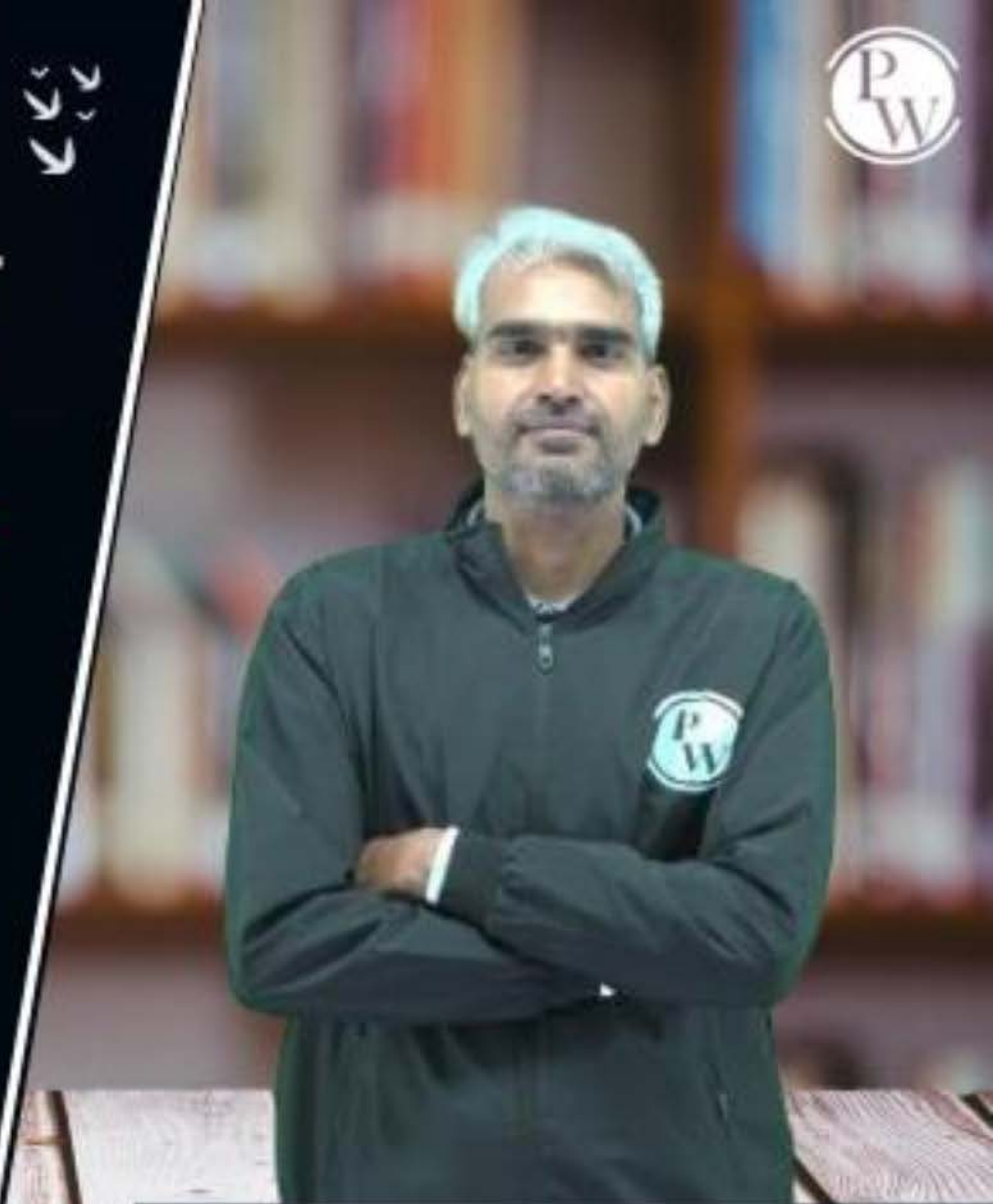
Computer Science & IT

Discrete Mathematics

Graph Theory

Lecture No. 11



By- Vishal Sir

Recap of Previous Lecture

Topic Line / Edge Covering

Topic Minimal & Minimum line covering

Topic Line independent set

Topic Maximal & Maximum line independent set

Slide

Topics to be Covered

- **Topic** Vertex Covering
- **Topic** Minimal & Minimum vertex covering
- **Topic** Vertex independent set
- **Topic** Maximal & Maximum vertex independent set
- **Topic** Spanning tree

Slide

$$\boxed{\chi(G) \leq \Delta(G) + 1}$$

always true

In a graph G,

$$\boxed{\chi(G) \geq \delta(G)}$$

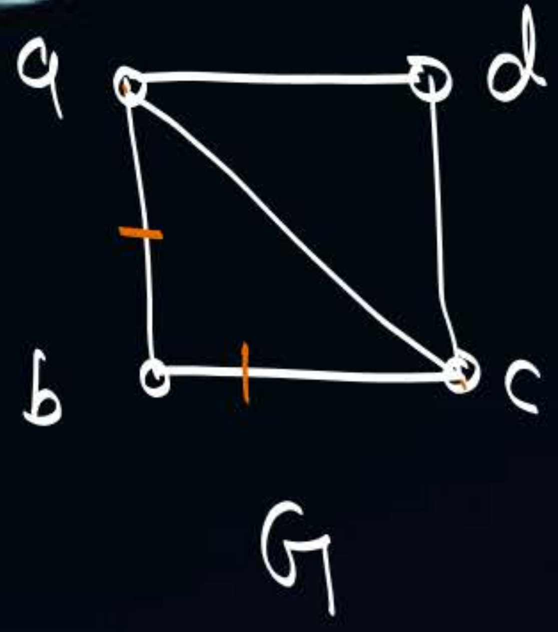
need not be true

Topic : Vertex Covering

Let G=(V,E) be a graph.

A subset K of set of vertices V is called vertex covering of graph G if every edge of the graph is incident with at least one vertex of set K

Topic : Example of vertex covering



$K_1 = \{a, b, c, d\}$

$K_2 = \{a, c\}$

$K_3 = \{b, d, a\}$

$K_4 = \{b, d, c\}$

Minimal Vertex Covering

$K_5 = \{b, d\}$ ← Not a vertex covering of graph G.

Topic : Minimal vertex covering

{No vertex can be deleted from the set}

A vertex covering from which no vertex can be deleted without destroying the ability to cover all the edges is called a minimal vertex covering

In the above example K_2, K_3, K_4 are minimal vertex covering

Topic : Minimum vertex covering

/ Smallest minimal vertex covering

A vertex covering of graph G with minimum number of vertices is called minimum vertex covering.

A graph may have more than one Minimum Vertex Covering

In the above example 'K_2' is the minimum vertex covering

Topic : Vertex covering number (α_2)

Vertex covering No. of graph G = No. of vertices in any one of the minimum vertex covering of graph G

for the above example

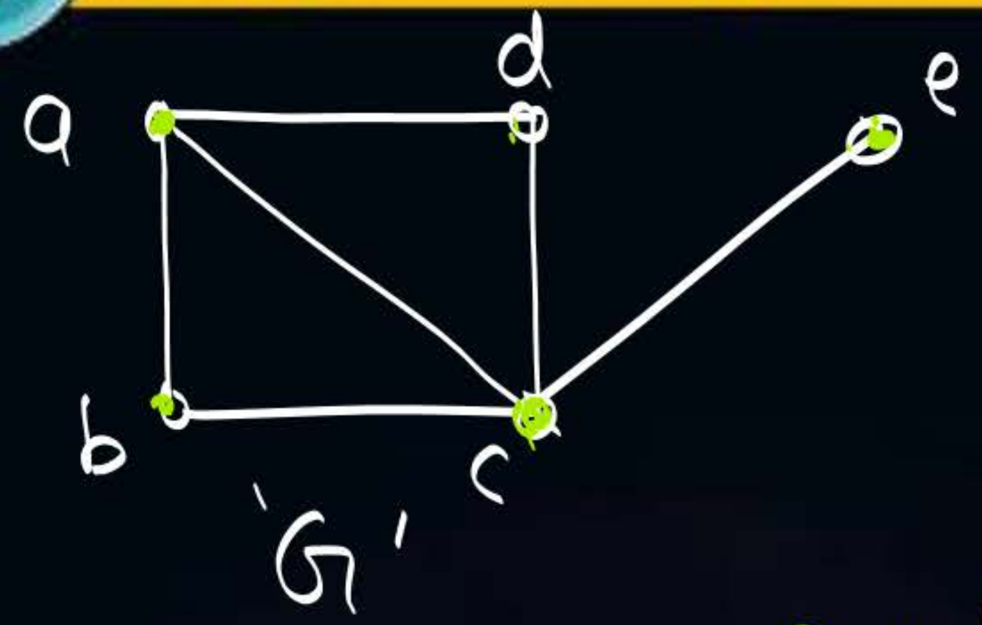
$\alpha_2 = 2$ {it is w.r.t. vertex covering K_2}

Topic : Vertex independent set

Let $G=(V,E)$ be a graph,

A subset 'S' of set of vertices V is called a vertex independent set if no two vertices of set S are adjacent to each other

Topic : Example of vertex independent set



$S_1 = \{a\}$

$S_2 = \{c\}$

Any subset of vertices with only one vertex is always a vertex independent set.

$S_3 = \{a, e\}$

$S_4 = \{b, d, e\}$

$S_5 = \{b, e\}$

S_2, S_3, S_4 are Maximal vertex independent set.

$S_6 = \{a, c\}$ is not a vertex independent set.

Topic : Maximal vertex independent set

A vertex independent set in which no other vertex can be added without destroying its property of being a vertex independent set, is called a maximal vertex independent set.

In the above eg,

S_2, S_3 & S_4 are maximal vertex independent set.

Topic : Maximum vertex independent set

/ Largest Maximal Vertex independent Set

→ A vertex independent set with maximum number of vertices is called a maximum vertex independent set

May be more than one

In the above example
S_4 is the maximum vertex independent set.

Topic : Vertex independence number (β_2)

Vertex independence No. of graph G = No. of vertices in any one of the Maximum vertex independent set of graph G.

In the above example

$\beta_2 = 3$ {w.r.t. S_4}

Topic : NOTE

for a graph $G = (V, E)$

$$\boxed{\alpha_2 + \beta_2 = |V(G)|}$$

Topic : NOTE

In a graph $G = (V, E)$

(1) If 'S' is the vertex independent set of graph G then 'V-S' will be the vertex covering of graph G

(2) If 'K' is the vertex covering of graph G, then 'V-K' will be vertex independent set of graph G

Topic : NOTE

In a graph $G = (V, E)$

(3) If 'S' is the maximal vertex independent set, then 'V-S' will be minimal vertex Covering

(4) If 'K' is the minimal vertex covering, then 'V-K' is the maximal Vertex independent set.

Topic : NOTE

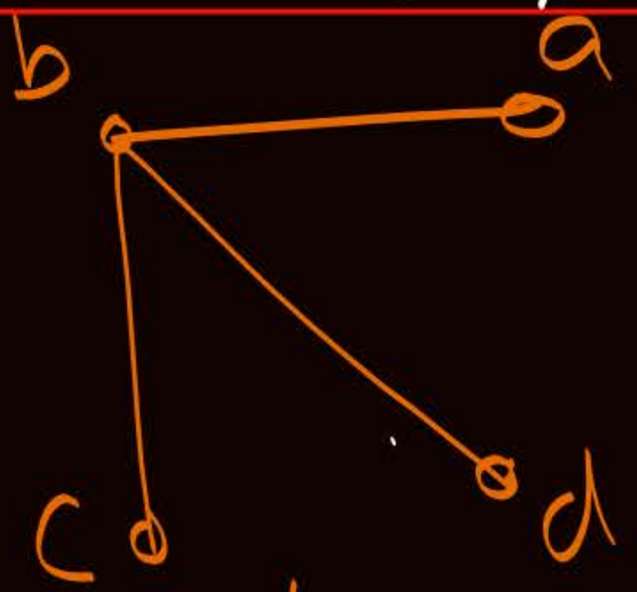
In a graph $G = (V, E)$

(5) If 'S' is the maximum vertex independent set, then 'V-S' is the minimum vertex covering

(6) If 'K' is the minimum vertex covering, then 'V-K' is the maximum vertex independent set.

IMP **Note:** If 'L' is a line independent set of graph $G=(V,E)$ then 'E-L' need not be line covering of graph G.

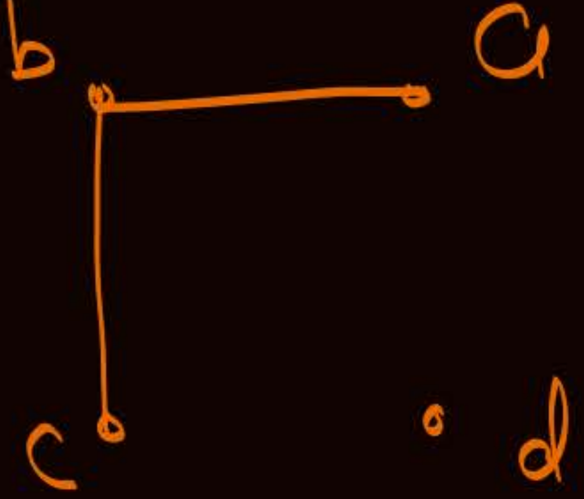
eg.



'G'

$L_1 = \{b,d\}$ is a line independent set

$'E-L_1' =$



is not a line covering of graph G

$C_1 = \{\{b,a\},\{b,c\}\}$

HW
Zoom
Last Class

Q: Find α_1 & β_1 for the following graphs.

	α_1	β_1
K_n	$\lceil \frac{n}{2} \rceil$	$\lfloor \frac{n}{2} \rfloor$
C_n	$\lceil \frac{n}{2} \rceil$	$\lfloor \frac{n}{2} \rfloor$
W_n	$\lceil \frac{n}{2} \rceil$	$\lfloor \frac{n}{2} \rfloor$
$K(m,n)$	Max(m,n)	Min(m,n)
Star graph with n-vertices	$(n-1)$	1

Q: Find α_2 & β_2 for the following graphs

	α_2	β_2
K_n	$(n-1)$	1
C_n	$\lceil \frac{n}{2} \rceil$	$\lfloor \frac{n}{2} \rfloor$
W_n	$\underbrace{1}_{\text{w.r.t. hub to cover all spokes}} + \underbrace{\lceil \frac{n-1}{2} \rceil}_{\text{to cover all edges of cycle } C_{n-1}} = \lceil \frac{n+1}{2} \rceil$	$\lfloor \frac{n-1}{2} \rfloor$
$K(m,n)$	$\min(m,n)$	$\max(m,n)$
Star graph with n-vertices	1	$(n-1)$

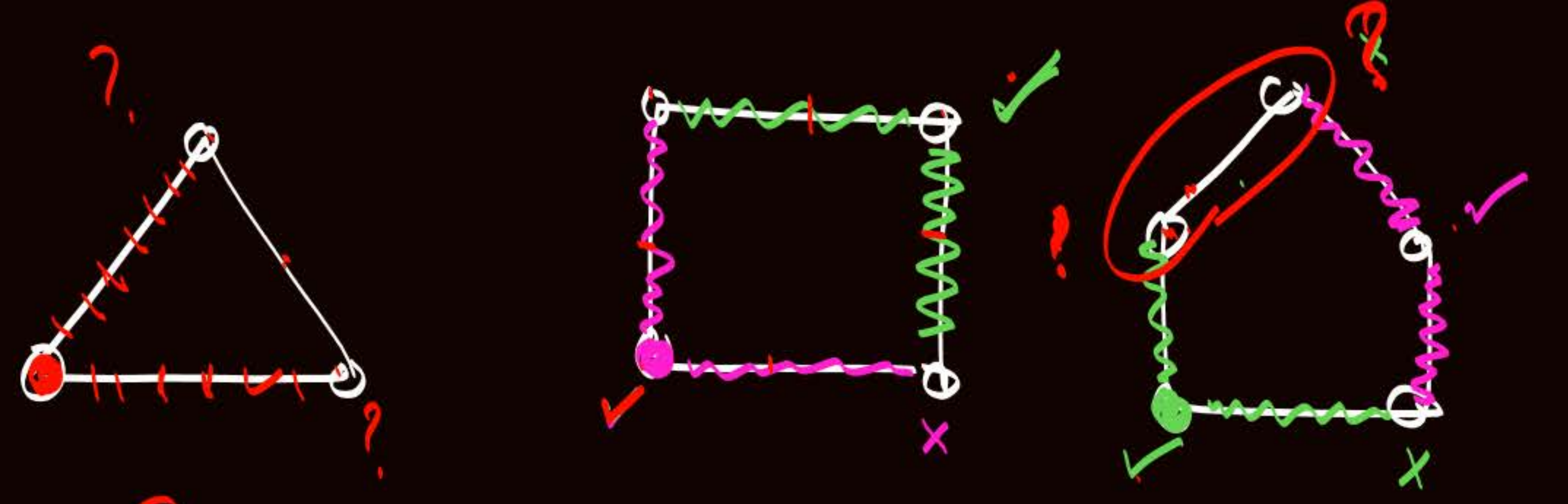
$\alpha_2 + \underset{1}{\beta_2} = n$

$\alpha_2 = n-1$

$\alpha_2 + \beta_2 = n$

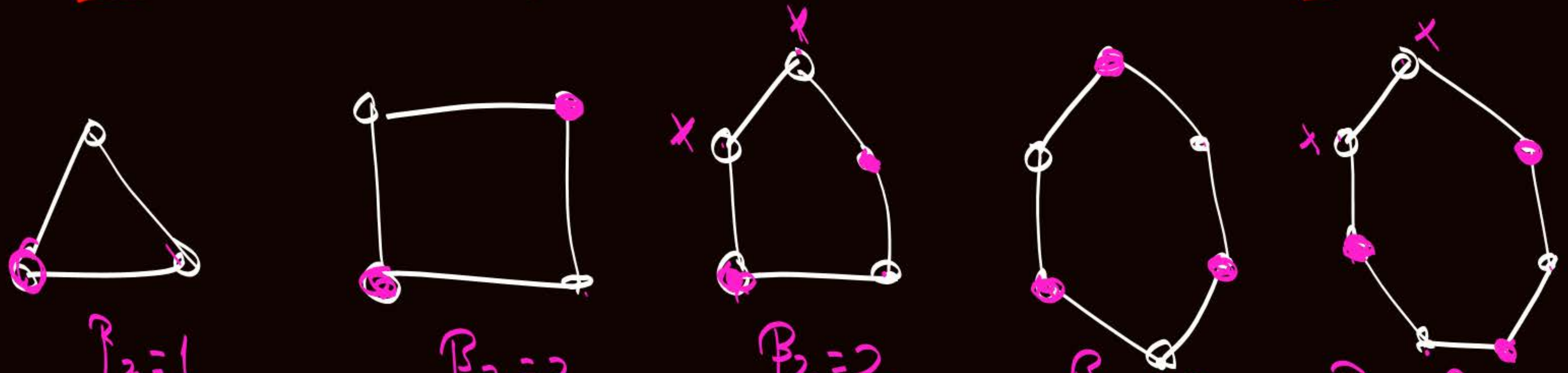
$\alpha_2 + \lfloor \frac{n-1}{2} \rfloor = n$

$\alpha_2 = \lceil \frac{n+1}{2} \rceil$

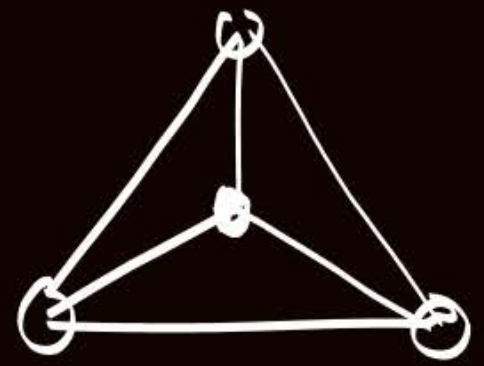


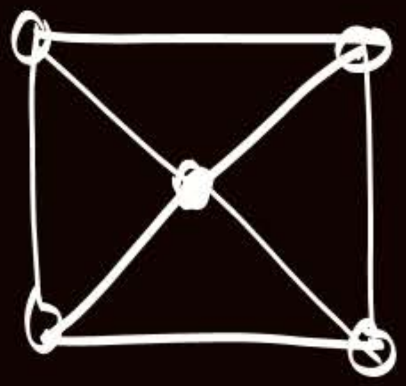
C_3 C_4 C_5 C_6

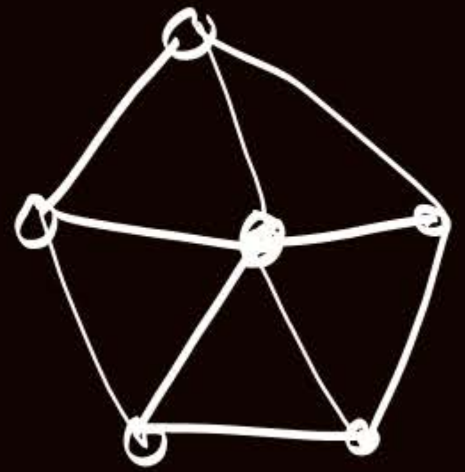
$\underline{\alpha_2 = 2}$ $\alpha_2 = 2$ $\alpha_2 = 3$ $\alpha_2 = 3$

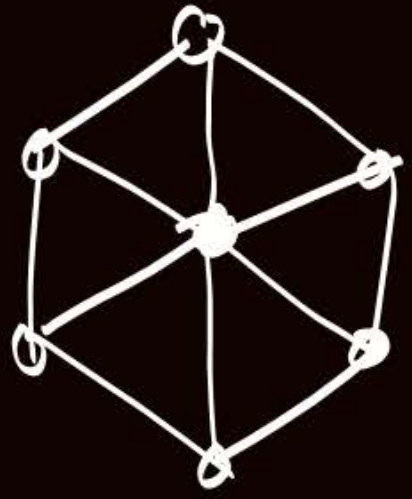


$\beta_2 = 1$ $\beta_2 = 2$ $\beta_2 = 2$ $\beta_2 = 3$ $\beta_2 = 3$









$W_n = C_{n-1} + 1$ hub

$W_4 = C_3 + 1 \Rightarrow \alpha_2 :$

$\beta_2 = \left\lfloor \frac{3}{2} \right\rfloor = \left\lfloor \frac{n-1}{2} \right\rfloor$

$W_5 = C_4 + 1$

$\beta_2 = \left\lfloor \frac{4}{2} \right\rfloor = \left\lfloor \frac{n-1}{2} \right\rfloor$

$W_6 = C_5 + 1$

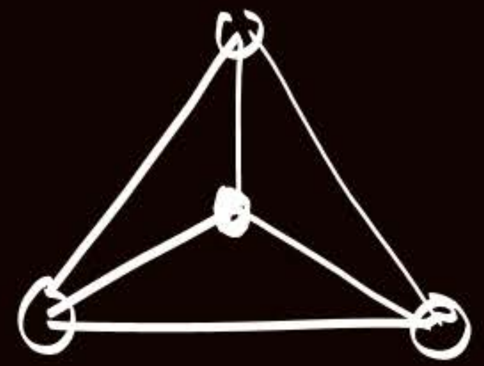
$\beta_2 = \left\lfloor \frac{5}{2} \right\rfloor = \left\lfloor \frac{n-1}{2} \right\rfloor$

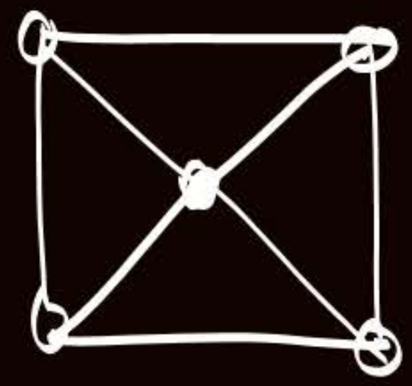
$W_7 = C_6 + 1$

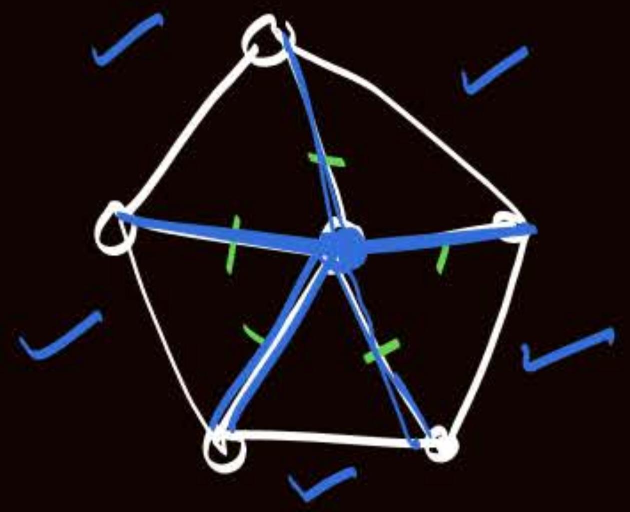
$\beta_2 = \left\lfloor \frac{6}{2} \right\rfloor = \left\lfloor \frac{n-1}{2} \right\rfloor$

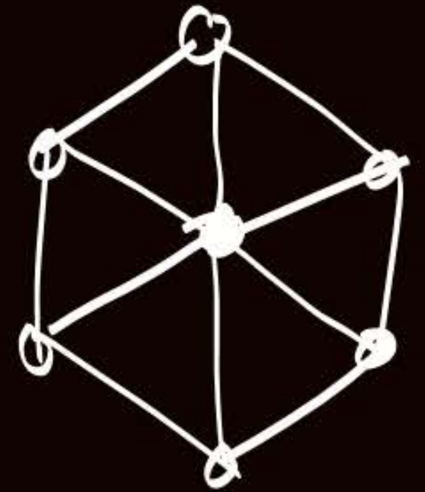
W.r.t. Maximum vertex independent set, hub vertex should not be selected because it is adjacent to all other vertices.

To select max. non-adjacent vertices from a wheel graph, they must be from C_{n-1} of wheel graph









$W_n = C_{n-1} + 1$ hub

$W_4 = C_3 + 1 \Rightarrow \alpha_2 =$

$\beta_2 = \left\lfloor \frac{3}{2} \right\rfloor = \left\lfloor \frac{n-1}{2} \right\rfloor$

$W_5 = C_4 + 1$

$\beta_2 = \left\lfloor \frac{4}{2} \right\rfloor = \left\lfloor \frac{n-1}{2} \right\rfloor$

$W_6 = C_5 + 1$

$\beta_2 = \left\lfloor \frac{5}{2} \right\rfloor = \left\lfloor \frac{n-1}{2} \right\rfloor$

$W_7 = C_6 + 1$

$\beta_2 = \left\lfloor \frac{6}{2} \right\rfloor = \left\lfloor \frac{n-1}{2} \right\rfloor$

$\alpha_2 = \underbrace{1}_{\text{hub}} + \left\lceil \frac{n-1}{2} \right\rceil = 1 + \left\lceil \frac{4-1}{2} \right\rceil$

To, cover all the edges of the graph Wn, using minimum number of vertices we must select hub vertex to cover all the edges w.r.t spokes of wheel graph

Topic : Spanning Tree

Let G be a connected graph,

A sub-graph H of graph G is called a spanning tree if,

- H is a tree {Acyclic}
- H contains all vertices of graph G

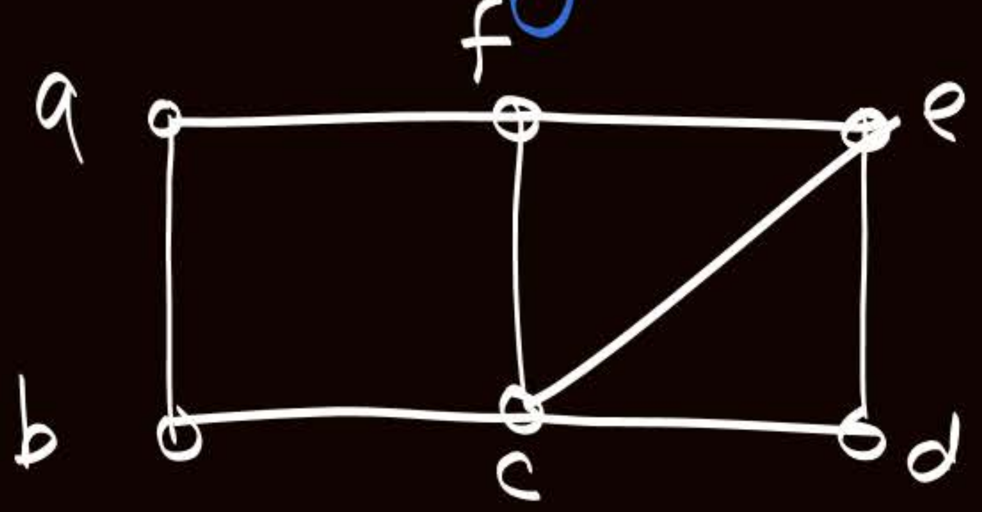
Acyclic + Connected

Graph G has a spanning tree if and only if graph G is a Connected graph

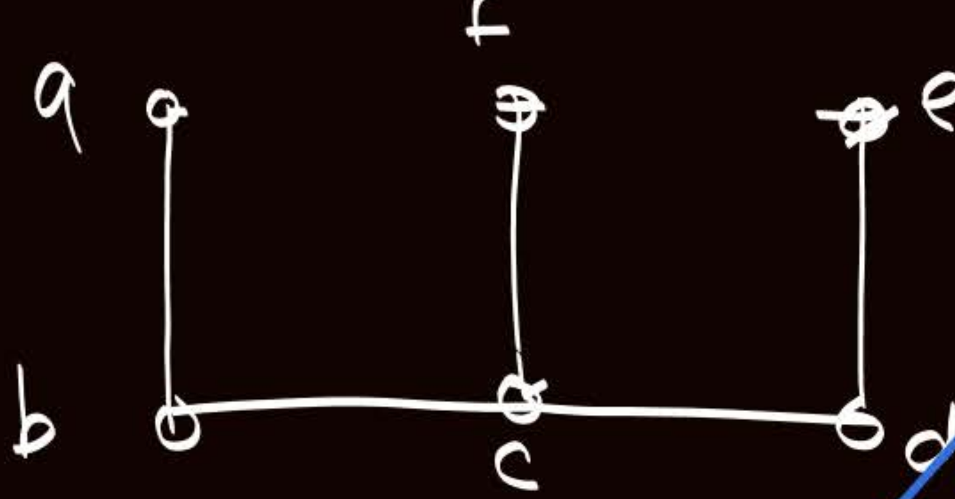
Let G be a Connected graph, A connected acyclic subgraph of graph G, that contain all vertices of graph G is called spanning tree of graph G

Note: There may be more than one spanning tree for a given connected graph

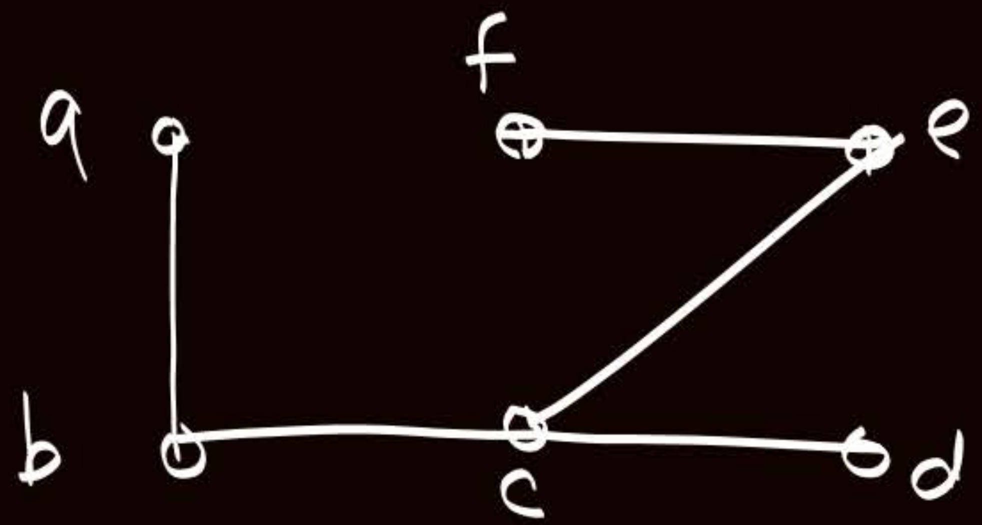
eg.



G



H_1



H_2

Both H_1 & H_2 are spanning tree of graph G

Note: Let G be a connected graph with n-vertices, then spanning tree of graph G will contain exactly 'n' vertices & exactly '$n-1$' edges.

Note: If the given graph is a connected acyclic graph (i.e., tree) then there will be only one spanning tree for that graph, and that spanning tree will be given graph itself.

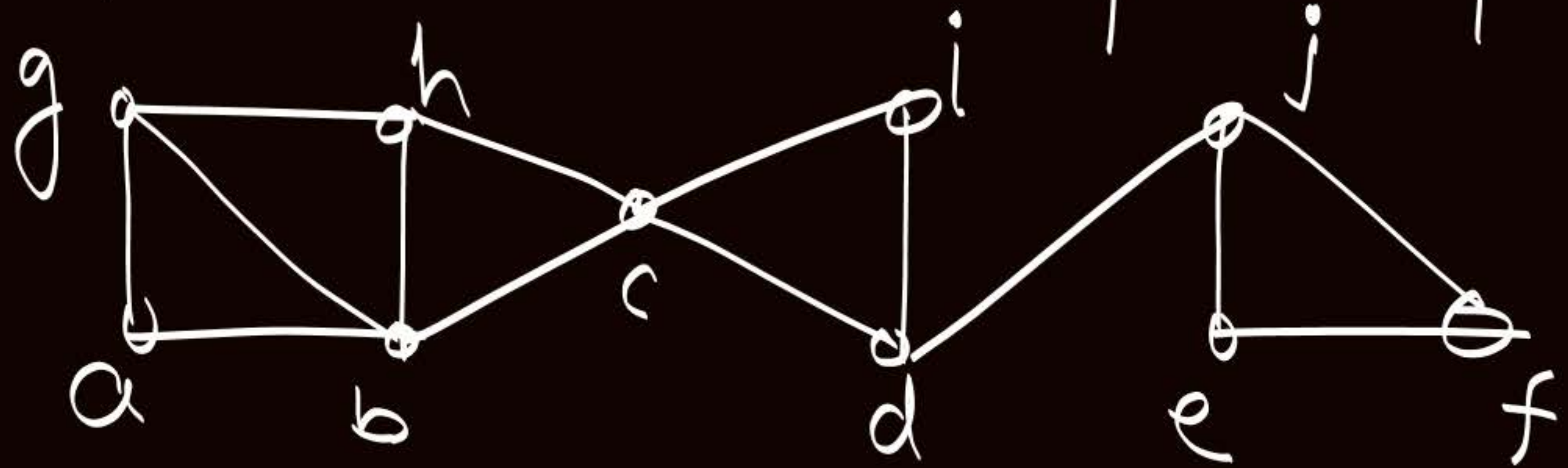
Topic : Circuit Rank

Number of edges that must be deleted from a connected graph G in order to obtain a spanning tree of G is called circuit rank of graph G.

→ Let G is a connected graph with n-vertices & 'm' edges then,

$$\text{Circuit rank of graph } G = \overbrace{m}^{\text{no. of edges in } G} - \underbrace{(n-1)}_{\text{No. of Edges required in spanning tree}} = m-n+1$$

Q: Find the Circuit rank of the following graph 'G'



$n = 10$

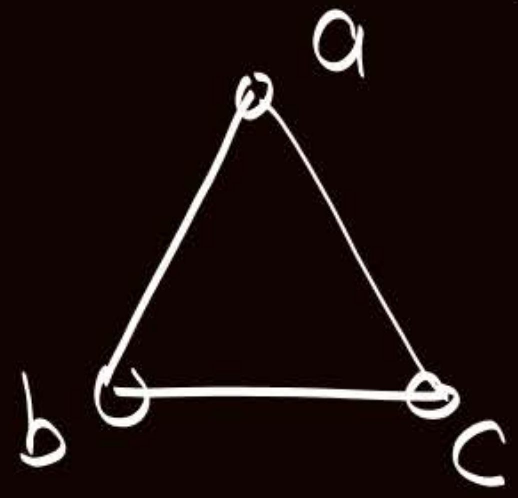
$|E|$ in $G = m = 14$

No. of Edges in spanning tree of G: $n-1 = 10-1 = 9$

$\therefore$ Circuit rank of $G = 14-(9) = 5$ } We can not delete this edges randomly to Obtain a Spanning tree.

Q: Find the number of spanning trees possible for the following graphs.

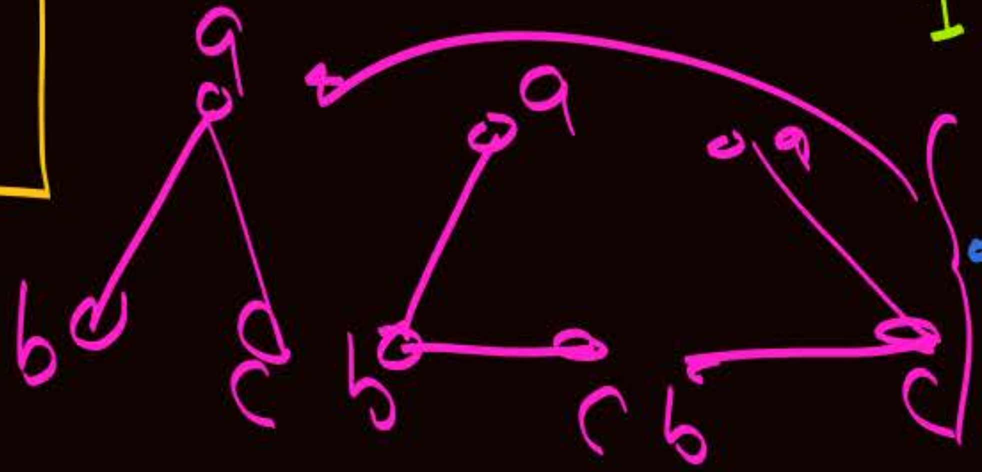
(1)



$$\boxed{\begin{aligned}\text{Circuit rank} &= m-(n-1)\\ &= 3-(3-1)\\ &= \underline{1}\end{aligned}}$$

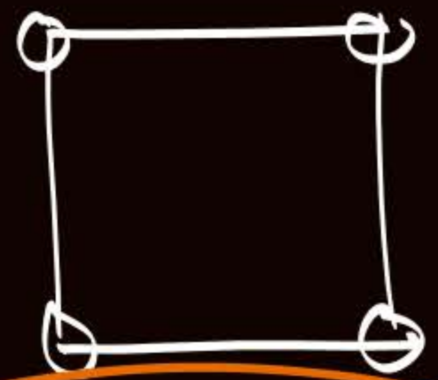
⇒ From a cycle we can delete any edge in order to obtain a spanning tree

→ from a cycle of length '3' one edge that needs to be deleted can be choosen in $^3C_1 = 3$ ways



∴ 3 different spanning trees are possible

Q: Find the number of spanning trees possible for the following graphs.

② 

No. of spanning trees possible $= {}^4C_1 = 4$

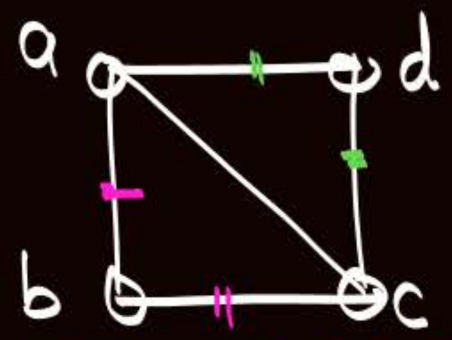
Circuit Rank $= m-(n-1)$

$= 4-(4-1)$

$= 1$

Q: Find the number of spanning trees possible for the following graphs.

③



Circuit Rank: $m-(n-1)$
$= 5-(4-1)$
$= 2$

* Out of 5 edges. two edges Can be choosen in ${}^5C_2 = 10$ ways.

Case ① If deleted edges are {a,b} & {b,c} then resulting graph is not a spanning tree

Case ② If deleted edges are {a,d} & {c,d} then resulting graph is not a spanning tree

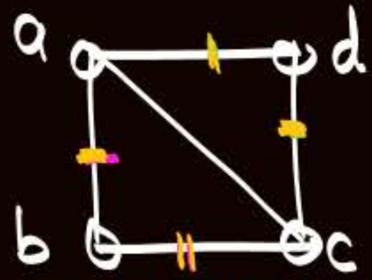
→ If we delete any other Combination of two edges then resulting graph will be a spanning tree

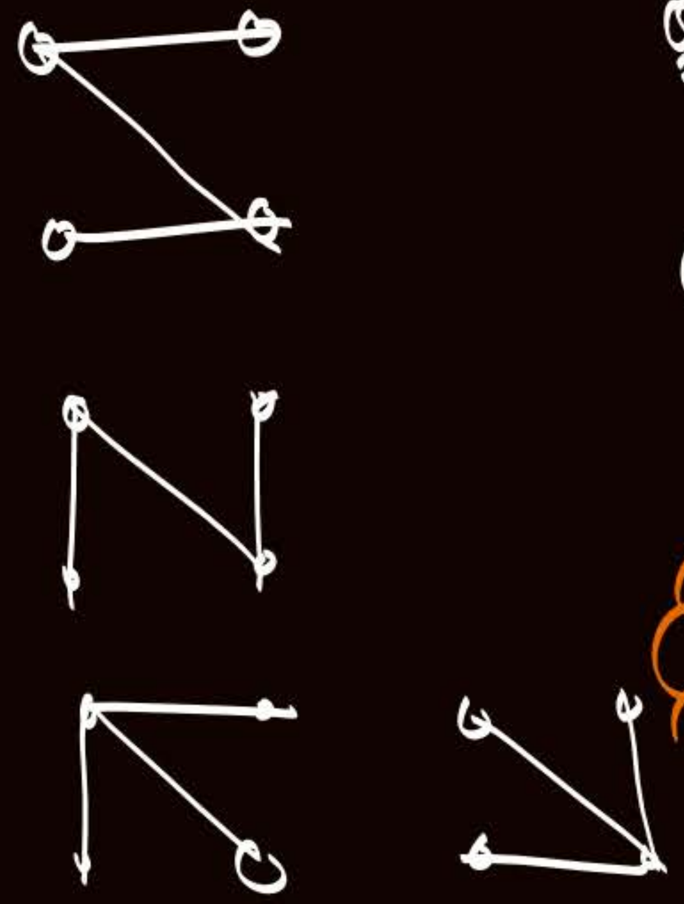
∴ No. of Spanning trees = Total no. of ways to delete 2-edges − No. of invalid ways to delete two edges

$= 10 - 2 = 8$

Q: Find the number of spanning trees possible for the following graphs.

③





Circuit Rank: $m-(n-1)$
$= 5-(4-1)$
$= 2$

∴ Case ① When edge {a,c} is deleted.

- We can choose & delete {a,c} in only one way. After deletion of {a,c} there is a cycle of length '4', to break cycle we can choose one edge in $^4C_1 = 4$ ways.

∴ No of spanning trees when edge {a,c} is not present $= 1 * {}^4C_1 = 4$

Case-② When edge {a,c} is not deleted.

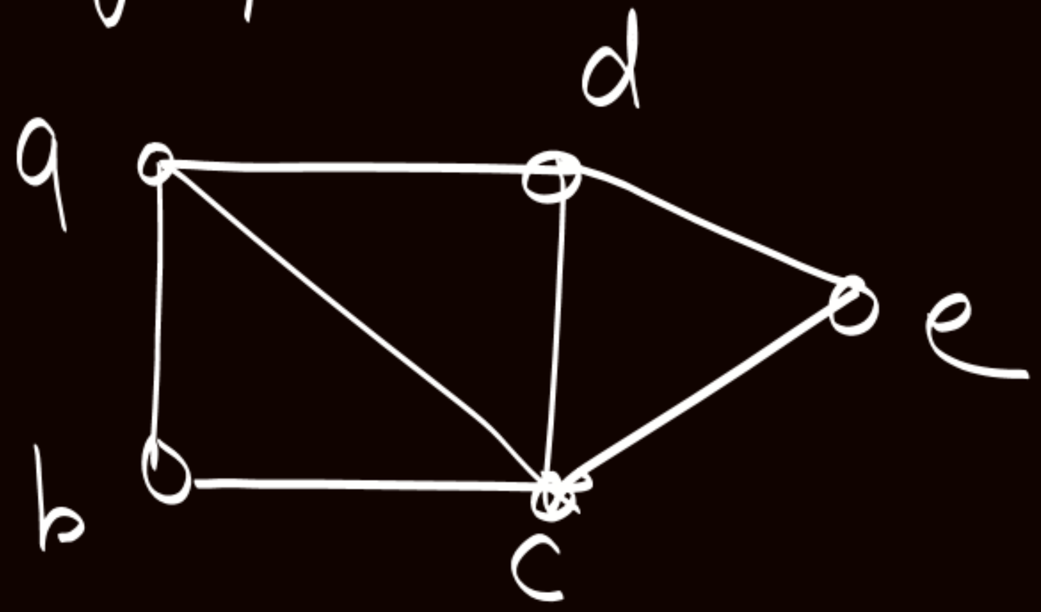
When edge {a,c} is present then in order to obtain a spanning tree one edge must be deleted out of {a,b} & {b,c} and edge must be deleted out of {a,d} & {c,d}

∴ # Spanning trees when edge {a,c} is present $= {}^2C_1 * {}^2C_1 = 2*2 = 4$

Case ① & Case ② are mutually exclusive & mutually exhaustive.

∴ Total no. of Spanning tree = Spanning tree using Case ① + Spanning tree using Case ②
$= 4+4 = 8$

H.W Find the No. of spanning trees for the following graph



2 mins Summary

- **Topic** Vertex Covering
- **Topic** Minimal & Minimum vertex covering
- **Topic** Vertex independent set
- **Topic** Maximal & Maximum vertex independent set
- **Topic** Spanning tree

THANK - YOU

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