Computing a stiffness matrix

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1 Introduction

The computation of a stiffness matrix is a combination of computing the derivatives of an indirectly defined function, a geometric map and a numerical integration. Is the core of any finite element program and what makes it a very general tool.

2 Steps in computing a stiffness matrix

Given a set of shape functions ϕ_i which belong to the appropriate function space (generally $H^1(\Omega)$, the element (i,j) of an element stiffness matrix is computed as

$$K_{ij}^e = \int_{\Omega_e} \nabla_x \phi_i(x) \cdot \nabla_x \phi_j(x)\ dx$$

Using the known map $x = x(\xi)$, this integral is transformed in an integral over the master element

$$K_{ij}^e = \int_{\hat{\Omega}} \nabla_x \phi_i(x(\xi)) \cdot \nabla_x \phi_j(x(\xi)) |\frac{dx}{d\xi}| d\xi$$

The integral over the master element domain is then approximated by numerical integration using points and weights (ξ_p, w_p)

$$K_{ij}^e = \Sigma_p \nabla_x \phi_i(x(\xi_p)) \cdot \nabla_x \phi_j(x(\xi_p)) || \frac{dx}{d\xi}|_p w_p$$

3 Integrating over a mapped element

In finite element computations, we need to integrate over a mapped domain Ω_e where

$$x \in \Omega_e \Longleftrightarrow x = x_e(\xi)\ \xi \in \hat{\Omega}$$

where $x_e(\xi)$ is the function that maps the master element to the actual element. In most cases

$$x_e(\xi) = \Sigma_i x_i^e \mathcal{N}_i(\xi)$$

The integral of a function $f(x)$ over the domain Ω_e

$$\int_{\Omega_e} f(x_e(\xi))dx = \int_{\hat{\Omega}} f(x_e(\xi)) \left| \frac{dx_e}{d\xi} \right| d\xi$$

where $|\frac{dx_e}{d\xi}|$ is the determinant of the gradient of x_e

The integral over $\hat{\Omega}$ can then be approximated by a numerical integration.

$$\int_{\hat{\Omega}} f(x_e(\xi)) \left| \frac{dx_e}{d\xi} \right| d\xi = \Sigma_{i_p} f(x_e(\xi_p)) \left| \frac{dx_e}{d\xi} \right|_{\xi_p} w_p$$

3.1 Exercise

Taking the one dimensional function

$$\mathcal{N}_0(\xi) = \frac{1-\xi}{2}$$
$$\mathcal{N}_1(\xi) = \frac{1+\xi}{2}$$

Define functions $r(\xi)$ and $\theta(\xi)$ such that

$$r(\xi) = \mathcal{N}_0(\xi) + 5\mathcal{N}_1(\xi)$$
$$\theta(\eta) = \frac{\pi}{2}\mathcal{N}_1(\eta)$$

and the map

$$x(\xi,\eta) = r(\xi)\cos(\theta(\eta))$$
$$y(\xi,\eta) = r(\xi)\sin(\theta(\eta))$$

then

$$\begin{aligned} \begin{bmatrix} \nabla x \\ \nabla y \end{bmatrix} &= \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} \\ &= \begin{bmatrix} r'(\xi)\cos(\theta(\eta)) & -r(\xi)\sin(\theta(\eta))\theta'(\eta) \\ r'(\xi)\sin(\theta(\eta)) & r(\xi)\cos(\theta(\eta))\theta'(\eta) \end{bmatrix} \end{aligned}$$

$$\int_{\Omega} f(x,y)|\frac{\partial(x,y)}{\partial(\xi,\eta)}|d\xi d\eta = \Sigma_p|\frac{\partial(x,y)}{\partial(\xi,\eta)}|_p w_p$$

Use this Jacobian to integrate the area of Ω numerically. The correct value $f(x,y) \equiv 1$ is

$$A(\Omega) = 6\pi$$

4 Computing derivatives $\nabla_x \phi_i(x(\xi))$

By definition or construction

$$\phi_i(x(\xi)) = \hat{\phi}_i(\xi)$$

and by the chain rule

$$\frac{d\phi_i(x)}{d\xi} = \frac{d\phi_i(x)}{dx}\frac{dx}{d\xi}$$

which leads to

$$\nabla_x \phi_i(x) = (\frac{dx}{d\xi})^{-1} \nabla_\xi \hat{\phi}_i(\xi)$$

4.1 Extending the computation of derivative in higher dimensions

The gradient of a function $\phi(x, y)$ is defined as

$$\nabla \phi(x, y) = \left[\begin{array}{c} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{array} \right]$$

Analogously, the gradient of the mapping function $(x(\xi, \eta), y(\xi, \eta))$ is computed as

$$\begin{aligned} \nabla_{\xi\eta} \left[\begin{array}{c} x(\xi, \eta) \\ y(\xi, \eta) \end{array} \right] &= \left[\begin{array}{cc} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{array} \right] \\ &= J(\xi, \eta) \end{aligned}$$

Note that, in finite elements $x(\xi, \eta) = \Sigma_i x_i \mathcal{N}_i(\xi, \eta)$, the gradient of x is easy to compute.

By the chain rule (extended to 2 dimensions)

$$\nabla_{\xi\eta} \phi(x(\xi, \eta), y(\xi, \eta)) \cdot (\delta\xi, \delta\eta) = \nabla_{xy} \phi(x, y) \cdot \nabla_{\xi\eta} (x(\xi, \eta), y(\xi, \eta))(\delta\xi, \delta\eta)$$

But, by the definition of $\phi(x, y)$:

$$\phi(x(\xi, \eta), y(\xi, \eta)) = \hat{\phi}(\xi, \eta)$$

We deduce

$$\nabla_{\xi\eta} \phi(x(\xi, \eta), y(\xi, \eta)) = \nabla_{\xi\eta} \hat{\phi}(\xi, \eta)$$

such that

$$\nabla_{\xi\eta} \hat{\phi}(\xi, \eta) \cdot (\delta\xi, \delta\eta) = \nabla_{xy} \phi(x, y) \cdot \left[\begin{array}{cc} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{array} \right] \left[\begin{array}{c} \delta\xi \\ \delta\eta \end{array} \right]$$

As this relation holds for any $(\delta\xi, \delta\eta)$, we choose an increment such that

$$\begin{bmatrix} \delta\xi \\ \delta\eta \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}^{-1} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$$

then

$$\nabla_{\xi\eta}\hat{\phi}(\xi,\eta) \cdot \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}^{-1} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \nabla_{xy}\phi(x,y) \cdot (\delta x, \delta y)$$

$$(a \cdot Mb) = (M^T a \cdot b)$$

$$\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}^{-T} \begin{bmatrix} \frac{\partial \hat{\phi}}{\partial \xi} \\ \frac{\partial \hat{\phi}}{\partial \eta} \end{bmatrix} \cdot \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \nabla_{xy}\phi(x,y) \cdot (\delta x, \delta y)$$

or

$$\begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}^{-T} \begin{bmatrix} \frac{\partial \hat{\phi}}{\partial \xi} \\ \frac{\partial \hat{\phi}}{\partial \eta} \end{bmatrix}$$