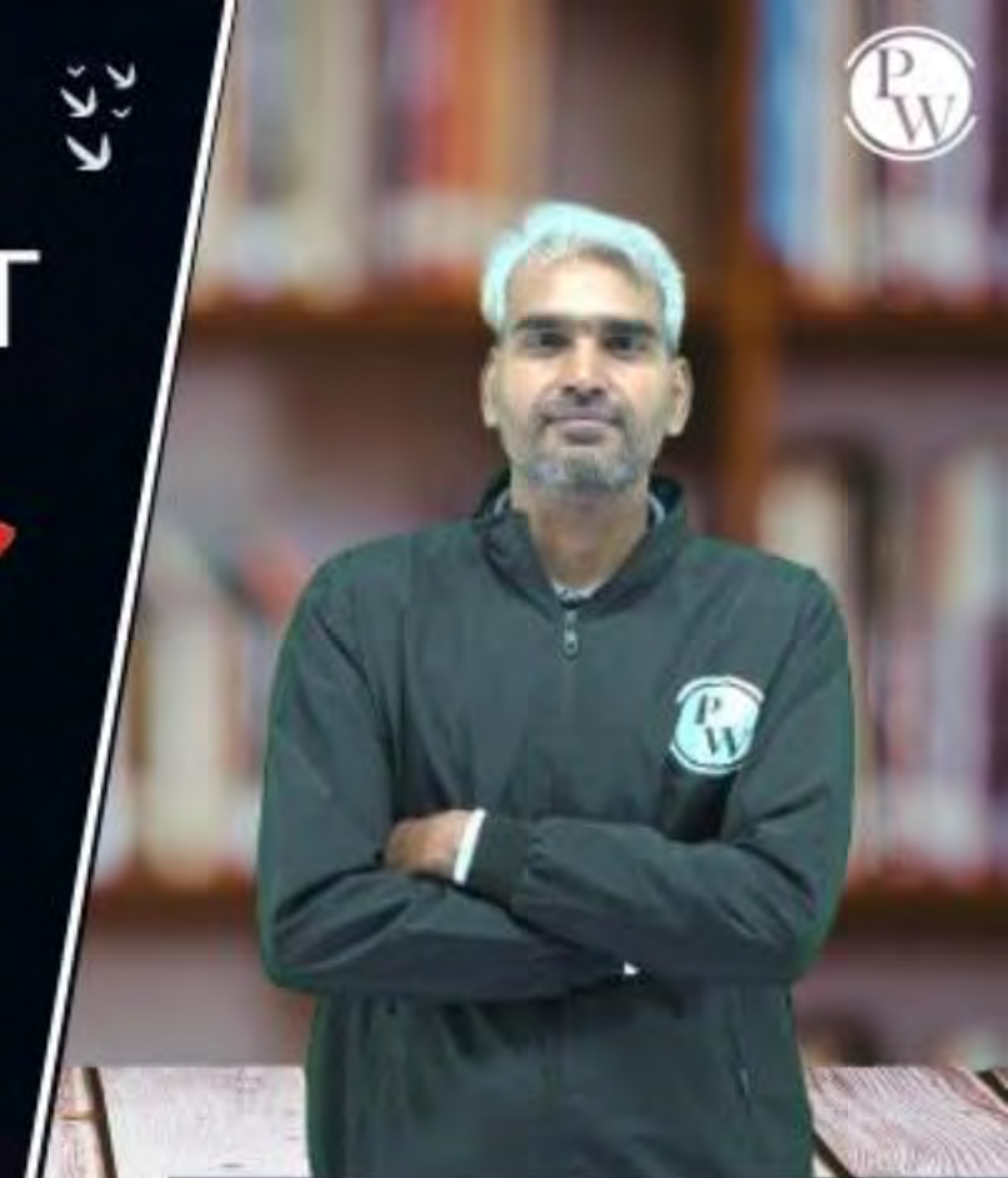
Computer Science & IT

Discrete Mathematics

Graph Theory

Lecture No. 03



By- Vishal Sir

Recap of Previous Lecture

Topic — Some important terminologies

Topic — Different types of graphs

Slide

Topics to be Covered

Topic Different types of graphs

Topic Sum of degree theorem

Topic Degree Sequence

Topic Havel Hakimi's algorithm

Slide

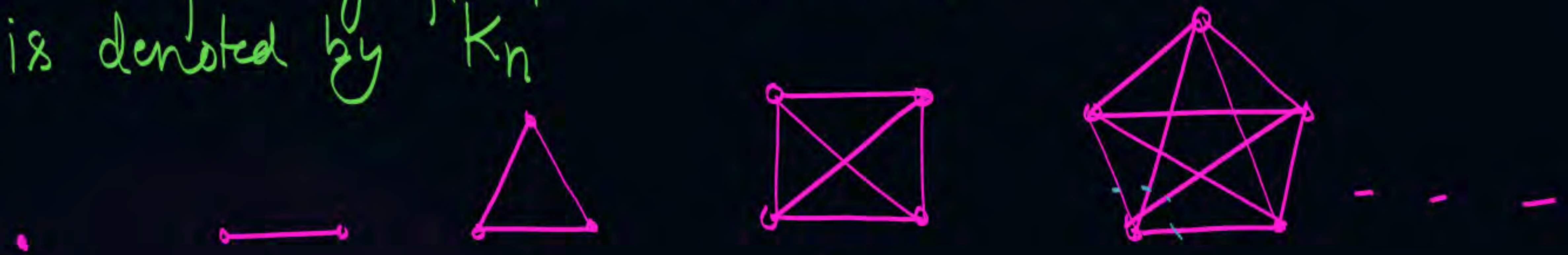
Topic : Types of graphs

1. Complete Graph
2. Cycle Graph
3. Wheel Graph
4. Connected Graph
5. Cyclic Graph
6. Acyclic Graph
7. Tree
8. Bipartite Graph
9. Complete Bipartite Graph

Topic : Complete Graph

{ A simple graph with all possible edges is called a Complete graph }

* A Complete graph with n-vertices is denoted by 'K_n'



K_1 K_2 K_3 K_4 K_5

* A simple graph in which all vertices are mutually adjacent, is Called a Complete graph

* A Complete graph K_n is a simple graph with n-vertices & all possible edges

$$\therefore \boxed{|E(K_n)| = {}^nC_2 = \frac{n(n-1)}{2}}$$

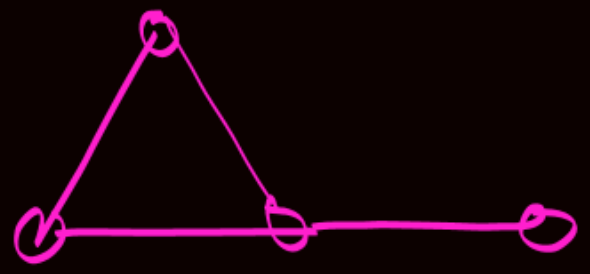
* In a Complete graph K_n, degree of each vertex is '$n-1$', $\therefore$ Complete graph K_n is a '$(n-1)$-regular' graph

Topic : Cycle Graph

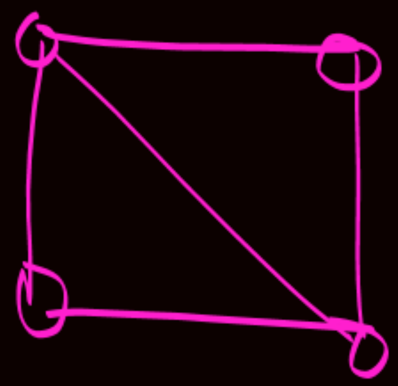
* A simple graph with n-vertices ($n \geq 3$), where all 'n' vertices form a cycle of length $= n$ is called a cycle graph.
* A cycle graph with n-vertices is denoted by C_n.

C_3 C_4 C_5 . . .

* length of a cycle is equal to the number of edges in that cycle
* $\boxed{|E(C_n)| = n}$
* Every cycle graph is a '2-regular' graph



Not a cycle graph

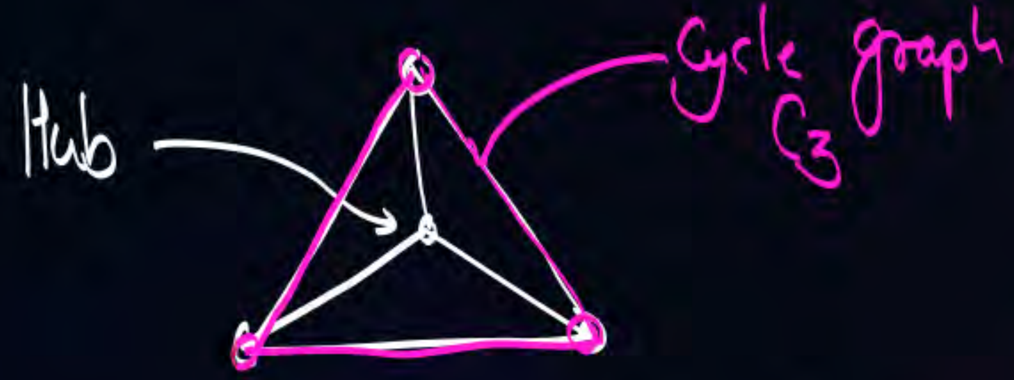


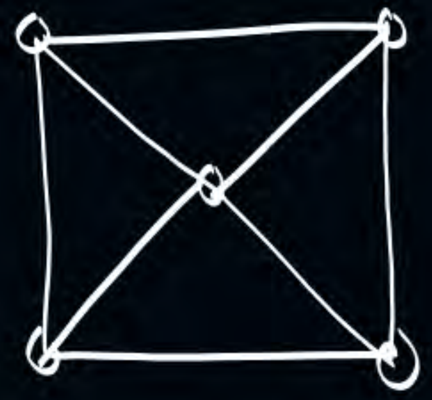
Not a cycle graph

Topic : Wheel Graph

* A wheel graph with n-vertices is denoted by W_n. *

A wheel graph with 'n' vertices can obtained by Connecting all the vertices of a cycle graph C_{n-1}, with a new vertex called hub vertex





$W_4 = C_3 + 1\text{-hub}$

$W_5 = C_4 + 1\text{-hub}$

Find the no. of edges in wheel graph W_n = ?

$$|E(W_n)| = \underbrace{(n-1)}_{\text{w.r.t. } C_{n-1}} + \underbrace{(n-1)}_{\text{to connect hub vertex with } (n-1) \text{ vertices of } C_{n-1}}$$

$$\boxed{|E(W_n)| = 2(n-1) = 2n-2}$$

- In a wheel graph W_n
  - (i) degree of hub-vertex $= (\underline{n}-1)$
  - (ii) degree of vertices at cycle $= \underline{3}$

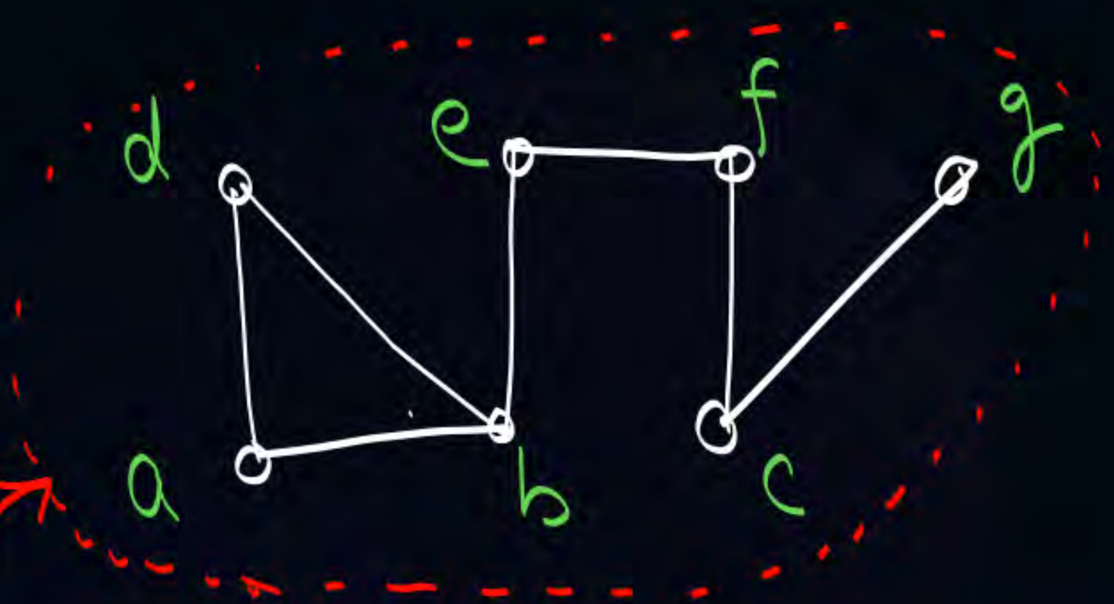
- A wheel graph W_n can be a regular graph if and only if $(n-1) = 3$

  i.e. $\boxed{n=4}$ ✓

  The only wheel graph which is also a regular graph is 'W_4'.

Topic : Connected Graph

A graph is said to be connected if there exist a path between every pair of vertices



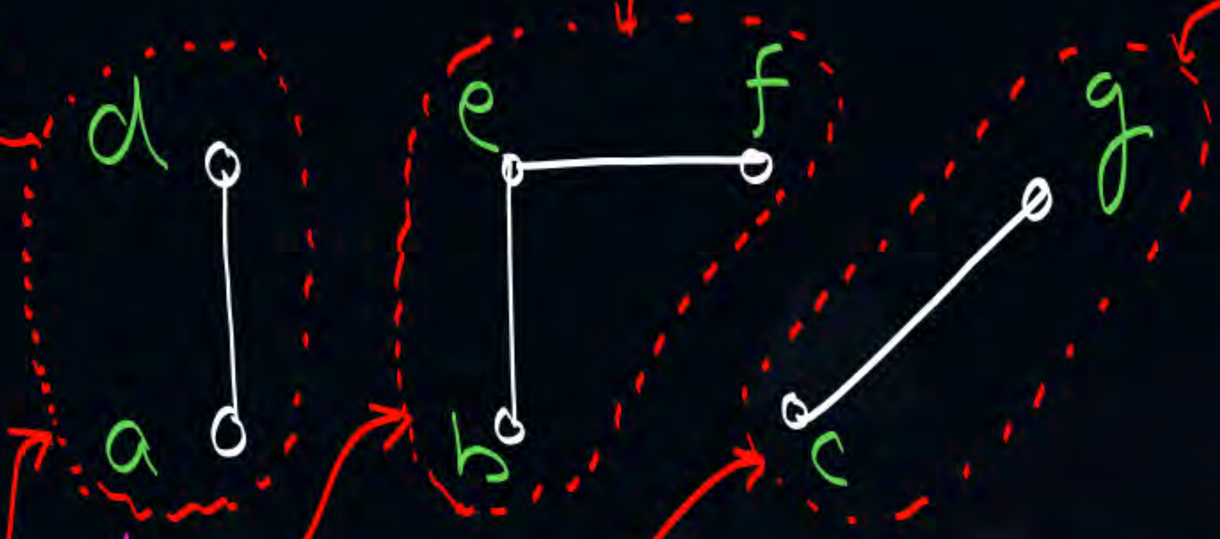
A single Connected Component

it is a Connected graph

Connected

Connected

Connected



Three Connected Components in this graph

G_2

G_2 is a disconnected graph

- In a Connected graph there will be only one Connected Component.
- In a disconnected graph there exist two or more Connected Components
- K_n, C_n, & W_n are always Connected graphs
- Connected graph may or may not be simple

Connected but not simple

Topic : Cyclic Graph

- A graph which contains at least one cycle in it is called a cyclic graph

eg.

Contains a cycle

∴ Cyclic graph

as well as Connected

eg.

G

It is disconnected graph

but

a Cyclic graph

* Every cycle graph is a cyclic graph, but Every cyclic graph need not be a cycle graph.

* A cyclic graph may <u>or</u> may not be connected

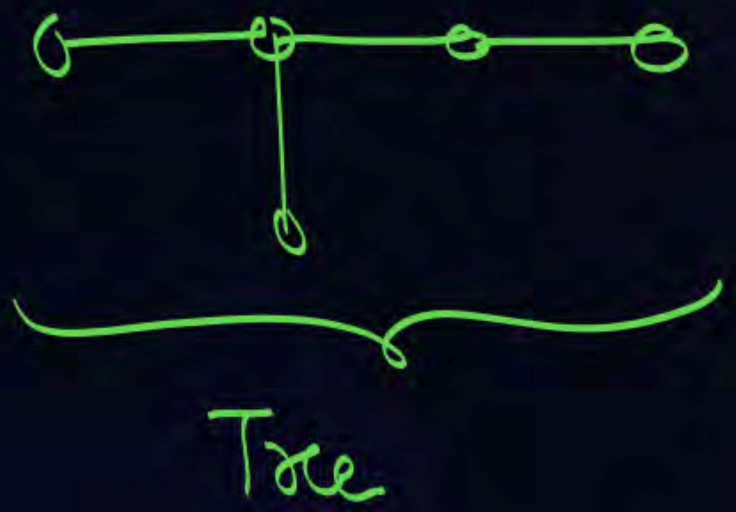
Topic : Acyclic Graph

- A graph with no cycle in it is called an acyclic graph

Topic : Tree

+ A tree is an acyclic connected graph

+ A tree with 'n' vertices will have exactly (n-1) edges.



A collection of tree is called forest

Note:- 1. A tree is a connected graph with minimum number of edges

2. A graph with n vertices and less than $(n-1)$ edges is always a disconnected graph

3. A graph with n vertices and no. of edges $\geq (n-1)$ may or may not be connected

4. A simple graph with n vertices and more than $(n-1)$ edges $(>(n-1))$ is always cyclic

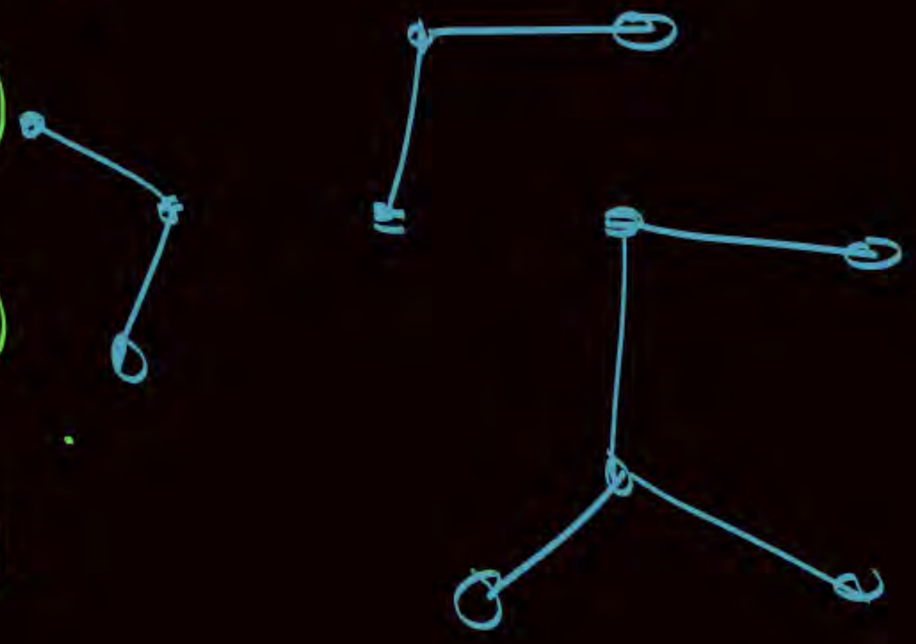
(5) A graph with n-vertices and no. of edges $\leq (n-1)$ then graph may or may not be cyclic

H.W: A forest with n-vertices & K connected components will have how many edges?

Soln

All components must be tree i.e must be acyclic

No. of vertices	No. of components	No. of edges
n	1 {single tree in forest}	$(n-1)$
n	2	$(n-2)$
n	3	$(n-3)$
n	4	$(n-4)$
$\vdots$		
n	K	$(n-K)$



Topic : Bi-partite Graph

A bipartite graph (or bigraph) is a graph whose vertices can be partitioned into two sets V_1 and V_2 such that every edge of the graph connects a vertex in set V_1 to a vertex in set V_2.

$V_1 \cap V_2 = \phi$

$V_1 \cup V_2 = V$

{ No edge should Connect a vertex of set V_1 to another vertex of V_1 and No edge should Connect a vertex of set V_2 with another vertex of set V_2 }

i.e No two vertices of set V_1 should be adjacent to each other & No two vertices of set V_2 should be adjacent to each other

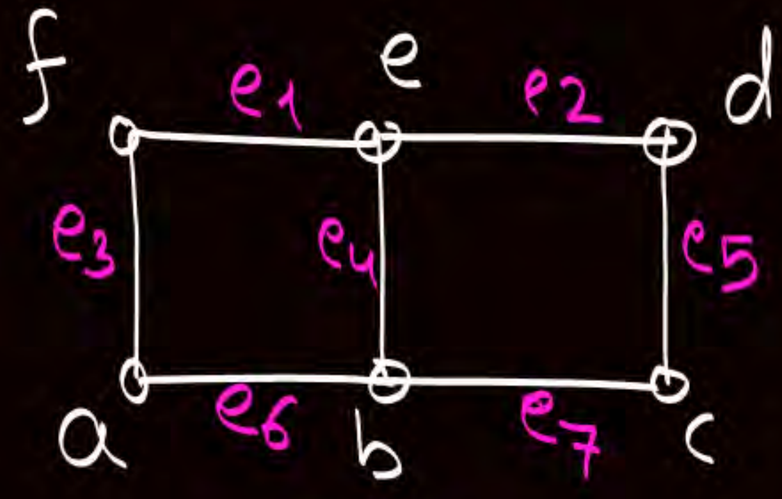
order triple

A bipartite graph can be represented using the notation $G=(V_1, V_2, E)$ where,

V_1 is the first set of the partition

V_2 is the second set of the partition and

E is the set of edges of graph G



G

$V = \{a, b, c, d, e, f\}$

$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

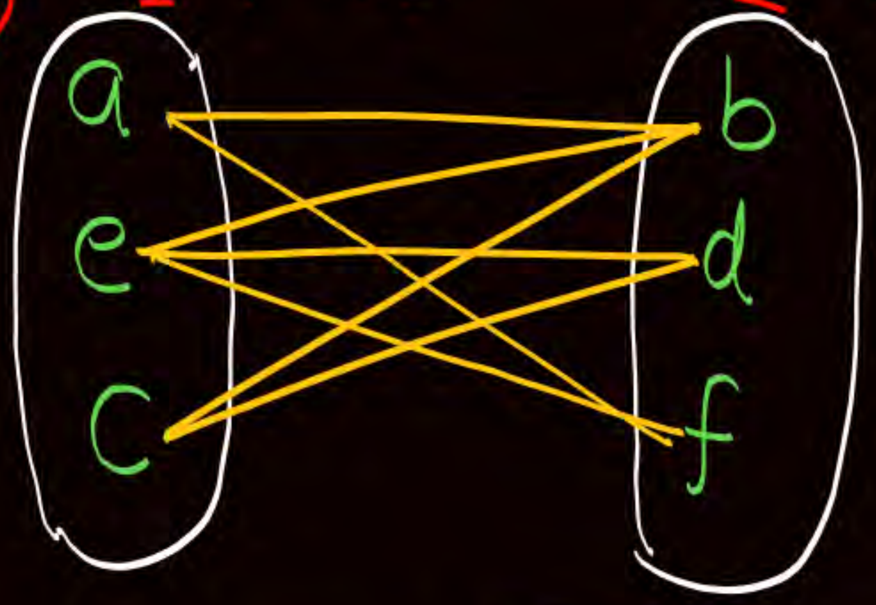
Partition of $V = \{\{a, c, e\}, \{b, d, f\}\}$

$V_1 = \{a, e, c\}$ $\quad$ $V_2 = \{b, d, f\}$

$V_1 \cap V_2 = \emptyset$

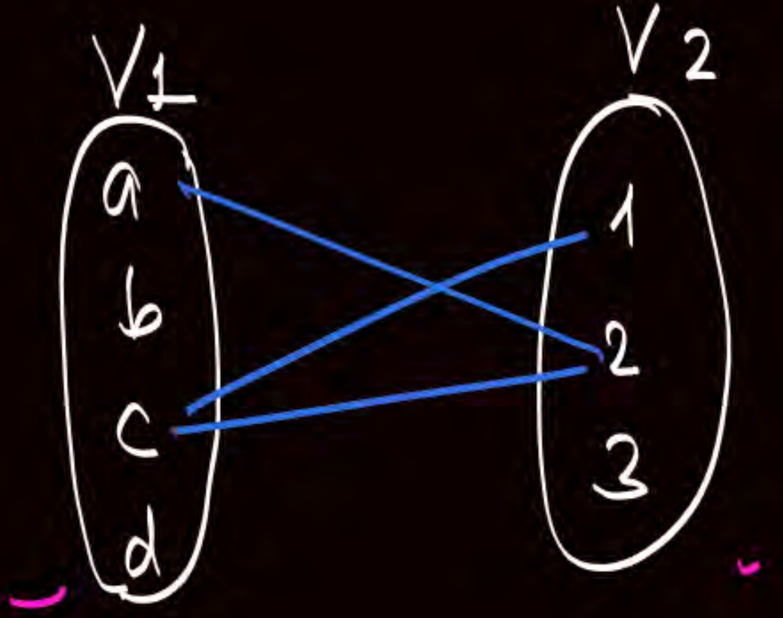
& $V_1 \cup V_2 = V$

$\therefore$ it is a Partition of V

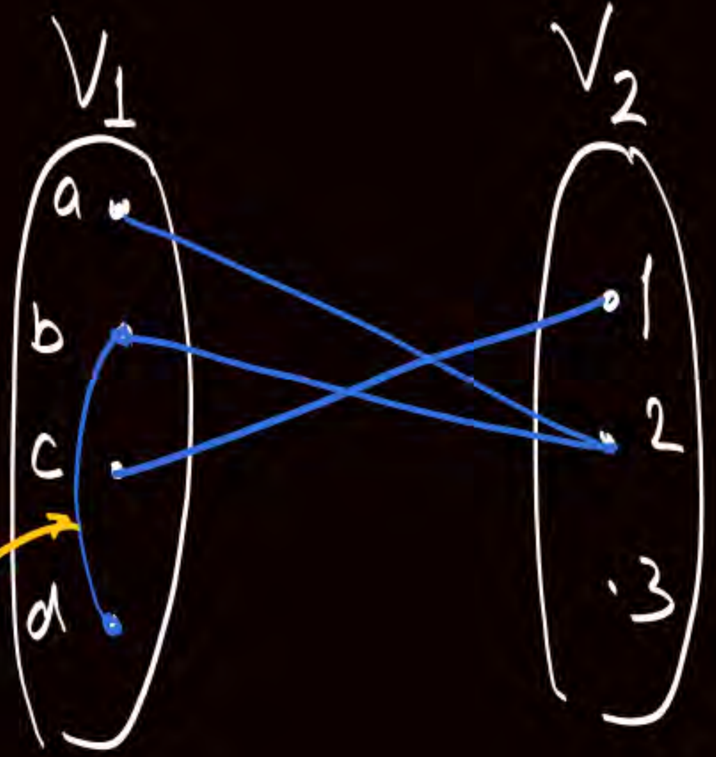


We can partition the set of vertices of graph G into two subsets V_1 & V_2 such that every edge of graph G connects a vertex of set V_1 with a vertex of set V_2

$\therefore$ G is a bipartite graph



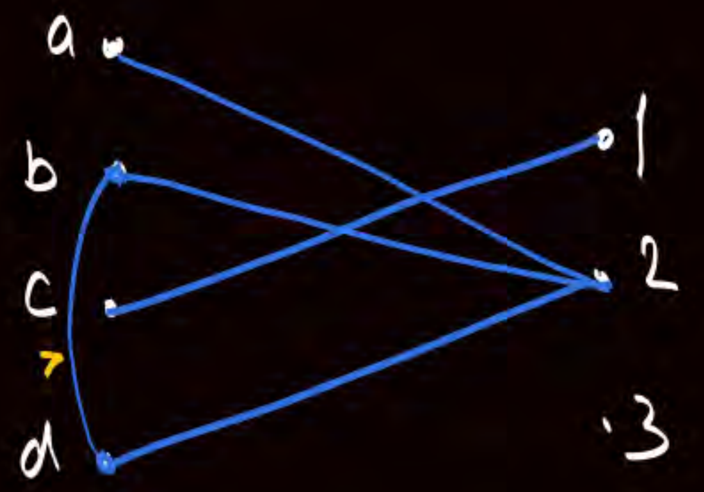
it is a bi-partite graph



two vertices of the same subset of partition are adjacent to each other

∴ it is not a correct partition

if d is moved from V_1 to V_2 then it will become correct partition & Hence graph is a bipartite graph



graph 'G'

it is not a bi-partite graph

Not a bipartite graph

G_1

Cycle of length = 0

G_2

Cycle of length = 4

G_3

Cycle of length = 6

All are bi-partite graphs

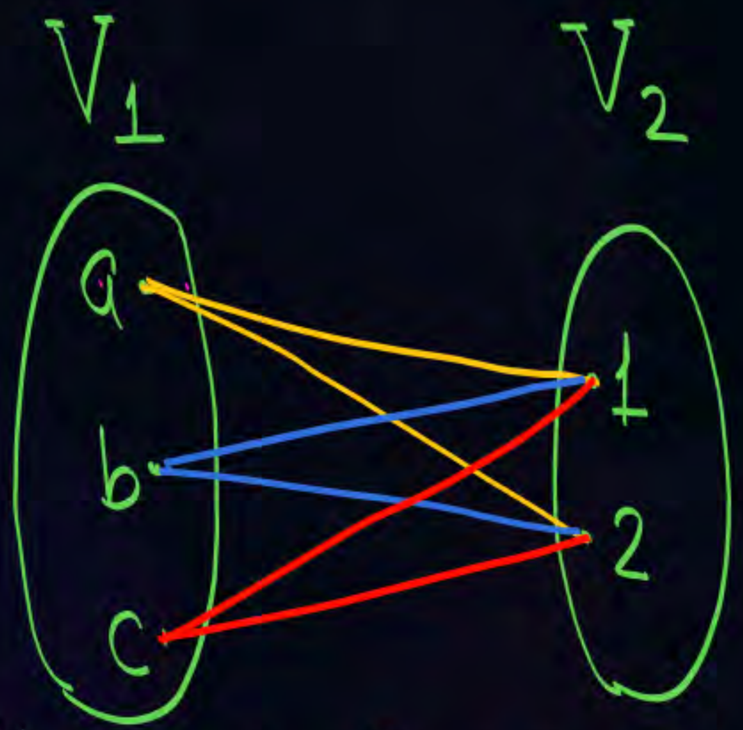
Note:- In a graph G, if all the cycles are of even length then graph G is a bi-partite graph

Note: Tree is an acyclic graph,
i.e. all cycles are of length = 0 (even)
∴ Every tree is a bipartite graph

Note: If there exist any cycle of odd length then it can not be a bi-partite graph

Topic : Complete bipartite Graph

A bipartite graph $G=(V_1, V_2, E)$ is called a complete bipartite graph if every vertex in set V_1 is adjacent to every vertex of set V_2.



it is a Complete bipartite graph

Every Complete bipartite graph is a bi-partite graph, but Every bi-partite graph need not be Complete bipartite graph

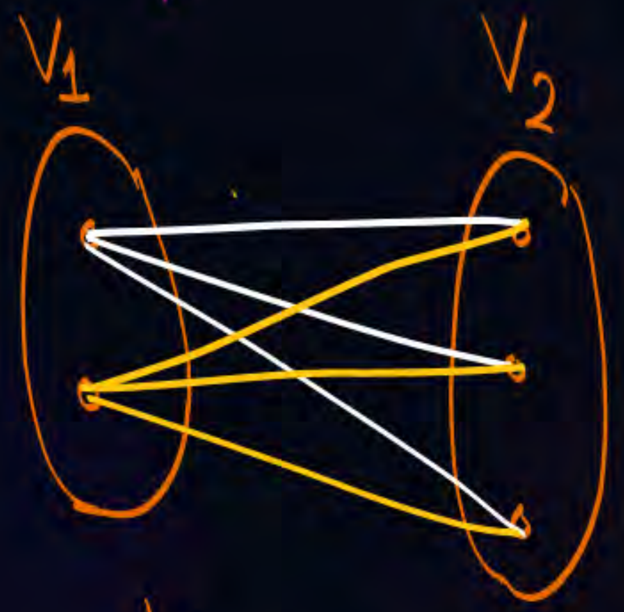
Topic : Complete bipartite Graph

Note: In a Complete bi-partite graph $G = (V_1, V_2, E)$

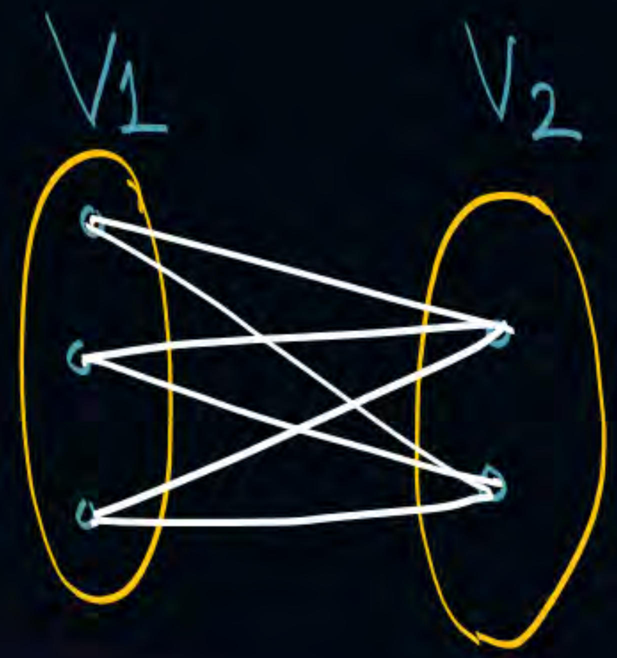
if $|V_1| = m$ & $|V_2| = n$, then

Complete bipartite graph is denoted by $K_{m,n}$

e.g.



$K_{2,3}$



$K_{3,2}$

Topic : Complete bipartite Graph

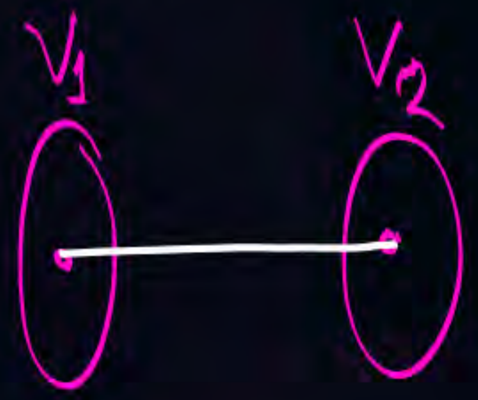
Note:- For a complete bi-partite graph.

$$\boxed{|E(K_{m,n})| = m \cdot n}$$

Topic : Complete bipartite Graph

* In general Complete bi-partite graph is not a Complete graph

* A Complete bipartite graph $K_{m,n}$ is a Complete graph if and only if $m = n = 1$



$K_{1,1} \equiv K_2$

Q: Let G be a bi-partite graph with 'n' vertices, find the maximum number of edges possible in graph G.

$|V| = n$

This are the total No. of vertices

$V = \{V_1, V_2\}$

$V_1 \cap V_2 = \emptyset$

then $V_1 \cup V_2 = V$

$\therefore |V_1| + |V_2| = |V| = n$

V_1 $\lvert V_1\rvert$	V_2 $\lvert V_2\rvert$	Max No. of edges $\lvert V_1\rvert \cdot \lvert V_2\rvert$
1	$(n-1)$	$(n-1)$
2	$(n-2)$	$2.(n-2)$
3	$(n-3)$	$3.(n-3)$
$(n/2)$	$(n/2)$	$\frac{n^2}{4}$
$(n-3)$	3	$3.(n-3)$
$(n-2)$	2	$2.(n-2)$
$(n-1)$	1	$1.(n-1)$

Q: Let G be a bi-partite graph with 'n' vertices, find the maximum number of edges possible in graph G.

This are the total No. of vertices

$|V| = n = 10$

$V = \{V_1, V_2\}$

$V_1 \cap V_2 = \emptyset$

then $V_1 \cup V_2 = V$

$\therefore |V_1| + |V_2| = |V| = n$

V_1 $\lvert V_1 \rvert$	V_2 $\lvert V_2 \rvert$	Max No. of edges
1	9	9
2	8	16
3	7	21
4	6	24
$\boxed{5}$	$\boxed{5}$	$\boxed{25}$
6	4	24
7	3	21
8	2	16
9	1	9

Q: Let G be a bi-partite graph with 'n' vertices, find the maximum number of edges possible in graph G.

$|V| = n = 9$

This are the total NO. of vertices

$V = \{V_1, V_2\}$

$V_1 \cap V_2 = \emptyset$

then $V_1 \cup V_2 = V$

$\therefore |V_1| + |V_2| = |V| = n$

V_1 $\lvert V_1 \rvert$		V_2 $\lvert V_2 \rvert$	Max No. of edges
1		8	8
2		7	14
3		6	18
4 $\left(\frac{n-1}{2}\right) = \left(\frac{9-1}{2}\right)$	$\left(\frac{9+1}{2}\right)$ $\frac{(n+1)}{2}$	5	20
5 $\frac{(n+1)}{2}$	$\left(\frac{n-1}{2}\right)$	4	20
6		3	18
7		2	14
8		1	8

Q: Let G be a bi-partite graph with 'n' vertices, find the maximum number of edges possible in graph G.

This are the total No. of vertices

$V = \{V_1, V_2\}$

$V_1 \cap V_2 = \emptyset$

then $V_1 \cup V_2 = V$

$\therefore |V_1| + |V_2| = |V| = n$

$$\text{Max No. of edges} = \begin{cases} \frac{n}{2} * \frac{n}{2} & \text{if } n = \text{even} \\ \left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}\right) & \text{if } n = \text{odd} \end{cases}$$

$$= \left\lfloor \frac{n^2}{4} \right\rfloor \quad \text{for any random 'n'}$$

Topic : Sum of degree theorem

Let $G = (V, E)$ be a graph

Where $V = \{v_1, v_2, v_3, \ldots, v_n\}$

$\boxed{|V| = n}$

then,

$$\boxed{\sum_{i=1}^{n=|V|} \deg(v_i) = 2 * |E|}$$

$\left\{ \text{Handshaking Lemma} \right\}$

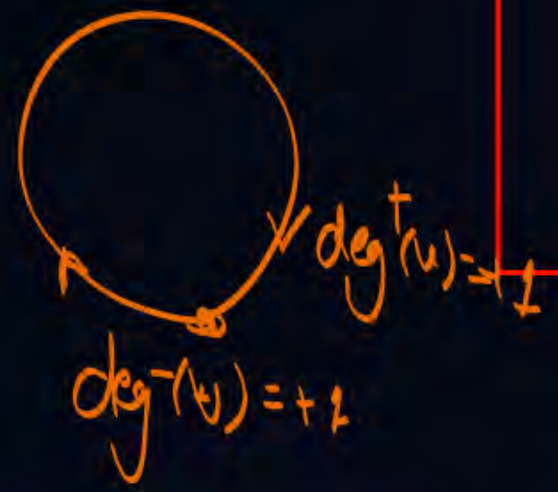
$$\boxed{\deg(v_1) + \deg(v_2) + \ldots + \deg(v_n) = 2 * |E|}$$

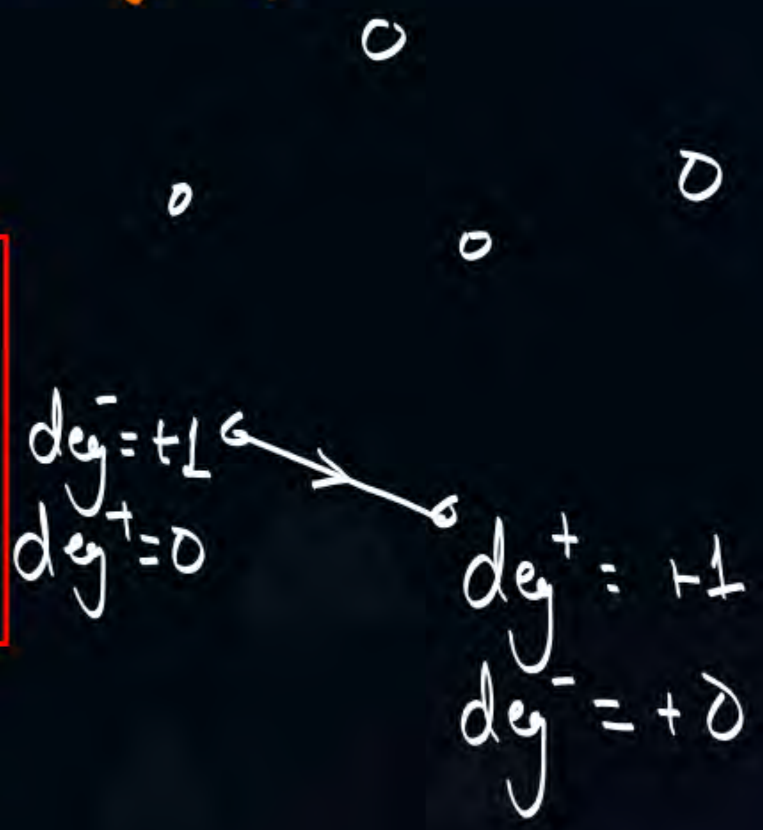
Topic : Sum of degree theorem

Corollary 1 :- If $G = (V, E)$ is a directed graph,

then,

$$\boxed{\sum_{i=1}^{|V|} \deg^{+}(v_i) = \sum_{i=1}^{|V|} \deg^{-}(v_i) = |E|}$$





Topic : Sum of degree theorem

Corollary 2 :-

We know $\sum_{i=1}^{|V|} deg(v_i) = \underbrace{2|E|}_{\text{it is an even number}}$

$E+E=E$

$\boxed{E+O=O}$ x

$O+O=E$

$\boxed{O+O+O=O}$

If odd numbers added odd number of times, then result will be odd.

In any graph G, number of vertices with odd degree must be "even".

Topic : Sum of degree theorem

Corollary 3 :-

In a graph G if degree of each vertex is exactly 'k' $\{$ i.e. $deg(v) = k, \forall v \in G \}$, then

$$\boxed{k \cdot |V| = 2|E|}$$

$$deg(v_1) + deg(v_2) + \cdots + deg(v_n) = 2|E|$$

$$\underbrace{k + k + \cdots + k}_{k \text{ added } 'n_{|V|}' \text{ times}} = 2|E|$$

$$\therefore \boxed{k|V| = 2|E|}$$

Topic : Sum of degree theorem

Corollary 4 :-

In a graph G, if degree of each vertex is at least 'K' $\{ \text{i.e.} \geq K \}$ then

$$\boxed{K|V| \leq 2|E|}$$

$$deg(v_1) + deg(v_2) + \cdots + deg(v_{|V|}) = 2|E|$$

$x_1, x_2, \ldots x_{|V|} \geq 0$

$$(k + x_1) + (k + x_2) + \cdots + (k + x_{|V|}) = 2|E|$$

$$K \cdot |V| + \sum_{i=1}^{|V|} x_i = 2|E|$$

≥ 0

$$\therefore \boxed{K|V| \leq 2|E|}$$

Topic : Sum of degree theorem

Corollary 5 -

In a graph G, if degree of each vertex is at most 'K' {i.e. $\leq K$} then

$$\boxed{K|V| \geq 2|E|}$$

$$deg(v_1) + deg(v_2) + \cdots + deg(v_{|V|}) = 2|E|$$

$x_1, x_2, \ldots x_{|V|} \geq 0$

$$(k - x_1) + (k - x_2) + \cdots + (k - x_{|V|}) = 2|E|$$

$$K|V| - \left(\sum_{i=1}^{|V|} x_i\right) = 2|E|$$

≥ 0

$$\boxed{K|V| \geq 2|E|}$$

Note: In a graph $G = (V, E)$,

- degree of each vertex is at least $\delta(G)$

and,

- degree of each vertex is at most $\Delta(G)$

$\therefore$ $\delta(G) \cdot |V| \leq 2|E|$

& $2|E| \leq \Delta(G) \cdot |V|$

i.e.

$$\boxed{\delta(G) \cdot |V| \leq 2|E| \leq \Delta(G) \cdot |V|}$$

?

deg of each vertex = k

i.e $|V| = n$

#Q. The number of edges in a k - regular graph with n vertices is

$$k|V| = 2|E|$$

$$|E| = \frac{k|V|}{2} = \frac{k \cdot n}{2}$$

$$\boxed{|E| = \frac{nk}{2}}$$

$|E| = 16$

#Q. A non-directed graph contains 16 edges and all vertices are of degree 2. Then the number of vertices in G is ______.

$k = 2$

$$k \cdot |V| = 2|E|$$

$$|V| = \frac{2 \cdot |E|}{k} = \frac{\cancel{2} * 16}{\cancel{2}}$$

$$|V| = 16$$

$|E| = 25$

#Q. G is undirected graph with n vertices and 25 edges such that each vertex of G has degree at least 3. Then the maximum possible value of n is 16.

$K \leq 3$

$K|V| \leq 2|E|$

$|V| \leq \frac{2|E|}{K}$

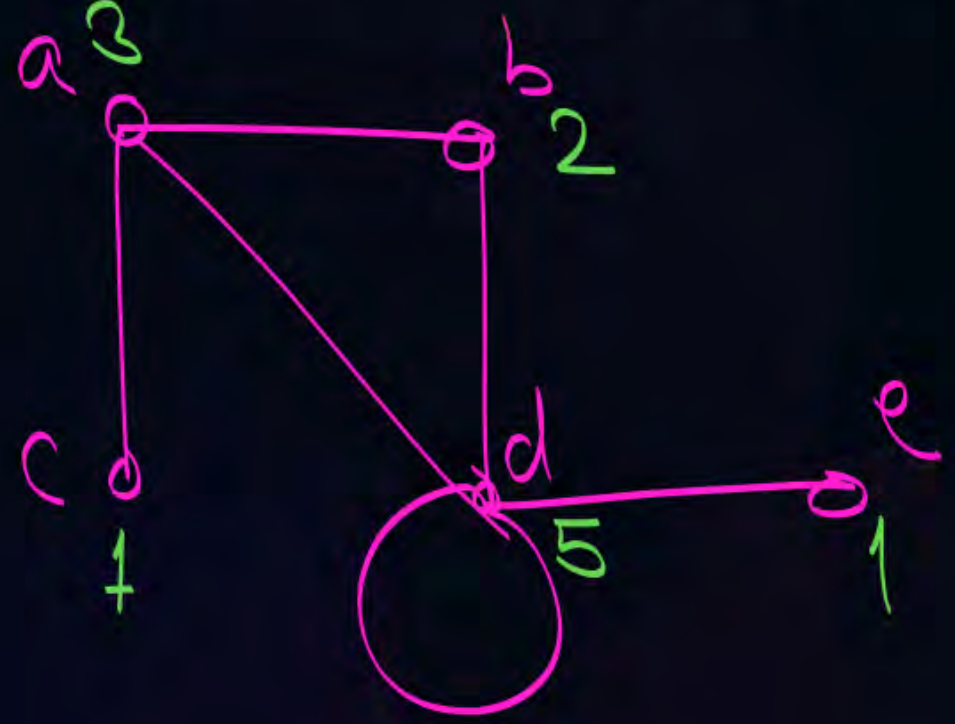
$|V| \leq \frac{2 * 25}{3}$

$|V| \leq 16.6$

$|V| \leq 16$ → Max possible value of $|V|$ is 16

Topic : Degree sequence

In a graph G if degrees of all the vertices are arranged in non-increasing or non-decreasing order, then it is called degree sequence of graph G.



degrees in non-increasing order = $\{5, 3, 2, 1, 1\}$

degrees in non-decreasing order = $\{1, 1, 2, 3, 5\}$

Graphic :-

If a degree sequence can represent a simple non-directed graph, then that degree sequence is called a "graphic"

H.W.

#Q. Which of the following degree sequences can represent a simple non-directed graph?

1. {2, 3, 3, 4, 4, 5}
2. {2, 3, 4, 4, 5}
3. {1, 3, 3, 3}
4. {0, 1, 2, 3,............,n-1}
5. {1, 3, 3, 4, 5, 6, 6}
6. {3, 3, 3, 3, 2}

#Q. Which of the following degree sequences represent a simple non directed graph?

$S1 = \{6, 6, 6, 6, 4, 3, 3, 0\}$

$S2 = \{6, 5, 5, 4, 3, 3, 2, 2, 2\}$

A. Only S1

B. Only S2

C. Both S1 and S2

D. Neither S1 nor S2

2 mins Summary

$\leq$

- Topic — Different types of graphs
- Topic — Sum of degree theorem
- Topic — Degree Sequence
- Topic — Havel Hakimi's algorithm

THANK - YOU

Slide