Problem 1.29b (Learning outcome: Limits of Finite Automata).
Use the pumping lemma to show that the language $A_2 = \{www \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\}$ is not regular.

A_2 recognizes a string containing any combination of a's and b's if they can be broken into three identical substrings. We wish to show that there exists a string $s \in A_2$ that cannot be pumped according to the pumping lemma.

Assume A_2 is regular

Let p be the pumping length for A_2.

Let $s = a^p b a^p b a^p b$. Since s is composed of 3 identical combinations of a, b it is in s. $|s| > p$ so by the pumping lemma can be split as so, $s = xyz$.

The pumping lemma states that $|xy| \leq p$. So xy must be constrained to the a^p of s. This means that pumping s involves repeating some number of a's

However repeating any number of a's in the beginning of s makes it so that the beginning of the string is no longer identical to the other two pieces of the string.

$$xy^i z = a^p a^j b a^p b a^p b \text{ for } 1 \leq j \leq p$$

$$w = a^p b \neq a^p a^j b$$

Therefore we have a contradiction and A_2 cannot be regular.

Problem 1.30 (Learning outcome: Proof debugging).
Describe the error in the following "proof" that $\mathtt{0}^*\mathtt{1}^*$ is not a regular language.

The proof is by contradiction.

Assume that $\mathtt{0}^*\mathtt{1}^*$ is regular.

Let p be the pumping length for $\mathtt{0}^*\mathtt{1}^*$ given by the pumping lemma.

Choose s to be the string $\mathtt{0}^p\mathtt{1}^p$.

We know that s is a member of $\mathtt{0}^*\mathtt{1}^*$, but Example 1.73 shows that s cannot be pumped.

The problem with this proof lies in this statement. There is a faulty assumption here that if s cannot be pumped in another language ($0^n 1^n$ in Example 1.73) then it cannot be pumped in a different language. The string s is indeed a member of $\mathtt{0}^*\mathtt{1}^*$ but the string s can be pumped to result in a string that is still in $\mathtt{0}^*\mathtt{1}^*$ given that the number of $\mathtt{1}$'s and $\mathtt{0}$'s do not need to be equal. Dividing this string into $s = xyz$ and pumping it results in pumping some number of zeros and $\mathtt{0}^*\mathtt{1}^*$ can have any number of zeroes insofar as they all precede the $\mathtt{1}$'s (which they do).

Thus we have a contradiction and $\mathtt{0}^*\mathtt{1}^*$ is not regular.

Problem 1.46a (Learning outcome: Limits of Finite Automata).
Prove that the following language is not regular. You may use the pumping lemma and the closure of the class of regular languages under union, intersection, and complement.
$A = \{\mathtt{0}^n\mathtt{1}^m\mathtt{0}^n \mid m, n \geq 0\}$

Assume A is regular.

Let p be the pumping length for A.

Let $s = 0^p10^p$. Given that $|s| > p$ we can apply the pumping lemma and s can be split into three pieces $s = xyz$.

The pumping lemma states that $|xy| \leq p$, therefore y can only be a series of 0's

Since y can only be a series of 0's any repetitions of y yields a string not in A given that the number of zeroes in the beginning and end of the string have to be equal.

Problem 1.53 (Learning outcome: Limits of Finite Automata).
Let $\Sigma = \{\mathtt{0}, \mathtt{1}, \mathtt{+}, \mathtt{=}\}$ and

$$ADD = \{x\mathtt{=}y\mathtt{+}z \mid x, y, z \text{ are binary integers, and } x \text{ is the sum of } y \text{ and } z.\}$$

Show that ADD is not regular.

Assume ADD is regular.

Let p be the pumping length for ADD.

Let s be $10^{p-1}1 = 10^{p-1}0 + 00^{p-1}1$. Given that $|s| > p$ we can apply the pumping lemma and s can be split into three pieces $s = xyz$.

The pumping lemma states the $|xy| \leq p$, therefore xy can be at least 0 and at most 10^{p-1} because $|y| \geq 1$.

No matter what y is, if we pump s by repeating y then s will fail to be in ADD given that it will no longer be the sum $10^{p-1}0 + 00^{p-1}1$.

Therefore we have a contradiction and ADD cannot be regular.

Problem 1.71 (Learning outcome: Distinguishing Regular and Irregular Languages).

Let $\Sigma = \{0, 1\}$.

a. Let $A = \{0^k u 0^k \mid k \geq 1 \text{ and } u \in \Sigma^*\}$. Show that A is regular.

A language is regular if and only if it can be described by a regular expression.

To prove that this is true we must show that both sides of the if and only if statement imply the other. Let R be the collection of all strings described by our regular expression. We wish to show that:

$$R \subseteq A \wedge A \subseteq R$$

.

Consider the regular expression $E = 0\Sigma^*0$, and let R be the set of all strings described by E.

A string is recognized by A if it can have the beginning and end of it broken into an equal number of 0's

Take $t \in R$. We know that it must have at least one 0 at both its beginning and end. Therefore no matter what Σ^* equals the string can be divided so that there are an equal number of 0's in the beginning and end, namely one pair of 0's. Therefore $R \subseteq A$.

A string is in R if it has a 0 in both the beginning and end.

Take $s \in A$. We know that s will have at least one 0 in both the beginning and end of the string because $k \geq 1$. Therefore s must also belong to R. Thus, $A \subseteq R$.

We have shown that $0\Sigma^*0$ is a regular expression which describes the language. Therefore A is regular.

b. Let $B = \{0^k1u0^k \mid k \geq 1 \text{ and } u \in \Sigma^*\}$. Show that B is not regular.

Assume B is regular. Let p be the pumping length of B.

Let $s = 0^p10^p$. $s \in B$ given that u can be $\varepsilon \in \Sigma *$. Since $s \in B$ and $|s| > p$ it can be divided into three parts, $s = xyz$.

The pumping lemma states that $|xy| \leq p$ therefore y must be series of 0's

If we pump s by repeating y then there will no longer be an equal number of 0's in both the beginning and end of the string. therefore xy^iz for $i > 0$ is not in B

Thus we have a contradiction and B cannot be regular.