{F}_p} c_x v_x(f) = \frac{p+1}{12}k \tag{1}$$

where

$$c_x = \begin{cases} \frac{1}{3} & \text{if } x = \frac{q}{2p} + \frac{\sqrt{3}}{2p}i \text{ for some odd } q \\ \frac{1}{2} & \text{else if } x \in \mathcal{B}_p \\ 1 & \text{otherwise} \end{cases}$$

Eisenstein Series

We studied the modular form, the Eisenstein series of $\Gamma_0(p)$ centered at the cusp at ∞. This is defined to be:

$$E_{k,p}^\infty(z) = \frac{1}{2} \sum_{\substack{\gcd(c,d)=1 \\ p \nmid c}} (cz+d)^{-k}$$

Important Results

- For even $k \ge 4$ and $p \ge 3$, there are at least $\lfloor \frac{k}{3} \rfloor - 1$ zeros of $E_{k,p}^\infty$ on the arc $\{\frac{e^{i\theta}}{p} + \frac{1}{p} \mid \theta \in (0, 2\pi/3)\}$
- If $z = x + iy$ is a zero of $E_{k,p}^\infty$, then for each k there is a small fixed ε_k such that y must satisfy the following inequality:

$$y^k \le \frac{p(1+\varepsilon_k)}{p^k - 1} \cdot \frac{\zeta(k-1)}{\zeta(k)} \tag{2}$$

Proof Outline of Result 1

First we define the function

$$\begin{aligned} F_{k,p}(\theta) &= e^{ik\theta/2} E_{k,p}^\infty(e^{i\theta}/p - 1/p) \\ &= \frac{1}{2} \sum_{\substack{\gcd(c,d)=1 \\ p \nmid d}} (ce^{i\theta/2} + (-c+d)e^{-i\theta/2})^{-k} \end{aligned}$$

Then we note that every zero of $F_{k,p}$ is a zero of $E_{k,p}^\infty$. In addition, the image of $F_{k,p}$ are real numbers. Now, we will have to split this into two cases: when $4 \le k \le 88$ and when $k > 88$.

1. $4 \le k \le 88$:
   - Write $F_{k,p}(\theta) = 2\cos(k\theta/2) + R$
   - Show that $|R| < 2$ for $\theta \in (\alpha_k, 2\pi/3)$ where $\alpha_k < \frac{2\pi}{k}$ by approximating the tail end of the series as $\sum_{N \ge M} 2\sqrt{N}(N/m)^{-k/2}$ (Rankin and Swinnerton-Dyer Method)
   - By the intermediate value theorem, we know there must be $\lfloor \frac{k}{3} \rfloor - 1$ zeros
2. $k > 88$:
   - Write $F_{k,p}(\theta) = 2\cos(k\theta/2) + D + S$, where

   $$\begin{aligned} D = \frac{1}{2}\sum_{n=1}^{N} &(ne^{i\theta/2} + (n+1)e^{-i\theta/2})^{-k} + (ne^{i\theta/2} - (n+1)e^{-i\theta/2})^{-k} \\ &+((n+1)e^{i\theta/2} + ne^{-i\theta/2})^{-k} + ((n+1)e^{i\theta/2} - ne^{i\theta/2})^{-k} \end{aligned}$$

   and $S = R - D$
   - Bound $|S| < 2$ using the Rankin and Swinnerton-Dyer Method
   - Show that $\operatorname{sgn}(D) = \operatorname{sgn}(\cos(m\pi))$ at $\theta = 2\pi m/k$ for integer m
   - Use the intermediate value theorem to show that $F_{k,p}$ will have $\lfloor \frac{k}{3} \rfloor - 1$ zeros on the arc.

More Applications of the Rankin and Swinnerton-Dyer Method

We consider the following function where $r \ge 1$:

$$F_{k,p,r}^q(\theta) = r^{k/2} e^{ik\theta/2} E_{k,p}\left(\frac{re^{i\theta}}{p} + \frac{q}{p}\right)$$

We find that:
For $\theta \in (\frac{\pi}{2} - \alpha, \frac{\pi}{2} + \alpha)$, $r > 1$, $F_{k,p,r}^q(\theta)$ will be at most ϵ away from the description of an elliptic function and have no zeros, where α is a function of ϵ and k.

Proof Outline of Result 2

Consider $E(z) = (p^k - 1)E_{k,p}^\infty(z) = p^k - 1 - \lambda_k \sum_{n=1}^\infty a_n q^n$ where $\lambda_k = \frac{(2\pi i)^k}{(k-1)!\zeta(k)}$ and

$$a_n = \begin{cases} \sigma_{k-1}(n) & p \nmid n \\ \sigma_{k-1}(n) - p^k \sigma_{k-1}(n/p) & p \mid n \end{cases}$$

Note that we can bound $|a_n| < p\sigma_{k-1}(n) < pn^{k-1}\zeta(k-1)$. Now, assume that $z = x + iy$ is a zero of $E_{k,p}^\infty$. Then we have

$$\begin{aligned} 0 = |E_{k,p}^\infty(z)| &= |E(z)| \\ &\ge p^k - 1 - \frac{(2\pi)^k}{(k-1)!\zeta(k)} \sum_{n=1}^\infty p\zeta(k-1)n^{k-1}e^{-2\pi yn} \\ p^k - 1 &\le \frac{(2\pi)^k p}{(k-1)!} \frac{\zeta(k-1)}{\zeta(k)} \sum_{n=1}^\infty n^{k-1} e^{-2\pi yn} \end{aligned}$$

Then we can find ε_k such that

$$\sum_{n=1}^\infty n^{k-1}e^{-2\pi yn} < (1+\varepsilon_k)(2\pi y)^{-k}\Gamma(k) = (1+\varepsilon_k)(2\pi y)^{-k}(k-1)!$$

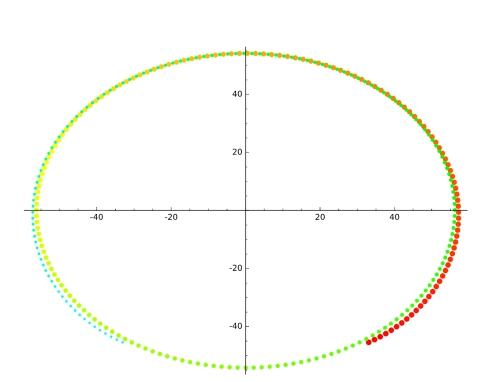
So,

$$\begin{aligned} p^k - 1 &\le \frac{(2\pi)^k p}{(k-1)!} \frac{\zeta(k-1)}{\zeta(k)} (1+\varepsilon_k)(2\pi y)^{-k}(k-1)! \\ &= \frac{(1+\varepsilon_k)p}{y^k} \frac{\zeta(k-1)}{\zeta(k)} \end{aligned}$$

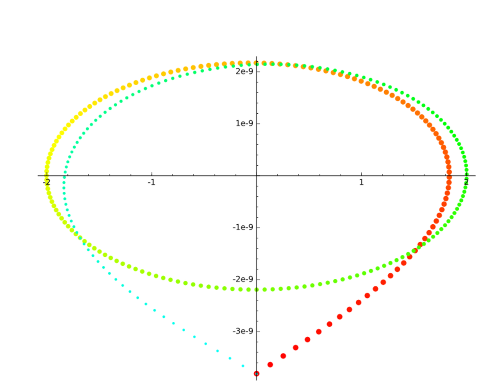
So, this implies that

$$y^k \le \frac{p(1+\varepsilon_k)}{p^k - 1} \frac{\zeta(k-1)}{\zeta(k)}$$

Graphs of $F_{k,p,r}^q(\theta)$



(a) $r = 1.2$



(b) $r = 1 + 10^{-10}$

Figure: Graphs of weight 22 level 7 $F_{k,p,r}^q(\theta)$ on arc 3 for different radii for $\theta \in (\pi/3, 2\pi/3)$

Inspecting the Boundary of $\mathcal{F}_p$

We observe that the only zeros that seem to lie on the boundary of the fundamental domain are on the arc $\{e^{i\theta}/p - 1/p \mid \theta \in (0, 2\pi/3)\}$ and the elliptic points of $\Gamma_0(p)$. Defining, $F_{k,p}^q(\theta) = e^{ik\theta/2}E_{k,p}(\frac{e^{i\theta}}{p} + \frac{q}{p})$, we plot the graphs for zeros along the arcs:

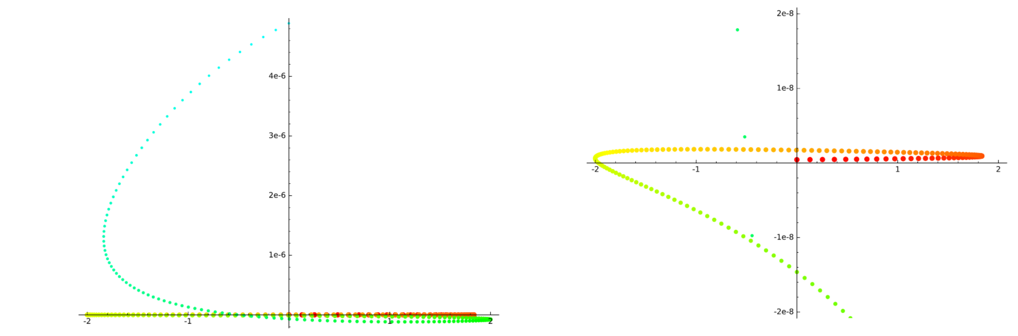


Figure: $F_{22,13}^2(\theta)$, at various zooms

Conclusion

Using a generalization of the Rankin and Swinnerton-Dyer Method, we can locate some zeros of $E_{k,p}^\infty$. By bounding the y component and inspecting $F_{k,p,r}^q$, we can bound the zeros closer to the boundary of $\mathcal{F}_p$. However, we conjecture that the zeros do not actually lie on the boundary of $\mathcal{F}_p$.

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