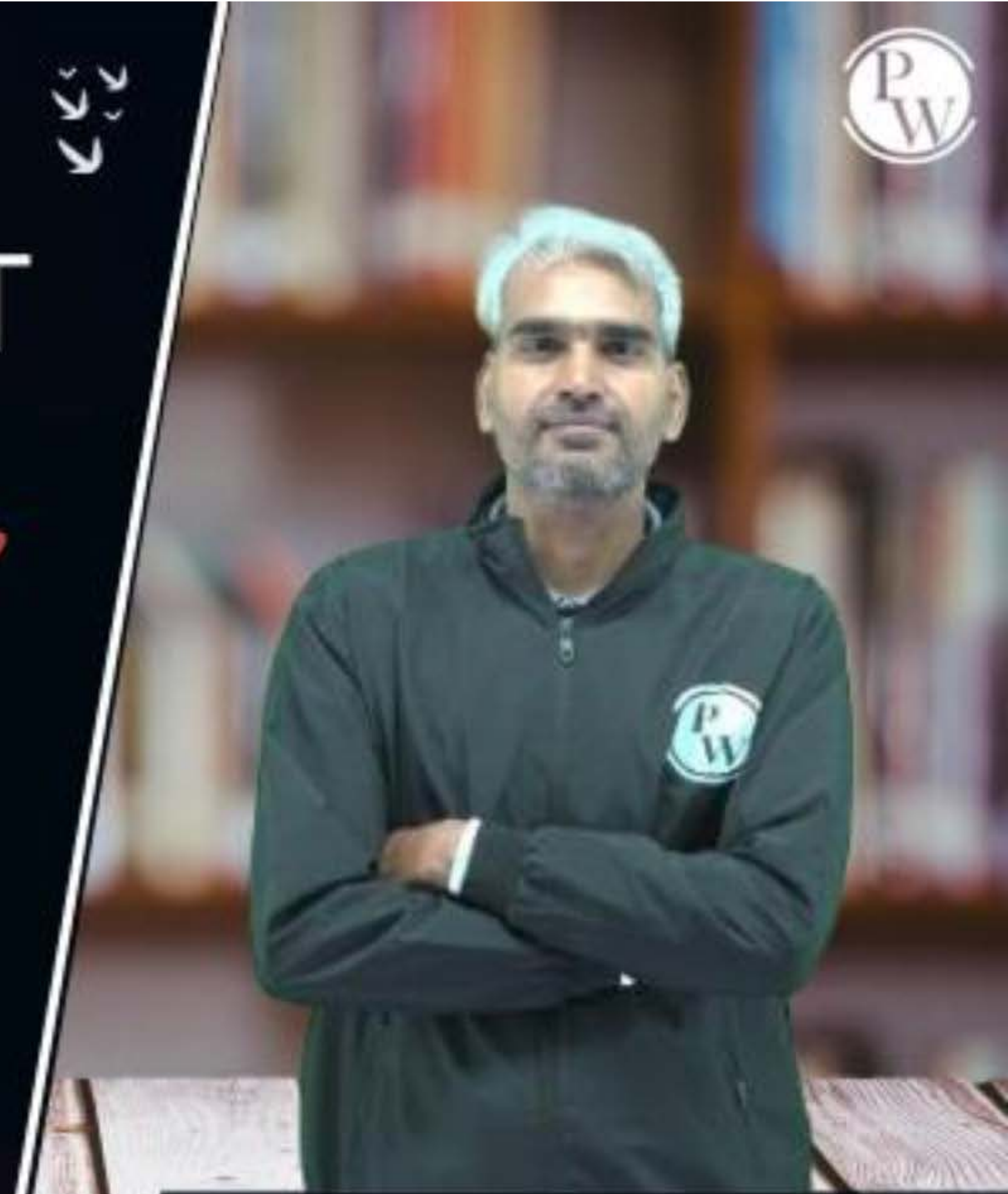
Computer Science & IT

Discrete Mathematics

Set Theory & Algebra

Lecture No. 03



By- Vishal Sir

Recap of Previous Lecture

Topic Types of sets

Slide

Topics to be Covered

Topic Venn diagram

Topic Set operations and properties of set operations

Topic Multi-set

Slide

Topic : Cardinality of a set

* $|A|$ = No. of elements in set A.

Slide

Topic : Types of sets

Empty Set: A set with no elements in it

denoted by ϕ or $\{\ \}$

$$\boxed{\{\ \} = \phi}$$

$|\phi| = 0$

Slide

Topic : Types of sets

Singleton Set: A set with exactly one element in it

eg: $A = \{1\}$

$B = \{a\}$

$C = \{January\}$

$D = \{\underbrace{\{1, 2, \{3\}\}}_{\text{One single element}}\}$

All are singleton sets

Slide

Topic : Types of sets

Finite Set: Any set with finite number of element in it is called a finite set.

eg: $A = \{1, 2, 3\}$

$B = \{1, a, b, 2, d\}$

$C = \{\ \}$ ← Empty set is also a finite set.

Slide

Topic : Types of sets

Infinite Set: A set with infinite number of elements in it is called an infinite set.

eg: $A = \{x \mid x \in R \text{ and } 0 \le x \le 1\}$

B = Set of all Natural Numbers = N

Slide

Another Classification of sets

- Countable Set
  - Finite Set
    - { Finite sets are always Countable }
  - Infinit Set
    - eg: Set of all Natural Numbers
- Uncountable Set
  - Uncountable Sets are always infinite
    - Set $\{x \mid x \in R \text{ and } 0 \le x \le 1\}$

Topic : Types of sets

Equal Sets: Two sets A & B are said to be equal if they have exactly same elements

eg: $A = \{1, 2, 3\}$ & $B = \{3, 2, 1\} \Rightarrow \boxed{A = B}$

$A = \{1, 2, 3\}$ & $B = \{1, a, 2\} \Rightarrow \boxed{A \neq B}$

$A = \{1, 2, 3, 4\}$ & $B = \{1, 2, 3\} \Rightarrow \boxed{A \neq B}$

Slide

Topic : Types of sets

Equivalent Sets: Two sets A & B are said to be equivalent if their cardinality is same

eg. $A = \{1, 2, 3\}$ & $B = \{1, 2, 3\}$

$|A| = 3$ & $|B| = 3 \implies \therefore A \cong B$

$A = \{1, 2, 3\}$ & $B = \{1, a, 2\}$

$|A| = 3$ & $|B| = 3 \implies \therefore A \cong B$

$A = \{1, 2, 3, 4\}$ & $B = \{1, 2, 3\}$

$|A| = 4$ & $|B| = 3 \implies \therefore A \ncong B$

Slide

- If $A = B$ then $A \cong B$
- <u>But</u> if $A \cong B$ <u>then</u> A may <u>or</u> may not be equal to B.
- if $A \ncong B$ then $A \neq B$

Universal Set: A set of all elements w.r.t. problem under Consideration is called universal set.

It is generally denoted by "U"

Slide

Topic : Types of sets

Subset & Superset: Let A & B are two sets,

$\subseteq$ $\supseteq$

If every element of set A is also a member of set B, then we say A is subset of B {i.e $A \subseteq B$}

If A is a subset of B then B is a superset of A. {i.e. $B \supseteq A$} {Converse of the statement is also true}

Slide

eg: let $A = \{1, 2, 3, a, b\}$ $B = \{1, a, b\}$

$B \subseteq A$ but $\underbrace{A \nsubseteq B}$ $\Rightarrow$ $\boxed{A \neq B}$ but $\boxed{B \subset A}$

elements 2 & 3 of A are not member of B

eg: let $A = \{1, 2, 3\}$ $B = \{1, 2, 3\}$

$A \subseteq B$ & $B \subseteq A$

we also know $A = B$

$\boxed{A = B \text{ if and only if } A \subseteq B \text{ \& } B \subseteq A}$

$$A = \{1, 2, \{1, 2\}\} \qquad B = \{\{1\}\}$$

$$\boxed{1 \in A}$$

$$\boxed{\cancel{1 \in B}}$$ False

element of set B

$$\{1\} \notin A$$

$$\therefore \boxed{B \not\subseteq A}$$

$$1 \notin \{\{1\}\}$$

$$A \not\subseteq B$$

$$\{1\} \in \{\{1\}\}$$

Subsets of A

$\{\ \}$

$\{1\}$

$\{2\}$

$\{\{1, 2\}\}$

$\{1, 2\}$

$\{1, \{1, 2\}\}$

$\{2, \{1, 2\}\}$

$\{1, 2, \{1, 2\}\}$

Note: ① for any set A, ϕ is always a subset of set A

i.e. ϕ is a subset of every set

② for any set A, set A itself is always a subset.

i.e. Every set is a subset of itself

Topic : Types of sets

"$\subset$"

Proper Subset:

Let A & B are two sets such that every element of set A is a member of set B, but every element of set B is not a member of set A then A is a proper subset of B.

i.e. if $A \subseteq B$ & $B \nsubseteq A$ then $A \subset B$

Every subset of a set 'A' except the set itself are called proper subsets of set A.

eg. let $A = \{1, 2, 3\}$

Subsets of set A are $= \{\ \}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$

Proper subsets of A $= \{\ \}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \cancel{\{1,2,3\}}$

Slide

Q: Let A is a set of size = 3, then how many subsets of set A are possible?

let $A = \{1, 2, 3\}$ ⇒ Subsets of set A

of size = 0	of size = 1	of size = 2	of size = 4
$\{\ \}$	$\{1\}, \{2\}, \{3\}$	$\{1,2\}, \{1,3\}, \{2,3\}$	$\{1,2,3\}$
# subsets of size = 0	# subsets of size = 1	# subsets of size = 2	# subsets of size = 3

Total 8 subsets

$$\text{Total No. of subsets} = 1 + 3 + 3 + 1 = 8$$

$$\text{Total No. of subsets} = {}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3$$

Topic : Number of subsets of a set 'A' of cardinality 'n'

let $A = \{1, 2, 3, \ldots n\}$

No. of Subsets of set A of

size = 0   size = 1   size = 2   size = k $\{k < n\}$ size = n

$$\text{Total No. of subsets} = \underbrace{{}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_k}_{\text{No. of subsets of A of size atmost 'k' i.e. } \leq k} + \cdots + {}^nC_n = 2^n$$

No. of subsets of A of size atmost 'k' i.e. $\leq k$.

Number of subsets of size exactly 'k'

No. of subsets of A of size at least k i.e. $\geq k$

Slide

$$(1+x)^n = {}^nC_0 \cdot x^0 + {}^nC_1 \cdot x^1 + {}^nC_2 \cdot x^2 + {}^nC_3 \cdot x^n + \cdots + {}^nC_n \cdot x^n$$

Put $\underline{x = 1}$

$$(1+1)^n = {}^nC_0 \cdot (1)^0 + {}^nC_1 \cdot (1)^1 + {}^nC_2 \cdot (1)^2 + \cdots + {}^nC_n \cdot (1)^n$$

$$\boxed{2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots + {}^nC_n}$$

Q:- No. of subsets of set

$$A = \{1, 2, 3, 4, 5, \ldots, (n-1), n\}$$

No. of subsets of A $= \underbrace{2 * 2 * 2 * 2 * 2 * \cdots * 2 * 2}_{\text{"n times 2"}} = 2^n$

Q: Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ $\{a\}$

How many subsets of set A are possible such that all elements of the subset are even and every element of the subset is greater than '4'.

$A : \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

No. of subsets satisfying the Cond^n $= 1 * 1 * 1 * 1 * 1 * 2 * 1 * 2 * 1 * 2 = 2^3 = 8$

only one choice i.e. don't select

odd $\therefore$ don't select

{ }

Check if

x "<" atleastone of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, \dots 10\}$

some of

any of

$4 < 5$ is true $\therefore$ will not be Compared with remaining element

$x = 4$,

Check if

$x > \text{all of} \{1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots 10\}$

$4 \not> 4$

$\therefore$ false

will not be compared with remaining element

$x = 4,$

$x > \text{all of} \{ \}$ ✓.

$x = 4$

Q: let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

and $A = \{1, 3, 5\}$ is a subset of universal set U

find the no of supersets of set A.

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

No. of supersets of $A = \{1, 3, 5\}$ $= 1 \times 2 \times 1 \times 2 \times 1 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7$

$= 128$ Ans

In every superset of A, all elements of set A must be present.

Must be present.

∴ Only one choice

Note: Let A is a set with 'n' elements,

(i) Number of subsets of $A = 2^n$

(ii) Number of proper subsets of $A = 2^n - 1$

Subset

$\subseteq$

Equal sets are also considered

Proper subset

$\subset$

Equal not considered

Topic : Power Set

Power Set: Let 'A' is a finite set, then Power set of set A $\underbrace{\text{is a set}}$ containing all subsets of set A

→ Power set of set 'A' is denoted by $P(A)$ or 2^A

Slide

eg: $A = \{1, 2, 3\}$

$$2^A \equiv P(A) = \{\underbrace{\{\ \}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}}\}$$

set of → all subsets of set A

Topic : Cardinality of Power Set

Let $P(A)$ is the power set of set A.
then Cardinality of set $P(A)$ is denoted by $|P(A)|$

$|P(A)| =$ Number of $\underbrace{\text{elements in } P(A)}$

Elements of $P(A)$ are subsets of set A

$=$ Number of subsets of set A.

$$\boxed{|P(A)| = 2^{|A|}}$$

Slide

Q: let $\emptyset$ be an empty set, then

$$|P(P(\emptyset))| = ?$$

$$|P(P(\emptyset))| = 2^{|P(\emptyset)|} = 2^{\left(2^{|\emptyset|}\right)} \qquad |\emptyset| = 0$$

$$= 2^{(2^0)} = (2)^{(2^0)} = 2^1$$

$$= 2$$

- given set is $\emptyset = \{\ \}$
- the only subset of $\emptyset$ is $\emptyset$ itself.

$\therefore P(\emptyset) =$ Set of subsets of $\emptyset$

$$= \{\ \{\ \}\ \}$$

$$\boxed{P(\emptyset) = \{\{\ \}\} = \{\emptyset\}} \qquad |\{\{\ \}\}| = 1$$

$$\underbrace{\{\ \}}_{\text{Empty set}} \neq \underbrace{\{\{\ \}\}}_{\text{set Containing an element}}$$

element is an empty set

$P(\emptyset) = \{ \{ \} \}$

Subsets of $P(\emptyset) = \underbrace{\{ \}}, \quad \underbrace{\{ \{ \} \}}$

empty set is subset of every set

every set is a subset of itself

$P(P(\emptyset))$ = Set of subsets of $P(\emptyset)$

$= \{ \ \{ \}, \ \{ \{ \} \} \ \}$

$= \{ \emptyset, \{ \emptyset \} \}$

Q: Let $A = \{1, 2, \{1,2\}, \{\{2,3\}\}\}$

Which of the following is/are true

$\{\{\{2,3\}\}\} \subseteq A$

(a) $1 \in A$

(b) $1 \subseteq A$

(c) $\emptyset \in A$

(d) $\emptyset \subseteq A$

$\underline{\{\ \}} \subseteq A$

(e) $\{1\} \in A$

(f) $\{1\} \subseteq A$

(g) $\{1,2\} \in A$

(h) $\{1,2\} \subseteq A$

(i) $\{\{1,2\}\} \in A$

(j) $\{\{1,2\}\} \subseteq A$

(k) $\{2,3\} \in A$

(l) $\{2,3\} \subseteq A$

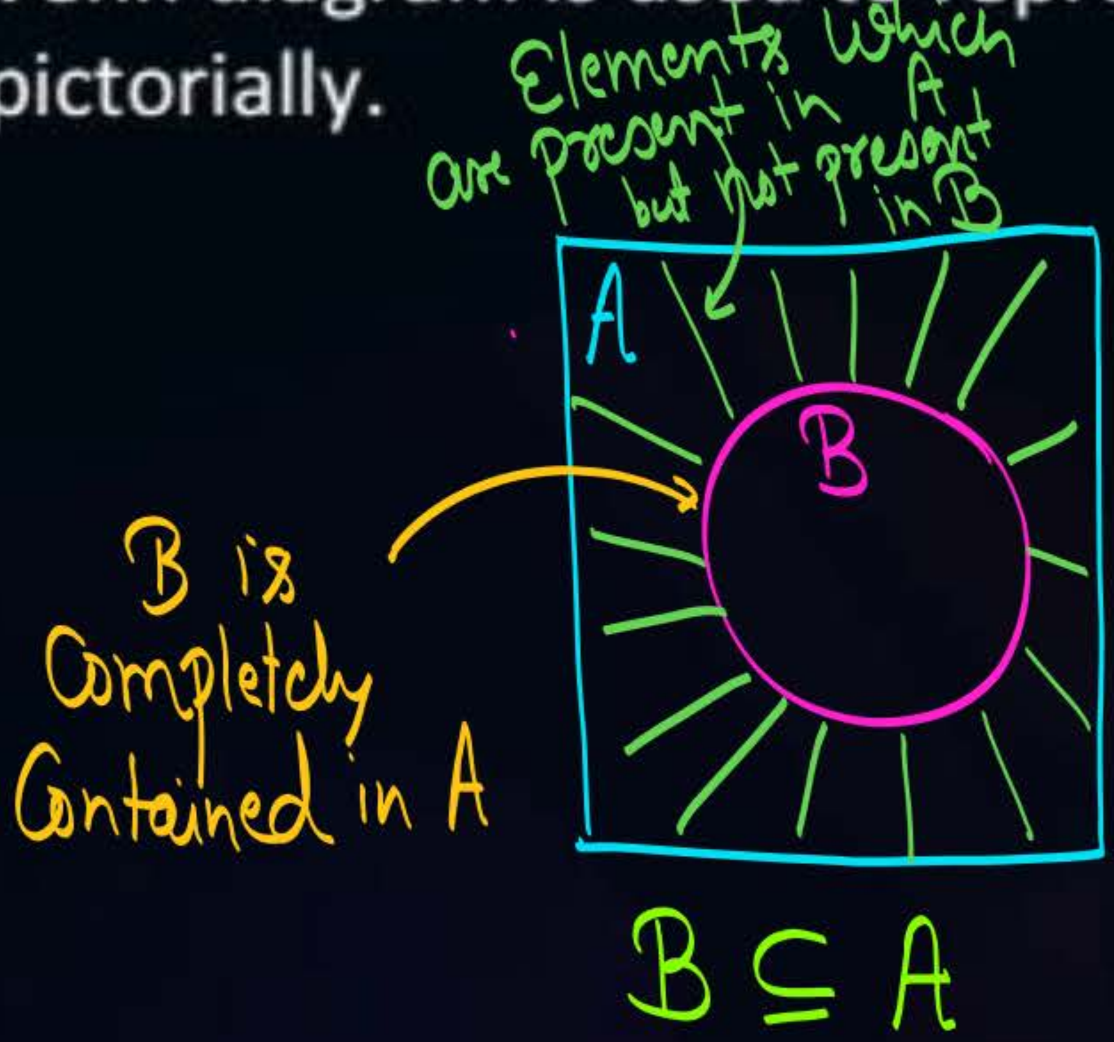
(m) $\{\{2,3\}\} \in A$

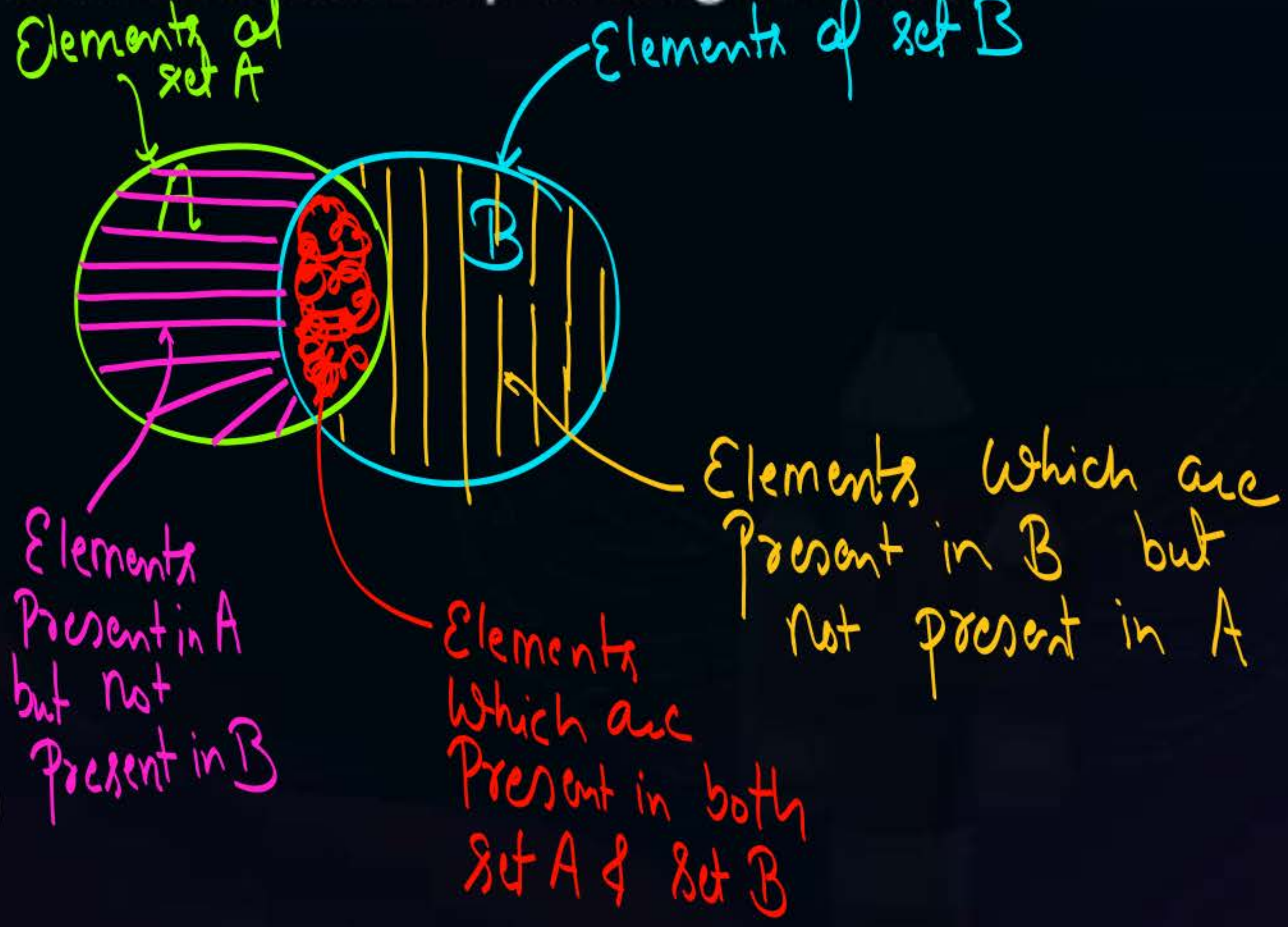
(n) $\{\{2,3\}\} \subseteq A$

Subset of A $= \underset{\text{Set}}{\{}\ \underbrace{\qquad\qquad}_{\text{elements from set}}\ \}$

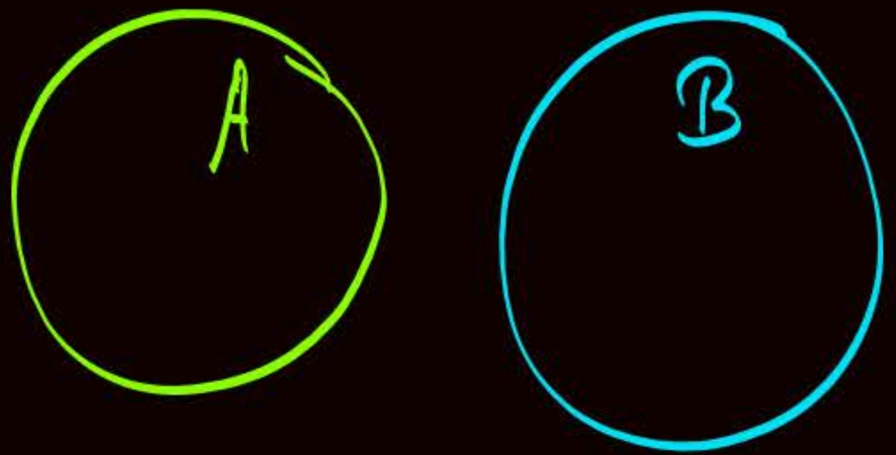
Topic : Venn Diagram

- Venn diagram is used to represent the relationship among the sets pictorially.





Slide



No shared region b/w A & B

$\therefore$ No common elements between A & B

Topic : Set Operations

- ✓ Complement of a set
- ✓ Union of two sets
- ✓ Intersection of two sets
- ✓ Set difference
- ✓ Symmetric difference of two sets

Slide

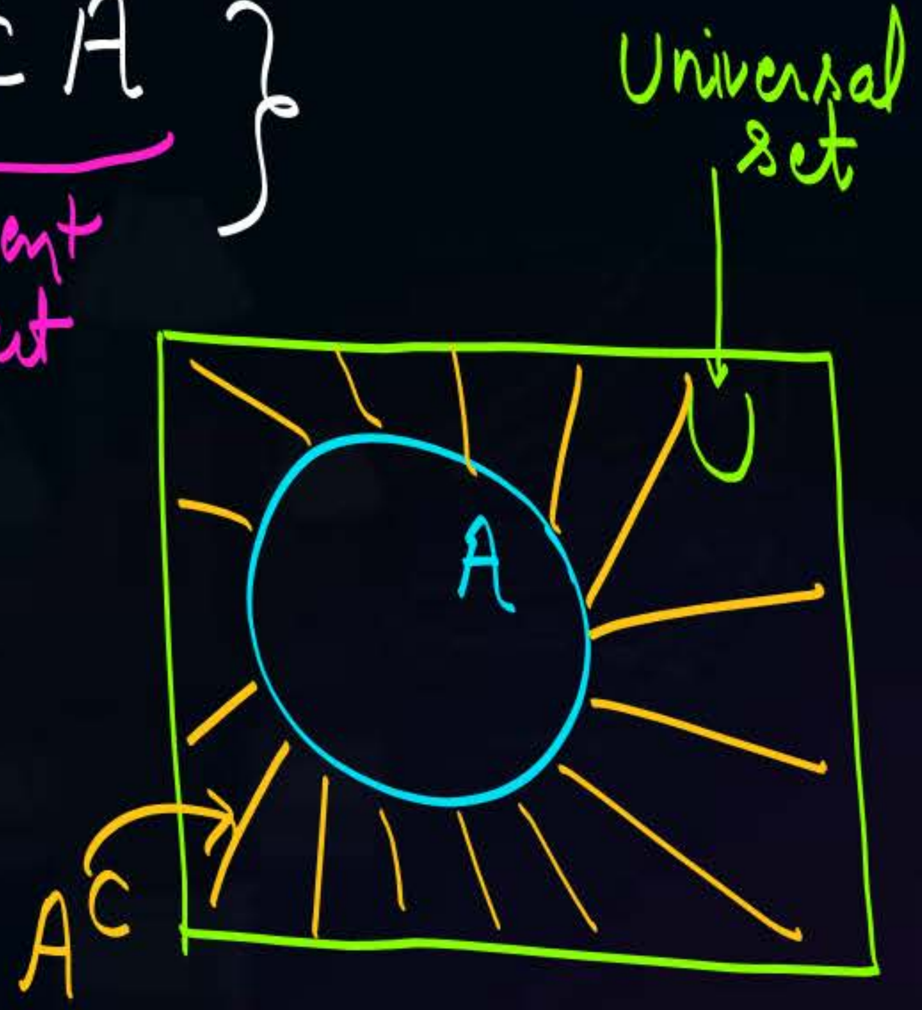
Topic : Complement of a set

Complement of a set is always defined w.r.t. universal set U

For set A,

$$A^c = A' = \bar{A} = \left\{x \mid \underbrace{x \in U \text{ and } x \notin A}_{\text{Elements which are present in the universal set, but not present in set A}}\right\}$$

Complement of set A



eg: let $U = \{1, 2, 3, a, b, c\}$ & $A = \{1, 3, b\}$

then $A^c = \{2, a, c\}$

Slide

Topic : Union of two sets

Let A & B are two sets,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

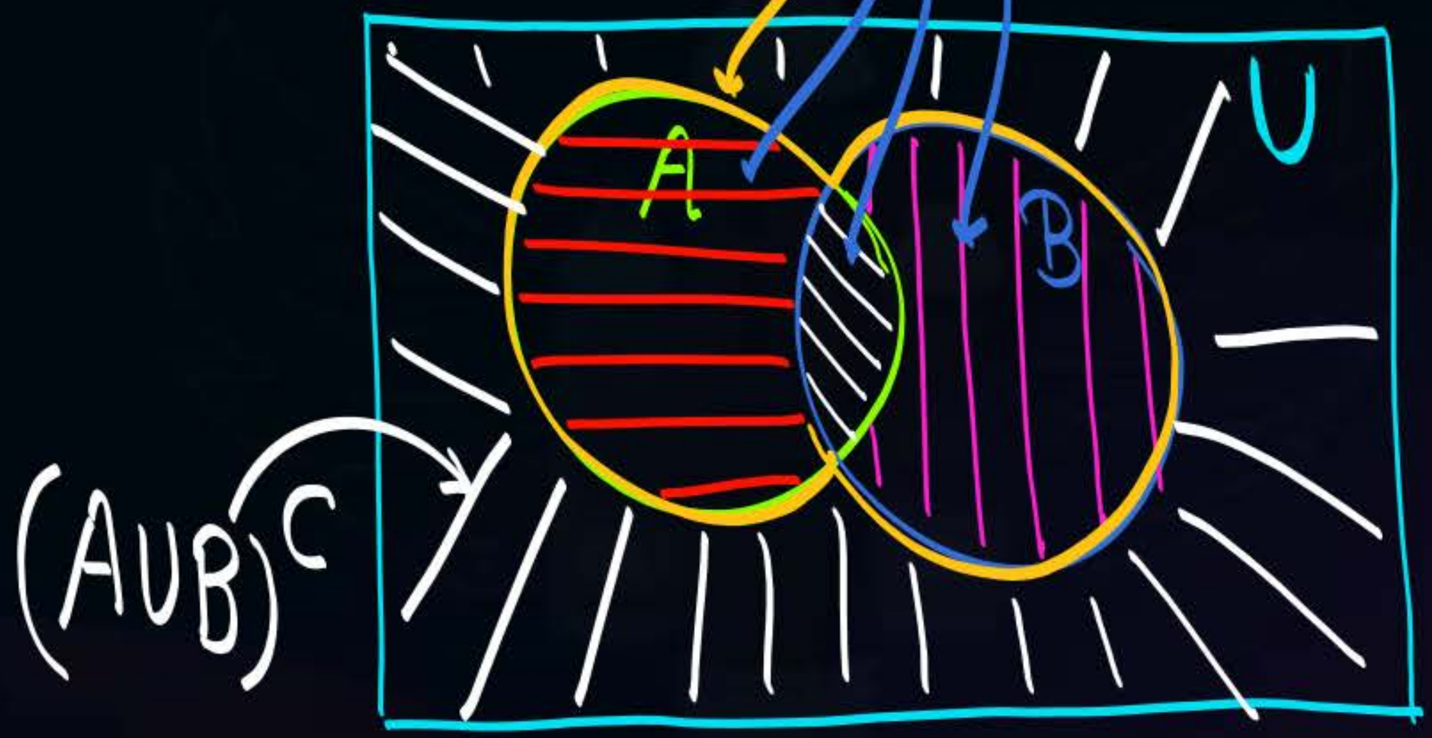
eg: Let $A = \{1, 2, 3, a, b, c\}$

$B = \{2, 3, a, d, e\}$

$A \cup B = \{1, 2, 3, a, b, c, d, e\}$

$\boxed{\underline{\text{Note}}: A \cup B = B \cup A}$

$A \cup B = B \cup A$



Slide

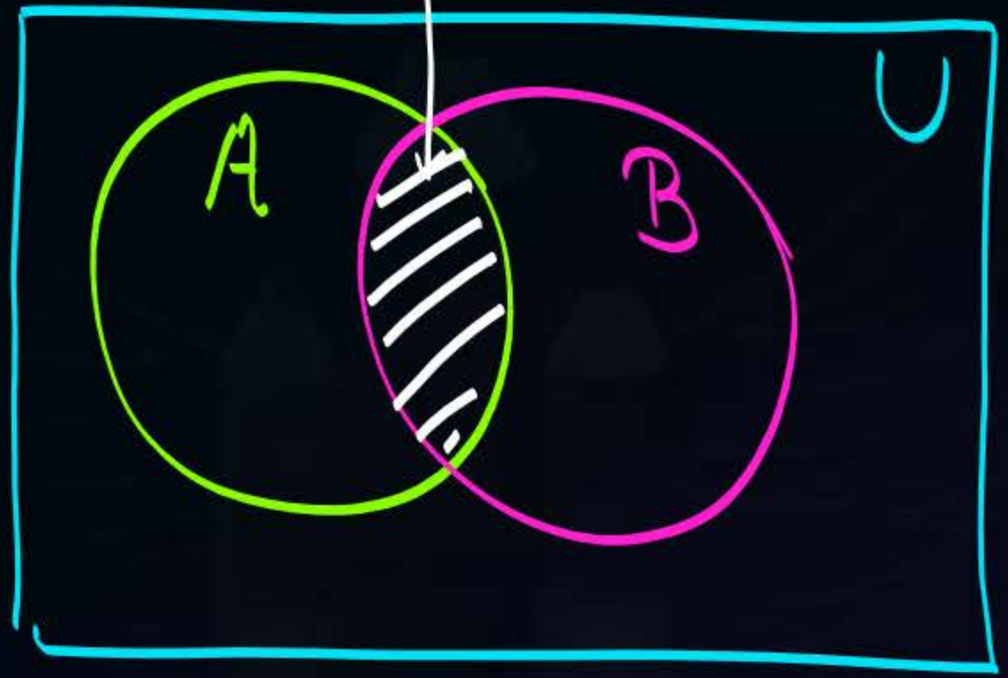
Topic : Intersection of two sets

Let A & B are two sets,

{ It gives the Common elements of two sets }

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

$A \cap B = B \cap A$



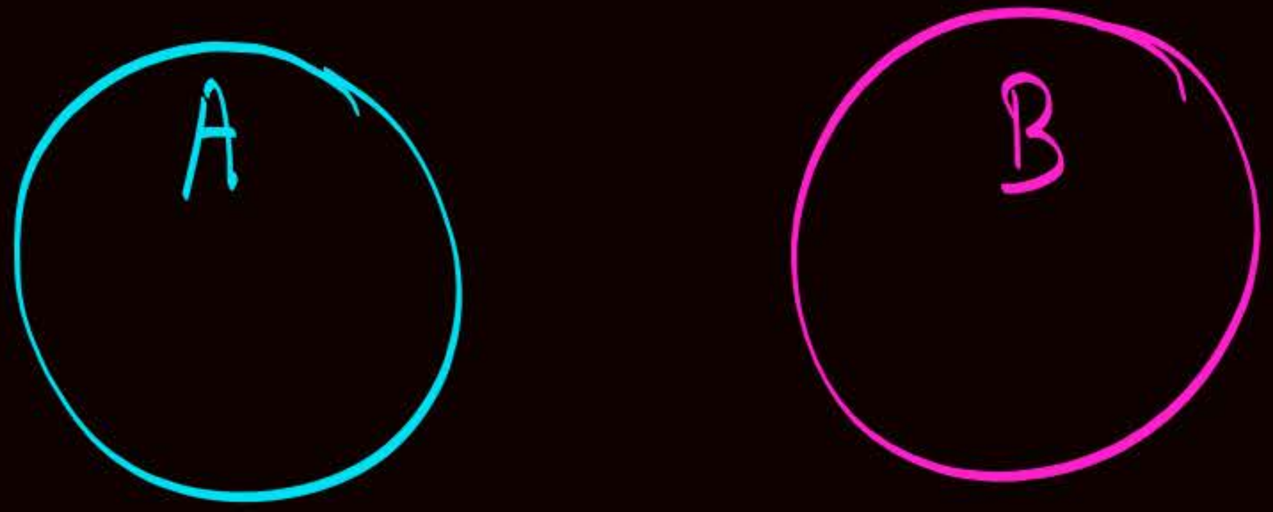
eg. $A = \{1, 2, 3, a, b, c\}$

$B = \{2, 3, b, d, e\}$

$A \cap B = \{2, 3, b\}$

$\boxed{\text{Note : } A \cap B = B \cap A}$

Slide



Representation of disjoint sets

No common elements b/w A & B

$\therefore A \cap B = \phi$

if $A \cap B = \phi$, then A & B are disjoint sets

Topic : Set difference

Let A & B are two sets,

$$\underbrace{A-B}_{\text{Elements which are present in set } A \text{, but not present in set } B} = \{x \mid x \in A \text{ and } x \notin B\}$$

$A-B \neq B-A$

Note: $\boxed{A-B \neq B-A \text{ in general}} \Rightarrow \boxed{A-B = B-A \text{ if and only if } A=B}$

Slide

Q: Let $A = \{2, 3, a, b, 7, 6, c\}$

$B = \{1, 3, 4, b, 8, c, 9, 10\}$

$A - B = \{2, a, 7, 6\}$

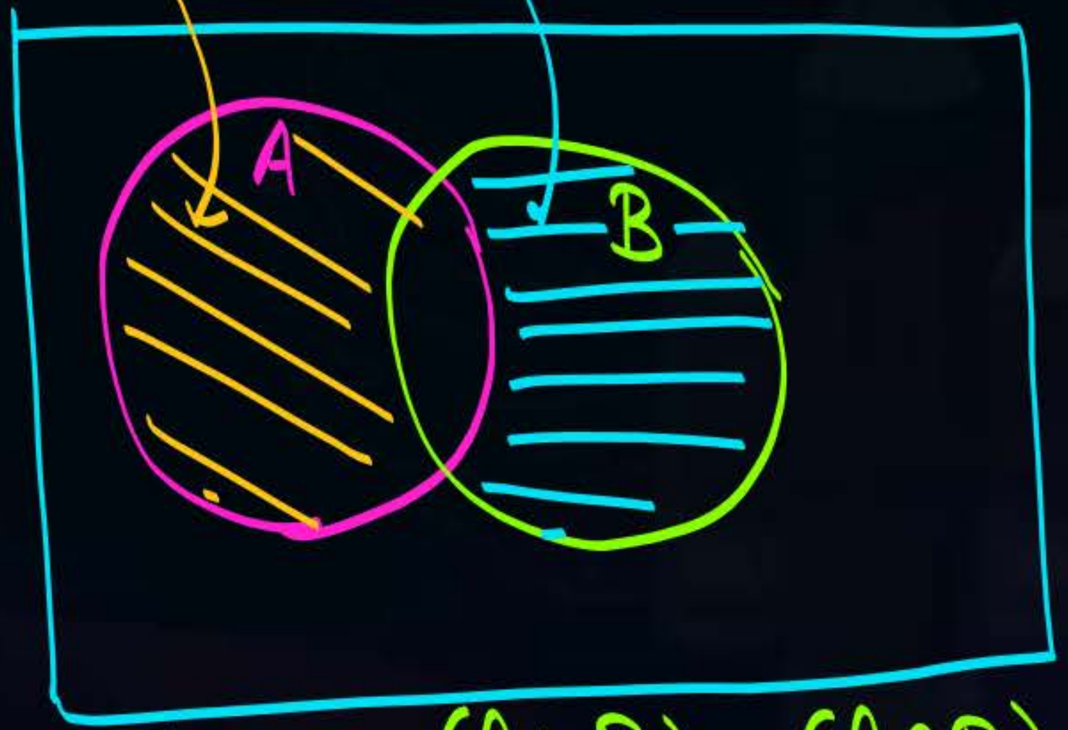
Topic : Symmetric difference of two sets

Let A & B are two sets,

$$A \Delta B = \{x \mid (x \in A \text{ or } x \in B) \text{ and } x \notin A \cap B\}$$

Elements which are either present in set A or present in set B, but not common, {i.e. not present in $A \cap B$}

$$(A-B) \cup (B-A) = A \Delta B = B \Delta A$$



Note: $\boxed{A \Delta B = B \Delta A}$

$$A \Delta B = (A \cup B) - (A \cap B)$$

Slide

Topic : Properties of Set Operations

1. Idempotent:

   a. $A \cap A = A$

   b. $A \cup A = A$

2. Identity:

   a. $A \cup \emptyset = A$

   b. $A \cap U = A$

Slide

Topic : Properties of Set Operations

3. Domination:

a. $A \cap \emptyset = \emptyset$

b. $A \cup U = U$

Slide

Topic : Properties of Set Operations

4. Complementation:

   a. $A \cup A^c = U$

   b. $A \cap A^c = \emptyset$

5. Double Complement:

   a. $(A^c)^c = A$

Slide

Topic : Properties of Set Operations

6. Commutative

   a. $A \cup B = B \cup A$

   b. $A \cap B = B \cap A$

   c. $A \Delta B = B \Delta A$

   $\boxed{\text{Note } A - B \neq B - A}$

7. Associative

   a. $A \cup (B \cup C) = (A \cup B) \cup C$

   b. $A \cap (B \cap C) = (A \cap B) \cap C$

Slide

Topic : Properties of Set Operations

8. Absorption

can not contain any element which is not present in A

i.e. $\underset{\sim\sim\sim}{A \cap B \subseteq A}$ & $A \cap B \subseteq B$

a. $A \cup (\underbrace{A \cap B}_{\text{some element of set A}}) = A$

b. $A \cap (A \cup B) = A$

will contain all elements of set A

$\therefore A \cup B \supseteq A$

9. DeMorgan's

a. $(A \cup B)^C = A^c \cap B^c$

b. $(A \cap B)^C = A^c \cup B^c$

Slide

Topic : Properties of Set Operations

10. Distributive

a. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

b. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Set Theory $\rightarrow \left\{ \begin{array}{l} \cap \equiv \cdot \\ \cup \equiv + \end{array} \right\} \leftarrow$ digital

Slide

Topic : Multi-set

An unordered Collection of elements/objects in which an element may appear more than once is called a multiset

eg. $A = \{a, a, b, b, b, c, d, d\}$ is a multiset

(Or)

$= \{2 \cdot a, 3 \cdot b, 1 \cdot c, 2 \cdot d\}$

Multiplicities of corresponding elements

$\boxed{\text{Multiplicity represent the number of times that element appears in the multiset}}$

Slide

Topic : Multi-set

- Let multiset $P = \{m_1 \cdot a_1, m_2 \cdot a_2, m_3 \cdot a_3, \ldots, m_k \cdot a_k\}$
  where m_i is the multiplicity of element a_i.

- and let multiset $Q = \{n_1 \cdot a_1, n_2 \cdot a_2, n_3 \cdot a_3, \ldots, n_k \cdot a_k\}$
  where n_i is the multiplicity of element a_i

Find the multiplicity of a_i in.

(i) $P \cup Q$

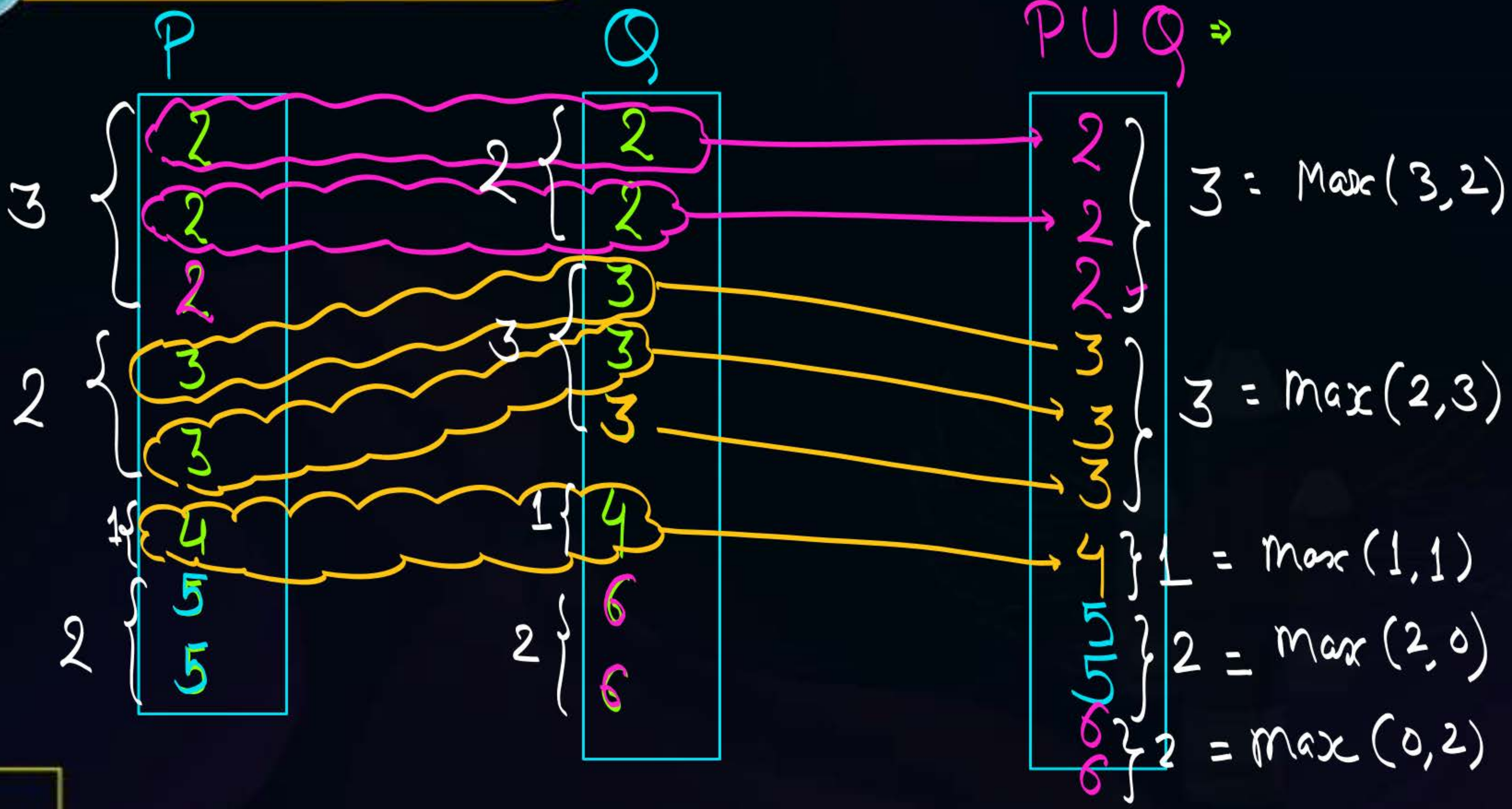
(ii) $P \cap Q$

(iii) $P - Q$

All operations are w.r.t. multiset.

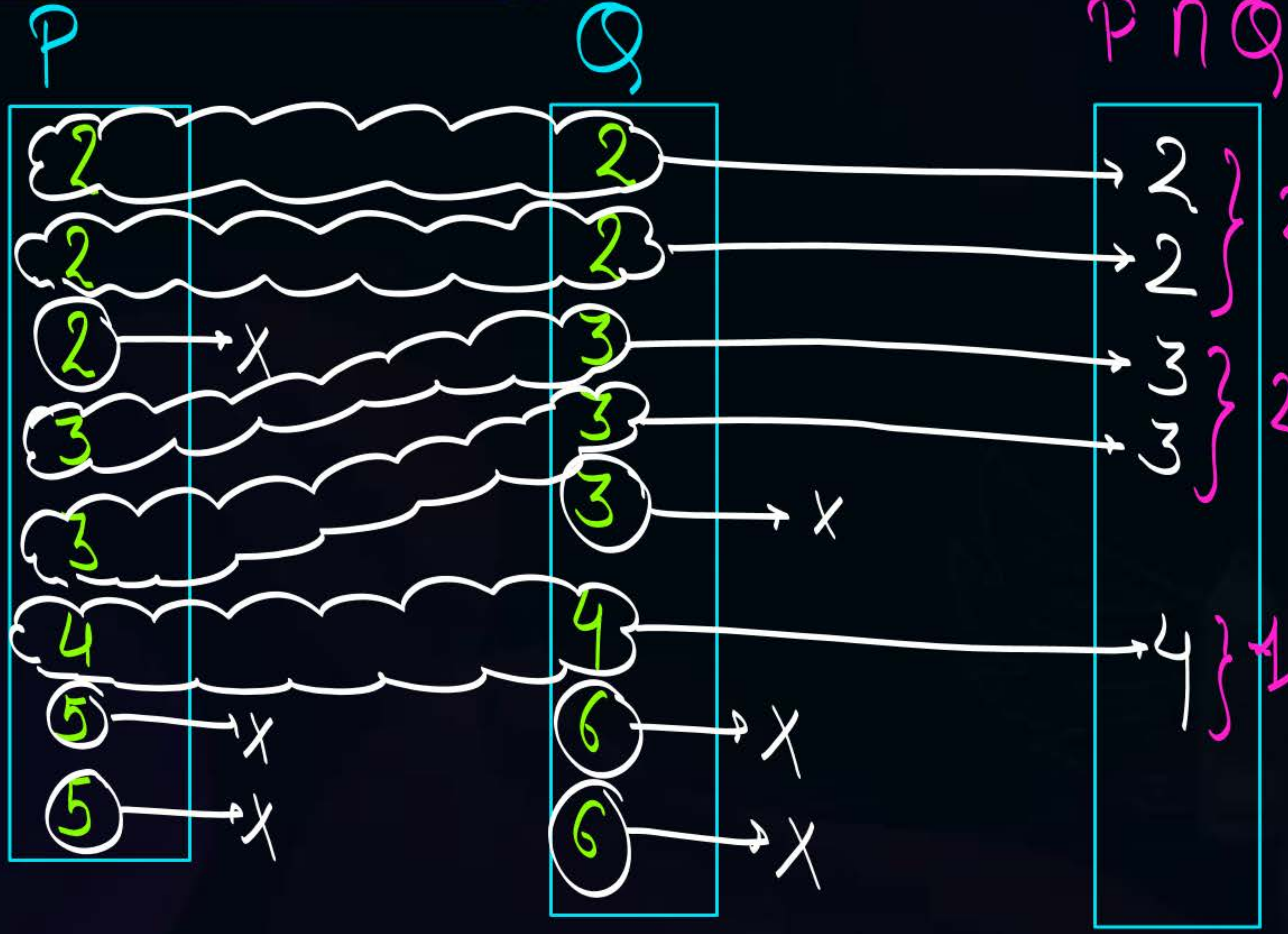
Slide

Topic : Multi-set



Slide

Topic : Multi-set



$2 = \text{Min}(3, 2)$

$2 = \text{Min}(2, 3)$

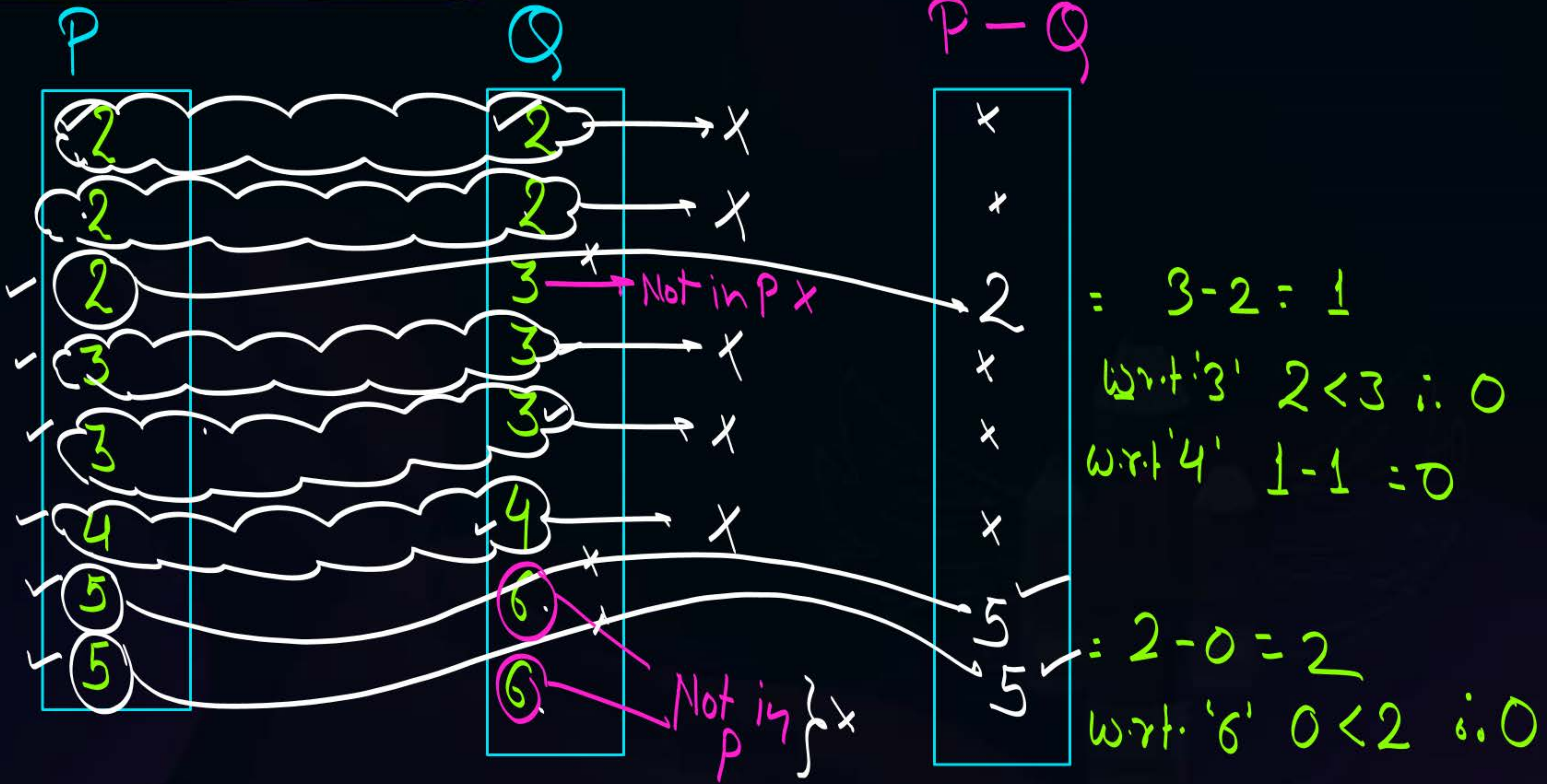
$1 = \text{Min}(1, 1)$

w.r.t. $5 = \text{Min}(2, 0) = 0$

w.r.t. $6 = \text{Min}(0, 2) = 0$

Slide

Topic : Multi-set



Slide

Multiplicity of a_i in $P \cup Q$ = $\max(m_i, n_i)$

Multiplicity of a_i in $P \cap Q$ = $\min(m_i, n_i)$

Multiplicity of a_i in $P - Q$ = $\begin{cases} m_i - n_i & \text{if } m_i \geq n_i \\ 0 & \text{, otherwise} \end{cases}$

Topic : Multi-set

H.W. : Let $A = \{1, 2, 3, 4, \ldots\ldots n\}$

How many multisets are possible with elements of set A

{ Note: each element of A can be taken any no. of times in multiset }

Topic : Multi-set

- H.W. : Let $A = \{1, 2, 3, 4, \ldots\ldots n\}$

How many multisets of size = 4 are possible with the elements of set A, such that at least one element appear exactly twice in the multiset?

{ Note: each element of A can be taken any no. of times in multiset }

Size of $\{1, 2, 3, 4\} = 4$

Size of $\{1, 1, 1, 2\} = 4$

$\{3 \cdot 1, 1 \cdot 2\}$ $3+1=4$

Slide

H.W:

Q1. Let $P(S)$ denote the power set of a set S. Which of the following is always true?

A. $P(P(S)) = P(S)$

B. $P(S) \cap P(P(S)) = \{\emptyset\}$

C. $P(S) \cap S = P(S)$

D. $S \notin P(S)$

Slide

H.W.

Q2. For a set A, the power set of A is denoted by 2^A. If $A = \{5, \{6\}, \{7\}\}$ which of the following options are true?

1. $\emptyset \in 2^A$
2. $\emptyset \subseteq 2^A$
3. $\{5, \{6\}\} \in 2^A$
4. $\{5, \{6\}\} \subseteq 2^A$

Slide

2 mins Summary

- **Topic** Venn diagram
- **Topic** Set operations and properties of set operations
- **Topic** Multi-set

Slide

THANK - YOU

Slide