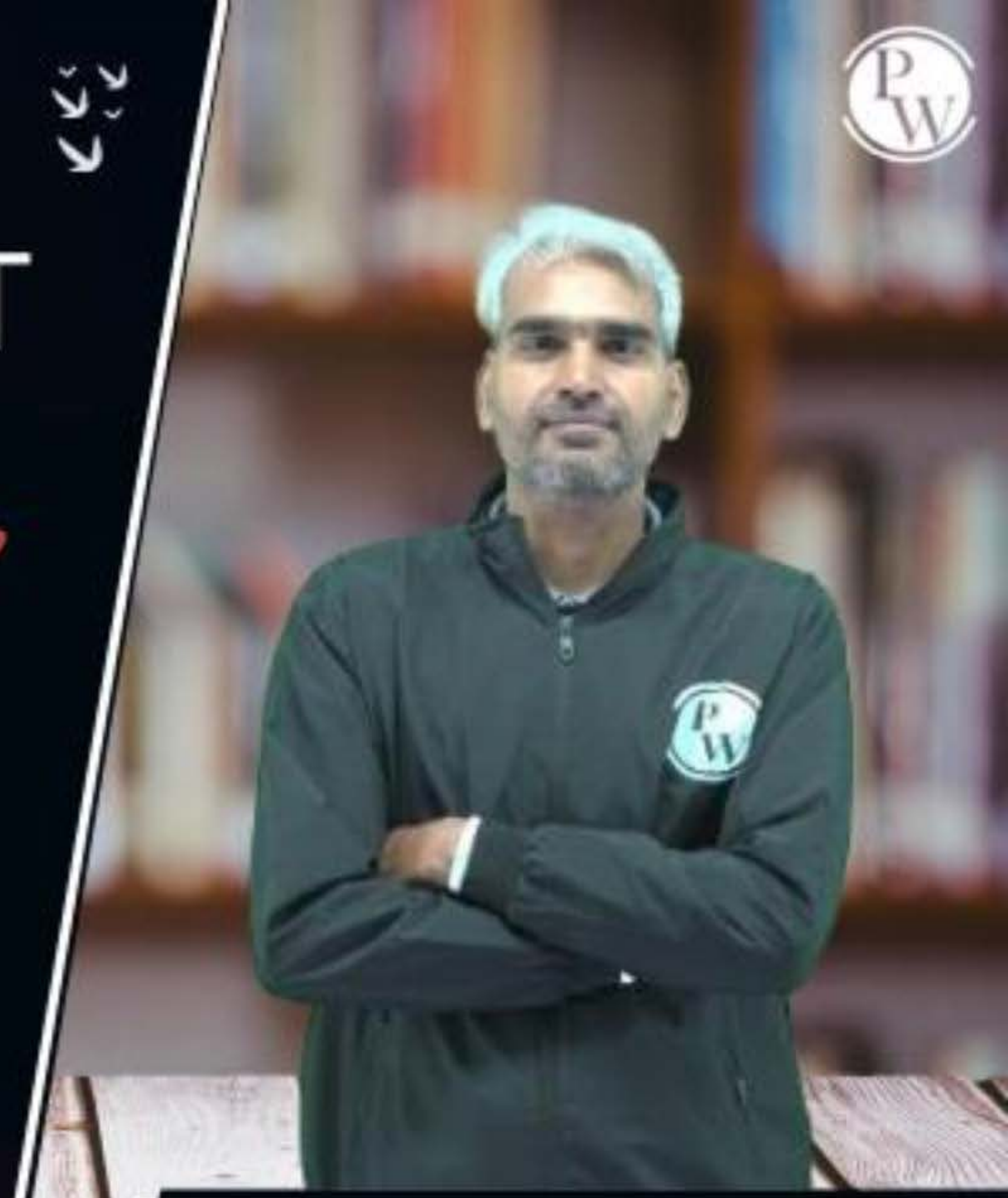
Computer Science & IT

Discrete Mathematics

Set Theory & Algebra

Lecture No. 09



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Recap of Previous Lecture

Topic Partition of a set

Topic Number of equivalence relation

Topic Bell number

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Topics to be Covered

Topic — Partial Order Relation and Partially Ordered Set

Topic — Total Order Relation and Totally Ordered Set

Topic — Least Upper Bound & Greatest Lower Bound

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Topic : Partial Order Relation

A relation R on set A is said to be partial order relation if and only if relation R is

(1) Reflexive

& (2) Anti-symmetric

& (3) Transitive

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eg: Let $A = \{1, 2, 3\}$

$R_1 = \Delta_A = \{(1,1), (2,2), (3,3)\}$

Reflexive ✓
Anti-symmetric ✓
Transitive ✓

$\Rightarrow \therefore$ Partial order relation

Diagonal relation on set A is the only relation which is equivalence relation as well as Partial order relation

$R_2 = \{(1,1), (2,2), (3,3), (1,2), (2,3), (1,3)\}$

Reflexive ✓
Anti-symmetric ✓
Transitive ✓

$\therefore$ Partial order Relation

Topic : Partially Ordered Set

(P.O. SET)

Let R be a partial order relation on set A.
then set A along with this partial order relation R
is called partially ordered set.

It is denoted by (A, R)

Must be a partial order relation

eg: Let 'A' is any set of real numbers, then
$(A, \leq)$ is a POSET.

let $A = \{1, 2, 3\}$

$\leq = \{(1,1), (1,2), (1,3),$
$\cancel{(2,1)}, (2,2), (2,3),$
$\cancel{(3,1)}, \cancel{(3,2)}, (3,3)\}$

Antisymmetric

transitive

Reflexive

eg: let A is any set of non-zero positive integers

then $(A, \div)$ is a POSET

divides

{ '$\div$' is a partial order relation on any set of non-zero positive integers }

$A = \{1, 3, 5\}$

$\div = \{(1,1), (1,3), (1,5),$

$\cancel{(3,1)}, (3,3), \cancel{(3,5)},$

$\cancel{(5,1)}, \cancel{(5,3)}, (5,5)\}$

eg: Let 'C' be any collection of sets and "$\subseteq$" is set containment opⁿ (i.e. Subset opⁿ) then $(C, \subseteq)$ is a POSET.

$$C = \{ \overset{P}{\{\ \}}, \overset{Q}{\{1,2\}}, \overset{R}{\{1,2,3\}}, \overset{S}{\{2,3\}} \}$$

$$\begin{aligned}\subseteq = \{ &(P,P), (P,Q), (P,R), (P,S) \\ &\cancel{(Q,P)}, (Q,Q), (Q,R), \cancel{(Q,S)} \\ &\cancel{(R,P)}, \cancel{(R,Q)}, (R,R), \cancel{(R,S)} \\ &\cancel{(S,P)}, \cancel{(S,Q)}, (S,R), (S,S) \}\end{aligned}$$

Topic : Comparability

Let (A, R) be a POSET, $\{$ i.e. R is a partial order relⁿ $\}$

For any pair of elements $a, b \in A$,

"a & b" are said to be comparable w.r.t.

Partial order relation R if $a R b$ or $b R a$

i.e. $(a,b) \in R$ or $(b,a) \in R$

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eg. Let $A = \{1, 2, 3, 4\}$
and $(A, \underset{R}{\div})$ is a POSET

1 divides every element of the set, ∴ '1' is comparable with all elements of this set.

$(1,1) \in R$ ∴ '1 & 1' are comparable
$(1,2) \in R$ ∴ 1 & 2 are — " —
$(1,3) \in R$, ∴ 1 & 3 — " —
$(1,4) \in R$, ∴ 1 & 4 — " —

$(2,1) \notin R$ but $(1,2) \in R$ ∴ "2 & 1" are Comparable
$(2,2) \in R$ ∴ '2 & 2' are Comparable
$(2,4) \in R$ ∴ '2 & 4" are Comparable

Neither 2 divides 3
nor 3 divides 2
i.e neither $(2,3) \in R$ nor $(3,2) \in R$
∴ "2 & 3" are not comparable w.r.t. relation '$\div$' (divides)

Similarly "3 & 4" are also not Comparable w.r.t. relation divides ($\div$)

Topic : Totally Ordered Set

(Linearly ordered Set)

A POSET (A, R) is called a totally ordered set only if each pair of element of set A is comparable w.r.t. relation R. {and relation R is called Total order relation}

eg. let $A = \{1, 2, 4, 8\}$

$(A, \div)$ is a POSET

Every pair of elements of set A is comparable w.r.t. relation $\div$ (divides)

$\therefore$ above POSET $(A, \div)$ is also a Totally Ordered Set (TOSET)

eg. let $A = \{1, 2, 3, 4\}$ and $(A, \div)$ is a POSET.

We know '2 & 3' are not comparable because neither 2 divides 3 (i.e. $(2,3) \notin \div$) nor 3 divides 2 (i.e. $(3,2) \notin \div$)

$\therefore (A, \div)$ is just a POSET

it is not a Totally Ordered Set (TOSET)

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eg: POSET $(A, \leq)$ is a TOSET for any set A of real numbers

eg: POSET $(A, \div)$ where A is any set of non-zero Positive integers may or may not be a TOSET

eg: if $A = \{1, 2, 4, 8\}$ then $(A, \div)$ is a TOSET

& if $A = \{1, 2, 3, 4\}$ then $(A, \div)$ is not a TOSET

Note: Every TOSET is a POSET, but every POSET need not be a TOSET.

- If $a \, R \, b$ i.e. $(a, b) \in R$

then a is said to be at lower side
& b is said to be at upper side

$$(a, b) \in R$$

lower side

upper side

Topic : Least Upper Bound

/ lub / Join / Supremum

L.U.B.

it can be any partial order Relⁿ

Let $(A,\leq)$ be a POSET, for any two elements $a,b\in A$ if there exists an element $c\in A$

such that,

$a\leq c$ and $b\leq c$, then c is called upper bound of a and b,

'c' is at the upper side of both a & b

And if there exists no element $d\in A$ such that

$\underbrace{a\leq d \text{ and } b\leq d}$ and $\underbrace{d\leq c}$,

There exists no element 'd' ∈ A s.t. ⇒ d is also an upper bound of a & b ⇒ and d is at lower side of c

then c is called least upper bound of a and b

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Topic : Least Upper Bound

/ lub / Join / Supremum
L.U.B.

Let (A,R) be a POSET, for any two elements $a,b \in A$ if there exists an element $c \in A$ such that,

$a \overset{R}{\leq} c$ and $b \overset{R}{\leq} c$, then c is called upper bound of a and b,

i.e. $(a,c) \in R$ & $(b,c) \in R$

And if there exists no element $d \in A$ such that

$a \overset{R}{\leq} d$ and $b \overset{R}{\leq} d$ and $d \overset{R}{\leq} c$,

i.e. $(a,d) \in R$ & $(b,d) \in R$ & $(d,c) \in R$ then c is called least upper bound of a and b

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Least upper bound of elements a & b is denoted by $lub(a, b)$ <u>or</u> $a \vee b$

Symbol used to denoted <u>lub</u>.

Topic : Greatest Lower Bound

/glb/ Meet / Infimum

Let $(A, \leq)$ be a POSET, for any two elements $\underline{a}, b \in A$ if there exists an element

$\underline{c} \in A$ such that,

$c \leq a$ and $c \leq b$, then c is called lower bound of a and b,

'C' is at lower side of 'a' as well as 'b'

And if there exists no element $d \in A$ such that

$\underbrace{d \leq a \text{ and } d \leq b}$ and $\underbrace{c \leq d}$, then c is called greatest lower bound of a and b

no element 'd' s.t.	d is at lower side of a & b	and	d is at upper side of c

Topic : Greatest Lower Bound

/glb / Meet / Infimum

Let $(A, \overset{R}{\leq})$ be a POSET, for any two elements $a,b \in A$ if there exists an element $c \in A$ such that,

$c \overset{R}{\leq} a$ and $c \overset{R}{\leq} b$, then c is called lower bound of a and b,

$(c,a) \in R$ & $(c,b) \in R$

And if there exists no element $d \in A$ such that

$d \overset{R}{\leq} a$ and $d \overset{R}{\leq} b$ and $c \overset{R}{\leq} d$,

$(d,a) \in R$ & $(d,b) \in R$ & $(c,d) \in R$ then c is called greatest lower bound of a and b

Slide

greatest lower bound of a & b is denoted by glb(a, b) <u>or</u> $a \wedge b$

symbol to represent glb

eg: let $A = \{1, 2, 3, 4\}$
and $(A, \leq)$ is a POSET.

find lub & glb of elements $2, 3 \in A$

$2 \not\leq 1$ $\quad$ $2 \leq 2$ $\quad$ $2 \leq 3$ $\quad$ $2 \leq 4$

$3 \not\leq 1$ $\quad$ $3 \not\leq 2$ $\quad$ $3 \leq 3$ $\quad$ $3 \leq 4$

'1' & '2' can never be the upper bound of 2 & 3

Both 3 & 4 are upper bounds of 2 & 3

but $3 \leq 4$
i.e. 3 is at lower side of 4
$\therefore$ 4 can not be the least upper bound
$\therefore$ $lub(2,3) = 3$

eg: let $A = \{1, 2, 3, 4\}$

and $(A, \leq)$ is a POSET.

find lub & glb of elements $2, 3 \in A$

$1 \leq 2$ $\quad$ $2 \leq 2$ $\quad$ $3 \nleq 2$ $\quad$ $4 \nleq 2$

$1 \leq 3$ $\quad$ $2 \leq 3$ $\quad$ $3 \leq 3$ $\quad$ $4 \nleq 3$

Both 1 & 2 are lower bounds of 2 & 3

$\therefore$ 3 & 4 can not be the lower bounds of 2 & 3

but $1 \leq 2$

i.e. '1' is at lower side of another lower bound '2'

$\therefore$ 1 can not be the greatest lower bound of 2 & 3

$\therefore \text{glb}(2,3) = 2$

Note:

+ Let A is any set of real numbers and $(A, \leq)$ is a POSET,

less than or equal

for element $a, b \in A$

identify

(i) $lub(a,b) = ?$ $\Rightarrow$ $lub(a,b) = Max(a,b)$

(ii) $glb(a,b) = ?$ $\Rightarrow$ $glb(a,b) = Min(a,b)$

Note:

Let is a set of all natural numbers and $(A, \div)$ is a POSET.

for elements $a, b \in A$

(i) $lub(a,b) = LCM(a,b)$

(ii) $glb(a,b) = GCD(a,b)$

it is the set of all natural no.s

$\therefore$ LCM as well as GCD of any pair of natural No.s will also be present in the set of all natural numbers.

Q. let $A = \{2, 3, 4\}$
and $(A, \div)$ is a POSET
What will be

lub (2,3) : No element in the set which is divided by 2 as well as by 3
∴ lub (2,3) does not exists in the above POSET

glb (2,3) = No element in the set which divides element '2' as well as element '3'
∴ glb (2,3) does not exists in the above POSET

Note: Lub and/or glb for a pair of elements of the set can not be out side the set

Q:- Let A is any set of non-zero positive integers and $(A, \div)$ is a POSET, than for elements $a, b \in A$,

$lub(a,b)$ = We don't know the elements of the set
$\therefore$ Nothing can be said

$glb(a,b)$ = We don't know the elements of the set
$\therefore$ Nothing can be said

2 mins Summary

Topic Partial Order Relation and Partially Ordered Set

Topic Total Order Relation and Totally Ordered Set

Topic Least Upper Bound & Greatest Lower Bound

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THANK - YOU

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