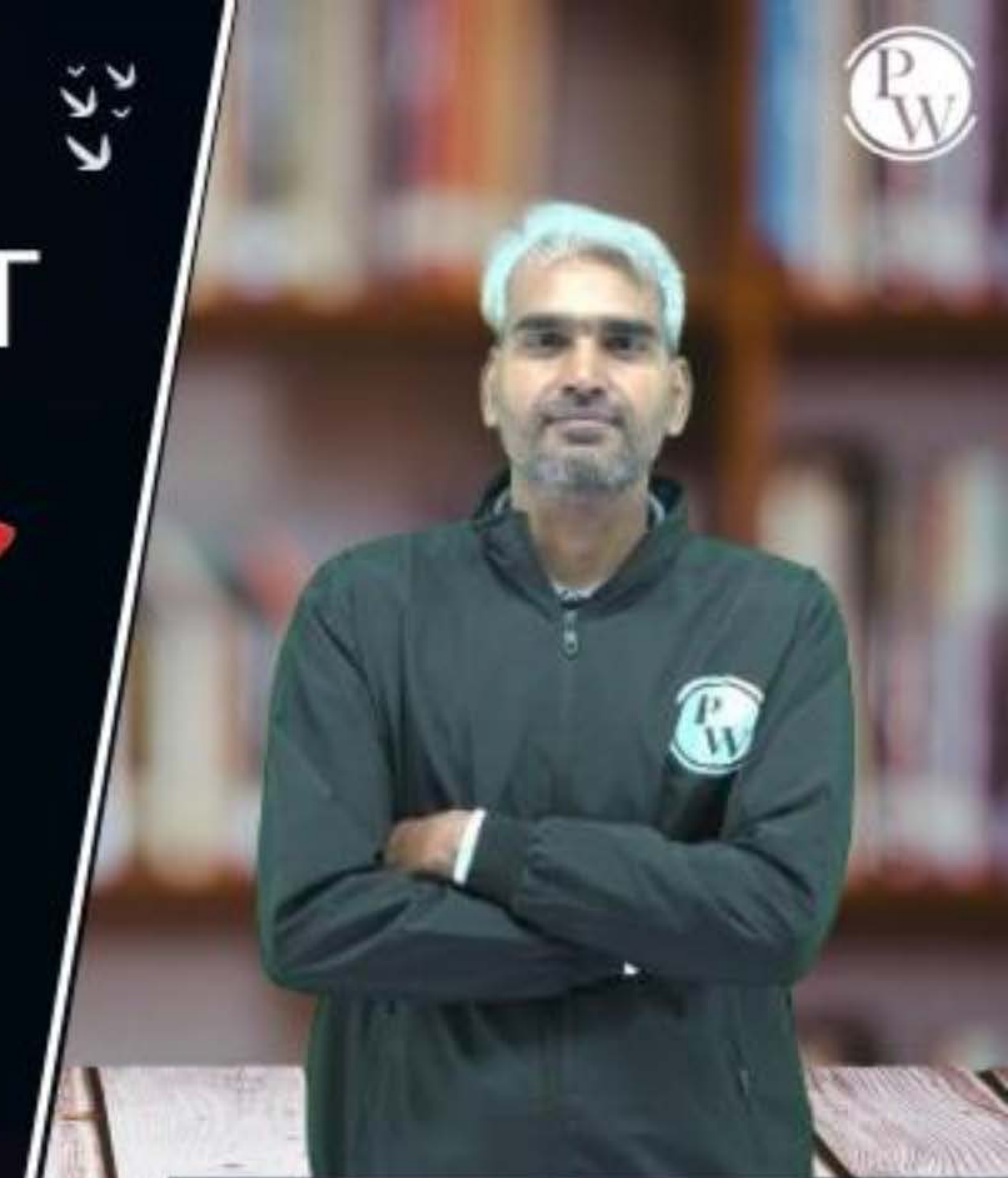
Computer Science & IT

Discrete Mathematics

Set Theory & Algebra

Lecture No. 08



By- Vishal Sir

Recap of Previous Lecture

Topic	Composite of two relations
Topic	Reflexive, symmetric and transitive closure
Topic	Equivalence relation
Topic	Equivalence class
Topic	Partition of a set

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Topics to be Covered

Topic Partition of a set

Topic Number of equivalence relation

Topic Bell number

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Topic : Equivalence relation

A relation R on set A is said to be an Equivalence relation if and only if relation is

1. Reflexive
2. & Symmetric
3. & Transitive

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eg. let $A = \{1, 2, 3\}$

① $\Delta_A = R_1 = \{(1,1), (2,2), (3,3)\}$ Reflexive ✓ Symmetric ✓ Transitive ✓ $\Rightarrow \therefore$ Equivalence Relⁿ

Diagonal relation on set A is the smallest equivalence relation on set A

② $A \times A = R_2 = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

A x A is the largest equivalence relation on set A

Reflexive ✓ Symmetric ✓ Transitive ✓

③ $R_3 = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$ Reflexive ✓ Symmetric ✓ transitive ✓ $\Rightarrow \therefore$ Equivalence Relⁿ

④ $R_4 = \{(1,1), (2,2), (3,3) (1,3) (3,1)\}$ is Equivalence relⁿ

⑤ $R_5 = \{(1,1), (2,2), (3,3) (2,3), (3,2)\}$ is Equivalence Relⁿ

$R_6 = \{ (1,1), (2,2), (3,3), (1,2), (2,1), (1,3), (3,1) \}$

R_6 is Reflexive ✓
Symmetric ✓
but not transitive

$\therefore (2,3)$, but $(2,3) \notin R_6$
$\therefore$ Not transitive

to make it transitive if we add (2,3) then for symmetricity we will have add (3,2) as well

And it will become $A \times A$.

$\therefore$ No other equivalence rel^n possible on set $A = \{1, 2, 3\}$

Topic : Equivalence Class

Let 'R' be an equivalence relation on set A,
for any element $x \in A$ the equivalence class of
element 'x' w.r.t. equivalence relation R can be
denoted by $[x]$, and it is defined as.

$$[x] = \{ y \mid x R y \}$$

i.e. $(x, y) \in R$

i.e. Equivalence class of element 'x' is a set of all elements which are related with x

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eg. let $A = \{1, 2, 3, 4, 5\}$
and let R is an Equivalence relation on set A.

$\rightarrow$ $R = \{ \underbrace{(1,1), (2,2), (3,3), (4,4), (5,5)}_{\text{reflexive}}, \underbrace{(1,3), (3,1)}, \underbrace{(4,5), (5,4)} \}$

symmetric

as well as transitive

define Equivalence of every element of set A w.r.t. equivalence Relⁿ 'R'

$[1] = \{1, 3\}$

$[2] = \{2\}$

$[3] = \{3, 1\} = \{1, 3\}$

$[4] = \{4, 5\}$

$[5] = \{5, 4\} = \{4, 5\}$

same

same

There are three distinct equivalence Classes for elements of set A

i.e. $\{1,3\}$ & $\{2\}$ & $\{4,5\}$

Note :-

1. Equivalence class of element x {i.e $[x]$} may be same as equivalence class of element y {i.e. $[y]$} even if $x \neq y$.

{ in the above eg. $1 \neq 3$, but $[1] = [3] = \{1, 3\}$ }

2. The set of all distinct equivalence classes of elements of set A w.r.t. an Equivalence relation R Creates a partition of set A.

i.e $\{\underbrace{\{1,3\}, \{2\}, \{4,5\}}\}$ in above example is a partition of set $A = \{1,2,3,4,5\}$

set of distinct equivalence classes of elements of set A

Topic : Partition of a set

Partition of a set A is a set of non-empty subsets of set A such that each element of set A is present in exactly one of those non-empty subsets.

(or)

let A be a non-empty set, and $A_1, A_2, A_3, \ldots A_k$ are non-empty subsets of set A, then

$\{A_1, A_2, A_3, \ldots, A_k\}$ is a partition of set A

if and only if ① $A_i \cap A_j = \emptyset$, $\forall\, i \neq j$

& ② $\bigcup_{i=1}^{k} A_i = A$

{i.e. $A_1 \cup A_2 \cup \ldots \cup A_k = A$}

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Q: Let $A = \{1, 2, 3, 4, 5\}$

Which of the following is/are partitions of set A.

ⓐ $\{\{1,2\}, \{3,4\}, \{4,5\}\}$ $\quad \{3,4\} \cap \{4,5\} = 4 \neq \emptyset$ {∴ Not a partition}

ⓑ $\{\{1,2\}, \{3\}, \{4,5\}, \{\ \}\}$ Empty subsets are not allowed {∴ Not a Partition}

ⓒ $\{\{1\}, \{2,3\}, \{4\}\}$ $\quad \{1\} \cup \{2,3\} \cup \{4\} = \{1,2,3,4\} \neq A$ {∴ Not a Partition}

ⓓ $\{\{1,2\}, \{3,4,5\}\}$ $\quad \left.\begin{array}{l} \cap = \emptyset \\ \cup = A \end{array}\right\}$ ✓ ∴ Partition of A

ⓔ $\{\{1,2\}, \{3\}, \{4\}, \{5\}\}$ $\quad \left.\begin{array}{l} \cap = \emptyset \ \forall i \neq j \\ \& \cup = A \end{array}\right\}$ ∴ Partition of A

There may be more than one Partition of same set

H.W. Find the no. of Partitions of set A:

1. when $|A| = 0$
2. when $|A| = 1$
3. when $|A| = 2$
4. when $|A| = 3$
5. when $|A| = 4$
6. when $|A| = 5$
7. when $|A| = 6$

Q.1 Note: Let $|A| = 0$

i.e. $A = \{\ \}$

How many partitions of set A are possible

Solⁿ. $\underbrace{\text{Partition of } A = \{\ \}}$

this is the only partition Possible w.r.t set A, if $|A| = 0$

No non-empty subset of set $A = \{\ \}$

$\therefore$ No element will be present in the set w.r.t. partition.

Partition of a set can be an Empty set, but Empty set can never be a member of partition

$$\boxed{\text{if } |A| = 0, \text{ then Number of Partitions of set } A = 1}$$

Q.2 :- Let $A = \{a\}$, i.e. $|A| = 1$
then find all partitions of set A.

$$\underbrace{\text{Partition of set } A = \{\ \{a\}\ \}}$$

w.r.t. set of cardinality = 1, Only one non-empty subset is possible.

this is the only partition possible w.r.t. set A s.t. $|A| = 1$

$$\boxed{\text{if } |A| = 1, \text{ then Number of Partitions of set } A = 1}$$

Q.3 Let $A = \{a, b\}$, i.e. $|A| = 2$

Find all partitions of set A.

Soln. Partitions of set A are $= \underbrace{\{\{a\}, \{b\}\} \ \& \ \{\{a, b\}\}}_{\text{This are the only two partition possible w.r.t. set 'A'}}$

$$\boxed{\text{if } |A| = 2, \text{ then Number of Partitions of set } A = 2}$$

Q.3 Let $A = \{a, b\}$, i.e. $|A| = 2$

Find all partitions of set A.

Soln. Partitions of set A are $= \{\underbrace{\{a\}, \{b\}}_{\text{two subsets of A of size = 1}}\}$ & $\{\underbrace{\{a, b\}}_{\text{one subset of A of size = 2}}\}$

$|A| = 2 \rightarrow$

$\rightarrow 1 + 1$ — Two subsets of size = 1

$\{a, b\} \quad \{b, a\}$

$^2C_1$ and $^1C_1$ $= \frac{^2C_1 \times {^1C_1}}{2!} = \frac{2 \times 1}{2} = 1$

{Only '1' such Partition of set A is Possible which Contains two subsets of size = 1}

$\rightarrow 2$

Only one subset of size = 2

$\{\_, \_\}$ $\quad {^2C_2} = 1$

{Only one partition of set A is possible that Contains one subset of size = 2}

{Total No. of Partitions of a set A such that $|A| = 2$ is $1 + 1 = 2$

Q.14 let $A = \{a, b, c\}$, i.e $|A| = 3$

How many partitions of set A are possible

Partitions of set A are $= \{\{a\}, \{b\}, \{c\}\}$, $\{\{a\}, \{b, c\}\}$ & $\{\{b\}, \{a, c\}\}$ & $\{\{c\}, \{a, b\}\}$, $\{\{a, b, c\}\}$

if $|A| = 3$, then 5 different partitions are possible of set "A"

$|A| = 3 \Rightarrow \underset{\sim\sim}{1+2} \Rightarrow$

$\rightarrow 1 + 1 + 1$

$\{a\}, \{b\}, \{c\} \Rightarrow \frac{{}^3C_1 * {}^2C_1 * {}^1C_1}{3!} = 1.$

{Only one partition of set A is possible that contains three subsets of size = 1}

$\rightarrow 1 + 2$

$\{\_\} \; \& \; \{\_, \_\} \Rightarrow {}^3C_1 * {}^2C_2 = 3 * 1 = 3$

{Three different partitions of set A are possible that contains one subset of size = 1 & another subset of size = 2}

$\rightarrow 3$

$\{\_, \_, \_\} = {}^3C_3 = 1$

Total No. of partition $= 1 + 3 + 1 = \underline{5}$

Q.5 Let $A = \{a, b, c, d\}$, i.e. $|A| = 4$

How many partitions of set A are possible

Partitions of set A are = $\{\{a\},\{b\},\{c\},\{d\}\}$

$\{\{a\}, \{b\}, \{c,d\}\}$

$\{\{a\}, \{c\}, \{b,d\}\}$

$\{\{a\}, \{d\}, \{b,c\}\}$

$\{\{b\}, \{c\}, \{a,d\}\}$

$\{\{b\} \{d\}, \{a,c\}\}$

$\{\{d\}, \{c\}, \{a,b\}\}$

$\{\{a\}, \{b,c,d\}\}$

$\{\{b\}, \{a,c,d\}\}$

$\{\{c\}, \{a,b,d\}\}$

$\{\{d\}, \{a,b,c\}\}$

$\{\{a,b\}, \{c,d\}\}$

$\{\{a,c\}, \{b,d\}\}$

$\{\{a,d\}, \{b,c\}\}$

$\{\{a,b,c,d\}\}$

$|A| = 4 = 1 + 3 \Rightarrow$

$1 + 1 + 1 + 1 \Rightarrow \dfrac{{}^4C_1 * {}^3C_1 * {}^2C_1 * {}^1C_1}{4!} = 1$

$1 + 1 + 2 = \dfrac{{}^4C_1 * {}^3C_1 * {}^2C_2}{2!} = \dfrac{4*3*1}{2} = 6$

$\{a\}\ \{b\}\ \{c,d\}$

$\{b\}\ \{a\}\ \{c,d\}$

$1 + 3 \Rightarrow {}^4C_1 * {}^3C_3 = 4 * 1 = 4$

$2 + 2 = \dfrac{{}^4C_2 * {}^2C_2}{2!} = \dfrac{6*1}{2} = 3$

$\{a,b\}\ \{c,d\}$

$\{c,d\}\ \{a,b\}$

$4 \Rightarrow {}^4C_4 = 1$

Total No. of Partitions of A

$= 1 + 6 + 4 + 3 + 1$

$= 15$

Q.6 $A = \{a, b, c, d, e\}$ i.e. $|A| = 5$

How many Partitions of set A are possible

$|A| = 5 = 1 + 4 \Rightarrow$

$1+1+1+1+1 = 1$

$1+1+1+2 = \frac{{}^5C_1 * {}^4C_1 * {}^3C_1 * {}^2C_2}{3!} = 10$

$1+1+3 = \frac{{}^5C_1 * {}^4C_1 * {}^3C_3}{2!} = 10$

$1+2+2 = {}^5C_1 * \frac{{}^4C_2 * {}^2C_2}{2!} = 15$

$1+4 = {}^5C_1 * {}^4C_4 = 5$

$2+3 = {}^5C_2 * {}^3C_3 = 10$

$5 = {}^5C_5 = 1$

Total no. of partition

$$\begin{array}{r} 1 \\ +10 \\ +10 \\ +15 \\ +5 \\ +10 \\ +1 \\ \hline 52 \end{array}$$

Q.7. let $A = \{a, b, c, d, e, f\}$, i.e $|A| = 6$ $\{$ H.W. $\}$

How many Partitions are possible of set A.

NOTE

(1) If we know the equivalence relation on set A, then we can obtain the partition of set A w.r.t. given equivalence relation

Partition of set A can be obtained by defining a set of all distinct equivalence classes of elements of set A. w.r.t. given equivalence relation eg: Let $A = \{1, 2, 3, 4, 5\}$

& Equivalence relⁿ $\underline{R} = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,2), (2,1), (1,3), (3,1), (2,3), (3,2), (4,5), (5,4)\}$

$[1] = \{1, 2, 3\}$

$[2] = \{2, 1, 3\} = \{1, 2, 3\}$

$[3] = \{3, 1, 2\} = \{1, 2, 3\}$

$[4] = \{4, 5\}$

$[5] = \{5, 4\} = \{4, 5\}$

two distinct equivalence classes.

$\therefore$ Partition of A, w.r.t equivalence relⁿ R

$= \{ \{1, 2, 3\}, \{4, 5\} \}$

NOTE

(2) If we know the Partition of set A, then we can obtain the equivalence relation on set A Corresponding to the given partition

If we perform the self cross product of subsets in the Partition of set A & if we union the result of all those Cartesian products, then the result will be required equivalence relation.

eg: Let $A = \{1, 2, 3, 4, 5\}$ ✓

and $\{\{1,2,3\}, \{4,5\}\}$ is a partition of set A.

Obtain the equivalence relⁿ on set A Corresponding to the given Partition.

Solⁿ

$$\{1,2,3\} \times \{1,2,3\} \cup \{4,5\} \times \{4,5\}$$

$$\Downarrow$$

$$\{(1,1),(1,2),(1,3),(2,1)(2,2)(2,3),(3,1),(3,2),(3,3)\} \cup \{(4,4),(4,5),(5,4),(5,5)\}$$

$$= \{(1,1),(2,2),(3,3),(4,4),(5,5),(1,2),(1,3),(2,1),(2,3),(3,1),(3,2),(4,5),(5,4)\}$$

For a given partition, there unique equivalence relation, and for a given equivalence relation there will be unique partition of the set

NOTE

There is one-one Correspondance between set of equivalence relation on set A and set of partitions of set A.

i.e | Set of Equivalence relation on set A | = | Set of Partitions of set A |

There is one-one Correspondance between set A & B, if we can define a bijective function from set A to set B (and vise-versa)

i.e $|A| = |B|$

function which is one-one as well as onto

$\therefore$ No of Equivalence relation on set A = No. of Partitions of set A

Topic : Number of equivalence relation on a set

Number of equivalence relation on a set A will be exactly same as number of partitions of that set A.

1. If $|A| = 0$, then no. of Equivalence relation on set A = 1
2. If $|A| = 1$, — " — " — " — " — " — = 1
3. If $|A| = 2$, — " — " — " — " — " — = 2
4. If $|A| = 3$, — " — " — " — " — " — = 5
5. If $|A| = 4$, — " — " — " — " — " — = 15
6. If $|A| = 5$, — " — " — " — " — " — = 52
7. If $|A| = 6$, — " — " — " — " — " — = 203

Topic : Bell Number

{ Bell number 'B_n' defines the number of partitions of set A (or Number of Equivalence relⁿ on set A) where $|A| = n$ }

$B_0 =$	1					
$B_1 =$	1	2				
$B_2 =$	2	3	5			
$B_3 =$	5	7	10	15		
$B_4 =$	15	20	27	37	52	
$B_5 =$	52	67	87	114	151	203
$B_6 =$	203					

and so on

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Q: Find the number of equivalence relation on set $A = \{1, 2, 3, 4, 5\}$, such that the equivalence relation contains exactly "9" order pairs.

Soluⁿ of above Problem

$\{1\}\ \{2\}\ \{3\}\ \{4\}\ \{5\}$

$\times\ \cup\ \times\ \cup\ \times\ \cup\ \times\ \cup\ \times$

$\{1\}\ \{2\}\ \{3\}\ \{4\}\ \{5\}$

$1\times1+1\times1+1\times1+1\times1+1\times1=5$

$|A|=5 = 1+4 \Rightarrow$

$\to 1+1+1+1+1 = 1 \Longrightarrow$ Resulting Equivalence relⁿ will Contain "5" order pairs

$\to 1+1+1+2 = \frac{{}^5C_1 * {}^4C_1 * {}^3C_1 * {}^2C_2}{3!} = 10$ $\quad (1\times1)+(1\times1)+(1\times1)+(2\times2) = 7$ order pairs

$\to 1+1+3 = \frac{{}^5C_1 * {}^4C_1 * {}^3C_3}{2!} = 10$ $\quad (1\times1)+(1\times1)+(3\times3) = 11$ order pairs

$\boxed{\to 1+2+2 = \frac{{}^5C_1 * {}^4C_2 * {}^2C_2}{2!} = 15 \quad (1\times1)+(2\times2)+(2\times2) = 9 \text{ order pairs}}$

15 such relation on set A that contains exactly "9" order pairs

$\to 1+4 = {}^5C_1 * {}^4C_4 = 5$ $\quad (1\times1)+(4\times4) = 17$ order pairs

$\to 2+3 = {}^5C_2 * {}^3C_3 = 10$ $\quad (2\times2)+(3\times3) = 13$ order pairs

$\to 5 = {}^5C_5 = 1$ $\quad (5\times5) = 25$ order pairs

2 mins Summary

Topic Partition of a set

Topic Number of equivalence relation

Topic Bell number

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THANK - YOU

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