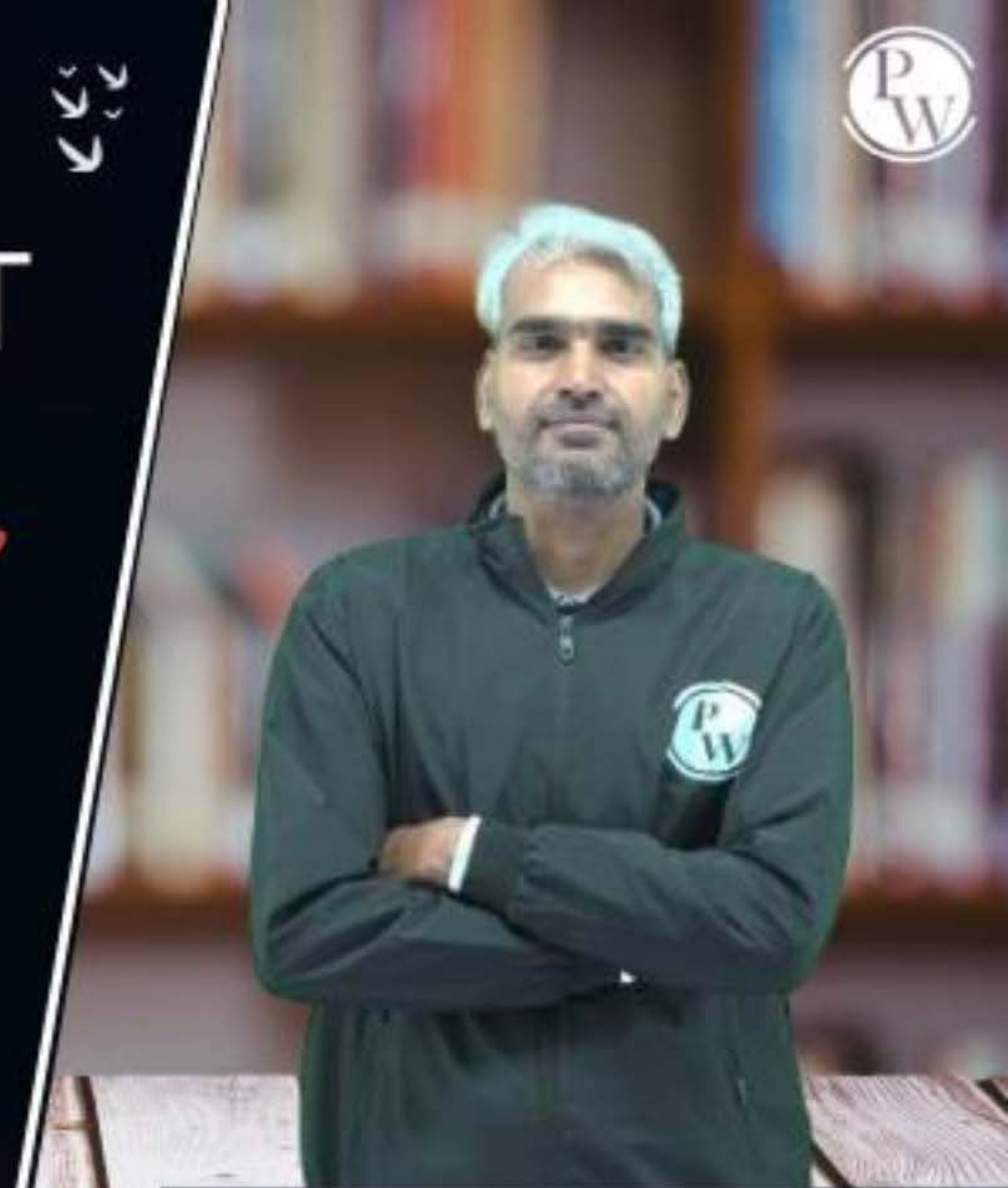
Computer Science & IT

Discrete Mathematics

Graph Theory

Lecture No. 01



By- Vishal Sir

Recap of Previous Lecture

Topic Cyclic group

Topic Properties w.r.t. cyclic groups

Topic Some important properties

let $(G, *)$ be a cyclic group s.t. $O(G) = n$

then No. of generators $= \phi(n)$

$= n \cdot \left\{\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\right\}$

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Topics to be Covered

Graph Theory

Topic Introduction to Graph Theory

Topic Representation of graph

Topic Simple graph, multi-graph and pseudo graph

Topic Maximum number of edges in a simple graph with n vertices

Topic Number of simple graphs possible with n labeled vertices

Topic Degree of a vertex and terminologies associated with it

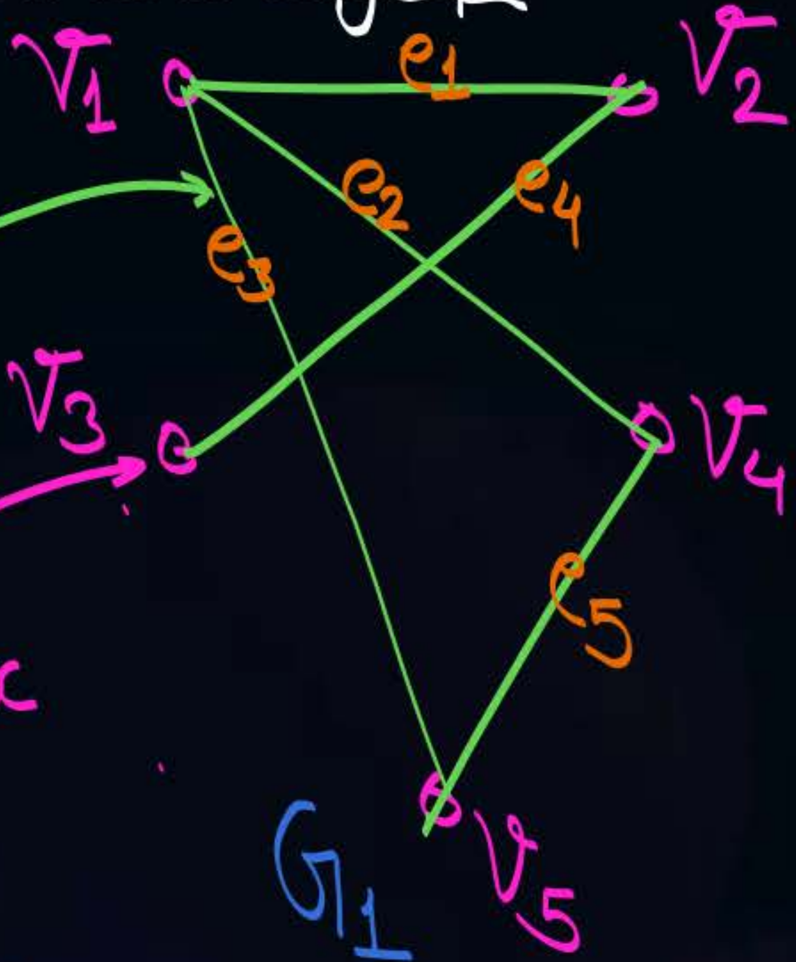
Slide

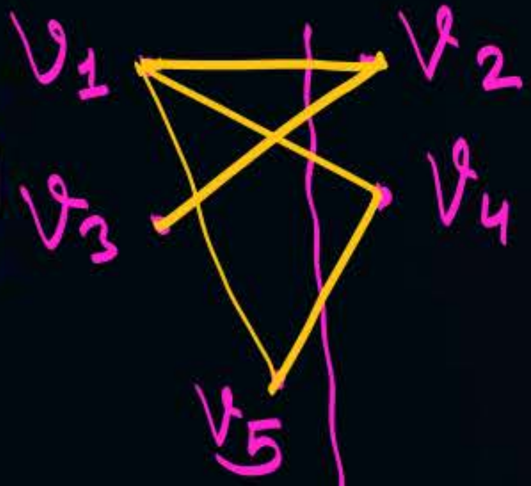
Topic : Graph

* Undirected graph:-

line/edge

node/vertex





$e_1 = \{v_1, v_2\}$

$e_2 = \{v_1, v_4\}$

$e_3 = \{v_1, v_5\}$

$e_4 = \{v_2, v_3\}$

$e_5 = \{v_4, v_5\}$

$G_1 = (V_1, E_1)$

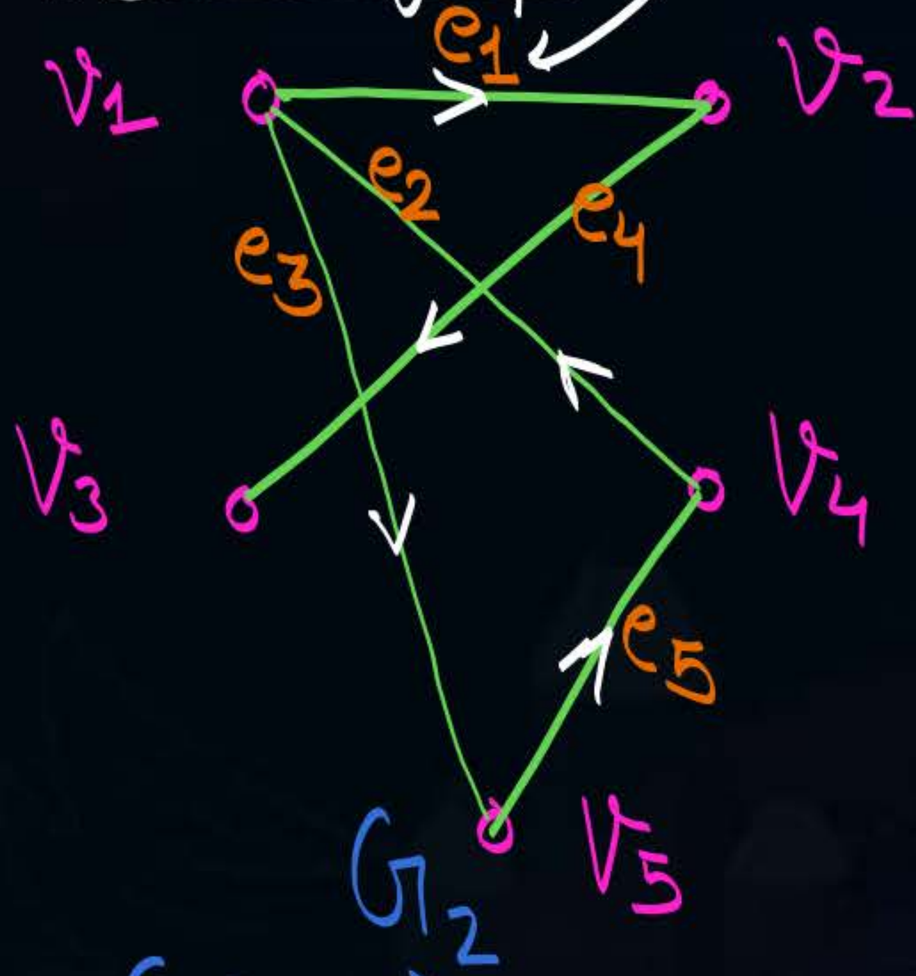
$V_1 = \{v_1, v_2, v_3, v_4, v_5\}$

$E_1 = \{e_1, e_2, e_3, e_4, e_5\}$

$E_2 = \{\{v_1, v_2\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_2, v_3\}, \{v_4, v_5\}\}$

Directed graph:

edge is from v_1 to v_2



$e_1 = (v_1, v_2) \neq (v_2, v_1)$

$e_2 = (v_4, v_1)$

$e_3 = (v_1, v_5)$

$e_4 = (v_2, v_3)$

$e_5 = (v_5, v_4)$

$G_2 = (V_2, E_2)$

$V_2 = \{v_1, v_2, v_3, v_4, v_5\}$

$E_2 = \{e_1, e_2, e_3, e_4, e_5\}$

Topic : Graph

order in which elements appear is important

A Graph is defined as an ordered pair of two sets,

Set of vertices & Set of edges

i.e., $G = (V, E)$

Name of graph

Set of vertices

Set of edges

{edges are used to define that how exactly vertices of the graph are connected with each other}

$|V|$ = Number of vertices in set V ($|V|$ = Order of graph)

$|E|$ = Number of edges in set E ($|E|$ = Size of graph)

Topic : Representation of edge

- In a non-directed graph an edge is represented by a set of two vertices

  e.g., {vi, vj} which represents an edge between vertices vi and vj

  $= \{v_j, v_i\}$

  v_i & v_j are called end-vertices associated with that edge

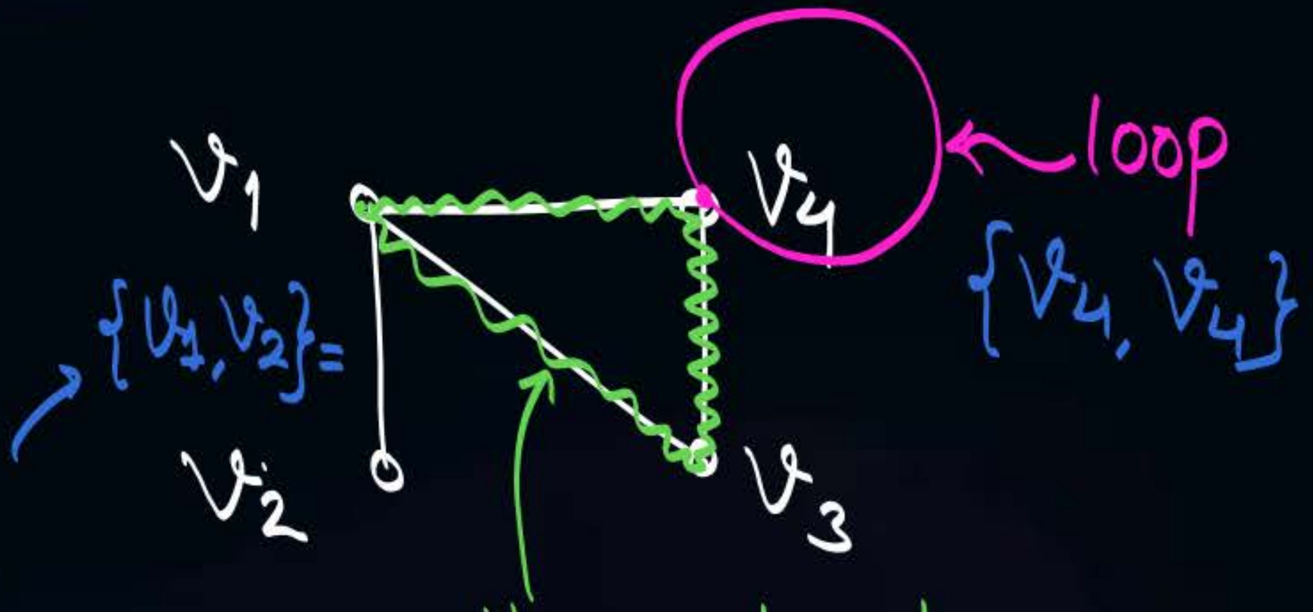
- In a directed graph an edge is represented by an order pair of two vertices

  e.g., (vi, vj) which represents an edge from vertex vi to vertex vj

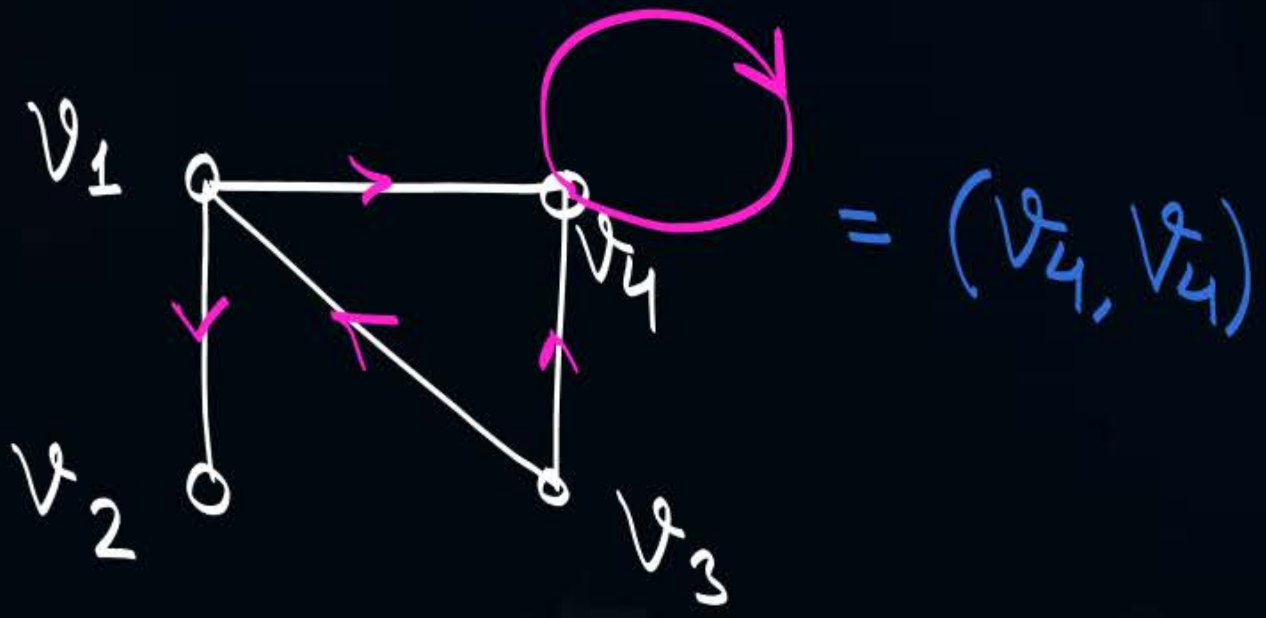
  $\neq (v_j, v_i)$

Topic : Loop

Self-loop :-



- it is not a loop
  it is called
  Cycle in the graph

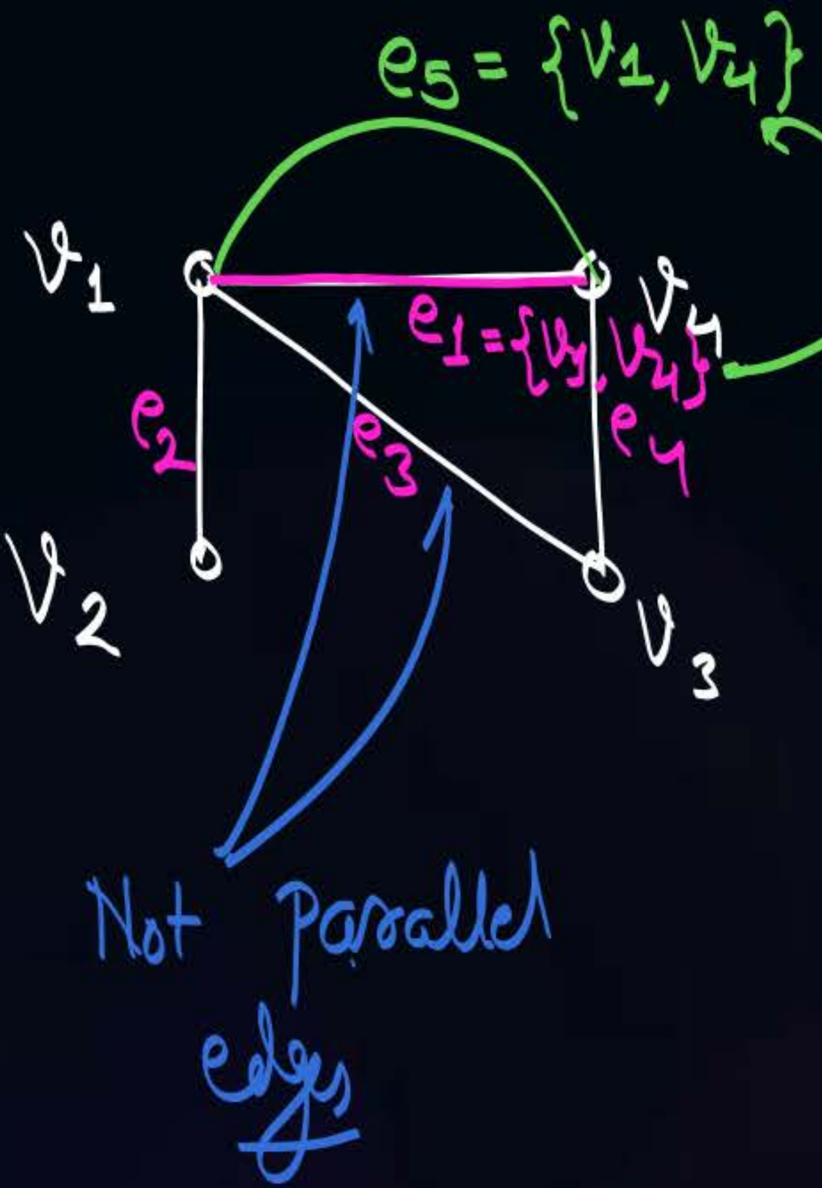


Topic : Loop

both

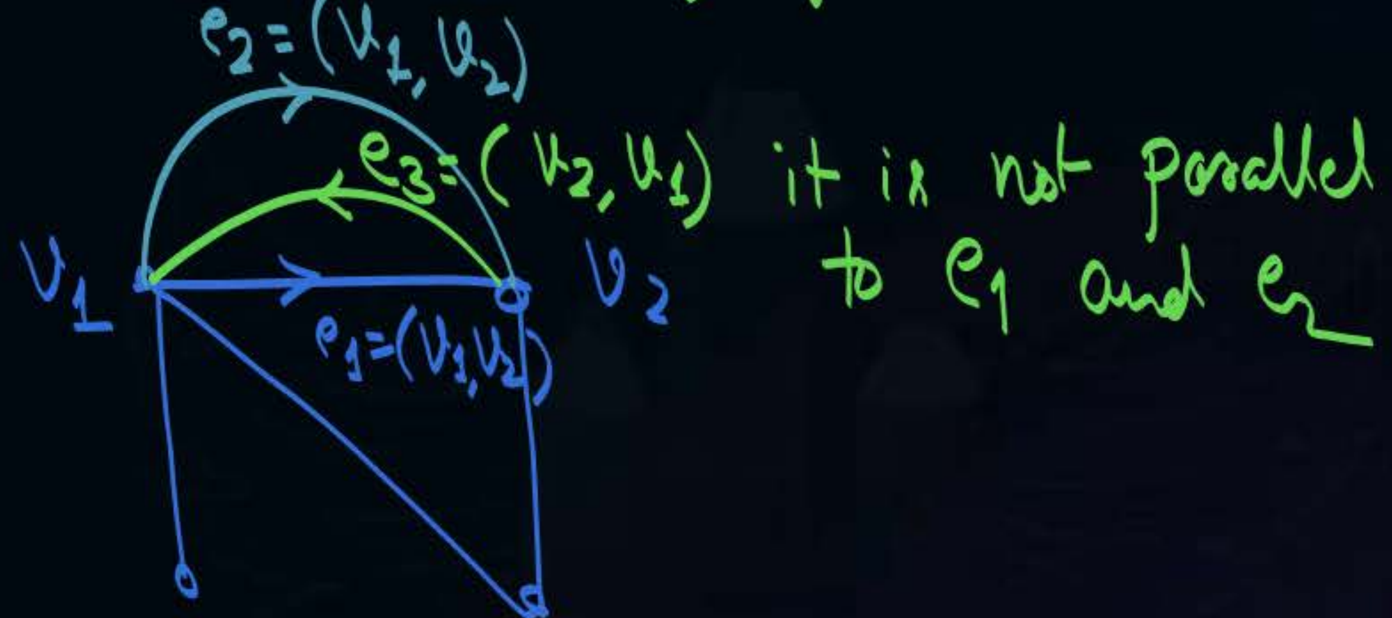
An edge in which end vertices associated with the edge are same is called as loop or self-loop.

Topic : Parallel Edges



Parallel edges exist

$\therefore$ Graph is a multi-graph

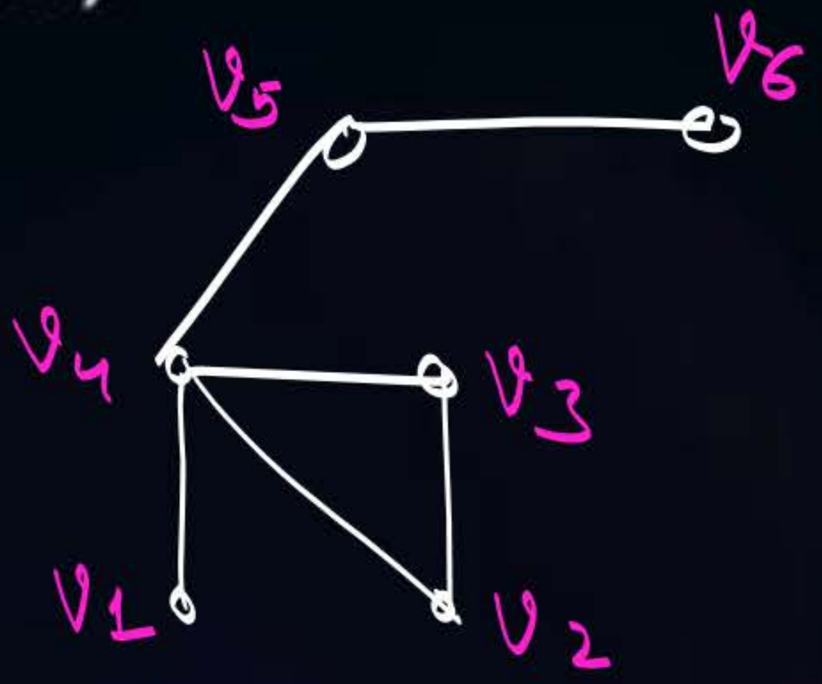


Topic : Parallel Edges

If two or more edge are associated with same end vertices, then edges are called parallel edges and resulting graph is called multi-graph.

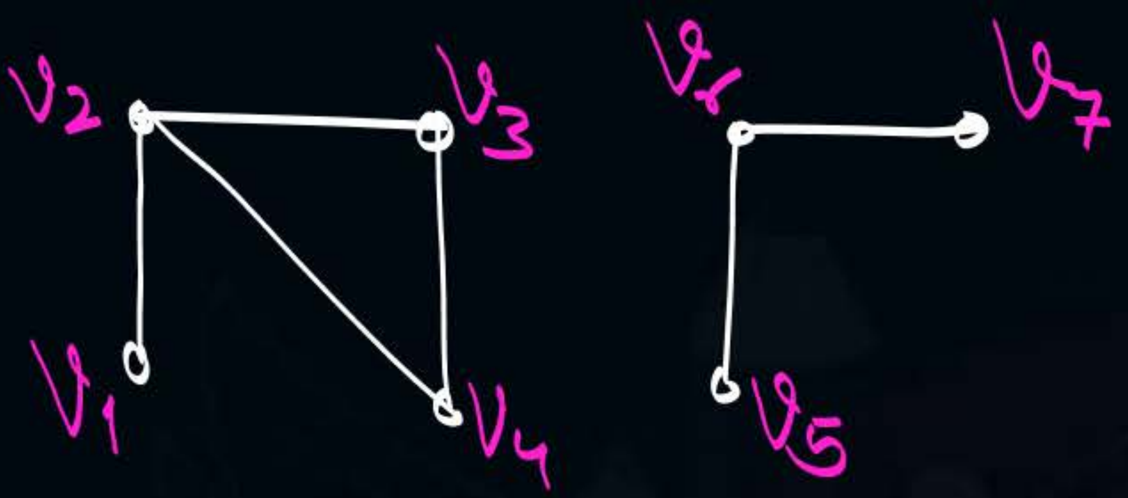
Topic : Simple graph

A graph with no self-loop and no parallel edge is called simple graph,



G_1

It is a simple graph as well as connected graph



graph G_2

It is also a simple graph but it is a dis-connected graph

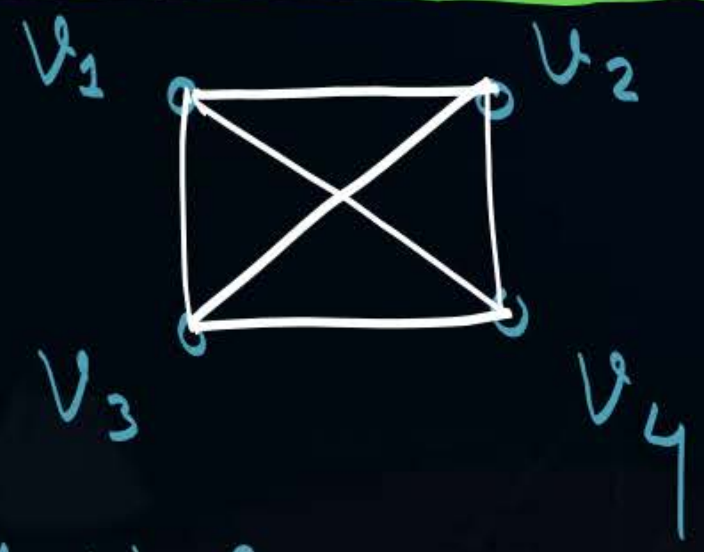
Topic : Classification of graph

	Loop	Parallel edge
Simple Graph	Not allowed	Not allowed
Multi-graph	Not allowed	Allowed
Pseudo graph	Allowed	Allowed

Topic : Simple Graph

Maximum number of edges possible in "Simple Graph" With n-vertices

labeled



* In a simple graph every edge will be between two distinct vertices of the graph

- $\therefore$ Maximum No. of edges Possible in a simple graph with n-vertices = No. of ways in which two distinct vertices can be choosen out of n distinct vertices = $^nC_2$

$\{v_1, v_2\} = \{v_2, v_1\}$

$\{v_1, v_3\} = \{v_3, v_1\}$

$\{v_1, v_4\} = \{v_4, v_1\}$

$\{v_2, v_3\} = \{v_3, v_2\}$

$\{v_2, v_4\} = \{v_4, v_2\}$

$\{v_3, v_4\} = \{v_4, v_3\}$

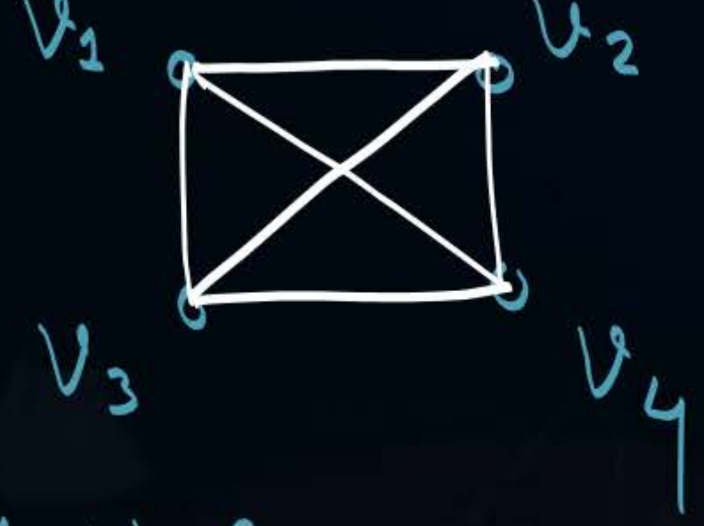
$$\boxed{\text{Maximum no. of edges Possible in a simple graph with n-vertices} = {}^nC_2 = \frac{n(n-1)}{2} = \frac{n^2-n}{2}}$$

Topic : Simple Graph

labeled

Maximum number of edges possible in "Simple Graph" With n-vertices

In a simple graph with n-vertices number of edges will always be $\leq \dfrac{n(n-1)}{2}$



$\{v_1, v_2\} = \{v_2, v_1\}$

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$\{v_1, v_4\} = \{v_4, v_1\}$

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$\{v_2, v_4\} = \{v_4, v_2\}$

$\{v_3, v_4\} = \{v_4, v_3\}$

Number of "Simple Graphs" possible With 'n' labeled vertices

If vertices are not labeled, then both the graphs are same

If vertices are labeled then graphs are different

Number of "Simple Graphs" possible With 'n' labeled vertices

In a simple graph with n-vertices there will be exactly n-vertices and at most $^{n}C_2$ edges

i.e. A simple graph with n-vertices may have

n-vertices & 0-edges (or) n-vertice & 1-edge (or) n-vertices & 2-edges (or) . - - - (or) n-vertices & all $^{n}C_2$ edges

Topic : Simple Graph

Number of "Simple Graphs" possible With 'n' labeled vertices

If vertices are labeled, then all $^nC_2$ edges are different from each other.

Number of simple graph possible with n-labeled vertices

= Choose all n-vertices and {Choose '0' out of $^nC_2$ edges (or) Choose '1' out of $^nC_2$ edges (or) Choose '2' out of $^nC_2$ edges - - - (or) Choose all $^nC_2$ edges out of $^nC_2$ edges}

$$= \underbrace{\cancel{^nC_n}}_{1} * \left\{ {}^{^nC_2}C_0 + {}^{^nC_2}C_1 + {}^{^nC_2}C_2 + \cdots + {}^{^nC_2}C_k + \cdots {}^{^nC_2}C_{^nC_2} \right\} = 1 \cdot 2^{^nC_2} = 2^{\frac{n(n-1)}{2}}$$

Topic : Simple Graph

Number of "Simple Graphs" possible With 'n' labeled vertices

No. of simple graphs possible with n-labeled vertices

$$= {}^{{}^nC_2}C_0 + {}^{{}^nC_2}C_1 + \cdots + {}^{{}^nC_2}C_k + \cdots + {}^{{}^nC_2}C_{{}^nC_2} = 2^{{}^nC_2} = 2^{\frac{n(n-1)}{2}}$$

No. of simple graphs with n-labeled vertices and at most 'K' edges

No. of simple graphs Possible with n-labeled vertices & Exactly K-edges

No. of simple graphs Possible with n-labeled vertices & at least 'K' edges

Q: How many simple graphs are possible with '5' labeled vertices

Solⁿ Maximum edges possible = $^5C_2 = 10$

& No. of simple graphs possible = $2^{10} = \underline{1024}$

Q: How many simple graphs are possible with 4 labeled vertices & at most '2' edges.

Solⁿ:- Maximum No. of edges possible in a simple graph with 4 vertices $= {}^4C_2 = 6$

&

No. of simple graphs possible with all 4-vertices & at most '2' edges $= {}^4C_4 * \left\{ {}^{{}^4C_2}C_0 + {}^{{}^4C_2}C_1 + {}^{{}^4C_2}C_2 \right\}$

$= 1 * \left\{ {}^6C_0 + {}^6C_1 + {}^6C_2 \right\}$

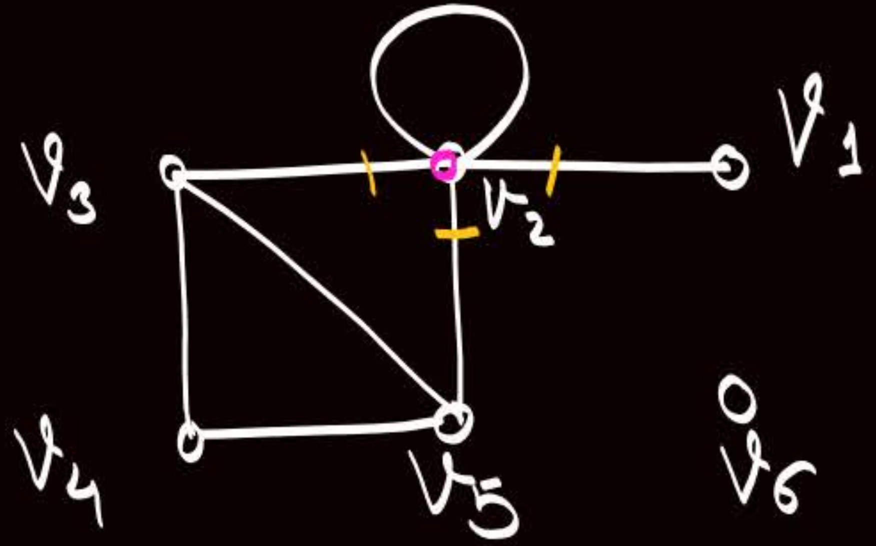
$= 1 + 6 + 15 = 22$ Ans

Topic : Degree of a vertex in an Undirected Graph

In an undirected graph, degree of a vertex 'v' is denoted by "deg(v)" and it is defined as

$$\boxed{deg(v) = \text{Number of edges incident with vertex 'v'}}$$

eg:



$deg(v_1) = 1$ { A vertex with degree = '1' is called Pendant vertex }

$deg(v_3) = 3$

$deg(v_4) = 2$

$deg(v_5) = 3$

$deg(v_6) = 0$ { A vertex with degree = '0' is called an isolated Vertex }

Note: In an undirected graph a loop at a vertex is counted as two edges, to obtain the degree of that vertex

$\therefore deg(v_2) = \underset{\text{w.r.t. simple edges}}{3} + \underset{\text{w.r.t. loop}}{2} = 5$

Topic : Degree of a vertex in directed Graph

In a directed graph two types of degrees are defined for each vertex of the graph

(1) In-degree

(2) Out-degree

Topic : Degree of a vertex in directed Graph

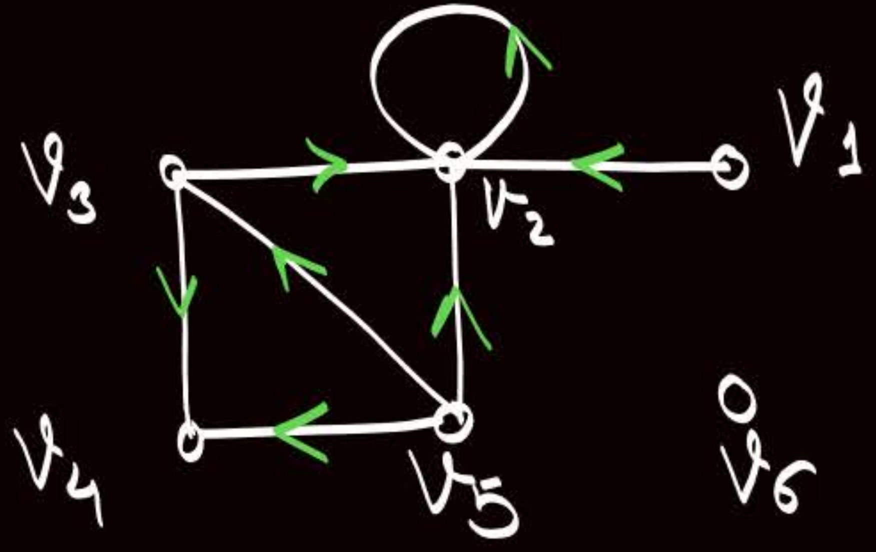
① In-degree :- In a directed graph in-degree of vertex 'v' is denoted by $deg^+(v)$, and it is defined as

$deg^+(v)$ = No. of edges incoming to vertex v.

② Out-degree :- In a directed graph Out-degree of vertex 'v' is denoted by $deg^-(v)$, and it is defined as

$deg^-(v)$ = No. of edges out-going from vertex 'v'.

eg:



Note: In a directed graph a loop at a vertex is counted as one edge to obtain the in-degree of that vertex & counted as one edge to obtain out-degree of that vertex

$\deg^+(v_1) = 0$
$\deg^-(v_1) = 1$
$\Sigma = 1 = \deg(v_1)$

$\deg^+(v_3) = 1$
$\deg^-(v_3) = 2$
$\Sigma = 3 = \deg(v_3)$

$\deg^+(v_4) = 2$
$\deg^-(v_4) = 0$
$\Sigma = 2 = \deg(v_4)$

$\deg^+(v_5) = 0$
$\deg^-(v_5) = 3$
$\Sigma = 3 = \deg(v_5)$

$\deg^+(v_6) = 0$
$\deg^-(v_6) = 0$
$\Sigma = 0 = \deg(v_6)$

$\deg^+(v_2) = 3 + 1 = 4$

w.r.t. loop

$\deg^-(v_2) = 0 + 1 = 1$

w.r.t. loop

$\Sigma = 5 = \deg(v_2)$

→ Summation of in-degree and out-degree of a vertex in directed graph is defined as overall degree of that vertex

i.e. $\boxed{\deg(v) = \deg^+(v) + \deg^-(v)}$

Topic : Null Graph

A graph with no edge in it is called a null graph

{In a null graph vertices may exist but edges can not exist}

In a graph $G = (V, E)$
if $|E| = 0$ then
graph 'G' is a Null graph

Vertices may exist without edges, but edges can not exist without vertices

If $deg(v) = 0, \forall v \in G$
then 'G' is a Null graph

Note:- A graph with no vertex and no edge is called an "Empty graph"

2 mins Summary

Topic	Introduction to Graph Theory
Topic	Representation of graph
Topic	Simple graph, multi-graph and pseudo graph
Topic	Maximum number of edges in a simple graph with n vertices
Topic	Number of simple graphs possible with n labeled vertices
Topic	Degree of a vertex and terminologies associated with it

THANK - YOU

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