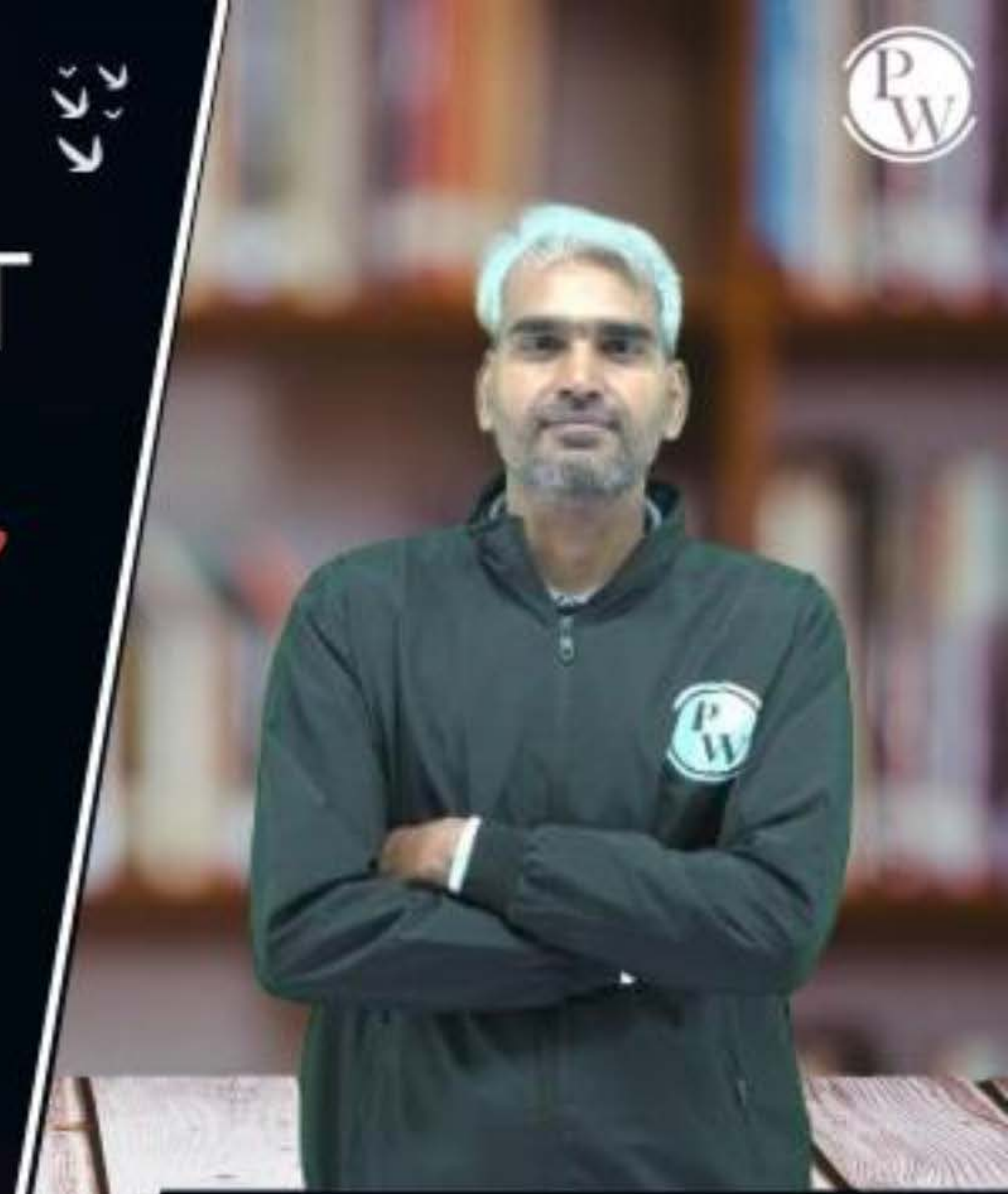
Computer Science & IT

Discrete Mathematics

Set Theory & Algebra

Lecture No. 12



By- Vishal Sir

Recap of Previous Lecture

Topic Lattice $[L, \vee, \wedge]$

Topic Properties of a lattice

Topic Hasse diagram

Slide

Topics to be Covered

{Types of Lattice}

- Topic: Hasse diagram
- Topic: Sublattice
- Topic: Bounded lattice
- Topic: Complements of an element in a lattcie
- Topic: Complemented lattice

Slide

Topic : Hasse Diagram / POSET Diagram

In a Hasse diagram of a POSET, (A, R)

1. There is a vertex corresponding to every element of set.

2. There is an edge from vertex a to vertex b only if a relates to b and there is no element x in the set such that a relates to x, and x relates to b.(Transitivity is implied in the Hasse diagram not represented explicitly)

   $(a,b) \in R$ $(a,x) \in R$ $(x,b) \in R$

3. No self-loop on the vertices (i.e. reflexivity is implied in the Hasse diagram not represented explicitly).

4. It is not directed but it uses implied upward orientation.

if $(a,b) \in R \Rightarrow$ upper lower

b

a

Slide

Topic : Hasse Diagram / POSET Diagram

Draw the hasse diagram for the following POSETs

1) $(\{-1, 0, 2.5, 4, 6\}, \leq)$
2) $(D_6, \div)$
3) $(D_{12}, \div)$
4) $(\{2, 3, 4, 6\}, \div)$
5) $(\{2, 3, 6, 12\}, \div)$
6) $(\{1, 2, 3, 4, 6, 9\}, \div)$

Slide

Topic : Hasse Diagram / POSET Diagram

Draw the hasse diagram for the following POSET

$(\{-1, 0, 2.5, 4, 6\}, \leq)$

It is a totally ordered set

$(\{-1, 0, 2.5, 4, 6\}, \leq)$ is a totally ordered set.

Hasse diagram for a TOSET will always look like a linear chain

$\therefore$ Totally ordered set are also known as "Linearly ordered Set"

Slide

Q: Let $A = \{a, b, c, d\}$

How many total orders (total order relation) are possible on set A.

Solu^n:- Hasse diagram of a totally ordered set will always look like a linear chain.

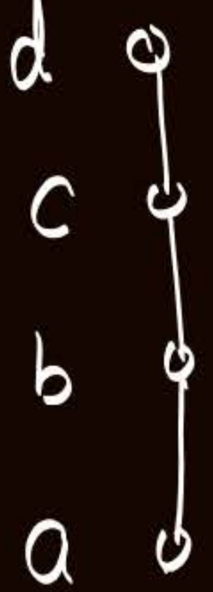
- Hasse diagram for a TOSET with 4 elements will look like

← This 4 vertices are w.r.t. 4 elements of set.

We can place 4 elements of the set in 4! ways, in Hasse diagram of 4 vertices

$\therefore$ $\underline{\underline{4!}}$ total order relation are possible

① let w.r.t total order = R_1



Total order

$\Rightarrow R_1 = \{(a,a), (a,b), (a,c), (a,d),$
$(b,b), (b,c), (b,d),$
$(c,c), (c,d),$
$(d,d)\}$

② Hasse diagram w.r.t. total order R_2

$\Rightarrow R_2 = \{(b,b), (b,c), (b,a), (b,d),$
$(c,c), (c,a), (c,d),$
$(a,a), (a,d),$
$(d,d)\}$

Similarly $4!$ different total order relation are possible on set $A = \{a,b,c,d\}$

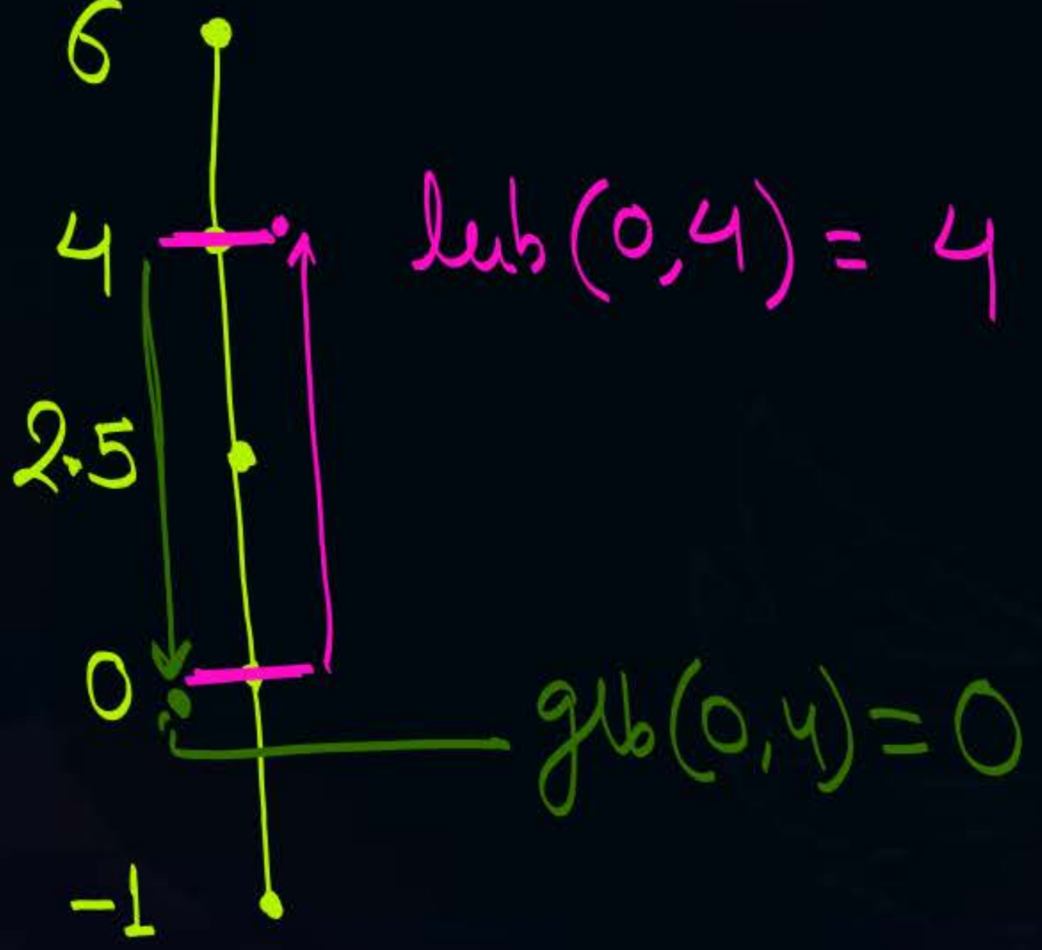
Note:

If A is any set of cardinality 'n', then $n!$ total order relation are possible on set A.

Topic : Hasse Diagram / POSET Diagram

Draw the hasse diagram for the following POSET

$(\{-1, 0, 2.5, 4, 6\}, \leq)$

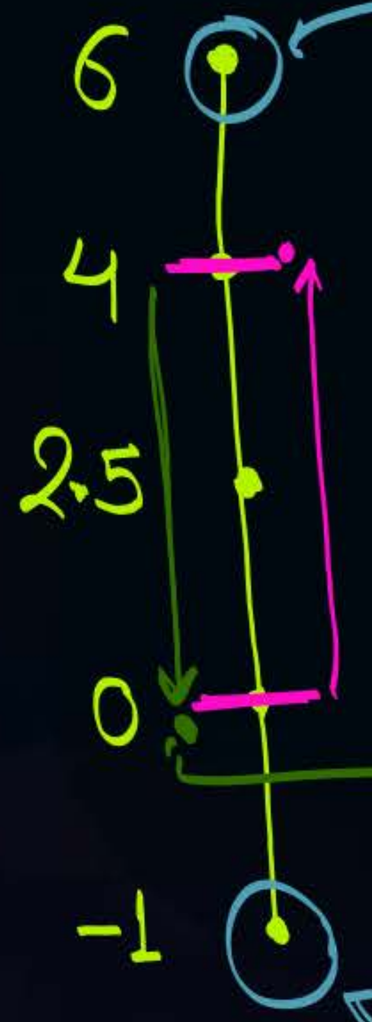


Slide

Topic : Hasse Diagram / POSET Diagram

Draw the hasse diagram for the following POSET

$(\{-1, 0, 2.5, 4, 6\}, \leq)$



Maximal element

{Unique maximal, ∴ also maximum element}

$lub(0,4) = 4$

$glb(0,4) = 0$

Minimal element

{Unique Minimal, ∴ It is minimum element as well}

lub as well as glb exist for every Pair of elements.

∴ It is a lattice

i.e Join semi lattice as well as Meet semi lattice

Slide

Topic : Hasse Diagram / POSET Diagram

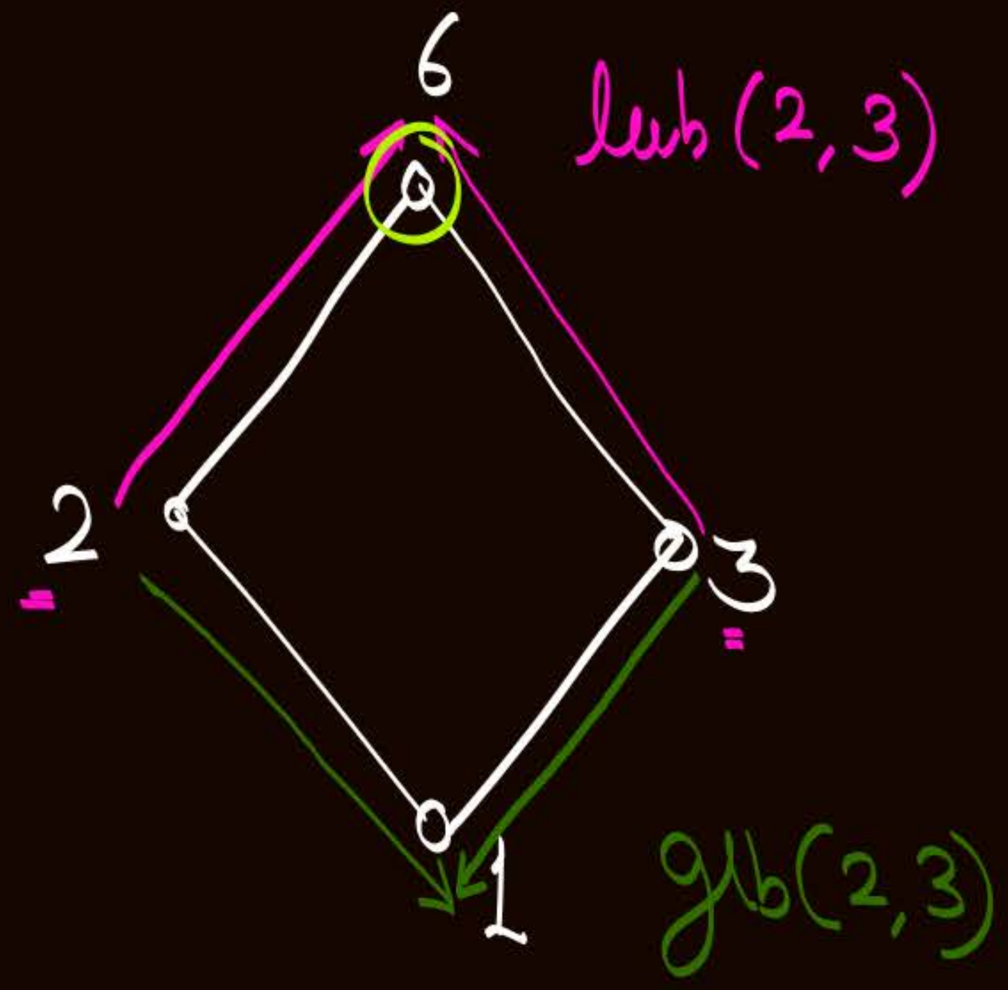
Draw the hasse diagram for the following POSET

$(D6, \div)$

Slide

$(D_6, \div)$

$D_6 = \{1, 2, 3, 6\}$

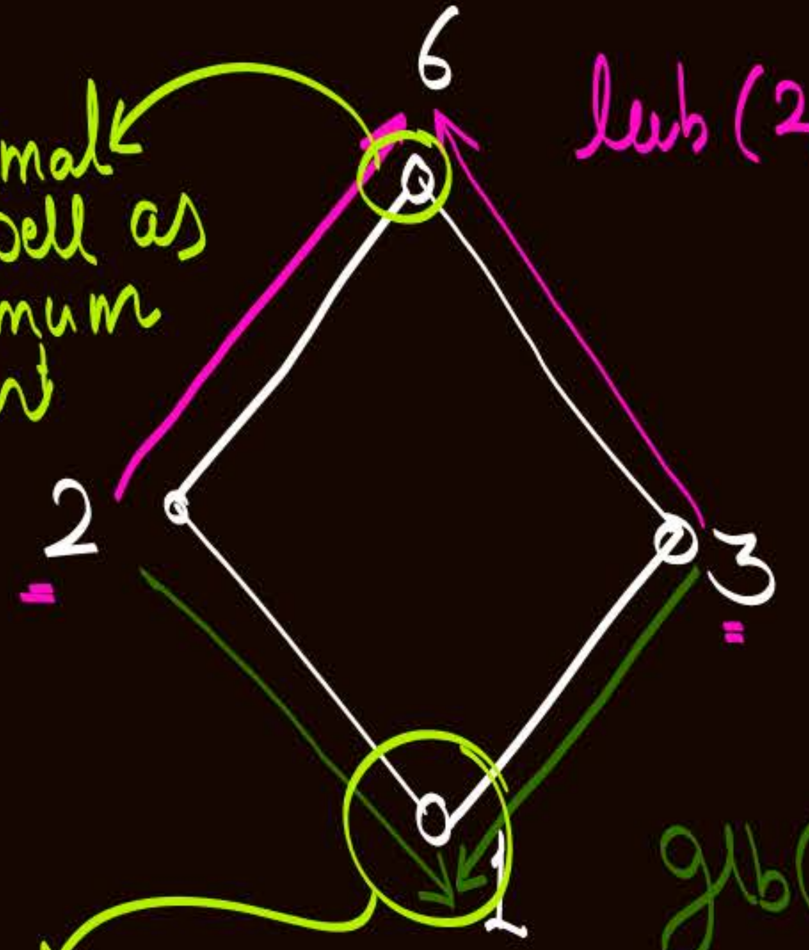


$(D_6, \div)$

$D_6 = \{1, 2, 3, 6\}$

Maximal as well as Maximum element

lub(2,3)



glb(2,3)

Minimal as well as Minimum element

lub as well as glb exist for every pair of elements,

$\therefore$ It is a lattice

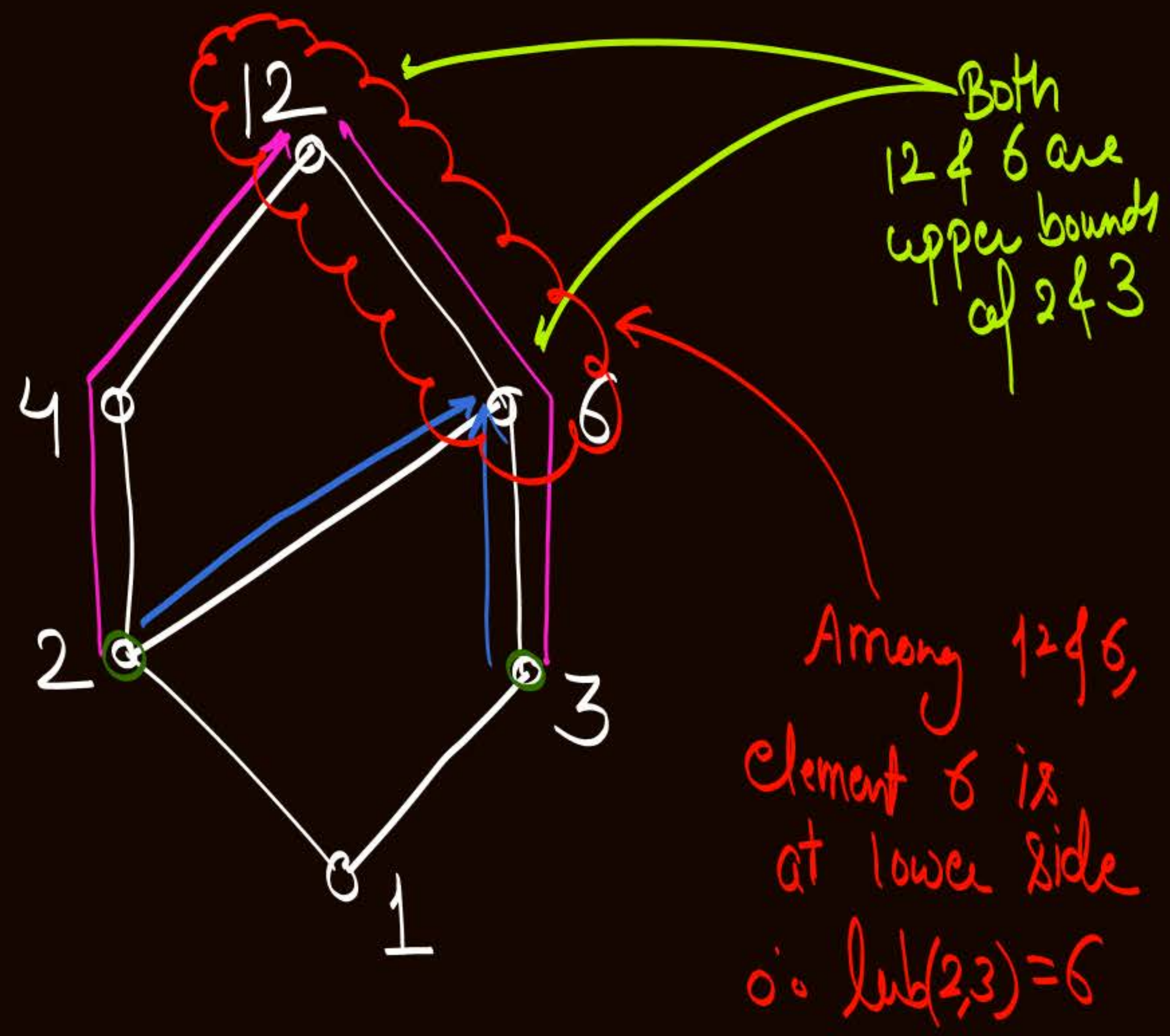
- i.e. Join semi lattice as well as Meet semi lattice

Draw the hasse diagram for the following POSET

$(D12, \div)$

$(D_{12}, \div)$

$D_{12} = \{1, 2, 3, 4, 6, 12\}$

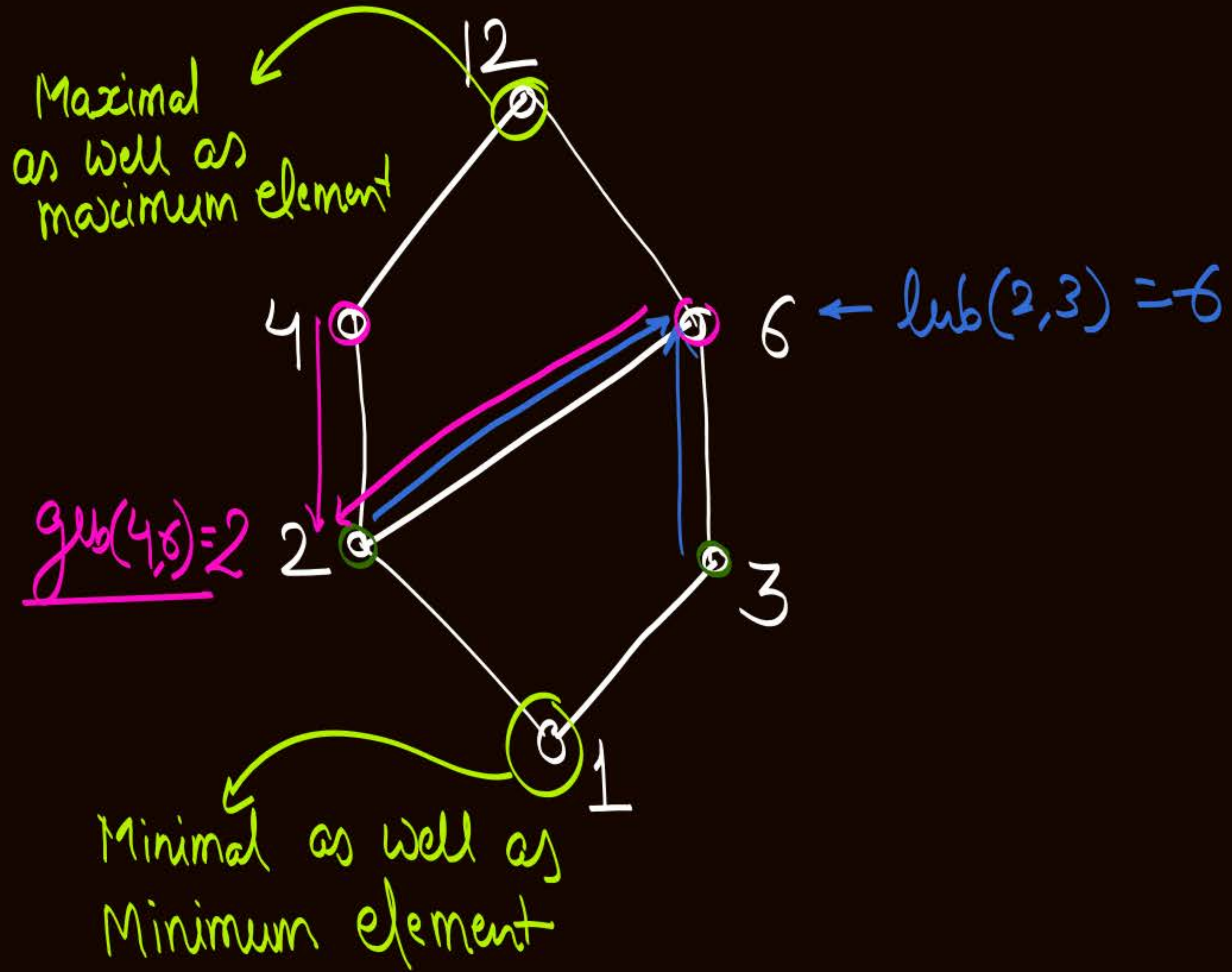


$(D_{12}, \div)$

$D_{12} = \{1, 2, 3, 4, 6, 12\}$

Join semi lattice
as well as
Meet semi lattice

$\therefore$ lattice



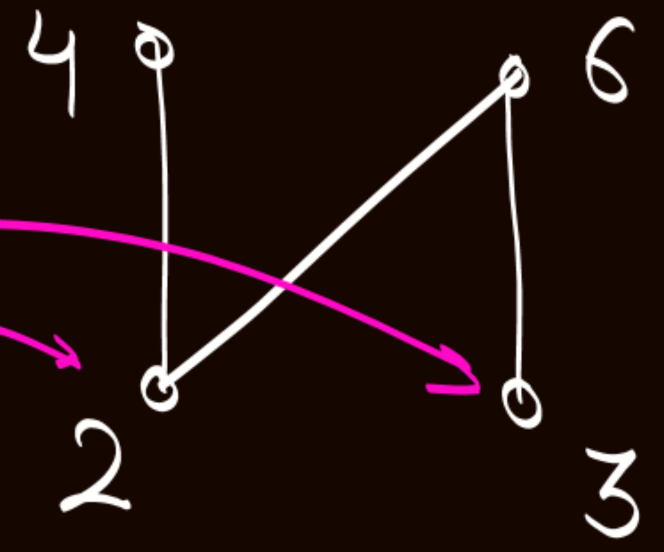
Topic : Hasse Diagram / POSET Diagram

Draw the hasse diagram for the following POSET

$(\{2,3,4,6\},\div)$

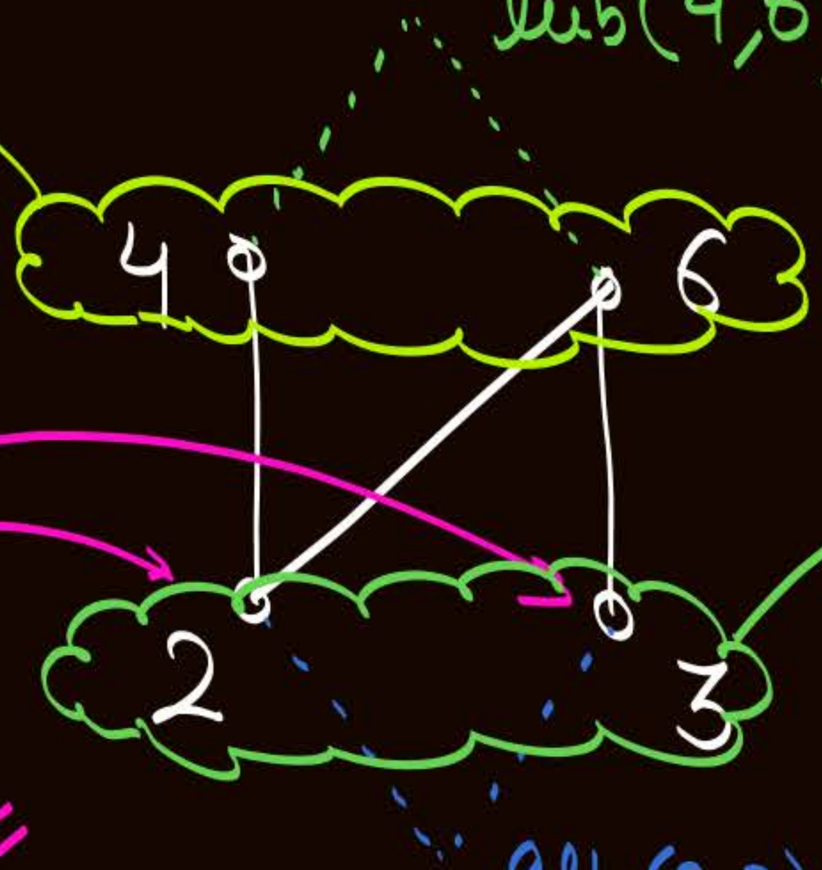
Slide

$(\{2, 3, 4, 6\}, \div)$



if elements are not comparable, then they may be at same level

$(\{2,3,4,6\}, \div)$



$lub(4,6)$ = does not exist

∴ Not a Join semi lattice

Both are Maximal elements

↓

Two or more maximal

∴ No Maximum element

Both are Minimal elements

↓

Two or more minimal

∴ No minimum element

$glb(2,3)$ = does not exist

∴ Not a Meet semi lattice

if elements are not Comparable, then they may be at same level

Neither a Join semi lattice nor a Meet semi lattice

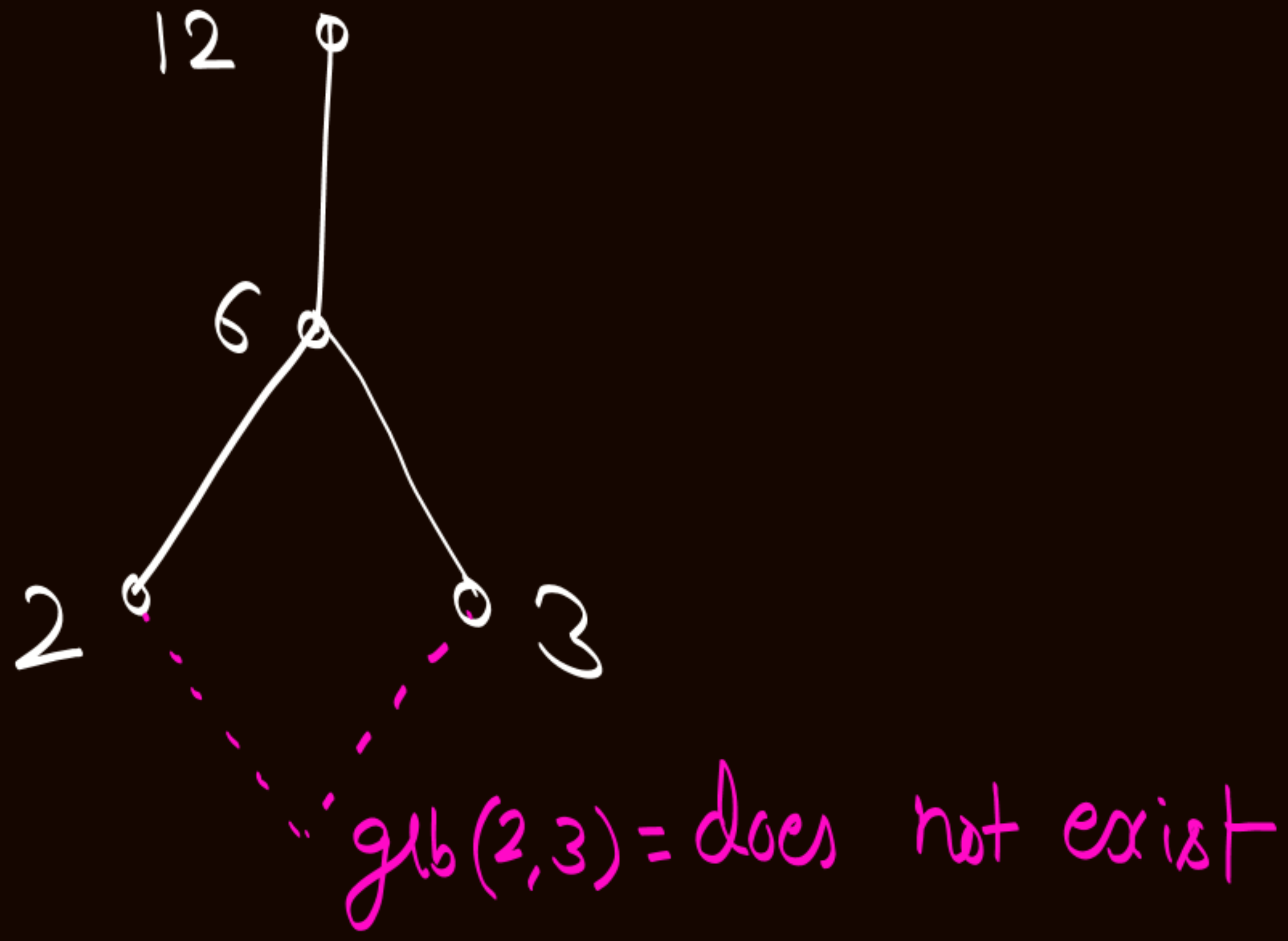
Topic : Hasse Diagram / POSET Diagram

Draw the hasse diagram for the following POSET

$(\{2,3,6,12\},\div)$

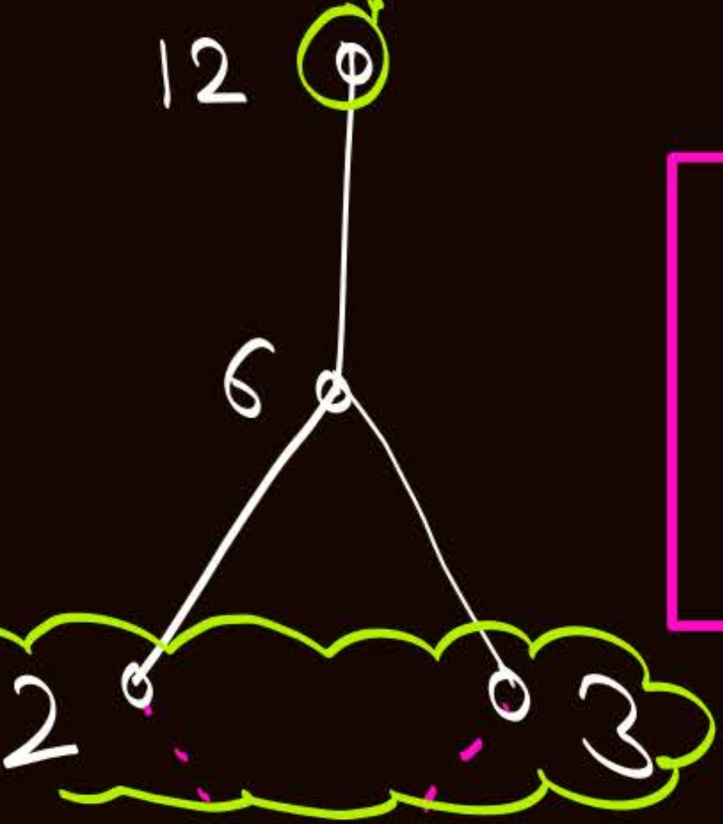
Slide

$(\{2, 3, 6, 12\}, \div)$



$(\{2, 3, 6, 12\}, \div)$

It is a Join semi lattice but, not a Meet semi lattice



Maximal as well as Maximum element

lub exist for every pair of elements, $\therefore$ It is a Join semi lattice

Two minimal elements

$\downarrow$

$\therefore$ No minimum element

glb(2,3) = does not exist

$\downarrow$

$\therefore$ Not a Meet semi lattice

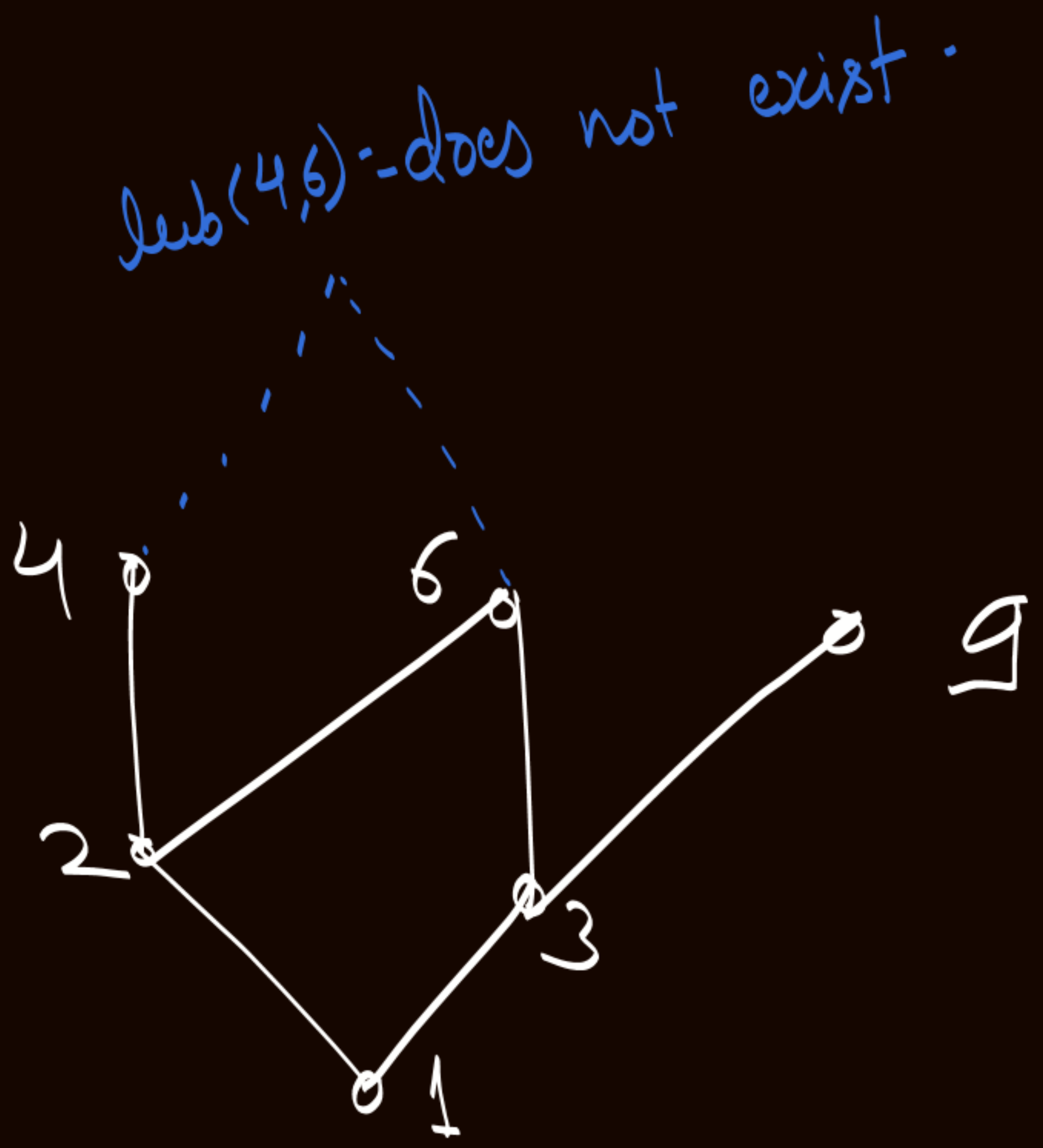
Topic : Hasse Diagram / POSET Diagram

Draw the hasse diagram for the following POSET

$(\{1,2,3,4,6,9\}, \div)$

Slide

$(\{1, 2, 3, 4, 6, 9\}, \div)$



$(\{1, 2, 3, 4, 6, 9\}, \div)$

lub(4,6) - does not exist.

∴ Not a Join semi lattice

Three maximal elements

∴ No maximum element

glb exist for every Pair of elements,
∴ It is a Meet semi lattice

Minimal as well as Minimum element

It is not a Join semi lattice, but it is a Meet semi lattice

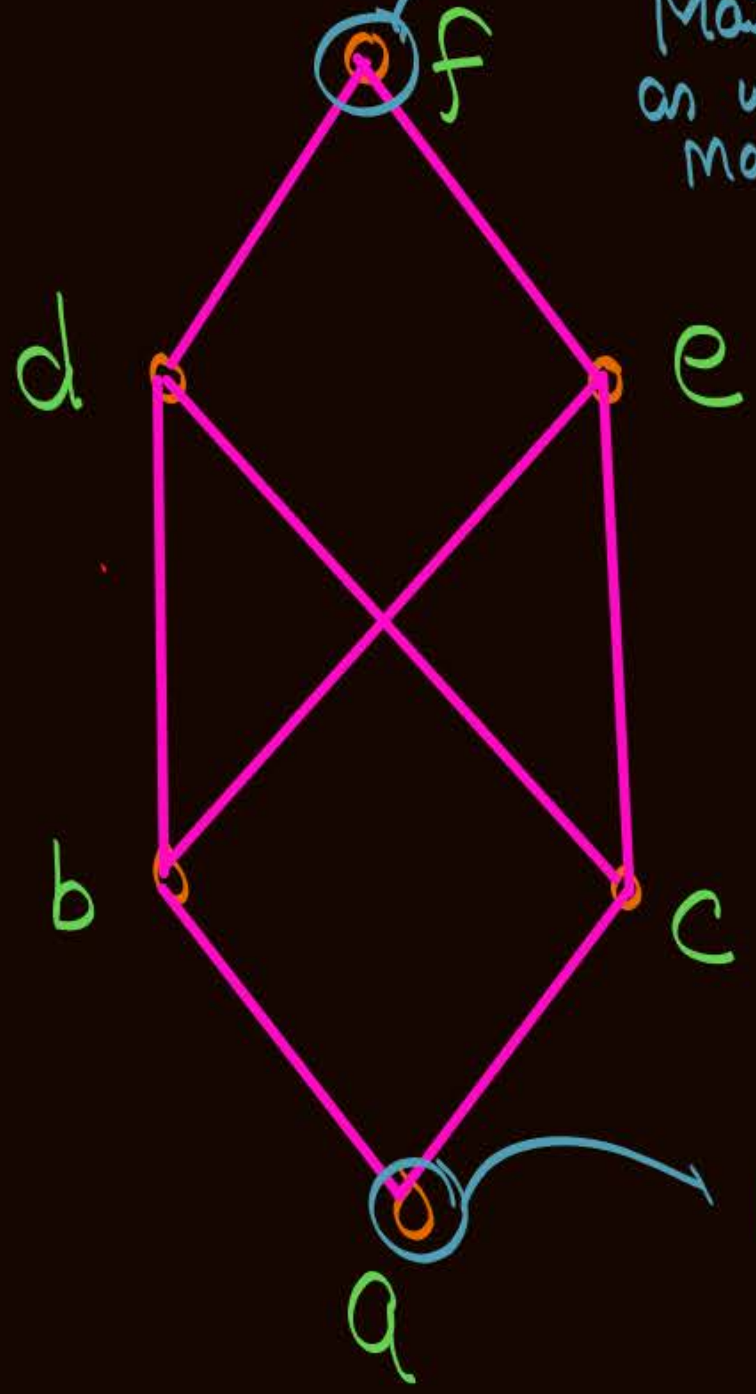
Note:-

1. If there exist two or more maximal elements in a POSET, then it can not be a Join semi lattice

2. If there exist two or more minimal elements in a POSET, then it can not be a Meet semi-lattice

NOTE

eg:



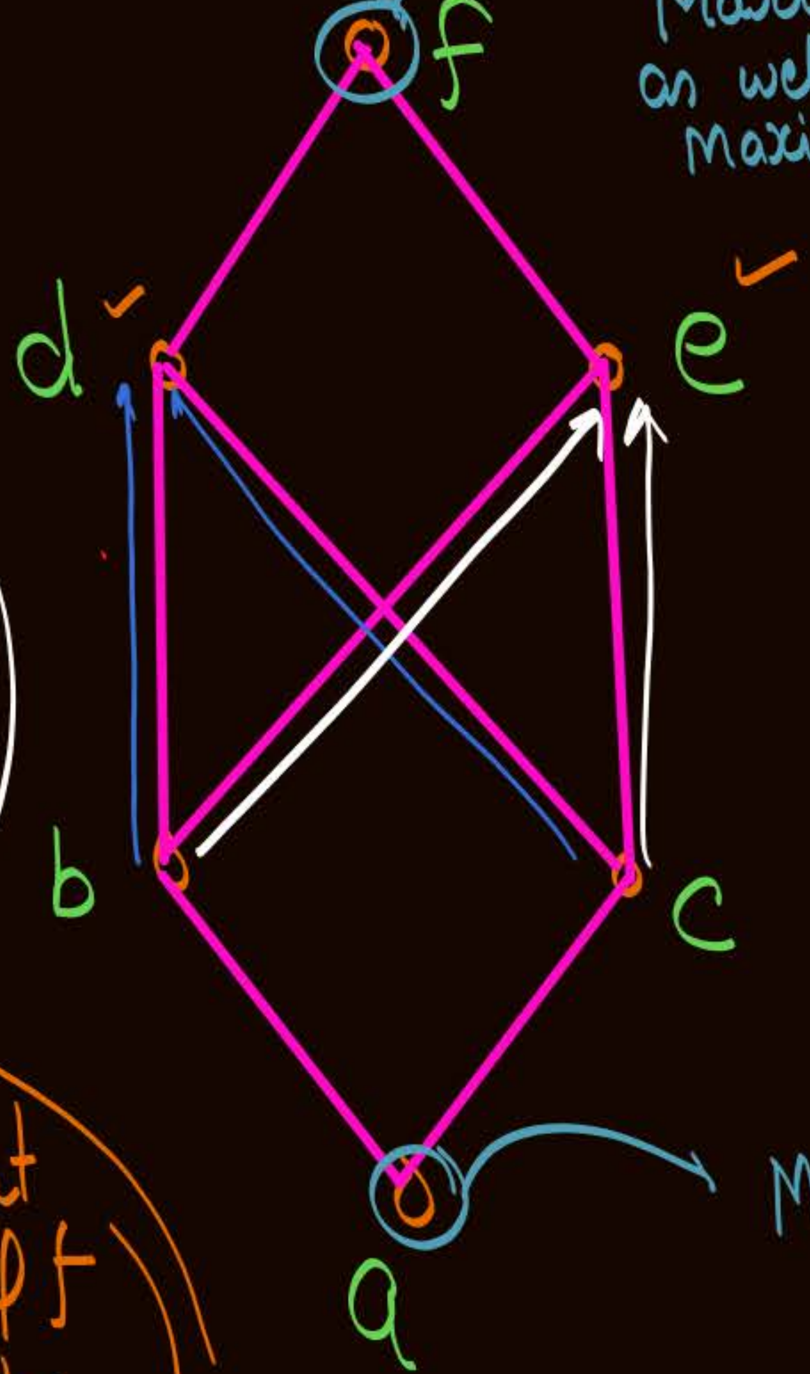
Maximal as well as maximum element

Minimal as well as Minimum element

$R = \{ \underline{(a,a), (b,b), (c,c), (d,d), (e,e), (f,f)}$
$(a,b), (a,d), (a,e), (a,f), (a,c)$
$(b,d), (b,e), (b,f),$
$(c,d), (c,e), (c,f),$
$(d,f), (e,f) \}$

NOTE

eg:



Maximal as well as maximum element

Minimal as well as Minimum element

d, e, & f all are upper bounds of b & c

Out of them d & e are at lower side of f
$\therefore$ f can not be least upper bound

d & e are not Comparable, $\therefore$ we can not decide the lub of (b & c)

We can not decide among 'd' & 'e'

lub(b, c) = does not exist

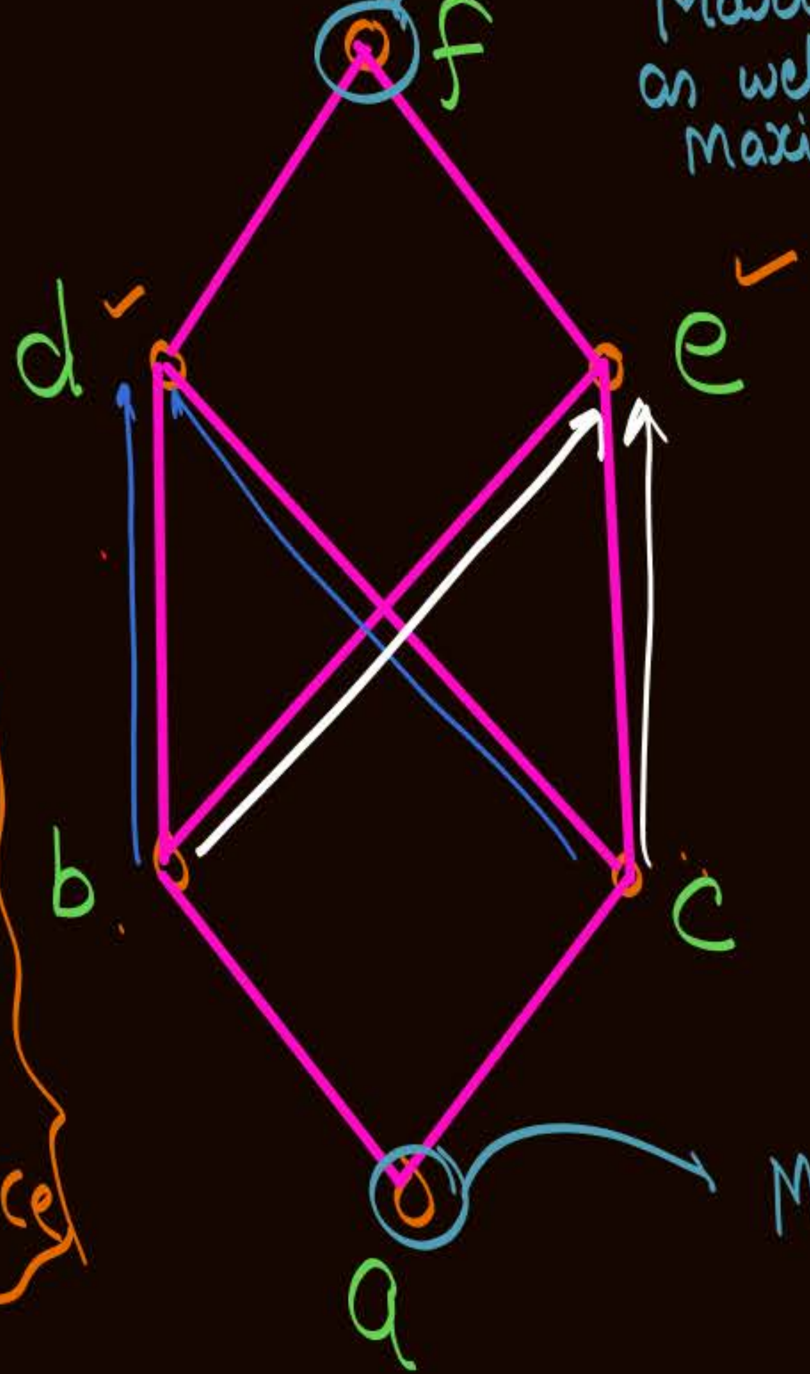
└ In a lattice lub as well as glb should exist for every pair of elements & it must be unique

└ $\therefore$ given POSET is not a lattice

Similarly, glb(d, e) = does not exist.

NOTE

eg:



Maximal as well as maximum element

Minimal as well as Minimum element

* Minimum as well as maximum element exist in this POSET, but this POSET is not a lattice

We can not decide among 'd' & 'e'

lub(b, c) = does not exist

└ In a lattice lub as well as glb should exist for every pair of elements & it must be unique

└ ∴ given POSET is not a lattice

similarly, glb(d, e) = does not exist.

2 mins Summary

Topic Hasse diagram

Topic Sublattice

Topic Bounded lattice

Topic Complements of an element in a lattcie

Topic Complemented lattice

Slide

THANK - YOU

Slide