Geometric Brownian motion (GBM)

What is an appropriate model of the stock price? Assuming certain conditions on the behaviour of stock price we arrive at a model called the Geometric Brownian motion. The assumptions are

i) Returns over a short period are normally distributed

$$\left.\frac{\Delta S}{S}\right|_t = \frac{S(t+\Delta t)-S(t)}{S(t)} \sim N\left(\mu\Delta t, \overset{\sigma^2\Delta t}{\cancel{\sigma\sqrt{\Delta t}}}\right)$$

ii) Expected return is independent of the stock price $E\left(\frac{\Delta S}{S}\right) = \mu\Delta t$

iii) The volatality of the stock (σ) is also independent of the stock price

The discrete time version of the model is

$$\frac{\Delta S}{S} = \mu\Delta t + \sigma z\sqrt{\Delta t}$$

$z \to N(0,1)$

or $\Delta S = \mu S\Delta t + \sigma S z\sqrt{\Delta t}$

Consider a stock that has volatality 30% per annum providing an expected return 15% per annum then

$$\frac{\Delta S}{S} = 0.15\Delta t + 0.30 z\sqrt{\Delta t}$$

If a time interval of a week or 0.0192 year is given then $\Delta t = 0.0192$

$$\therefore \quad \frac{\Delta S}{S} = 0.15 \times 0.0192 + 0.30 \times \sqrt{0.0192}\, z$$

$$\text{or} \quad \Delta S = 0.00288\, S + 0.0416\, S\, z$$

How do we simulate the stock price using random variable generated on a computer? Well, using the above eqn. all we need to do is to generate a $N(0,1)$ r.v. Lets say the stock price in the beginning was 100 and we sampled from a normal distribution to get value 0.52

then $\Delta S = 0.00288 \times 100 + 0.0416 \times 100 \times 0.52$

$\Delta S = 2.45$

New stock price is 102.45

Stock price	Random sample	Change in price
100	0.52	2.45
102.45	1.44	6.43
108.88	−0.86	−3.58
10.5.30	1.46	6.70
112.00	−0.69	−2.89
109.11	−0.74	−3.04

Monte Carlo simulation of stock price following GBM

What if you did not know the parameters μ and σ. You can then estimate μ and σ by using historical stock data.

Estimating μ

We can use the fact that $E\left(\frac{\Delta S}{S}\right) \simeq \mu$ to do this estimate. For example using the table the values of $\frac{\Delta S}{S}$ for various periods are $\left\{ \frac{2.45}{100}, \frac{6.45}{102.45}, -\frac{3.58}{108.88}, \dots \right\}$

The average of these values should give a good estimate of μ

Estimating σ

Define

$n+1$: Number of observations

S_i : Stock price at the end of interval $i = 0, 1, \dots n$

τ : Length of time interval in years.

Then let

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right) \quad \text{for } i = 1, 2, \dots n$$

Let

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (u_i - \bar{u})^2}$$

then the estimate $\hat{\sigma}$ of σ is

$$\hat{\sigma} = \frac{s}{\sqrt{\tau}}$$

Log normal property of stock prices

We have

$$\frac{\Delta S}{S} \sim N(\mu \Delta t, \sigma^2 \Delta t)$$

This implies (we will see this using Ito's lemma in the next lecture)

$$\ln S_T - \ln S_0 \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$$

OR

$$\ln\left(\frac{S_T}{S_0}\right) \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$$

OR

$$\ln S_T \sim N\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$$

or

$$S_T \sim e^{N\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)}$$

So GBM implies stock price is not normal but exponential of a normal random variable.