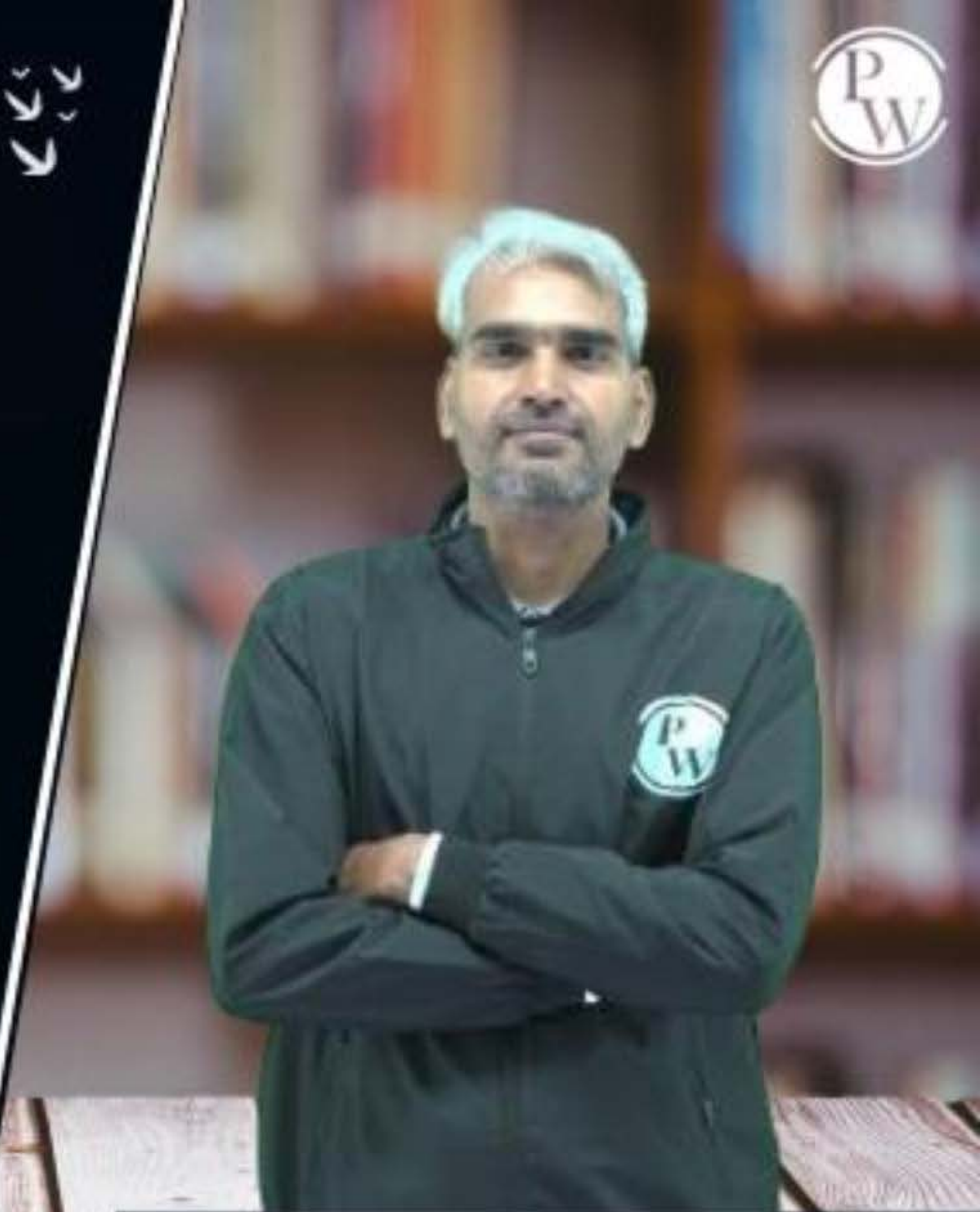
Computer Science & IT

Discrete Mathematics

Set Theory & Algebra

Lecture No. 05



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Recap of Previous Lecture

$A \times B = \{(a,b) \mid a \in A \;\&\; b \in B\}$

Topic Cartesian product

Topic Different types of relations

Topic Diagonal relation (Identity relation)

Slide

Topics to be Covered

Topic Reflexive relation and irreflexive relation

Topic Symmetric, anti-symmetric and asymmetric relation

Topic Transitive relation

Slide

(Cross Product)

Topic : Cartesian Product

Let A & B are two sets, Cartesian product of A & B is denoted by "$A \times B$" and it is defined as,

$$A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}$$

Order pair

∴ Order in which elements appear is important.

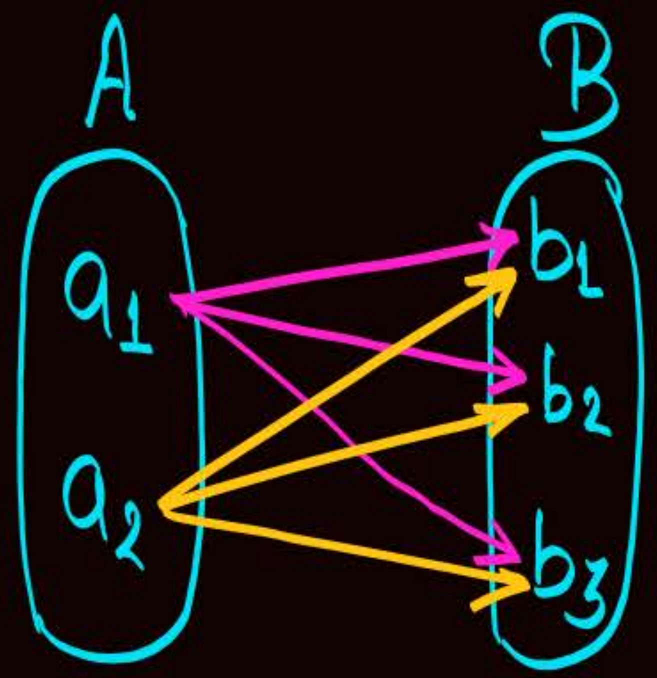
i.e; $(a, b) \neq (b, a)$

Slide

Let $A = \{a_1, a_2\}$

$B = \{b_1, b_2, b_3\}$

$\therefore A \times B = \{(a_1, b_1), (a_1, b_2), (a_1, b_3),$
$(a_2, b_1), (a_2, b_2), (a_2, b_3)\}$



$A \times B$

$B \times A = \{(b_1, a_1), (b_1, a_2),$
$(b_2, a_1), (b_2, a_2),$
$(b_3, a_1), (b_3, a_2)\}$

We know,
$(a_1, b_1) \neq (b_1, a_1)$
$\therefore A \times B \neq B \times A$

- In general,

$$A \times B \neq B \times A$$

- If $A \times B = B \times A$, then

  either $A = B$

  <u>or</u> at least one of A or B is an empty set.

- If A <u>or</u> B is an empty set

  then $A \times B = B \times A = \phi = \{\ \}$

- In $A \times B$, every element of set A relates with every element of set B.

$\boxed{\begin{array}{l} \cdot \text{ If } |A| = m \text{ \& } |B| = n \\ \text{then } |A \times B| = |A| \cdot |B| \\ \qquad |A \times B| = m \cdot n \end{array}}$

Topic : Relation

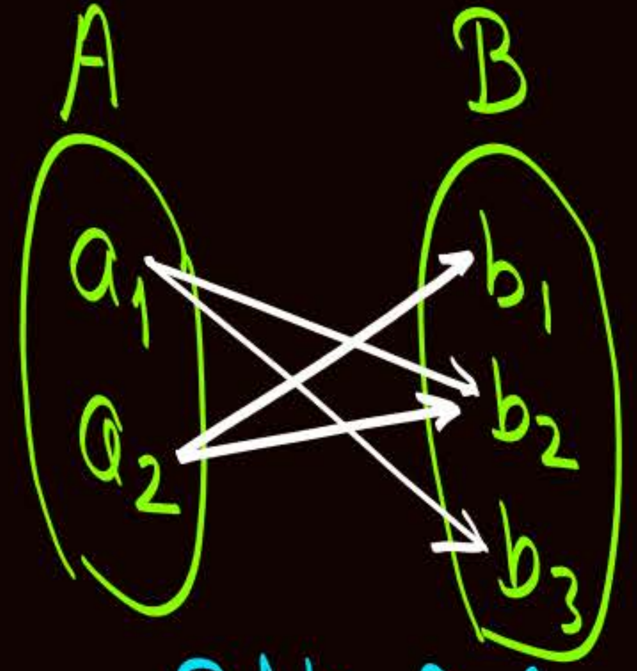
A relation from set A to set B defines that how exactly elements of set A relates with elements of set B

Note: Every relation from set A to set B is a subset of '$A \times B$'.

Slide

eg: let $A = \{a_1, a_2\}$, $B = \{b_1, b_2, b_3\}$

and let R_1 is relation which defines that how exacty elements of set A relates with set B.



$$\Rightarrow \underbrace{R_1 = \{(a_1, b_2), (a_1, b_3), (a_2, b_1), (a_2, b_2)\}}$$

we can observe that it is a subset of '$A \times B$'

Relation R_1 from A to B.

$R_1 : A \to B$

Q: Let $|A| = m$ & $|B| = n$, then
how many different relation are possible from set A to set B.

Solu^n :- Every relation from A to B is a subset of $A \times B$

$\therefore$ Number of relation from A to B = Number of subsets of $A \times B$

$$= 2^{|A \times B|} = 2^{|A| \cdot |B|}$$

$$= 2^{m \cdot n}$$

Note :- One of the subset of '$A \times B$' is an Empty set,
that Empty set is also a relation from A to B,
and that relation is called "Empty relation"

* A relation from set A to set A is called a relation on set A.

- If $|A| = m$, then

No. of relations possible on set $A = 2^{|A \times A|} = 2^{|A|\cdot|A|} = 2^{m \cdot m}$

$= 2^{(m^2)}$

Topic : Types of Relations

All this relations are defined from set A to same set A {i.e. on set A}

1. Diagonal Relation (Identity relation)
2. Reflexive Relation
3. Irreflexive Relation
4. Symmetric Relation
5. Anti-symmetric Rel^n
6. Asymmetric Rel^n
7. Transitive Relation

8. Complement of a relation
9. Inverse of a relation
10. Composite relation

Slide

Topic : Diagonal Relation

(Identity Relation)

Diagonal relation on set A is denoted by Δ_A.
and it is defined as,

$$\Delta_A = \{(a,a) \mid a \in A\}$$

it is definition of a set

e.g. let $A = \{1,2,3\}$

$R_1 : \{(1,1), (2,2)\}$ $(3,3) \notin R_1$ $\therefore R_1$ is not a diagonal relation on set A.

$R_2 = \{(1,1), (2,2), (3,3), (1,2)\}$ + order pair $(1,2)$ can never be an element of diagonal relⁿ
$\therefore R_2$ is not a diagonal relⁿ.

$R_3 = \Delta_A = \{(1,1), (2,2), (3,3)\}$

Slide

Let $A = \{1, 2, 3, 4, \ldots\ldots, n\}$

non-diagonal order pairs

diagonal order pairs

$$A \times A = \left\{ \begin{matrix} (1,1) & (1,2) & (1,3) & \cdots & (1,n) \\ (2,1) & (2,2) & (2,3) & \cdots & (2,n) \\ (3,1) & (3,2) & (3,3) & \cdots & (3,n) \\ \vdots & \vdots & \vdots & & \vdots \\ (n,1) & (n,2) & (n,3) & \cdots & (n,n) \end{matrix} \right\}$$

In '$A \times A$',

(i) Total no. of order pairs $= n^2$

(ii) No. of diagonal order pairs $= n$

(iii) No. of non-diagonal order pairs $= n^2 - n$

Topic : Reflexive Relation

{ All diagonal order pairs must be present }

A relation R on set A is called reflexive relation if and only if,

$a \, R \, a, \ \forall a \in A$

i.e., $(a,a) \in R, \ \forall a \in A$

It is a constraint for a relation to be reflexive

$a \, R \, a \Rightarrow$ a relates to a w.r.t. relation R

i.e. $(a,a) \in R$

$a \not{R} a \Rightarrow$ 'a' does not relate with 'a' w.r.t. relation R

i.e. $(a,a) \notin R$

Slide

eg. Let $A = \{1, 2, 3\}$

$R_1 = \{(1,1), (2,2), (3,3)\}$ $\left\{ \begin{array}{l} R_1 \text{ is a diagonal relation} \\ \text{as well as reflexive relation} \end{array} \right\}$

$R_2 = \{(1,1), (2,2), (3,3), (2,3)\}$ $\left\{ \begin{array}{l} R_1 \text{ is a reflexive relation,} \\ \text{but it is not a diagonal relation} \end{array} \right\}$

$R_3 = \{(1,1), (2,2)\}$ $(3,3) \notin R_3$ $\therefore$ Not a reflexive relation on set A.

$R_4 = \{(1,1), (2,2), (1,3)\}$ Not a reflexive relation on set A.

Note:

1. Every diagonal relation on set A is also a reflexive relation on set A, but every reflexive rel^n on set A need not be a diagonal rel^n on set A.

2. Diagonal relation on set A is the smallest reflexive relation on set A.

3. Largest reflexive relation on set A is "$A \times A$"

eg: Relation "$\leq$" is a reflexive relation on any set of real numbers.

let $A = \{1, 2.5, 3\}$

$\boxed{(a, b) \in \text{"}\leq\text{"} \text{ iff } a \leq b}$

$\leq = \{(1,1), (1,2.5), (1,3)$
$\cancel{(2.5,1)}, (2.5,2.5), (2.5,3)$
$\cancel{(3,1)}, \cancel{(3,2.5)}, (3,3)\}$

All diagonal order pair are Present, $\therefore$ Reflexive Rel^n.

eg: Relation "$\div$" (divides) is a reflexive relation on any set of non-zero positive integers. $\{ (a,b) \in \text{"}\div\text{"}$ iff a divides $b \}$

eg: Relation "$\subseteq$" (subset) is a reflexive relation on any Collection of sets. {Every set is a subset of itself, $\therefore$ Every set will relate with itself hence reflexive}

Q: Let $A = \{1, 2, 3, 4, - - - - - , n\}$

How many reflexive relation are possible on set A.

Let $A = \{1, 2, 3, 4, \ldots\ldots, n\}$

diagonal order pairs

non-diagonal order pairs

$$A \times A = \left\{ \begin{matrix} (1,1) & (1,2) & (1,3) & \cdots\cdots & (1,n) \\ (2,1) & (2,2) & (2,3) & \cdots\cdots & (2,n) \\ (3,1) & (3,2) & (3,3) & \cdots\cdots & (3,n) \\ \vdots & \vdots & \vdots & & \vdots \\ (n,1) & (n,2) & (n,3) & \cdots\cdots & (n,n) \end{matrix} \right\}$$

In "$A \times A$",

(i) Total no. of order pairs $= n^2$

(ii) No. of diagonal order pairs $= n$

(iii) No. of non-diagonal order pairs $= n^2 - n$

Q: Let $A = \{1, 2, 3, 4, \ldots, n\}$

How many reflexive relation are possible on set A. {0 to all}

Reflexive Relation = (All diagonal order Pairs must be present) and (apart from that any number of non-diagonal order pairs may be Present)

Choose all 'n' diagonal order pairs

Choose any no. of non-diagonal order pairs out of (n^2-n)

∴ Number of Reflexive Relation $= {}^{n}C_{n} * \left\{ {}^{n^2-n}C_0 + {}^{n^2-n}C_1 + {}^{n^2-n}C_2 + \cdots + {}^{n^2-n}C_{n^2-n} \right\}$

$= 1 * 2^{n^2-n}$

$= 2^{n^2-n}$

Another way :-

diagonal order pairs

non-diagonal order pairs

$$A \times A = \left\{ \begin{array}{ccccc}
(1,1) & (1,2) & (1,3) & \cdots & (1,n) \\
1 \times & 2 \times & 2 \times & \cdots \times & 2 \\
(2,1) & (2,2) & (2,3) & \cdots & (2,n) \\
2 \times & 1 \times & 2 \times & \cdots \times & 2 \\
(3,1) & (3,2) & (3,3) & \cdots & (3,n) \\
2 \times & 2 \times & 1 & \cdots \times & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(n,1) & (n,2) & (n,3) & \cdots & (n,n) \\
2 \times & 2 & 2 & & 1
\end{array} \right\}$$

In "$A \times A$",

(i) Total no. of order pairs $= n^2$

(ii) No. of diagonal order pairs $= n$

(iii) No. of non-diagonal order pairs $= n^2 - n$

Topic : Irreflexive Relation

{ none of the diagonal order pair should be present }

A relation R on set A is called irreflexive relation if and only if.

$$a \not R\, a, \quad \forall a \in A$$

i.e. $(a,a) \notin R, \ \forall a \in A$

Slide

eg: Let $A = \{1, 2, 3\}$

$R_1 = \{ \quad \}$ Empty relation on any set A is the smallest irreflexive relation on set A.

Empty relation

{it is not reflexive}

$R_2 = \{(1,2), (3,1), (2,3)\}$ { No order of type (x,x) ∴ irreflexive relation }

{it is not reflexive}

$R_3 = \{(1,2), (2,3), (3,3)\}$

$(3,3) \in R_3$ s.t. $3=3$ ∴ Not an irreflexive relation

{neither reflexive nor irreflexive}

Note:

1. Empty relation is the smallest irreflexive relation

2. let $|A| = n$, then largest irreflexive relation on set A will contain $\underline{(n^2 - n)}$ order pairs.

3. These may be some relations on set A which are neither reflexive nor irreflexive

Q: Let $A = \{1, 2, 3, 4, - - - -, n\}$

How many irreflexive relation are possible on set A.

Q: Let $A = \{1, 2, 3, 4, \ldots, n\}$

How many irreflexive relation are possible on set A. {0 to all}

Irreflexive Relation = (None of the diagonal order pairs should be present) and (apart from that any number of non-diagonal order pairs may be present)

Choose zero diagonal order pairs

Choose any no. of non-diagonal order pairs out of (n^2-n)

$\therefore$ Number of irreflexive Relation $= {}^{n}C_0 * \left\{ {}^{n^2-n}C_0 + {}^{n^2-n}C_1 + {}^{n^2-n}C_2 + \cdots + {}^{n^2-n}C_{n^2-n} \right\}$

$= 1 * 2^{n^2-n}$

$= 2^{n^2-n}$

Topic : Symmetric Relation

A relation R on set A is called symmetric relation if and only if,

$\left(\text{if } a R b \text{ then } b R a, \ \forall a, b \in A\right)$ $\left\{ \begin{array}{l} \forall a, b \in A, \\ \text{if - then statement} \\ \text{should be true} \end{array} \right\}$

i.e., if $(a, b) \in R$ then $(b, a) \in R$, $\forall a, b \in A$

Slide

eg: let $A = \{1, 2, 3\}$

$R_1 = \{ \quad \}$ it is smallest symmetric relation on set A.

Empty relation

$R_2 = \{(1,1)\}$

w.r.t. diagonal order pair $a = b$

$\therefore$ if (a,b) is true then (b,a) is automatically true

Hence because of presence or absence of diagonal order pair there will be no problem w.r.t symmetric property

Hence R_2 is a symmetric relation.

$R_3 = \{(1,1), (1,2), (2,1)\}$

diagonal order pair $\therefore$ No problem

both $(1,2)$ & $(2,1) \in R_3$ $\therefore$ No problem

R_3 is symmetric Relation

$R_4 = \{(1,1), (2,3), (3,1), (3,2)\}$

$(3,1) \in R_4$ but $(1,3) \notin R_4$

if is true then is false

Cond^n is false for at least one order pair

$\therefore$ R_4 is not symmetric Rel^n.

$R_5 = A \times A$ it is the largest symmetric relation on set A.

Topic : Anti-Symmetric Relation

* A relation R on set A is called anti symmetric relation if and only if,

$$\text{if } (a \, R \, b \text{ and } b \, R \, a) \text{ then } a = b, \quad \forall a, b \in A$$

i.e., if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$, $\forall a, b \in A$

"$a = b$" means diagonal order pair

"if" if cond^n is true then order pair must be diagonal order pair,
Otherwise, if cond^n itself must be false

Slide

Note: In anti-symmetric relation there will be no problem because of presence or absence of diagonal order pairs because w.r.t. diagonal order pair "a = b" will always be true

∴ if - then statement will be true w.r.t. diagonal order pair.

Note: In case "$a \neq b$" we know that then condⁿ is false,

∴ if "then" condⁿ is false then for statement to be true "if" statement should be false eg. let $A = \{1, 2, 3\}$

$R_1 = \{\ \}$ it is the smallest anti-symmetric relation

Empty relⁿ

$R_2 = \{\underbrace{(\overset{a}{1}, \overset{b}{1})}_{(a,b) \in R_2 \,\&\, (b,a) \in R_2}\}$

Presence or absence of diagonal order pair does not create any problem w.r.t. anti-symmetric Relⁿ.

but $1 = 1$ i.e $a = b$

$\therefore$ No problem

$R_3 = \{(1,1), (1,3), (2,3)\}$

$(2,3) \in R_3$ but $(3,2) \notin R_3$ } $\therefore$ if itself is false

Relⁿ R_3 is anti-symmetric

$(1,3) \in R_3$ but $(3,1) \notin R_3$ } $\therefore$ if Condⁿ itself is false we don't need to worry about them

$R_4 : \{(1,2), (3,1), (2,3), (3,2)\}$

$(2,1) \notin R_4$ $(1,3) \notin R_4$

$(2,3) \in R_4$ & $(3,2) \in R_4$ ← if is true

but $2 \neq 3$ ← then is false

∴ if-then false

Hence, Not anti-symmetric

Topic : Asymmetric Relation

A relation R on set A is asymmetric relation if and only if,

if $a R b$ then $b \cancel{R} a$, $\forall a, b \in A$

i.e., if $(a,b) \in R$ then $(b,a) \notin R$, $\forall a, b \in A$

it simply says if $(a,b) \in R$ then (b,a) must not belong to R no matter $a=b$ or not.

i.e., even diagonal order pairs are not allowed in asymmetric relation

Slide

eg: let $A = \{1, 2, 3\}$

$R_1 = \{\ \}$ ← Smallest asymmetric relⁿ on set A.

Empty relation

$R_2 = \{(1,1)\}$ ✗

diagonal order pairs are not allowed in asymmetric relation

$R_3 = \{(1,2), (3,1), (1,3)\}$

$(2,1) \notin R_3$
$\therefore$ OK

$\underbrace{(3,1) \in R_3 \text{ as well as } (1,3) \in R_3}_{\therefore \text{ Not asymmetric}}$

$R_4 = \{(1,2), (3,1), (3,3)\}$

Not asymmetric

Not allowed

$R_5 = \{(1,2), (1,3), (2,3)\}$ it is an asymmetric relation.

H.W.

Q: Let $A = \{1, 2, 3, 4, \dots, n\}$

How many Symmetric relation are possible on set A.

H.W.

Q: Let $A = \{1, 2, 3, 4, - - - - , n\}$

How many anti-symmetric relation are possible on set A.

H.W.

Q: Let $A = \{1, 2, 3, 4, \ldots, n\}$

How many asymmetric relation are possible on set A.

THANK - YOU

Slide