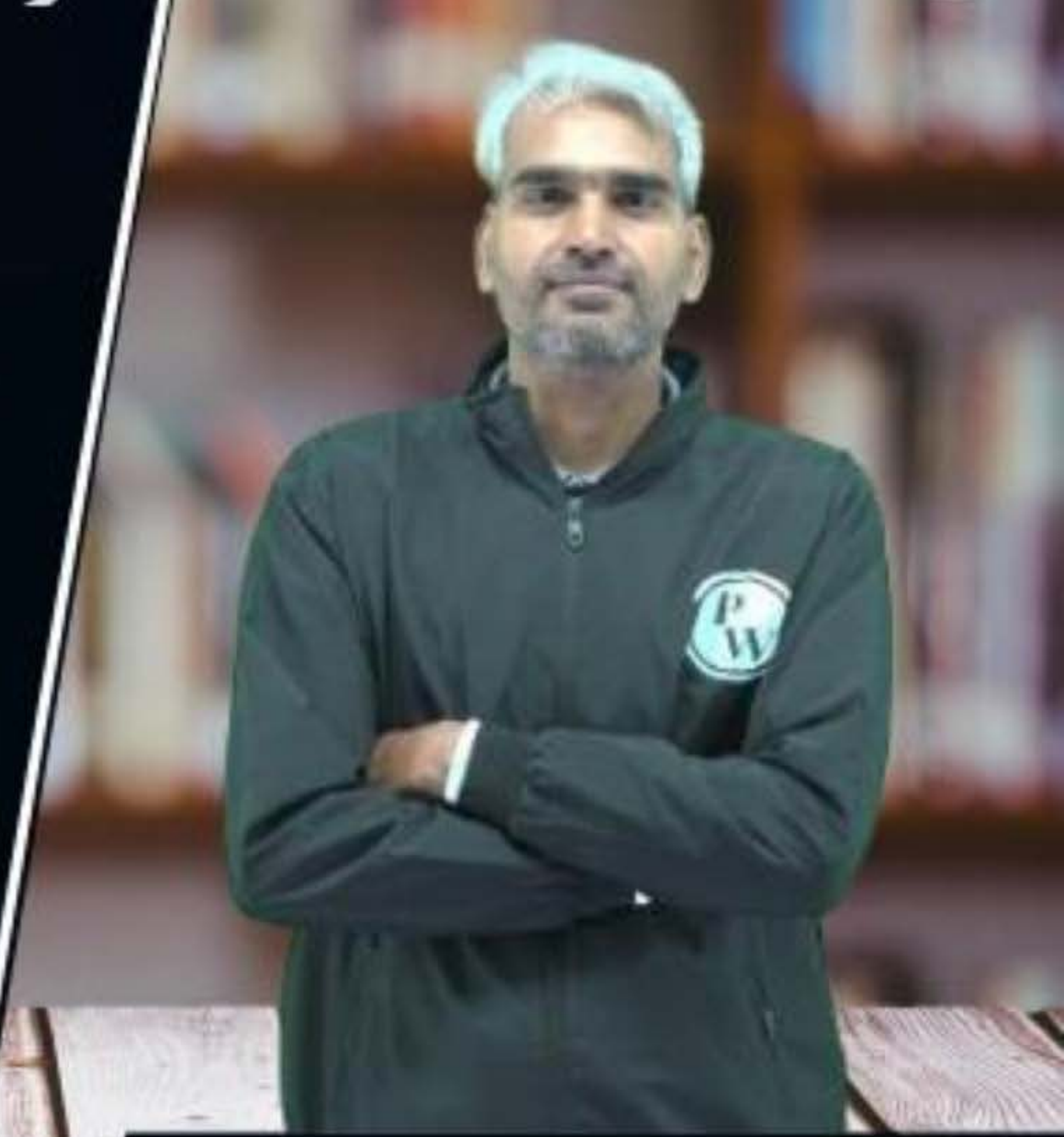
Computer Science & IT

Discrete Mathematics

Graph Theory

Lecture No. 08



By- Vishal Sir

Recap of Previous Lecture

Topic Planar graph

Slide

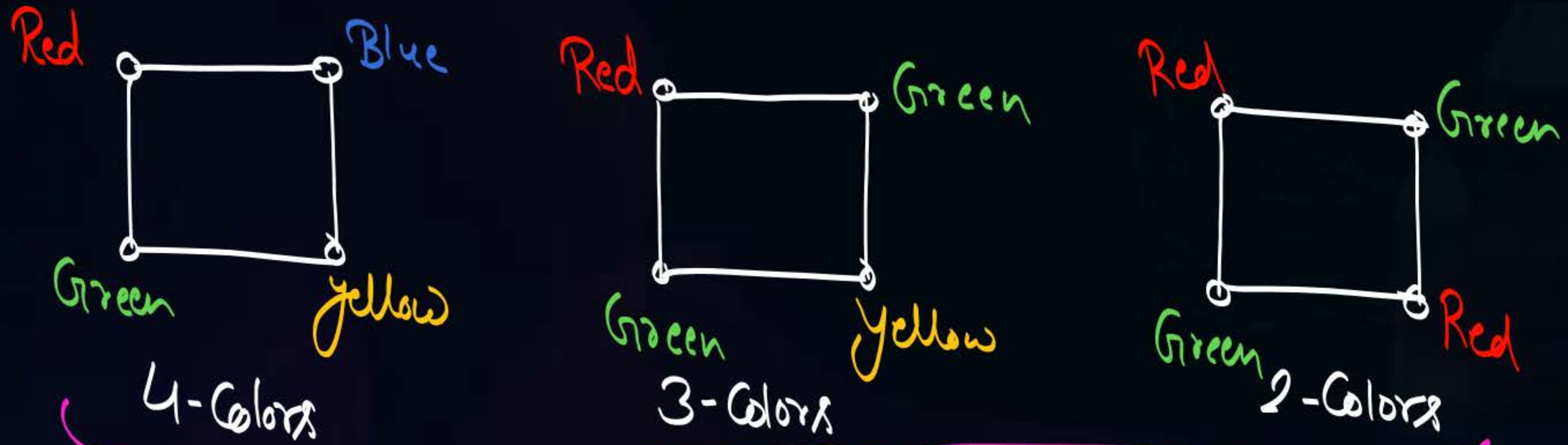
Topics to be Covered

Topic Vertex Coloring and Chromatic Number

Topic Welsh Powell's Algorithm

Topic : Vertex coloring

An assignment of colors to the vertices of graph G, such that no two adjacent vertices of the graph have the same color is called vertex coloring of graph G.



All are Valid Vertex Coloring of the graph

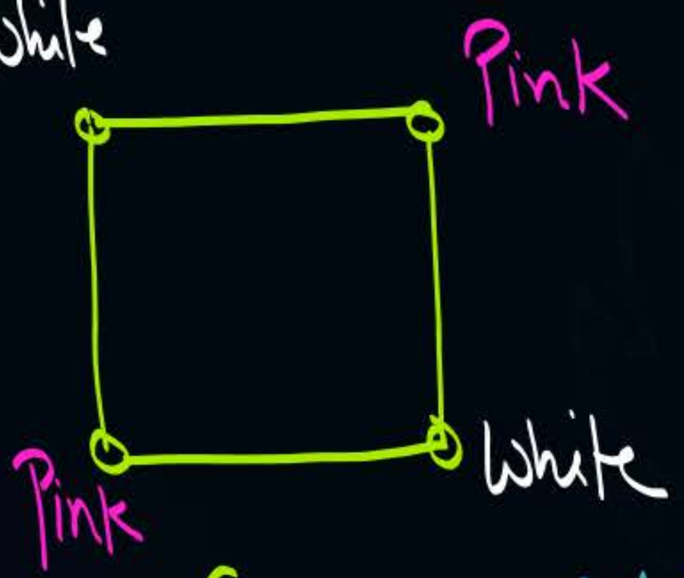
Topic : Chromatic number

No two adjacent vertices should have same color.

Minimum number of colors needed for the vertex coloring of graph G is called chromatic number of graph G,

It is denoted by $\chi(G)$.

eg:-



$G \Rightarrow \chi(G) = 2$

Topic : K-colorable vs K-chromatic

K-Colorable

For a graph G if there exist a vertex coloring of graph G that uses at most 'K'- colors, then graph will be called "K-colorable".

and

$$\boxed{\text{if graph } G \text{ is K-colorable, then } \chi(G) \leq K}$$

K-chromatic

- For a graph G, if Chromatic number of graph G is 'K' then graph is called K-chromatic

$$\boxed{\text{if graph } G \text{ is K-chromatic then } \chi(G) = K}$$

Topic : Four-Color Theorem

- Every planar graph G is four colorable.
- For any planar graph G, $\chi(G) \leq 4$

Planar : $\chi(G) \le 4$

K_1

$\chi(K_1) = 1$

$\Delta(G) = 0$

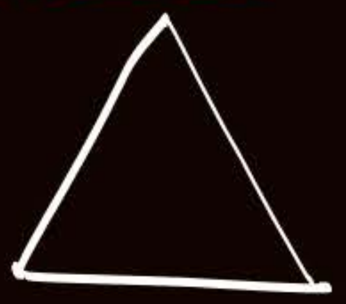
$\chi(G) = \Delta(G) + 1$

K_2

$\chi(K_2) = 2$

$\Delta(G) = 1$

$\chi(G) = \Delta(G) + 1$

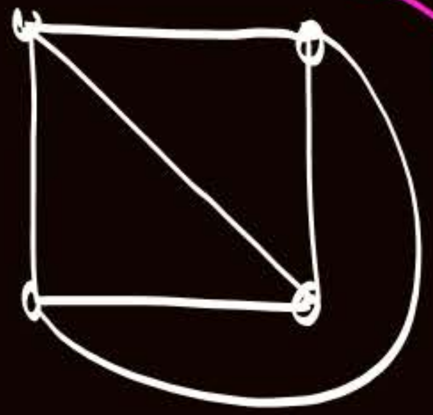


K_3

$\chi(K_3) = 3$

$\Delta(G) = 2$

$\chi(G) = \Delta(G) + 1$

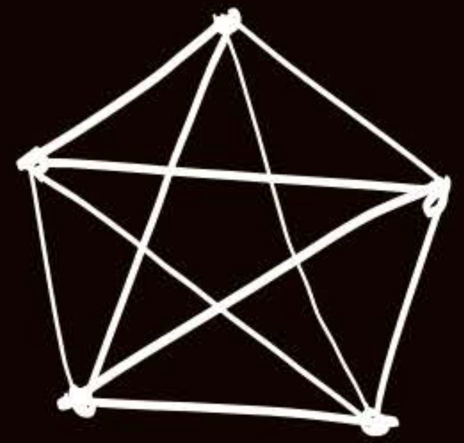


K_4

$\chi(K_4) = 4$

$\Delta(G) = 3$

$\chi(G) = \Delta(G) + 1$



K_5

$\chi(K_5) = 5$

$\chi(K_5) = 5 > 4$

$\Rightarrow$ K_5 is not planar

$\Delta(G) = 4$

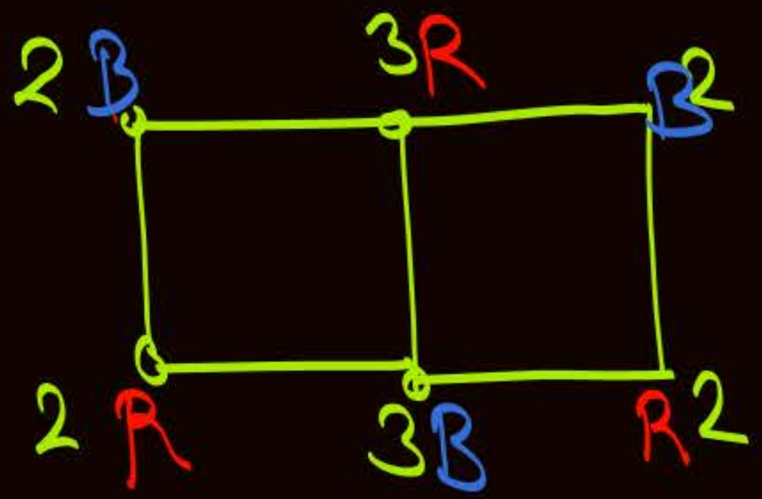
$\chi(G) = \Delta(G) + 1$

* $\boxed{\text{for any complete graph } K_n,\ \chi(K_n) = n}$

Note :-

For any graph G,

$$\chi(G) \leq \Delta(G) + 1$$



'G'

$\Delta(G) = 3$

$\chi(G) = 2$

here, $\boxed{\chi(G) < \Delta(G)}$

Note:

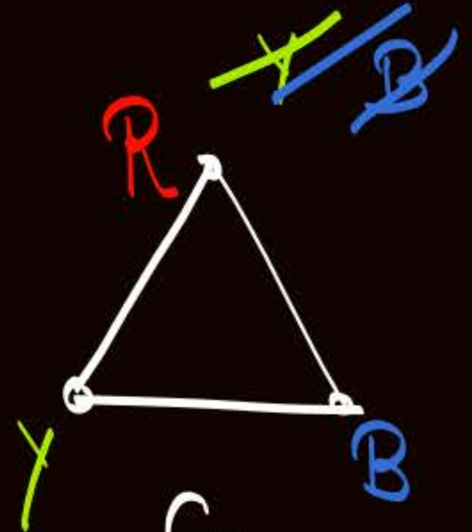
Chromatic number of graph G can be '1' if and only if graph G is a NULL graph { i.e. No edge is present in the graph }

Note: If graph G is not a NULL graph then $\chi(G) \geq 2$

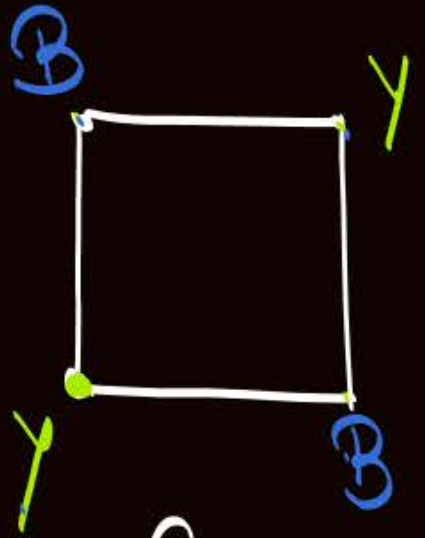
Note:- In a graph G,

$$\chi(G) \geq \delta(G)$$

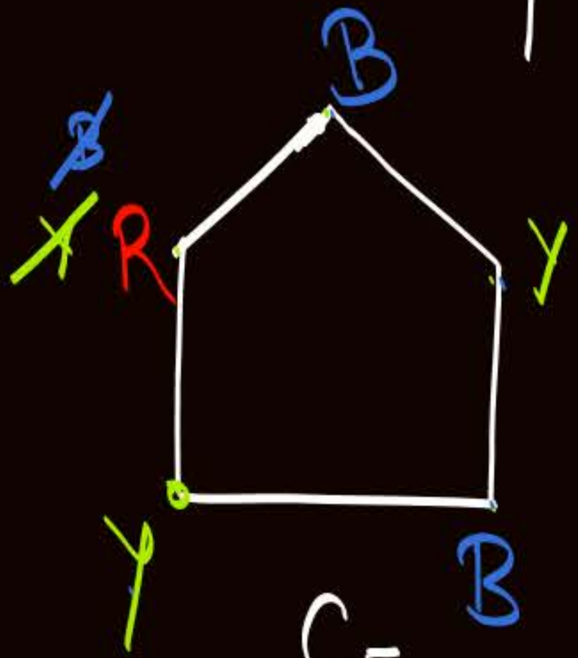
Q: Find the Chromatic number of Cycle graph C_n



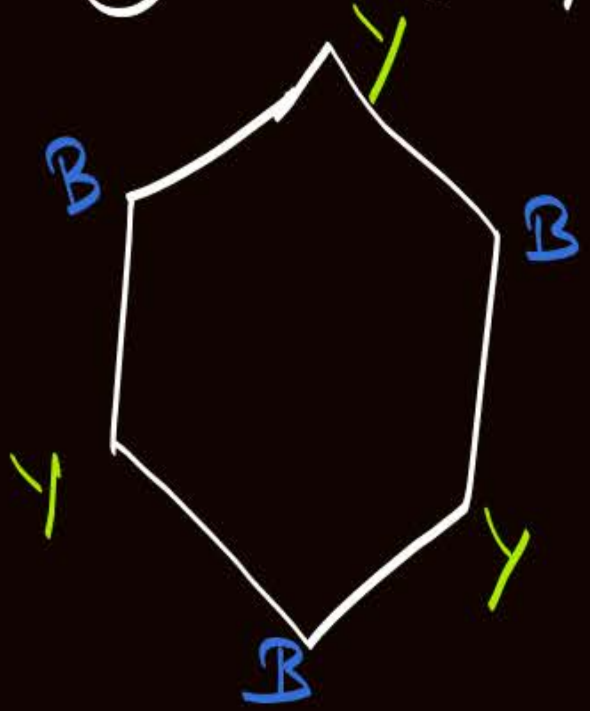
C_3



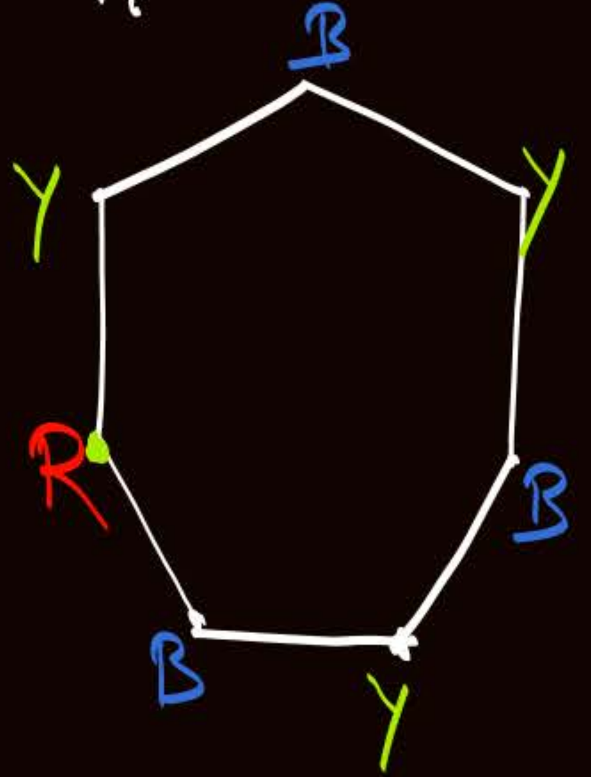
C_4



C_5



C_6



C_7

$\chi(C_3) = 3$

$\chi(C_4) = 2$

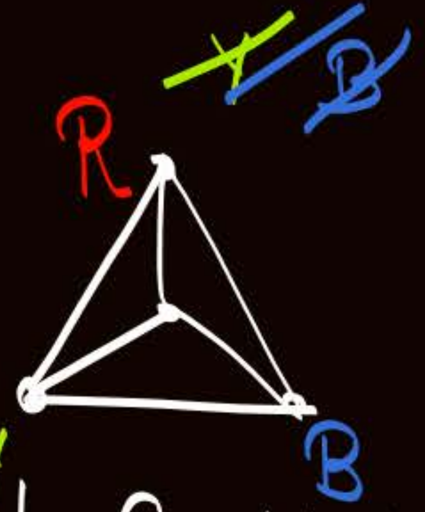
$\chi(C_5) = 3$

$\chi(C_6) = 2$

$\chi(C_7) = 3$

$$\chi(C_n) = \begin{cases} 2, & \text{if } n = \text{even} \\ 3, & \text{if } n = \text{odd} \end{cases}$$

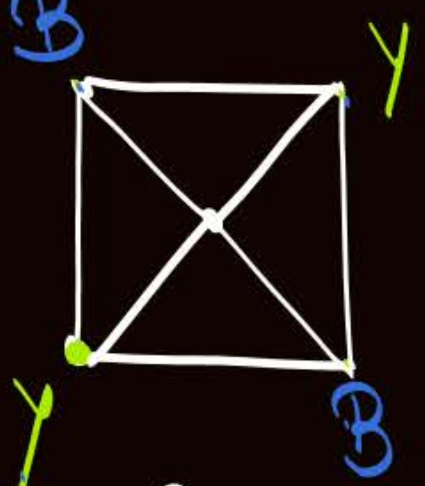
Q: Find the Chromatic number of Wheel graph W_n



$W_4 = C_3 + 1$ hub

3-color + 1 color

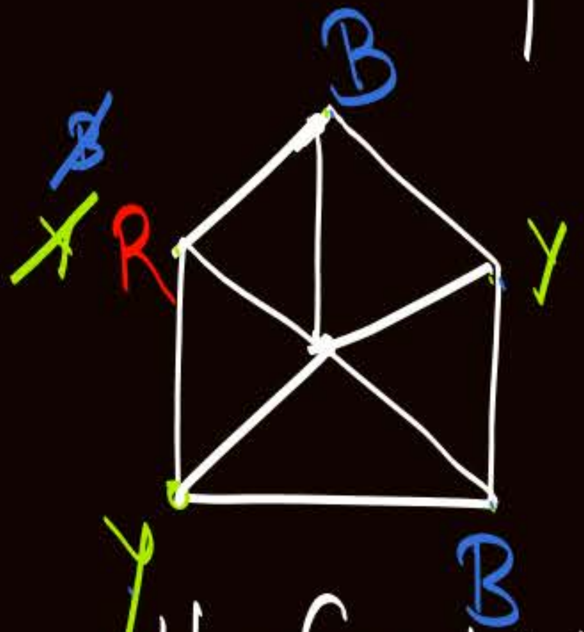
$\chi(W_4) = 4$



$W_5 = C_4 + 1$ hub

2 color + 1 color

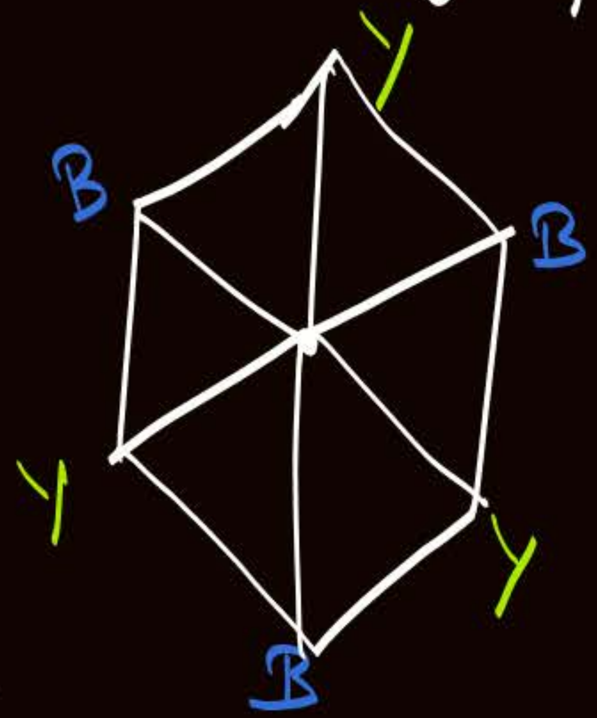
$\chi(W_5) = 2 + 1 = 3$



$W_6 = C_5 + 1$ hub

3 Color + 1 Color

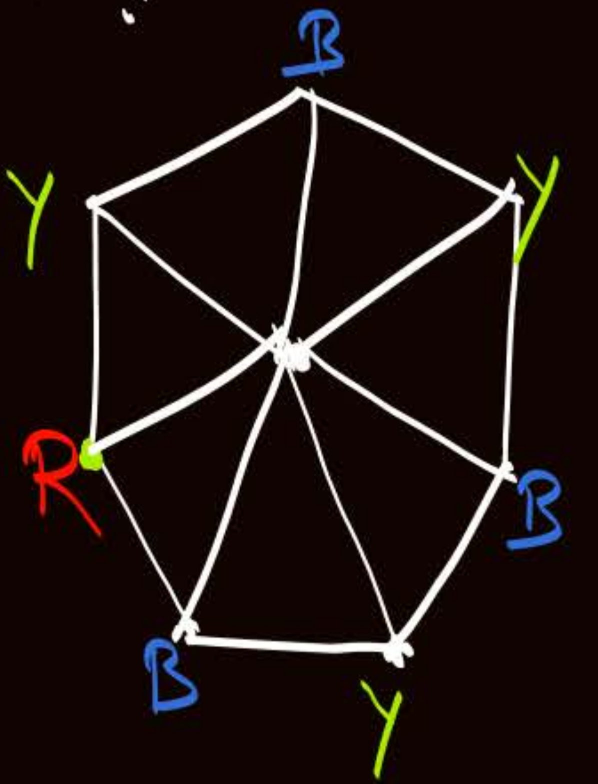
$\chi(W_6) = 3 + 1 = 4$



$W_7 = C_6 + 1$

2 Color + 1 Color

$\chi(W_7) = 2 + 1 = 3$



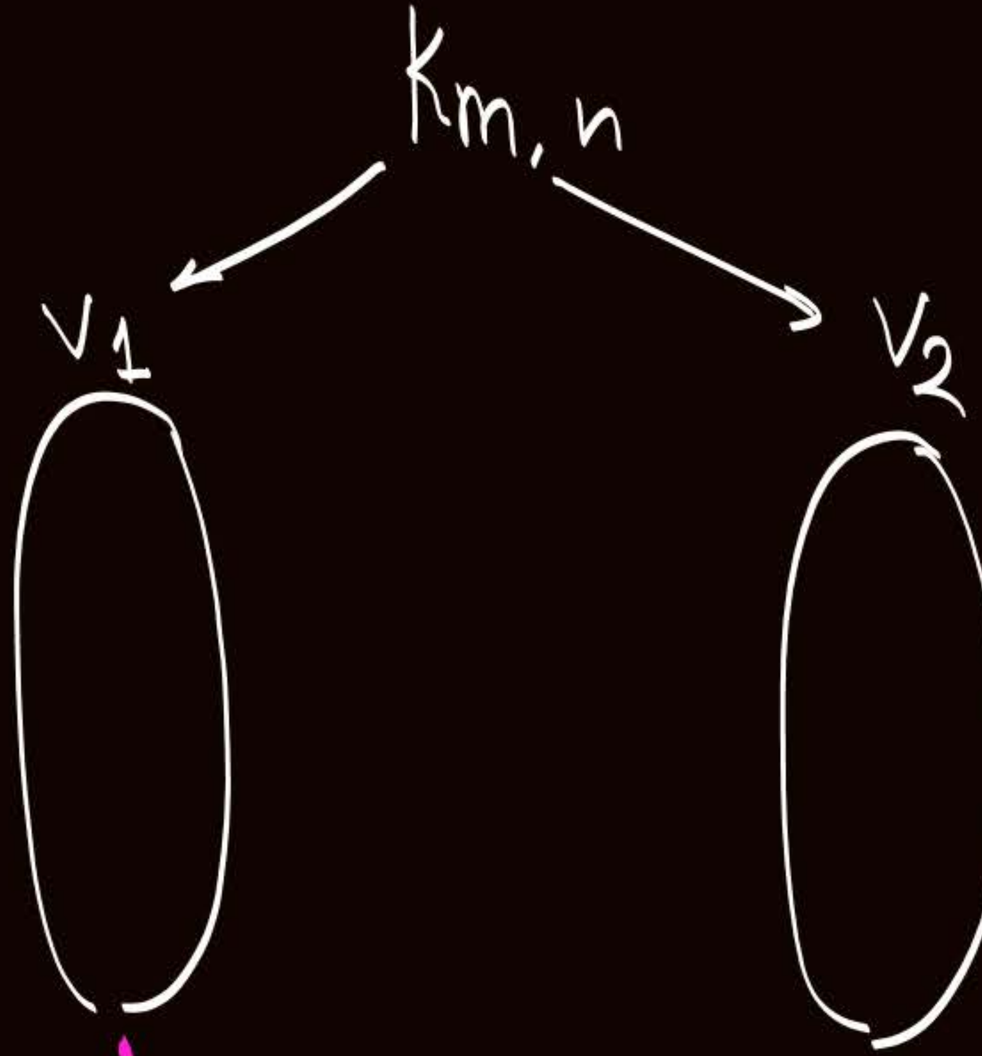
$W_8 = C_7 + 1$

3-Color + 1 Color

$\chi(W_8) = 4$

$$\boxed{\chi(W_n) = \begin{cases} 4, & \text{if } n = \text{even} \\ 3, & \text{if } n = \text{odd} \end{cases}}$$

Q: Find the chromatic number of Complete bipartite graph $K_{m,n}$



$$\boxed{\chi(K_{m,n}) = 2}$$

Assign one color to all vertices of set V_1 & assign some other color to all vertices of set V_2

Note: For any bi-partite graph G, with at least one edge

$$\chi(\text{Bipartite graph}) = 2$$

Bipartite graphs are also known as bi-Colorable graph

Note: If Chromatic number of a graph G is '2', then G is a bipartite graph.

i.e. if $\chi(G)=2$, then G is bartite graph

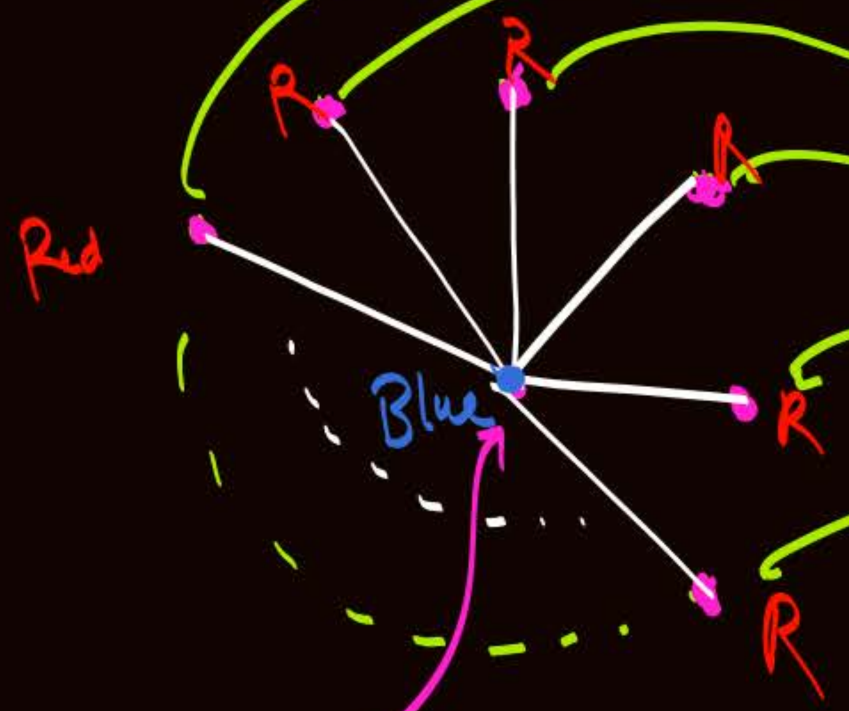
Note 1: In a graph G (Not NULL graph) if all the cycles are of even length then $\chi(G) = 2$

Note 2: If $\chi(G) = 2$, then graph G is a bi-partite graph

Note 3: In a graph G (Not NULL graph) If all the cycles are of even length then graph G is a bipartite graph

Note 4: Every tree (Except a single isolated vertex) is a bipartite graph, and hence Chromatic number of every tree is '2'

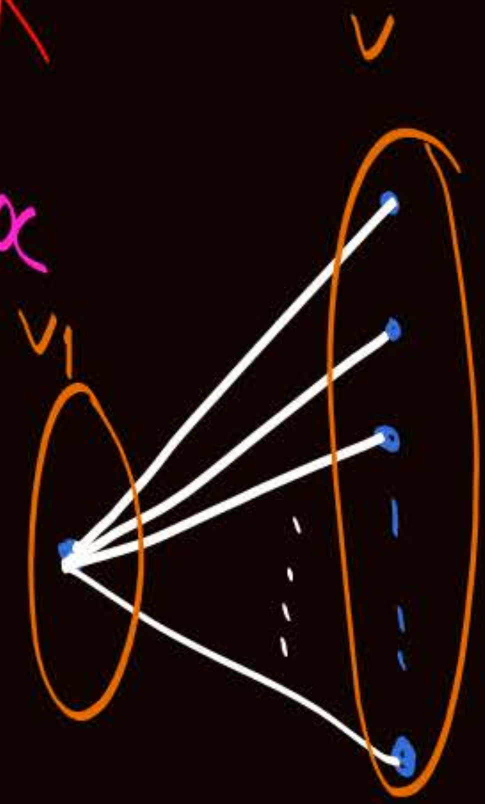
Star graph with n-vertices is defined as



$(n-1)$ vertices, not adjacent to each other, but all $(n-1)$ vertices are adjacent to hub vertex

One hub vertex

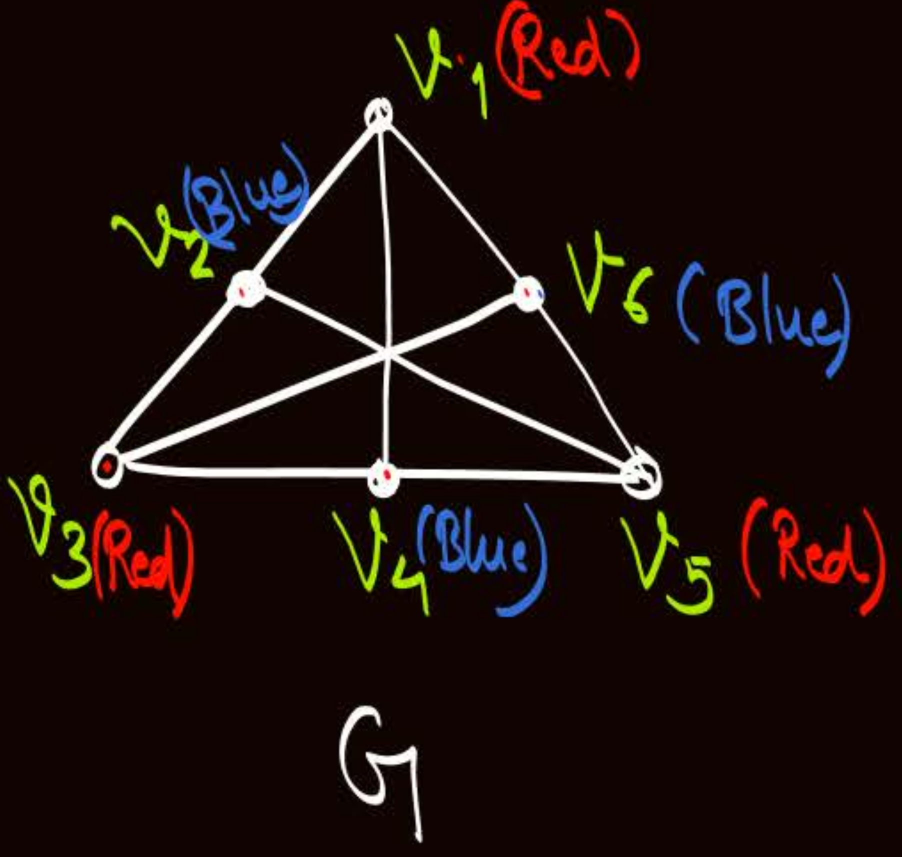
Star graph with n-vertices



$\equiv K_{1,n-1}$ or $K_{n-1,1}$

$$\boxed{\chi(\text{Star-graph with } n\text{-vertices}) = 2}$$

Q1: Find the chromatic number of the following graph



We can color graph G using '2' colors

$\therefore$ 'G' is 2-Colorable, $\therefore \chi(G) \leq 2$ — eqn (1)

G is not a NULL graph $\therefore \chi(G) \geq 2$ — eqn (2)

By eqn (1) & eqn (2)

$$\boxed{\chi(G) = 2}$$

Topic : Welsh Powell's Algorithm

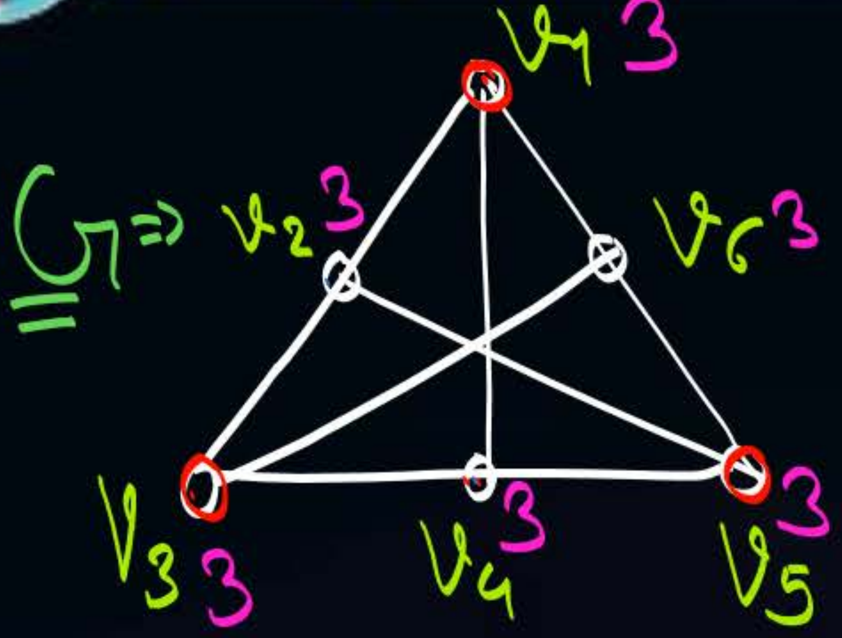
{It is a greedy algorithm, ∴ Need not produce optimal result}

It is used to check the upper bound on chromatic number of a graph.

1. Arrange the vertices of the graph in ~~descending~~ (non-increasing) order of their degrees.
2. If two or more vertices are having the same degree then use alphanumeric order.
3. Assign the colors to the vertices in ~~decreasing~~ non-increasing order of their degrees such that no two adjacent vertices have the same color.
4. If Welsh Powell's algorithm uses 'm' colors for a graph G then $\chi(G) \leq m$. {i.e. G is m-colorable}

Topic : Welsh Powell's Algorithm

all vertices are having same degree ∴ Alphanumeric order



Vertices in non-increasing order of degree	v_1	v_2	v_3	v_4	v_5	v_6
Colors assigned	Red	Blue	Red	Blue	Red	Blue

Welsh Powell's algorithm uses two colors.

$\therefore \chi(G) \le 2$ — eq^n (1)

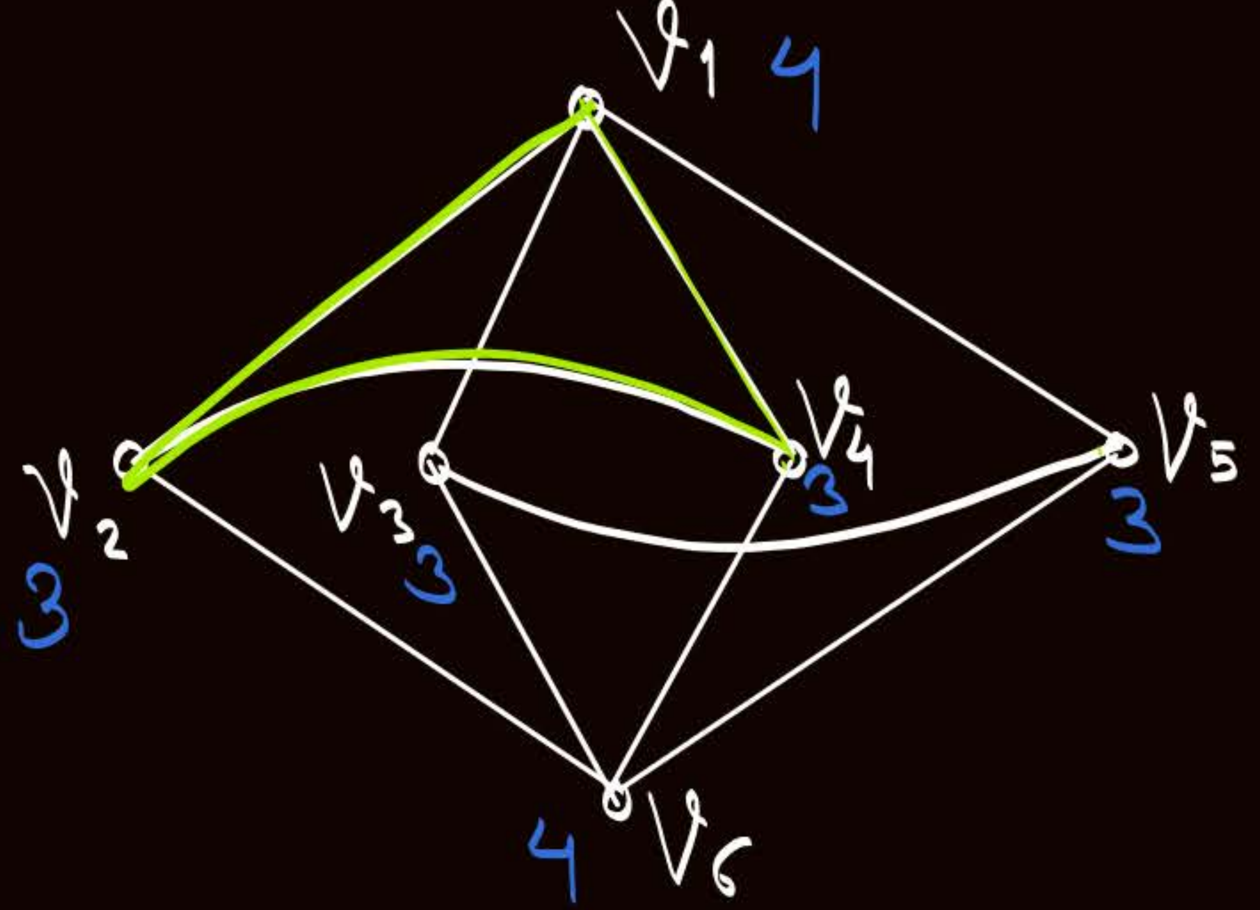
G is not a NULL graph.

$\therefore \chi(G) \ge 2$ — eq^n (2)

⇒ By eq^n (1) & eq^n (2)

$\chi(G) = 2$

Q. Find the chromatic number of following graph.



same degree (v_1, v_6) same degree (v_2, v_3, v_4, v_5)

Vertices	v_1	v_6	v_2	v_3	v_4	v_5
Colors	Red	Red.	Green	Green	yellow	yellow

- Algorithm uses '3' colors.

$\therefore \chi(G) \leq 3$ —— eqⁿ (1)

- v_1, v_2, v_4 form a cycles of length = 3 (odd)

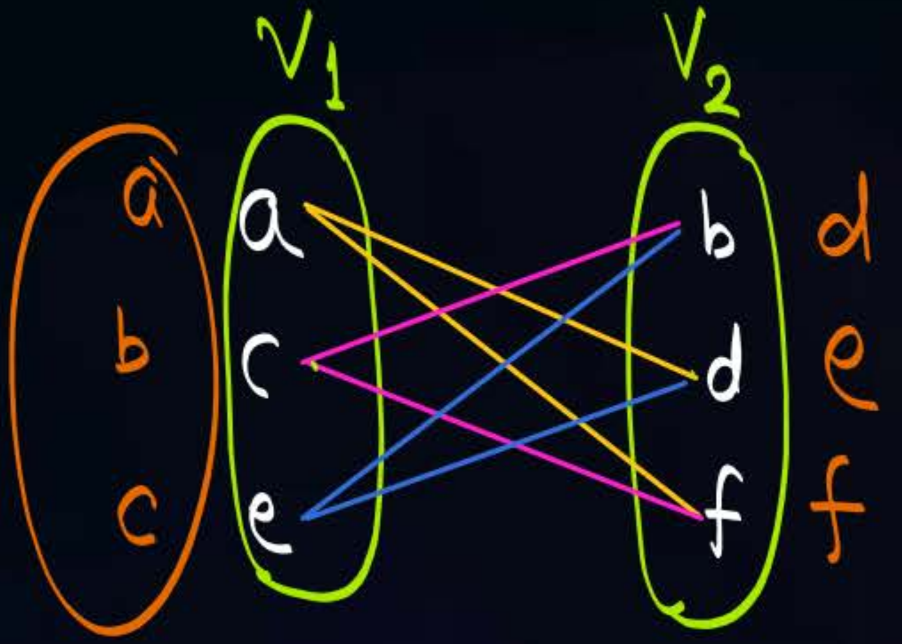
$\therefore \chi(G) \geq 3$ —— eqⁿ (2)

By eqⁿ (1) & (2)

$\chi(G) = 3$

Topic : Welsh Powell's Algorithm

Q. Find the no. of colors used of following graph using Welsh Powell's algo.



all are of same degree {∵ alpha-numeric order}

Vertex	a	b	c	d	e	f
	C_1	C_1	C_2	C_2	C_2	C_2
Color	C_1	C_1	C_2	C_2	C_3	C_3

{Welsh Powell uses three color for a bi-partite graph}

$\chi(G) \leq 3$

i.e., Welsh Powell's algo need not produce Optimal number of colors required.

Edge Coloring :- An assignment of Colors to the edges of the graph such that no two adjacent edges of the graph are Colored using same Color, is "edge coloring" of the graph.

2 mins Summary

Topic Vertex Coloring and Chromatic Number

Topic Welsh Powell's Algorithm

THANK - YOU

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