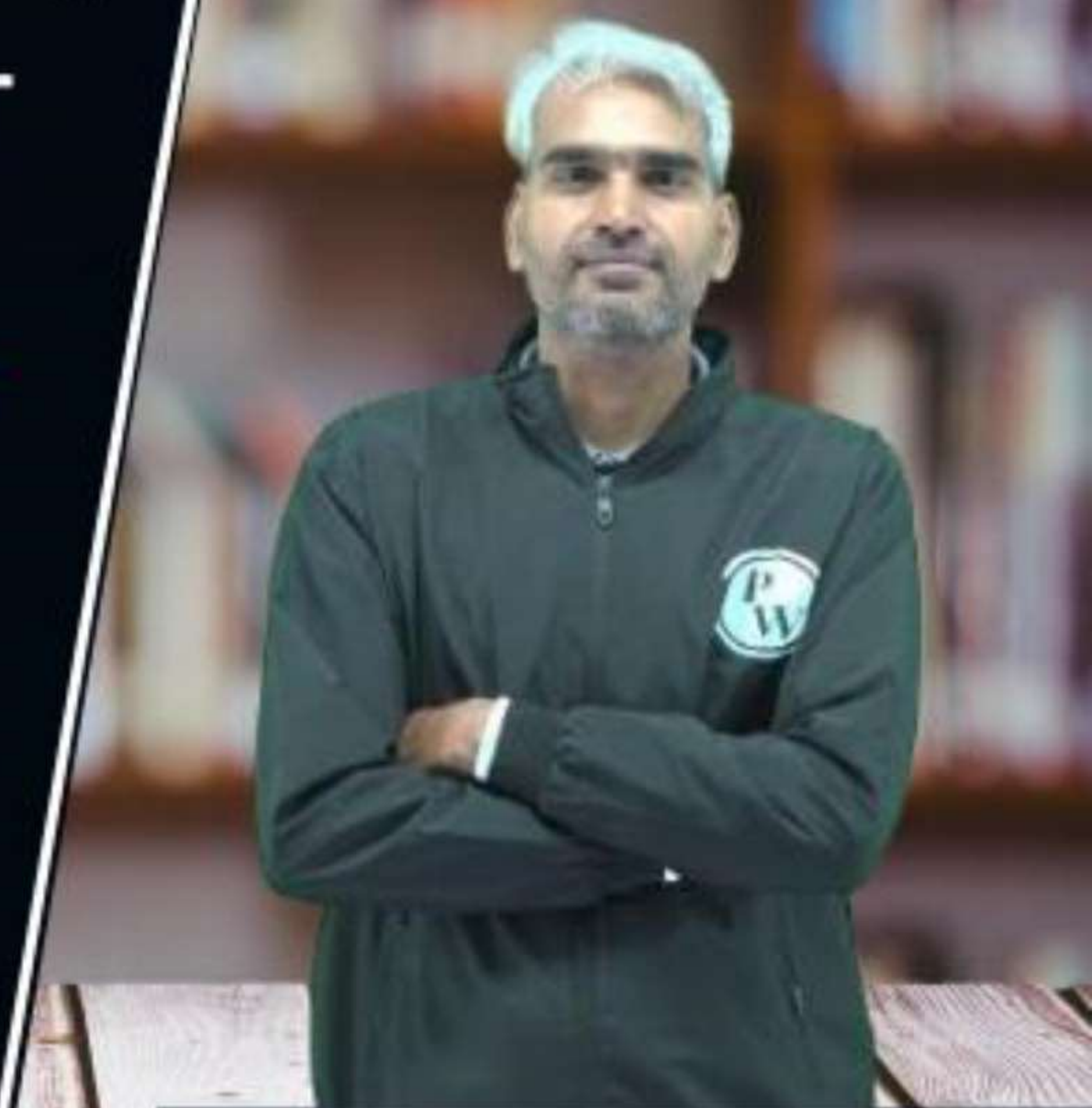
Computer Science & IT

Discrete Mathematics

Set Theory & Algebra

Lecture No. 15



By- Vishal Sir

Recap of Previous Lecture

- Topic — Complemented lattice {1 or more Complements}
- Topic — Distributive lattice {0 or 1 Complement}
- Topic — Boolean lattice / Boolean algebra {Exactly one Complement}

Slide

Topics to be Covered

Topic Boolean lattice / Boolean algebra

Topic Functions

Topic Range of a function

Topic Injective (one-one) function

Topic Surjective (onto) function

Slide

Topic : Boolean lattice

- A lattice which is both Complemented as well as distributive is called a Boolean algebra <u>or</u> Boolean lattice

- A lattice is called boolean lattice if and only if every element has exactly one complement

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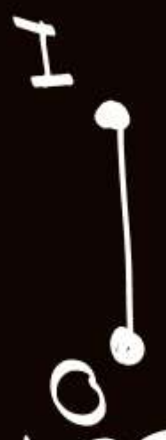
$2^0 = 1$

$\bar{a} = a$

Boolean lattice



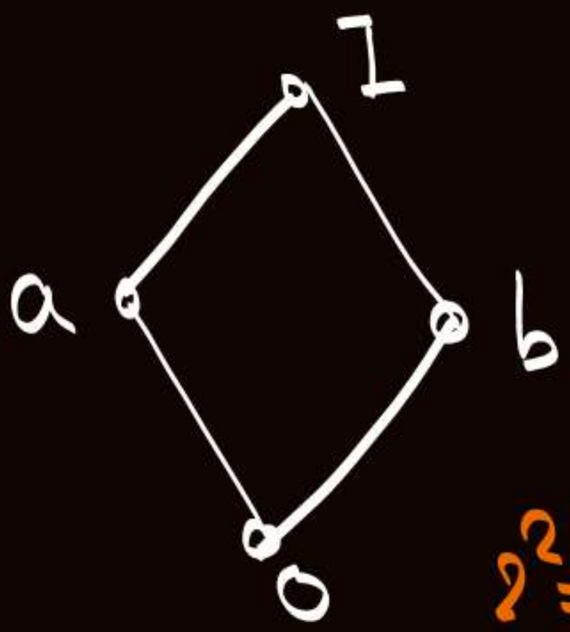
$2^1 = 2$

$|E| = 1 \cdot 2^{1-1} = 1$

$\bar{0} = I$

$\bar{I} = 0$

Boolean lattice



$2^2 = 4$

$|E| = 2 \cdot 2^{2-1} = 4$

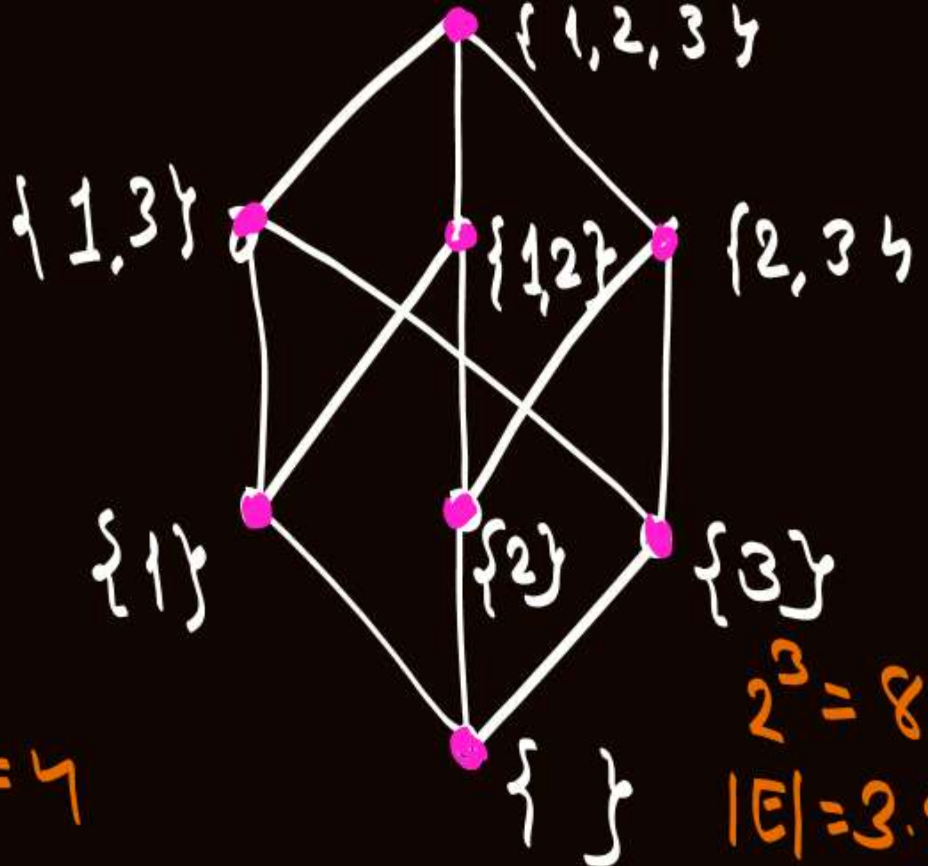
$\bar{0} = I$

$\bar{I} = 0$

$\bar{a} = b$

$\bar{b} = a$

Boolean lattice



$2^3 = 8$

$|E| = 3 \cdot 2^{3-1} = 12$

$\overline{\{\ \}} = \{1,2,3\}$	
$\overline{\{1\}} = \{2,3\}$	$\overline{\{1,2\}} = \{3\}$
$\overline{\{2\}} = \{1,3\}$	$\overline{\{1,3\}} = \{2\}$
$\overline{\{3\}} = \{1,2\}$	$\overline{\{2,3\}} = \{1\}$
	$\overline{\{1,2,3\}} = \emptyset$

Boolean lattice

Note:- A boolean lattice will have 2^n vertices, and $n \cdot 2^{n-1}$ edges

Note :- If 'n' is positive integer such that D_n {Set of all +ve divisors of 'n'} has no perfect square (Except '1'), then 'n' is called a square free integer.

eg: $D_{21} = \{\underbrace{1, 3, 7, 21}\}$

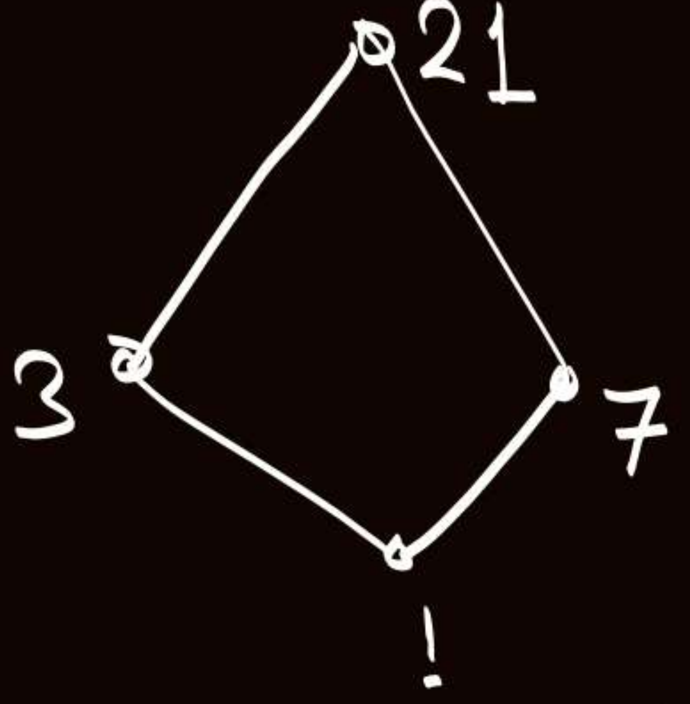
no element Except '1' is a perfect square, ∴ '21' is a square free integer

Note :- If 'n' is a square free integer, then $(D_n, \div)$ is a boolean lattice

and for any element $x \in D_n$, $\bar{x} = \dfrac{n}{x}$

$D_{21} = \{1, 3, 7, 21\}$

POSET $(D_{21}, \div)$ is a lattice



it is a Boolean lattice

$\bar{1} = 21 \Rightarrow \bar{1} = \frac{21}{1} = 21$

$\bar{3} = 7 \Rightarrow \bar{3} = \frac{21}{3} = 7$

$\bar{7} = 3 \Rightarrow \bar{7} = \frac{21}{7} = 3$

$\overline{21} = 1 \Rightarrow \overline{21} = \frac{21}{21} = 1$

Which of the following statements is/ are not true

True a) If A is any finite set then [P(A),⊆] is distributive lattice $\left\{ \begin{array}{l} \text{for set } X \in P(A) \\ \bar{X} = A - X \end{array} \right.$

b) Every sub lattice of a distributive lattice is also a distributive lattice

c) Every totally ordered set is a distributive lattice

d) Every totally ordered set is bounded

e) Every distributive lattice is bounded

f) Every distributive lattice is a complemented lattice

Slide

Which of the following statements is/ are not true

a) If A is any finite set then [P(A),⊆] is distributive lattice

for set $X \in P(A)$
$\overline{X} = A - X$

(True) → b) Every sub lattice of a distributive lattice is also a distributive lattice

c) Every totally ordered set is a distributive lattice

d) Every totally ordered set is bounded

e) Every distributive lattice is bounded

f) Every distributive lattice is a complemented lattice

let $[L, \vee, \wedge]$ be a distributive lattice, and $[M, \vee, \wedge]$ be a sublattice of L.

for any pair of elements $a, b, c \in M$ we know $a, b, c \in L$ as well

L is a distributive lattice
$\therefore\ a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
& $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
holds true in lattice L

$\Rightarrow$ we know $\vee$ & $\wedge$ represent lub & glb $\Rightarrow$

In a Sublattice lub & glb for every pair of elements will be same as original lattice
$\therefore$ Distributive Property will hold in sublattice as well.

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Which of the following statements is/ are not true

a) If A is any finite set then $[P(A), \subseteq]$ is distributive lattice

b) Every sub lattice of a distributive lattice is also a distributive lattice

(True) c) Every totally ordered set is a distributive lattice

False d) Every totally ordered set is bounded — $(N, \leq)$ is Totally ordered set, but not bounded

False e) Every distributive lattice is bounded — lattice w.r.t. $(N, \leq)$ is distributive but not bounded

False f) Every distributive lattice is a complemented lattice

eg: $(\{1, 2, 3, 4\}, \leq)$

it is distributive, but not complemented

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Which of the following lattice is /are not distributive?

(A) $(D_{125}, \div)$
(B) $(P(A), \subseteq)$
(C) $(D_{12}, \div)$
(D) $(\{1, 2, 3, 5, 30\}, \div)$
(E) $(\{1, 2, 3, 4, 12\}, \div)$

(A) $D_{125} = \{1, 5, 25, 125\}$

Distributive ✓
Complemented ✗

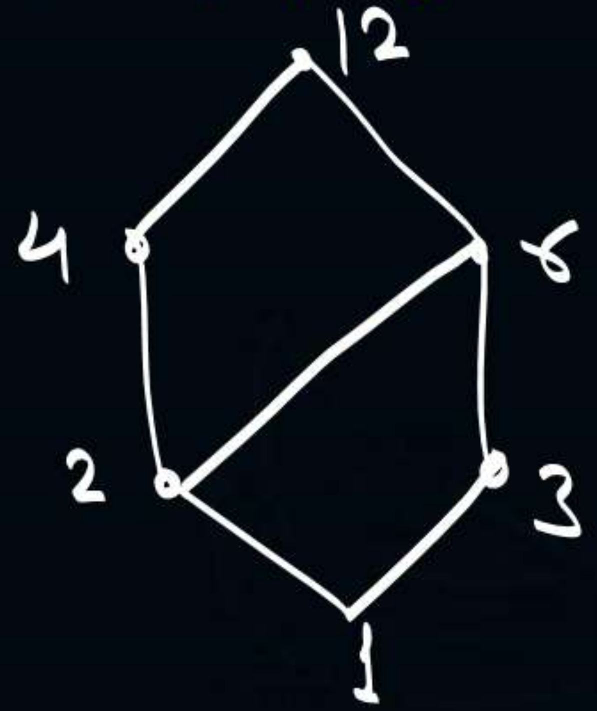
Every element has unique Complement
∴ Boolean lattice
↳ i.e, Complemented ✓
Distributive ✓

Slide

Which of the following lattice is /are not distributive?

(A) $(D_{125}, \div)$
(B) $(P(A), \subseteq)$
(C) $(D_{12}, \div)$ $\longrightarrow$ $D_{12} = \{1, 2, 3, 4, 6, 12\}$
(D) $(\{1, 2, 3, 5, 30\}, \div)$
(E) $(\{1, 2, 3, 4, 12\}, \div)$

$(D_{12}, \div) \Rightarrow$



$\Rightarrow$ Every element has at most one complement
$\therefore$ May or may not be distributive
$\Downarrow$
No sublattice which is isomorphic to L_1^* or L_2^*
$\therefore$ Distributive ✓

$\bar{2}$ = does not exist, $\therefore$ Not Complemented

Slide

Which of the following lattice is /are not distributive?

(A) $(D_{125}, \div)$

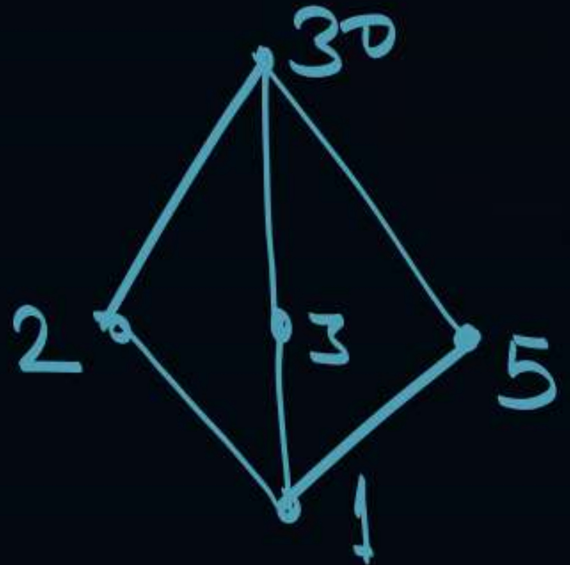
(B) $(P(A), \subseteq)$

(C) $(D_{12}, \div)$

(D) $(\{1, 2, 3, 5, 30\}, \div)$

(E) $(\{1, 2, 3, 4, 12\}, \div)$

(D)



isomorphic to L_2^*

$\therefore$ Not distributive

$\bar{1} = 30$, $\overline{30} = 1$, $\underbrace{\bar{2} = 3, 5, \ \bar{3} = 2, 5, \ \bar{5} = 2, 3}$

$\therefore$ Not distributive

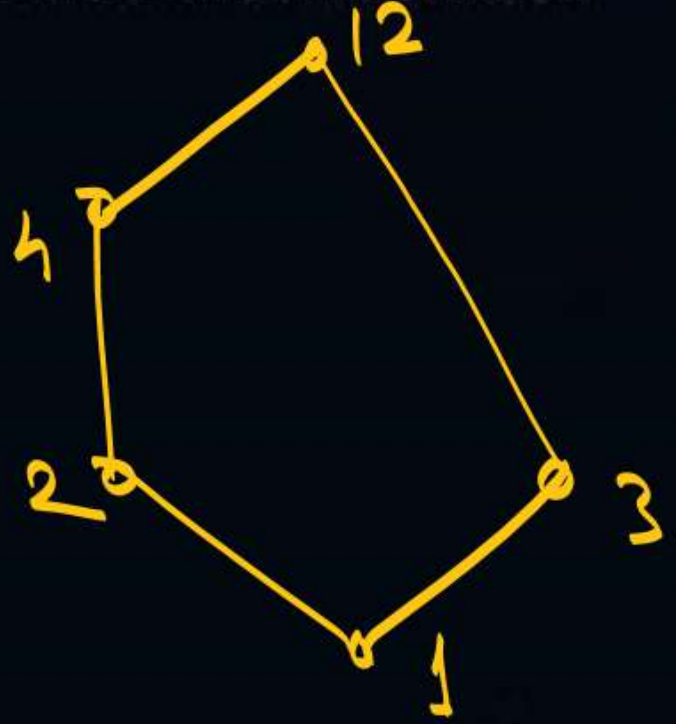
but Complemented

Slide

Which of the following lattice is /are not distributive?

(A) $(D_{125}, \div)$
(B) $(P(A), \subseteq)$
(C) $(D_{12}, \div)$
(D) $(\{1, 2, 3, 5, 30\}, \div)$
(E) $(\{1, 2, 3, 4, 12\}, \div)$

(E)



but Complemented

$\bar{1} = 12$
$\overline{12} = 1$
$\bar{2} = 3$
$\bar{4} = 3$
$\bar{3} = 2, 4$ → Not distributive

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Which of the following lattice is /are Boolean Algebra?

(A) $(D_{125}, \div)$ Distributive Complemented

(B) $(P(A), \subseteq)$

(C) $(D_{12}, \div)$

(D) $(\{1, 2, 3, 5, 30\}, \div)$

(E) $(\{1, 2, 3, 4, 12\}, \div)$

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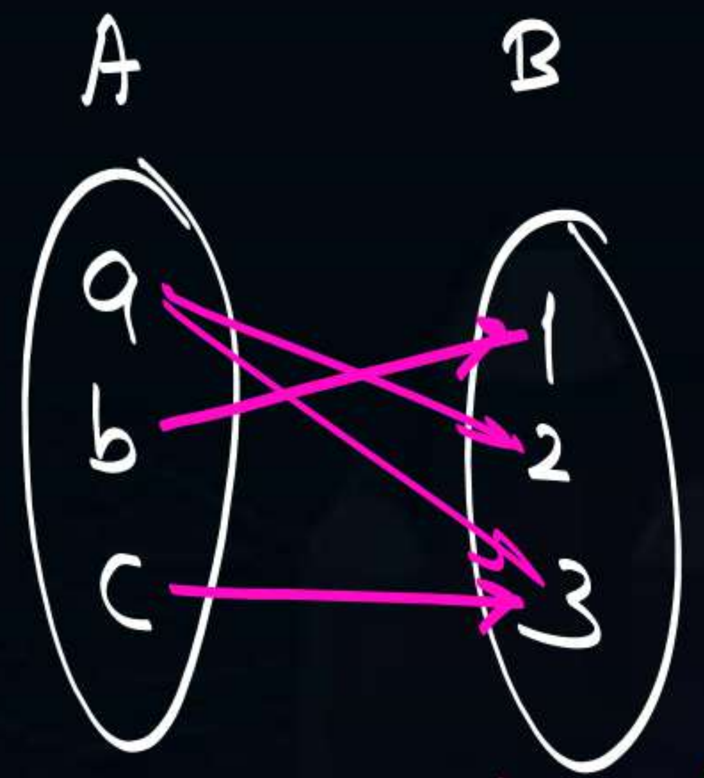
Topic : Function

A relation from set A to set B is called a function from set A to set B if and only if

Every element of set A relates with exactly one element of set B

{ Function from set A to set B is also a relation from set A to set B }

{ Every function is a relation, but every relation need not be a function }



It is a valid relation from A to B but is not a function from A to B

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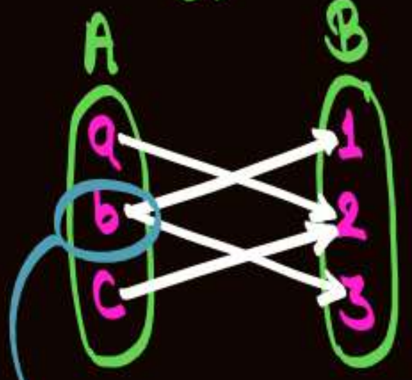
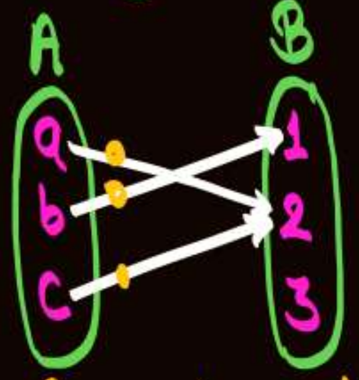
(i)



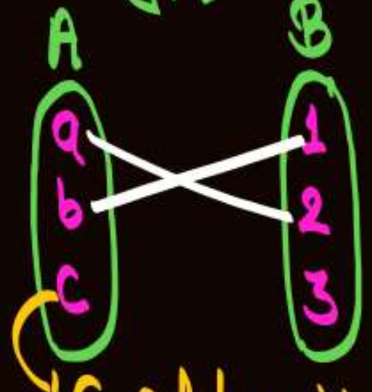
element 'b' of set A relates with two elements of set B
$\therefore$ Not a function from A to B

(ii)



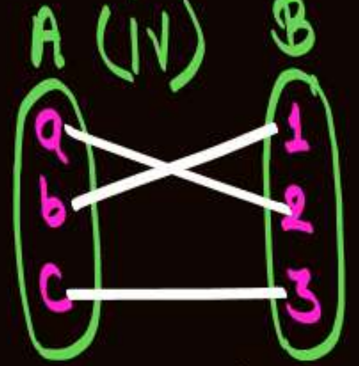
Every element of set A relates with exactly one element of set B
$\therefore$ It is a function from A to B

(iii)



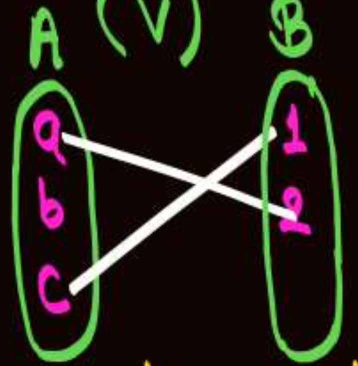
C relates with '0' elements of set B.
$\therefore$ Not a function from A to B

(iv)



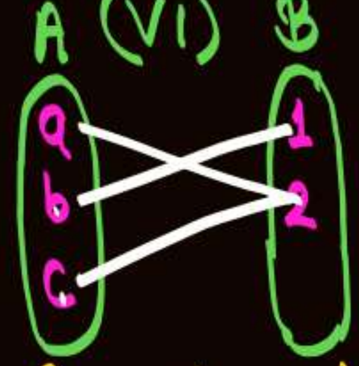
Every element of set A relates with exactly one element of set B
$\therefore$ It is a function from A to B

(v)



'b' does not relate with any element of set B
$\therefore$ Not a function from A to B

(vi)



Every element of set A relates with exactly one element of set B
$\therefore$ It is a function from A to B

(i) Neither a function from A to B nor a function from B to A

(ii) function from A to B but not a function from B to A

(iii) Neither a function from A to B nor a function from B to A

(iv) Function from A to B as well as function from B to A.

(v) Not a function from A to B, but a function from B to A

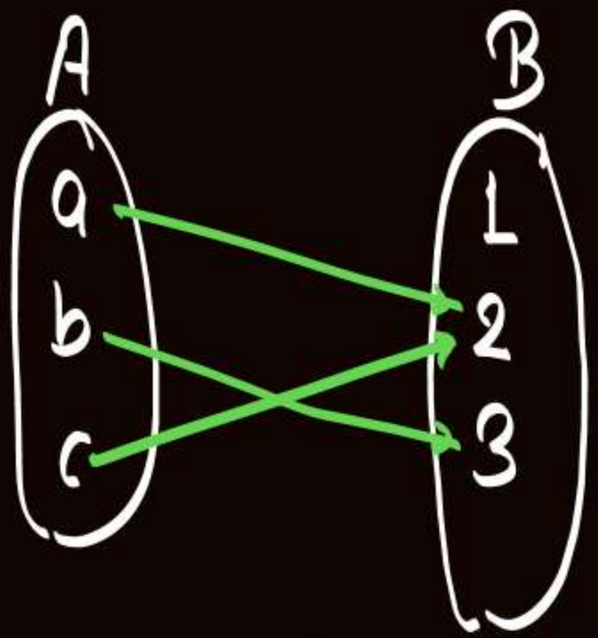
(vi) function from A to B, but not a function from B to A

- A function 'f' from set A to B is denoted by function $f : A \longrightarrow B$

- Let $f : A \rightarrow B$ is a function, then

  Set A is called domain of the function

  & Set B is called Co-domain of the function eg:



function $f: A \to B$

Domain = $\{a, b, c\}$

Co-domain = $\{1, 2, 3\}$

In a function

it is not necessary for every element of the co-domain to be mapped by at least one element of the domain

eg: '1' is not mapped by any element of domain

Topic : Range of a function

It is a set of all the elements of the co-domain that are mapped by at least one element of domain

In general

$$\boxed{\text{Range of function} \subseteq \text{Co-domain of function}}$$

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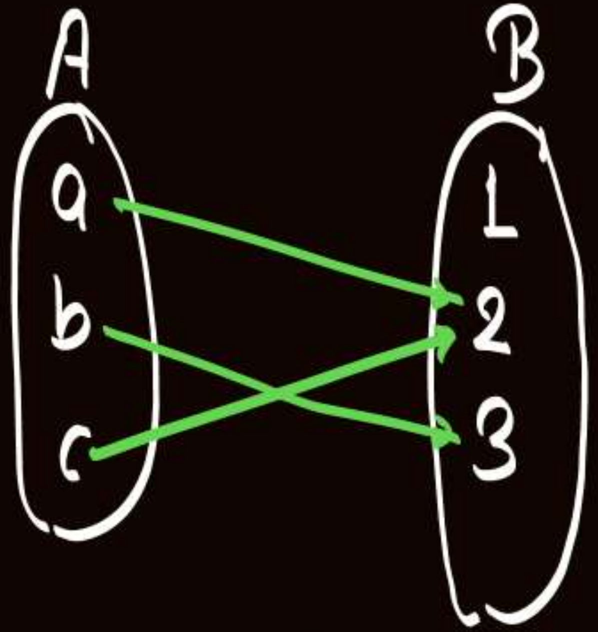
e.g.



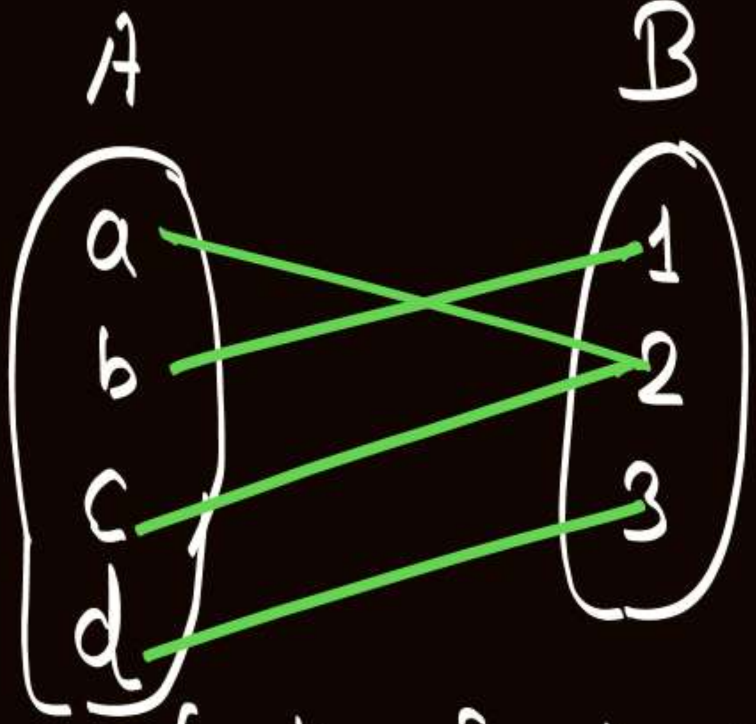
function $f_1 : A \to B$

Domain = $\{a, b, c\}$

Co-domain = $\{1, 2, 3\}$

Range = $\{2, 3\}$

Range $\subset$ Co-domain



function $f_2 : A \to B$

Domain = $\{a, b, c, d\}$

Co-domain = $\{1, 2, 3\}$

Range = $\{1, 2, 3\}$

Range = Co-domain

Note:-

1. Function must be defined for every element of the domain

2. The result of the function on the input from its domain can not acquire a value which is not present in Co-domain

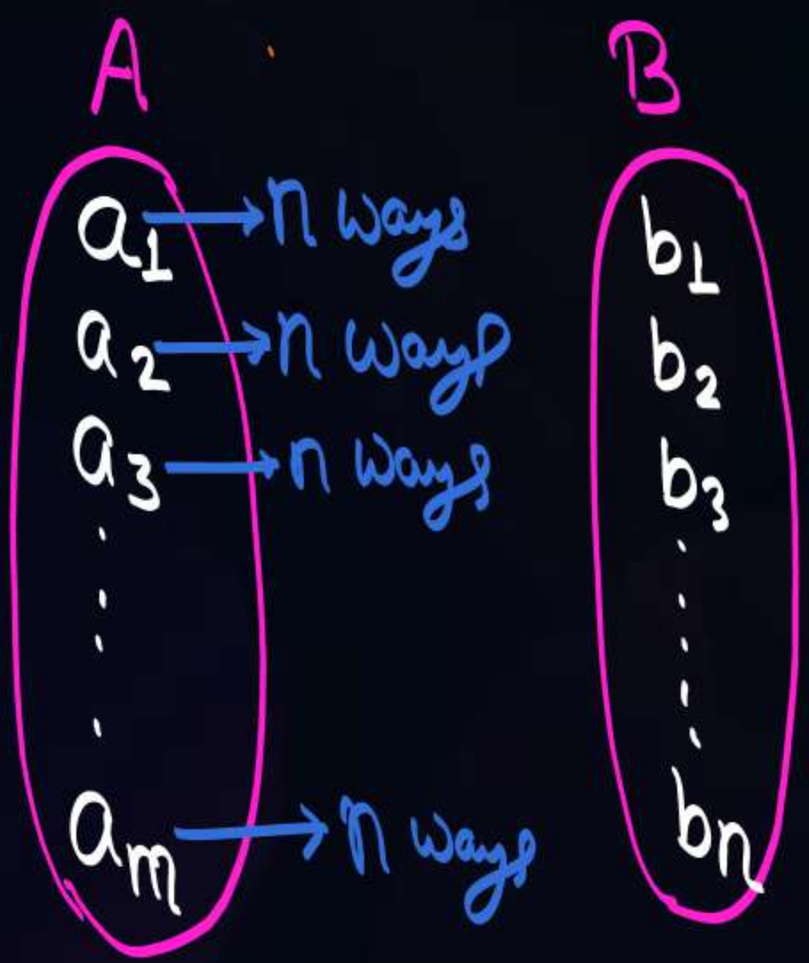
Topic : Total number of functions

let $|A| = m$ & $|B| = n$, then

total Number of functions Possible from set A to set B $= n^m = (\text{Size of Co-domain})^{\text{Size of domain}}$

$$= \underbrace{n * n * n * \cdots * n}_{m \text{ times}}$$

$$= n^m$$



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Note:- A function from set A to set A itself is called a function on set A.

Note: if $|A| = n$, then

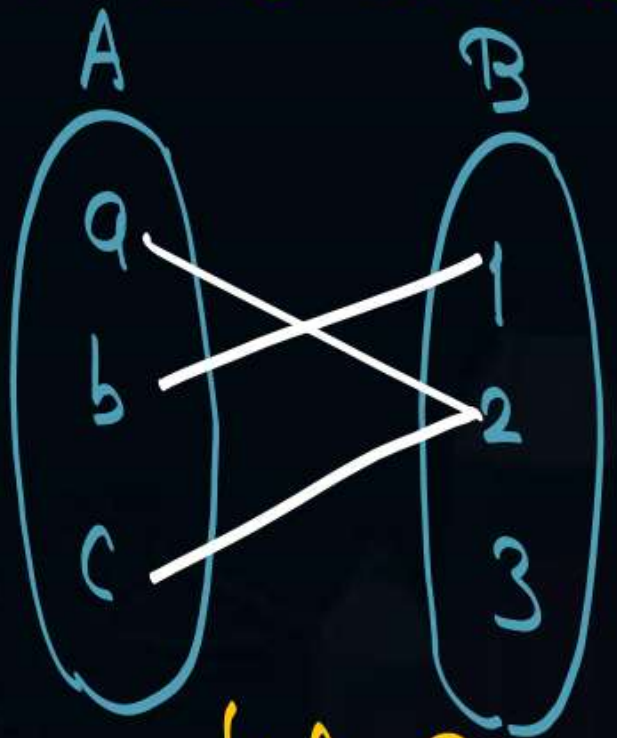
Total number of functions possible on set A : n^n

Topic : Injective (one-one) function

A function $f: A \to B$ is called an injective (one-one) function Only if distinct elements of domain have distinct images in co-domain

i.e, if $a \neq b$, then $f(a) \neq f(b)$

$\underbrace{a \neq b}_{\text{i.e, a \& b are distinct}} \longrightarrow \text{then} \longrightarrow \underbrace{f(a) \neq f(b)}_{\text{images of 'a' \& 'b' must be distinct}}$



f(a) is called image of element 'a' w.r.t function 'f'

$f: A \to B$

it is a function but not a one-one function

as $a \neq c$ but $f(a) = f(c) = 2$

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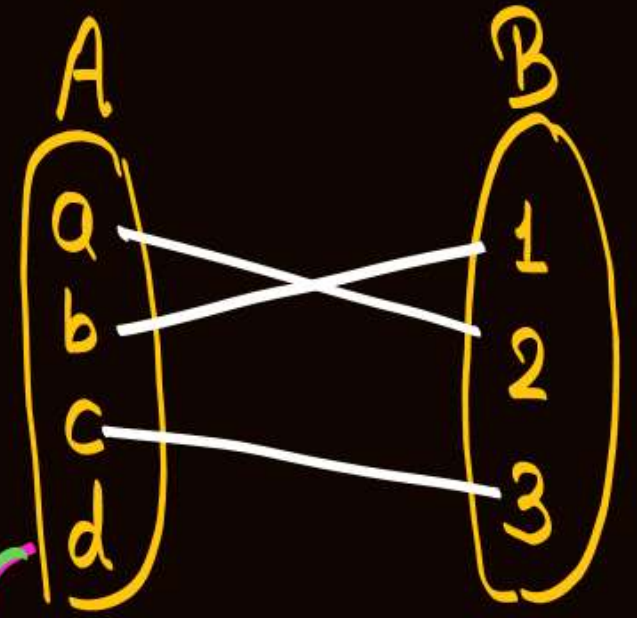


if there is no image of 'd' then it will not be a function

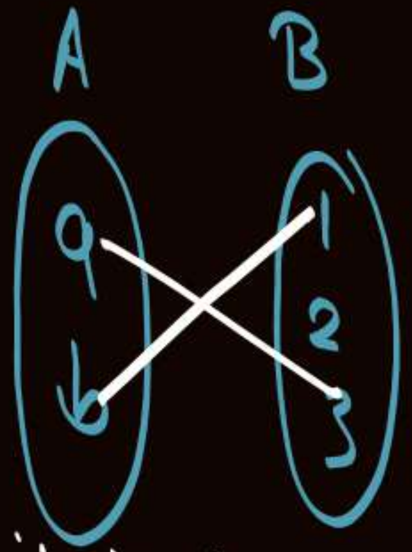
if there is an image of 'd' then it will be same as one of a, b or c

$\therefore$ it will not be one-one function

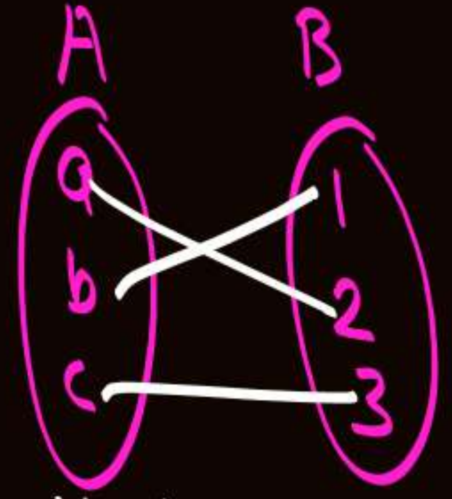
If $|A| > |B|$, then one-one function is not possible from set A to set B

A one-one function from set A to set B is possible only if,

$$|A| \leq |B|$$



it is one-one function



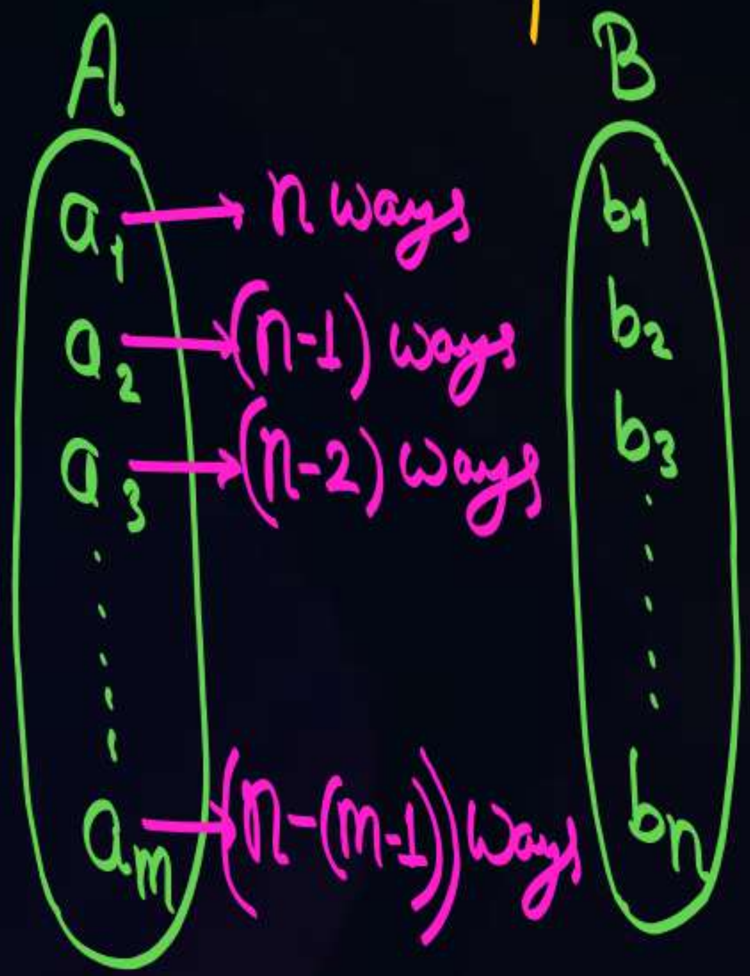
it is one-one function

Topic : Number of one-one function

let $|A| = m$ & $|B| = n$ such that $m \leq n$,

then,

total number of one-one functions possible from A to B $= {}^{n}P_{m} = \dfrac{n!}{(n-m)!}$



$$= n \times (n-1) \times (n-2) \times \dots (n-m+1) \times \frac{(n-m) \times (n-m-1) \times \dots 3 \times 2 \times 1}{(n-m) \times (n-m-1) \times \dots 3 \times 2 \times 1}$$

$$= \frac{n!}{(n-m)!} = {}^{n}P_{m}$$

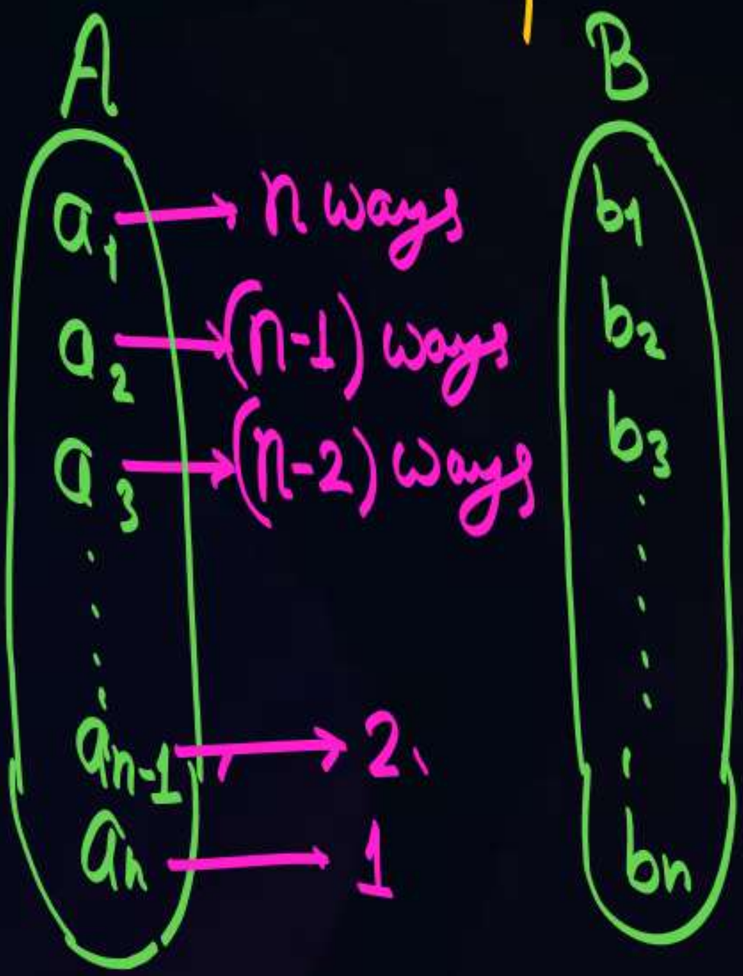
Slide

Topic : Number of one-one function

let $|A| = |B| = n$,

then,

total number of one-one functions possible from A to B $= n! \Rightarrow \boxed{{}^{n}P_{n} = \frac{n!}{(n-n)!} = n!}$



$= n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$

$= n!$

Slide

H.W. Read about onto function :-

Imp. H.W. let $|A| = m$ & $|B| = n$ s.t $m \geq n$,
then,
How many onto functions are possible from A to B.

2 mins Summary

- **Topic** Boolean lattice / Boolean algebra
- **Topic** Functions
- **Topic** Range of a function
- **Topic** Injective (one-one) function
- **Topic** Surjective (onto) function

Slide

THANK - YOU

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