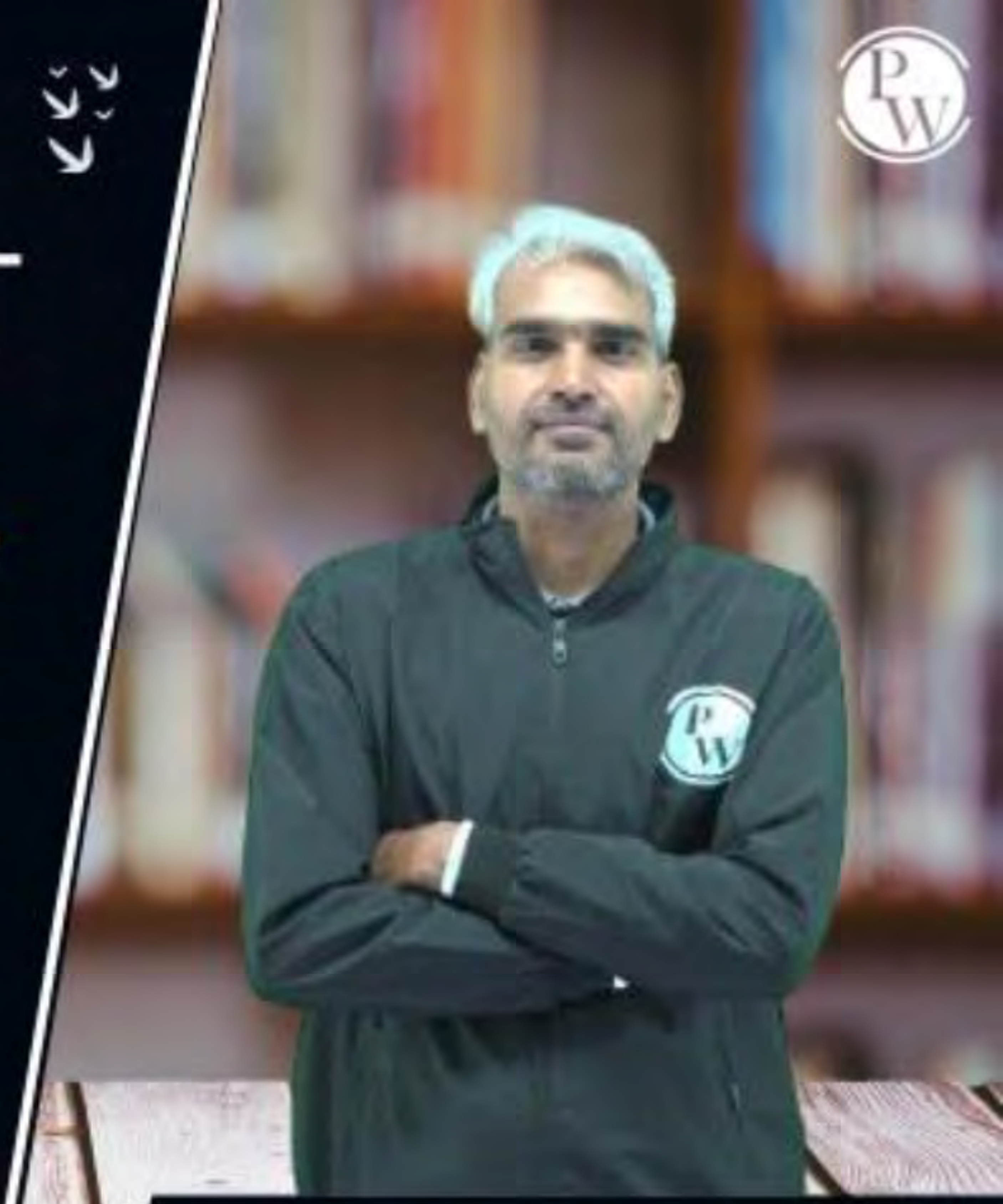
# Computer Science & IT

## Discrete Mathematics

**Set Theory & Algebra**

**Lecture No. 13**



By- Vishal Sir

# Recap of Previous Lecture

**Topic** Hasse diagram

Slide

# Topics to be Covered

**Topic** Sublattice

**Topic** Bounded lattice

**Topic** Complements of an element in a lattcie

**Topic** Complemented lattice

Slide

# Topic : Sublattice

- Let $[L, \vee, \wedge]$ be a lattice, and M is a subset of set L.
- "M" w.r.t. same relation is called a sublattice of lattice L

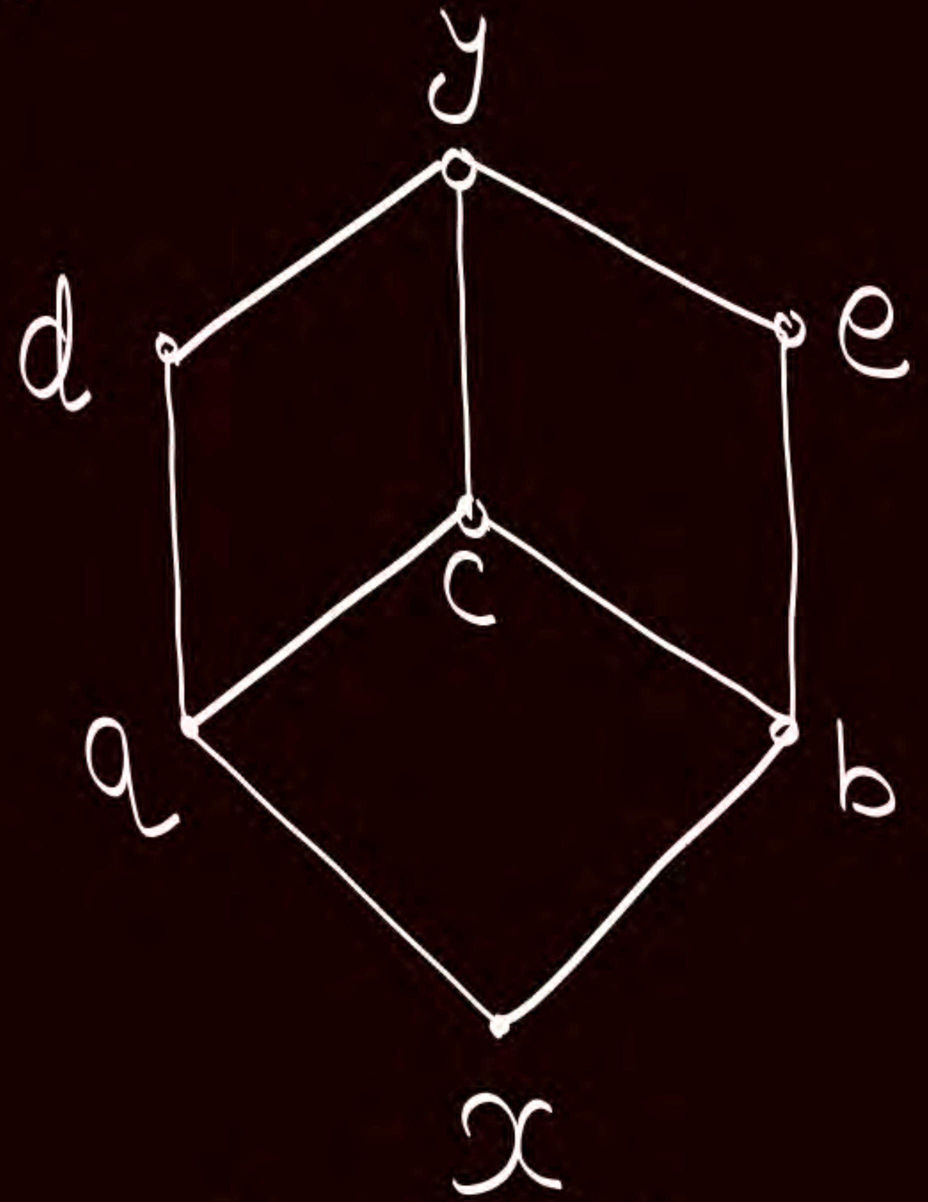
Only if, (1) $[M, \vee, \wedge]$ must be a lattice {i.e. for every pair of elements of subset M lub & glb must exist in subset M.}

and (2) For every pair of elements of subset M, lub & glb must be same as lub & glb for that pair of elements in Lattice $[L, \vee, \wedge]$

i.e. $\forall a, b \in M$. $\underset{\text{in } M}{lub(a,b)} = \underset{\text{in } L}{lub(a,b)}$ & $\underset{\text{in } M}{glb(a,b)} = \underset{\text{in } L}{glb(a,b)}$

Slide

eg: Consider the following Lattice 'L'




$L = \{x, a, b, c, d, e, y\}$

Which of the following is/are sublattice of lattice L

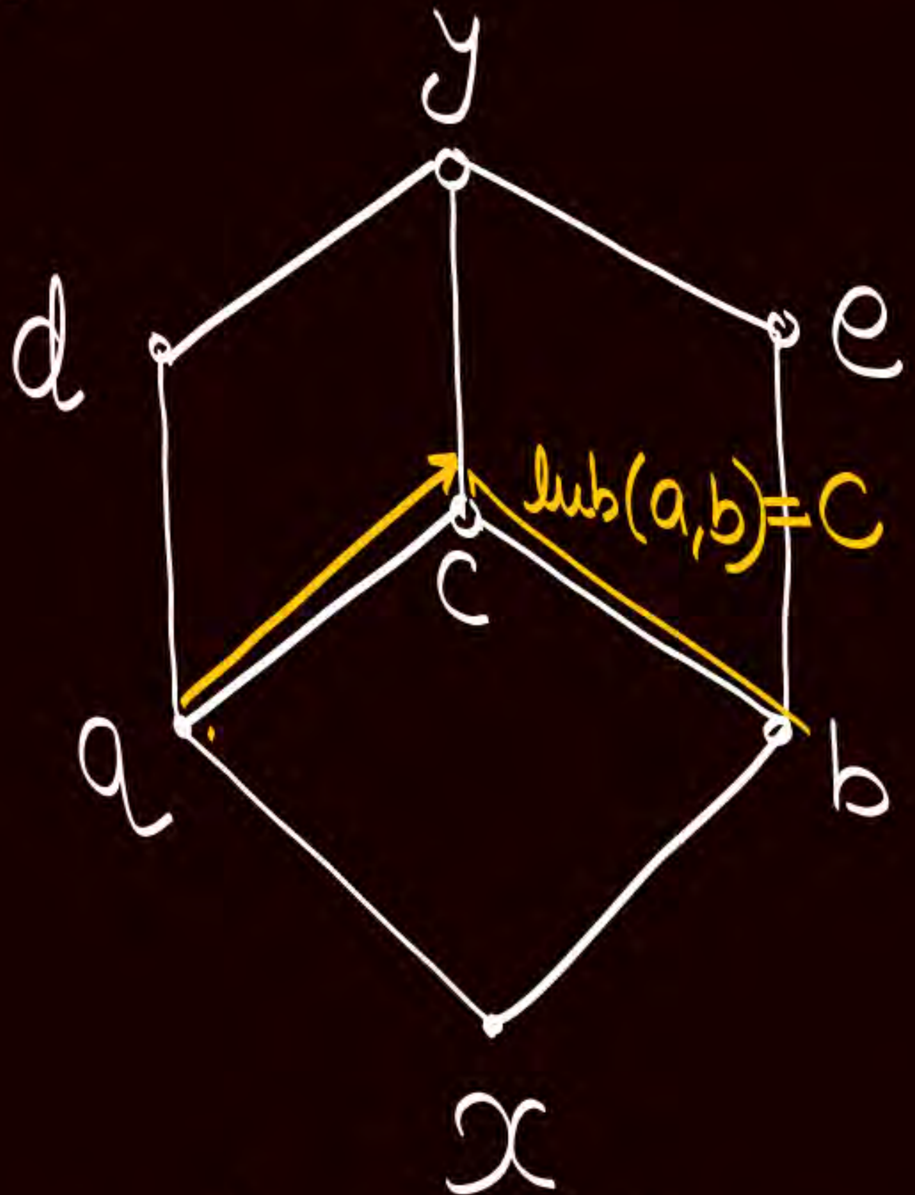
ⓐ $L_1 = \{x, a, b, y\}$

ⓑ $L_2 = \{x, a, c, y\}$

ⓒ $L_3 = \{x, c, d, y\}$

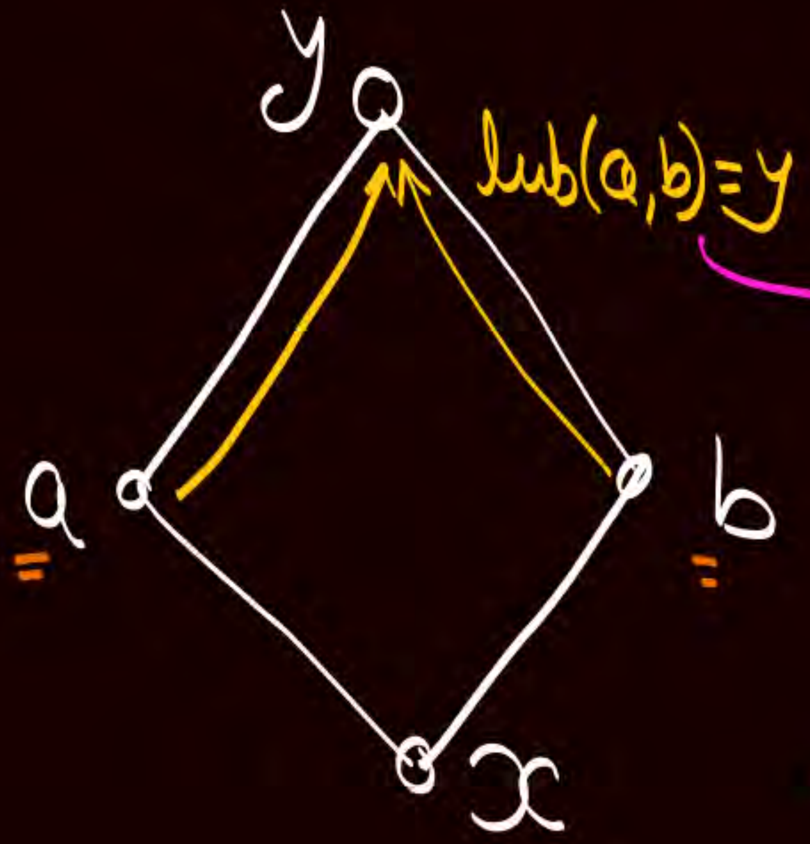
ⓓ $L_4 = \{x, a, e, y\}$

eg: Consider the following Lattice 'L'




$L = \{x, a, b, c, d, e, y\}$

ⓐ $L_1 = \{x, a, b, y\}$

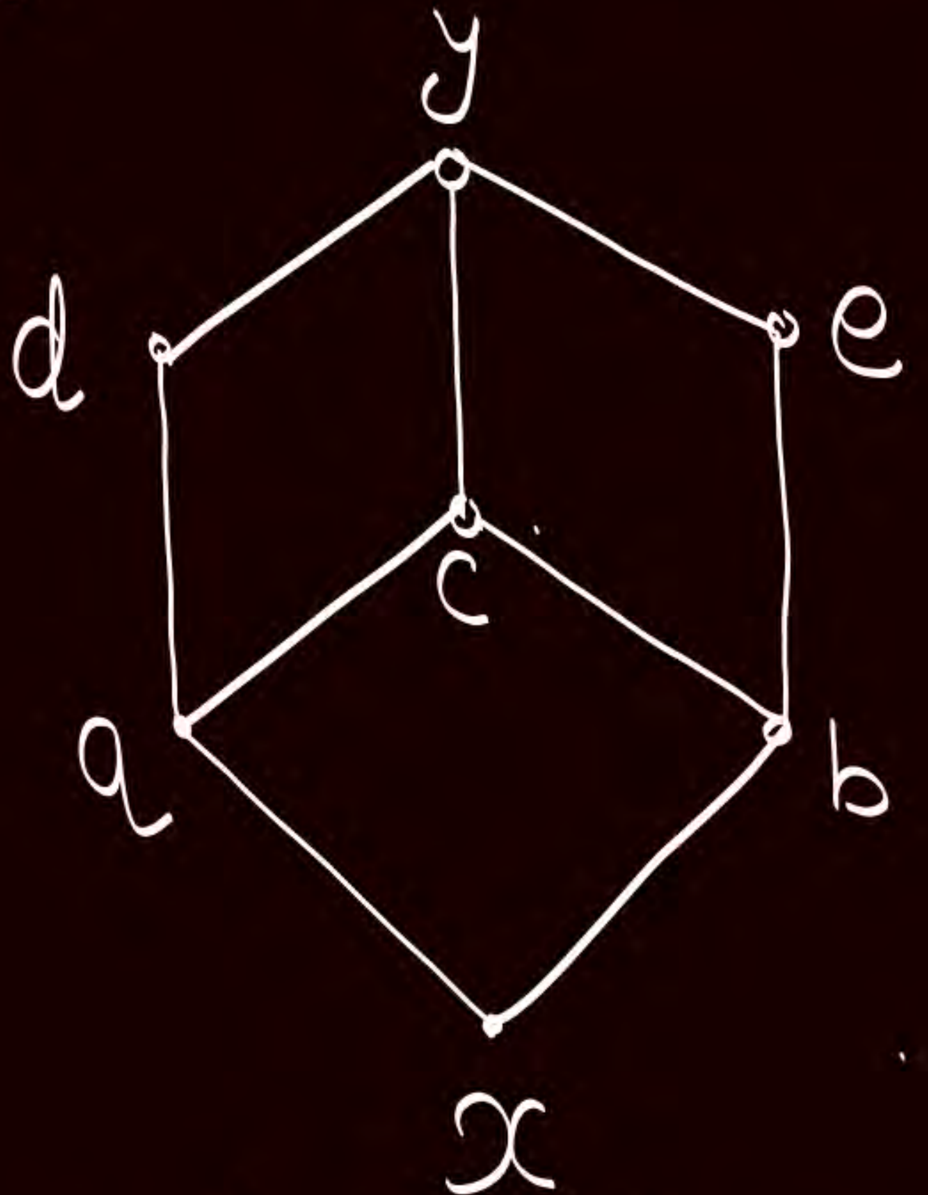



① $L_1$ is a lattice

Not same as lattice L

$\therefore$ $L_1$ is not a sublattice of lattice L

eg: Consider the following Lattice 'L'




$L = \{x, a, b, c, d, e, y\}$

ⓑ $L_2 = \{x, a, c, y\}$

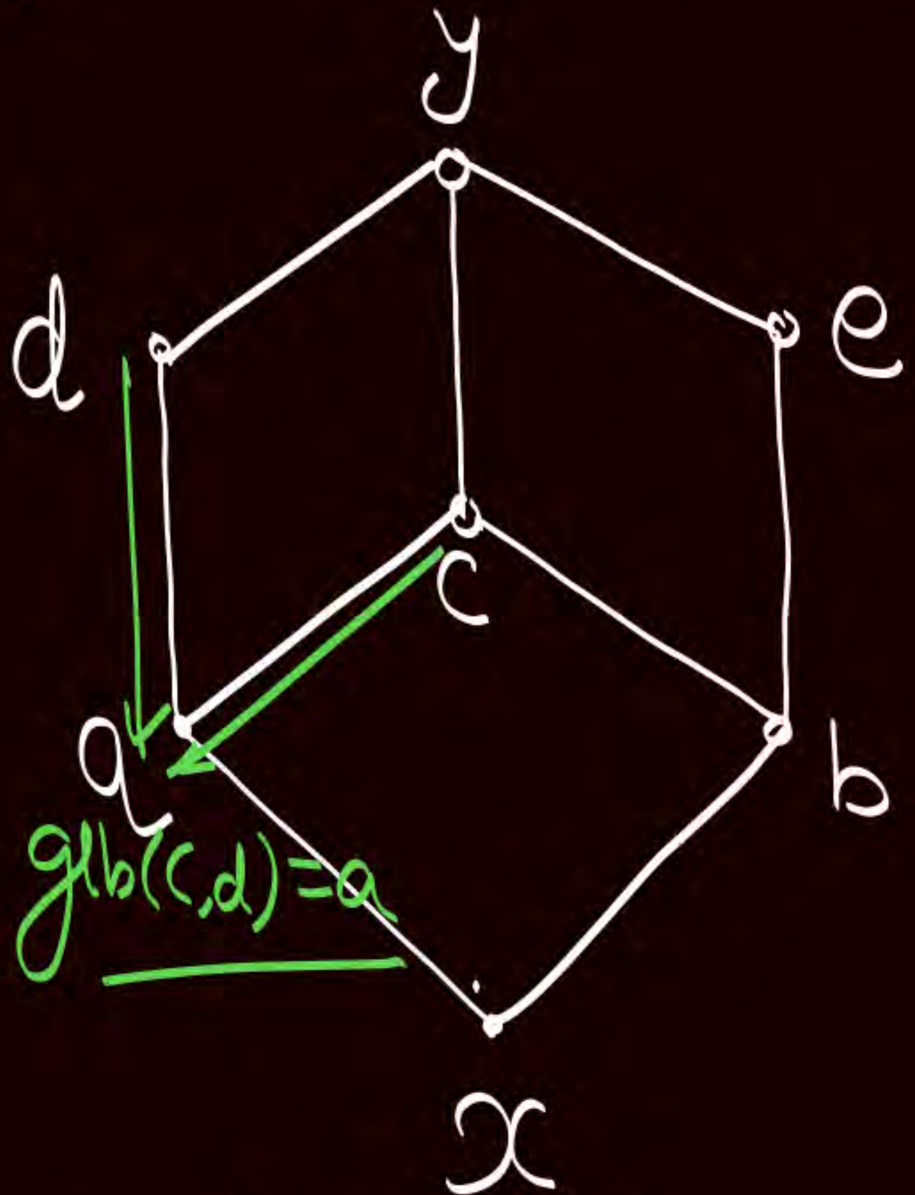

① $L_2$ is a lattice

② $x, a, c$ & $y$ are in a line, in lattice $L$ as well as in lattice $L_2$

lub & glb for every Pair of elements in $L_2$ will be same as $L$

$L_2$ is a Sublattice of lattice $L$

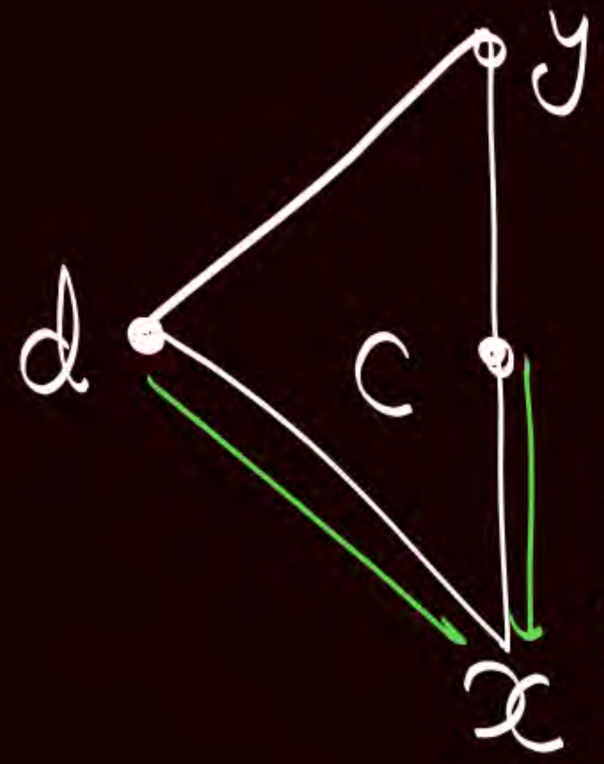
eg: Consider the following Lattice 'L'




$glb(c,d) = a$

$L = \{x, a, b, c, d, e, y\}$

(i) $L_3 = \{x, c, d, y\}$

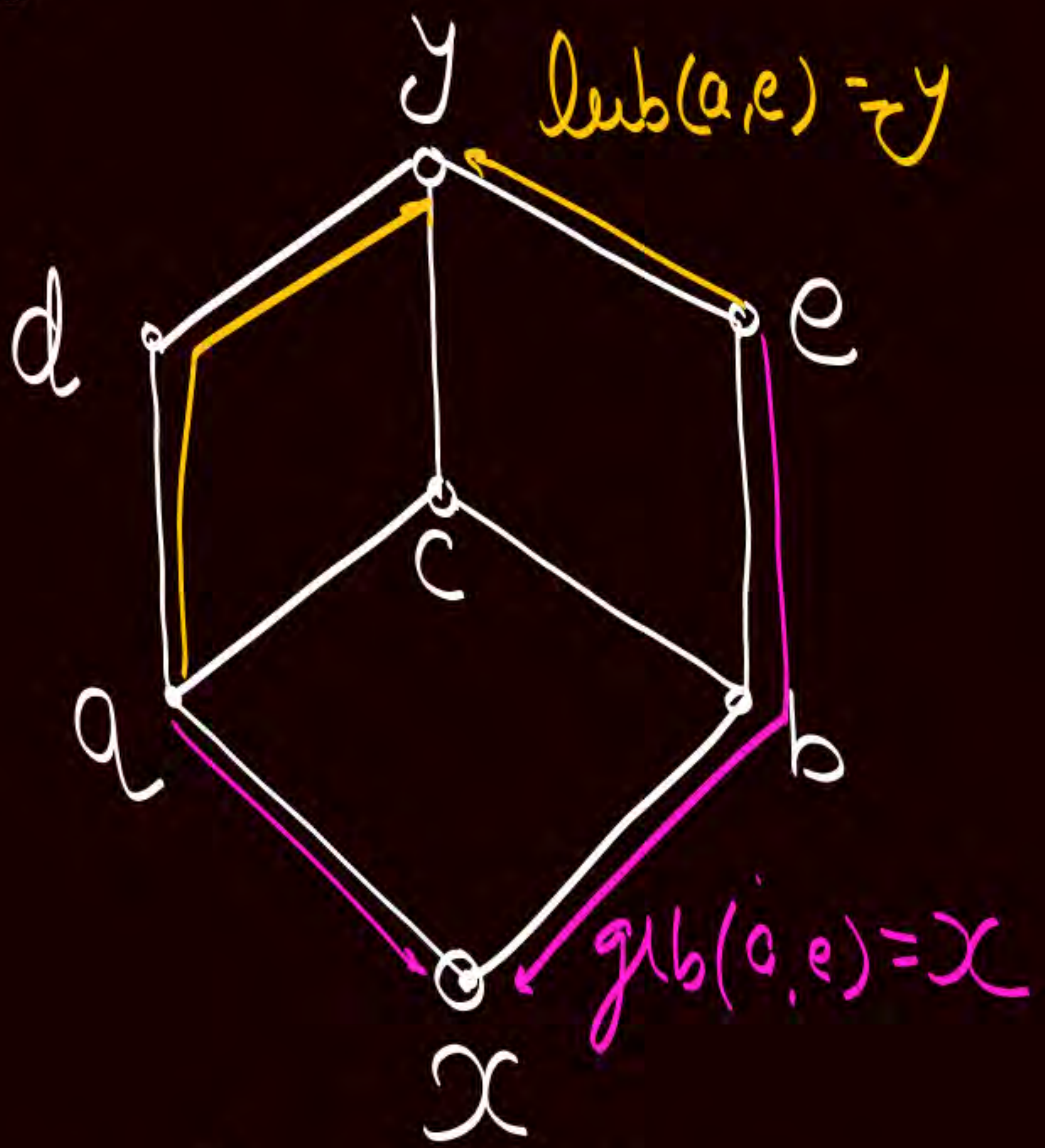



① $L_3$ is a lattice

$\underbrace{glb(c,d)}_{\text{Not same as lattice } L} = x$

$\therefore L_3$ is not a Sublattice of lattice $L$

eg: Consider the following Lattice 'L'




$L = \{x, a, b, c, d, e, y\}$

(d) $L_4 = \{x, a, e, y\}$

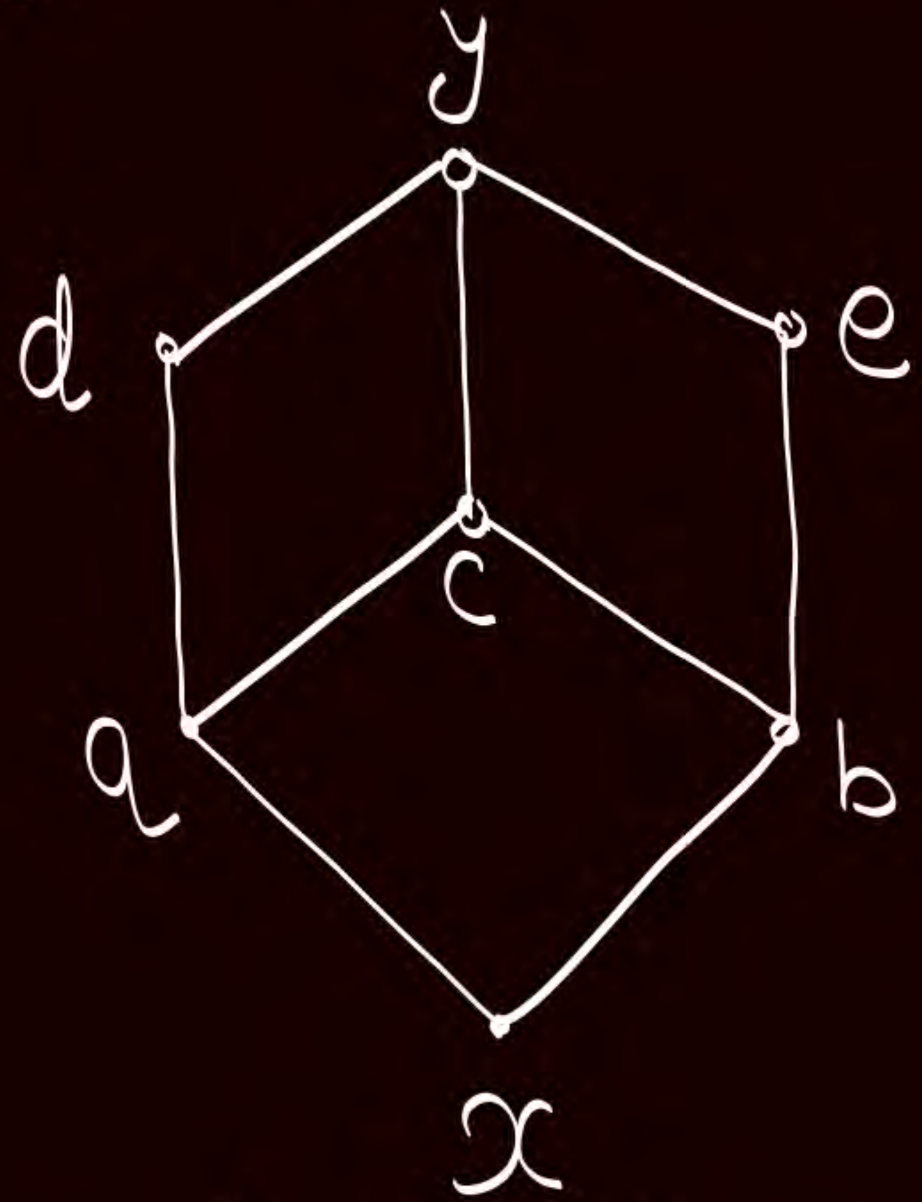

(1) $L_4$ is a lattice

same as L

(2) lub & glb for every pair of elements of $L_4$ is same as L

∴ $L_4$ is a sublattice of L

eg: Consider the following Lattice 'L'




$L = \{x, a, b, c, d, e, y\}$

Which of the following is/are sublattice of lattice L

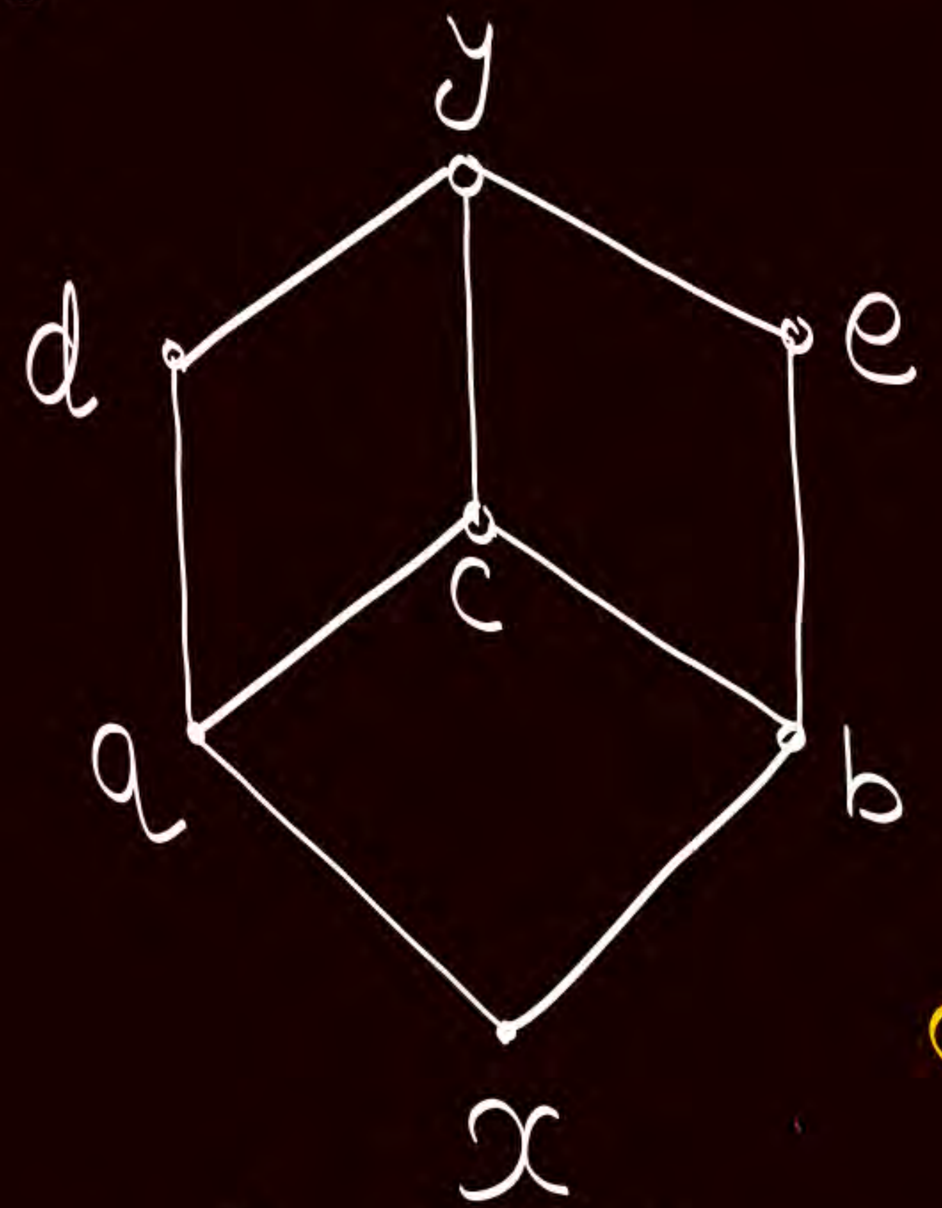
(a) $L_1 = \{x, a, b, y\}$

(b) $L_2 = \{x, a, c, y\}$

(c) $L_3 = \{x, c, d, y\}$

(d) $L_4 = \{x, a, e, y\}$

eg: Consider the following Lattice 'L'




$L = \{x, a, b, c, d, e, y\}$

Which of the following is/are sublattice of lattice 'L'

None of them

$L_5 = \{x, a, b, e\} =$


$L_6 = \{a, b, c, y\} =$



$= L_7 = \{c, d, e, y\}$

$L_8 = \{a, b, c, d, e\} =$

# Topic : Bounded Lattice

Let $[L, \vee, \wedge]$ be a lattice

(1) If there exist an element $I \in L$, such that

$$\underbrace{a \vee I}_{lub(a, I)} = I, \quad \forall a \in L$$

then, I is called upper bound of lattice L (or) (Universal upper bound)

It is actually the maximum element of POSET w.r.t. lattice

Slide

# Topic : Bounded Lattice

Let $[L, \vee, \wedge]$ be a lattice

(2) If there exist an element $0 \in L$, such that

$$\underset{glb(a,0)}{a \wedge 0} = 0, \quad \forall a \in L$$

then, $0$ is called lower bound of lattice L
(or)
(Universal lower bound)

It is actually the minimum element of POSET, w.r.t lattice

Slide

# Topic : Bounded Lattice

Let $[L, \vee, \wedge]$ be a lattice

If both I (universal upper bound) & O (universal lower bound) exist in lattice 'L', then L is called a bounded lattice

$I$ = Maximum (Greatest) element of POSET w.r.t. lattice

$O$ = Minimum (least) element of POSET w.r.t. lattice

Slide

# Topic : Bounded Lattice

Let $[L, \vee, \wedge]$ be a lattice

If both Minimum as well as Maximum element exist in lattice, then 'L' is a bounded lattice.

otherwise, unbounded lattice

Slide

**Note** ① There are some lattice which are not bounded.

eg: $(\mathbb{N}, \leq)$ is a lattice, but it is not a bounded lattice

set of natural No.

POSET diagram $\Rightarrow$


'1' is the universal lower bound

**but** universal upper bound does not exist.

eg $(\mathbb{Z}, \leq)$ is also a lattice, but not bounded

set of all integers

$I$ = Universal upper bound = does not exist

$O$ = Universal lower bound = does not exist.

**Note:** ② If lattice $[L, \vee, \wedge]$ is an unbounded lattice, then underlying set 'L' is an infinite set, but converse of the statement need not be true

eg: let $A = \{x \mid x \in R \text{ and } 0 \leq x \leq 1\}$

it is an infinite set

but $(A, \leq)$ is a bounded lattice

Where $I$ = universal upper bound = '1'

$O$ = universal lower bound = '0'

Note:- 3 Every finite lattice is a bounded lattice.

# Topic : Complement of an element in a lattice

**Note:-** Complement of an element in a lattice can be defined only if lattice is a bounded lattice.

{i.e., if lattice is not a bounded lattice then Complement does not exist for any element of that lattice}

Let $[L, \vee, \wedge]$ be a bounded lattice

where, $I$ = universal upper bound & $0$ = universal lower bound

for an element $a \in L$ if there exist any element $b \in L$

such that (i) $a \vee b = I$ {i.e. lub$(a,b)$ = Maximum element}

& (ii) $a \wedge b = 0$ {i.e. glb$(a,b)$ = Minimum element}

then 'a' & 'b' are called Complement of each other

Slide

## Topic : Complement of an element in a lattice

① In a bounded lattice,

'I' & '0' are always Complement of each other

The only complement of element $I = 0$

& only complement of element $0 = I$

② In a bounded lattice,

Complement need not exist for every element of lattice,

and if exist for any element then it need not be unique

i.e. In a bounded lattice number of complements of an element can be 0 or more

Slide

Note:

1. A totally ordered set is always a lattice.

2. A totally ordered set may or may not be a bounded lattice,

   e.g. $(\{1,2,3,4,5\}, \leq)$ is a bounded lattice

   e.g. $(N, \leq)$ is an unbounded lattice.

   Both are totally ordered set

3. If totally ordered set is an unbounded lattice then complement does not exist for any element of that lattice

**Note: (4)** If totally ordered set is bounded lattice, then

(i) Complement of $I = 0$

& (ii) Complement of $0 = I$

& (iii) Complement does not exist for any other element of totally ordered set


← Maximum element = $I$

Minimum element = $0$

bounded totally ordered set. {Bounded lattice}

if an element 'a' is at upper side of $x$ then $glb(a, x) = x \neq 0$

∴ 'a' can never be complement of $x$

$x \vee x = x \neq I$

∴ $x$ can not be complement of $x$

Similarly if element 'b' is at lower side of $x$, then $lub(b, x) = x \neq I$

∴ 'b' can never be complement of $x$

**Note:** Two elements on the same line in the Hasse diagram can never be complement of each other, Except for minimum & maximum elements

## Topic : Hasse Diagram / POSET Diagram

Draw the hasse diagram for the following POSET

$(\{-1, 0, 2.5, 4, 6\}, \leq)$


$\overline{-1} = 6$ & $\overline{6} = -1$

$\overline{0}$ = does not exist

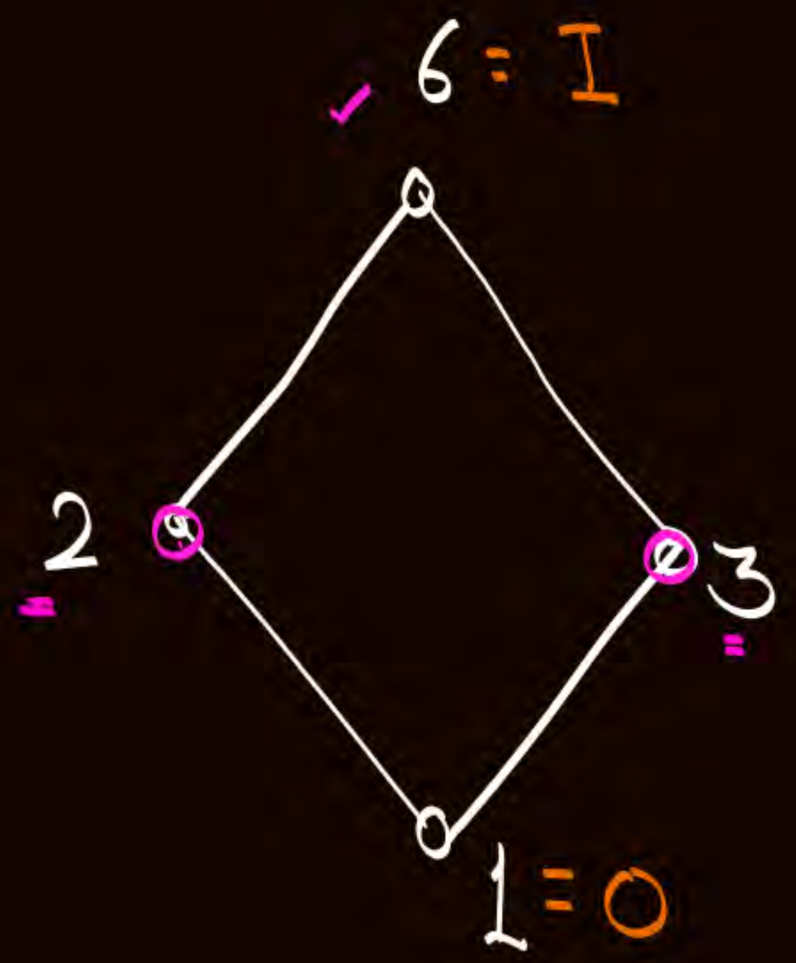
$\overline{2.5}$ = does not exist

$\overline{4}$ = does not exist

Slide

$(D_6, \div)$

$D_6 = \{1, 2, 3, 6\}$




$\boxed{\bar{I} = 0 \ \& \ \bar{0} = I}$

$\bar{6} = 1 \ \& \ \bar{1} = 6$

$2 \vee 3 = 6 = I$

$2 \wedge 3 = 1 = 0$

∴ 2 & 3 are complement of each other

i.e. $\bar{2} = 3$

& $\bar{3} = 2$

$(D_{12}, \div)$

$D_{12} = \{1, 2, 3, 4, 6, 12\}$

$\bar{2}$ = does not exist

$12 = I \Rightarrow \overline{12} = 1$


$1 = 0 \Rightarrow \bar{1} = 12$

Complement of 2 can never exist on this Yellow color line or Red color line $\Rightarrow$ $\therefore$ Only possibility = 3

$2 \vee 3 = 6 \neq I$

$\therefore$ '3' is not Complement of 2

$(D_{12}, \div)$

$D_{12} = \{1, 2, 3, 4, 6, 12\}$

$\bar{2}$ = does not exist

$\bar{3} = 4$

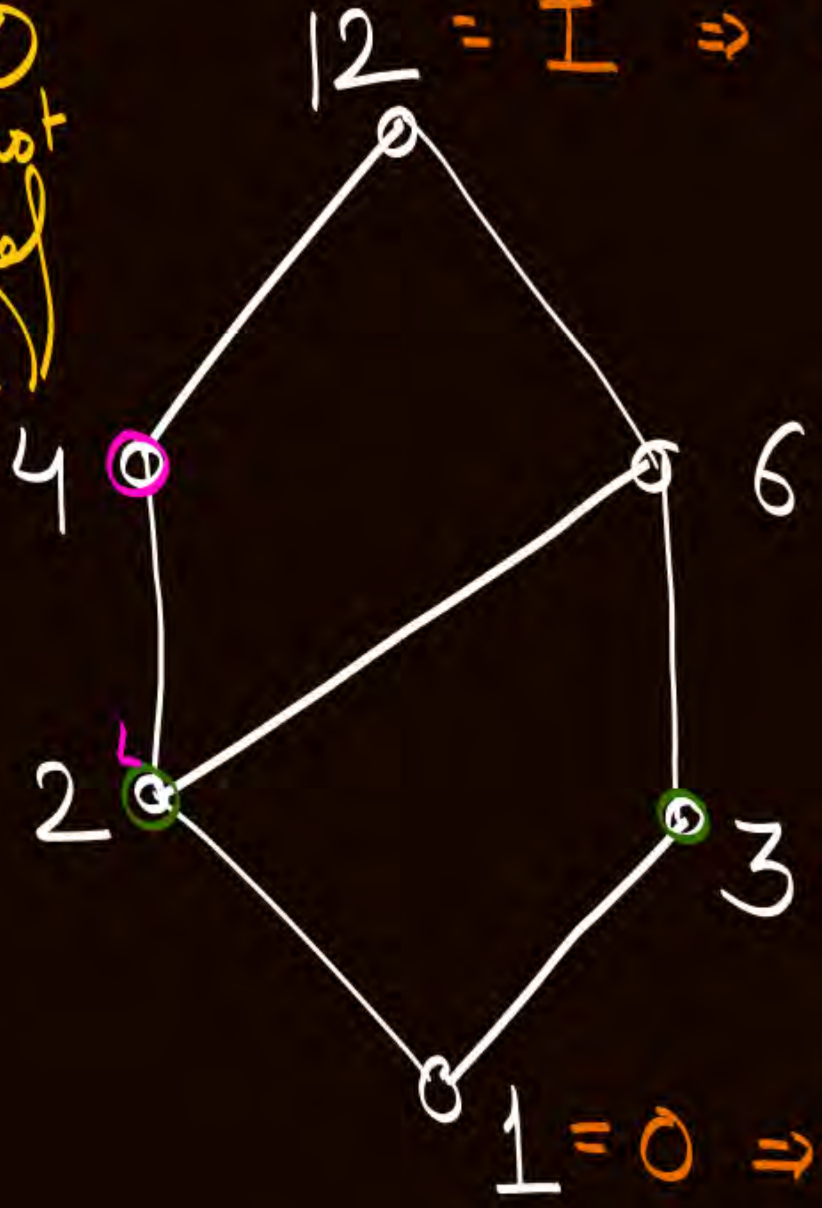
$\bar{4} = 3$

$\bar{6}$ = does not exist

$glb(4,6) = 2 \neq 0$

$\therefore$ 4 & 6 can not be complement of each other

$12 = I \Rightarrow \overline{12} = 1$




$1 = 0 \Rightarrow \bar{1} = 12$

$\Rightarrow$ $3 \vee 4 = 12 = I$

$3 \wedge 4 = 1 = 0$

$\therefore$ 3 & 4 complement of each other

# 2 mins Summary

**Topic** Sublattice

**Topic** Bounded lattice

**Topic** Complements of an element in a lattcie

**Topic** Complemented lattice

Slide

# THANK - YOU

Slide