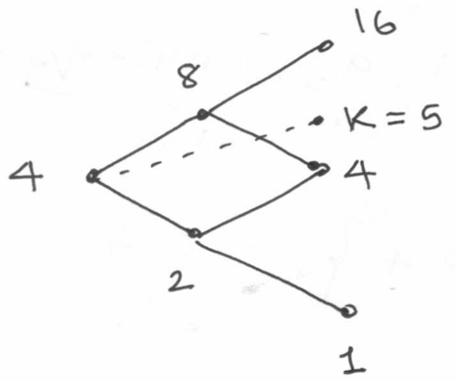
American options

An American option has the same pay-off as a European option but can be exercised at any time prior to expiry. Consider a two period American put option



$S_0 = 4$

$K = 5$

$r = \frac{1}{4}$ $\longrightarrow$ which gives risk neutral. prob. $q_u = q_d = \frac{1}{2}$

$u = 2$

$d = 1/2$

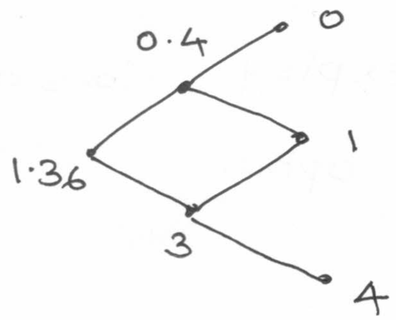
Since an American option can be exercised at any time it is worth at least $K - S_n$. This will be called the intrinsic value of the option. At the same time one can argue (hedging argument) just like the European option and hence the worth of the option is at least

$$V_n(\omega_1 \dots \omega_n) = \frac{q_u V_{n+1}(\omega_1 \dots \omega_n H) + q_d V_{n+1}(\omega_1 \dots \omega_n T)}{1+r}$$

Therefore

$$V_n = \max\left\{ K - S_n, \frac{q_u V_{n+1}(H) + q_d V_{n+1}(T)}{1+r} \right\}$$

Using this we can compute the value of the American option in the example. As usual we work backward from $T = 2$



$$V_2(HH) = 0$$
$$V_2(HT) = V_2(TH) = 1$$
$$V_2(TT) = 4$$

Now $V_1(H) = \max\left\{ K - S_1(H),\ \frac{q_u V_2(HH) + q_d V_2(HT)}{1+r} \right\}$

$$= \max\{-3,\ 0.4\} = 0.4$$

$$V_1(T) = \max\left\{ K - S_1(T),\ \frac{q_u V_2(TH) + q_d V_2(TT)}{1+r} \right\}$$

$$= \max\{3,\ 2\} = 3$$

And finally

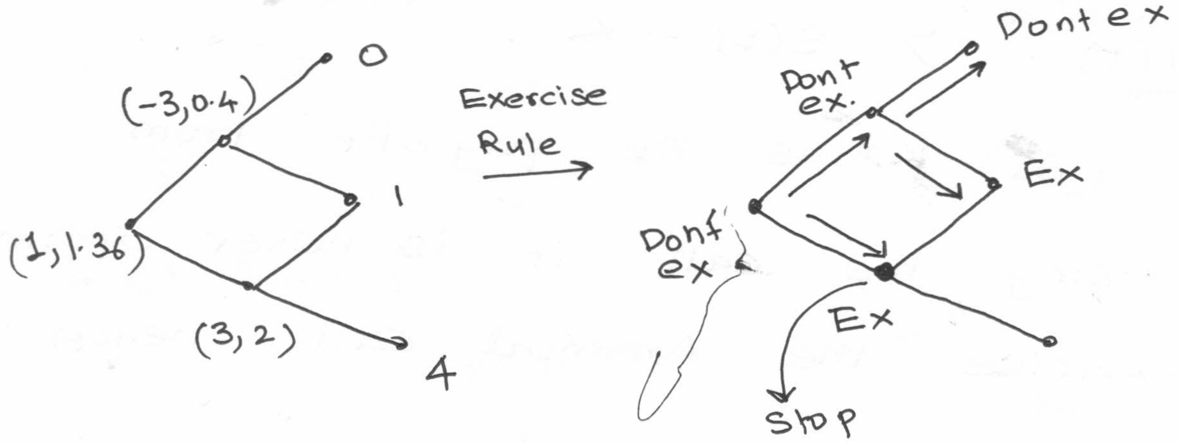
$$V_0 = \max\left\{ K - S_0,\ \frac{q_u V_1(H) + q_d V_1(T)}{1+r} \right\}$$

$$= \max\{1,\ 1.36\} = 1.36$$

An important question as far as American options are concerned is the best time to exercise it. We will not go into the details of this since it requires knowledge of stopping times, sub and super Martingales and Jensen's inequality. We will just state the theorem.

Theorem: The optimal time to exercise an American option is the first time the intrinsic value exceeds the value of the derivative security (value of the hedged portfolio)

In the previous example, the above thm. implies the following exercise rule



Another important fact about American options is that there is no advantage to an early exercise of an American call option that pays no dividend.

$$\boxed{V_{call}^{American} = V_{call}^{Euro}}$$

Proof:

First note that

$$V_{call}^{Euro} \geq S - Ke^{-rT}$$

This is because R.H.S. is the value of ~~the~~ a forward contract presently. Since the

European call option has superior payoff than a forward contract at time T, hence the present situation the inequality must hold. In fact it must be true that

$$V^{Euro}_{call}(t) \geq S(t) - Ke^{-r(T-t)}$$

and ~~hence~~ since $V^{American}(t) \geqslant V^{Euro}(t)$ (American option is worth atleast as much as European)

$$V^{American}_{call}(t) \geqslant S(t) - Ke^{-r(T-t)}$$

~~Since~~ ~~|S(t)-K|~~ ~~is the~~ ~~payoff~~

$$\therefore \quad V^{American}_{call}(t) > S(t) - K.$$

Since $(S(t) - K)$ is the payoff from exercising the call it is never optimal to exercise the Americal call. Hence

$$V^{American}_{call} = V^{Euro}_{call}$$

~~Next we do~~

In the continuous situation unlike the European option there is no formula like the Black-Scholes for the American option. Next we describe two methods to value American option 1. Finite difference

2. Monte carlo simulation.

Methods for valuing American options (Put)

1. Finite difference method

We have the B-S-M PDE for the option $f(t, S)$ given by

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

Let the maturity time be T. Divide T into N equally spaced intervals of length

$$\Delta t = \frac{T}{N}$$

Total of $N+1$ time intervals

$0, \Delta t, 2\Delta t, \ldots, T$

Suppose S_{max} is a stock price that is high enough so that when reached the put has no value. Divide S_{max} into M intervals.

Let $\Delta S = \dfrac{S_{max}}{M}$ giving a total of

$0, \Delta S, 2\Delta S, \ldots, S_{max}$. $(M+1)$ values

Total grid size is $(M+1)(N+1)$

We denote

$$f_{i,j} = f(i\Delta t, j\Delta S)$$

For an interior point on the grid

$$\frac{\partial f}{\partial S} \cong \frac{f_{i,j+1} - f_{i,j}}{\Delta S}, \quad \frac{\partial f}{\partial S} \cong \frac{f_{i,j} - f_{i,j-1}}{\Delta S}$$

(forward difference) (backward difference)

We take $\frac{\partial f}{\partial S}$ to be the average of the two

that is

$$\boxed{\frac{\partial f}{\partial S} = \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S}} \quad \text{— } ①$$

Next consider $\frac{\partial^2 f}{\partial S^2}$, the backward difference

at point (i,j) $= \left.\frac{\partial f}{\partial S}\right|_{(i,j)} = \frac{f_{i,j} - f_{i,j-1}}{\Delta S}$

Backward diff at $(i,j+1)$ $= \left.\frac{\partial f}{\partial S}\right|_{(i,j+1)} = \frac{f_{i,j+1} - f_{i,j}}{\Delta S}$

So we approximate

$$\left.\frac{\partial^2 f}{\partial S^2}\right|_{(i,j)} = \frac{\left.\frac{\partial f}{\partial S}\right|_{(i,j+1)} - \left.\frac{\partial f}{\partial S}\right|_{(i,j)}}{\Delta S}$$

$$\boxed{\frac{\partial^2 f}{\partial S^2} = \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{\Delta S^2}} \quad \text{— } ②$$

Also,

$$\boxed{\frac{\partial f}{\partial t} = \frac{f_{i+1,j} - f_{i,j}}{\Delta t}} \quad — (3)$$

Now we substitute three equations in the B-S-M PDE

$$\frac{f_{i+1,j} - f_{i,j}}{\Delta t} - r j \Delta S \left(\frac{f_{i,j+1} - f_{i,j-1}}{2 \Delta S} \right) + \frac{1}{2} \sigma^2 j^2 \Delta S^2 \left(\frac{f_{i,j+1} - 2 f_{i,j} + f_{i,j-1}}{\Delta S^2} \right)$$

$$= r f_{i,j}$$

where we have used $S = j \Delta S$

and $j = 1, \ldots, M-1$, $i = 0, \ldots N-1$

Rearranging we get

$$a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} = f_{i+1,j} \quad — (4)$$

where

$$a_j = \frac{1}{2} r j \Delta t - \frac{1}{2} \sigma^2 j^2 \Delta t$$

$$b_j = 1 + \sigma^2 j^2 \Delta t + r \Delta t$$

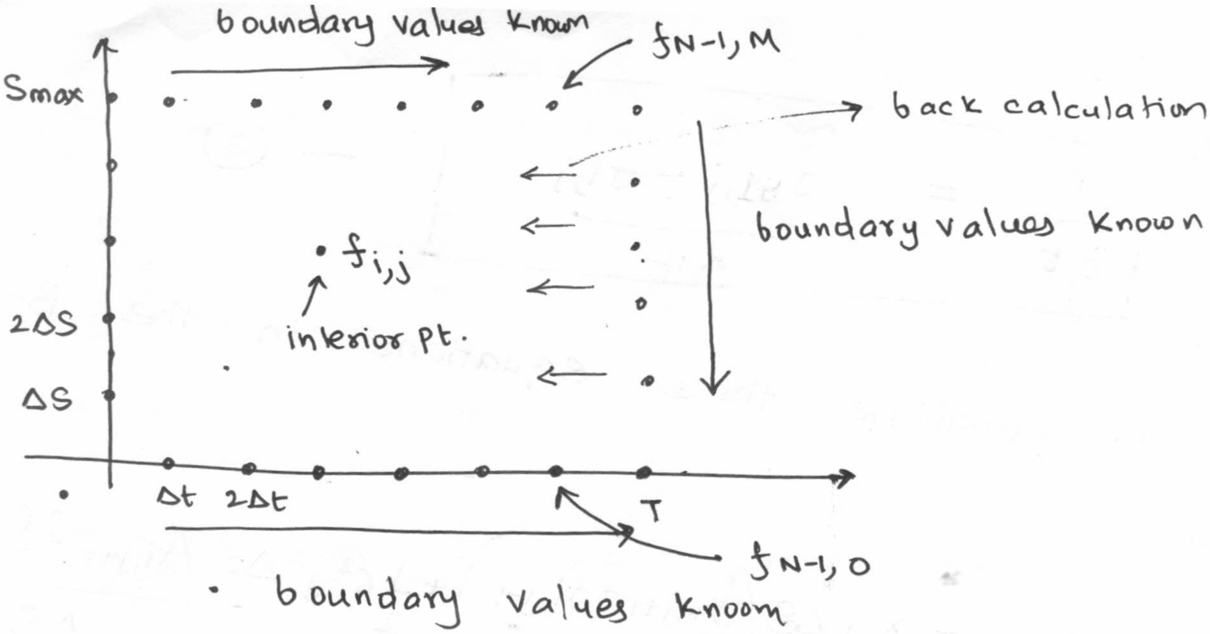
$$c_j = -\frac{1}{2} r j \Delta t - \frac{1}{2} \sigma^2 j^2 \Delta t$$

Now for boundary values

$$f_{N,j} = \max(K - j \Delta S, 0) \quad \text{for } j = 0, 1, \ldots M$$

$$f_{i,0} = K \quad \text{for } i = 0, 1, \ldots N \quad — (5)$$

$$f_{i,M} = 0 \quad \text{for } i = 0, \ldots N$$



Further more

$$\boxed{f_{N-1,0} = K, \quad f_{N-1,M} = 0} \quad — \quad (6)$$

We start from grid line corresponding to $t = T$ and back calculate

for $i = N-1$

$$a_j f_{N-1,j-1} + b_j f_{N-1,j} + c_j f_{N-1,j+1} = f_{N,j}$$

for $j = 1, 2, \ldots M-1$

These are $M-1$ equations and due to (6) there are $M-1$ unknowns $f_{N-1,1}, f_{N-1,2}, \ldots f_{N-1,M-1}$

Thus we can solve them. Now work your way backward (towards left vertical lines) to arrive at $f_{0,s}$ (where s is current stock price)

Also note
At any point if $f_{i,j} < K - j\Delta S$ replace $f_{i,j}$ by $K - j\Delta S$

Valuing American options by Monte Carlo simulations: A least squares approach

Lets assume that we have an American put option with the following parameters

$S(0) = 1.00$, $K = 1.10$, $T = 3$ yrs $r = 6\%$ p.a.

The option can be exercised only at $t=1$, $t=2$ and $t=3$. Lets also assume that the a Monte Carlo simulation generated the following 8 paths.

Path	t = 0	t = 1	t = 2	t = 3
1	1.00	1.09 (0.01)	1.08 (0.02)	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07 (0.03)	1.03 (0.07)
4	1.00	0.93 (0.17)	0.97 (0.13)	0.92 (0.18)
5	1.00	1.11	1.56	1.52
6	1.00	0.76 (0.34)	0.77 (0.33)	0.90 (0.20)
7	1.00	0.92 (0.18)	0.84 (0.26)	1.01 (0.09)
8	1.00	0.88 (0.22)	1.22	1.34

At time $t=3$ paths 3, 4, 6, 7 are in the money and should be ex. ~~be~~ ~~For~~ these Now, ~~paths~~ we look at $t=2$ and device a method to see if at $t=2$ the options ~~should be~~ that are in the money should be exercised or are they worth continuing. The paths that are in the money at $t=2$ are 1, 3, 4, 6, 7. We compute the value of continuing by a least squares approach.

Lets assume there is a relationship between the worth of the option (for continuing) and the stock price as

$$V = a + bS + cS^2 \quad \text{where } a, b, c$$

are constants to be determined. We have 5 observations for the 5 paths, as follows.

Paths	Observed value of continuing (discounted)	
1	0.00	(V_1)
3	$0.07 \times e^{0.06\times 1}$	(V_2)
4	$0.18 \times e^{-0.06\times 1}$	(V_3)
6	$0.20 \times e^{-0.06\times 1}$	(V_4)
7	$0.09 \times e^{-0.06\times 1}$	(V_5)

We use least square to minimize

$$\sum_{i=1}^{5} (V_i - a - bS_i - cS_i^2)^2$$ to determine

a, b, c. and we get $a=-1.07$, $b=2.983$, $C=-1.813$

Thus $V = -1.07 + 2.983S - 1.813S^2$

This gives the value at the 2 year point of continuing for paths 1, 3, 4, 6 7 as 0.0369, 0.0461, 0.1176, 0.1520 and 0.1565

We compare these values against value of exercising and note that paths 4, 6 7 should be exercised at $t=2$.

Next lets look at $t=1$. Paths 1, 4, 6, 7, 8 are in the money. Again as before we compute their worth of continuing by a least squares approach

Path	$t=1$		$t=2$		$t=3$	
→ 1	0.01		0.00	Dont ex	0.00	Dont ex
2	NA	Dont ex	0.00	Dont ex	0.00	Dont ex
3	NA		0.00		0.07	Ex
→ 4	0.17		0.13	Ex	0.00	
5	NA ~~0.34~~		0.00		0.00	
→ 6	0.34		0.33	Ex	0.00	
→ 7	0.18		0.26	Ex	0.00	
→ 8	0.22		0.00		0.00	

Again we compute the value of continuing for these paths (1, 4, 6, 7, 8) at $t=1$ assuming relationship

$V = k + lS + mS^2$. using observations

1	0
4	$0.13e^{-0.06}$
6	$0.33 \times e^{-0.06}$
7	$0.26e^{-0.06}$
8	0

We get by least squares

$k = 2.038$

$l = -3.35$

$m = 1.356$

Values for continuing for paths 1, 4, 6, 7 8 are thus 0.0139, 0.01092, 0.2866, 0.1175 and 0.1533. Based on this ~~for~~ we see that paths 4, 6, 7, 8 should be exercised at t=1. We finally get the following table

Paths	t = 1	t = 2	t = 3	
1	0.00	0.00	0.00	Dont Ex
2	0.00	0.00	0.00	Dont Ex
3	0.00	0.00	0.07	Ex t=3
4	0.17	0.00	0.00	Ex t=1
5	0.00	0.00	0.00	Dont ex
6	0.34	0.00	0.00	Ex t=1
7	0.18	0.00	0.00	Ex t=1
8	0.22	0.00	0.00	Ex t=1

Hence path 3 must be exercised at t=3 and paths 4, 6, 7, 8 at t=1 and the remaining should never be exercised.

Value of option at t=0

$$\frac{1}{8}\left[0.07 \times e^{-0.06\times3} + 0.17e^{-0.06\times1} + 0.34\times e^{-0.06\times1} + 0.18e^{-0.06\times1} + 0.22e^{-0.06\times1}\right]$$

$$= 0.1144$$