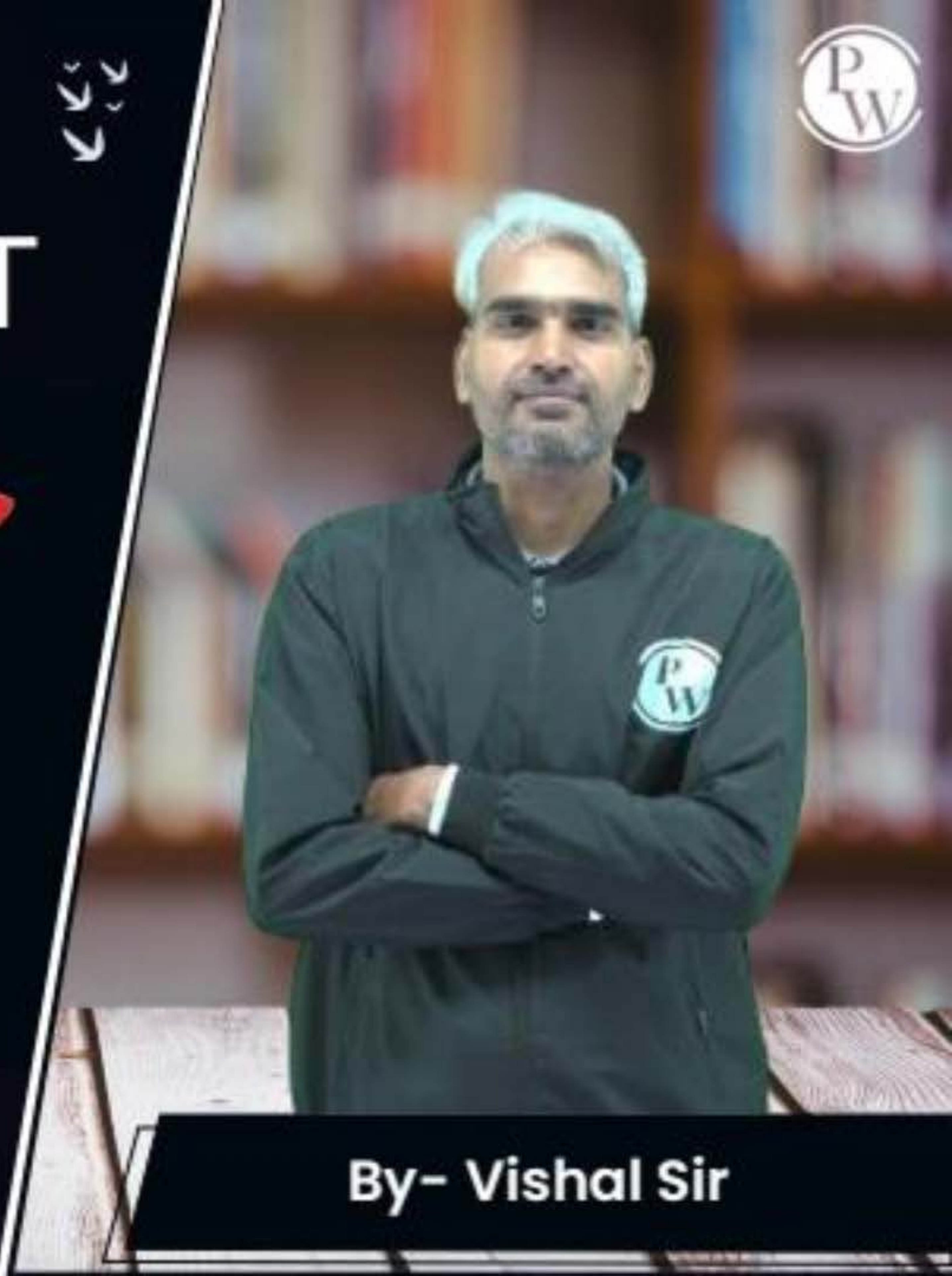
# Computer Science & IT

## Discrete Mathematics

**Set Theory & Algebra**

**Lecture No. 07**



**By- Vishal Sir**

# Recap of Previous Lecture

**Topic** Symmetric, anti-symmetric and asymmetric relation

$2^n \cdot 2^{\left(\frac{n^2-n}{2}\right)}$ $\quad$ $2^n \cdot 3^{\frac{n^2-n}{2}}$ $\quad$ $3^{\frac{n^2-n}{2}}$

**Topic** Transitive relation

if $(aRb \,\&\, bRc)$ then $aRc$, $\forall a,b,c \in A$

**Topic** Complement of a relation

**Topic** Inverse of a relation

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# Topics to be Covered

- **Topic** Composite of two relations
- **Topic** Reflexive, symmetric and transitive closure
- **Topic** Equivalence relation
- **Topic** Equivalence class
- **Topic** Partition of a set

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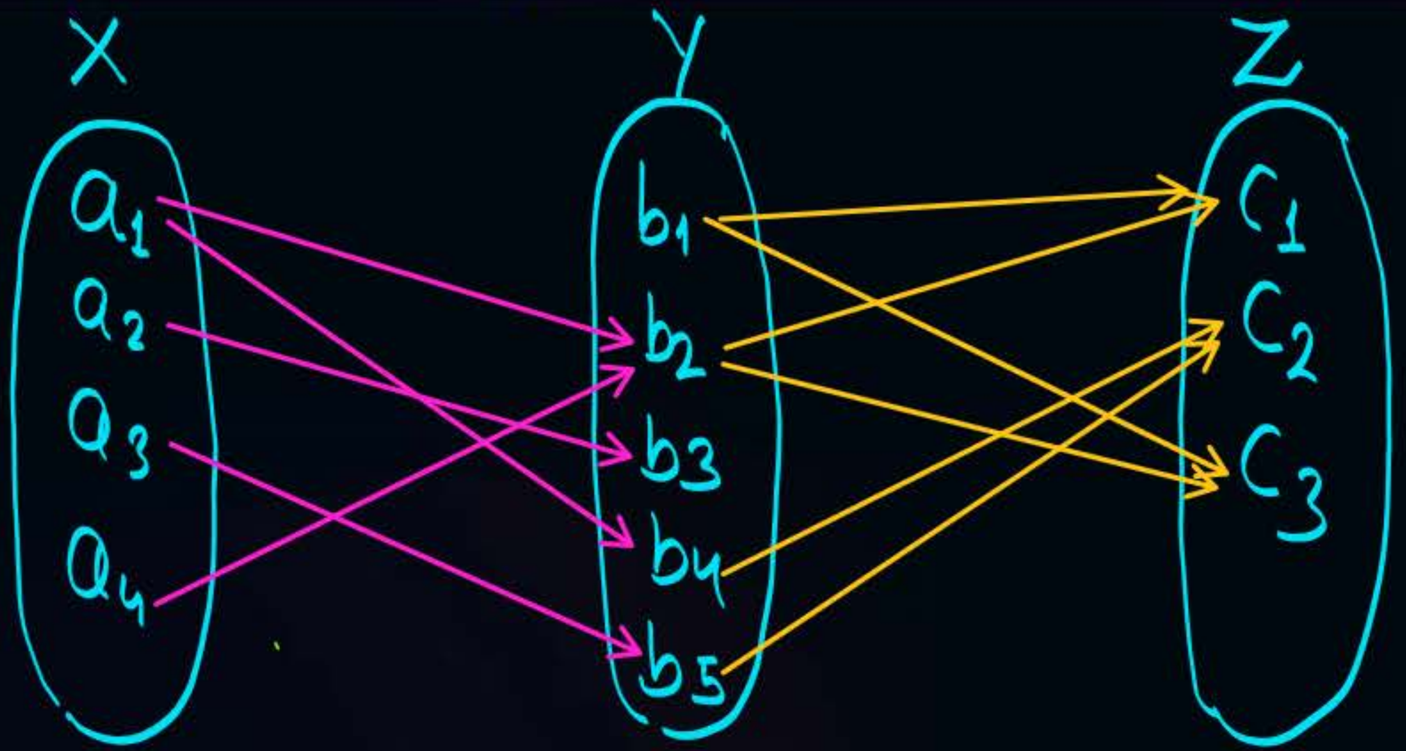
# Topic : Composite of two relations

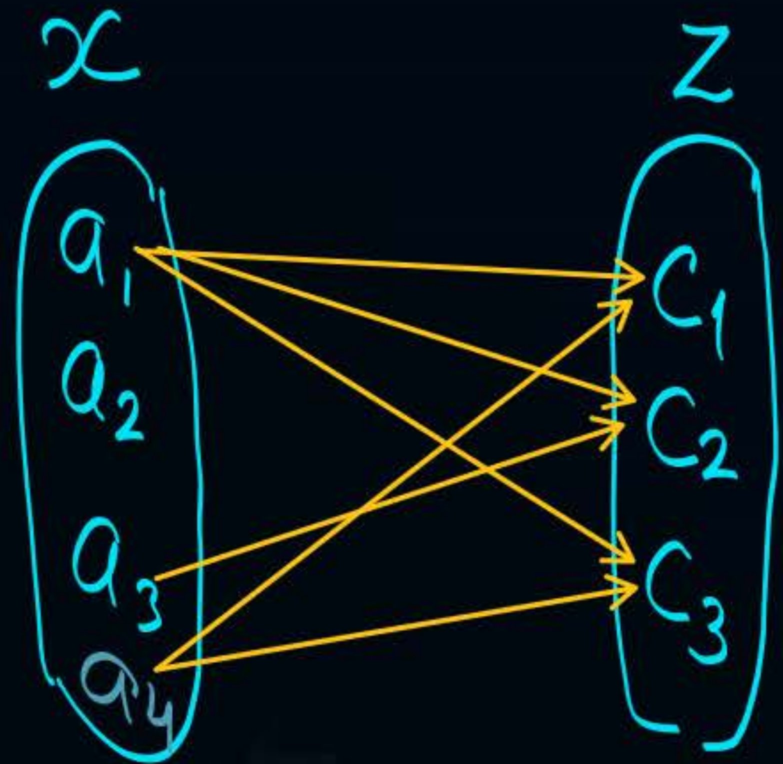



R is relation from X to Y

i.e. $R: X \rightarrow Y$

S is a relation from Y to Z

i.e. $S: Y \rightarrow Z$




Composite Relation

$R;S : X \longrightarrow Z$

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$R = \{(a_1, b_2), (a_1, b_4), (a_2, b_3), (a_3, b_5), (a_4, b_2)\}$

& $S = \{(b_1, c_1), (b_1, c_3), (b_2, c_1), (b_2, c_3), (b_4, c_2), (b_5, c_2)\}$

Identify $\underline{R;S} = ?$

# Topic : Composite of two Relations

Some author define Composite of two relations R and S as "RoS" but there is an ambiguity by different authors

∴ we use 'R;S' or simply 'RS'

Let X,Y and Z be three sets, and

If R is a relation from X to Y, and S is a relation from Y to Z, then their composition R;S is the relation defined as,

$$R;S = \{ (x,z) \mid \underbrace{(x,z) \in X \times Z}_{\text{i.e. } x \in X \,\&\, z \in Z} \text{ and there exists } y \in Y \text{ such that } (x,y) \in R \text{ and } (y,z) \in S\}$$

i.e., R;S is a relation from X to Z defined by the rule that $(x,z) \in R;S$ if and only if there is an element

$y \in Y$ such that $(x,y) \in R$ and $(y,z) \in S$

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# Topic : Reflexive Closure

Let R be a relation on set A,
Reflexive closure of R is the smallest reflexive relation on set A that contains R.

eg. let $A = \{1, 2, 3\}$

$R = \{(1,1), (1,2), (2,1), (2,2), (3,1)\}$

Find Reflexive closure of R

Reflexive Closure of $R = \{\underbrace{(1,1), (1,2), (2,1), (2,2), (3,1)}_{\text{Contain } R}, \underbrace{(3,3)}_{\text{for reflexive}}\}$

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**Note** : Reflexive closure of $R$ will be $= R \cup \Delta_A$

* If relation $R$ is a reflexive relation on set $A$, then reflexive closure of $R$ will be relation '$R$' itself

# Topic : Symmetric Closure

Let R be a relation on set A,
Symmetric closure of R is the smallest symmetric Relⁿ on set A that contains R.

eg: Let $A = \{1, 2, 3\}$

$R = \{(1,1), (1,2), (2,1), (3,1)\}$

Symmetric Closure of $R = \{\underbrace{(1,1), (1,2), (2,1), (3,1)}_{\text{Contain } R}, \underbrace{(1,3)}_{\text{for Symmetric}}\}$

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* **Note:-** * Symmetric closure of $R = R \cup R^{-1}$

* If relation $R$ is a symmetric relation, then symmetric closure of $R$ will be relation $R$ itself.

# Topic : Transitive Closure

Let R be a relation on set A,

Transitive Closure of R is the smallest transitive relation on set A that contains R.

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Q: Let $A = \{1, 2, 3\}$

$R = \{(1,1), (1,2), (1,3), (2,2), (3,1)\}$

Find transitive closure of R.

because of (3,1) & (1,2)

because of (3,1) & (1,3)

1st iteration $R^* = \{(1,1), (1,2), (1,3), (2,2), (3,1)\ (3,2), (3,3)\}$

all pairs of $R^*$

2nd iteration $R^{**} = \{(1,1), (1,2), (1,3), (2,2), (3,1), (3,2), (3,3)\}$

Nothing new is added

$R^* = R^{**}$, $\therefore$ $R^*$ is transitive closure of R

**Note:-** If relation R is a transitive relation, then transitive closure of relation R will be relation R itself

**Reflexive transitive closure:→** Reflexive transitive closure of relation R will be, the relation which is smallest relation on the set such that it is reflexive as well as transitive and contains R.

Q: let $A = \{1, 2, 3\}$

$R = \{(1,1), (1,3), (2,2), (3,1), (3,2)\}$

find transitive closure of $R$

$R^* = \{(1,1), (1,3), (2,2), (3,1), (3,2), (1,2), (3,3)\}$

$R^{**} = \{(1,1), (1,2), (1,3), (2,2), (3,1), (3,2), (3,3)\}$

Nothing new is added

$R^* = R^{**}$

$\therefore R^*$ is our transitive closure

Q. $A = \{a, b, c, d\}$

$R = \{(a,d), (b,a), (b,c), (c,a), (c,d), (d,c)\}$

Find transitive closure of R.

$R^* = \{(a,d), (b,a), (b,c), (c,a), (c,d), (d,c), \; (a,c), (b,d), (c,c), (d,a), (d,d)\}$

Rearranged $R^* = \{(a,c), (a,d), (b,a), (b,c), (b,d), (c,a), (c,c), (c,d), (d,a), (d,c), (d,d)\}$

$R^{**} = \{(a,c), (a,d), (b,a), (b,c), (b,d), (c,a), (c,c), (c,d), (d,a), (d,c), (d,d), \; (a,a)\}$

Rearranged $R^{**} = \{(a,a), (a,c), (a,d), (b,a), (b,c), (b,d), (c,a), (c,c), (c,d), (d,a), (d,c), (d,d)\}$

$R^{***}$ = Same as $R^{**}$, i.e. $R^{**} = R^{***}$ $\therefore$ $R^{**}$ is transitive closure of R

**Note:-** Warshall's algorithm can be used to identify the transitive closure of a given relation

# Topic : Equivalence relation

A relation R on set A is said to be an Equivalence relation if and only if relation is

① Reflexive

, ② Symmetric

& ③ Transitive

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eg. let $A = \{1, 2, 3\}$

① $\Delta_A = R_1 = \{(1,1), (2,2), (3,3)\}$

Reflexive ✓
Symmetric ✓
Transitive ✓

$\Rightarrow$ ∴ Equivalence Relⁿ

$\boxed{\text{Diagonal relation on set } A \text{ is the smallest equivalence relation on set } A}$

② $A \times A = R_2 = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

$\boxed{A \times A \text{ is the largest equivalence relation on set } A}$

Reflexive ✓
Symmetric ✓
Transitive ✓

③ $R_3 = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$

Reflexive ✓
Symmetric ✓
transitive ✓

$\Rightarrow$ ∴ Equivalence Relⁿ

④ $R_4 = \{(1,1), (2,2), (3,3)\ (1,3)\ (3,1)\}$ is Equivalence relⁿ

⑤ $R_5 = \{(1,1), (2,2), (3,3)\ (2,3), (3,2)\}$ is Equivalence Relⁿ

$R_6 = \{ (1,1), (2,2), (3,3), (1,2), (2,1), (1,3), (3,1) \}$

$\underline{R_6 \text{ is}}$ Reflexive ✓

Symmetric ✓

but $\underline{\text{not transitive}}$

$\because (2,3)$, but $(2,3) \notin R_6$

$\therefore$ Not transitive

to make it transitive if we add (2,3) then for symmetricity we will have add (3,2) as well

And it will become $A \times A$.

$\therefore$ No other equivalence $rel^n$ possible on set $A = \{1, 2, 3\}$

# Topic : Equivalence Class

Let 'R' be an equivalence relation on set A,
for any element $x \in A$ the equivalence class of element '$x$' w.r.t. equivalence relation R can be denoted by $[x]$, and it is defined as,

$$[x] = \{ y \mid x R y \}$$

i.e. $(x, y) \in R$

i.e. Equivalence class of element '$x$' is a set of all elements which are related with $x$

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eg. let $A = \{1, 2, 3, 4, 5\}$
and let $R$ is an Equivalence relation on set $A$.

$$R = \{ \underbrace{(1,1), (2,2), (3,3), (4,4), (5,5)}_{\text{reflexive}}, \underbrace{(1,3), (3,1)}, \underbrace{(4,5), (5,4)} \}$$

symmetric

as well as transitive

define Equivalence of every element of set A, w.r.t. equivalence Rel$^n$ 'R'

$[1] = \{1, 3\}$

$[2] = \{2\}$

$[3] = \{3, 1\} = \{1, 3\}$

same

$[4] = \{4, 5\}$

$[5] = \{5, 4\} = \{4, 5\}$

same

There are three distinct equivalence classes for elements of set A

i.e. $\{1, 3\}$ & $\{2\}$ & $\{4, 5\}$

**Note:-** (1) Equivalence class of element $x$ {i.e $[x]$} may be same as equivalence class of element $y$ {i.e. $[y]$} even if $x \neq y$.

{ in the above eg.
$1 \neq 3$,
but $[1] = [3] = \{1, 3\}$ }

(2) The set of all distinct equivalence classes of elements of set A w.r.t. an Equivalence relation R Creates a partition of set A.

i.e $\{\underbrace{\{1,3\}, \{2\}, \{4,5\}}_{\text{distinct equivalence classes of elements of set A}}\}$ in above example is a Partition of set $A = \{1,2,3,4,5\}$

set of

# Topic : Partition of a set

Partition of a set A is a set of non-empty subsets of set A such that each element of set A is present in exactly one of those non-empty subsets.

(or)

Let A be a non-empty set, and $\underbrace{A_1, A_2, A_3, \ldots A_k}$ are non-empty subsets of set A, then

$\{A_1, A_2, A_3, \ldots, A_k\}$ is a partition of set A

if and only if ① $A_i \cap A_j = \emptyset, \forall\, i \neq j$

& ② $\bigcup_{i=1}^{k} A_i = A$ $\{$i.e. $A_1 \cup A_2 \cup \ldots \cup A_k = A\}$

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Q: Let $A = \{1, 2, 3, 4, 5\}$

Which of the following is/are partitions of set A.

(a) $\{\{1,2\}, \{3,4\}, \{4,5\}\}$ $\quad \{3,4\} \cap \{4,5\} = 4 \neq \emptyset$ {∴ Not a partition}

(b) $\{\{1,2\}, \{3\}, \{4,5\}, \{\ \}\}$ $\quad$ Empty subsets are not allowed {∴ Not a Partition}

(c) $\{\{1\}, \{2,3\}, \{4\}\}$ $\quad \{1\} \cup \{2,3\} \cup \{4\} = \{1,2,3,4\} \neq A$ {∴ Not a Partition}

(d) $\{\{1,2\}, \{3,4,5\}\}$ $\quad \left.\begin{matrix} \cap = \emptyset \\ \cup = A \end{matrix}\right\}$ ✓ ∴ Partition of A

(e) $\{\{1,2\}, \{3\}, \{4\}, \{5\}\}$ $\quad \left.\begin{matrix} \cap = \emptyset \ \forall i \neq j \\ \& \cup = A \end{matrix}\right\}$ ∴ Partition of A

There May be More than one Partition of same set

H.W. Find the no. of Partitions of set 'A':

1. when $|A| = 0$
2. when $|A| = 1$
3. when $|A| = 2$
4. when $|A| = 3$
5. when $|A| = 4$
6. when $|A| = 5$
7. when $|A| = 6$

# 2 mins Summary

**Topic** Composite of two relations

**Topic** Reflexive, symmetric and transitive closure

**Topic** Equivalence relation

**Topic** Equivalence class

**Topic** Partition of a set

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# THANK - YOU

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