Control of inverted pendulum-wheeled cart systems using nonlinear observers

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Antonio Moran
Pontifical Catholic University of Peru

Control of Inverted Pendulum - Wheeled Cart Systems Using Nonlinear Observers

N. Murata, H. Matsuno, A. Moran, and M. Hayase

Tokyo University of Agriculture and Technology
Koganei-shi, Naka-machi, Tokyo 184, JAPAN

Abstract: This paper presents the dynamical modeling and nonlinear controller design of inverted pendulum-wheeled cart systems. The cntroller is a nonlinear observer-based compensator with linear feedback gains. Using a linearized model, the linear feedback gains were determined according to H_∞ and H_2 control theories. A Luenberger-like nonlinear observer was designed to estimate the unmeasured variables required for control. The performance of the proposed controller was experimentally verified. It was found that the nonlinear controller is able to stabilize the pendulum in a short period of time. Also the controller with H_∞ gains requires shorter range of variations of pendulum angle and cart displacement than the controller with H_2 gains.
Keywords: Inverted pendulum, Nonlinear observers, H_∞ control

1. Introduction

Control of the inverted pendulum system is a classical problem in control theory. Although the system is easy to be described it is not easy to be controlled given its inherent unstability and nonlinear characteristics. It is not easy to autonomously move the pendulum to its vertical position and to keep it from arbitrary initial positions. The difficulty and complexity of the control problem increases if a wheeled cart is used as moving base of the pendulum: slip and skid of the cart wheels are not easy to estimate and the actual traction (friction) forces applied on the cart, which nonlinearly depend on the slip ratio of the traction wheels, can not be exactly modeled.

The analysis and control of inverted pendulum - wheeled cart systems are not only interesting from the viewpoint of control theory but also because it allows the analysis of other problems related to robot balancing and walking, and traction control systems.

This study analyses experimentally and by computer simulation the modeling and control of inverted pendulum - wheeled cart systems. A novel nonlinear controller composed of linear feedback gains and nonlinear observer is proposed. The effectiveness of the proposed controller has been experimentally verified using an experimental pendulum-cart model equipped with a DC motor.

2. Dynamical Modeling

2.1 Motion equations

Figure 1 shows the structure of the inverted pendulum and wheeled cart as well as all internal and external forces and moments acting on the pendulum and cart. Assuming the mass of the pendulum concentrates in its upper extreme and neglecting skid and/or slip of the traction wheels, the equations describing the motion of the pendulum and cart are:

$$\ddot{\phi} = \frac{-T/r\cos\phi + \bar{M}g\sin\phi - ml\dot{\phi}^2\sin\phi}{L(\bar{M} - m\cos^2\phi)} \quad (1)$$

$$\ddot{x} = \frac{T/r - mg\sin\phi\cos\phi + ml\dot{\phi}^2\sin\phi}{\bar{M} - m\cos^2\phi} \quad (2)$$

ϕ represents the angle between the pendulum and a vertical reference, x is the longitudinal displacement of the cart, T is the torque applied by a DC motor on the traction wheels, m and L are the lumped mass and length of the pendulum. $\bar{M}$ is an equivalent mass $\bar{M}$ defined as

$$\bar{M} = m + M + I_f/r_f^2 + (I_m k_g^2 + I_r)/r_r^2 \quad (3)$$

where M is the mass of the cart body including traction system and wheels, I_f and I_r represents the inertia of front and rear wheels with radius r_f and r_r, respectively. I_m represents the inertia of the electric motor and k_g represents the ratio of the gear box connecting the DC motor and rear wheels axis.

Given that friction forces acting on the cart are bounded, the maximum acceleration the cart can develop is:

$$\ddot{x}_{max} = \mu g \quad (4)$$

where μ is the friction coefficient and g is the acceleration of gravity. By the reason of bounded acceleration $\ddot{x}_{max}$, the range of stabilizable initial positions of the pendulum is also bounded. It is easy to show that the pendulum is stabilizable if

$$|\phi| \leq \arctan(\mu) \quad (5)$$

2.2. Linear Model

Linearizing Eqs.(1) and (2) around the equilibrium point $\phi \approx 0$, and considering that DC motor torque T and current i are proportional

$$T = k_t i \quad (6)$$

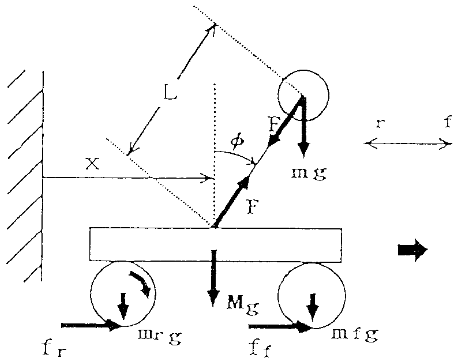

Figure 1: Inverted pendulum - wheeled cart system.

then, the approximate linear state-space equation of the pendulum-cart system can be written as:

$$\dot{\mathrm{x}} = -\mathrm{A}\,\mathrm{x} + \mathrm{B}_1 u + \mathrm{B}_2 w \tag{7}$$

where the state vector is:

$$\mathrm{x} = [\phi\,,\ \dot{\phi}\,,\ x\,,\ \dot{x}]^T \tag{8}$$

$u = i$, and w is a disturbance signal representing the efffect of the neglected slip and/or spin of the traction wheel. Matrices A, B_1 and B_2 are:

$$\mathrm{A} = -\begin{bmatrix} 0 & 1 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ a_2 & 0 & 0 & 0 \end{bmatrix} \quad \mathrm{B}_1 = \begin{bmatrix} 0 \\ b_1 \\ 0 \\ b_2 \end{bmatrix} \quad \mathrm{B}_2 = \begin{bmatrix} 0 \\ w_1 \\ 0 \\ w_2 \end{bmatrix} \tag{9}$$

here

$$\begin{array}{ll} a_1 = \bar{M}g/L(\bar{M} - m) & a_2 = -mg/(\bar{M} - m) \\ b_1 = -k_t k_g / r_r L(\bar{M} - m) & b_2 = k_t k_g / r_r(\bar{M} - m) \\ w_1 = 1/L & w_2 = 1 \end{array} \tag{10}$$

3. Experimental Set-Up

The structure of the experimental pendulum-cart system is shown in Fig.2. The cart is composed of two front and two rear wheels connected to a DC motor. By controlling the current intensity of the DC motor, the traction torque applied to the rear wheels is controlled so that the cart develops the required longitudinal accelerations $\ddot{x}$ to move the pendulum to its vertical position. One frictionless potentiometer (PM) is used to measure the vertical angle ϕ of the pendulum and a rotary encoder (RE) is used to measure the angle of rotation θ of the front wheels. The signals from these two sensors are input to a microcomputer (CPU) through an I/O board and a A/D converter. Using the algorithm of the nonlinear controller, the microcomputer calculates a control voltage which is applied to a power amplifier through a D/A converter. The power amplifier provides the desired current i to control the torque T of the DC motor.

The sampling time of control is chosen to be 10 msec which is long enough to compute the required control signal and fast enough to avoid the undesired effects of sampling.

4. Design of Nonlinear Controller

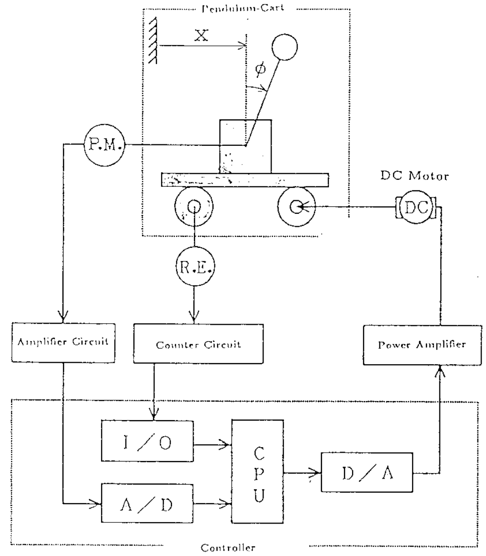

Figure 2: Experimental pendulum-cart model and control system

4.1 Controller Structure

The controller of the pendulum cart system has been designed having the structure of observer based feedback compensators. In this way, the control signal u is calculated from the equation

$$u = -\mathrm{K}\,\hat{\mathrm{x}} \tag{11}$$

$\mathrm{K} = [k_1,\ k_2,\ k_3,\ k_4]$ is the feedback gain vector determined using the linearized model of Eq.(7). $\hat{\mathrm{x}} = [\hat{\phi}\ \ \hat{\dot{\phi}}\ \ \hat{x}\ \ \hat{\dot{x}}]^T$ is the state vector estimated by a nonlinear observer. The process to design the nonlinear observer will be presented in Section 5.

4.2 Linear Feedback Gains

Before describing the process to calculate the feedback gain K, the performance variables of the control problem will be presented.

Performance Variables. Since the control objective is to move the pendulum to its vertical position ($\phi \approx 0$, $\dot{\phi} \approx 0$) with bounded displacements of the cart (x small and $\dot{x} \approx 0$) and requiring practically realizable control signals (u small), the performance vector z is:

$$\mathrm{z} = \mathrm{C}_2\,\mathrm{x} + \mathrm{D}_{21}\,u \tag{12}$$

where

$$\mathrm{C}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_3 & 0 \\ 0 & 0 & 0 & \rho_4 \end{bmatrix} \quad \mathrm{D}_{21} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{13}$$

ρ_1, ρ_2, ρ_3 and ρ_4 are weighting coefficients determined by trial and error so that acceptable performance is achieved.

Feedback Gain. The feedback gain vector K has been determined using linear H_2 and H_∞ control theories. H_∞ controllers are designed to minimize the H_∞ norm of the transfer function T_{zw} from disturbances w to performance variables $\mathbf{z}$. In this way, H_∞ controllers are designed to satisfy the following inequality:

$$||\gamma \mathrm{T}_{zw}||_\infty < 1 \tag{14}$$

where γ is a given scalar.
The feedback gain K is calculated from the equation

$$\mathrm{K} = \mathrm{B}_1^T \mathrm{P} \tag{15}$$

where P is the positive solution of the following algebraic Riccati equation[1]:

$$\mathrm{A}^T\mathrm{P} + \mathrm{PA} + \mathrm{P}(\mathrm{B}_1\mathrm{B}_1^T - \gamma^2\mathrm{B}_2\mathrm{B}_2^T)\mathrm{P} - \mathrm{C}_2^T\mathrm{C}_2 = 0 \tag{16}$$

Optimal H_∞ feedback gain controllers are designed for the maximum possible value of γ, optimal H_2 controllers are designed considering $\gamma = 0$ which can be interpreted as the fact that H_2 controllers are designed without imposing any constraint on the H_∞ norm of the transfer function T_{zw}.

5. Design of Nonlinear Observer

5.1 Nonlinear State Equation

Considering the state vector x of Eq.(8), Eqs.(1) to (5) may be combined to formulate the nonlinear state-space equation of the pendulum-cart system:

$$\dot{\mathrm{x}} = \mathrm{f}(\mathrm{x}) + \mathrm{g}u \tag{17}$$

5.2 Measured-Variables

Considering that one angular potentiometer measuring ϕ and a rotary encodar measuring (approximately) x are available, the measurement equation is

$$\mathrm{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathrm{x} \tag{18}$$

5.3 Observability

In order to design a nonlinear observer for the the system of Eqs.(17) and (18), first the observability conditions should be verified. Several criterions have been formulated to determine the observability of nonlinear systems. In the following, the simple criterion of Ref.2 will be considered.
Consider the single-input/single-output nonlinear system:

$$\begin{aligned} \dot{\mathrm{x}} &= \mathrm{f}(\mathrm{x}) + \mathrm{g}\,u \qquad \mathrm{x}(0) = \mathrm{x}_0 \\ y &= h(\mathrm{x}) \end{aligned} \tag{19}$$

with state $\mathrm{x} \in R^n$, h a C^∞ real-valued function and $\mathrm{f}, \mathrm{g} \in C^\infty$ real-valued vector fields. For the following it is necessary to recall the definition of the Lie derivative $L_f h$ of the function $h(\mathrm{x})$ with respect to the vector field f:

$$L_f h(\mathrm{x}) = \sum_{i=1}^{n} \frac{\partial h(\mathrm{x})}{\partial x_i} f_i(\mathrm{x}) \tag{20}$$

Moreover the symbols $L_f^k h(\mathrm{x})$ means the k-times repeated iteration of $L_f h(\mathrm{x})$

$$\begin{aligned} L_f^k h(\mathrm{x}) &= L_f(L_f^{k-1} h(\mathrm{x}))(\mathrm{x}) \\ L_f^0 h(\mathrm{x}) &= h(\mathrm{x}) \end{aligned} \tag{21}$$

The so-called observability matrix $\mathrm{Q}(\mathrm{x})$ is defined as follows:

$$\mathrm{Q}(\mathrm{x}) = \frac{d}{d\mathrm{x}} \begin{bmatrix} h(\mathrm{x}) \\ L_f h(\mathrm{x}) \\ \vdots \\ L_f^{n-1}(\mathrm{x}) \end{bmatrix} \tag{22}$$

The system will be observable if:

(1) the observability matrix $\mathrm{Q}(\mathrm{x})$ has full rank,

(2) the triple $[\mathrm{f}(\mathrm{x}), \mathrm{g}(\mathrm{x}), h(\mathrm{x})]$ has relative degree n.

5.4 Luenberger-like Nonlinear Observer

If the observability conditions are satisfied then it is possible to design a nonlinear observer for the system of Eq.(17). Several methods have been proposed to design observer for nonlinear systems. In this study, it has been designed a Luenberger-like nonlinear observer which is computationally simple and easily implementable[2].
The observer is given by the equation:

$$\dot{\hat{\mathrm{x}}} = \mathrm{f}(\hat{\mathrm{x}}) + \mathrm{g}\,u + [\mathrm{Q}(\hat{\mathrm{x}})]^{-1}\,\mathrm{L}\,[y - h(\hat{\mathrm{x}})] \tag{23}$$

where L is a finite gain vector. For the practical computation of L it is enough to choose $Real(\lambda_i) < -\gamma$, $i = 1, 2, \ldots, n$, for a suitable λ, and to compute the coefficients of the polynomial

$$(\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n) = \lambda^n + L_1\lambda^{n-1} + L_2\lambda^{n-2} + \ldots + L_n \tag{24}$$

and

$$\mathrm{L} = [L_1\ L_2 \ldots L_n]^T \tag{25}$$

For the exponential convergence of the estimation error to zero, it is necessary that $\|\ \hat{\mathrm{x}}(0) - \mathrm{x}(0)\ \| \leq \eta$ for a suitable $\eta > 0$.

6. Experimental Results

After determining the feedback gain vector K and designing the nonlinear observer, the performance of the proposed nonlinear controller was analyzed experimentally and by computer simulation. Also the performance and characteristics of the controller with H_∞ feedback gains and H_2 feedback gains were compared.

Figure 3 shows the time history of the pendulum angle ϕ and cart displacement x corresponding to H_∞ feeback gains determined considering a set of weighting coefficients with relatively low values of the weights ρ_3 and ρ_4 affecting the cart displacement and velocity x and $\dot{x}$, respectively. In Fig. 3(a) it is noted that for the initial position $\phi = 10^o$ the pendulum is driven to its vertical position and vibrates around it: ϕ is in the range $[-3^o,\ 3^o]$ for the most of time. In Fig. (b) it is noted that the cart displacement x required to stabilize the pendulum are in the range $[-0.3m,\ 0.3m]$ which can be considered relatively big.

Figure 4 shows the time history of the pendulum angle ϕ and cart displacements x corresponding to H_∞ feeback gains determined considering a second set of weighting coefficients with relatively high values of the weights ρ_3 and ρ_4. In Fig. 4(a) it is noted that for the initial position $\phi = 10^o$ the pendulum is driven to its vertical position with significant overshoot (around 4^o), but after that the angle ϕ remains inside the range $[-0.5^o\ 0.5^o]$ for the most of time. In Fig. 4(b) it is noted that the maximum cart displacement x required to stabilize the pendulum is less than $0.15m$. Also it is noted that after the pendulum has been driven to its vertical position, the cart almost does not move ($x \approx 0$) for the most of time. Since the results obtained with the second set of weigthing coefficients (Fig.4) are better than those obtained with the first set of weights (Fig.3), it is suggested that when designing

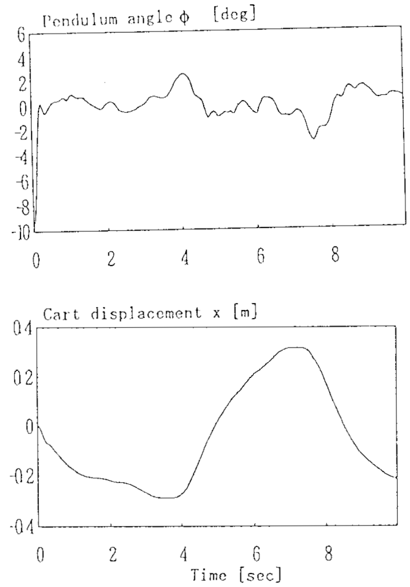

Figure 3: Time history of (a) pendulum angle ϕ and (b) cart displacement x . H_∞ gains (low ρ_3 , ρ_4)

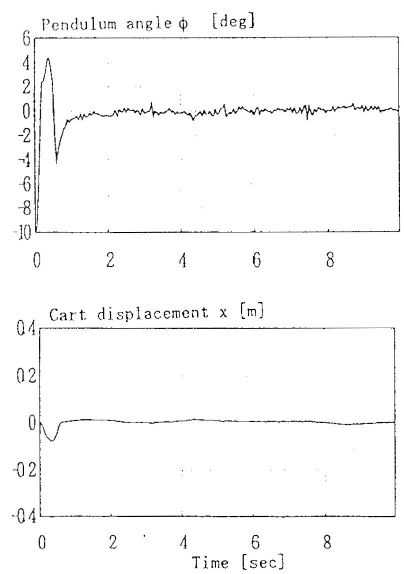

Figure 4: Time history of (a) pendulum angle ϕ and (b) cart displacement x . H_∞ gains (high ρ_3 , ρ_4)

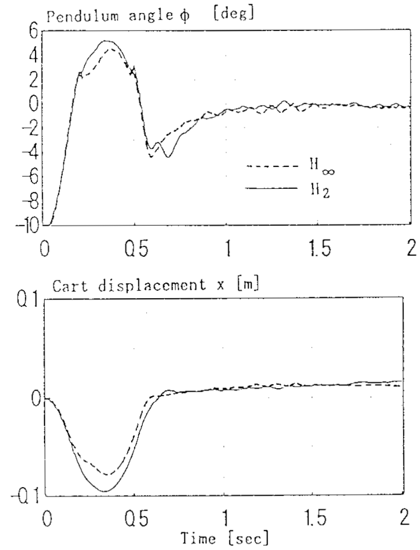

Figure 5: Time history of (a) pendulum angle ϕ and (b) cart displacement x . H_∞ and H_2 gains

linear controllers to stabilize the pendulum, not only ϕ and $\dot{\phi}$ should have priority but also x and $\dot{x}$ should be properly weighted.

Figure 5 compares the time history of pendulum angle ϕ and cart displacement x corresponding to controllers with H_∞ and H_2 feedback gains computed with the second set of weighting functions. It can be noted that for the controller with H_∞ gains, the range of variations of ϕ and x are shorter than the range of variations corresponding to the controller with H_2 feedback gains.

7. Conclusions

This paper has presented the dynamical modeling and design of nonlinear controller for stabilizing pendulum-wheeled cart systems. The controller is a observer-based compensator with linear feedback gains. Using a linearized model, the linear feedback gains were determined according to H_∞ and H_2 control theories. Also a Luenberger-like nonlinear observer was designed to estimate the unmeasured variables required for control. The performance of the proposed controller was experimentally verified. It was found that the controller with H_∞ gains requires shorter range of variations of angle ϕ and displacement x than the controller with H_2 gains.

8. References

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