Lecture 5: Random Walks

Computational Finance

Prof. Jaideep Mulhelkar

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1 Introduction

Start with a person at the origin on a lattice (Z). A coin is flipped if head occurs then person moves one step to right otherwise one step to left.

$$S_n \sim \text{Position of random walker at time n} \tag{1}$$

$$S_n = \sum_{i=1}^{n} X_i \tag{2}$$

where,

$$X_i = \begin{cases} 1 & if\ H \\ -1 & if\ T \end{cases}$$

$P(H) = p, P(T) = 1 - p = q$. If p=q=$\frac{1}{2}$ then it will be called a symmetric random walk

2 Properties

Random walks are quite important in engineering,physics and finance. Mathematicians also study properties of random walks.Some properties of random walk are:

1. First passage time τ_m
   $\tau_m = \{\min\ \text{n} : S_n = m\}$

2. Maximum of random walk M_n
   $M_n = \{\max S_i | i = 0, ...n\}$

3. Return time
   $P(S_n = 0\,\text{for some n}|S_0 = 0)$

2.1 First Passage Time

We are interested int he probability distribution $P(\tau_m = k)$
Let m=1 (First time that a random walker hits level 1)

$$P(\tau_1 = 2j + 1)\ for\ j = 0, 1, 2...$$

(Note $P(\tau_1 = 2j) = 0$)
$P(\tau_1 = 1) = \frac{1}{2}$

We will solve the general case of $P(\tau_1 = 2j+1)$ .It is convenient to look at $P(\tau_1 \leq 2j+1)$ (First time a random walk hit level 1 before time 2j+1) and observe that

$$P(\tau_1 = 2j+1) \; = P(\tau_1 \leq 2j+1) \; - P(\tau_1 \leq 2j-1) \qquad (why?)$$

Now consider the following figures:
Consider all paths such that $P(\tau_1 \leq 2j+1)$ they are the union(disjoint) of paths of type A,B,C.

$A \sim$ *Paths that are at level* 1 *at* $2j+1$
$B \sim$ *Paths that are strictly above level* 1 *at time* $2j+1$
$C \sim$ Paths that have touched or crossed level 1 at time before 2j+1 but are strictly below level 1 at time 2j+1

$$P(\tau_1 \leq 2j+1) = P(A) + P(B) + P(C)$$

Claim : #B = #C (No of paths of type of B and C are equivalent)

Proof : As shown in the figure draw a reflected path from the first time the random walk hits level 1. Paths of type C are in 1-1 correspondence with the reflected paths (Paths of type B).This is called reflection principle.

$$P(\tau_1 \leq 2j+1) = \; P(A) + P(B)$$

$$P(\tau_1 \leq 2j+1) = \; P(S_{2j+1} = 1) + P(S_{2j+1} > 1)$$
$$P(\tau_1 \leq 2j+1) = \; P(S_{2j+1} = 1) + P(S_{2j+1} > 1) + P(S_{2j+1} < -1)$$
$$P(\tau_1 \leq 2j+1) = \; 1 - P(S_{2j+1} = -1)$$

$$P(\tau_1 = 2j+1) = \; P(\tau_1 \leq 2j+1) - P(\tau_1 \leq 2j-1)$$
$$P(\tau_1 = 2j+1) = \; P(S_{2j-1} = -1) - P(S_{2j+1} = 1)$$
$$P(S_{2j-1} = -1) = \; \binom{2j-1}{j-1}\frac{1}{2^{2j-1}} \qquad (j \; tails \; , j-1 \; heads \;)$$
$$P(S_{2j+1} = -1) = \; \binom{2j+1}{j}\frac{1}{2^{2j+1}} \qquad (j+1 \; tails \; , j \; heads \;)$$
$$P(\tau_1 = 2j+1) = \; P(S_{2j-1} = -1) - P(S_{2j+1} = 1) = (\frac{1}{2})^{2j+1}\frac{(2j)!}{j!(j+1)!}$$

2.2 Maximum of Random walk in n-tosses

$$M_n = max(S_i) \quad 0 \leq i \leq n$$

What is the connection between $\tau_m \, and M_n$
$P(\tau_m = n) = P(M_{n-1} = m-1\,, S_{n-1} = m-1\,, S_n = m)$
We will exploit this to find an alternate derivation of τ_m for all m.
Again note that,

$$P(M_n = r) = P(M_n \geq r) - P(M_n \geq r+1)$$

- Lemma 1:

$$P(M_n \geq r, S_n = m) = \begin{cases} P(S_n = m) & if\; m \geq r \\ P(S_n = 2r - m) & if\; m < r \end{cases}$$

Now we give alternate derivation of $P(\tau_m = n)$:

$$\begin{aligned} P(\tau_m = n) &= P(M_{n-1} = S_{n-1} = m-1\,, S_n = m) \\ &= \frac{1}{2}[P(M_{n-1} \geq m-1\ \ , S_{n-1} = m-1) - P(M_{n-1} \geq m\ \ , S_{n-1} = m-1)] \end{aligned}$$

then applying lemma 1 we get

$$\begin{aligned} P(\tau_m = n) &= \frac{1}{2}[]P(S_{n-1} = m-1) - P(S_n = 2m - (m-1))] \\ &= \frac{1}{2}[P(S_{n-1} = m-1) - P(S_n = m+1))] \end{aligned}$$

Further Simplification,

$$P(\tau_m = n) = \frac{m}{n} P(S_n = m)$$

- Lemma 2: $P(M_n \geq r) = P(S_n = r) + 2P(S_n \geq r+1)$

$$\begin{aligned} P(M_n \geq r) &= P(M_n \geq r, S_n \geq r) + P(M_n \geq r, S_n < r) \\ &= \sum_{k=r}^{\infty} P(M_n \geq r, S_n = k) + \sum_{j=-\infty}^{r-1} P(M_n \geq r, S_n = j) \\ &= \sum_{K=r}^{\infty} P(S_n = k) + \sum_{j=-\infty}^{r-1} P(S_n = 2r - j) \quad \text{(By Lemma 1)} \\ &= P(S_n = r) + \sum_{K=r+1}^{\infty} P(S_n = k) + \sum_{u=r+1}^{\infty} P(S_n = u) \quad \text{(change of vriable u=2r-j)} \end{aligned}$$

$P(M_n \geq r) = P(S_n = r) + 2P(S_n \geq r+1)$

- Theorem: $P(M_n = r) = P(S_n = r) + P(S_n = r+1)$

$$P(M_n = r) = P(M_n \geq r) - P(M_n \geq r+1)$$

From lemma 2 we get

$$\begin{aligned} P(M_n = r) &= [P(S_n = r) + 2P(S_n \geq r+1)] - [P(S_n = r+1) + 2P(S_n \geq r+2)] \\ &= P(S_n = r) + 2[P(S_n \geq r+1) - P(S_n \geq r+2)] - P(S_n = r+1) \\ P(M_n = r) &= P(S_n = r) + P(S_n = r+1) \end{aligned}$$

2.3 Return to origin of Symmetric Random walk

$U_{2m} \sim$ Probability of equalization
$U_{2m} = P(S_{2m} = 0|S_0 = 0)$
$f_{2k} \sim$ Probability of first return to origin occuring at time 2k
$f_{2k} = P(S_2 \neq 0, S_4 \neq 0, ..., S_{2k-2} \neq 0, S_{2k} = 0|S_0 = 0)$
$W_{2n} = \displaystyle\sum_{i=1}^{n} f_{2i}$ is the probability that a random walker return to origin occurs no later than 2n

$W^* = \lim_{n\to\infty} W_{2n} = \sum_{i=1}^{\infty} f_{2i} \equiv$ Probability of eventual return

- Theorem 1: $U_{2m} = \binom{2m}{m}\frac{1}{2^{2m}}$
  Proof: Obvious
- Theorem 2: For n≥ 1 :
  $\mathrm{U}_{2n} = f_0U_{2n} + f_2U_{2n-2} + ... + f_{2n}U_0 \quad where\, U_0 = 1, f_0 = 0$
  Proof:
  $U_{2n} \sim$ Probability that a random walk returns to origin in 2n steps
  Paths that return for the first time at time 2 and then at the origin after 2n-2 steps

  $$\oplus$$

  Paths that return to origin for first time in 4 steps and then are back at origin in remaining 2n-4 steps

  $$\oplus$$

  $$...$$

  $$\oplus$$

  Paths that return to origin for first time in 2n steps.
  Hence,
  $$U_{2n} = f_0U_{2n} + f_2U_{2n-2} + ... + f_{2n}U_0$$

  Next we introduce the concept of generating function
  $U(x) = \sum_{m=0}^{\infty} U_{2m}x^m, F(x) = \sum_{m=0}^{\infty} f_{2m}x^m$

- Theorem 3: $U(x) = 1 + U(x)F(x)$
  Proof:
  $$F(x) = \frac{U(x) - 1}{U(x)}$$
  we are interested in
  $$W_* = F(1) = \frac{U(1) - 1}{U(1)}$$
  now if $U(1) < \infty$ then $F(1) < 1$ and if $U(1) = \infty$ then $F(1) = 1$

Therefore return to origin probability depends on $U(1) = \sum_{m=0}^{\infty} U_{2m}$

If $\sum_{m=0}^{\infty} U_{2m} \longrightarrow W_* < 1$ *otherwise* $W_* = 1$

$U_{2m} = \binom{2m}{m} \frac{1}{2^{2m}}$
we use Stirlings approximation to $m! \sim (\frac{m}{e})^m \sqrt{2\pi m}$

we get $U_{2m} \sim \frac{1}{\sqrt{\pi m}}$
$W^* = 1$

For 2-D,

$$U_{2m}^{(2)} = \sum_{k=0}^{2m} \frac{2m!}{k!k!(m-k)!(m-k)!} \quad Check!!$$

(Hint : For return to origin L=R, U=D ,P(L)=P(U)=P(R)=P(D)=$\frac{1}{4}$)

3 Scaled Random Walk

3.1 Introduction

Position of a random walker at time K

$$S_K = \sum_{i=1}^{K} X_i$$

where,

$$X_i = \begin{cases} 1 & if\ H\ occurs \\ -1 & if\ T\ occurs \end{cases}$$

To go to the scaled random walk we toss the coin more and more frequently(instead of 1 toss per unit time we toss n times per unit time).To avoid blow-up we also scale the random walk appropriately. If we toss the coin n times per unit interval,the position of the random walker in t units of time(Assuming nt is an integer) is

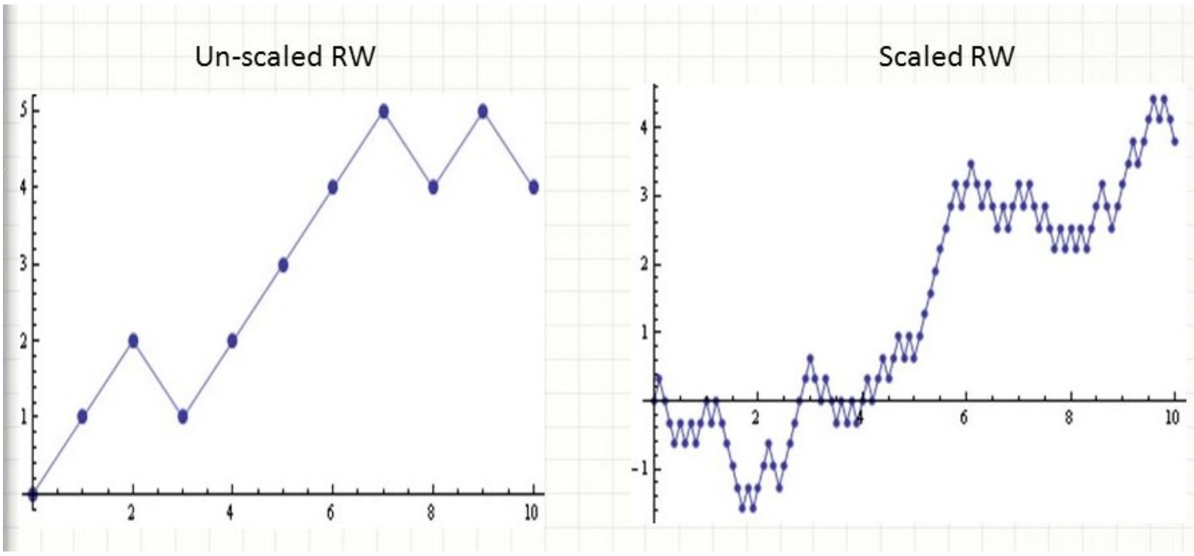
$$S_{nt} = \sum_{i=1}^{nt} X_i$$

To avoid the blow-up of the random walkers position we scale by a factor of $\sqrt{n}$ .So the scaled random walk denoted by $W_{(n)}(t)$ is given by:

$$W_{(n)}(t) = \frac{S_{nt}}{\sqrt{n}} = \frac{\sum_{i=1}^{nt} X_i}{\sqrt{n}}$$

Here n denotes the number of tosses per unit time and t denotes the current time. Lets take an example,we would like to calculate $(P(W^{(100)}(0.25) = 0.1)$
The values of $W^{(100)}(0.25)$ range from -2.5(all tails) to 2.5(all heads) (-2.5,-2.3,...,-0.1,0.1,0.3,...,2.5)
The value 0.1 occurs when there are 13 heads and 12 tails so

$$P(W^{(100)}(0.25) = 0.1) = \binom{25}{13} \frac{1}{2^{25}} \cong 0.155$$



The mathematical way of arriving at a Brownian motion is taking the limit as n$\rightarrow \infty$
$W(t) = \lim_{n\to\infty} W^{(n)}(t) = lim_{n\to\infty} \frac{S_{nt}}{\sqrt{n}}$

3.2 Expectation and variance of scaled Random walk

$$W_{(n)}(t) = \frac{S_{nt}}{\sqrt{n}} = \frac{\sum_{i=1}^{nt} X_i}{\sqrt{n}}$$

$$E[W_{(n)}(t)] = E[\frac{\sum_{i=1}^{nt} X_i}{\sqrt{n}}] = \frac{\sum_{i=1}^{nt} E[X_i]}{\sqrt{n}} = 0$$

$$Var(W_{(n)}(t)) = Var(\frac{\sum_{i=1}^{nt} X_i}{\sqrt{n}}) = \frac{1}{n}\sum_{i=1}^{nt} Var(X_i) = \frac{1}{n}\sum_{i=1}^{nt} 1 = \frac{nt}{n} = t$$

thus,

$$E[W_{(n)}(t)] = 0 \quad Var[W_{(n)}(t)] = t$$

We defined the Brownian motion as the limit of the scaled random walk as $n \to \infty$ Lets look at some of the properties of the Brownian motion. These properties carried over from the scaled random walk.

3.3 Properties of W(t) (Brownian motion)

1. $W(0) = 0$
2. $E[W(t)] = 0\,,\ Var(W(t)) = t$
3. $W(t) \sim N(0, t)$ (This is the consequence of the central limit theorem) where N is the Normal Random Variable.
4. $0 = T_0 < T_1 < T_2 < T_3 < ... < T_m$
   $W(t_1) - W(t_0), W(t_2) - W(t_1), W(t_3) - W(t_2), ..., W(t_m) - W(t_{m-1})$ are independent of each other
5. $W(t) - W(s) \sim N(0, t - s)$

6. W(t) is continuous everywhere but differential nowhere