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A general regression model for lifetime evaluation and prediction

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Abstract. A particular form of a probabilistic model for materials under fatigue which embodies Weibull features and the size effect in a weakest-link framework is derived. The parametric and functional form of the model arises from a certain set of assumptions, as the weakest-link principle, stability, limit behavior, limited range and compatibility, which can be justified as being consistent with experimental features of fatigue (mainly of highly drawn steel wires) and the mathematics of extreme value theory. These assumptions, which are discussed, can be used to rule out other possible forms as being fundamentally inconsistent. The authors also discuss estimation procedures for the parameters based on two steps: a non-linear regression step, in which the threshold lifetime and stress range values are determined, and a second step in which the Weibull parameters are estimated by pooling data from different stress levels and using a probability-weighted moments approach or the Castillo-Hadi estimators. Next, the damage accumulation problem is dealt with and two different proposals for the damage index are given. The model, originally developed to handle a fixed load parameter (such as the stress range in cyclic fatigue), is extended to handle a block load sequence involving many load levels, as well as random load programs. Some formulas for calculating the accumulated damage index for constant, block and random loading are given. Finally, the model and methods are applied to a particular fatigue program on concrete to illustrate all concepts and the practical use of formulas.

Key words: Accumulated damage, block loading, dimensional analysis, fatigue, *S–N* field, random loading, Weibull model.

1. Introduction

In the evaluation or prediction of lifetime results the search for a general statistical model, including the consideration of the size effect, and a successful estimation of the corresponding parameters represents one of the most difficult and attracting tasks, which remain, so far, not satisfactorily solved.

Depending on the specific research program undertaken and the S-N field region covered by the experimentation, different intuitive models (parabolic, hyperbolic, linear, etc.) have been proposed in the literature (see references in Castillo et al. 1985) to fit experimental data. At most, the use of these models, together with a good knowledge of physical or metallurgical aspects of the lifetime phenomenon, could represent an adequate approach for tackling the particular problem under consideration, and for a limited judgment or interpretation on the experimental results obtained, but it would neither afford an extrapolation of results outside the parameter range considered, nor contribute to provide an overview about the general treatment of fatigue evaluation and prediction, indispensable to progress in a better understanding of this complex phenomenon or to develop appropriate strategies and adequate test planning, as an alternative to a simple data fitting. Additionally, the possible shortage of results represents a constant feature in the fatigue experimentation due to economical or time reasons and the physical impossibility of testing specimens over a certain size, and must be taken into consideration.

Thus, critical points in this modelling are: the general capability of the model to be applied on all possible cases of lifetime problems (fatigue, creep, corrosion fatigue, dielectric stress breakdown, dock damage, road surface damage, etc.) and on different types of materials (metallic, cementitious, polymers, etc.), irrespective of the failure mechanisms (dominant crack or generalized microcracks growth), the estimation of the quantiles and the determination of the confidence intervals from data fitting and, finally, the extrapolation to out-of range S-N field regions or sizes (size-effect). All these questions are of practical paramount significance because they exert a strong influence on the quality control by the acceptance tests as well as in the structural or mechanical design.

In this paper, a short description of a general model based on consistent physical and statistical considerations is presented and its applicability to practical cases is discussed.

Since the Weibull model will be systematically used in the following sections and is selected as the most adequate for dealing with fatigue problems, it is introduced in Section 2, where some of its properties related to the lifetime problem are discussed. In Section 3 we identify the main variables involved in the problem of fatigue, and use the well known II-theorem of dimensional analysis to obtain a smaller equivalent set of non-dimensional variables. This allows reducing the complexity of our models and working with adimensional variables and parameters, a convenient way of avoiding inconsistencies and problems. Section 4 justifies the proposed model based on physical and engineering considerations. Section 5 discusses some weaknesses of the proposed model. In Sections 6 and 7 we give one estimation method for obtaining the threshold values of lifetime and stress level, and we describe the probability weighted (PWM) and the Castillo-Hadi (1994) methods to estimate the location, scale and shape parameters of the Weibull distribution. Section 8 deals with the problem of damage accumulation, and in Section 9 one example of application is given. Finally, we conclude with a section devoted to conclusions.

2. The Weibull model

In this section we introduce the Weibull model with some of its properties, applicable to lifetime problems, because it is systematically used in the following sections.

The cumulative distribution function (cdf) of the three-parameter Weibull family is given by:

$$\begin{aligned}&F(x;\lambda,\delta,\beta)=1-\exp\left[-\left(\frac{x-\lambda}{\delta}\right)^{\beta}\right],\\&x\geq\lambda;\ -\infty<\lambda<\infty,\ \delta>0,\ \beta>0,\end{aligned}\tag{1}$$

where, $F(x;\lambda,\delta,\beta)$ represents the probability of the event $X \leq x$, and δ, λ and β are the scale, the location (minimum possible value of the random variable X), and the shape parameter, respectively. When X has the cumulative distribution function in (1) we write $X \sim W(\lambda,\delta,\beta)$.

Its mean and variance are:

$$\begin{aligned}\mu &= \lambda+\delta\Gamma[1+1/\beta],\\ \sigma^2 &= \delta^2\left[\Gamma[1+2/\beta]-\Gamma^2[1+1/\beta]\right],\end{aligned}\tag{2}$$

and its percentiles:

$$x_p = \lambda + \delta[-\log(1-p)]^{1/\beta}, \quad 0 \le p \le 1. \tag{3}$$

Two important properties of the Weibull family are:

(1) It is stable with respect to location and scale transformations. More precisely:

$$X \sim W(\lambda, \delta, \beta) \Leftrightarrow \frac{X-a}{b} \sim W\left(\frac{\lambda - a}{b}, \frac{\delta}{b}, \beta\right), \tag{4}$$

where the new parameters are given in terms of the old parameters and the transformation constants a and b. This means that if a Weibull random variable is transformed by location and scale transformations, the resulting variable is also a Weibull random variable.

(2) It is stable with respect to minimum operations. In other words, if the random variables X_i; $i = 1, 2, \ldots, m$ are independent and identically distributed, then

$$X_i \sim W(\lambda, \delta, \beta) \Rightarrow \min(X_1, X_2, \ldots, X_m) \sim W(\lambda, \delta m^{-1/\beta}, \beta) \tag{5}$$

In other words, if a set of independent and identically distributed random variables are Weibull, its minimum is also a Weibull random variable.

Since the cdf $F_{\min}(x)$ of the minimum of a set of independent and identically distributed random variables $X_1, X_2, \ldots, X_m$, with common cdf $F(x)$ is:

$$F_{\min}(x) = 1 - [1 - F(x)]^m, \tag{6}$$

it follows that:

$$\begin{aligned} F_{\min}(x) &= 1 - \left\{1 - \left(1 - \exp\left[-\left(\frac{x-\lambda}{\delta}\right)^{\beta}\right]\right)\right\}^m \\ &= 1 - \exp\left[-\left(\frac{x-\lambda}{\delta m^{-1/\beta}}\right)^{\beta}\right] \end{aligned} \tag{7}$$

which proves (5).

3. Dimensional analysis

Before selecting a model to solve an engineering problem, the relevant variables involved have to be identified. By previous experience, accumulated in the study of the fatigue phenomenon, we know that the 5 variables initially involved in the fatigue problem are: P, N, N_0, $\Delta\sigma$ and $\Delta\sigma_0$, where P is the probability of fatigue failure of a piece when subject to N cycles at a stress range $\Delta\sigma_0$, N_0 is the threshold value for N, the minimum lifetime for any $\Delta\sigma$, and $\Delta\sigma_0$ is the endurance limit, below which fatigue failure does not occur.

However, this initial number of variables can be reduced. The II-Theorem allows representing any existing relation among the initial variables in terms of another smaller set of non-dimensional variables. In fact, a dimensional analysis of the initial set of 5 variables leads to the matrix of dimensions, with rank 2, given in Table 1, where M, L and T refer to mass, length and time, respectively. Thus, the initial set of 5 variables reduces to a set of 3 adimensional variables. It seems convenient to choose N/N_0, $\Delta\sigma/\Delta\sigma_0$ and P as these variables.

Table 1. Dimensional analysis of the initial set of variables involved in the fatigue problem

	N	N_0	$\Delta\sigma$	$\Delta\sigma_0$	P
M	0	0	1	1	0
L	0	0	−1	−1	0
T	1	1	−2	−2	0

This means that any existing relation $r(N, N_0, \Delta\sigma, \Delta\sigma_0, P) = 0$ among the independent 5 variables can be written in terms of only these three non-dimensional variables:

$$r(N, N_0, \Delta\sigma, \Delta\sigma_0, P) = 0 \equiv f\left(\frac{N}{N_0}, \frac{\Delta\sigma}{\Delta\sigma_0}, P\right) = 0. \tag{8}$$

Since we are interested in P, (8) can be written as:

$$P = q\left(\frac{N}{N_0}, \frac{\Delta\sigma}{\Delta\sigma_0}\right), \tag{9}$$

where $q(\,)$ is a function to be determined.

The important result is that only the adimensional quotients N/N_0 and $\Delta\sigma/\Delta\sigma_0$ have influence on the probability of failure P, so that either N/N_0 and $\Delta\sigma/\Delta\sigma_0$, or any monotone functions of them, as $h(N/N_0)$ and $g(\Delta\sigma/\Delta\sigma_0)$ have to be considered. For the sake of notation simplicity, in the following sections we denote:

$$N^* = h(N/N_0); \quad \Delta\sigma^* = g(\Delta\sigma/\Delta\sigma_0). \tag{10}$$

4. Justification of the Weibull *S–N field*

Consider a series of fatigue tests carried out at a constant stress range $\Delta\sigma_i$ (see Figure 1), which yields the following results for the number of cycles to failure on a natural scale:

$$N_{i,1}, N_{i,2}, \ldots, N_{i,k}, \ldots, N_{i,n_i} \tag{11}$$

or taken on a logarithmic scale:

$$\log N_{i,1}, \log N_{i,2}, \ldots, \log N_{i,k}, \ldots, \log N_{i,n_i}. \tag{12}$$

Following the methodology proposed by Castillo, Fernández-Canteli, Esslinger and Thürlimann (1985), the selection of the model for the S–N field is based on the following considerations (see Figure 2):

(1) *Weakest link principle*: This principle establishes that the fatigue lifetime of a longitudinal element is the minimum fatigue life of its constituting pieces. Thus, we are dealing with a minimum model. As shown in Figure 3 the actual longitudinal element of length $L = m\ell$ can be supposedly subdivided in m pieces of length ℓ. Thus, taking into account (6), we have:

$$F_{\min}(x) = F_{m\ell}(x) = 1 - [1 - F_\ell(x)]^m, \tag{13}$$

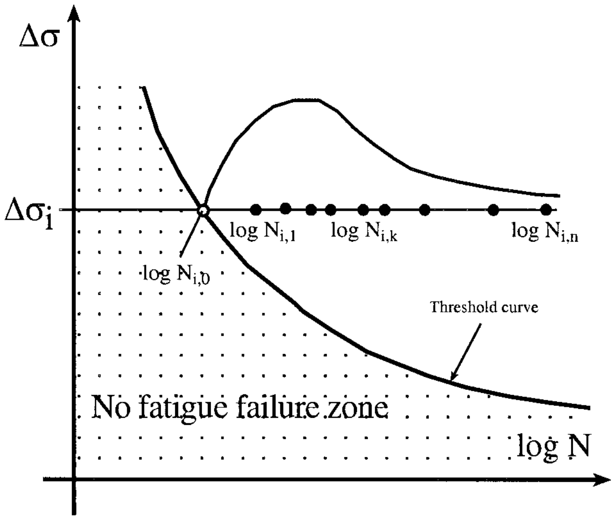

Figure 1. S-N field with fatigue results at stress range $\Delta\sigma_i$.

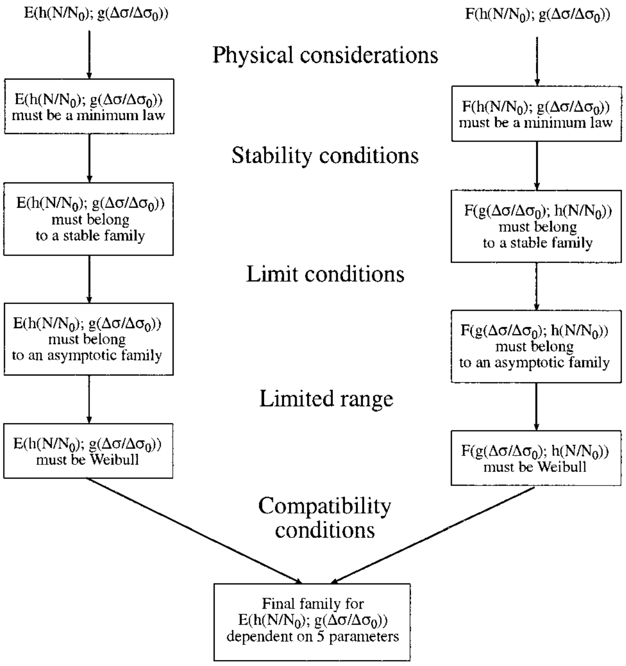

Figure 2. Illustration of the selection procedure for the cumulative distribution function of the lifetime.

where $F_\ell(x)$ is the cumulative distribution function of the fatigue lifetime of an element of length ℓ.

(2) *Stability*: The selected distribution function type must hold (be valid) for different lengths. If a parametric cdf $F(x; \lambda(\ell), \delta(\ell), \beta(\ell))$ is used to represent the cdf for fatigue lifetime of a longitudinal element of length ℓ, then, according to (13), the cdf for an element of length $m\ell$ must be

$$F(x; \lambda(m\ell), \delta(m\ell), \beta(m\ell)) = 1 - [1 - F(x; \lambda(\ell), \delta(\ell), \beta(\ell))]^m. \tag{14}$$

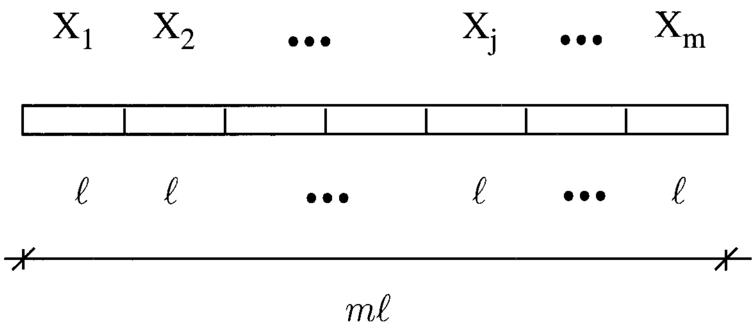

Figure 3. Pieces in which a longitudinal element can be supposedly subdivided.

This is a functional equation, where the unknowns are $F(x; \lambda, \delta, \beta)$, $\lambda(\ell)$, $\delta(\ell)$, and $\beta(\ell)$, that only some families of distributions, as the Weibull, Gumbel and Frechet families, satisfy.

(3) *Limit behavior*: To include the extreme case of the size of the supposed pieces constituting the element going to zero, or the number of pieces going to infinity, it is convenient for the distribution function family to be an asymptotic family.

It is well known that in the case of independence, there are only three limit distributions, namely, Weibull, Gumbel and Frechet (see Castillo (1988)).

(4) *Limited range*: Experience shows that the selected non-dimensional variables N^* and $\Delta\sigma^*$ have a finite lower end, which must coincide with the theoretical end of the selected cdf. This implies that the Weibull distribution is the only one satisfying this requirement. If the variable were unlimited in the left tail, the Gumbel and the Frechet models would still be possible.

Since $N \geq N_0$ and $\Delta\sigma \geq \Delta\sigma_0$, we have for the adimensional variables $N/N_0 \geq 1$ and $\Delta\sigma/\Delta\sigma_0 \geq 1$, and $\log(N/N_0) \geq 0$ and $\log(\Delta\sigma/\Delta\sigma_0) \geq 0$. Thus, selection of $h(x) = x$, or $h(x) = \log x$ leads to a limited variable in the lower tail, and then Weibull is the only adequate family.

(5) *Compatibility*: In the $S-N$ field, the cumulative distribution function of lifetime given stress range, $E(N^*; \Delta\sigma^*)$, should be compatible with the cumulative distribution function of stress range given lifetime, $F(\Delta\sigma^*; N^*)$.

The compatibility condition can be written as the following functional equation:

$$E(N^*; \Delta\sigma^*) = F(\Delta\sigma^*; N^*), \tag{15}$$

which in the case of selecting the Weibull distribution becomes

$$[(\Delta\sigma^* - \lambda(N^*))/\delta(N^*)]^{\beta(N^*)} = [(N^* - \lambda(\Delta\sigma^*))/\delta(\Delta\sigma^*)]^{\beta(\Delta\sigma^*)}, \tag{16}$$

where $\lambda(N^*)$, $\delta(N^*)$, $\beta(N^*)$, $\lambda(\Delta\sigma^*)$, $\delta(\Delta\sigma^*)$ and $\beta(\Delta\sigma^*)$, are the unknown functions to be obtained.

It is interesting to remark that the basic hypotheses of this model can also be established on the basis of microstructural properties of the material (see Bolotin, 1981).

Two general solutions of functional equation (16) are possible (see Castillo and Galambos, 1987):

Model I: The unknown functions are:

$$\lambda(N^*) = \frac{\lambda}{N^* - B}; \qquad \lambda(\Delta\sigma^*) = \frac{\lambda}{\Delta\sigma^* - C};$$

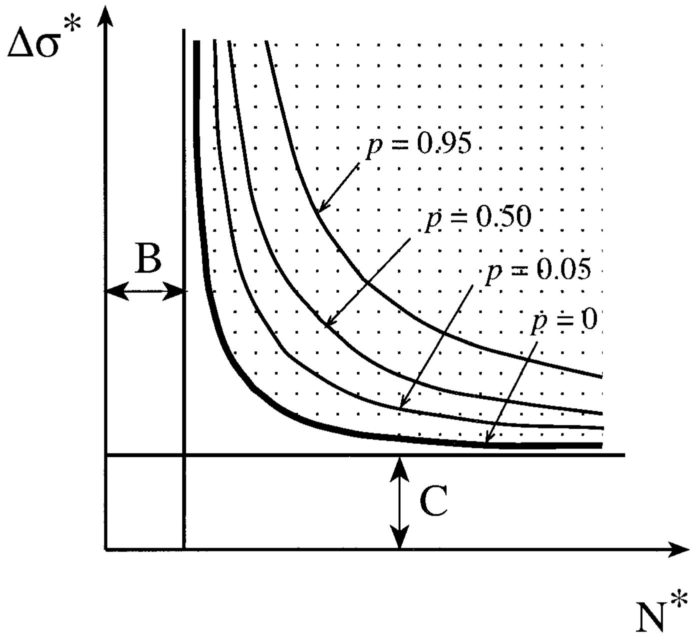

Figure 4. Percentiles curves representing the relationship between lifetime, N^*, and stress range, $\Delta\sigma^*$, in the $S-N$ field for Model I.

$$\begin{aligned}\delta(N^*) &= \frac{\delta}{N^*-B}; \qquad \delta(\Delta\sigma^*) = \frac{\delta}{\Delta\sigma^*-C};\\ \beta(N^*) &= \beta; \qquad\qquad \beta(\Delta\sigma^*) = \beta,\end{aligned} \tag{17}$$

that leads to:

$$\begin{aligned}F[N^*;\Delta\sigma^*] &= E[\Delta\sigma^*;N^*]\\ &= 1-\exp\left\{-\left[\frac{(N^*-B)(\Delta\sigma^*-C)-\lambda}{\delta}\right]^{\beta}\right\}; \quad N^* \geq B+\frac{\lambda}{\Delta\sigma^*-C'},\end{aligned} \tag{18}$$

where B, C, λ, δ and β are the non-dimensional model parameters. Their physical meanings (see Figure 4) are the following:

B: treshold value of lifetime.

C: endurance limit.

λ: defines the position of the corresponding zero-percentile hyperbola.

δ: scale factor.

β: Weibull shape parameter of the whole cdf in the $S-N$ field.

Since for $\Delta\sigma^* \to \infty$ a lower end for $N^* - h(N_0/N_0) = h(1)$ must exist, we have $B = h(1)$. Similarly, since for $N^* \to \infty$ a lower end for $\Delta\sigma^* = g(\Delta\sigma_0/\Delta\sigma_0) = g(1)$ must exist, we have $C = g(1)$.

Thus, the corresponding percentile curves are given by:

$$\frac{N^*\Delta\sigma^*-\lambda}{\delta} = \text{constant} \Leftrightarrow N^*\Delta\sigma^* = \text{constant}. \tag{19}$$

The zero-percentile curve represents the minimum possible number of cycles of fatigue failure for different values of $\Delta\sigma$, and happens to be a hyperbole (thick line in Figure 4). For such a curve, the minimum number of cycles to fatigue failure decreases with increasing $\Delta\sigma$, in agreement with experimental results.

Model II: The unknown functions are:

$$\begin{aligned} &\lambda(N^*) = C; && \lambda(\Delta\sigma^*) = B; \\ &\delta(N^*) = [\delta(N^* - B)^\beta]^{-1/\gamma}; && \delta(\Delta\sigma^*) = [\delta(\Delta\sigma^* - C)^\gamma]^{-1/\beta}; \\ &\beta(N^*) = \gamma; && \beta(\Delta\sigma^*) = \beta; \end{aligned} \tag{20}$$

that leads to:

$$F[N^*, \Delta\sigma^*] = 1 - \exp[-[\delta(N^* - B)^\beta(\Delta\sigma^* - C)^\gamma]]; \quad N^* \geq B. \tag{21}$$

where B, C, γ, δ and β are the non-dimensional model parameters. The physical meaning of these parameters (see Figure 5) is the following:

B: minimum lifetime associated with all values of $\Delta\sigma$.

C: endurance limit.

γ: shape parameter associated with $\Delta\sigma^*$.

δ: scale factor.

β: shape parameter associated with N^*.

The corresponding percentile curves are given by:

$$N^*\Delta\sigma^* = \text{constant}. \tag{22}$$

Contrary to Model I, in this model the zero-percentile curve degenerates to the two asymptotes (thick lines in the Figure 5). This means that the minimum number of cycles to fatigue failure remains constant with increasing $\Delta\sigma$, contradicting the Fracture Mechanics practice, particularly, in what concerns the general shape of the crack growth curve $\mathrm{d}a/\mathrm{d}N - \Delta K$. Consequently, this model has to be rejected, and Model I is adopted as the most convenient.

In addition, following the tradition of using logarithms for the number of cycles to failure and the stress range, we choose $h(x) = g(x) = \log x$, then:

$$N^* = \log(N/N_0); \qquad \Delta\sigma^* = \log(\Delta\sigma/\Delta\sigma_0), \tag{23}$$

and $B = C = h(1) = g(1) = 0$, so that Model (18) becomes:

$$F[N^*; \Delta\sigma^*] = 1 - \exp\left\{-\left[\frac{(N^*\sigma^* - \lambda}{\delta}\right]^\beta\right\}; \quad N^* \geq \frac{\lambda}{\Delta\sigma^*}. \tag{24}$$

which, returning to the original variables can be written as:

$$\begin{aligned} F(\log(N/N_0); \delta, \lambda, \beta) &= 1 - \exp\left[-\left(\frac{[\log(N/N_0)][\log(\Delta\sigma/\Delta\sigma_0)] - \lambda}{\delta}\right)^\beta\right] \\ &= 1 - \exp\left[-\left(\frac{\log(N/N_0) - \lambda/\log(\Delta\sigma/\Delta\sigma_0)}{\delta/\log(\Delta\sigma/\Delta\sigma_0)}\right)^\beta\right] \\ &\qquad \log(N/N_0) \geq \frac{\lambda}{\log(\Delta\sigma/\Delta\sigma_0)}, \end{aligned} \tag{25}$$

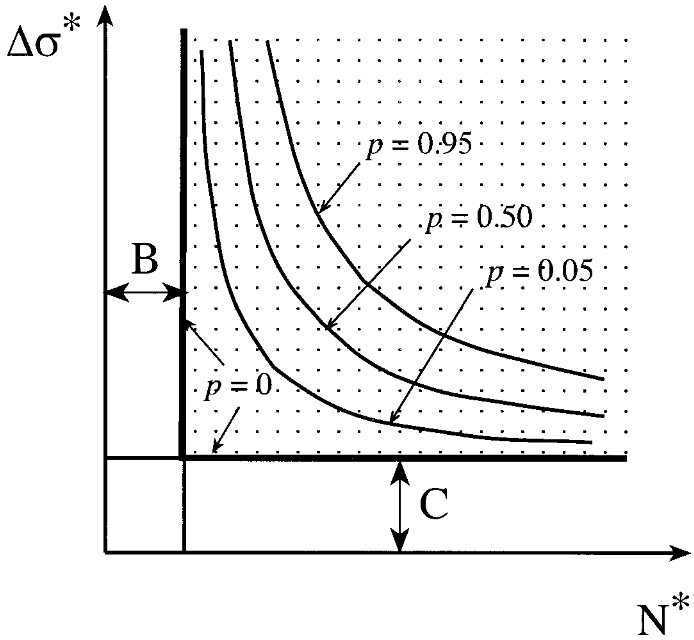

Figure 5. Percentiles curves representing the relationship between lifetime, N^*, and stress range, $\Delta\sigma^*$, in the $S-N$ field for Model II.

from which, the *p*-percentile curves become:

$$\log(\Delta\sigma/\Delta\sigma_0) = \frac{\lambda + \delta(-\log(1-p))^{1/\beta}}{\log(N/N_0)}; \qquad 0 \le p \le 1. \tag{26}$$

The model (18) has been subsequently studied and successfully applied to different cases of lifetime problems, as for instance prestressing wires and strands with different lengths, plain concrete, etc. (see Castillo et al. (1985) or Holmen (1979)). It seems specially suitable for the study of the size effect, particularly for evaluation of results for different lengths and extrapolation to long lengths, provided the experimental program is carried out with specimens long enough to ensure independence or weak dependence. Recently, it has been advantageously applied to cumulative damage assessment by block and random loading.

Expression (25) implies that:

$$[\log(N/N_0)][\log(\Delta\sigma/\Delta\sigma_0)] \sim W(\lambda, \delta, \beta), \tag{27}$$

and

$$\log(N/N_0) \sim W(\lambda(\Delta\sigma), \delta(\Delta\sigma), \beta), \tag{28}$$

where

$$\lambda(\Delta\sigma) = \frac{\lambda}{\log(\Delta\sigma/\Delta\sigma_0)}, \tag{29}$$

$$\delta(\Delta\sigma) = \frac{\delta}{\log(\Delta\sigma/\Delta\sigma_0)}. \tag{30}$$

It is interesting to note that (25) is in adimensional form, and reveals that the probability of failure of a piece subject to a stress range $\Delta\sigma$ during N cycles, depends only on the product

$[\log(N/N_0)][\log(\Delta\sigma/\Delta\sigma_0)]$. Thus, $V = [\log(N/N_0)][\log(\Delta\sigma/\Delta\sigma_0)]$ is useful to compare fatigue strength at different, but constant, stress levels, and can be considered as a normalizing variable.

5. Some weaknesses of the proposed model

Some of the above five properties that led to the proposed model can be questioned. In fact the following are pertinent comments:

(1) The weakest-link assumption, stated in the form of Equation (13)), implies that the fatigue lifes of its different pieces are assumed to be independent. This can hold approximately for long pieces.

(2) Though the limit behavior is a convenient property for the fatigue model, it is not necessary. Furthermore weak convergence in increasing L is not concerned with preserving lower tail behavior in terms of relative error. So, other models, different from the one proposed here, can be obtained, as a consequence of dropping this assumption.

Some researchers, as the referee, could see no a priori reason to rule out model forms satisfying

$$f(t; s) = 1 - \exp\{-LG(t; s)\}, t, s > 0,$$

where G is an appropriate increasing function of time t and stress s, and L is length or volume of material. There are perfectly justifiable models, from the point of view of random defects, micromechanical stress redistribution and catastrophic crack growth that are not Weibull in form and do not lead to Weibull distributions. A discussion of possible forms for G can be found in a classic paper of Coleman (1958).

(3) The limited range assumption implies that a fatigue limit exists. Since there is no universal agreement in the concept of a fatigue limit for steels and much less for other materials such as aluminum, N_0 and $\delta\sigma_0$ can be viewed as scale constants with these ratios greater than or equal to zero (as for instance might occur when using two parameter Weibull distributions with $\lambda = 0$). In these cases, other alternatives are also possible.

In summary, the proposed model is a convenient and practical model that arise as the only model satisfying the five assumptions, but relaxing some of these assumptions, other models are possible.

6. Estimation of the threshold values

The regression curve of $\log(N/N_0)$ on $\log(\Delta\sigma/\Delta\sigma_0)$ is given by the mean value of $\log(N/N_0)$ as a function of $\log(\Delta\sigma/\Delta\sigma_0)$. Since, as indicated in (2), the mean of a Weibull $W(\lambda, \delta, \beta)$ distribution is $\mu = \lambda + \delta\Gamma(1 + 1/\beta)$, from (28) we have:

$$E[\log(N/N_0) \mid \log(\Delta\sigma/\Delta\sigma_0)] = \frac{\lambda + \delta\Gamma(1 + 1/\beta)}{\log(\Delta\sigma/\Delta\sigma_0)}, \tag{31}$$

which is equivalent to:

$$E[\log(N) \mid \log(\Delta\sigma/\Delta\sigma_0)] = \log N_0 + \frac{K}{\log \Delta\sigma - \log \Delta\sigma_0}, \tag{32}$$

where

$$K = \lambda + \delta\Gamma(1 + 1/\beta)$$

is a constant.

The regression equation (32) suggests estimating N_0 and $\Delta\sigma_0$ by minimizing, with respect to N_0, $\Delta\sigma_0$ and K:

$$Q = \sum_{i=1}^{m}\sum_{j=1}^{n_i}\left(\log N_{ij} - \log N_0 - \frac{K}{\log\Delta\sigma_i - \log\Delta\sigma_0}\right)^2, \tag{33}$$

where m is the number of different stress ranges used in the tests, $\Delta\sigma_i$ are the associated stress ranges, n_i is the number of specimens tested at the stress range $\Delta\sigma_i$, and N_{ij} is the number of cycles to failure of the j-th sample tested at stress range $\Delta\sigma_i$.

Once N_0 and $\Delta\sigma_0$ have been estimated, all the data points can be pooled together by calculating the values of

$$v_{ij} = [\log(N_{ij}/N_0)][\log(\Delta\sigma_i/\Delta\sigma_0)],$$

to estimate δ, λ and β, since all they follow a Weibull distribution $W(\lambda, \delta, \beta)$.

6.1. Initial estimates

Since the function in (33) is non-linear, it is convenient to obtain some initial estimates to avoid convergence problems. One possibility consists of using three different stress ranges (the first three for example) $(\Delta\sigma_i; i = 1, 2, 3)$, obtaining the corresponding means, μ_i of $\log N$ and choose the values of N_0, $\Delta\sigma_0$, and K such that the regression curve (32) passes through those mean points, as the initial estimates, i.e., solving in N_0, $\Delta\sigma_0$, and K the system of equations

$$\mu_i = \frac{1}{n_i}\sum_{j=1}^{n_i}\log N_{ij} = \log N_0 + \frac{K}{\log\Delta\sigma_i - \log\Delta\sigma_0}; \qquad i = 1, 2, 3. \tag{34}$$

7. Estimation of the Weibull parameters

Several methods have been proposed for estimating the parameters of the Weibull distribution (see references in Castillo and Hadi (1994)). Jenkinson (1969) uses the method of sextiles. The maximum likelihood method (ML) has been considered by Jenkinson (1969) and Prescott and Walden (1980, 1983). Smith (1985) considers the applicability of ML and discusses non-regular cases. The maximum likelihood estimates (MLE) require numerical solutions and for some samples, the likelihood may not have a local maximum. Furthermore, for $\beta < 1$, the likelihood can be made infinite and hence the MLE does not exist. Hosking et al. (1985) suggest estimating the parameters and quantiles by the probability-weighted moments (PWM), introduced by Greenwood, Landwehr, Matalas and Wallis (1979). They find that the PWM outperform the ML in many cases. Hosking et al. (1985), however, consider only cases where the shape parameter β lies in the range $\beta < 2$ because it has been observed in practice that β usually lies in this range. While the PWM performs quite admirably within the above restricted range of k, it presents problems outside this range.

7.1. The PWM estimators

The PWM estimators are given by

$$\hat{\beta}_{\mathrm{PWM}} = (7.859c + 2.9554c^2)^{-1}, \tag{35}$$

$$\hat{\delta}_{\mathrm{PWM}} = \frac{(\bar{x} - 2b_1)}{\Gamma(1 + 1/\hat{\beta}_{\mathrm{PWM}})(1 - 2^{-1/\hat{\beta}_{\mathrm{PWM}}})}, \tag{36}$$

$$\hat{\lambda}_{\mathrm{PWM}} = \bar{x} - \hat{\delta}_{\mathrm{PWM}}\Gamma(1 + 1/\hat{\beta}_{\mathrm{PWM}}), \tag{37}$$

where $\bar{x}$ is the sample mean,

$$c = \frac{2b_1 - \bar{x}}{3b_2 - \bar{x}} - \frac{\log 2}{\log 3},$$

and

$$b_j = n^{-1}\sum_{i=1}^{n} \frac{(i-1)(i-2)\dots(i-j)}{(n-1)(n-2)\dots(n-j)} x_{n-i+1:n}, \qquad j = 1, 2.$$

7.2. The Castillo-Hadi estimators

Castillo and Hadi (1994) proposed a method based on a two-stage procedure for estimating the parameters and quantiles of the Weibull distribution. First, a set of initial estimates are obtained by equating the cdf evaluated at the observed order statistics to their corresponding percentile values (first stage). Next, these estimates are combined to obtain a statistically more efficient estimates of the parameters (second stage).

7.2.1. *The first stage: initial estimates*

Let $x_{i:n} \le x_{2:n} \le \dots \le x_{n:n}$ be the order statistics obtained from a random sample from $W(\lambda, \delta, \beta)$. Let $I = \{i, j, r\}$ be a set of three distinct indices, where $i < j < r \in \{1, 2, \dots, n\}$. Then, using (3), we write

$$\begin{aligned} x_{i:n} &\cong \lambda + \delta[-\log(1 - p_{i:n})]^{1/\beta}, \\ x_{j:n} &\cong \lambda + \delta[-\log(1 - p_{j:n})]^{1/\beta}, \\ x_{r:n} &\cong \lambda + \delta[-\log(1 - p_{r:n})]^{1/\beta}, \end{aligned} \tag{38}$$

where

$$p_{s:n} = \frac{s - 0.35}{n} \tag{39}$$

are suitable plotting positions. The system in (38) is a set of three independent equations in three unknowns, λ, δ, and β. Estimates of λ, δ, and β can then be obtained by solving (38) for λ, δ, and β.

The solution of (38) can be obtained by the elimination method as follows. Eliminating λ and δ, we obtain.

$$D_{ijr} = \frac{x_{j:n} - x_{r:n}}{x_{i:n} - x_{r:n}} = \frac{C_r^k - C_j^k}{C_r^k - C_i^k} = \frac{1 - A_{jr}^k}{1 - A_{ir}^k}, \tag{40}$$

where $k = 1/\beta$, $C_i = -\log(1 - p_{i:n})$ and $A_{ir} = C_i/C_r$. An initial estimate $\hat{k}_{ijr}$ of k, which depends on $x_{i:n}, x_{j:n}$ and $x_{r:n}$, is obtained by solving (40). Equation (40) involves only one variable, hence it can be easily solved using the bisection method.

To this end, Castillo and Hadi (1994) show that:

(1) If $D_{ijr} < \log(A_{jr})/\log(A_{ir})$, then $\hat{k}_{ijr}$ lies in the interval $(0, \frac{\log D_{ijr}}{\log A_{jr}})$.

(2) If $D_{ijr} > \log(A_{jr})/\log(A_{ir})$, then $\hat{k}_{ijr}$ lies in the interval $(\frac{\log(1-D_{ijr})}{\log A_{jr}}, 0)$.

Once $\hat{k}_{ijr}$ is obtained, $\hat{\beta}_{ijr}$, $\hat{\lambda}_{ijr}$, and $\hat{\delta}_{ijr}$ are obtained in a closed form as:

$$\hat{\beta}_{ijr} = 1/\hat{k}_{ijr}, \tag{41}$$

$$\hat{\delta}_{ijr} = \frac{x_{i:n} - x_{r:n}}{C_i^{k_{ijr}} - C_r^{k_{ijr}}}, \tag{42}$$

$$\hat{\lambda}_{ijr} = x_{i:n} - \hat{\delta}_{ijr} C_{ijr}^{k_{ijr}}. \tag{43}$$

7.2.2. *The second stage: final estimates*

The above initial estimates are based on only three order statistics. More statistically efficient estimates are obtained using other order statistics as follows.

(1) Let $i = 1$ and $r = n$ and compute $\hat{\beta}_{1jn}$, $\hat{\lambda}_{1jn}$ and $\hat{\delta}_{1jn}$, $j = 2, 3, \ldots, n-1$.

(2) Apply the robust median function to each of the above sets of estimates to obtain the corresponding overall estimates:

$$\begin{aligned} \hat{\beta}_{\text{MED}} &= \text{median}(\hat{\beta}_{1,2,n}, \hat{\beta}_{1,3,n}, \ldots, \hat{\beta}_{1,n-1,n}), \\ \hat{\lambda}_{\text{MED}} &= \text{median}(\hat{\lambda}_{1,2,n}, \hat{\lambda}_{1,3,n}, \ldots, \hat{\lambda}_{1,n-1,n}), \\ \hat{\delta}_{\text{MED}} &= \text{median}(\hat{\delta}_{1,2,n}, \hat{\delta}_{1,3,n}, \ldots, \hat{\delta}_{1,n-1,n}), \end{aligned} \tag{44}$$

where $\text{median}(y_1, y_2, \ldots, y_n)$ is the median of $\{y_1, y_2, \ldots, y_n\}$.

The reason for setting $i = 1$ and $r = n$ in Step 1 is that the range of the random variable in this case depends on the parameters. We, therefore, have to ensure that $x_{1:n} > \lambda$. In this way we force parameter estimates to be consistent with the observed data.

The quantile estimates for any desired p are then obtained by substituting the above parameter estimates in (3).

Note that since the parameter and quantile estimates are well defined for all possible combinations of parameter and sample values, the variances of these estimates (hence, confidence intervals for the corresponding parameter or quantile values) can be obtained using sampling based methods such as the jackknife and the bootstrap methods (Efron, 1979 and Diaconis and Efron, 1983).

8. Damage accumulation

The $S-N$ curves, discussed in previous sections, have been obtained under the assumption of loading under a constant stress range. For the analysis of data and fatigue lifetime prediction based on experimental programs under varying loading (block or random loading) a damage accumulation hypothesis is needed.

Establishing a damage accumulation model, suitable of taking into consideration the instability due to peak loads and the interaction effects influencing the crack growth process in the case of a general complex load, that is, crack closure, overloads, local plastification, etc., entails a considerable difficulty, assuming it to be feasible, and has to be, necessarily, complex (see Fernández-Canteli, 1982).

In the following, a fatigue damage accumulation model is derived from the relations developed in Section 4, which permits, in a consequent way, the definition of a damage index associated with the corresponding damage state, capable of being statistically interpreted in terms of a probability of failure. For the sake of simplicity, instability and interaction effects are not considered in the present stage of the proposed damage model, though they could be subsequently and gradually implemented.

The normalization concept can be advantageously applied to define the equivalence, in terms of damage, between two fatigue states owning to different stress ranges. A damage equivalence is ensured as long as the value of the normalized lifetime, identified as the damage index, is maintained unaltered in each stress range conversion. Thus, the case of multistep loading can be regarded as a simple extension of the one-step constant load case.

Since the quantile curves in the S–N field represent the number of cycles to failure to be conducted at different stress ranges yielding the same probability of failure, it follows that the quantile curves could be rather contemplated as the curves representing the damage state, or alternatively, showing the same maximum flaw size which would cause the failure, precisely, for the number of cycles given by the S–N associated curve. We speak of ‘equivalent’ damage, or, ‘equivalent’ maximum flaw size, to avoid consideration of the lifetime scatter due to the random character of the crack growth separately from the lifetime scatter due to the random character of the flaw size (see Figure 6). Thus, both are considered to act, provisionally, lumped together. Moreover, the Weibull distributions resulting for the different stress ranges in the S–N field yield a unique pooled Weibull distribution associated with the normalized variable, which can be univoquely related to the probability of failure. Therefore, the same value of the normalized variable for two different stress ranges implies the same probability of failure for both, or more properly, the same state of damage given by the same equivalent flaw size.

Using the normalizing concept, the case of multi-step loading, i.e., block loading, can be regarded as a simple extension of the one-step or constant load case, provided the number of cycles at a certain stress range can be replaced by an equivalent number of cycles at the onset of the subsequent stress range (see Figure 7). Though in Figure 7 the evolution in the number of cycles could be interpreted as having apparent sudden decreases, when changing the stress ranges, the correct interpretation is that the damage state is kept constant during the stress range changes, and increases continuously during the loading process. This figure only explains how to obtain equivalent number of cycles, in terms of damage, for two different stress ranges (the larger the stress range, the smaller the lifetime). This is accomplished as long as the damage index, identified with the normalized lifetime V^* or N^*, is maintained in each conversion. The non-dimensional values V^* or N^* can be associated with the probability of failure. The meaning, implications and limitations of this approach will be justified and discussed in other paper, currently in preparation.

This allows establishing the equivalence of the damage state for two or more different stress ranges, simply, by equating the normalized number of cycles to failure, that is:

$$V_i^* = V_j^*, \tag{45}$$

where V_i^* and V_j^* are the normalized number of cycles at the stress range i and j, respectively.

The generalization to random loading is straightforward. The conversion between stress levels is identical in blocks and in random loading, the only differences lying in the stress

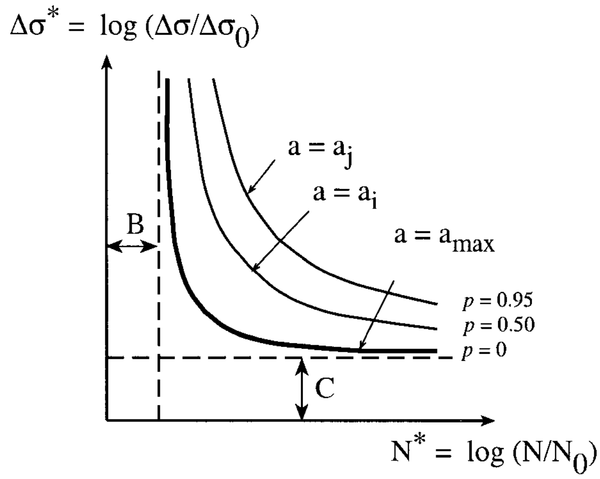

Figure 6. Identification of the quantile curves with lifetimes corresponding to the same initial flaw for different stress ranges.

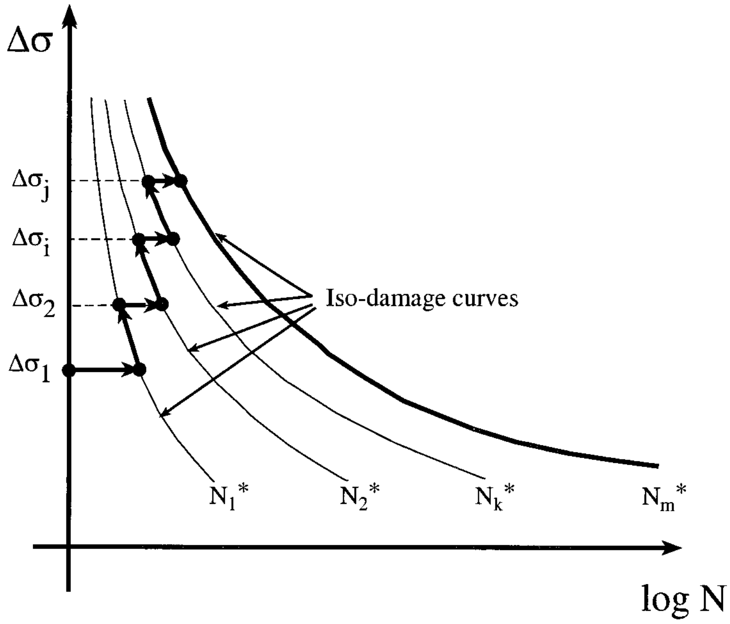

Figure 7. Schematic representation of the conversion process.

amplitude, deterministically known in the case of block loading but unknown in the random loading, which has to be generated from a load history, representative of the process.

Now, one relevant question is how to define the normalization. Two different proposals can be envisaged:

(1) Due to the stability of the Weibull distribution with respect to location and scale transformations we can proceed to normalize through the transformations:

$$V^* = (N^* - m)/s, \tag{46}$$

where m and s are the mean and the standard deviation of N^*, or, equivalently:

$$V^* = (\log N - \ell)/d, \tag{47}$$

where, ℓ and d are the mean and the standard deviation of $\log N$. Such transformations present the disadvantage that they need to make use, directly or indirectly, of the standard deviation of the stress ranges being considered, which, occasionally, as in the case of scarce results, makes difficult a subsequent reliable estimation of the S–N parameters.

(2) On the contrary, considering Expression (22), in Section 6 we make use of the proposed model for the S–N field and thus the following normalization can be suggested:

$$V^* = \log(N/N_0)\log(\Delta\sigma/\Delta\sigma_0) = N^*\Delta\sigma^*, \tag{48}$$

where $N^* = \log(N/N_0)$ and $\Delta\sigma^* = \log(\Delta\sigma/\Delta\sigma_0)$, which allows the advantageous application of the two-step parameter estimation method above mentioned.

In both cases, the normalized fatigue lifetime is an adimensional damage index representing a damage state and, therefore, an equivalent flaw or crack size, unknown for the time being. It can be easily proved that both normalizing procedures are thoroughly equivalent, even if they represent two quantitatively different magnitudes, whose statistical interpretation leads, consequently, to identical probabilities of failure.

We have justified the use of V^* as the basis for damage accumulation, together with its advantages. However, it must be clear that with this model it is not intended to exhaust such a complex subject as the cumulative damage problem, but merely proposing one extension of the many possible consistent with the rest of the model. Thus, other alternatives can also be justified.

8.1. Accumulated Damage after a Constant Stress Range Load

The damage accumulated after a constant load step can be calculated as follows:

Assume that a piece with accumulated damage V_0^* is subject to ΔN cycles at a constant stress range $\Delta\sigma$. To calculate the damage increment suffered by the piece (see Figure 8), we first calculate the equivalent number of cycles at the stress range $\Delta\sigma$ for damage V_0^* :

$$N_{\text{initial}} = N_0 \exp\left(\frac{V_0^*}{\log\left(\dfrac{\Delta\sigma}{\Delta\sigma_0}\right)}\right). \tag{49}$$

Then, the damage after ΔN more cycles is

$$\begin{aligned} V^*_{\text{final}} &= \log[(N_{\text{initial}} + \Delta N)/N_0]\log(\Delta\sigma/\Delta\sigma_0) \\ &= \log\left[\exp\left(\frac{V_0^*}{\log\left(\dfrac{\Delta\sigma}{\Delta\sigma_0}\right)}\right) + \frac{\Delta N}{N_0}\right]\log(\Delta\sigma/\Delta\sigma_0). \end{aligned} \tag{50}$$

Since, according to (25), V^* follows a Weibull $W(\lambda, \delta, \beta)$ distribution, the corresponding probability of failure p_F is

$$p_F = 1 - \exp\left[-\left(\frac{V^*_{\text{final}} - \lambda}{\delta}\right)^{\beta}\right]. \tag{51}$$

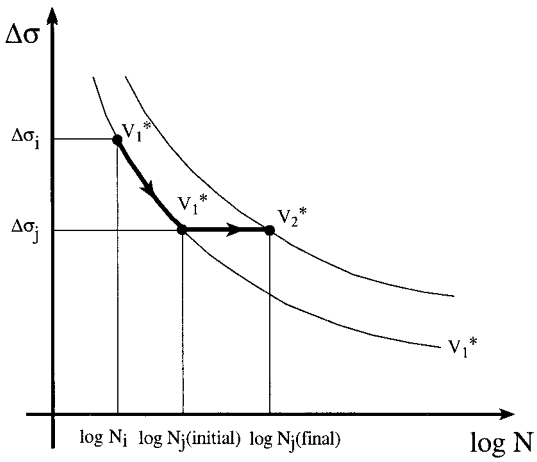

Figure 8. Illustration of the damage accumulation process.

8.2. ACCUMULATED DAMAGE AFTER BLOCK LOADING

Assume that a piece with initial damage V_0^* is subject to a block loading: $\{(\Delta\sigma_i, \Delta N_i); i = 1, 2, \ldots, n\}$, then from (50) we get the recurrence formula

$$V_i^* = \log\left[\exp\left(\frac{V_0^*}{\log\left(\dfrac{\Delta\sigma_i}{\Delta\sigma_0}\right)}\right) + \frac{\Delta N_i}{N_0}\right]\log(\Delta\sigma_i/\Delta\sigma_0); \qquad i = 1, 2, \ldots, n, \tag{52}$$

where V^* is the damage after *i*-th load step, and the associated probability of failure, at the end of the *i*-th block, is

$$p_{Fi} = 1 - \exp\left[-\left(\frac{V_i^* - \lambda}{\delta}\right)^{\beta}\right]; \qquad i = 1, 2, \ldots, n. \tag{53}$$

One can ask: what does the hazard rate look like in the transition from one block loading to the next. Is there a jump? One could argue on the basis of statistically distributed growing cracks and fracture mechanics (critical crack size vs. stress peak) that there should be a delta function in the hazard rate if the amplitude (or at least peak stress) increases from one block to the next as certain cracks suddenly become susceptible to catastrophic instability due to the increased peak load. Since sudden overloads can suddenly bring airplanes down as component fail instantaneously, so the issue is important. Of course the hazard rate will increase with a discontinuous jump motivated by an stress level discontinuous increase.

8.3. FATIGUE UNDER LOADING WITH BLOCKS IN A RANDOM SEQUENCE

In this case, the choice of the subsequent block unit follows not in a deterministic manner, but randomly, whereas the probability of occurrence can be considered to be equal for the constituting units of the blocks. Using simulation techniques, information about the influence of the randomness on the scatter of the results can be gained after a sufficiently high number of replications of the block units, so that a statistical judgment of the experimental results can be performed. The same holds for prediction of the number of replications (or time) needed to reach a certain probability of failure. In this case, a parallel calculation of the probability of failure follows associated with the calculation of the damage V^*.

8.4. Random loading

This is a mere extension of the foregoing procedure, the only difference being that the load varies in amplitude and sequence in a random way according to a statistical frequency distribution. Every cycle will produce a random increase of the normalized damage index, N^*, resulting in a quantity which can be identified, due to its uniqueness, with a defined probability of failure. The cases of random loading will be treated in a paper under preparation.

9. Example of application

In this section we use the Holmen (1979) data to illustrate the methods proposed in previous sections. These data consist of 75 fatigue tests, at 5 different stress levels as shown in Table II.

9.1. Parameter estimation

To estimate N_0 and $\Delta\sigma_0$, we have to minimize (33). However, we obtain first the initial estimates using the method given in subsection 6.1, taking as selected stress ranges $\Delta\sigma = 0.95$, 0.90 and 0.825, that leads to the system of equations

$$\begin{aligned}
\frac{1}{n_1}\sum_{j=1}^{n_1}\log N_{1j} &= -2.18364 = \log N_0 + \frac{K}{\log\Delta\sigma_1 - \log\Delta\sigma_0},\\
\frac{1}{n_2}\sum_{j=1}^{n_2}\log N_{2j} &= -0.99939 = \log N_0 + \frac{K}{\log\Delta\sigma_2 - \log\Delta\sigma_0},\\
\frac{1}{n_3}\sum_{j=1}^{n_3}\log N_{3j} &= \quad 1.01688 = \log N_0 + \frac{K}{\log\Delta\sigma_3 - \log\Delta\sigma_0},
\end{aligned} \tag{54}$$

with solution

$$\log N_0 = -57.4174; \qquad K = 142.27; \qquad \log\Delta\sigma_0 = -2.62708.$$

Using now these estimates we minimize (33) and obtain the final estimates:

$$\log N_0 = -20.7843; \qquad K = 19.731; \qquad \log\Delta\sigma_0 = -1.10607.$$

Once the threshold values have been calculated, we can pool de sample together by calculating the values of $\log(N/N_0)\log(\Delta\sigma/\Delta\sigma_0)$ for all sample data points, thus, getting a sample of size 75, which is shown in Figure 9 on a Weibull probability plot. The linear trend of the cumulative distribution function guarantees that the Weibull law assumption is reasonable.

Now, using the PWM estimates of the Weibull parameters we get:

$$\beta = 2.70123; \qquad \delta = 1.68844; \qquad \lambda = 18.2305,$$

and using the Castillo-Hadi estimates we obtain:

$$\beta = 3.40031; \qquad \delta = 2.42772; \qquad \lambda = 17.5225. \tag{55}$$

Table 2. Holmen data

$\Delta\sigma$	Lifetime (thousands of cycles)						
0.95	0.257	0.217	0.206	0.203	0.143	0.123	0.120	0.109
	0.105	0.085	0.083	0.076	0.074	0.072	0.037	
0.90	1.129	0.680	0.540	0.509	0.457	0.451	0.356	0.342
	0.311	0.295	0.257	0.252	0.226	0.216	0.201	
0.825	5.598	5.560	4.820	4.110	3.847	3.590	3.330	2.903
	2.590	2.410	2.400	1.492	1.460	1.258	1.246	
0.75	67.340	50.090	48.420	36.350	27.940	26.260	24.900	20.300
	18.620	17.280	16.190	15.580	12.600	9.930	6.710	
0.675	11748	11748	3295	1459	1400	1330	1250	1242
	896	659	486	367	340	280	103	

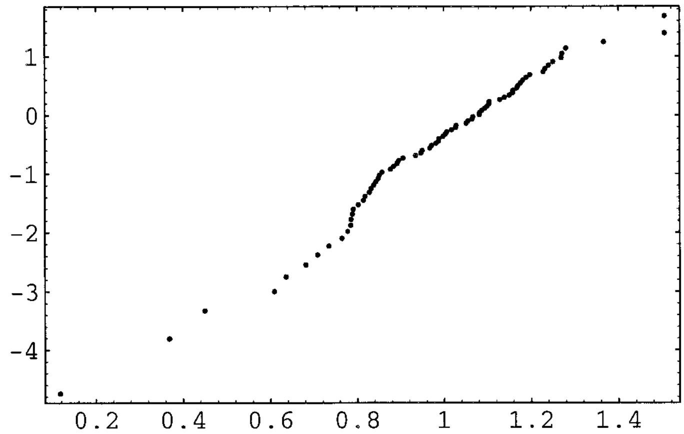

Figure 9. Pooled sample on a Weibull probability paper.

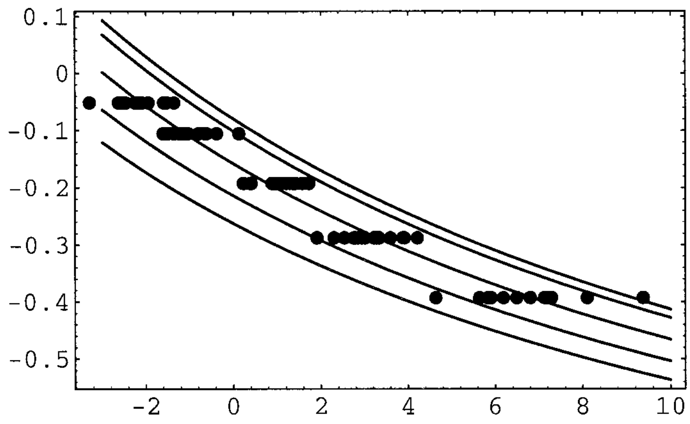

Figure 10. Data and fitted Whöhler field in terms of $\log(N)$ and $\log(\Delta\sigma)$.

Table 3. Blocks applied to the piece, and corresponding damage levels and probabilities of failure, when the initial damage level is $V_0^* = 18$

Step i	$\Delta\sigma_i$	ΔN_i	V_i^*	p_{F_i}
0	–	–	18.00	0.004
1	0.70	20	18.44	0.035
2	0.75	20	19.67	0.481
3	0.80	10	20.71	0.919
4	0.85	5	21.60	0.997
5	0.90	5	22.78	0.999

Figure 10 shows the data and the Whöhler field in terms of $\log(N)$ and $\log(\Delta\sigma)$, for these last estimates.

9.2. Damage accumulation

To illustrate the damage accumulation under block loading we present below a simple example.

The load block sequence is shown in Table 3, where the number of cycles ΔN_i (in thousands) applied to the piece at stress levels $\Delta\sigma_i$ are shown for $i = 1, 2, 3, 4, 5$.

The piece is assumed to have an initial damage level of 18.00, corresponding to a probability of failure of 0.004, and to come from a population such that its lifetime follows a Weibull distribution $W(\lambda, \delta, \gamma)$, where the parameters are those in (55).

Thus, using Expression (52), the accumulated damage values, and the corresponding probabilities of failure are those in columns 4 and 5 of Table 3.

10. Conclusions

From the sections above we can conclude the following:

(1) The use of non-dimensional variables simplifies the problem under consideration and clarifies what is the minimal set of variables, or functions of them, which are relevant to the problem, and allows working with non-dimensional parameters that have many important advantages, apart from independency of the set of selected units, as a better numerical behavior.

(2) Physical and engineering considerations allow rejecting many models non satisfying the associated constraints. These considerations can be written, in many cases, in terms of functional equations, that lead to explicit forms for the mathematical and statistical models.

(3) A Weibull based model for the S–N field has been obtained by solving a functional equation. This model is useful not only to fit fatigue data, but also to explain the fatigue behavior of longitudinal elements.

(4) There are two types of parameters. One is related to the non-dimensional variables, and used for normalization purposes, it includes the threshold parameters N_0 and $\Delta\sigma_0$. Other type of parameters are statistical parameters, as the location parameter λ, the scale parameter δ, and the shape parameter β.

(5) The model suggests the variable $\log(N/N_0)\log(\Delta\sigma/\Delta\sigma_0)$ as the base for damage accumulation. This allows measuring the damage caused by a given number of fatigue cycles at different stress levels by means of a common scale that is independent of the stress range.

(6) The probabilities of failure associated with a given load history can be easily calculated, as it has been shown in the illustrative example of Section 9.

(7) Relaxing some of the initial assumptions, apart from the suggested models for the S–N field, other are possible.

(8) In the proposed damage accumulation approach instability and interaction effects are not considered, though they could be implemented.

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