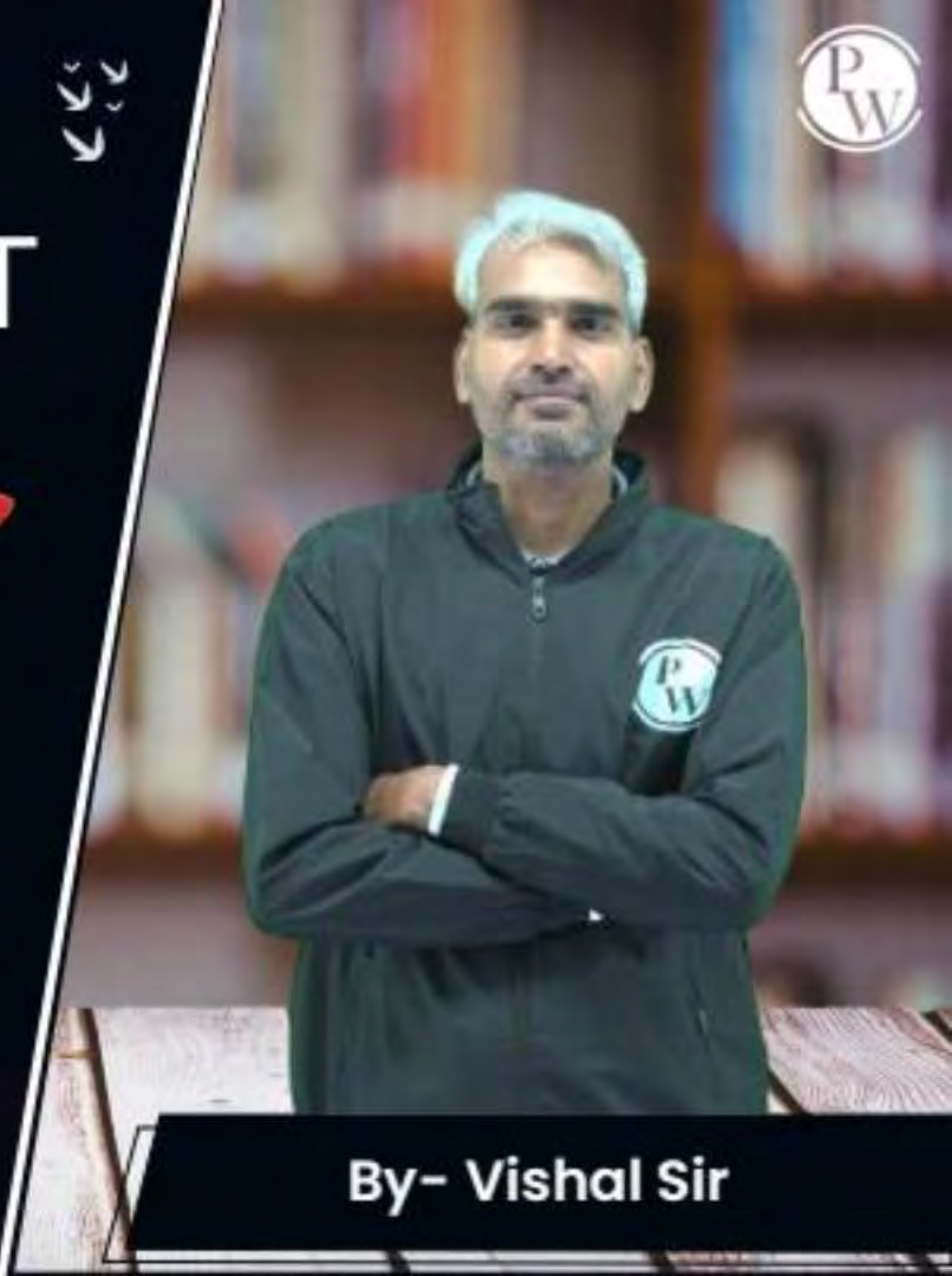
Computer Science & IT

Discrete Mathematics

Graph Theory

Lecture No. 05



By- Vishal Sir

Recap of Previous Lecture

Topic — Degree Sequence

Topic — Havel Hakimi's algorithm

Topic — Complement of a graph

$G \cup \bar{G} = K_n$

$|E(G)| + |E(\bar{G})| = |E(K_n)| = \frac{n(n-1)}{2}$

Topic — Graph isomorphism

Slide

Topics to be Covered

Topic Graph isomorphism

Topic Self-complementary graph

Topic Planar graph

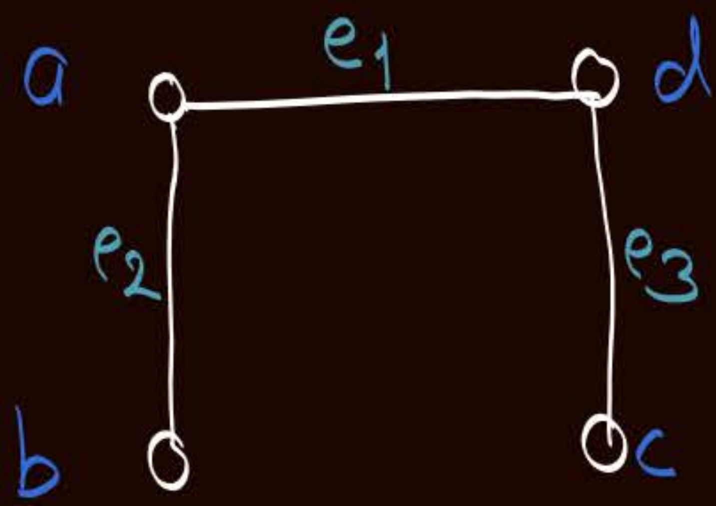
Slide

Topic : Complement of a graph

{ Complement is defined Only for simple graphs }

Let G be a simple graph with n-vertices, then complement of graph G is a simple graph with same n-vertices as of G, but an edge is present in complement of graph G if and only if that edge is not present in G.

Complement of graph G is denoted by $\overline{G}$.



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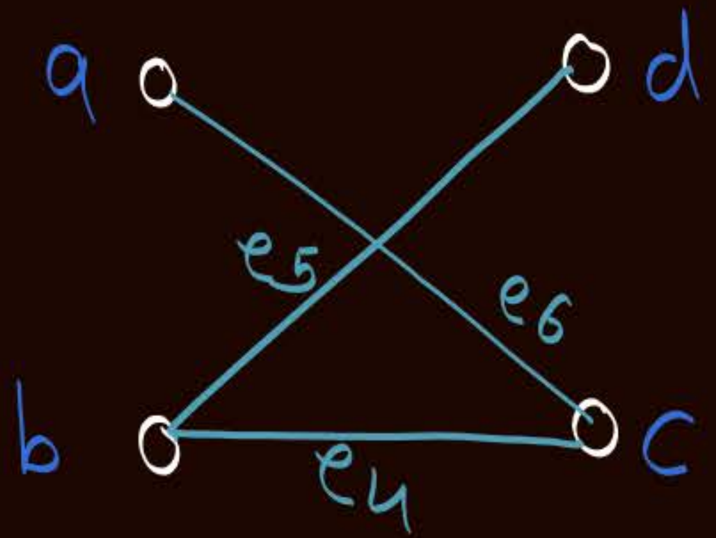

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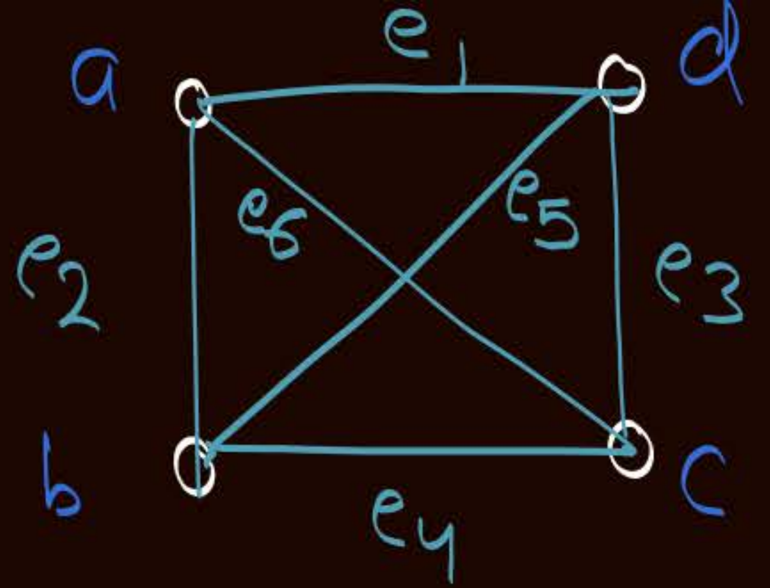


$G \cup \overline{G}$

$G \cup \overline{G} = \{$ it is a Complete graph $\}$

$G = (V, E_1)$

$V = \{a, b, c, d\}$

$E_1 = \{e_1, e_2, e_3\}$

$\overline{G} = (V, E_2)$

$V = \{a, b, c, d\}$

$E_2 = \{e_4, e_5, e_6\}$

$G \cup \overline{G} = (V \cup V, E_1 \cup E_2)$

$V \cup V = \{a, b, c, d\}$

$E_1 \cup E_2 = \{e_1, e_2, e_3, e_4, e_5, e_6\}$

Topic : Complement of a graph

- Let G be a simple graph with n- vertices, and $\overline{G}$ is Complement of graph G, then

$$\boxed{G \cup \overline{G} = K_n}$$

$$\boxed{E(G) \cap E(\overline{G}) = \phi}$$

$$\boxed{V(G) = V(\overline{G})}$$

$$|E(G)| + |E(\overline{G})| = |E(K_n)|$$

$$|E(G)| + |E(\overline{G})| = {}^{n}C_2 = \frac{n(n-1)}{2}$$

Q1: Let G be a simple graph with n-vertices & 21 edges, if there are 24 edges in the Complement of graph G. Find the number of vertices in graph G. ?

$$|E(G)| + |E(\bar{G})| = {}^nC_2 = \frac{n(n-1)}{2}$$

$$21 + 24 = \frac{n(n-1)}{2}$$

$$90 = n(n-1)$$

$$\boxed{n = 10}$$

Topic : Graph Isomorphism

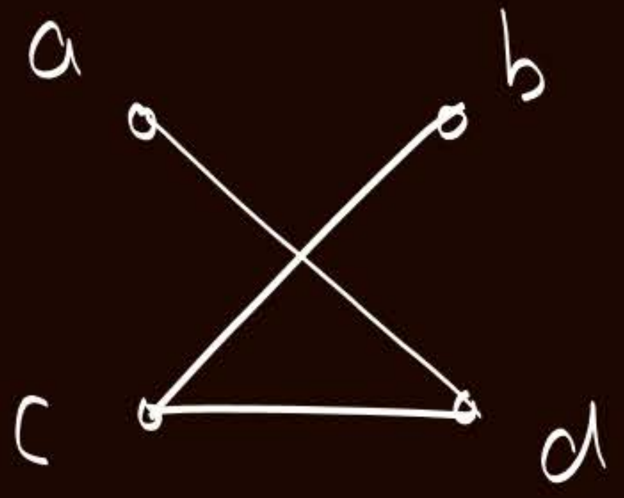
{ Two graphs are said to be isomorphic if they have exactly same properties }
{ Name of vertices & edges may be different }

Two graphs G and G' are said to be isomorphic iff there exists a function f: V(G)→V(G') such that

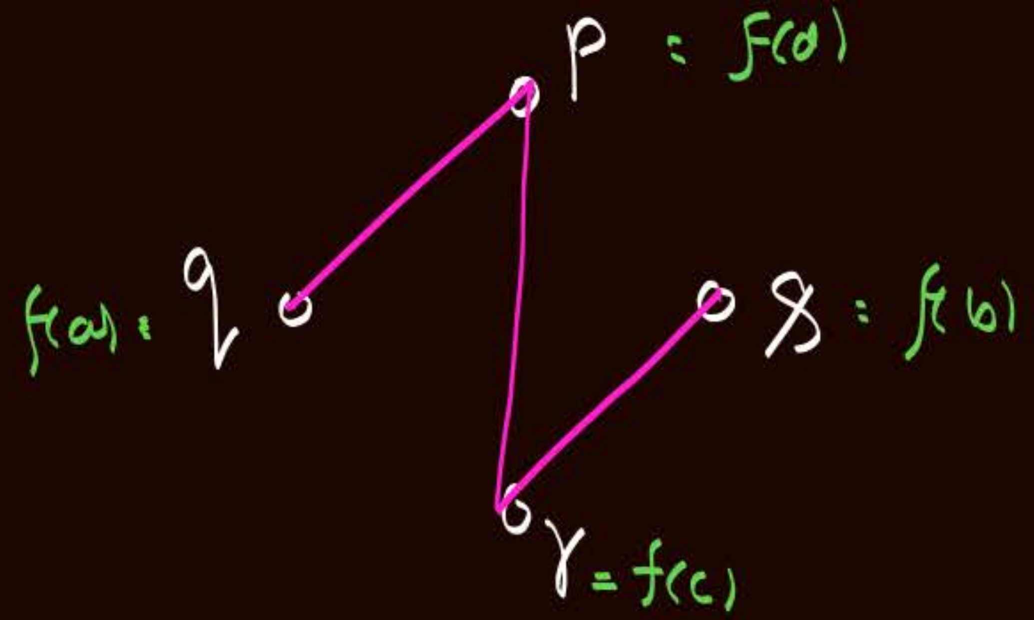
I. f is bijective { one-one + onto } $\Rightarrow |V(G)| = |V(G')|$

II. f preserves adjacency of vertices

If two vertices $a, b \in V(G)$ are adjacent to each other in graph G then their images in G', { i.e. $f(a), f(b) \in V(G')$ } should also be adjacent to each other in G'. $\Rightarrow |E(G)| = |E(G')|$



G



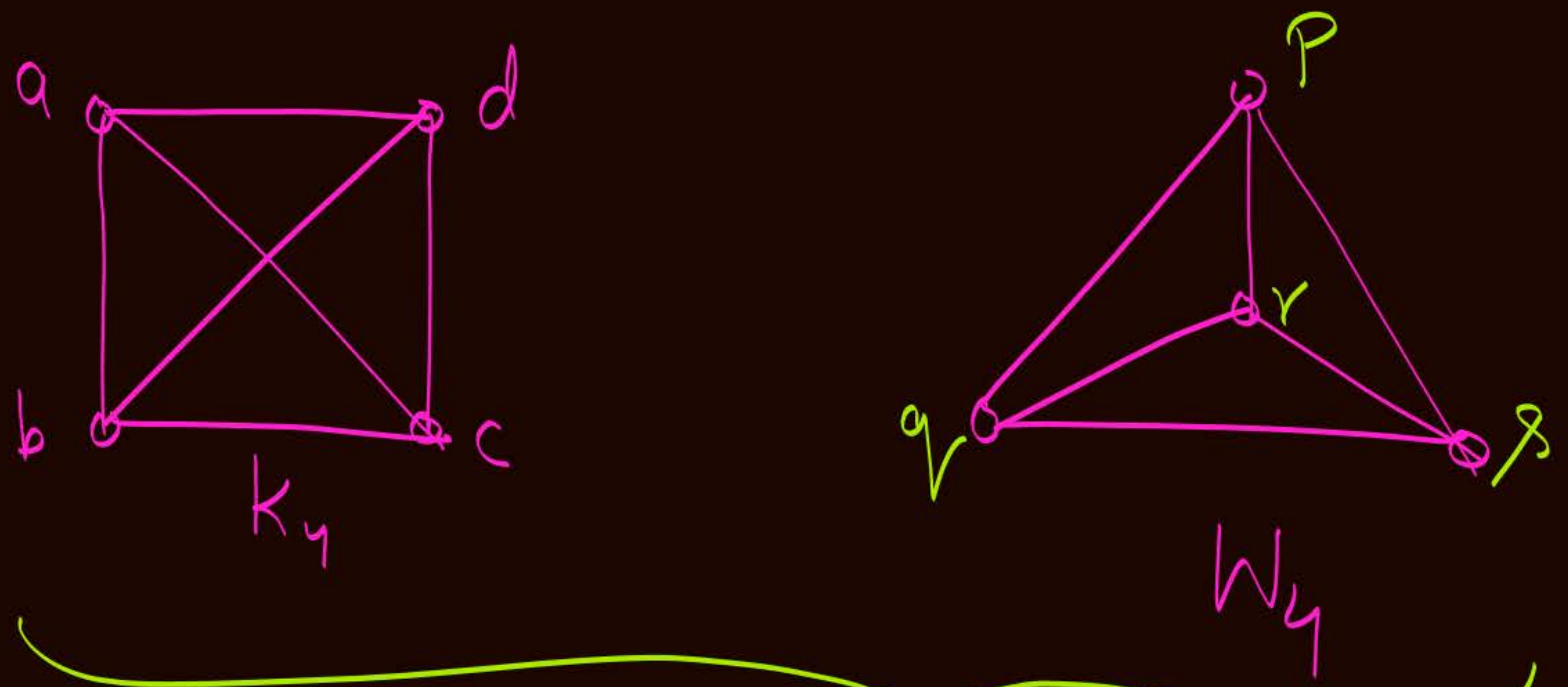
G'

let G & G'
are isomorphic

$\therefore |V(G)| = |V(G')|$

$\exists$ $f: V(G) \to V(G')$
s.t. f is bijective.

Let $f(a) = q$
$f(b) = s$
$f(c) = r$
$f(d) = p$



K_4 & W_4 are isomorphic
to each other

Topic : Graph Isomorphism

* If Graphs G & G' are isomorphic to each other then it is denoted by,

$$\boxed{G \cong G'}$$

Topic : Graph Isomorphism

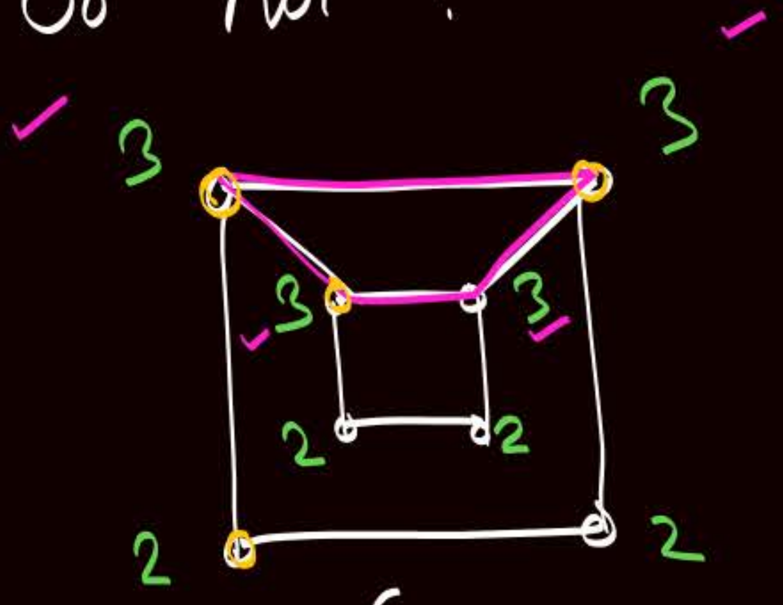
If $G \cong G'$ then following conditions must hold true :-

This 4 conditions are necessary conditions for two graphs to be isomorphic

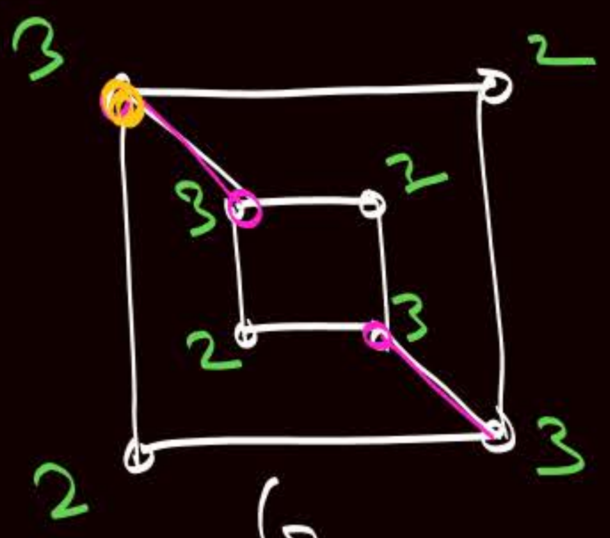
1. $|V(G)| = |V(G')|$
2. $|E(G)| = |E(G')|$
3. Degree sequence of G & G' must be same
4. If vertices $v_1, v_2, \ldots v_K$ of graph form a cycle of length 'K' in graph G, their corresponding images in G' i.e., $f(v_1), f(v_2), \ldots f(v_K)$ must form a cycle of length 'K' in graph G'.

i.e, If any of the above cond^n is dis-satisfied then graphs can not be isomorphic.

Q. Check whether the following graphs are isomorphic or not?



G_1



G_2

$\left\{ \begin{array}{l} G_1 \text{ \& } G_2 \\ \text{are not} \\ \text{isomorphic} \end{array} \right\}$

① $|V(G_1)| = 8$

② $|E(G_1)| = 10$

③ $\{3,3,3,3,2,2,2,2\}$ is the deg. seq

① $|V(G_2)| = 8$

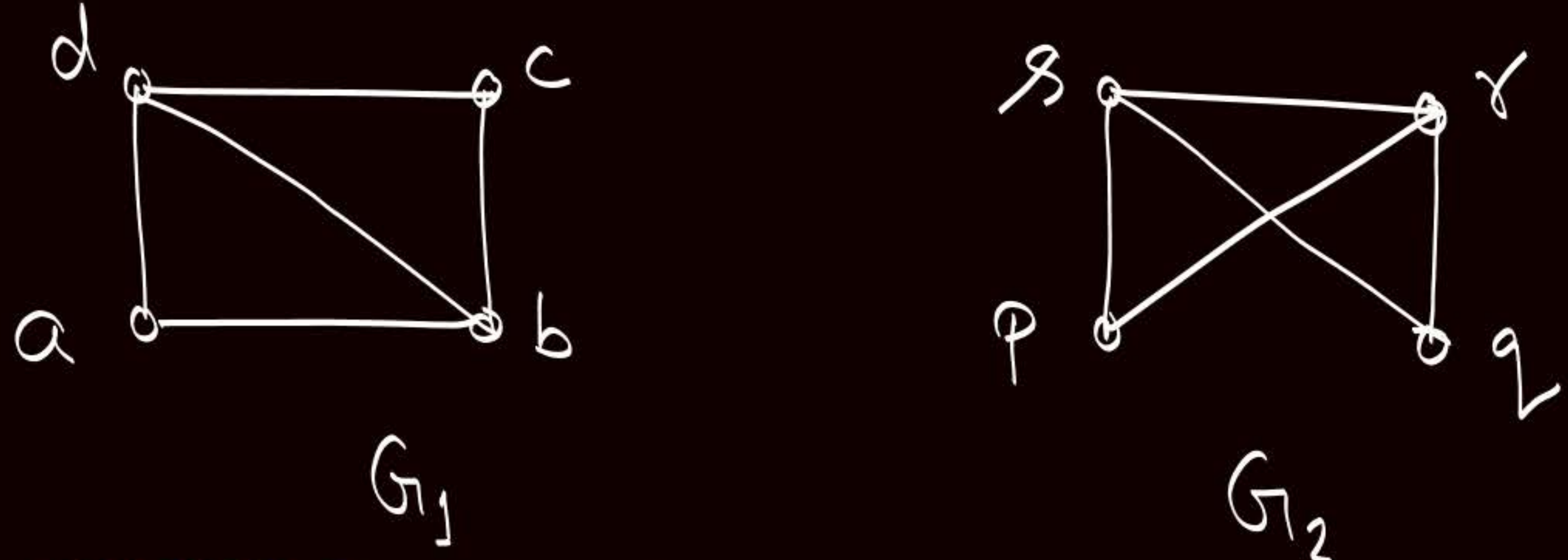
② $|E(G_2)| = 10$

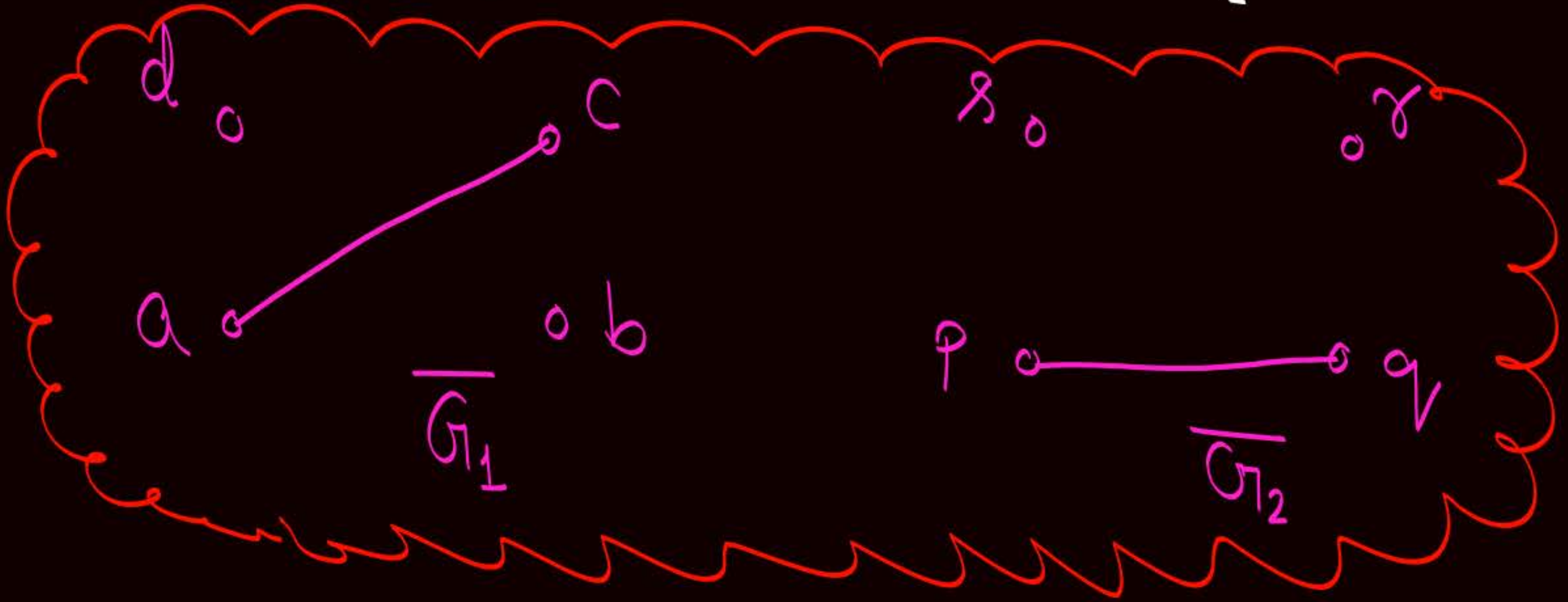
③ $\{3,3,3,3,2,2,2,2\}$ is the deg. sequence

Note:- Two simple graphs G_1 & G_2 are isomorphic to each other if and only if their Complements are isomorphic to each other

i.e. $G_1 \cong G_2$ iff $\overline{G_1} \cong \overline{G_2}$

Q. Check whether the graphs are isomorphic or not?

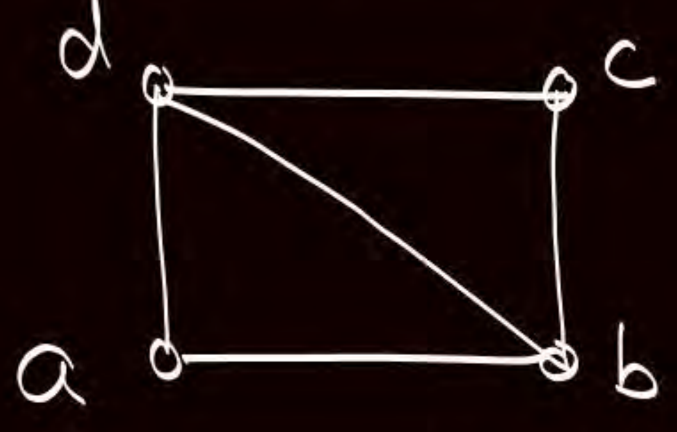




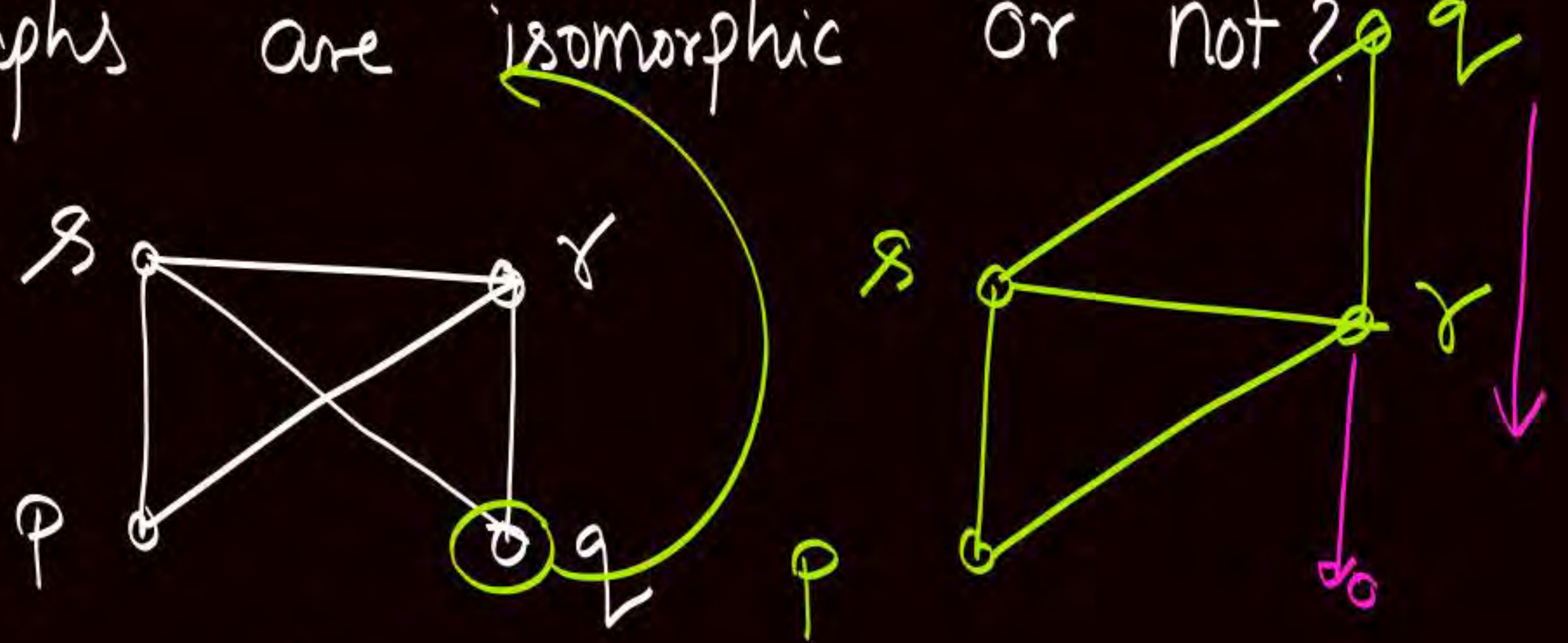
$\overline{G_1} \cong \overline{G_2}$

$\therefore G_1 \cong G_2$

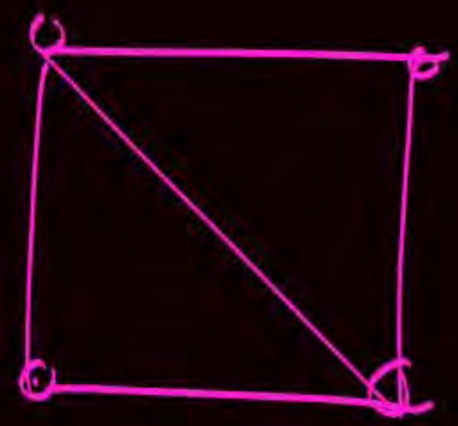
Q. Check whether the graphs are isomorphic or not?



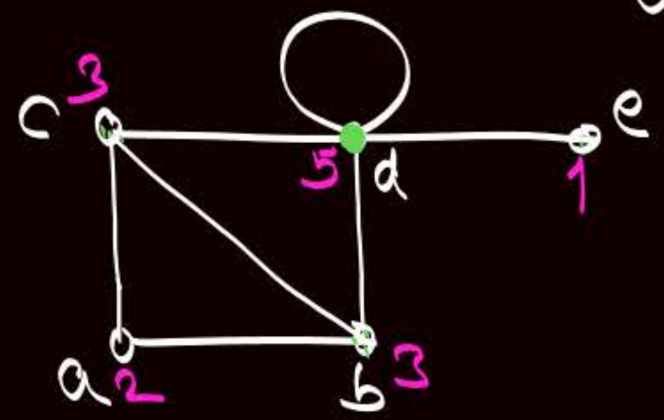
G_1



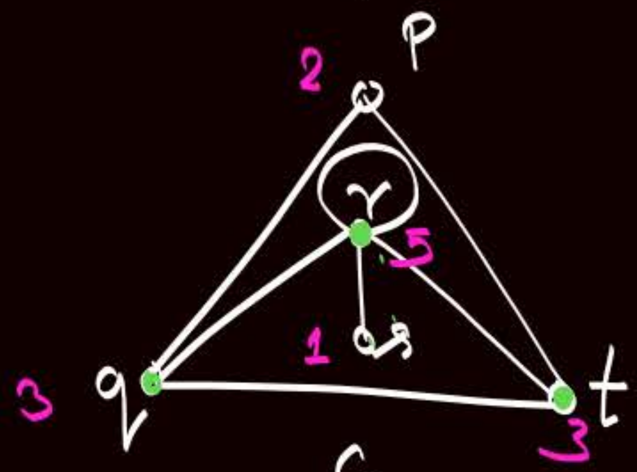
G_2



Note :- Two graphs G_1 & G_2 are isomorphic to each other if and only if their corresponding sub-graphs obtained on deleting a vertex 'v' from graph G_1 and on deleting the image of vertex 'v' {i.e. f(v)} from graph G_2 are isomorphic to each other eg: Check whether the graphs are isomorphic or not?



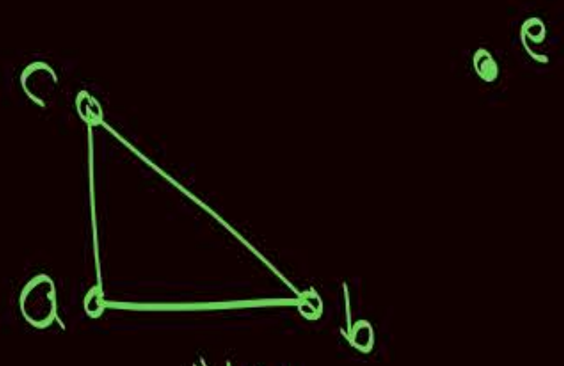
G_1



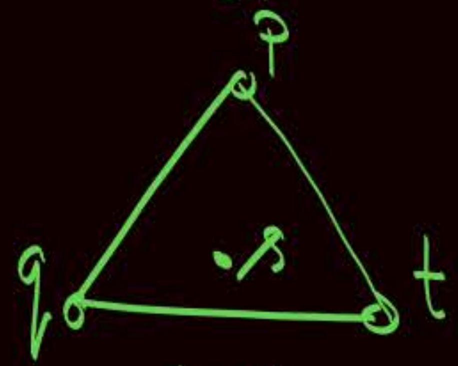
G_2

Image of vertex 'd' of graph G_1 in graph G_2 is vertex 'r' of graph G_2

delete vertex 'd' from G_1 delete vertex 'r' from G_2



"H_1"



"H_2"

We can observe $H_1 \cong H_2$

$\therefore G_1 \cong G_2$

Q. How many simple graphs are possible with 3- labeled vertices

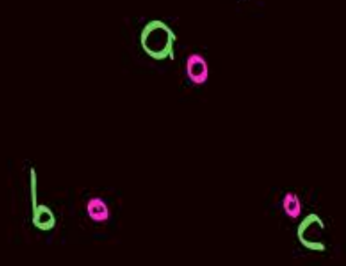
$\Rightarrow$ Total no. of simple graphs possible $= 2^{{}^3C_2} = 2^3 = 8$

Q. How many non-isomorphic simple graphs are possible with 3- labeled vertices

Max. No. of edges possible in a simple graph with 3 vertices are $= {}^3C_2 = 3$

Non-isomorphic simple graphs possible with '3' vertices

0-edges



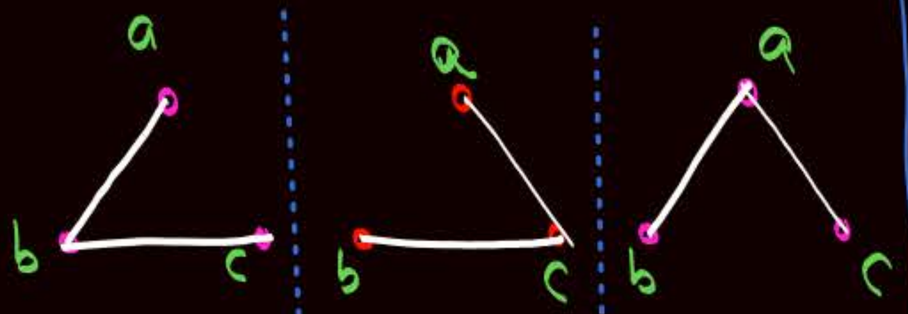
Only one simple graph is possible with 3-vertices & 0-edges

1-edge

'3' simple graphs are possible with '3' vertices & 1-edge, but all are isomorphic to each other

∴ Counted as only one non-isomorphic graph

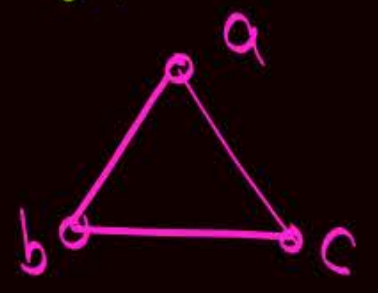
2-edges



'3' simple graphs are possible with '3' vertices & 2-edges but all are isomorphic to each other

∴ Counted as only one non-isomorphic graph

3-edges



Only one simple graph is possible with 3-vertices & 3-edges

* Total no. of non-isomorphic simple graphs possible with 3-vertices $= \underset{\text{3-vertices, 0-edge}}{1} + \underset{\text{3-vertices, 1-edge}}{1} + \underset{\text{3-vertices, 2-edges}}{1} + \underset{\text{3-vertices, 3-edges}}{1} = 4$

Q: How many simple graphs are possible with 4-labeled vertices

$\Rightarrow$ Total no. of simple graphs possible with 4-vertices $= 2^{^4C_2} : 2^6 = 64$

Q. How many non-isomorphic simple graphs are possible with 4-vertices

Maximum No. of edges possible in a simple graph with 4-vertices = ${}^4C_2 = 6$.

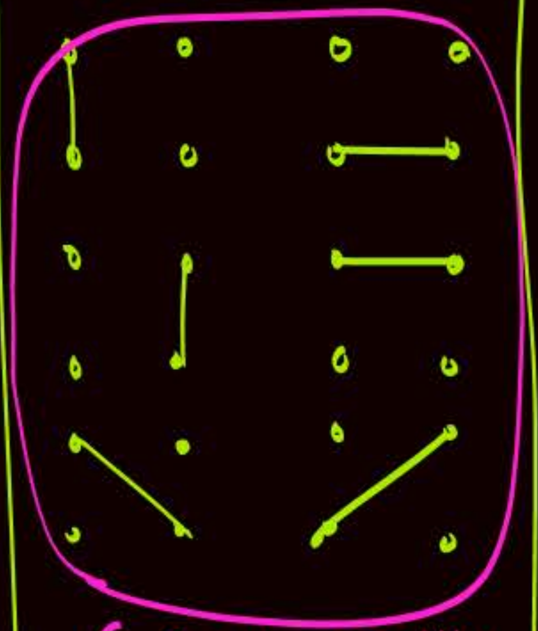
Total no. of non-isomorphic simple graphs possible with 4-vertices = Non-isomorphic graphs with 4 vertices 0-edges + Non-isomorphic graphs with 4 vertices 1-edges + Non-isomorphic graphs with 4 vertices 2-edges + Non-isomorphic graphs with 4 vertices 3-edges + Non-isomorphic graphs with 4 vertices 4-edges + Non-isomorphic graphs with 4 vertices 5-edges + Non-isomorphic graphs with 4 vertices 6-edges

$= 1 + 1 + 2 + 3 + 2 + 1 + 1$

$= 11$ Ans

4-V
0-E

4-V
1-E

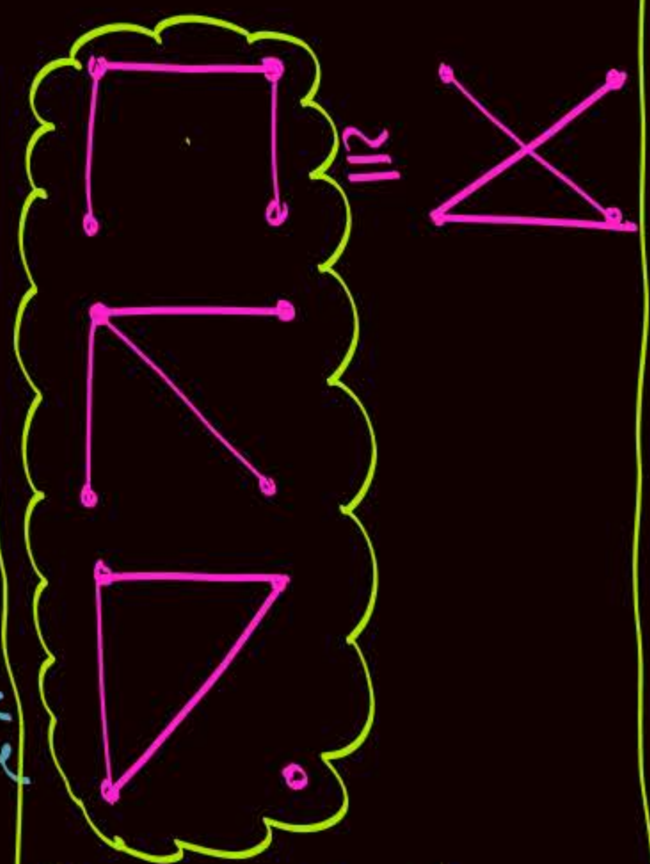


6-graphs possible
all are isomorphic.

4-V
2-E

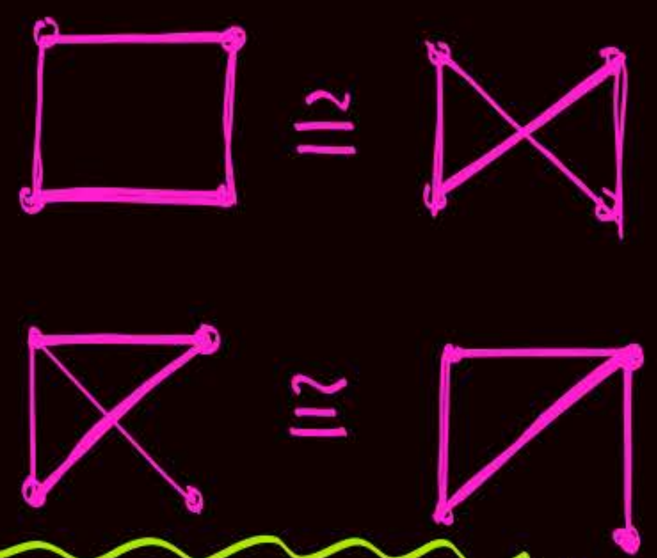
Two non-isomorphic
graphs are possible
with 4-vertices
& 2-edges

4-V
3-E



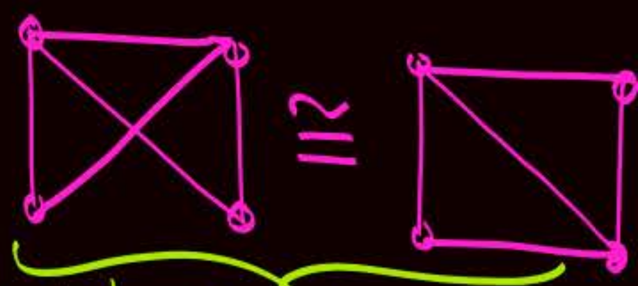
Three non-isomorphic
graphs possible
with 4-vertices
& 3-edges

4-V.
4-E



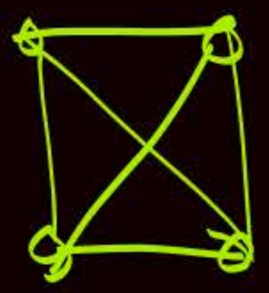
Two non-isomorphic
graphs are possible
with 4-vertices
& 4-edges

4-V
5-E



only one non-isomorphic
graph possible
with 4-vertices
& 5-edges

4-V
6-E



only one with
4-vertices
& All 6-edges

Q. How many non-isomorphic simple graphs are possible with 5-vertices & 2-edges.

Ans = 2

No. of non-isomorphic simple graphs possible with 3-vertices & 2-edges = 1

No. of non-isomorphic simple graphs possible with n-vertices ($n \geq 4$) & 2-edges = 2

'1' when edges are not adjacent

'1' when edges are adjacent.

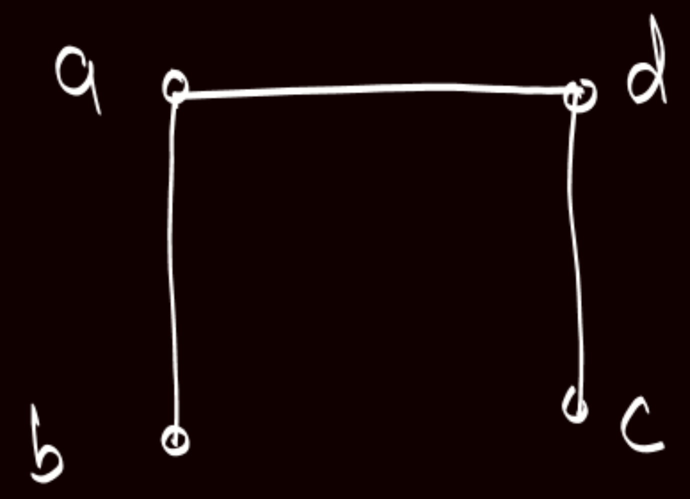
Topic : Self-Complementary Graph

A simple graph G is said to be self-complementary if complement of G is isomorphic to G itself.

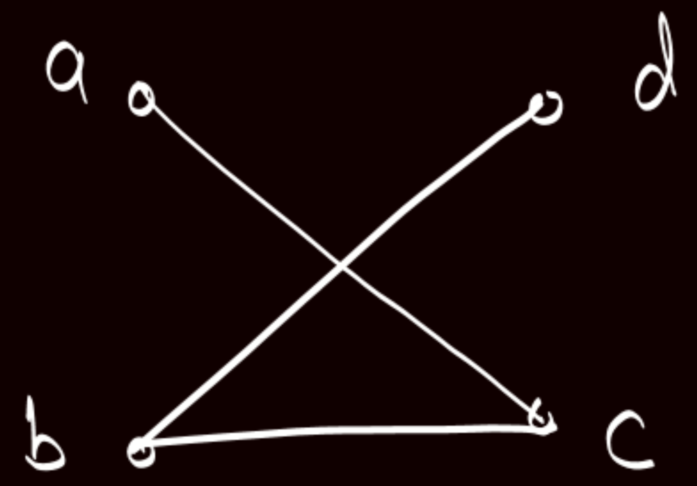
i.e. In a simple graph G,

if $G \cong \overline{G}$

then G is called a self-complementary graph eg



G



$\overline{G}$

We can observe $G \cong \overline{G}$

$\therefore$ Graphs are self-complementary.

Topic : Self-Complementary Graph

In a self complementary graph G, {i.e. if $G \cong \overline{G}$}

① Because $G \cong \overline{G}$

$\therefore \; |E(G)| = |E(\overline{G})|$ —— ①

{By the property of isomorphism}

② By the property w.r.t. Complement of a graph

$$|E(G)| + |E(\overline{G})| = \frac{n(n-1)}{2} \quad \text{—— ②}$$

By eqn ① & Eqn ②

$$|E(G)| + \underbrace{|E(\overline{G})|}_{\equiv |E(G)|} = \frac{n(n-1)}{2}$$

$$2\,|E(G)| = \frac{n(n-1)}{2}$$

$$\boxed{|E(G)| = \frac{n(n-1)}{4}}$$

Topic : Self-Complementary Graph

* If graph G is a self complementary graph, then $\boxed{|E(G)| = \frac{n(n-1)}{4}}$, but converse of the statement need not be true

It is necessary cond^n for graph to be self-complementary, but not sufficient

* In a self complementary graph G, $|E(G)| = \frac{n(n-1)}{4}$.

No. of edges in graph G can not be in fraction $\rightarrow$ $\therefore$ $\frac{n(n-1)}{4}$ must be an integer

i.e. n = (multiple of 4)
(or)
((multiple of 4)+1)

<u>i.e.</u>, In a self complementary graph,

No. of vertices = $4K$ <u>or</u> $4K+1$

where 'K' is any +ve integer

Q:- Let C_n is cycle graph with n-vertices, such that Complement of C_n is isomorphic to C_n {i.e. $\overline{C_n} \cong C_n$}

Find the value of n?

Solu^n C_n is a cycle graph

$\therefore \; |E(C_n)| = n \quad \text{--- (1)}$

Given C_n is a self-complementary graph.

$\therefore \; |E(C_n)| = \frac{n(n-1)}{4} \quad \text{--- (2)}$

By eq^n (1) & eq^n (2)

$$n = \frac{n(n-1)}{4}$$

$$n = 0 \text{ or } 5$$

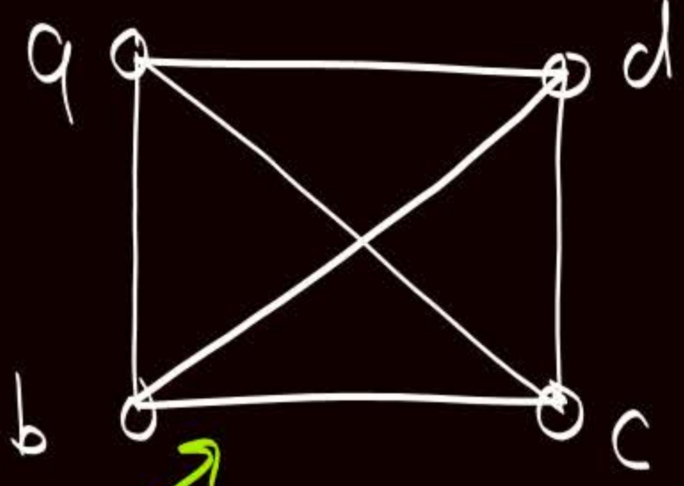
0: Empty graph

5: valid cycle graph $\Rightarrow \; \therefore \; C_5$ $\quad n = 5$

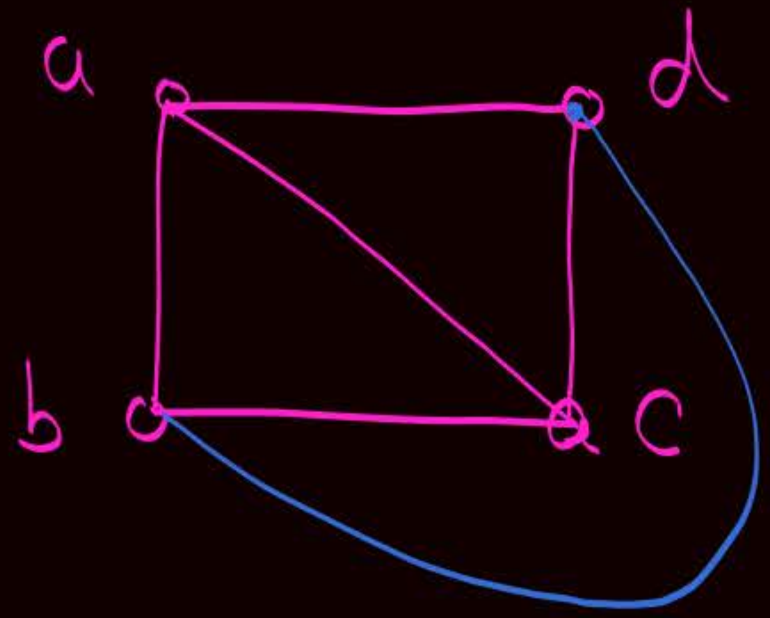
Topic : Planar graph

- A graph G= (V, E) is said to be planar if it can be drawn in the plane such that no two edges of G intersect each other at a non-vertex point.
  Such a drawing of a planar graph is called a planar embedding of the graph.

* Check whether the graph is planar or not?



$\Longrightarrow$



Graph can be drawn in a plane s.t. no two edges cross each other at a non-vertex point

∴ Given graph is a planar graph

2 mins Summary

Topic Graph isomorphism

Topic Self-complementary graph

Topic Planar graph

THANK - YOU

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