1 Linear Algebra

Objective

Upon completion of this chapter, you will be able to determine:

- Determinant of a matrix
- Adjoint and inverse of a matrix
- Rank of a matrix
- Solution of homogeneous and non-homogeneous linear equations.
- Eigen values and eigen vectors of a matrix
- Higher powers of matrix using Cayley Hamilton theorem

Introduction

Linear algebra is a branch of mathematics concerned with the study of vectors, with families of vectors called vector spaces or linear spaces and with functions that input one vector and output another, according to certain rules. These functions are called linear maps or linear transformations and are often represented by matrices.

Matrices are rectangular arrays of numbers or symbols, and matrix algebra or linear algebra provides the rules defining the operations that can be formed on such an object. An elementary application of linear algebra is to the solution of a system of linear equations in several unknowns, which often result when linear mathematical models are constructed to represent physical problems.

Matrix

A system of mn numbers arranged in the form of a rectangular array having m rows and n columns is called a matrix of order m×n

If A = $[aij]_{m\times n}$ be any matrix of order m×n then it is written in the form:

$$A = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \text{\_\_\_\_\_\_\_\_\_\_} & a_{1n} \\ a_{21} & a_{22} & \text{\_\_\_\_\_\_\_\_\_\_} & a_{2n} \\ & & \text{\_\_\_\_\_\_\_\_\_\_} & \\ a_{m1} & a_{m2} & \text{\_\_\_\_\_\_\_\_\_\_} & a_{mn} \end{bmatrix}$$

Horizontal lines are called rows and vertical lines are called columns.

Types of matrices:

The square matrix: The matrix in which the number of rows is same as the number of columns is called as square matrix. If it has n rows, then it is called as n-rowed square matrix. The elements $a_{ij/i=j}$ i.e., a_{11}, a_{22} ... are called diagonal elements and the line along which the elements lie is called principal diagonal of matrix. Elements other than a_{11}, a_{22}, etc. are called off – diagonal elements i.e. $a_{ij/i\neq j}$.

Example: $A = \begin{bmatrix} 1 & 9 & 4 \\ 7 & 2 & 8 \\ 10 & 78 & 45 \end{bmatrix}_{3\times 3}$ is a square matrix

The diagonal elements of this matrix are {1, 2, 45}

Note:

A square sub-matrix of a square matrix A is called a "principle sub-matrix" if its diagonal elements are also the diagonal elements the matrix A. So $\begin{bmatrix} 1 & 9 \\ 7 & 2 \end{bmatrix}$ is a principle sub matrix of the matrix A given above, but $\begin{bmatrix} 9 & 4 \\ 2 & 8 \end{bmatrix}$ is not.

Diagonal matrix: A square matrix in which all off-diagonal elements are zero is called a diagonal matrix. The diagonal elements may or may not be zero.

Example: $A = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 89 \end{bmatrix}$ is a diagonal matrix

The above matrix can also be written as A = diag [11 , 15 ,89]

Properties of diagonal matrix:

diag [x, y, z] + diag [p, q, r] = diag [x + p, y + q, z + r]

diag [x, y, z] × diag [p, q, r] = diag [xp, yq, zr]

$$\left(\text{diag}\left[x, y, z\right]\right)^{-1} = \text{diag}\left[\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right]$$

$$\left(\text{diag}\left[x, y, z\right]\right)^{t} = \text{diag}\left[x, y, z\right]$$

$$\left(\text{diag}\left[x, y, z\right]\right)^{n} = \text{diag}\left[x^n, y^n, z^n\right]$$

Eigen values of $\text{diag}\left[x, y, z\right] = x, y$ and z

determinant of $\text{diag}\left[x, y, z\right] = \left[x, y, z\right] = xyz$

Scalar matrix: A scalar matrix is a diagonal matrix with all diagonal elements being equal.

Example : $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a scalar matrix

Unit matrix or identity matrix: A diagonal matrix whose each diagonal element is 1 is called as identity matrix.

Thus a square matrix $A = [a_{ij}]$ is a unit matrix if $a_{ij} = 1$ when $i = j$ and $a_{ij} = 0$ when $i \neq j$

Example : $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Properties of identity matrix:

1. AI = IA = A
2. $I^n = I$
3. $I^{-1} = I$
4. |I| = 1

Null matrix: The m×n matrix whose elements are all zero is called null matrix. Null matrix is Denoted by null matrix need not be square.

Example : $O_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, O_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, O_{2\times1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Properties of null matrix:

1. A + O = O + A = A

   So, O is additive identity.

2. A + (–A) = O if

Upper triangular matrix: An upper triangular matrix is a square matrix whose lower off-diagonal elements are zero, i.e. $a_{ij} = 0$ whenever $i > j$

It is denoted by U. The diagonal and upper off-diagonal elements may or may not be zero.

Example : $U = \begin{bmatrix} 79 & 45 & -76 \\ 0 & 54 & 87 \\ 0 & 0 & 0 \end{bmatrix}$

Lower triangular matrix: A lower triangular matrix is a square matrix whose upper off-diagonal triangular elements are zero. i.e. $a_{ij} = 0$ whenever $i < j$. The diagonal and lower off-diagonal elements may or may not be zero. It is denoted by L.

Example : $L \begin{bmatrix} 17 & 0 & 0 \\ 85 & 0 & 0 \\ 42 & 34 & 62 \end{bmatrix}$

Idempotent matrix: A matrix A is called Idempotent if and only if $A^2 = A$.

Example : $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

are examples of idempotent matrices.

Involutory matrix: A matrix A is called involutory if and only if $A^2 = I$.

Example : $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is Involutory

Also $\begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$ is Involutory since $A^2 = I$.

Nilpotent matrix: A matrix A is said to be nilpotent of class x or index x if $A^x = O$ and $A^{x-1} \neq O$ i.e. x is the smallest index which makes $A^x = O$

Matrix Algebra

Equality of matrices:

Two matrices $A = [a_{ij}]$ and $B = [b^{ij}]$ are said to be equal if.

1. They are of same size.

2. The elements in the two corresponding places of two matrices are the same i.e., $a_{ij} = b_{ij}$ for each pair of subscripts i and j.

Example : Let $\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 42 & 55 \\ & \end{bmatrix}$

$\Rightarrow$ x = 42 , y = 55 , p = 31 and q = 19

Addition of matrices:

Two matrices A and B are compatible for addition only if they both have the same size say m×n . Then their sum is defined to be the matrix of the type m×n obtained by adding corresponding elements of A and B

i.e. adding the elements that lie in the same position.

Thus if, $A = [a_{ij}]_{m\times n}$ & $B = [b_{ij}]_{m\times n}$ then $A + B = [a_{ij} + b_{ij}]_{m\times n}$

Example : $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 6 \\ 7 & 8 \end{bmatrix}$

$$A + B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ 7 & -8 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 10 & 13 \end{bmatrix}$$

Properties of matrix addition:

1. Matrix addition is commutative A+B = B+A
2. Matrix addition is associative (A+B) + C = A + (B+C)
3. Cancellation laws holds good in case of addition of matrices, which is X = –A

   $A + X = B + X \Rightarrow A = B$

   $X + A = X + B \Rightarrow A = B$
4. The equation A + X = 0 has a unique solution in the set of all m×n matrices.

Subtraction of two matrices:

If A and B are two m×n matrices, then we define, A – B = A + (B)

Thus, the difference A – B is obtained by subtracting from each element of A corresponding elements of B.

Note:

Subtraction of matrices is neither commutative nor associative

Multiplication of a matrix by a scalar:

Let A be any m×n matrix and k be any real number called scalar. The m×n matrix obtained by multiplying every element of the matrix A by k is called scalar multiple and is denoted by kA.

If $A = [a_{ij}]_{m\times n}$ then $Ak = kA = [kA]_{m\times n}$

If $A = \begin{bmatrix} 5 & 6 & 1 \\ 2 & -5 & 3 \\ 1 & 1 & 6 \end{bmatrix}$ then, $4A = \begin{bmatrix} 20 & 24 & 4 \\ 8 & -24 & 12 \\ 4 & 4 & 24 \end{bmatrix}$

Properties of multiplication of a matrix by a scalar:

1. Scalar multiplication of matrices distributes over the addition of matrices i.e., k (A + B) = kA + kB
2. If p and q are two scalars and A is any m×n matrix then, (p + q) A = pA + qA
3. If p and q are two matrices and $A = [a_{ij}]_{-m\times n}$ then, p (qA) = (pq) A
4. If $A = [a_{ij}]_{-m\times n}$ be a matrix and k be any scalar then, (–k) A = –(kA) = k (–A)

Multiplication of two matrices:

Let $A = (a_{ij})_{-m\times n}$; $B = [b_{jk}]_{-n\times p}$ be two matrices such that the number of columns in A is equal to the number of rows in B.

Then the matrix $C = [C_{ik}]_{-m\times p}$ such that $Cik = \sum_{j=1}^{n} a_{ij}b_{jk}$ is called the product of matrices A and B.

Properties of matrix multiplication:

1. Multiplication of matrices is not commutative. If AB exits, then it is not guaranteed that BA also exists.

   For example, $A_{2\times 2} \times B_{2\times 7} = C_{2\times 7}$ but $B_{2\times 7} \times A_{2\times 2}$ does not exist since these are not compatible for multiplication.
2. Matrix multiplication is associative. i.e., A (BC) = (AB)C where A, B , C are m×n, n×p, p×q matrices respectively.
3. Multiplications of matrices are distributive i.e., A (B + C) = AB + AC
4. The equation AB = O does not necessarily imply that at least one of matrices A and B must be a zero matrix

   For example, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

5. AB = AC ⇒ B = C (iff A is non-singular matrix)

6. BA = CA ⇒ B = C (iff A is non – singular matrix)

Trace of a matrix:

The sum of the diagonal elements of a square matrix is termed as trace of a matrix.

Thus if $A = [a_{ij}]_{m \times n}$ then, $\text{Tr}(A) = \sum_{j=1}^{n} a_{jj} = a_{11} + a_{22} + \dots a_{nn}$

$$\text{Let } A = \begin{bmatrix} 11 & 2 & 5 \\ -2 & -23 & 1 \\ -1 & 6 & 35 \end{bmatrix}$$

Then, trace (A) = tr (A) = 11 + (–23) + 35 = 23

Properties of trace of a matrix:

Let A and B be two square matrices of order n and k be a scalar. Then,

1. tr (kA) = k tr A
2. tr (A + B) = tr A + trB
3. tr (AB) = tr (BA)

Transpose of a matrix:

Let $A = [a_{ij}]_{m \times n}$. Then the n×m matrix obtained from A by changing its rows into columns and its columns into rows is called the transpose of A and is denoted by A′ or A^T.

$$\text{Let } A = \begin{bmatrix} 6 & 8 \\ 3 & 4 \\ 2 & 1 \end{bmatrix} \text{ then, } A^T = A' = \begin{bmatrix} 6 & 3 & 2 \\ 8 & 4 & 1 \end{bmatrix}$$

If $B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ then

$$B' = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}' = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^t = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Properties of Transpose of a matrix:

If A^T and B^T be transposes of A and B respectively then,

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(kA)^T = kA^T$. k being any complex number
4. $(AB)^T = B^T A^T$
5. $(ABC)^T = C^T B^T A^T$

Conjugate of a matrix:

The conjugate of a matrix is obtained by taking the conjugate of each element of the matrix. For real matrices,

Conjugate will yield the same.

$$\text{Let } A = \begin{bmatrix} 6 & 8 \\ 3 & 4 \\ 2 & 1 \end{bmatrix} \text{ then, } A^T = A = \begin{bmatrix} 6 & 3 & 2 \\ 8 & 4 & 1 \end{bmatrix}$$

If $B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ then

$$\text{Example : If } A = \begin{bmatrix} 2+3i & 4-7i & 8 \\ -i & 6 & 9+i \end{bmatrix}$$

$$A \begin{bmatrix} 2-3i & 4+7i & 8 \\ i & 6 & 9-i \end{bmatrix}$$

Properties of conjugate of a matrix:

If $\bar{A}$ & $\bar{B}$ be the conjugates of A & B respectively. Then

1. $\overline{(\bar{A})} = A$
2. $\overline{(A+B)} = \bar{A} + \bar{B}$
3. $\overline{(KA)} = \bar{K}\,\bar{A}$, k being any complex number
4. $\overline{(AB)} = \bar{A}\bar{B}$, if A & B can be multiplied
5. $\bar{A} = A$ if and only if A is real matrix
6. $\bar{A} = -A$ if and only if A is purely imaginary matrix

Transposed conjugate of matrix:

The transpose of the conjugate of a matrix A is called transposed conjugate of A and is denoted by

A^θ or A^* or $(\bar{A})^T$. It is also called conjugate transpose of A.

$$\textbf{Example : } \text{If } A = \begin{bmatrix} 2+i & 3-i \\ 4 & 1-i \end{bmatrix}$$

To find A^θ, we first find $\underline{A} = \begin{bmatrix} 2-i & 3+i \\ 4 & 1+i \end{bmatrix}$

Then $A^\theta = (\bar{A})^T = \begin{bmatrix} 2-i & 4 \\ 3 & \end{bmatrix}$

Properties:

If A^θ & B^θ be the transposed conjugates of A and B respectively then,

1. $(A^\theta)^\theta = A$
2. $(A + B)^\theta A^\theta + B^\theta$

3. $\left(KA\right)^{\theta} = \bar{K}A^{\theta}$, K → complex number

4. $(AB)^{\theta} = B^{\theta} A^{\theta}$

Classification of Real Matrices Symmetric Matrices:

If the transpose of a matrix is identical to the original matrix, then the matrix is called as symmetric matrix.

$A^T = A$

Note: AAT and $\frac{A+A^T}{2}$ are always symmetric matrices.

Skew symmetric matrices:

If the transpose of a matrix is negative of the original matrix, then the matrix is termed as a skew symmetric matrix.

$A^T = -A$

The diagonal elements of a skew symmetric matrix are always zero.

Here, the first term represents a symmetric matrix, and the second term represents skew symmetric matrix.

Orthogonal matrices:

If the product of a matrix and its transpose is equal to the identity matrix or the inverse of a matrix is identical to its transpose, then the matrix is termed as orthogonal matrix.

$A^T = A^{-1}$ or $AA^T = I$

$\left|AA^T\right| = \left|I\right| = 1$

Thus, $\left|A\right|\left|A^T\right| = \left(\left|A\right|\right)^2 = 1$

$\left|A\right| = \pm 1$

Classification of complex matrices:

Hermitian matrix:

If the conjugated transpose of a matrix is identical to the original matrix, then the matrix is termed as Hermitian matrix.

$A^{\theta} = A$

Skew-hermitian matrix:

If the conjugated transpose of a matrix is negative of the original matrix, then the matrix is termed as skew Hermitian matrix.

$A^{\theta} = -A$

A complex matrix can be resolved into Hermitian and skew hermitian matrix,

$$A = \frac{A+A^{\theta}}{2} + \frac{A-A^{\theta}}{2}$$

Here, the first term represents a Hermitian matrix, and the second term represents Skew Hermitian Matrix.

Unitary matrix:

If the product of matrix and its conjugated transpose is the identity matrix. Thus, inverse of matrix is identical to its conjugated transpose of a matrix.

$A^{\theta} = A^{-1}$ or $AA^{\theta} = I$

Let a_{11}, a_{12}, a_{21}, a_{22} be any four numbers. The symbol $\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ represents the number $a_{11}\,a_{22} - a_{12}\,a_{21}$ and is called determinants of order 2.

Similarly, $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ represent a determinant of order 3.

Minors and cofactors:

Consider the determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Leaving the row and column passing through the elements a_{ij} , then the second order determinant thus obtained is called the minor of element aij and we will be denoted by M_{ij}.

As for example, the minor of the element

a_{11} is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \left(a_{22}\,a_{33} - a_{23}\,a_{32}\right)$

Similarly, the minor of the element

a_{12} is $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = \left(a_{21}\,a_{33} - a_{23}\,a_{31}\right)$

Minor of the element

a_{31} is $M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = \left(a_{12}\,a_{23} - a_{22}\,a_{13}\right)$

Cofactors:

The minor M_{ij} multiplied by $(-1)^{i+j}$ is called the cofactor of element a_{ij} . We shall denote the cofactor of an element by corresponding capital letter.

Example: Cofactor of $a_{ij} = A_{ij} = (-1)^{i+j} M_{ij}$

Cofactor element

$$a_{12} = A_{12} = (-1)^{1+2} M_{12}$$

$$= -\begin{vmatrix} a_{12} & a_{23} \\ a_{13} & a_{33} \end{vmatrix} = -(a_{21}\, a_{33} - a_{23}\, a_{31})$$

Cofactor element

$$a_{31} = A_{31} = (-1)^{3+1} M_{31} = M_{31}$$

$$= \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = (a_{12}\, a_{23} - a_{22}\, a_{13})$$

We define for any matrix, the sum of the products of the elements of any row or column with corresponding cofactors is equal to the determinant of the matrix.

Thus, the determinant of a 3x3 matrix is, $\Delta = a_{11} \times A_{11} + a_{12} \times A_{12} + a_{13} \times A_{13}$

Example : If $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 6 & 1 \\ 2 & 0 & 2 \end{bmatrix}$

$$\mathrm{Cof}(A) = \begin{bmatrix} 12 & 4 & -12 \\ -4 & 2 & 4 \\ 2 & -1 & 8 \end{bmatrix}$$

$|A| = (1 \times 12) + (2 \times 4) + (0 \times -12) = (-1 \times -4) + (6 \times 2) + (1 \times 4) = (2 \times 2) + (0 \times -1) + (2 \times 8) = 20$

Note: The determinant can only be computed for square matrices, and it can be calculated along any row or column.

Properties of determinants:

1. The value of the determinant will not be changed if rows and columns are interchanged.

   i.e. $|A| = |A|$

   $$|A| = \begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} = 10 - 4 = 6$$

   $$|A|^T = \begin{vmatrix} 5 & 1 \\ 4 & 2 \end{vmatrix} = 10 - 4 = 6$$

2. The value of determinant is multiplied by -1 if two rows or two columns are interchanged.
3. The value of determinant can be zero in the following cases.
   a) The elements in two rows or two columns are identical

   $$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 0$$

   b) The elements in two rows and two columns are proportional to each other.

   $$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 5 & 3 \\ 2 & 5 & 6 \end{vmatrix} = 0 \qquad \because C_3 = 3C_1$$

   c) All elements in any row or any column are zeros.

   $$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

   d) If the elements in the determinant are of consecutive order (valid for 3rd and higher order)

   $$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$$

   e) The first row of each element starts from the 2nd element of previous row such that the elements in that determinant are of consecutive order.

   $$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

4. The determinant of upper triangle, lower triangle, diagonal, scalar or identity matrix is the product of its diagonal elements.

   $$\text{Upper triangle} = |A| = \begin{vmatrix} 10 & 7 & 5 \\ 0 & 13 & 1 \\ 0 & 0 & 7 \end{vmatrix}$$

   $$= 10 \times 13 \times 7 = 910$$

Lower triangle $= |A| = \begin{vmatrix} 11 & 0 & 0 \\ 21 & 7 & 0 \\ 54 & 5 & 9 \end{vmatrix}$

$= 11 \times 7 \times 9 = 693$

Diagonal matrix $= \begin{vmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3 \times 6 \times 1 = 18$

Scalar matrix: A diagonal matrix with same diagonal elements.

$$|A| = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2 \times 2 \times 2 = 8$$

- If A is n × n matrix then
  $|KA| = K^n |A|$

  Consider the matrix $|A| = \begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix}_{2\times 2}$

  $\therefore |3A| = 3^2 \times \begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix} = 9 \times 12 = 108$

- If each element of a determinant contains sum of two elements, then that determinant should be expressed as sum of two determinants of the same order.

Eg : $|A| = \begin{vmatrix} a & a^2 & a^3+3 \\ b & b^2 & b^3+3 \\ c & c^2 & c^3+3 \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} a & a^2 & 3 \\ b & b^2 & 3 \\ c & c^2 & 3 \end{vmatrix}$

Solved Examples

Example: If $A = (a)_{ij\ 3\times3}$, $B = (b)_{ij\ 3\times3}$ such that $b_{ij} = 2^{i+j} a_{ij}\ \forall_{i,j}$ and $|A| = 2$ then $|B| =$ ________?

Solution: Given, $b_{ij} = 2^{i+j}\ a_{ij} \forall\ ij$

$$\therefore |B| = \begin{vmatrix} 2^2a_{11} & 2^3a_{12} & 2^4a_{13} \\ 2^3a_{21} & 2^4a_{22} & 2^5a_{23} \\ 2^4a_{31} & 2^5a_{32} & 2^6a_{33} \end{vmatrix} = 2^2 \times 2^3 \times 2^4$$

$$\begin{vmatrix} a_{11} & 2a_{12} & 2^2a_{13} \\ a_{21} & 2a_{22} & 2^2a_{23} \\ a_{31} & 2a_{32} & 2^2a_{33} \end{vmatrix}$$

$$|B| = 2^2 \times 2^3 \times 2^4 \times 2 \times 2^2 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$|A| = 2 \Rightarrow |B| = 2^{12} \times 2 = 2^{13}$

Example: If $A = a_{ij\ m\times n}$ such that $a_{ij} = i + j\ \forall i, j$ then sum of all elements of A is ?

Solution: The determinant of A can be expressed as,

$$|A| = \begin{vmatrix} a_{11} & a_{12} \ldots\ldots\ldots\ldots\ldots & a_m \\ a_{12} & a_{22} \ldots\ldots\ldots\ldots\ldots & a_m \\ : & & \\ : & \ldots\ldots\ldots\ldots\ldots & a_m \\ a_{m1} & a_{m2} \ldots\ldots\ldots\ldots\ldots & a_{mn} \end{vmatrix}$$

Since, $a_{ij} = i + j\ \forall i, j$

$$|A| = \begin{bmatrix} 1+1 & 1+2 \ldots\ldots\ldots\ldots & 1+n \\ 2+1 & 2+2 \ldots\ldots\ldots\ldots & 2+n \\ : & & \\ : & & \\ m+1 & m+2 \ldots\ldots\ldots\ldots & m+n \end{bmatrix}_{m\times n}$$

$$= \begin{vmatrix} 1 & 1 & 1 \ldots\ldots\ldots\ldots & 1 \\ 2 & 2 & 2 \ldots\ldots\ldots\ldots & 2 \\ : & & & \\ : & & & \\ m & m & \ldots\ldots\ldots\ldots & m \end{vmatrix}_{m\times n} + \begin{vmatrix} 1 & 2 & 3 \ldots\ldots\ldots\ldots & n \\ 1 & 2 & 3 \ldots\ldots\ldots\ldots & n \\ : & & & \\ : & & & \\ 1 & 2 & 3 \ldots\ldots\ldots\ldots & n \end{vmatrix}$$

Taking the sum of each column in first matrix. It will be equal to sum of first 'm' natural numbers

$$\frac{m(m+1)}{2}$$

Sum of each column = 2

Sum of 'n' columns = $\dfrac{mn(m+1)}{2}$

Taking sum of each row in second matrix. Each row sum is equal to sum of 'n' natural numbers.

Sum of each row = $\dfrac{n(n+1)}{2}$

Sum of ‘m’ rows = $\frac{mn(n+1)}{2}$

Total Sum = $\frac{mn}{2}[m+n+2]$

Example: Given $A = (a_{ij})$ such that $a_{ij} = i^2 - j^2\ \forall$ i, j . Find sum of the elements of the matrix.

Solution: The elements of matrix are,

$$A = \begin{bmatrix} 0 & -3 & -8 \ldots\ldots\ldots 1 & -n^2 \\ 3 & 0 & -5 \ldots\ldots\ldots 2^2 & -n^2 \\ 8 & 5 & 0 \ldots\ldots\ldots\ldots\ldots & -n^2 \\ n^2-1^2 & n^2-2^2 & \ldots\ldots\ldots & 0 \end{bmatrix}$$

$\Rightarrow$ Skew symmetric matrix

$\therefore$ sum of elements = 0

Example: Matrix A has m rows and (m + 5) columns and B has n rows and (11 – n) columns, the orders of A & if AB and BA are defined is?

Solution: Since AB is defined m + 5 = n Since BA is defined, 11 – n = m

Solving above two equations, m = 3 and n = 8

Example: The following represents equation of a straight line. $\begin{vmatrix} x & 2 & 4 \\ y & 8 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0.$ The line passes through,

(A) 0, 0 (B) 3, 4

(C) 4, 3 (D) 4, 4

Solution: The determinant is given by 8x + 2y – 32 = 0 8x + 2y = 32

Substituting the options we get (3, 4)

Solution:

$$|A| = 5\times(3-0) + 2\times(0-2\times 3) = 15-12 = 3$$

Note: A 4x4 determinant can be calculated as,

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}$$

$$+a_{13}\begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} + a_{14}\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

By using either row or column or both operations make all elements above and below or to the left and right of the selected element are zeros. And then expand the determinant along the row or column.

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & b_{22} & b_{23} & b_{24} \\ 0 & b_{32} & b_{33} & b_{34} \\ 0 & b_{42} & b_{43} & b_{44} \end{vmatrix}$$

$$|A| = a_{11}\begin{vmatrix} b_{22} & b_{23} & b_{24} \\ b_{32} & b_{33} & b_{34} \\ b_{42} & b_{43} & b_{44} \end{vmatrix}$$

b_{ij} are the new elements after row or column transformation.

Example: Determine the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 4 & 5 & 7 \\ 1 & 1 & 1 & 3 \\ 2 & 3 & 4 & 9 \end{bmatrix}$$

Solution: Applying the row transformations,

$R_3 \to R_3 - R_1$

$R_4 \to R_4 - 2R_1$

The resultant matrix is, $A = \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 4 & 5 & 7 \\ 0 & -1 & 2 & -2 \\ 0 & -1 & 6 & -1 \end{bmatrix}$

Determinant is given by,

$$|A| = 1\times\begin{bmatrix} 4 & 5 & 7 \\ -1 & 2 & -2 \\ -1 & 6 & -1 \end{bmatrix} = 17$$

Example: Find the value of the determinant

$$A = \begin{vmatrix} 1+b & b & 1 \\ b & 1+b & 1 \\ 1 & 2b & 1 \end{vmatrix}$$

Solution: Applying the column transformation, $C_1 \to C_1 + C_2$

$$\begin{vmatrix} 1+2b & b & 1 \\ 1+2b & 1+b & 1 \\ 1+2b & 2b & 1 \end{vmatrix}$$

Since, $C_1 = (1 + 2b)\, C_3$ i.e. both columns are proportional

$\therefore |A| = 0$

Adjoint of a matrix:

Let B be the cofactor element of the matrix A, then adj of A = B^T

Singular matrix:

A matric is said to be singular matrix if $|A| = 0$. It is called as nonsingular if $|A| \neq 0$

The inverse of only non-singular matrix can be computed. Thus, singular matrices are non-invertible.

A matrix A is used to be invertible if we can find some other matrix B such that AB=BA=I, then B is called Inverse of the matrix A.

Inverse of a matrix A is given by, $A^{-1} = \dfrac{\text{adj}A}{|A|}$

Properties of inverse of a matrix:

1. $AA^{-1} = A^{-1}A = I$
2. A and B inverse of each other iff $AB = BA = I$
3. $(AB)^{-1} = B^{-1}A^{-1}$
4. $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
5. If A be an n×n non-singular matrix, then $(AT)^{-1} = (A^{-1})^T$
6. If A be an n×n non-singular matrix, then $(A^{-1})^{\theta} = (A^{\theta})^{-1}$
7. For a 2 × 2 matrix there is a short-cut formula for inverse as given below

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

8. Assuming A is matrix of order "n x n"

$|A|\, A^{-1}$ adj A

$|A|\, |A^{-1}| = |\text{adj } A|$

Since, $|KA| = k^n\, |A|$

Thus, $|A|^n\, |A^{-1}| = |\text{adj } A|$

$|A|^n\, |A|^{-1} = |\text{adj } A|$

$|A|^{n-1} = |\text{adj } A|$

Replacing A by adj A

$\Rightarrow |\text{adj (adj } A)| = |\text{adj } A|^{n-1} = |A|^{n-1}$

i.e., $|\text{adj (adj } A)| = |A|^{(n-1)^2}$

Similarly $|\text{adj(adj(adj } A))| = |A|^{(n-1)^3}$

Solved Examples

Example: For the matrix m $\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{bmatrix}$ and $m^T = m^{-1}$. Find x?

Solution: If m is an orthogonal matrix, its rows and columns must be pairwise orthogonal and periodic orthogonal.

$$\frac{3}{5}x + \frac{4}{5} \times \frac{3}{5} = 0$$

$$\frac{3}{5}x = -\frac{4}{5} \times \frac{3}{5}$$

$$x = -\frac{4}{5}$$

Example: If $A = (a_{ij})^{5\times5}$ such that $a_{ij} = i - j$. Find A^{-1} in each case.

Solution: $aij = i - j = -(j - i)$

$a_{ij} = -a_{ji}$

Thus, A is a skew symmetric matrix

$A^T = -A$

$|A^T| = |-A|$

$|A| = (-1)^5\, |A| = -|A|$

$|A| + |A| = 0$

$2|A| = 0$

$|A| = 0$

Hence, A is a singular matrix and thus A^{-1} does not exist

Note: Every odd order skew symmetric matrix is a singular matrix $|A|_{n\times n} = 0$ Every even order skew symmetric matrix is a non-singular matrix $|A|_{n\times n} \neq 0$
The determinant of even order skew symmetric matrix is always perfect square.

$$\text{Eg}: \begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix} = 9 = 3^2 \neq 0$$

Example: If x and y are two non-zero matrices of the same order, such that xy=0, then

(A) $|x| \neq 0, |y| = 0$

(B) $|x| = 0, |y| \neq 0$

(C) $|x| \neq 0, |y| \neq 0$

(D) $|x| = 0, |y| = 0$

Solution : Let $|x| = 0$ and $|y| \neq 0 \Rightarrow y^{-1}$ exists.

$xy = 0$

$xyy^{-1} = 0y^{-1}$

$x = O$ is false since it is given x and y are non-zero matrices.

$\therefore |y| \neq 0$ is wrong $\Rightarrow |y| = 0$

Similarly, $|x| = 0$

Note: Product of two nonzero matrices is a null matrix if both of them are singular.

Example: Let k be a positive real number and let

$$A = \begin{bmatrix} (2k-1) & 2\sqrt{\cancel{k}} & 2\sqrt{\cancel{k}} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 2k-1 & \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0\sqrt{k} \end{bmatrix}$$

Find A) |adj B| B) |adj A| C) |adj A| $= 10^6$, the value of k is ________?

Solution: (A) Here B is an odd order skew symmetric matrix.

$\therefore |B| = 0 \Rightarrow |\text{adj } B| = |B^{n-1}| = 0$

(B) $|\text{adj } A| = |A|^{n-1} = |A|^2$

$$|A| = \begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{vmatrix}$$

$$R_2 \to R_2 + R_3 \Rightarrow \begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 0 & 1+2k & -(1+2k) \\ -2\sqrt{k} & 2k & -1 \end{vmatrix}$$

$$C_2 \to C_2 \to C_3 \Rightarrow \begin{vmatrix} 2k-1 & 4\sqrt{k} & 2\sqrt{k} \\ 0 & 0 & -(1+2k) \\ -2\sqrt{k} & 2k-1 & -1 \end{vmatrix}$$

Expanding the determinant along the second row,

$|A| = (1+2k)[(2k-1)+8]$

$= (1+2k)[4k^2 - 4k + 1 + 8k]$

$= (1+2k) = (2k+1)$

$\therefore |\text{adj } A| = |A|^2 \left[(2k+1)^3\right]^2 = (2k+1)^6$

(C) $(2k+1)^6 = 10^6$

ie. $2k + 1 = 10 \Rightarrow 2k = 9 \Rightarrow k = \frac{9}{2}$

Example: Given an orthogonal matrix A

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}. \text{ The } (A \times A^T)^{-1} \text{ is } \_\_\_\_\_\_?$$

Solution: $(A \times A^T) = I$

$\therefore (A \times A^T)^{-1} = I^{-1} = I$

Example: Find the inverse of matrix.

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}?$$

Solution:

Since its rows and columns are pairwise orthogonal. Thus, A is an orthogonal matrix. $A^T = A^{-1}$

$$\therefore A^{-1} = \begin{bmatrix} \text{os}\,\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank of a Matrix

Submatrix:

A matrix is obtained after deleting some rows or columns is called a submatrix.

$$\text{Eg} : A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 1 & -1 & 0 & 2 \end{bmatrix}$$

Some of the submatrices of A are

$$B_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times3}, \qquad B_2 = \begin{bmatrix} 3 & 4 \\ 6 & 7 \end{bmatrix}_{2\times2}$$

$$B_3 = \begin{bmatrix} 5 & 6 & 7 \\ -1 & 0 & 2 \end{bmatrix}_{2\times3}$$

Thus, minor can also be defined as the determinant of a square sub-matrix.

If the determinant of at-least one highest possible square sub matrix is nonzero then order of the determinant is called rank of a matrix.

Solved Examples

Example: Find the rank of the matrix,

$$A = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & -1 & 4 & 5 \\ 1 & -11 & 14 & 5 \end{bmatrix}_{3\times4} \quad ?$$

Solution: Highest possible square matrix is 3×3.

$$|B| = \begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{vmatrix}$$

$$= (44 - 14) - 2(42 - 22) + (12 - 2) = 0$$

$$|B| = \begin{vmatrix} 3 & -2 & 1 \\ -1 & 4 & 5 \\ -11 & 14 & 5 \end{vmatrix}$$

$$= 3(20 - 70) + 1(-10 - 14) - 11(-10 - 4) \neq 0$$

$$\therefore \text{Rank}(A) = 3$$

Properties of rank of a matrix:

- Rank $(O_{n\times n}) = 0$
- Rank $(I_{n\times n}) = n$
- Rank $(a_{ij}\,(I_{n\times n})) = n$
- Rank $(A + B) \leq$ rank (A) + rank (B)
- Rank $(A - B) \geq$ rank (A) – rank (B)

If A is an $m \times n$ matrix, then rank $(A) \leq \min(m,n)$

- Rank $(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$
- Rank $(AB) \geq$ rank (A) + rank $(B) - n$. If A and B are $n \times n$ matrices
- Rank (A) = rank (A^T)
- If rank $(A)_{n\times n} = n$ then rank (adj A) = n
- If rank $(A_{n\times n}) = n - 1$ then rank (adj A) = 1

Example: Determine the rank of adjoint of the matrix $A = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & -1 \\ 0 & 0 & 0 \end{vmatrix}$

Solution: Adjoint of A is given by,

$$\text{adj}(A) = \begin{bmatrix} 0 & 0 & -17 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

Applying row transformations, $R_1 \to R_1 \to 17R_2$, $R_3 \to R_3 \to 5R_2$ as rank is invariant under row transformation.

$$\text{adj}\,A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ rank of (adj A) = number of non-zero rows = 1

If rank $(A_{n\times n}) \leq n - 2$, then rank of (adj A) = 0

Echelon form:

Row Echelon Form: A matrix is said to be in echelon form if it satisfies the following conditions.

- All zero rows should occupy last rows if any.
- The number of zeros before a non-zero entry of each row is less than the number of such zeros before a non- zero entry of

the next row. i.e., zeros increases row by row.

- Rank of a matrix in Echelon form = number of non-zero rows.
- To reduce any matrix to row echelon form we use only row operations.
- The number of non-zero rows in row echelon Form are also called linearly independent vectors or LI rows.

Note: Every upper triangular matrix will be a row Echelon matrix, but converse may or may not be true. Echelon form as it satisfies the conditions.

$$A = \begin{bmatrix} 0 & -2 & 5 & 6 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{Echelon form}$$

But not upper triangular matrix as the diagonal elements are all zero.

Example: Determine the rank of matrix A

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & 1 & 2 & 0 \\ 2 & 2 & 1 & 3 \end{bmatrix}$$

Solution: Applying row transformation

$R_2 \rightarrow R_2 \rightarrow 2R_1$

$R_3 \rightarrow R_3 \rightarrow 2R_1$

$$\therefore A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & -3 & -2 & 2 \\ 0 & -2 & -3 & 5 \end{bmatrix}$$

Now $R_3 \rightarrow 3R_3 - 2R_2$

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & -3 & -2 & 2 \\ 0 & 0 & -5 & 11 \end{bmatrix}$$

There are three linearly independent rows and thus rank(A) = 3

Example: Determine the rank of

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 5 \\ 1 & 1 & 2 & 3 \end{bmatrix}$$

Solution: Applying row transformation

$R_3 \rightarrow 2R_3 - R_1$

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 5 \\ 0 & 1 & 3 & 6 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

Since, there are three linearly independent rows, rank (A) = 3

Note: If rank(A) = n then it has n linearly independent rows and linearly independent columns.

Example: If $A = (a_{ij})$ such that $a_{ij} = i \times j \ \forall \ i, j$ then rank(A) = ?

Solution:

The matrix A when expanded is,

$$A = \begin{bmatrix} 1 & 2 & 3 & \ldots\ldots & n \\ 2 & 4 & 6 & \ldots\ldots & 2n \\ \vdots & & & & \\ m & 2m & 3m & \ldots\ldots & nm \end{bmatrix}$$

$$= 1 \times 2 \times 3 \times \ldots\ldots\ldots\ldots \times m \begin{bmatrix} 1 & 2 & 3 \ldots\ldots & n \\ 1 & 2 & 3 \ldots\ldots & n \\ \vdots & & & \\ 1 & 2 & 3 \ldots\ldots & n \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$

$R_3 \rightarrow R_3 - R_1$

$Rm \rightarrow Rm - R_1$

$$A = m! \begin{bmatrix} 1 & 2 & 3 \ldots\ldots\ldots & n \\ 0 & 0 & 0 \ldots\ldots\ldots & 0 \\ \vdots & & & \\ 0 & 0 & 0 \ldots\ldots\ldots & 0 \end{bmatrix}$$

Number of non-zero rows = 1

$\therefore$ Rank (A) = 1 as there is only 1 linearly independent vector.

Example: If $x = (x_1 \ x_2 \ \ldots \ x_n)^T$ is an n-tuple nonzero vector. Then find rank (xx^T)

Solution: Since x is a n-element column vector. Then, x^T is an n-element row vector.

Thus, both have rank of 1.

$$\rho\left(XX^T\right) \le \min\left\{\rho\left(X\right)\rho\left(X^T\right)\right\}$$
$$P\left(XX^T\right) \le \min\left\{1, 1\right\} = 1.$$

Here, ρ represents rank of matrix.

Example: The rank of 5×6 matrix is 4. Then which of the following statement is true?

(A) Q has 4 linearly independent rows and 4 linearly independent columns.

(B) Q has 4 LI rows and 5 LI columns

(C) $Q \cdot Q^T$ is invertible

(D) $Q^T \cdot Q$ is invertible

Solution: Option (A) is correct

If rank of a matrix is 4 then it have 4 LI rows and 4 LI columns. These 4 LI rows can be obtained by reducing the matrix into echelon form and 4 LI columns can be obtained by reducing the matrix into echelon form.

Linearly dependent and independent vectors:

If the elements are written in a horizontal line or in vertical line is called as vector. This means row matrix, or a column matrix can be termed as a vector.

Example: (1 2 3) $\rightarrow$ vector

Two vectors x_1 and x_2 are said to be linearly dependent if it is possible to express one of the vectors as multiple of another vector.

Suppose, one vector is $x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$X_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2x_1 \text{ or } x_1 = \frac{1}{2_2}x$$

Two vectors in R^2 (two dimensional) are said to be linearly dependent if they are collinear.

Similarly 3 vectors in R^3 (3-D) are said to be linearly dependent if they are coplanar i.e. lying in the same plane.

If it is not possible to express 1 vector as multiple of other vector the two vectors are said to be linearly independent.

Suppose, $x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Hence, $x_1 \ne kx_2$. Thus, x_1 & x_2 are linearly independent Vectors.

Linearly dependent vectors:

A set of r n-vectors (vectors having n-element each), x_1, x_2 ... x_r are said to be linearly dependent if there exists r scalars K_1, K_2 ... k_r such that $K_1x_1 + K_2x_2 + \ldots + K_rx_r = 0$ where K_1, K_2, ... K_r not all zeros. (At least 1 K should be a non-zero number)

Linearly independent vectors:

'r' vectors x_1, x_2 ... x_r are linearly independent vectors if there exist 'r' scalars. K_1, K_2 ... K_r such that $K_1X_1 + K_2X_2 + \ldots + K_rX_r = 0$ where K_1, K_2 ... K_r are all zeros.

- If the rank of matrix is less than the dimension of the matrix, then the vectors are said to be linearly dependent. Thus, if the determinant of the matrix is zero the vectors are said to be linearly dependent.
- If the rank of matrix is equal to the dimension of the matrix, then the vectors are said to be linearly independent. Thus, if determinant of the matrix is non-zero the vectors are said to be linearly independent.
- If the number of components of the vectors is more than the number of vectors, then the vectors are said to be linearly independent but if the number of components is less than the number of vectors then the vectors are said to be linearly dependent.

Example: Consider vectors (1 2 3 4)
(2 1 –1 7)
(2 3 4 5)

Number of vectors r = 3 Number of component n = 4

Since r < n, the vectors are linearly independent.

If the vectors are expressed in Matrix form,

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 7 \\ 2 & 3 & 4 & 5 \end{bmatrix}_{3\times4}$$

Applying row transformations,

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 2R_1$

$$A\begin{bmatrix}1 & 2 & 3 & 4\\0 & -3 & -7 & -1\\0 & -1 & -2 & -3\end{bmatrix}$$

$R_3 \to 3R_3 - R_2$

$$A\begin{bmatrix}1 & 2 & 3 & 4\\0 & -3 & -7 & -1\\0 & 0 & 1 & 8\end{bmatrix}$$

$\rho(A) = 3$

Thus, the rank of matrix is same as number of vectors. Thus, the vectors are linearly Independent. But if we consider the vectors X_1 = (1 2 3); X_2 = (1 0 3); X_3 = (1 1 – 1); X_4 = (0 1 2) Number of vectors r = 4

Number of components n = 3

Since r > n, so the given vectors are linearly dependent.

Suppose the vectors are linearly dependent, then any 1 of the vectors can be expressed as a linear combination of other vectors.

If the given vectors are linearly independent, then it is impossible to express any of the vectors as a linear combination of other vectors.

Example: Determine whether the given vectors are linearly independent or Dependent X_1 = (1 0 0); X_2 = (1 0 0); X_3 = (0 0 1)

Solution: Expressing the linear combination of vectors and equating them to zero, K_1X_1 + K_2X_2 + K_3X3 = 0

Substituting the vectors, K1(1 0 0) + K_2(0 1 0) + K_3(0 0 1) = (0 0 0)

i.e. (K_1 K_2 K_3) = (0 0 0) K_1 = K_2 = K_3 = 0

Since, all coefficients are zero. Thus, all vectors are linearly independent.

Alternate method:

Expressing the vectors in matrix form,

$$|A| = \begin{vmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{vmatrix} = 1 \neq 0$$

$\rho(A) =$ number of non-zero rows = 3

Since, rank is same as number of vectors. The vectors are linearly independent.

The set of vectors having at least one vector is a null vector, the vectors are said to be linearly dependent.

Example: Consider the vectors X_1 = (2 3 4); X_2 = (1 –1 2); X_3 = (0 0 0). Determine whether these vectors are linearly dependent or independent.

Solution: Expressing the vectors in matrix form. The determinant is given by,

$$|A| = \begin{vmatrix}2 & 3 & 4\\1 & -1 & 2\\0 & 0 & 0\end{vmatrix} = 0$$

Since |A| =0, rank(A) < 3, but the determinant of 2×2 square sub-matrix is nonzero

$|B| = \begin{vmatrix}2 & 3\\1 & -1\end{vmatrix} = -5 \neq 0$ Thus, $\rho(A) = 2$

Number of given vectors = 3. Thus, ρ(A) < n and the vectors are linearly dependent.

Orthogonal vectors:

For orthogonal Vectors dot product or inner product of the vectors must be zero. Each term in the row or column vector can be considered as a component of the vector. The dot product is obtained as sum of the product of corresponding components of both vectors.

In matrix form it can be expressed as, $X_1^TX_2 = X_1\,X_2^T = 0$

The set of orthogonal vectors in R^n (R vectors with n elements are linearly independent).

Dimension and basis:

- The number of linearly independent vectors in a vector space is called as dimension of vector space.
- For a three-dimensional space, there are only 3 linearly independent vectors and any fourth vectors can be expressed as a linear combination of the other three vectors.
- Number of linearly independent vectors can be determined by expressing the matrix in Row Echelon Form and the number of non-zero rows is equal to the number of linearly independent vectors.

- The set of linearly independent Vectors is called as basis of vector space. Any vector in vector space can be expressed as a linear combination of its basis vectors.

Nullity of a matrix:

Nullity of a matrix is denoted by N(A) and is defined as the difference between order and rank of the matrix.

i.e. $N(A)=n(A)-\rho(A)$

Nullity of a non-singular matrix of any order is always zero as the rank of the matrix is same as the order since the determinant of the matrix is non-zero.

Connection between rank and span:

A set of n vectors $X_1, X_2, X_3 \ldots X_n$ spans R_n iff they are linearly independent which can be checked by constructing a matrix with $X1, X_2, X_3 \ldots X_n$ as its rows (or columns) and checking that the rank of such a matrix is indeed n. If however the rank is less than, n say m, then the vectors span only a subspace of R^n

Solved Examples

Example: Test whether the following vectors are linearly dependent or linearly independent. Also find dimensions and basis?

(1 1 –1 0) (4 4 –3 1) (–6 2 2 2) (9 9 –6 3)

Solution: Expressing the vectors in matrix form,

$$A=\begin{bmatrix}1 & 1 & -1 & 0\\ 4 & 4 & -3 & 1\\ -6 & 2 & 2 & 2\\ 9 & 9 & -6 & 3\end{bmatrix}$$

Applying row transformation, $R_2 \to R_2 - 4R_1$; $R_3 \to R_3 + 6R_1$; $R_4 \to R_4 - 9R_1$

$$A=\begin{vmatrix}1 & 1 & -1 & 0\\ 0 & 0 & 1 & 1\\ 0 & 8 & -4 & 2\\ 0 & 0 & 3 & 3\end{vmatrix}$$

Interchanging row 2 and row 3.

$$R_2 \Rightarrow R_3 \Rightarrow A=\begin{bmatrix}1 & 1 & -1 & 0\\ 0 & 8 & -4 & 2\\ 0 & 0 & 1 & 1\\ 0 & 0 & 3 & 3\end{bmatrix}$$

Applying, $R_4 \to R_4 \to 3R_1$

$$A=\begin{bmatrix}1 & -1 & 0 & 1\\ 0 & 8 & -4 & 2\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

In row echelon form, the number of non-zero rows is 3. Thus, rank of the matrix is 3. Thus, the dimension is 3.

Basis is $\{$(1 –1 0 1), (0 8 –4 2), (0 0 1 1)$\}$

Example: Determine the value of k if nullity of $A=\begin{vmatrix}k & 1 & 2\\ 1 & -1 & -2\\ 2 & 1 & 1\end{vmatrix}$ is 1?

Solution: The order of matrix A is n(A) = 3 nullity of matrix A is, $N(A)=n(A)-\rho(A)$ $1=3-\rho(A)$

$\therefore\ \rho(A) = 3 - 1 = 2$

Since, Rank is 2. Thus, $|A| = 0$

$$|A|=0 \Rightarrow \begin{vmatrix}K & 1 & 2\\ 1 & -1 & -2\\ 2 & 1 & 1\end{vmatrix}=0$$

$$K(-1+2)-1(1+4)+2(1+2)=0$$

$$K=-1$$

Example: The rank of a matrix is 5 and nullity of the matrix is 3. Then what is the order of the matrix.

Solution: $N(A) = n(A) - \rho(A)$

$3 = n(A) - 5$

$n(A) = 8$

Example: Check if the vectors [1 2 –1], [2 3 0], [–1 2 5] span R^3

Solution: Constructing $A=\begin{bmatrix}1 & 2 & -1\\ 2 & 3 & 0\\ -1 & 2 & 5\end{bmatrix}$

Finding determinant to determine the rank,

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ -1 & 2 & 5 \end{vmatrix} = 1(15-0)-2(10-0)-1(4+3)$$

$= 15 - 20 - 7 = -12 \neq 0$

So, rank = 3

$\therefore$ The vectors are linearly independent and hence span R^3

System of Non-Homogenous and Homogeneous Linear Equations

Consider a non-homogenous system of equations, containing two equations and two variables.

$ax + by = e$

$cx + dy = f$

The above system of equations has

- No solution if $\frac{a}{c} = \frac{b}{d} \neq \frac{e}{f}$. Lines parallel to each other.
- Unique solution if $\frac{a}{c} \neq \frac{b}{d}$. Lines intersect at 1 particular point only.
- Infinite number of solutions if $\frac{a}{c} = \frac{b}{d} = \frac{e}{f}$

  The given system of equations can be brought to the form of number of equations < number of variables. In this case, both the lines are identical and hence overlap or intersect at infinite number of points.

Example: The system of equations $4x + 2y = 7$, $2x + y = 6$ have?

Solution: For the given set of equations,

$$\frac{a}{c} = \frac{b}{d} \neq \frac{e}{f} \text{ as } \frac{4}{2} = \frac{2}{1} \neq \frac{7}{6}$$

$\therefore$ No solution exists for the following system of equations.

Example: The system of equations $x + 3y = 5$, $2x + 5y = -3$ have.

Solution: For the following system of equations,

$$\frac{1}{2} \neq \frac{3}{5} \text{ as } \frac{a}{c} \neq \frac{b}{d}$$

Thus, the following system has a unique solution.

System of Equations with 'n' variables

Consider a system of m equations and n variables.

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{32}x_2 + a_{21}x_3 + \dots + a_{2n}x_n = b_2$

:

:

$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_n$

The above system of equation can be put in matrix form

$$\begin{bmatrix} a_{11} & a_{12} \dots\dots\dots & a_{1n} \\ a_{21} & a_{22} \dots\dots\dots & a_{2n} \\ \vdots & & \\ a_{21} & a_{m2} \dots\dots\dots & a_{mn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$AX = B$

Where $X \rightarrow$ solution matrix.

If we write the elements of matrix B in the last column of matrix A, the resulting matrix is called augmented matrix and is denoted by (A | B)

$$[A|B] = \begin{bmatrix} a_{11} & a_{12} \dots\dots\dots & a_1 & b_1 \\ a_{22} & a \dots\dots\dots & a_n & b_2 \\ \vdots & & & \\ am_1 & am_2 \dots\dots\dots & a_{mn} & b_m \end{bmatrix}_{(m+1)\times n}$$

Procedure to determine the solution:

1. Reduce (A | B) to row echelon form
2. Find (A| B) and $\rho(A)$
3. If $\rho(A) < \rho(A \mid B)$ or $\rho(A \mid B) \neq \rho(A)$ the given system of equations has no solution, system is inconsistent.
4. If $\rho(A \mid B) = \rho(A)$ = number of unknowns, the given system of equations has unique solution i.e. system is consistent.
5. If $\rho(A \mid B) = \rho(A)$ < number of unknowns, the given system of equations has infinite number of solutions.

6. If the number of equations < number of variables (r < n), the given system of equations will have infinite number of non-zero solutions. This non-zero solution can be found by assigning (n – r) variables or arbitrary constants. These n – r solutions are said to be linearly independent solutions.

r=rank of matrix & n=order of matrix

Inconsistent: The given system of equations is said to be inconsistent if they have no solution.

Consistent: The given system of equations is said to be consistent if they have a solution (unique or infinite).

Note: If the number of equations < number of variables (or) number of variables exceeds the number of equations, the system of equations have ∞ number of solutions.

Homogeneous system of linear equation:

Consider the following homogenous system of linear equations of m equations in n variables.

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0$

:

:

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0$

In matrix form

$$\begin{bmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{12} \ldots a_{2n} \\ : & \\ a_{m1} & a_{m2} \ldots a_{mn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ : \\ X_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ : \\ 0 \end{bmatrix}$$

The following system can be represented as, AX=O.

The solution of this system can have following cases:

Case-1: Inconsistency: This is not possible in a homogeneous system since such a system is always consistent since the trivial solution C $[0, 0, 0\ldots\ldots]^t$ always exists for a homogeneous system.

Case-2: Consistent unique solution: If r = n ; the equation AX = O will have only the trivial unique solution X = [0, 0, 0 ...]

Note: That $r = n \Rightarrow |A| \neq 0$ i.e., A is non-singular.

Case-3: Consistent infinite solution: If r <n we shall have n – r linearly independent non-trivial infinite solutions. Any linear combination of these (n – r) solutions will also be a solution of AX = 0

Thus in this case, the equation AX = O will have infinite solutions.

Note: If $r < n \Rightarrow |A|\ 0$ i.e, A is a singular matrix.

Solved Examples

Example: How many solutions does the following system of equations have?

$-x + 5y = -1,\ x - y = 2,\ x + 3y = 3$

Solution: The matrix A and B are given as,

$$A = \begin{bmatrix} -1 & 5 \\ 1 & -1 \\ 1 & 3 \end{bmatrix}; B = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

The augmented matrix is given by,

$$A|B = \begin{bmatrix} -1 & 5 & -1 \\ 1 & -1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$$

Determinant of augmented matrix is,

$$|A|B| = -1(-3-6) - 5(3-2) - 1(3+1) = 0$$

So rank (A|B) = 2 = rank (A) = number of variables. Thus, this system has unique solution.

Example: Find the value of λ and μ for the system of equation.

$x + y + z = 6,\ x + 4y + 6z = 20,\ x + 4y\ \lambda\ z = \mu$ to have

(A) No solution

(B) Unique solution

(C) ∞ number of non-zero solution

Solution: The augmented matrix is given by,

$$[A|B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 4 & 6 & 20 \\ 1 & 4 & \lambda & \mu \end{bmatrix}$$

Applying row transformation,

$R_2 \to R_2 \to R_1$ and $R_3 \to R_3 \to R_1$

$$[A|B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 3 & \lambda-1 & \mu-6 \end{bmatrix}$$

$R_3 \to R_3 \to R_2$

$$[A|B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 0 & \lambda-6 & \mu-20 \end{bmatrix}$$

(A) For No solution

$\rho(A) < \rho(A \mid B)$

i.e. $\lambda - 6 = 0$ so that last row of A is reduced to zero. Then, $\rho(A) = 2$

$\mu - 20 \neq 0$

Thus, last row of [A|B] is non-zero and $\rho(A \mid B) = 3$

$\lambda = 6$ and $\mu \neq 20$

(B) Unique solution

$\rho(A \mid B) = \rho(A)$ = number of unknowns.

$\rho(A \mid B) = \rho(A) = 3$

$\lambda - 6 \neq 0$, $\mu - 20 \neq 0$ or $\mu - 20 = 0$

$\therefore\ \lambda \neq 6$, μ = any value

Thus, last row will be non-zero for both A and A | B.

(C) Infinite number of solutions

$\rho(A \mid B) = \rho(A) <$ number of unknowns.

$\lambda - 6 = 0$ and $\mu - 20 = 0$

$\therefore\ \lambda = 6$ and $\mu = 20$

Example: For what value of α and β, the following system of equations

$x + y + z = 5$, $x + 3y + 3z = 9$, $2x + 2y + \alpha z = \beta$ have infinite number of solutions.

Solution: For infinite number of solutions $\rho(A \mid B) = \rho(A) <$ number of unknowns

$$[A|B] = \begin{bmatrix} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 2 & 2 & \alpha & \beta \end{bmatrix}$$

Apply Row Transformations, $R_2 \to R_2 - R_1$; $R_3 \to R_3 - 2R_1$

$$[A|B] = \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & \alpha-2 & \beta-10 \end{bmatrix}$$

$\alpha = 2$, $\beta = 10$

This ensures that $\rho(A \mid B) = \rho(A) = 2$

Example: Find the condition on a, b, c for which the following system of equations

$3x + 4y + 5z = a$

$4x + 5y + 6z = b$

$5x + 6y + 7z = c$

do not have a solution?

Solution:

The augmented matrix is given by,

$$[A|B] = \begin{bmatrix} 3 & 4 & 5 & a \\ 4 & 5 & 6 & b \\ 5 & 6 & 7 & c \end{bmatrix}$$

Applying row transformations, $R_2 \to 3R_2 - 4R_1$ and $R_3 \to 3R_3 - 5R_1$

$$[A|B] = \begin{bmatrix} 3 & 4 & 5 & a \\ 0 & -1 & -2 & 3b-4a \\ 0 & -2 & -4 & 3c-5a \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$[A|B] = \begin{bmatrix} 3 & 4 & 5 & a \\ 0 & -1 & -2 & 3b-4a \\ 0 & 0 & 0 & 3(a+c-2b) \end{bmatrix}$$

From above matrix $\rho(A) = 2$

If $3(a + c - 2b) \neq 0$, then $\rho(A \mid B) = 3 \neq \rho(A)$. Thus, no solution.

Example: For what values of λ does the system of equations have two linearly independent solutions.

$x + y + z = 0$, $(\lambda + 1)y + (\lambda + 1)z = 0$, $(\lambda^2 - 1)z = 0$

Solution: Since there are two linearly independent solutions $n - r = 2$

$3 - r = 2$

$r = 3 - 2 = 1$

If rank is 1, determinant of the matrix should be zero.

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & \lambda+1 & \lambda+1 \\ 0 & 0 & \lambda^2-1 \end{vmatrix}_{3\times 3} = 0$$

$$1\left[(\lambda+1)(\lambda^2-1)-0\right] = 0$$

$$(\lambda+1)(\lambda+1)(\lambda-1) = 0$$

$$\lambda = +1, -1$$

For λ = –1, rank = 1 as the determinant of order 2 square sub-matrix goes to 0. If λ = 1, rank = 2

$\therefore$ λ = –1 is correct.

Example: The rank of $A_{3\times 3}$ is 1. The system of equations AX=0 has how many linearly independent Solutions?

Solution: Order of Matrix n=3

Rank of Matrix r =1

Number of linearly independent Solutions = n-r = 2

Example: The system of equations $x + 3y - 2z = 0$, $2x - y + 4z = 0$, $x - 11y + 14z = 0$

trivial solutions?

Solution: The coefficient matrix A is given by,

$$A \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$

Determinant of A is

$$|A| - 1(-14+44) - 3(28-4) - 2(-22+1) = 0$$

Thus, rank(A) $<$ 3 = number of unknowns

Hence, this system has infinite non-trivial solutions.

Example: Find the value of k for which the system of equations.

$(3k - 8) \times 3y + 3z = 0$

$3x + (3k - 8)\, y + 3z = 0$

$3x + 3y + (3k - 8)\, z = 0$

has nontrivial solutions?

Solution: For non-trivial solutions, rank (A) < 3. Thus |A| = 0

$$\begin{vmatrix} 3k-8 & 3 & 3 \\ 3 & 3k-8 & 3 \\ 3 & 3 & 3k-8 \end{vmatrix} = 0$$

$$R_1 \to R_1 + R_2 + R_3,\ R_2 \to R_2 - R_3$$

$$(3k-8+6)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 3k-8-3 & 3-(3k-8) \\ 3 & 3 & 3k-8 \end{vmatrix} = 0$$

i.e. $(3k-8+6)(3k-8-3)(3k-8-3) = 0$

$$k = \frac{2}{3}, k = \frac{11}{3}, \frac{11}{3}$$

Eigen Values and eigen Vectors

Let A be an n $\times$ n matrix and λ is a scalar. The matrix (A – λI) is called a characteristic equation matrix.

|A – λI| is called characteristic determinant or characteristic polynomial. The equation |A – λI| = 0 , is called as characteristic equation.

The roots of the characteristic equation are called characteristic roots of eigen value or proper values or latent roots.

The set of eigen values of a matrix 'A' is called as spectrum of A.

Solved Examples

Example: Determine the eigen values of the matrix $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

Solution: The characteristic matrix is

$$A - \lambda I = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$$

Characteristic polynomial is

$$|A - \lambda| = \begin{vmatrix} 4-\gamma & 2 \\ 2 & 4-\lambda \end{vmatrix}$$

Characteristic equation is $|A - \lambda| = 0$

$$\begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 - 4 = 0$$

$16 + \lambda^2 - 8\lambda - 4 = 0$

$\lambda^2 - 8\lambda + 12 = 0$

$\lambda = 6, 2$

Eigen Vectors: If λ is an eigen value of the matrix A, then there exists a non-zero vector X such that AX = λX. The X is called an eigen vector corresponding to the eigenvalue λ.

From Previous Example, consider the matrix

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4+2 \\ \end{bmatrix} = \begin{bmatrix} 6 \\ \end{bmatrix} = 6\begin{bmatrix} 1 \\ \end{bmatrix}$$

$\therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigen vector corresponding to the eigen value λ = 6.

Similary, $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4+4 \\ 8+2 \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$\therefore \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not an Eigen vector.

Example: Find the eigen values and eigen vectors of matrix $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

Solution: The characteristic equation is,

$$|A - \lambda| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0$$

$$-(3-\lambda)(3+\lambda) - 16 = 0$$

$$\lambda^2 - 9 - 16 = 0$$

$$\lambda^2 - 25 = 0$$

$$\lambda^2 = 25$$

$$\lambda = \pm 5$$

Case-1: When $\lambda = 5$

AX = λX

AX = λXI

AX – λXI = 0

(A – λI) X = 0

$$\begin{bmatrix} 3-\lambda & 4 \\ 4 & -3\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\lambda = 5$

$$\begin{bmatrix} -2 & 4 \\ 4 & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \end{bmatrix}$$

$-2x_1 + 4x_2 = 0 \rightarrow (1)$

$4x_1 - 8x_2 = 0 \rightarrow (2)$

Equation (2) is obtained from Equation (1) by multiplying a factor of '-2'. Thus, unique solution is impossible.

$$\frac{x_1}{x_2} = \frac{2}{1}$$

Therefore the eigen vector corresponding to eigen value λ = 5

$$X_1 = \begin{bmatrix} x \\ x_2^1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Case-2: When λ = –5

AX – λIX = 0

$$\begin{bmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\lambda = -5$

$$\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \end{bmatrix}$$

$8x_1 + 4x_2 = 0$

$4x_1 + 2x_2 = 0$

The first equation in this case can be obtained from second by multiplying a factor of '2'

i.e. $4x_1 = -2x_2$

$$x_1 = \frac{-x_2}{2}$$

$$\therefore \frac{x_1}{x_2} = \frac{-1}{2}$$

$$\therefore X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$A^T = \begin{bmatrix} 3 & 4 \\ 4 & \end{bmatrix} = $ A. Thus, the matrix A is symmetric.

For the two eigenvectors obtained,

$x_1x_1^T = 0$ & ${}^T_1x_2 = 0$. Thus, both eigen vectors are orthogonal.

Remark:

The eigen vectors corresponding to distinct eigen values of a real symmetric matrix are always orthogonal.

$$|x_1 \quad x_2| = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 4 - (-1) = 5 \neq 0 \Rightarrow \text{L.I.}$$

The eigen vectors corresponding to distinct eigen values of any square matrix are always linearly independent. The eigen vectors corresponding to repeated eigen values may be linearly independent or linearly dependent vector.

Example: Find the eigen values and corresponding eigen vectors of the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: The characteristic matrix is,

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

Characteristic equation is,

$$|A - \lambda| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 = 0$$

$\lambda = 1,1,1$ are the eigenvalues of A.

To determine the eigen-vectors, $(A - \lambda I)\, X = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since, $\lambda = 1$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 = x_3 = 0$, x_1 can be any value.

Number of equations < number of variables Infinite number of nonzero solutions, r = 2, n =3. If r < n, put n – r = 3 – 2 = 1

This means one variable can be assigned any arbitrary value.

Note: Zero vectors cannot be eigen vectors.

$\therefore$ Eigen vectors $= \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ c can be 1, 2

Corresponding to 3 repeated eigen values, there exists only 1. Linearly independent eigen vector.

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Example: Find the eigen values and corresponding eigen vectors of the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: For any upper triangular matrix, diagonal elements represent the eigen values as seen in previous example.

Thus, $\lambda = 1,1,1$

To determine the eigen-vectors, $(A - \lambda I)\, x = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} [x] = 0$$

Since, $\lambda = 1$; $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

In this case, $x_2 = 0$ but x_1 & x_3 can have any values. r < n, put n – r = 3 – 1 = 2

Two variables can be assumed as arbitrary values.

Put $x_1 = c_1$, $x_2 = 0$, $x_3 = c_2$

$$X = \begin{bmatrix} c_1 \\ 0 \\ c_2 \end{bmatrix} = c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The two linearly independent eigen-vectors are,

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \; \& \; \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Both these vectors cannot be expressed as multiple of one another. Thus, both are linearly independent. Therefore, corresponding to 3 equal eigen values, there exist two linearly independent eigen vectors.

Example: Determine the eigen values and eigen vectors of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: The characteristic equation is,

$$|A - \lambda I| = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^3$$

$\lambda = 1,1,1$

Thus, eigen values of a diagonal matrix are same as its diagonal elements.

To determine eigen vectors, $(A - \lambda I)\,X = 0$

$\lambda = 1$

$$\text{i.e.} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$r = 0,\ n = 3$

$r < n$, put $n - r = 3 - 0 = 3$

Three variables can assume arbitrary values.

$x_1 = c_1,\ x_2 = c_2,\ x_3 = c_3$

$$x = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\downarrow\ X_1} + c_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\downarrow\ X_2} + c_3 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\downarrow\ X_3}$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

The vectors are linearly independent.

There exists 3 linearly independent eigen vectors.

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Corresponding to 3 equal eigen values.

Note: If an eigen value λ is repeated m times, then the corresponding number of linearly independent eigen Vectors is given by

$P = n - r$ where $1 \le P \le m$

P = Number of linearly independent eigen vectors. n = order of matrix or number of variables.

$r = \rho(A - \lambda I)$ i.e. Rank of characteristic Matrix.

m is the number of times an eigen value is repeated.

If some eigen values are repeated and some are non-repeated, then the corresponding eigen vectors may be linearly independent or linearly dependent.

Example: The number of linearly independent eigen vector of

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Solution: Since the matrix is upper triangular, diagonal elements represent eigenvalues λ = 2,2

To determine eigen vectors, $(A - \lambda I)\,X = 0$

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & \end{bmatrix} = A - \lambda I$$

Since, $\lambda = 2$

Characteristic Matrix becomes, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$P = n - r$, where $1 \le P \le 2$

Since, there is only one non-zero row, r =1

P = 2 – 1 = 1 linearly independent eigen vector

Example: The number of LI eigen vector of

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Solution: Since, this is a diagonal matrix, the diagonal elements represent eigen values.

$\lambda = 3,3$

To determine eigen vectors, $(A - \lambda I)\,X = 0$

$$\begin{bmatrix} 3-\lambda & 0 \\ 0 & 3-\kappa \end{bmatrix}$$

Since $\lambda = 3$

Characteristic Matrix becomes, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$P = n - r = 2 - 0 = 2$

Thus, there are 2 linearly independent eigen vectors.

Properties of eigen values and eigen vectors:

Let A be on $n \times n$ matrix. The eigen values of A are $\lambda_1, \lambda_2 \ldots \lambda_n$. Trace $(A) = \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n$

Product of the eigen values = Determinant of a matrix.

- The eigen values of a upper triangle or lower triangle or diagonal or scalar or identity matrix is its diagonal elements.
- The eigen values of A and A^T are same.
- The eigen values of A and $P^{-1}AP$ (P is a non-singular matrix) are same.
- The eigen values of orthogonal matrix are unit modular i.e. they have a magnitude of 1.
- If λ is an eigen value $|\lambda| = 1$ of an orthogonal matrix, then $1/\lambda$ is also one of its eigen value.
- The eigen values of real symmetric matric are real.
- The eigen value of skew symmetric matric are purely imaginary or zeros.
- For a real matrix if a+ib is an eigen value then a-ib is also another eigen value of the same matrix.
- The eigen vector of A and A^{-1} are same.
- The eigen vectors of A and A^m are same.
- If λ is an eigen value of a matrix A then $K\lambda$ is an eigen value of kA.
  - $\frac{1}{\lambda}$ is an Eigen value of A^{-1}
  - λ^2 is an eigen value of A^2
  - λ^m is an eigen value of A^m
  - $\frac{|A|}{\lambda}$ is an Eigen value of Adj A
  - $\lambda \pm K$ is an eigen value of $A \pm KI$
  - $(\lambda \pm K)^2$ is an eigen value of $(A \pm KI)^2$
  - $\frac{1}{\lambda \pm K}$ is an Eigen value of $\left(A \pm KI\right)^{-1}$

Note:

- A matrix A is said to be singular. i.e. Det A = 0; if one of its eigen value is zero. Its converse is also true.
- If one of the eigen values of a matrix A is zero, then the homogenous system of equations has infinite number of non-zero solutions.

Solved Examples

Example: The eigen value of

$$A = \begin{bmatrix} -1 & 3 & 5 \\ -3 & -1 & 6 \\ 0 & 0 & 3 \end{bmatrix} \text{ are}$$

(A) 3, 3+5j, –6 –j

(B) –6 + 5j, 3 – j, 3+j

(C) 3 – j, 5 + j, 3 + j

(D) 3, – 1 + 3j, – 1 – 3j

Solution: $\text{Tr}\,[A] = \lambda_1 + \lambda_2 + \lambda_3 = (-1) + (-1) + 3 = 1$

Verifying the options,

Only option (d) satisfies the above equation 3, – 1 + 3j, – 1 – 3j

Example: The eigen values and eigen vectors of a 2 × 2 matrix are given by eigen value.

$\lambda_1 = 8$ and $\lambda_2 = 4$

Eigen vectors $\gamma_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\gamma_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The matrix is

(A) $\begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}$ (B) $\begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix}$

(C) $\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ (D) $\begin{bmatrix} 8 & 8 \\ 8 & 4 \end{bmatrix}$

Solution: Since Trace of matrix is same as sum of eigen values Trace = 8+4 = 12

Options (A) and (D) have same trace.

Product of eigen values is same as determinant Product = 8 x 4 =32

Only option (A) has the determinant 32.

Example: The vector $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is an Eigen vector of

$$A = \begin{bmatrix} -2 & 2 \\ 2 & 1 \\ -1 & -2 \end{bmatrix}$$. The eigen value corresponding to the eigen vector is ?

Solution: If X is the eigen-vector of matrix A then $(A - \lambda I)x = 0$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-1 - 2\times 2 + -\lambda \times -1 = 0$$

$$-1 - 4 + \lambda = 0$$

$$\lambda = 5$$

Example: For the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$, the eigen value corresponding to the eigen vector $\begin{bmatrix} 101 \\ 101 \end{bmatrix}$ is ________________ ?

Solution: For any eigenvector X,

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}\begin{bmatrix} 101 \\ 101 \end{bmatrix} = 0$$

$2 \times 101 + (4 - \lambda)\, 101 = 0$

$101\,(2 + 4 - \lambda) = 0$

$6 - \lambda = 0$

$\lambda = 6$

Example: Given $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ The eigen values of $3A^3 + 5A^2 - 6A + 2I$ is?

Solution: Since the matrix is upper triangular, diagonal elements represent eigen values.

$\lambda = 1, 3, -2$

If λ is eigen value of A, then eigen value of $3A^3 + 5A^2 - 6A + 2I$ is $3\lambda^3 + 5\lambda^2 - 6\lambda + 2$

$\lambda = 1 \Rightarrow 3(1)^3 + 5(1)^2 - 6\,(1) + 2 = 4$

$\lambda = 3 \Rightarrow 3(3)^3 + 5(3)^2 - 6 \times 3 + 2 = 110$

$\lambda = -2 \Rightarrow 3(-2)^3 + 5(-2)^2 - 6 \times -2 + 2 = 10$

Example: The eigen value of a 3 × 3 matrix are given by 1, 2, 3.

Find

(A) $Tr(A^2 + A^{-1} + adj\ A)$

(B) $Det(A^2 + A^{-1} + adj\ A)$

Solution: eigen Values of A are $\lambda = 1, 2, 3$

$$|A| = 1 \times 2 \times 3 = 6$$

Eigen Values of $A^{-1} = \frac{1}{\lambda} = 1, \frac{1}{2}, \frac{1}{3}$

Eigen Values of $A^2 = \lambda^2 = 1, 4, 9$

Eigen Values of Adj(A)= $= \frac{|A|}{\lambda} = 6, 3, 2$

(A) EigenValues of $A^2 + A^{-1} + adj\ A$

$$= \lambda^2 + \frac{1}{\lambda} + \frac{|A|}{\lambda}$$

Eigen Values = 8 + 7.5 +11.33 = 26.83

Thus, $Tr(A^2 + A^{-1} + adj\ A) = 26.83$

(B) Det $\left(A^2 + A^{-1} + adj\ A\right) = 8 \times \frac{15}{2} \times \frac{34}{3} = 680$

Example: The eigen vectors of a 2 × 2 matric $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ are given by $\begin{bmatrix} 1 \\ a \end{bmatrix}\begin{bmatrix} 1 \\ b \end{bmatrix}$ What is a + b?

Solution: Since the matrix is upper triangular, the diagonal values represent eigen values.

$\lambda = 1, 2$

Since, $(A - \lambda I)X = 0$

$$\begin{bmatrix} 1-\lambda & 2 \\ 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

For $\lambda = 1$

$$\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x_2 = 0$ ...(1)

$x_2 = 0$ ...(2)

Thus, there is only one independent variable x_1

Since only one non-zero column, r = 1, n = 2

If r < n, n–r = 2 – 1 =1

Thus, only one variable can 1 have arbitrary value.

$$x_1 = \begin{bmatrix} c \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ a \end{bmatrix} \text{or} \begin{bmatrix} 1 \\ b \end{bmatrix} a = 0 \text{ or } b = 0$$

For $\lambda = 2$, $(A - \lambda I)X = 0$

$$\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 + 2x_2 = 0$

$$\frac{x_1}{x_2} = \frac{2}{1}$$

If $x_1 = 1$, then $x_2 = \frac{1}{2}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

i.e. $b = \frac{1}{2}$

$\therefore a + b = 0 + \frac{1}{2} = \frac{1}{2}$

Method 2:

If X is the eigen vector then, AX = λX

λ = 1, 2

If $\lambda = 1 \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ a \end{bmatrix} = 1\begin{bmatrix} 1 \\ \end{bmatrix}$

$$\begin{bmatrix} 1+2a \\ 2a \end{bmatrix} = \begin{bmatrix} 1 \\ \end{bmatrix}$$

$1 + 2a = 1$

$a = 0$

If $\lambda = 2$

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ b \end{bmatrix} = 2\begin{bmatrix} 1 \\ \end{bmatrix}$$

$$\begin{bmatrix} 1+2b \\ 2b \end{bmatrix} = \begin{bmatrix} 2 \\ 2b \end{bmatrix}$$

$1 + 2b = 2$

$2b = 1$

$b = \frac{1}{2}$

$\therefore a + b = \frac{1}{2}$

Cayley Hamilton Theorem

Every square matrix satisfies its own characteristic equation using Cayley Hamilton theorem, we can find Inverse of a matrix or Higher powers of a matrix

Example: Consider matrix $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ Its characteristic equation is $\lambda^2 - 8\lambda + 12 = 0$

By Cayley Hamilton theorem we have $A^2 - 8A + 12I = 0$

Note: The constant term of any characteristic equation of any order square matrix is equal to the determinant of a matrix.

We can compute positive and negative powers of A as,

Positive Power of A	Negative Power of A
Consider A2 – 8A + 12I = 0 $A^2 = 8A - 12I$ $A^3 = 8A^2 - 12A$ $A^4 = 8A^3 - 12A^2$	$A^2 - 8A + 12I = 0$ $12I = 8A - A^2$ $I = \frac{1}{12}\left[-A^2 + 8A\right]$ $A^{-1} = \frac{1}{12}\left[-A + 8I\right]$ $A^{-2} = \frac{1}{12}\left[-I + 8A^{-1}\right]$ $A^{-3} = \frac{1}{12}\left[-A^{-1} + 8A^{-2}\right]$ ....

Table 1.1

Consider $A_{2\times 2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Characteristic equation is $|A - \lambda| = 0$

i.e. $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$

$(a-\lambda)(d-\lambda) - bc = 0$

$ad - a\lambda - \lambda d + \lambda^2 - bc = 0$

$\lambda^2 - \lambda(a + d) + (ad - bc) = 0$

$\lambda^2 - \lambda\left[Tr(A)\right] - |A| = 0$ which is the characteristic equation

According to Cayley Hamilton Theorem we have We can replace $\lambda \to A$

$A^2 - A(Tr(A)) + |A|I = 0$

Consider a 3 × 3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Characteristic equation:

$\lambda^3 - \lambda^2 [Tr(A)] + \lambda$

$\left\{ \begin{vmatrix} a_{11} & a_{12} \\ a_{22} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \right\} + |A| = 0$

Solved Examples

Example: Find A^8 if $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ using Cayley Hamilton theorem

Solution: $|A| = -1 - 4 = -5$

Characteristic equation:

$\lambda^2 - \lambda(1 + (-1)) + -5 = 0$

$\lambda^2 - 5 = 0$

By Cayley Hamilton theorem

$A^2 - 5I = 0$

$A^2 = 5I$

$(A^2)^4 = (5I)^4 = 625I = 625 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 625 & 0 \\ 0 & 625 \end{bmatrix}$

Example: The characteristic equation of a 3 × 3 matrix P is given by $\alpha(\lambda) = |\lambda I - P| = \lambda^3 + \lambda^2 + 2\lambda + 1 = 0$, where I Identity matrix. The inverse of matrix P will be ______________ ?

Solution: By cayley theorem theorem

$\lambda \to P$

$P^3 + P^2 + 2P + I = 0$

$I = -(P^3 + P^2 + 2P)$

$P^{-1}I = -(P^2 + P + 2I)$

$P^{-1} = -(P^2 + P + 2I)$

Example: Given $A = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Express A^3 as a linear polynomial?

Solution: Characteristic equation is $\lambda^2 + 5\lambda + 6I = 0$

$A^2 + 5A + 6I = 0$

$A^2 = -(5A + 6I)$ .... (1)

$A^3 = -(5A^2 + 6A) = -5(-5A - 6I) - 6A$ from (1)

$A^3 = 19A + 30I$

Practice Questions

1. If $A = \begin{vmatrix} 1 & -2 & -1 \\ 2 & 3 & 1 \\ 0 & 5 & -2 \end{vmatrix}$ and

$adj(A) = \begin{vmatrix} -11 & -9 & 1 \\ 4 & -2 & -3 \\ 10 & k & 7 \end{vmatrix}$ then $k = ?$

(A) -5

(B) 3

(C) -3

(D) 5

2. If $A^T = A^{-1}$,, where A iis a reall matriix ,, then A is

(A) Normall

(B) Symmetriic

(C) Hermiitiian

(D) Orthogonall

3. 3. If a matrix A is m x n and B is n x p, the number of multiplication operations and addition operations needed to calculate the matrix AB, respectively, are:

(A) mn^2p, mpn

(B) mpn (n-1)

(C) mpn, mp (n-1)

(D) mn^2p, (m + p) n

4. The matrices $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ & $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ commute under multiplication.

(A) If $a = b$ or $\theta = n\pi$, n is an integer

(B) Always

(C) Never

(D) If $a\cos\theta \neq b\sin\theta$

5. The determinant of the matrix

$$\begin{bmatrix} 6 & -8 & 1 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

(A) 11

(B) -48

(C) 0

(D) -24

6. If A and B are two matrices and if AB exists, ten BA exists

(A) Only if A has an many rows as B has columns

(B) Only if both A and B are square matrices

(C) Only if A and B are skew symmetric matrices

(D) Only if both A and B are symmetric

7. The inverse of the matrix $A = \begin{bmatrix} -3 & 5 \\ 2 & 1 \end{bmatrix}$

(A) $\begin{bmatrix} \frac{5}{13} & -\frac{1}{13} \\ \frac{2}{13} & \frac{3}{13} \end{bmatrix}$ (C) $\begin{bmatrix} -\frac{1}{13} & \frac{5}{13} \\ \frac{2}{13} & \frac{3}{13} \end{bmatrix}$

(B) $\begin{bmatrix} \frac{2}{13} & \frac{5}{13} \\ \frac{-1}{13} & \frac{3}{13} \end{bmatrix}$ (D) $\begin{bmatrix} \frac{1}{13} & -\frac{5}{13} \\ \frac{2}{13} & \frac{3}{13} \end{bmatrix}$

8. If A and B are real symmetric matrices of order n then which of the following is true.

(A) AAT = I

(B) $A = A^{-1}$

(C) AB = BA

(D) $(AB)^T = B^T A^T$

9. The matrix $\begin{bmatrix} 1 & -4 \\ 1 & -5 \end{bmatrix}$ is an inverse of the matrix $\begin{bmatrix} 5 & -4 \\ 1 & -1 \end{bmatrix}$

(A) True

(B) False

10. Give matrix $L = \begin{vmatrix} 2 & 1 \\ 3 & 2 \\ 4 & 5 \end{vmatrix}$ and $M = \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix}$ then $L \times M$ is

(A) $\begin{vmatrix} 8 & 1 \\ 13 & 2 \\ 22 & 5 \end{vmatrix}$ (B) $\begin{vmatrix} 6 & 5 \\ 9 & 8 \\ 12 & 13 \end{vmatrix}$

(C) $\begin{vmatrix} 2 & 13 \\ 5 & 22 \\ 6 & 2 \end{vmatrix}$ (D) $\begin{vmatrix} 9 & 4 \\ 0 & 5 \end{vmatrix}$

Answer Key

1 – (A)	2 – (D)	3 – (C)	4 – (A)	5 – (B)
6 – (A)	7 – (C)	8 – (D)	9 – True	10 – (B)

2 Calculus

Objective

Upon completion of this chapter, you will be able to:

- Determine the limit of a function.
- Check for continuity and differentiability of a function at any given point.
- Determine the derivative of any differentiable function.
- Apply derivative to determine nature of function and finding maxima and minima.
- Determine indefinite and definite integrals.
- Find area under the curve using integration.
- Determine multiple integrals.
- Apply vector calculus to determine properties like gradient, divergence and curl.
- Apply vector integral theorems to determine line and surface integrals.

Introduction

Calculus is the mathematical study of change like algebra is the study of operations and their applications to solving equations. It has two major branches differential calculus and integral calculus and both are related to each other by the fundamental theorem of calculus. calculus has widespread uses in science, engineering and economics.

Function

A function exists between A → B if Vx – ∈ A there exists a unique y ∈ B that f(x) = y. That is for a unique input there should be a unique output. There cannot be one-to-many relationship between input and output of a function.

The functions can be classified into two broad categories:

Explicit function

If the dependent variable 'y' is directly expressed in terms of the independent variable in terms of a mathematical expression.

Example: $y = x(x - 2)$

In other words, a relation of the form $y = f(x)$ exists.

Implicit function

If the dependent and independent variables cannot be separated from each other. Then it is termed as an implicit function. It can be expressed in the form $\phi\,(x, y) = C$

Example: $x^2 + xy + y^2 = 1$

Composite function

If a variable is dependent on more than one variable which itself can be represented as a function of some other variable. Then, it is termed as a composite function.

In other words, $z = f\,(x, y)$ where $x = g\,(t)$ and $y = h\,(t)$

Some special functions

Even function:

A function f(x) is said to be even function of x if $f(x) = f(-x)$. This means that function is symmetrical about y-axis.

Example: $|x|$

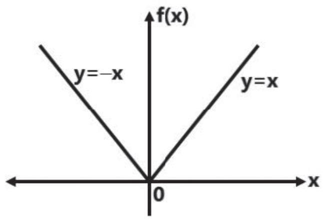


Fig. 2.1

Odd function:

A function f(x) is said to be odd function of x if $f(x) = -f(-x)$. This means that function is symmetrical in 1st and 3rd Quadrants.

Example: sin x

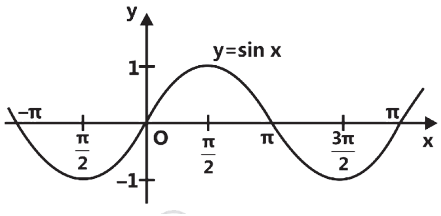


Fig. 2.2

Modulus function

Modulus of any number yields the magnitude of the number regardless of the sign.

It is defined as $f(x) = |x| = \begin{cases} x & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -x & \text{for } x < 0 \end{cases}$

The curve is not differentiable at x = 0.

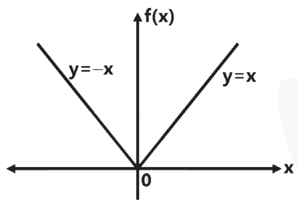


Fig. 2.3

Greatest integer function or step or bracket function

This function yields the highest integer value but less than the input real number.

It is represented as f(x) = [x] = n $\in$ Z, where n $\leq$ x $\leq$ n+1

Example: [7. 2] = 7, [7.999] = 7, [7] = 7, [-1.2]=−2

It can be plotted as shown below,

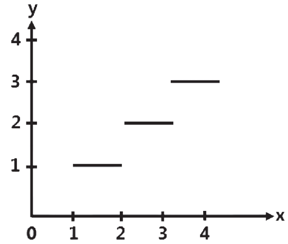


Fig. 2.4

The wave is a discontinuous function at every integer point (1, 2, ... n). So, it is not differentiable

Symmetric Properties of curves

Let f(x, y) = C be the equation of the curve. This is an implicit function in x and y.

1. If f(−x, y) = f(x, y), then it is symmetric about y axis

   Example: $x^2 - 4ay = 0$

   $f(-x, y) = x^2 - 4ay$

2. If f(x, y) = f(y, x) then the curve is symmetric about the line y=x

   Example: $x^3 + y^3 - 3axy = 0$

Limit of a function

Let f(x) be defined in deleted neighborhood of a $\in$ R, then $| \in$ R is said to be the limit of f(x) as 'x' approach 'a' for given $\in$ > 0 three exists δ > 0 such that f(x) $|-|<\in|$ whenever $|-x|$ a $\delta|$

Where a $\in$R, δ >0

Left and Right Hand Limit

When x tends to 'a' from the left side i.e. x<a and x → a Then, lim f(x) = lim f(a −h)

$x \to a^-$ h→0

When x tends to 'a' from the right side i.e. x>a and x→a Then, lim f(x) = lim f(a+h)

$x \to a^+$ h→0

When both the limits are equal, then limit of the function is equal to both the limits. If both the limits are unequal then the limit of the function is said does not exist.

Indeterminate Forms

If the limit of a function results in a form that is undefined, it is an indeterminate form. Some of the indeterminate forms are

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 0^0, 1^\infty \text{ and } \infty^0$$

- In such cases, we use L'hospital's rule if the indeterminate form is $\left[\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right]$

  $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'x}{g'(x)}$$

  where limf(x) = 0 and limg(x) = 0

  x→a x→a⁺

  Thus, limit of the function is the ratio of derivatives of both numerator and denominator.

- If the limit is of the form,
limf(x)g(x) where limf(x) = 0 and limg(x) = ∞.
x→a x→a x→a

This can be converted to previous form by taking $\phi(x)=\frac{1}{g(x)}$

$$\text{Thus, } \lim_{x\to a}(x)h(x)=\lim_{x\to a}\frac{f(x)}{\phi(x)}$$

where limf(x) = 0 and limϕ(x) =0
x→a x→a

- If the limit is of the form,

$$y=\lim_{x\to a}[\lfloor fx\rfloor]^{g(x)} \text{ where} \lim_{x\to a} f(x)=0$$

$$\text{or } 1\infty \text{ and} \lim_{x\to a} g(x)=0 \text{ or } 1 \text{ or } \infty$$

Then, the limit has an indeterminate form as, 0° , 1^∞ and ∞°.

This can be converted to previous form by taking

$$\log y=\lim_{x\to a}\left(g(x)\log[\lfloor fx\rfloor]\right)$$

Standard Limits

1. $\lim_{x\to a}\frac{x^n-a^n}{X-a}=na$
2. $\lim_{x\to 0}\frac{e-1}{x}=m$
3. $\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a$
4. $\lim_{x\to 0}[1+ax]^x=e^a$
5. $\lim_{x\to 0}\left[1+\frac{a}{x}\right]^x=e^a$
6. $\lim_{x\to 0}\left[\frac{\sin x}{x}\text{on}\frac{\tan x}{x}\right]=1$
7. $\lim_{x\to 0}\left[\frac{\sin mx}{x}\right]=m$
8. $\lim_{x\to\infty}\frac{\sin x}{x}=0$
9. $\lim_{x\to 0}\frac{a^x+b^x}{2}=\sqrt{ab}$
10. $\lim_{x\to 0}[\cos x+a\sin bx]^{1/x}=e^{ad}$
11. $\lim_{x\to a}\left[\frac{1-\cos-ax}{x}\right]=a\frac{2}{2}$

Solved Examples

Example: Determine $\lim_{x\to 0}\left[\frac{1-\cos 3x}{x\sin 2x}\right]$

Solution: If we substitute x = 0 then,

$$\lim_{x\to 0}\left[\frac{1-\cos 3x}{x\sin 2x}\right]=\left(\frac{0}{0}\right)\text{Form}$$

So, applying L'Hospital's rule

$$\lim_{x\to 0}\left[\frac{3\sin 3x}{\sin 2x+2x\cos 2x}\right]=\left(\frac{0}{0}\right)$$

So, applying L'Hospital's rule again

$$\lim_{x\to 0}\left[\frac{+9\cos 3x}{2\cos 2x+2\cos x-4x\sin 2x}\right]$$

$$=\left(\frac{9(1)}{2(1)+2(1)-4(0)}\right)=\frac{9}{4}$$

(OR) By using standard limit 11 and 8,

$$\lim_{x\to 0}\left[x^2\left(\frac{\sin 2}{x}\right)\right]=\left(\frac{1}{2}\right)=\frac{9}{4}$$

Example: Determine $\lim_{x\to 1}\frac{\cos\frac{\pi x}{2}}{1-\sqrt{x}}$

Solution: If we directly substitute x =1. Then,

$$\lim_{x\to 1}\frac{\cos\frac{\pi x}{2}}{1-\sqrt{x}}=\left(\frac{0}{0}\right)\text{Form}$$

Thus, applying L'Hospital's Rule,

$$\lim_{x\to 1}\frac{\cos\frac{\pi x}{2}}{1-\sqrt{x}}=\lim_{x\to 1}\frac{-\frac{\pi}{2}\sin\frac{\pi x}{2}}{-\frac{1}{2\sqrt{x}}}-\pi$$

Example: if $\lim_{x\to 0}\frac{\sin 2x+a\sin x}{x^3}=$ finite, than a = ______?

Solution: If we directly substitute x = 0. Then,

$$\lim_{x\to 0}\frac{\sin 2x+a\sin x}{x^3}=\left(\frac{0}{0}\right)$$

Applying L'Hospital's rule,

$$\lim_{x\to 0}\frac{2\cos 2x+a\cos x}{3x^2}=\frac{2+a}{0}=\text{finite (given)}$$

Thus, 2 + a = 0 ⇒ a = – 2 which will result in indeterminate form and we can then apply L'Hospital's rule which will result in a finite limit.

Example: Determine $\lim_{x\to 0}\frac{\log\sin 2x}{\log(\sin x)}=$ ________?

Solution: If we directly substitute, x = 0. Then,

$$\lim_{x\to 0}\frac{\log\sin 2x}{\log(\sin x)}=\frac{\infty}{\infty}$$

Applying L'Hospital's rule

$$\lim_{x\to 0}\frac{\log\sin 2x}{\log(\sin x)}=\lim_{x\to 0}\frac{\frac{1}{\sin 2x}\times\sin 2x}{\frac{1}{\sin x}\times\cos x}=\lim_{x\to 0}\frac{2\cos x}{2\cos x}=1$$

Example: Determine $\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{\log(\cos x)}$

Solution: If we directly substitute,

$x=\frac{\pi}{2}$. Then, $\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{\log(\cos x)}=\frac{\infty}{\infty}$

Applying L'Hospital's rule

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{\log(\cos x)}=\lim_{x\to\frac{\pi}{2}}\frac{\sec^2 x}{\sec x(-\sin x)}=$$

$$\lim_{x\to\frac{\pi}{2}}\frac{-1}{\sin x\cos x}=\frac{-1}{1\times 0}=-\infty$$

Example: Find $\lim_{x\to 0}x^2\log x$

Solution: If we directly substitute x = 0. Then,

$$\lim_{x\to 0}x^2\log x=0\times\infty$$

Converting the following limit to $\frac{\infty}{\infty}$ form

$$\lim_{x\to 0}x^2\log x=\lim_{x\to 0}\left[\frac{\log x}{\frac{1}{x^2}}\right]=\frac{\infty}{\infty}$$

Applying L'Hospital's rule,

$$\lim_{x\to 0}\left[\frac{\log}{\frac{1}{x^2}}\right]\lim_{x\to 0}\left(\frac{\frac{1}{x}}{\frac{-2}{x^3}}\right)=\lim_{x\to 0}\left(\frac{-x^2}{2}\right)=0$$

Example: Determine $\lim_{x\to 1}(x-1)\tan\frac{\pi x}{2}$?

Solution: If we directly subs titute x = 1. Then,

$$\lim_{x\to 1}(x-1)\tan\frac{\pi x}{2}=0\times\infty$$

Converting the following limit to $\frac{0}{0}$ form

$$\lim_{x\to 1}(x-1)\tan\frac{\pi x}{2}=\lim_{x\to 1}\frac{x-1}{\cot\frac{\pi x}{2}}\left(\frac{0}{0}\right)$$

Applying L'Hospital's rule

$$\lim_{x\to 1}(x-1)\tan\frac{\pi x}{2}=\lim_{x\to 1}\frac{x-1}{-\operatorname{cosec}^2\frac{\pi}{2}\left(\frac{\pi}{2}\right)}=-\frac{\pi}{2}$$

Example: Determine $\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{\tan x}\right]=?$

Solution: $\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{\tan x}\right]=\lim_{x\to 0}\left[\frac{\tan x-x}{x\tan x}\right]$

If we directly substitute x = 0. Then, this results in,

$$\lim_{x\to 0}\left[\frac{\tan x-x}{x\tan x}\right]=\frac{0}{0}$$

$$\lim_{x\to 0}\left[\frac{\tan x-x}{x\tan x}\right]=\lim_{x\to 0}\frac{\tan x-x}{\left(\frac{\tan x}{x}\right)}$$

$$\lim_{x\to 0}\frac{\tan x-x}{x^2}\times\lim_{x\to 0}\left(\frac{1}{\frac{\tan x}{x}}\right)=\lim_{x\to 0}\frac{\tan x-x}{x^2}\times 1$$

Applying L'Hospital's rule,

$$\lim_{x\to 0}\left[\frac{\tan x-x}{x\tan x}\right]\lim_{x\to 0}\left[\frac{sex^2\times x-1}{1}\right]=0$$

Applying L'Hospital's rule,

$$\lim_{x\to 0}\left[\frac{\tan x-x}{x\tan x}\right]\lim_{x\to 0}\frac{2\sec x(\sec x\tan x)}{2}=0$$

Example: Determine $\lim_{x\to 0}\left|\frac{x}{x}\right|$?

Solution: To find left hand limit,

$$\text{LL}\lim_{x\to 0}f(0-h)=\lim_{x\to 0}\left|\frac{h}{h}\right|=\lim_{x\to 0}\left|\frac{h}{-h}\right|=-1$$

To find right hand limit,

$$RL \lim_{x\to 0} f(0+h) = \lim_{x\to 0}\left|\frac{h}{h}\right| = \lim_{x\to 0}\left|\frac{h}{-h}\right| = 1$$

Since, both limits are unequal. Thus, $\lim_{x\to 0}\left|\frac{x}{x}\right|$ does not exist.

Example: $\lim_{x\to a}$ does not exist, when a is ?

(A) Integer

(B) Real number

(C) Rational number

(D) All of them

Solution: At any integer, we can determine the limit as, left hand limit is,

$LL \lim_{x\to a^-}[x] = a-1$

Right Hand Limit is, $\underset{x\to a}{Lt} f(x) = f(a)$

$RL \lim_{x\to a^-}[x] = a$

Since, both limits are unequal. Hence, limit does not exist for integers.

Example: Find $\lim_{x\to 0^-}{}^{\sin x}$?

Solution: If we directly substitute x = 0. Then,

$\lim_{x\to 0^-}{}^{\sin x} = 0^0$

let $y = x^{\sin x}$

$\log y = \sin x (\log x)$

$$\lim_{x\to 0}[\log y] = \lim_{x\to 0^-}\left[\frac{\sin x \log x}{x}\right]$$

$$\log\left[\lim_{x\to 0} y\right] = \lim_{x\to 0}\left(\frac{\log x}{\operatorname{cosec} x}\right) = \left(\frac{\infty}{\infty}\right)$$

Applying L'Hospital's rule,

$$\lim_{x\to 0} x^{\sin x} \lim_{x\to 0}\frac{1}{-\operatorname{cosec} x \cot x} \lim_{x\to 0} -\left[\frac{-\sin x}{x}\times \tan x\right]$$

$= -1\times 0 = 0$

$\log\left[\lim_{x\to 0} y\right] = 0 => \lim_{x\to 0} y = 1$

Continuity of a Function

- The function is said to be continuous at a point x=a if
- A function is said to be continuous in a interval (a, b) if it satisfied the condition

  (A) f(x) is continuous at -Vx ∈ (a, b)

  (B) $\underset{x\to a^+}{Lt} f(x) = f(a)$

  (C) $\underset{x\to a^-}{Lt} f(x) = f(b)$

This means that function should be continuous at any point in the open interval but it must be right side continuous at lower limit and left side continuous at upper limit.

Solved Examples

Example: Check the continuity of the function

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{for } x \neq 2 \\ & \text{at } x \neq 2 \\ 0 & \text{for } x = 2 \end{cases}$$

Solution: Given $\lim_{x\to 2} f(x) = \lim_{x\to 2}\left[\dfrac{f2(2=0)}{\dfrac{\dfrac{x-4}{x-2}}{x-4}}\right] = \dfrac{0}{0}$

Applying L'Hospital's rule

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\left[\frac{t-4}{x-2}\right] = \lim_{x\to 2} -4 \neq f(2)$$

∴ f(x) is discontinuous at x=2

Example: Determine the continuity of

$$f(x) = \begin{cases} 1 & \text{for } x \neq 0 \\ (1+3x)x^- & \text{at } x = 0\,? \\ e^3 & \text{for } x = 0 \end{cases}$$

Solution: Given $f(0) = e^3$

Determining the limit of the function at x = 0

$$\lim_{x\to 2}(1+3x)^{\frac{1}{x}} = e^3 = f(0)$$

∴ The function is continuous at x=0

Example: Let $f(x) = \begin{cases} \dfrac{\sin[x]}{[x]} & \text{for } [x] \neq 0 \\ 0 & \text{for } [x] = 0 \end{cases}$

The reason for f(x) to be discontinuous at x=0 is?

(A) f(0) is not defined

(B) f(0) is defined but $\underset{x\to 0}{\text{Lt}}\, f(x)$ does not exist

(C) $\underset{x\to 0}{\text{Lt}}\, f(x)$ exists, f(0) is defined

(D) F(x) is discontinuous at x=0, because

$$\underset{x\to 0}{\text{Lt}}\, f(x) \neq f(0)$$

Solution:

$$f(x) = \begin{cases} \dfrac{\sin(-1)}{-1} & \text{for } -1 \le x \le 0 \quad [x] = -1 \\ 0 & \text{for } -1 \le x < 0 \quad [x] = 0 \end{cases}$$

Thus, f(0) = 0

The left hand limit is sin 1 and right hand limit is 0.

Since, both limits are unequal. The limit does not exist. Hence, the function is discontinuous at x = 0. Hence (D) is correct.

Differentiability

A function f(x) is said to be differentiable at x=c, if $f'(c) = \lim\limits_{x\to c}\left[\dfrac{f(x)-f(c)}{x-c}\right]$ exists and is finite.

The left hand and right hand derivatives are given as,

$$\text{LHD} = \lim_{h\to 0}\left[\frac{f(x-h)-f(c)}{-h}\right]$$

$$\text{RHD} = \lim_{h\to 0}\left[\frac{f(c+h)-f(c)}{h}\right]$$

A function f(x) is said to be differentiable if the LHD, RHD exists and are finite and equal.

1. A function f is said to be differentiable in an open interval (a, b), if it is differentiable at each point of the open interval.
2. A function $f : [[a, b]] \to R$ is said to be differentiable in closed interval [a, b] if it is
   a) Differentiable from right at a [i.e. RHD exists] and
   b) Differentiable from left at b [i.e. LHD exists] and
3. Differentiable in the open interval (a, b)

Note: If a function is differentiable at any point, then it is necessarily continuous at that point but the converse is not true.

Solved Examples

Example: Let f(x) = x |x| where x ∈ R, then f(x) at x = 0 is

(A) Continuous and differentiable

(B) Continuous but not differentiable

(C) Differentiable but not continuous

(D) Neither differentiable nor continuous

Solution: The function can be expanded as,

$$f(x) = \begin{cases} x^2 & \text{for } x > 0 \\ -x^2 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases}$$

The left hand limit is LL = – 0^2 = 0

Right Hand Limit is RL = 0^2 = 0

f(0) = 0

Since LL = RL = f(0), the function is continuous at x = 0.

Calculating left hand and right hand derivatives,

$$\text{LHD} = \lim_{h\to 0}\left[\frac{-(-h)^2 - 0}{-h}\right] = 0$$

$$\text{RHD} = \lim_{h\to 0}\left[\frac{(+h)^2 - 0}{h}\right] = \lim_{h\to 0} h = 0$$

Since both derivatives are equal, the function is differentiable at x = 0.

Example: Check the continuity and differentiability of $f(x) = \dfrac{1}{1+|x|}$ at $x = 0$

Solution: We can express the function as,

$$f(x) = \begin{cases} \dfrac{1}{1+x} & \text{for } x > 0 \\ \dfrac{1}{1-x} & \text{for } x > 0 \\ 1 & \text{for } x = 0 \end{cases}$$

The left hand limit is, $LL = \lim_{x\to 0}\frac{1}{1-x} = \frac{1}{1-0} = 1$

The right hand limit is, $RL = \lim_{x\to 0}\frac{1}{1+x} = \frac{1}{1+0} = 1$

Since LL = RL = f(0), the function is continuous at x = 0. Calculating the Left Hand Derivative,

$$LHD = \underset{h\to 0}{\text{it}}\left[\frac{\frac{1-1}{1-(-h)}}{-h}\right] = \underset{h\to 0}{\text{it}}\frac{-h}{-h(1+h)} = 1$$

Calculating Right Hand Derivative,

$$RHD = \underset{h\to 0}{\text{it}}\frac{\frac{1}{(1+h)}}{h} = \underset{h\to 0}{\text{it}}\frac{-h}{h(1+h)} = -1$$

Since RHD $\neq$ LHD , the function is not differentiable at x = 0.

Mean Value Theorems

Rolle's theorem

Let f(x) be defined as [a, b], such that

1. f(x) is continuous in [a, b]
2. f(x) is differentiable in (a, b)
3. f(a) = f(b)

Then there exists at least one point c $\in$ (a, b) such that f'(c) = 0.

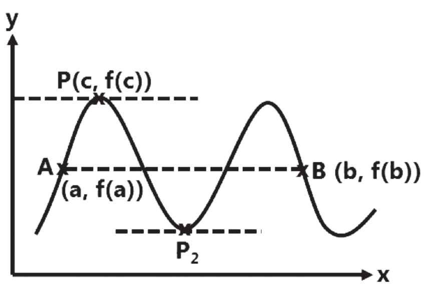


Fig. 2.5

Line joining A & B is parallel to x-axis. Since the function is differentiable so the Rolle's Theorem asserts that at some point in the interval, the derivative will go to zero.

Note: The converse of Rolle's theorem is not true that means if derivative goes to zero at some point in an interval, then it does not necessarily mean that function has same value at both end points.

Lagrange's mean value theorem

Let f(x) be defined in [a, b] such

1. f(x) is continuous in [a, b]
2. f(x) is differentiable in (a, b)

Then there exists at least one point

c $\in$ (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

In graphical form it can be represented as shown below,

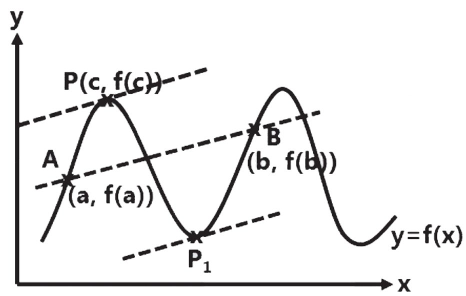


Fig. 2.6

Note: The converse of Lagrange Mean value Theorem may not be true.

Applications of LMVT

- If a function f(x) is
  - Continuous in [a, b]
  - Derivable in (a, b) and
  - f'(x) > 0 for all x in (a, b), then f(x) is strictly increasing function in [a, b]
- If a function f(x) is
  - Continuous in [a, b]
  - Derivable in (a, b) and
  - f'(x) > 0 for all x in (a, b), then f(x) is strictly decreasing function in [a, b]
- If the derivative of the function is constantly zero in the entire interval, then the function is constant in that interval.

Solved Examples

Example: The mean value c for the function

$f(x)^{x}\,[\sin x - \cos x]\text{in}\left[\frac{\pi}{4},\frac{4\pi}{4}\right]$ is ________.

Solution: The derivative of the function f(x) is given by,

$$f'(x) = e^x[[\sin x - \cos x]] + e^x[[\cos x + \sin x]]$$
$$= 2e^x \sin x$$

Value of f(x) at end points of the interval is,

$$f\left(\frac{\pi}{4}\right) = 0, f\left(\frac{5\pi}{4}\right) = 0$$

By Rolle's Theorem there exists $c \in \left(\frac{\pi}{4}, \frac{5}{4}\right)$ such that f'(c) =0

i.e. $2e^c \sin c = 0$

sin c = 0

$$c = 0, \pm\pi, \pm 2\pi, ...$$

$$c = \pi \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$$

Example: The mean value c for the function

f(x) = $f(x) = \sqrt{x^2 - 4}$ in the interval (2, 4) is?

$$\text{Then, } f'(x) = \frac{1}{2\sqrt{x^2 - 4}} \times 2 \times \frac{x}{\sqrt{x^2 - 4}}$$

f'(x) finite for –V (2, 4)

Computing the function values at end points of interval

$$f(2) = \sqrt{2^2 - 4} = 0$$

$$f(4) = \sqrt{12}$$

By Lagrange's theorem, there exists c ∊ (2, 4) such that

$$f'(c) = \frac{f(4) - f(2) 4 - 2}{4 - 2}$$

$$\frac{c}{\sqrt{c^2 - 4}} = \frac{\sqrt{12} - 0}{2}$$

$$\frac{c}{\sqrt{C^2 - 4}} = \sqrt{3}$$

$$c^2 = 3c^2 - 12$$

$$2c^2 = 12$$

$$c^2 = \sqrt{6} \in (2, 4)$$

Example: Find the value of ε such that

$$f(b) - f(a) = (b - a)\ f'(\varepsilon) \text{ for } f(x) = Ax^2 + Bx + C \text{ in } [a, b]$$

Solution: Any polynomial function is continuous and differentiable everywhere.

$f(x)=Ax^2+Bx+C$ is continuous and differentiable in [a, b].

$$f'(x) = 2Ax + B$$

$$f'(\varepsilon) = \frac{f(b) - f(a)}{b - a}$$

$$2A\varepsilon + B = \frac{f(b) - f(a)}{b - a} =$$

$$\frac{(Ab^2 + Bb + C) - (Aa^2 + Ba + C)}{b - a} =$$

$$\frac{A(b^2 - a^2) + B(b - a)}{b - a}$$

$$2A\varepsilon + = (b + a)A + B$$

$$\varepsilon = \frac{b + a}{2}$$

Example: The value of c ∊ (1 , e) for the function f(x) = logx in the interval [1, e] using Lagrange's mean value

theorem is ___________ ?

Solution: The derivative of function f(x) is,

$$f(x) = \frac{1}{x}$$

The value of function at interval end points is,

f(1) = 0, f(e) = 1

By Lagrange's Mean Value theorem

$$c \in (1, e), f'(c) = \frac{f(e) - f(1)}{e - 1}$$

$$\frac{1}{c} = \frac{1}{e = 1}$$

$$c = e - 1$$

Example: Lagrange's MVT cannot be applied

to $f(x) = 2(x - 1)^{\frac{1}{3}}$ in the interval [0,2] because?

(A) f(x) is not continuous in [0, 2]

(B) f(x) is not differentiable in (0, 2)

(C) f(0) ≠ f(2)

(D) both a ∊ b

Solution: The derivative of the function f(x) is given by,

$$f'(x) = 0 + \frac{2}{3}(x-1)^{\frac{1}{3}} = \frac{2}{3(x-1)^{\frac{1}{3}}}$$

$$f'(1) = \infty$$

Thus, f(x) is not differentiable in (0, 2) and it violates the condition of Lagrange Mean Value theorem. Thus, LMVT cannot be applied.

Cauchy's mean value theorem

Let f(x) and g(x) be defined in [a, b] such that

1. f(x) and g(x) are continuous in [a, b]
2. f(x) and g(x) are differentiable in (a, b)
3. $g'(x) \neq 0 \ \forall \ x \in (a, b)$

Then there exists at least a point $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Solved Examples

Example: The mean value 'c' for the function

$$f(x) - \frac{1}{x}c, g(x) = \frac{1}{x^2} \text{ in } [1,2] \text{ is}$$

Solution: The derivative of the function f(x) is, $f'(x) = \frac{1}{x^2}$

$$g'(x) = \frac{-2}{x^3} \neq 0 \ \forall \ x \in (-1, 2)$$

By Cauchy's Mean Value theorem,

$$\frac{f'(c)}{g'(c)} = \frac{f(2)-f(1)}{g(2)-g(1)}$$

$$\frac{-\frac{1}{c^2}}{\frac{2}{c^3}} = \frac{\frac{1}{2}-1}{\frac{1}{4}-1}$$

$$\frac{c}{2} = \frac{+\frac{1}{2}}{+\frac{3}{4}}$$

$$c = \frac{4}{3}$$

Example: The mean value 'c' for the functions f(x) = sinx, g(x) = cosx in $\left[-\frac{\pi}{2}, 0\right]$ is

Solution: Derivatives of both functions are,

$$f'(x) = \cos x, g'(x) = -\sin x \neq 0 x \in \left(\frac{\pi}{2}, 0\right)$$

By Cauchy's Mean Value theorem,

$$\frac{\cos}{-\sin c} = \frac{0+1}{1-0}$$

$$\frac{-\cos c}{\sin c} = 1$$

$$\tan c = -1$$

$$c = -\frac{\pi}{4} \in \left(-\frac{\pi}{2}, 0\right)$$

Rules of Differentiation

$(f+g)' = f' + g'$	(Sum rule)
$(f-g)' = f' - g'$	(Difference rule)
$(f.g)' = fg' + gf'$	(Product rule)
$\left(\frac{f}{g}\right) = \frac{gf' - fg'}{g^2}$	(Quotient rule)
$\frac{d}{dx}(f(g(x))) = \frac{df}{dg} \times \frac{dg}{dx}$	(Chain rule)

These rules are applicable when y is an explicit function of x.

Some of the common derivatives are,

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
$\log_a x$	$\log_a e.\left(\frac{1}{x}\right)$
e^x	e^x
a^x	$a^x \log_e a$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec^2 x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sinh x$	$\cosh x$

$f(x)$	$f'(x)$
$\cosh x$	$\sinh x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\operatorname{cosec}^{-1} x$	$-\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\sec^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\cot^{-1}$	$-\frac{1}{1+x^2}$
$\lvert x\rvert$	$\frac{x}{\lvert x\rvert}(x \neq 0)$

Differentiation by Substitution

For complicated functions, we make substitution and then apply chain rule to determine the derivative. If the function contains an expression of the form

1. $a^2 - x^2$, put x = asint or x = acost
2. $a^2 + x^2$, put x = atant or x = acot t
3. $x^2 - a^2$, put x = asect or x = acosect
4. $\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$, put $x = a\cos t$
5. $a\cos x \pm b\sin x$, put $a = r\cos\theta$ and $b = r\sin\theta$, $r > 0$

Solved Examples

Example: Differentiate the following function (by suitable substitutions) w.r.t. x.

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right), \text{put } x = \tan\theta \text{ i. e. } \theta = \tan^{-1} x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2.\frac{1}{1+x^2} = \frac{2}{1+x^2}$$

Example: Find $\frac{dy}{dx}$ when $x^2 + xy + y^2 = 100$

Solution: Given, $x^2 + xy + y^2 = 100$

Differentiating both sides w.r.t x, we get

$$2x + \left(x\frac{dy}{dx} + y.1\right) + 2y\frac{dy}{dx} = 0$$

$$(x+2y)\frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx} = -\frac{2x+y}{x+2y}$$

Example: if $X^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, find $\frac{dy}{dx}$

Solution: Given $X^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Differentiating both sides of w.r.t. x, regarding y as a function of x, we get

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}, \frac{dy}{dx} = 0$$

$$\frac{1}{x^{\frac{1}{3}}} + \frac{1}{y^{\frac{1}{3}}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} = -\sqrt[-3]{\frac{y}{x}}$$

Example:

if $y = \sqrt{\cos x\sqrt{\cos x + \sqrt{\cos x + ...\text{to}^{\infty}}}}$,

prove that $(1-2y)\frac{dy}{dx} = \sin x$

Solution:

$$\text{Given}, y = \sqrt{\cos x + y}$$

$$y^2 = \cos x + y$$

$$y^2 - y = \cos x$$

Differentiating w.r.t. x, we get

$$2y\frac{dy}{dx} - \frac{dy}{dx} = -\sin x$$

$$(1-2y)\frac{dy}{dx} = \sin x$$

Example: Differentiate the function f(x) = x^x

Solution: Let x y = x^2

Taking logarithm of both sides, we get

logy = xlogx

Differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = x\frac{1}{x} + \log x.1$$

$$\frac{dy}{dx} = y\left(1+\log x\right) = x^x\left(1+\log x\right)$$

Example: if $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{\left(1+\log x\right)^2}$

Solution: $x^y = e^{x-y}$ taking logarithm of both sides, we get

$$y\log x = \left(x - y\log e = x - y\right)$$

$$y + y\log x = x$$

$$\left(1+\log x\right)y = x$$

$$y = \frac{x}{1+\log x} \text{ differentiating w.r.t. x, we get}$$

$$\frac{dy}{dx} = \frac{\left(1+\log\right).1 - x\left(0+\frac{1}{x}\right)}{\left(1+\log x\right)} = \frac{1+\log x - 1}{\left(1+\log x\right)^2}$$

$$= \frac{\log x}{\left(1+\log x\right)^2}$$

Parametric Differentiation

If x and y are two variables such that both are explicitly expressed in terms of a third variable, say t, i.e., if x=f (t) and y =g (t) then such functions are called parametric functions and the third variable is called the parameter.

In order to find the derivative of a function in parametric form, we use chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \left(\text{provide } \frac{dx}{dt} \neq 0\right)$$

Solved Examples

Example: If x = a (t + sint), y = a (1 – cost),

find $\frac{dy}{dx}$ at $t = \frac{\pi}{2}$

Solution: Given, x = a (t + sint), y = a (1 -cost)

Differentiating both w.r.t 't', we get

$$\frac{dx}{dt} = a\left(a + \cos t\right)$$

$$\text{And } \frac{dy}{dt} = a\left(0 - \left(-\sin t\right)\right) = a\sin t$$

$$\text{We know that } \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{a\sin t}{a\left(1+\cos t\right)} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\cos^2\frac{t}{2}} = \tan\frac{t}{2}$$

$$\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{2}} = \tan\frac{\pi}{4} = 1$$

Example: Differentiate $\frac{x^3}{1-x^3}$ w.r.t. x^3

Solution: Let w.r.t.x^3

$$\frac{dy}{dx} = \frac{\left(1-x^3\right)3x^2 - x^3.\left(0-3x^2\right)}{\left(1-x^3\right)} = \frac{3x^2}{\left(1-x^3\right)^2}$$

$$\text{And } \frac{dz}{dx} = 3x^2$$

$$\text{Therefore, } \frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{1}{\left(1-x^3\right)^2}$$

Increasing and Decreasing functions

Let f be a real valued function defined in an interval D (a subset of R), then f is called an increasing function in an

interval D_1 (a subset of D) if

For all $x_1, x_2 \in D_1$

$$X_1 < X_2 \Rightarrow f\left(x_1\right) \leq f\left(x_2\right)$$

In other words, for an increasing function the function value f(x) increases.

And f is called a strict increasing function (or monotonically increasing function) in D_1 if for all

$x_1, x_2 \in D_1$

$$X_1 < X_2 \Rightarrow f\left(x_1\right) < f\left(x_2\right)$$

Analogously, f is called a decreasing function in an interval D_2 (a sunset of D) if for all

$x_1, x_2 \in D_2$

$X_1 < X_2 \Rightarrow f(x_1) \geq f(x_2)$

And f is called a strict decreasing function (or monotonically decreasing function) in 2 D if for all

$x_1, x_2 \in D_2$

$X_1 < X_2 \Rightarrow f(x_1) > f(x_2)$

Conditions for an Increasing or a Decreasing function

Applying Lagrange's Mean Value theorem,

$$f'(x) = \frac{f(x_2) - f(x_1)}{x_2 = x_1} \quad \text{where}\, x \in (x_1, x_2)$$

For an increasing function,

$f(X_1) \leq f(x_2)$. Thus, f '(x) ≥ 0 for the domain D_1

For a strictly increasing function,

$f(X_1) < f(x_2)$. Thus, f '(x) > 0 for the domain D_1

For a decreasing function,

$f(X_1) \geq f(x_2)$. Thus, f'(x) ≤ 0 for the domain D_1

For a strictly decreasing function

$f(X_1) > f(x_2)$. Thus, f'(x) < 0 for the domain D_1

This can be summarized as,

Theorem 1: If a function f is continuous is [a, b], and derivable in (a, b) and

1. f'(X) ≥ 0 for all x ∈ (a,b) , then f is increasing in [a, b]
2. f'(X) > 0 for all x ∈ (a,b), then f is strict increasing in [a, b]

Theorem 2: If a function f is continuous in [a, b] and derivable in (a, b) and

1. f'(X) ≤ 0 for all x in (a,b), then f(x) is decreasing in [a, b]
2. f'(X) < 0 for all x ∈ (a,b) then f(x) is strict decreasing in [a, b]

Solved Examples

Example: Determine the nature of the function $f(x) = \frac{3}{x} + 8$?

Solution: Let $f(x) = \frac{3}{x} + 8, D_f = R - [0]$

Diff. it w.r.t. x, we get

$$f'(x) = 3.(-1.x^{-2}) + 0 = -\frac{3}{X^2}$$

Since $x^2 > 0$ for all X∈R, x ≠ 0.

Therefore, f' (x) < 0 for all x∈R, x ≠ 0, i.e. for all X ∈ D_1

Thus, the given function is strictly decreasing.

Example: Determine the nature of the function $f(x) = \frac{e^x}{1+e^x}$

Solution: Differentiating the function we get,

$$f'(x) = \frac{e_x(1+ex) - e^{2x}}{(1+e^x)} = \frac{e^x}{(1+e^x)^2}$$

Since x e is positive for all values of x, f' (x) is positive for all values of x and hence f(x) monotonically increases.

Local Maxima and Minima

A function f(x) is said to be a local or relative maximum at x=a, if these exist positive number δ such that f(a + δ) < f (a) and f (a - δ) < f (a). In other words, function values on either side of maxima point must be less than the function value at that point.

A function f(x) is said to be a local or relative minimum at x = a, if there exists a positive number δ such that f(a + δ) > f(a) and f (a - δ) > f (a). In other words, function values on either side of minima point must be more than the function value at that point.

Properties of Relative Maxima and Minima

1. At least one maximum or one minimum must be lie between two equal values of a function.

2. Maximum and minimum values must occur alternatively.
3. There may be several maximum or minimum values of same function.
4. A function y = f(x) is maximum at x=a, if $\frac{dy}{dx}$ changes sign from +ve to –ve as x passes through a.
5. A function y = f(x) is maximum at x=a, if $\frac{dy}{dx}$ changes sign from -ve to +ve as x passes through a.
6. If the sign of $\frac{dy}{dx}$ does not change while x passes through a, then y is neither maximum nor minimum at x = a

Conditions from Maximum or Minimum Values

The necessary condition that f(x) should have a maximum or a minimum at x=a is that f'(a) = 0

There is a maximum of f(x) at x=a if f'(a) =0 and f"(a) is negative

Similarly there is a minimum of f(x) at x=a if f'(a) = 0 and f"(a) is positive.

Inflection point

An inflection point is a point on a curve at which the sign of the curvature (i.e., the concavity) changes. Inflection points may be stationary points, but are not local maxima or local minima. A necessary condition for 'x' to be an inflection point is f"(x)=0. A sufficient condition requires f"(x + ε) and f'' (x-ε) to have opposite signs in the neighborhood of x.

Note: If f"(a) is also equal to zero, then we can show that for a maximum or a minimum of f(x) at x = a . We must have f''(a) = 0. Then, if f'''(a) is negative, there will be a maximum at x = a and if f '''(a) is positive there will be minimum at x = a

In general if, f'(a)=f"(a)=f'''(a)=f^{n-1} (a)=0 and f^n (a) ≠ 0 then n must an even integer for maximum or minimum. Also for a maximum f" (a) must be negative and for a minimum f" (a) must be positive.

Absolute Maximum and Minimum in Range [a, b]

Absolute Maxima and Minima refers to a single maximum and minimum value of a function in a given range. If a function f is differentiable in [a, b] except (possibly) at infinitely many points, then to find (absolute) maximum and minimum values adopt the following procedure:

1. Evaluate f(x) at the points where f' (x) = 0
2. Evaluate f(x) at the points where derivative fails to exist.
3. Find f(a) and f(b)

Then the maximum of these values is the absolute maximum of the given function f and the minimum of these values is the absolute minimum of the given function f.

Solved Examples

Example: Find the absolute maximum and minimum values of:

$f(x) = 2x^3 - 9x^2 + 12x - 5 \text{ in } [0, 3]$

Solution: Given $f(x) = 2x^3 - 9x^2 + 12x - 5$

It is differentiable for all x in [0, 3], since it is a polynomial

Differentiating (i) w.r.t. x, we get

$f'(x) = 2 \times 3x^2 - 9 \times 2x + 12 = 6(x^2 - 3x + 2)$

Now, for extrema points f'(x) = 0

$6(x^2 - 3x + 2) = 0$

$x^2 - 3x + 2 = 0$

$(x - 1)(x - 2) = 0$

$x = 1, 2$

Also 1, 2 both are in [0, 3], therefore 1 and 2 both are stationary point Checking Function Values at Stationery Points and end points of interval Further,

$f(1) = 2.1^3 - 9.1^2 + 12.1 - 5 = 22 - 9 + 12 - 5 = 0$

$f(2) = 2.2^3 - 9.2^2 + 12.2 - 5 = 16 - 36 + 24 - 5$

$= -1$

$f(0) = -5$

And f

$(3) = 2.3^3 - 9.3^3 + 12.3 - 5 = 54 - 81 + 36 - 5 = 4$

Thus, maximum value is 4 which occurs at x = 3 and minimum value is -5 which occurs at x = 0.

Example: The maximum value of

$f(x) = x^3 - 9x^2 + 24x + 5$ in the interval [1, 6] is

Solution: We need absolute maximum of in the interval [1, 6]

$f(x) = x^3 - 9x^2 + 24x + 5$

First find local maximum if any by putting f'(x) = 0

i.e $f'(x) = 3x^2 - 18x + 24 = 0$

i.e. $x^2 - 6x + 8 = 0$

$x = 2, 4$

Now, $f''(x) = 6x - 18$

$f''(2) = 12 - 18 = -6 < 0$

$(\text{so } x = 2 \text{ is a point of local maximum})$

And $f''(4) = 24 - 18 = 6 > 0$

$(\text{so } x = 4 \text{ is a point of local minimum})$

Now tabulate the values of f at end point of interval and at local maximum point, to find absolute maximum in given range, as shown below:

x	f(x)
1	21
2	25
6	41

Clearly the absolute maxima is at x = 6

And absolute maximum value is 41.

$y = a\log|x| + bx^2 + x$ has extreme values at $x = +2$ and $x = -\frac{3}{4}$ then the values of a & b are ________.

Solution: $\frac{dy}{dx} = a \times \frac{1}{|x|} \times \frac{|x|}{x} + 2bx + 1 = 0$

$\frac{a}{x} + 2bx + 1 = 0$

$2bx^2 + x + a = 0$ ...(1)

$\left(x + \frac{3}{4}\right)(x - 2) = 0$

$4x^2 - 5x - 6 = 0$

$\frac{4}{5}x^2 + x + \frac{6}{5} = 0$ ...(2)

Equating (1) & (2)

$2b = \frac{4}{5} \Rightarrow b = \frac{2}{5}$

$a = \frac{6}{5}$

Example: Maximum of

$f(x) = \frac{e^{\sin x}}{e^{\cos x}}, x \in R$ is ________

Solution: $f(x) = e^{\sin - \cos x}$

Since, exponential is a monotonically increasing function. So maximum of sinx-cosx gives maximum value of function.

Let $g(x) = \sin x - \cos x$

$g'(x) = \cos x + \sin x = 0$

$\cos x = -\sin x$

$x = -\frac{\pi}{4}, \frac{3\pi}{4}$............

$g''\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} > 0 \rightarrow \min$

$g''\left(\frac{3\pi}{4}\right) = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} < 0 \rightarrow \max$

Maximum value $= f\left(\frac{3\pi}{4}\right) = e^{\frac{1}{\sqrt{2}} - \left(\frac{-1}{\sqrt{2}}\right)} = e^{\sqrt{2}}$

Taylor's theorem (Generalized MVT)

Let f(x) be defined in [a, a+h] such that it satisfies

1. $f(x), f'(x), f''(x)$ .... $f^{n-1}(x)$ are continuous in [a, a+h]

2. f(x), f'(x), f"(x) .... fn-1(x) are differentiable in (a, a+h) then there is at least one number $\theta \in (0,1)$ such that

$$f(a+h)=f(a)+hf'(a)=\frac{h^2}{2!}f''(a)+............+\frac{h^{n-1}}{(n-a)!}f^{n-1}(a)+R_n$$

$$\text{Where } R_n=\frac{h^n}{(n-1)!\times P}(1-\theta)^{n-p}f^n(a+\theta h)$$

Case 1: when P = n $R_n=\frac{h^n}{n!}f^n(a+h\theta)$

is called as Lagrange's form of remainder.

Case 2: when P = 1

$$R_n=\frac{h^n}{(n-1)!}(1-\theta)^{n-1}f^n(a+\theta h)$$

is called as Cauchy's form of remainder.

As $n\to\infty, R_n\to 0$ then,

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+............\text{is}$$

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+.......\infty$$

is a Taylor series expansion of f(x) about x=a

$$f(x)f(0)+xf'(a)+\frac{x^2}{2!}f''(0)+....................\infty\ \text{is a}$$

Taylor Series expansion of f(x) about x=0 which is also called as

Maclaurin's Series.

Some common Taylor Series expansions are,

1. $e^x=1+x+\frac{x^2}{2!}+.......\infty$

2. $\sin x=x\frac{x^3}{3!}+\frac{x^5}{5!}............\infty$

3. $\cos x=1\frac{x^2}{2!}+\frac{x^4}{4!}............\infty$

4. $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-............\infty$

5. $\log(1-x)=-\left[x+\frac{x^2}{2}+\frac{x^3}{3}+..........\infty\right]$

6. $\tan x=x\frac{x^3}{3}+\frac{2x^2}{15}+............\infty$

7. $(1+x)^{-1}=1-x+x^2-x^3+...........\infty$

Solved Examples

Example: The coefficient of $(x-2)^4$ in the Taylor's Series expansion of logx about x = 2 is

Solution: Coefficient of (x −2)

$$f'(x)=\frac{1}{x},\ f''(x)=-\frac{1}{x^2},\ f'''(x)=\frac{2}{x^3}$$

$$f''''(x)=-\frac{6}{x^4}$$

$$f''''(2)=\frac{-6}{16}=-\frac{3}{8}$$

Coefficient of

$$(x-2)^4=\frac{f^{(iv)}(2)}{4!}=\frac{-3/8}{4!}=\frac{-3}{8\times 24}=\frac{-1}{64}$$

Example: The Taylor Series expansion of tanx about $x=\frac{\pi}{4}$ is

Solution: $f(x)=f\left(\frac{\pi}{4}\right)+\left(x-\frac{\pi}{4}\right)$

$$f'\left(\frac{\pi}{4}\right)+\frac{\left(x-\frac{\pi}{4}\right)^2}{2!}f''\left(\frac{\pi}{4}\right)$$

.............∞

$$f(x)=\tan x$$

$$f'(x)=\sec^2 x$$

$$f''(x)=2\sec^2 x\tan x$$

$$f'\left(\frac{\pi}{4}\right)=2$$

$$f''\left(\frac{\pi}{4}\right)=4$$

$$f(x)=1+2\left(x-\frac{\pi}{4}\right)+2\left(x-\frac{\pi}{4}\right)^2+..............\infty$$

Example: Determine the first 3 non zero term in the expansion of $e^x\tan x$ is?

Solution: $f'(0)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+.....\infty$

$$f'(0)=0$$

$$f'(x)=e^x\left[\tan x+\sec^2 x\tan x\right]$$

$$f'(x)=e^x\left[\sec^2 x+\tan x+\sec^2 x+2\sec^2 x\tan x\right]$$

$$\Rightarrow f''(0)=2$$

$$f'''(x)=e^x$$

$$\left[\tan x + 2\sec^2 x + 2\sec^2 x \tan x + \sec^2 x + 4\sec^2 x \tan x + 2\sec^4 x + \sec^2 x \tan^2 x\right] f'''(0) = 5$$

$$f(x) = 0 + x(1) + \frac{x^2}{3!}(5) + ... + \infty = x + \frac{2x^3}{3} + x^2 + \frac{x^3}{2} + ...\infty$$

or

$$e^x \tan x = x + x^2 + \frac{5}{6}x^3 + ...\infty$$

Example: The Taylor Series Expansion of $\tan^{-1}x$ is

Solution:

$$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + ...$$

on integration

$$f(x) + c = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...\infty$$

Put $x = 0$

$0 + c = 0 \Rightarrow c = 0$

$$\therefore f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...\infty$$

Partial Differentiation

When a variable depends on more than one independent variable, then it is called as Multi-Variable Function.

When the derivative of such a variable is computed with respect to any of the independent variable then it is

called as Partial Derivative.

If z = f(x, y) then

Derivative of z w.r.t. x is given by,

$$z_x = \frac{\partial z}{\partial x} - \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Derivative of z w.r.t. y is given by,

$$z_x = \frac{\partial z}{\partial y} - \lim_{h \to 0} \frac{f(x, +h+k) - f(x, y)}{k}$$

Similarly $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}$, can be computed

To calculate any partial derivative w.r.t. any one variable then other independent variables are considered as constant.

The terms $\frac{\partial^2 z}{\partial y \partial y}$, are called as mixed partials and for a continuous function mixed partials are equal i.e.

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

Homogeneous function

If all the terms of a function have the same degree then it is called as homogenous function and degree of each term is called as order of the function.

Example:

1. 2x + 3y : Homogenous Function of order 1.
2. $x^3z+4x^2y^2-2xyz^2$: Homogenous Function of order 4.
3. $\frac{x^3y + z^2y^2}{4x - 3y}, n = 4 - 1 = 3$: Homogenous

   Function of order 3.

   n = order of the function = order of Numerator – Order of Denominator
4. $u = \cos^{-1}\left(\frac{x^2 + y^2}{2x - 3y}\right) \rightarrow$ Non homogenous function
5. $z = \log\left(\frac{x}{y}\right) \rightarrow$ Non homogenous function
6. Sinx → Non homogeneous function
   a) If the function of trigonometric, exponential and logarithmic function are homogeneous with degree 0 then the whole function will be homogenous.

Note:

1. If f(kx, ky) = kn f(x, y) then f(x, y) is a homogenous function with degree 'n'.
2. If f(x, y) is a homogeneous function with degree 'n' then $f(x, y) = \begin{cases} x^n\phi(y/x) \\ y^n\phi(x/y) \end{cases}$

Euler's Theorem

If f(x, y) is a homogeneous function with degree 'n' the

a. $x\frac{\partial f}{\partial x}+yx\frac{\partial f}{\partial x}=nf$

b. $x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2x\frac{\partial^2 f}{\partial y^2}=n(n-1)f$

Note: If u(x,y) = f(x,y) +g(x,y) where f and g are homogeneous function with degree m and n respectively, then

a) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=mf+ng$

b) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2x\frac{\partial^2 u}{\partial y^2}=m(m-1)f+$ $n(n-1)g$

Note: If f(u) is a homogeneous function in two variables x an y with degree 'n' then

a) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}=F(u)$

b) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=F(u)[F'(u)-1]$

Total Differentiation

If z=f(x, y) where x = g(t) and y = h(t) then the total derivative of 'z' w.r.t. $\frac{dz}{dt}=\frac{\partial f}{dt}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$

The total differential of z = f(x, y) is

$$dz=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$

Note: If f(x, y) = c is an implicit function then

$$\frac{dy}{dx}=\frac{\partial f/\partial x}{\partial f/\partial y}$$

If z = f(x, y) where x = g(u,v) and y = h(u,v) then

$$\frac{\partial z}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

Solved Examples

Example: If w = x^2 + y^2 where

$x=\frac{t^2-1}{t}$ and $y=\frac{t}{t^2+1}$

then $\frac{dw}{dt}\Big|_{t=1}=$ __________?

Solution: $\frac{dw}{dt}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial y}$

$$\frac{dw}{dt}=2x\left[1+\frac{1}{t^2}\right]+2y\left[\frac{(t^2+1)1-t(2t)}{(t^2+t)^2}\right]$$

At $t=1$

$$\frac{dw}{dt}=2\times(0)(1+1)+\left(\frac{1}{2}\right)\left(\frac{2-2}{4}\right)-0+0=0$$

Example:

If $v=x^2+y^2+z^2$ where $x=e^{2t}, y=e^{2t}$ $\sin 3t$ and $z=e^{2t}\cos 3t$, then $\frac{dv}{dt}=$ ____?

Solution:

$v=e^{4t}+e^{4t}\sin^2 3t+e^{4t}\cos^2 3t=e^{4t}[1+1]$

$v=e^{4t}$

$\frac{dc}{dt}=8e^{4t}$

Example: The total derivative of x^3 y^2 w.r.t. x where x and y are connected by the relation x^3 + y^3 – 3xy = 0 is

__________ ?

Solution: Let u = x^3 y^2

Since, x^3 + y^3 – 3xy = 0

Differentiate both sides w.r.t. x

$$3x^2+3y^2\frac{dy}{dx}-3y-3x\frac{dy}{dx}=0$$

Then, $\frac{dy}{dx}=\frac{3x^2-3y}{3x-3y^2}=\frac{x^2-y}{x-y^2}$

$$\frac{du}{dx}=\frac{\partial u}{\partial x},\frac{dx}{dx}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}=3x^2y^2(1)$$

$$+2x^3y\frac{dy}{dx}=3x^2y^2+2x^3y\left[\frac{x^2-y}{x-y^2}\right]$$

Example: If u = f (2x −3y, 3y − 4z,4z − 2x) then $6u_x + 4u_y$ = __________ ?

Solution: Let r = 2x −3y; s = 3y − 4z; t = 4z − 2x

Then, u = f(r,s,t)

$$u_x = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial x} =$$

$$f,\times + fs\times 0 + f_t \times (-2)$$

$$6u_x = 12f, -12f_t$$

$$u_y = f_t \times (-3) + f_s \times 3 + f_t \times 0$$

$$4u_y = -12f_r + 12f_s$$

$$\therefore 6u_x + 4u_y = 12f_s - 12f_t$$

$$u_z = f_r(0) + f_s(-4) + f_t(4)$$

$$-3u_z = 12f_s - 12f_t$$

$$\text{Thus}, 6u_x + 4u_y = -3u_z$$

Example: If $V = r^n, r = \sqrt{X^2 + y^2 + z^2}$ then $V_{xx} + V_{yy} + V_{zz} =$ ______

Solution: $\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$

$$V_x = \frac{\partial V}{\partial r}\times\frac{\partial r}{\partial x} = nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-1}, \frac{x}{r} = nr^{n-2}.x$$

$$V_{xx} = n\left[r^{n-2}.(1)\right] + xn(n-2)r^{n-3}.\frac{\partial r}{\partial x} = nr^{n-2} +$$

$$nx(n-2)r^{n-3}.\frac{x}{r} = nr^{n-2} + n(n-2)r^{n-4}x^2$$

Similarly

$$V_{yy} = nr^{n-2} + n(n-2)r^{n-4}.y^2$$

$$V_{zz} = nr^{n-2} + n(n-2)r^{n-4}.z^2$$

$$V_{xx} + V_{yy} + V_{zz} = 3nr^{n-2} + n(n-2)r^{n-4}(x^2+y^2+z^2)$$

$$= 3nr^{n-2} + n(n-2)r^{n-4}(r^2)$$

$$V_{xx} + Vyy + V_{zz} = r^{n-2}\left[3n + n^2 - 2n\right] = r^{n-2}\left[n^2 + n\right]$$

$$= n(n+1)r^{n-2}$$

Maxima & Minima for functions of two variables

$$\text{Let } z = f(x,y), p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2},$$

$$s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$$

Method

1. Find p, q, r, s & t
2. Equate p & q to zero for obtaining the stationary points.
3. At each stationary point find r, s, t
   a) If $rt - s^2 > 0$ and $r > 0 \Rightarrow$ Minimum at the stationary point.
   b) If $rt - s^2 > 0$ and $r < 0 \Rightarrow$ Maximum at the stationary point.
   c) If $rt - s^2 < 0$ then f(x, y) has no extreme at that stationary point and each points are SADDLE points.

Solved Examples

Example: The function $f(x,y) = 1-x^2-y^2$ has maximum at point?

Solution: p = −2x, q = −2y

r = −2, s = 0, t = −2

$$\text{For stationery point s,} \left.\begin{matrix} p=0 \\ q=0 \end{matrix}\right\} \Rightarrow \begin{matrix} x=0 \\ y=0 \end{matrix}$$

$$\text{At } (0,0)\, rt - s^2$$

$$\therefore \text{maximum at } (0,0)$$

Example: The minimum value of the function $x^3-3x^2+4y^2-10$ occurs at

Solution: $p=3x^2-6x$

q=8y

p=q=0 ⇒ (0,0) & (2,0) are the stationary point.

Determining double derivatives

r = 6x − 6

s = 0,t = 8

At (0, 0), r = −6, s = 0, t = 8

$rt - s^2 = -48 < 0$ → Saddle point

At (2, 0), r = 6,s = 0,t = 8

$rt - s^2 = 48 > 0$ and r = 6 > 0 min at (2, 0)

Example: A rectangular box open at the top is to have a volume of 32 cubic feet. Then the dimensions of the box

requiring least material for its construction are ________?

Solution: Let x, y, z be the dimensions. Since, it is open at the top,

$$s = xy + 2yz + 2xz$$

$$v = xyz = 32$$

$$f = xy + \frac{64}{x} + \frac{64}{y}$$

Calculating the partial derivatives

$$p = y - \frac{64}{x^2} \text{ and } q = x - \frac{64}{y^2}$$

Equating Partial Derivatives to zero to determine stationery points,

$$p = 0 => y = \frac{64}{x^2}$$

$$q = 0 => x = \frac{64}{y^2} = \frac{x^4}{64}$$

$$y = 0, y = 4$$

$$x = 0.x - 4$$

If x = 0 and y = 0 then z = inf

But dimension cannot be infinite so x =4 and y =4. Thus, z =2.

Example: The distance between origin and a point nearest to it on the surface $z^2 = (1 + xy)$ is __________ ?

Solution: Let P(x, y, z) be a point on the surface z2=(1+xy)

$$d = OP = \sqrt{x^2 + y^2 + z^2} = \sqrt{x^2 + y^2 + 1 + xy}$$

$$\text{Let } f = x^2 + y^2 + 1 + xy$$

Calculating partial derivatives,

p = 2x + y and q = 2y + x

Equating both to zero we get x = 0 and y=0 as the stationery point

Calculating double partial derivative r =2 and t=2 and s=1

At (0, 0) 2 rt – s^2 =3 > 0

r=2 > 0 so minima exists.

Minimum distance $= \sqrt{0 + 0 + 1 + 0} = 1$

Constrained Maxima & Minima

A constraint is a certain condition on independent variables that must be satisfied while optimizing a multivariable function.

To determine Constrained Maxima and Minima we use the method of Lagrange's Multipliers.

Suppose, we have to optimize the function f(x, y, z)

where $\phi(x, y, z) = c$ is the constraint

Let $F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$

Here, λ is called as Lagrange Multiplier

Now, we equate all partial derivatives of F to zero.

$$\text{i,e } \left.\begin{aligned} \frac{\partial F}{\partial x} + \lambda\frac{d\phi}{\partial x} = 0 \\ \frac{\partial F}{\partial y} + \lambda\frac{d\phi}{\partial y} = 0 \\ \frac{\partial F}{\partial z} + \lambda\frac{d\phi}{\partial z} = 0 \end{aligned}\right\} \text{ Lagrange's equation}$$

Solving equations (1) .... (4) we obtain x, y, z & λ

x, y, z → stationary point

f(x, y, z) → extreme value

Solved Examples

Example: The max-value of volume of parallel piped that can be inscribed in an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Solution: Constraint is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Function to be optimized is, v=8xyz

$$\text{Thus, } F = 8xyz, \phi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\text{Thus, } F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$$

Calculating the partial derivatives,

$$8yz + \lambda\left(\frac{2x}{a^2}\right) = 0 \Rightarrow \frac{-\lambda}{4} = \frac{a^2yz}{x}$$

$$8xz + \lambda\left(\frac{\partial y}{b^2}\right) = 0 \Rightarrow \frac{-\lambda}{4} = \frac{b^2xz}{y}$$

$$8xz + \lambda\left(\frac{\partial y}{c^2}\right) = 0 \Rightarrow \frac{-\lambda}{4} = \frac{c^2xz}{z}$$

$$\frac{a^2yz}{x} = \frac{b^2xz}{z} \Rightarrow \frac{y^2}{b^2} = \frac{x^2}{a^2}$$

$$\text{And}\ \frac{b^2xz}{y} = \frac{c^2xy}{z} \Rightarrow \frac{z^2}{c^2} = \frac{y^2}{b^2}$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow \frac{3x^2}{a^2} = 1$$

$$x = \frac{a}{\sqrt{3}},$$

$$\text{similarly}\ y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$$

$$\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right) \to \text{Stationary point}$$

$$\text{Extreme value}\ f = \left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right) = \frac{8abc}{3\sqrt{3}}$$

Indefinite Integration

Indefinite Integration is also called as Anti-derivative and it is the reverse process of differentiation.

$$\int f(x)dx = F(x) + c$$

Indefinite Integration is the process of finding F(x) for function f(x) such that

$F'(x) = f(x)$

The result of indefinite integration represents a family of curves. Here, constant 'c' is called as constant of Integration.

Basic Integration Formulas

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\int \frac{1}{x} dx = \log x$$

$$\int \sin x dx - \cos x$$

$$\int \cos x dx = \sin x$$

$$\int \sec^2 x dx = \tan x$$

$$\int \operatorname{cosec}^2 x dx = -\cot x$$

$$\int \sec x \tan x = \sec x$$

$$\int \operatorname{cosec} \cot x = -\operatorname{cosec} x$$

$$\int \cot x dx = \log|\sin x|$$

$$\int \tan x dx = \log|\sec x|$$

$$\int \operatorname{cosec} x = \log|\operatorname{cosec} x - \cot x|$$

$$\int \cosh x = \sinh x$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x$$

$$\int \sinh x dx = \cosh x$$

$$\int \frac{1}{1-x^2} dx = \sin^{-1} x$$

$$\int \sec x = \log|(\sec x + \tan x)|$$

$$\int \frac{1}{x\sqrt{x^2-1}} dx = \sin^{-1} x$$

$$\int \frac{1}{\sqrt{a^2+x^2}} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \log\left[x + \sqrt{x^2 - a^2}\right]$$

$$\int \frac{dx}{a^2+x^2} = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) = \log\left[x + \sqrt{x^2 - a^2}\right]$$

$$\int \sqrt{x^2+a^2} dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) \text{or}\ \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left\{x + \sqrt{x^2+a^2}\right\}$$

$$\int f'(ax+b) dx = \frac{f(ax+b)}{a}$$

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

$$\int \left[f(x)^n f'(x) dx = \frac{[f(x)]^{n+1}}{(n+1)} \text{when}\ n \neq 1 \right.$$

Solved Examples

Example: Evaluate $\int \frac{4x^3}{1+x^4}dx$

Solution:

Substitute $(1+x^4) = t \Rightarrow 4x^3dx = dt$

The integral reduces to

$$\int \frac{dt}{t} = \log t = \log(1+x^4)$$

Integration by Parts

Let u and v be two functions of x. then we have from differential calculus.

$$\frac{d}{dx}(uv) = u.\frac{dv}{dx} + v.\frac{du}{dx}$$

Integrating both sides of (1) with respect to x, we have

$$uv = \int u.\frac{dv}{dx}dx + \int v.\frac{du}{dx}dx$$

$$\int u\frac{dv}{dx}dx = uv - \int v.\frac{du}{dx}.dx$$

$$\text{i.e.,} \int udv = uv - \int v.du$$

This can also be written as

$$\int uvdx = u\int vdx = -\int\left[\frac{du}{dx}\int vdx\right]dx$$

This choice of which function will be u and which function will be v is very important in solving by integration by

parts.

The ILATE method helps to decide this.

ILATE stands for

I : Inverse trigonometric functions ($\sin^{-1} x$, $\cos^{-1}x$ etc)

L : Logarithmic functions (logx, In x etc)

A : Algebraic functions (x^2 , $x^3 + x^2 + 2$, etc)

T : Trigonometric functions (sinx, cosx etc)

E : Exponential function (e^x , a^x etc)

Whichever of the two functions comes first in ILATE, get designated as u and other function gets designated as v.

Based on above method we can define,

$$\int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$\int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx - b\sin bx)$$

Integration by Partial Fractions

$$I = \int \frac{1}{x^2-a^2}dx \quad (x > a)$$

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x-a}\right)$$

$$\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\left\{\int\frac{dx}{x-a} - \int\frac{dx}{x+a}\right\} =$$

$$\frac{1}{2a}\log(x-a) - \log(x+a) = \frac{1}{2a}\log\frac{x-a}{x+a}$$

$$\text{Thus,} \int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\frac{x-a}{x+a}, \ (x > a)$$

Some common integrals are,

(a) $\int \frac{1}{x^2-x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

(b) $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\frac{a+x}{a-x}$

(c) $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\frac{x-a}{x+a}$

(d) $\int \frac{1}{\sqrt{a^2+x^2}}dx = \sinh^{-1}\left(\frac{x}{a}\right)$

$= \log\left[x+\sqrt{x^2+a^2}\right]$

(e) $\int \frac{1}{a^2-x^2}dx = \sin^{-1}\left(\frac{x}{a}\right)$

(f) $\int \frac{1}{\sqrt{x^2-a^2}}dx = \cosh^{-1}\left(\frac{x}{a}\right)$

$$= \log\left[x + \sqrt{x^2 + a^2}\right]$$

(g) $\int \frac{1}{x\sqrt{a^2 - a^2}}dx = \frac{1}{a}\sec^{-1}\left(\left|\frac{x}{a}\right|\right)$

(h) $\int \sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

(i) $\int \sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

A few other useful integration formulae

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left[\frac{m+n+2}{2}\right]}$$

Where Γ(n+1) is called the gamma function which satisfies the following properties

Γ(n+1)=nΓn if n is a positive integer

$\Gamma(n+1) = n!, \Gamma(1) = 1$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Walle's formula

$$\int_0^{\pi/2} \sin^n x = \int_0^{\pi/2} \cos^n x = $$

$$\begin{cases} \frac{(n-n)(n-3)(n-5)}{(n)(n-2)(n-4)} \ldots\ldots\ldots\ldots \frac{2}{3}, \text{when n is odd} \\ \frac{\pi}{2} x \frac{(n-1)(n-3)(n-5)}{(n)(n-2)(n-4)} \ldots\ldots \frac{3}{4}, \frac{1}{2} \text{when n is even} \end{cases}$$

Note: The result of indefinite integration can be verified in exam by differentiating the options.

Definite Integral

Let f(x) be continuous in [a, b] and F be the antiderivative of the given function f(x) then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Note: $\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(x)dx\right] = f(v)\frac{dv}{dx} - f(u)\frac{du}{dx}$

Properties of Definite Integral

- $\int_a^b f(x)dx = -\int_b^a f(x)dx$

  If $c \in (a,b)$ then $\int_a^b f(x)dx = \int_a^b f(x)dx + \int_c^d f(x)dx$

- $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

- $\int_0^a \frac{f(x)}{f(x) + f(a+b-x)}dx = \frac{b-a}{2}$

- $\int_{-a}^a f(x)dx = \begin{cases} \int_0^a f(x)dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$

- $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$

- $\int_0^{na} f(x)dx = n\int_0^a f(x)dx \text{ if } f(x+a) = f(x)$

- $\int_0^a xf(x)dx = \frac{a}{2}\int_0^a f(x)dx \text{ if } (a-x) = f(x)$

- $\int_0^{\pi b} \sin^n xdx = \int_0^{\pi/2} \cos^n xdx =$

  $\left[\frac{n-1}{n} \times \frac{n-3}{n-2} \times \ldots\ldots\ldots\ldots \times \frac{2}{3}\left(\text{or } \frac{1}{2}\right)\right]K$

  $\text{where } K = \begin{cases} 1 & \text{if n is odd} \\ \frac{\pi}{2} & \text{if n is even} \end{cases}$

- $\int_0^{\pi/2} \sin^m x \cos^n dx$

  $\frac{[(m-a)(m-3)\ldots\ldots\ldots\ldots 2][(n-1)(n-3)\ldots\ldots\ldots 2(\text{or})1xK]}{[(m+n)(m+n-2)\ldots\ldots\ldots\ldots 2(\text{or}1)]}$

  $\text{where } K = \begin{cases} 1 & \text{if n is odd} \\ \frac{\pi}{2} & \text{if n is even} \end{cases}$

Solved Examples

Example: Evaluate $\int_0^{\pi/2} \frac{\tan x}{\tan x + \cot x} dx$

Solution: Let f(x) = tanx

$$f\left(0 + \frac{\pi}{2} - x\right) = \cot x$$

$$\int_0^{\pi/2} \frac{\tan x}{\tan x + \cot x} dx = \int_0^{\pi/2} \frac{f(x)}{f(x) + f\left(0 + \frac{\pi}{2} - x\right)} dx$$

$$= \frac{b-a}{2} = \frac{\pi}{4}$$

Example: Evaluate $\int_0^{\pi/2} \frac{1}{1 + \sqrt{\cot}} dx$

Solution:

$$\int_0^{\pi/2} \frac{1}{1+\sqrt{\cot}} dx = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} = \sqrt{\cos x}} dx$$

$$\int_0^{\pi/2} \frac{1}{1+\sqrt{\cot x}} dx = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{0 + \frac{\pi}{2} - x}} dx \frac{\frac{\pi}{2} - 0}{2} = \frac{\pi}{4}$$

Example: Determine the value of $\int_0^{\pi} |\cos x| dx$

Solution:

$$|\cos x| = \begin{cases} \cos x \le x \le \frac{\pi}{2} \\ -\cos x \frac{\pi}{2} \le x \le \pi \end{cases}$$

$$\int_0^{\pi} |\cos x| dx = \int_0^{2\pi} \cos x dx + \int_{\pi/0}^{\pi} -\cos x dx = [\sin x]_0^{\pi/2}$$

$$-[\sin x]_{\pi/2}^{\pi} = (1-0) - (0-1) = 2$$

Example:

Determine the value of $\int_0^2 |x^2 + 2x - 3| dx$

Solution:

$$I = \int_0^2 |x^2 + 2x - 3| dx = \int_0^2 |(x+3)(x-1)| dx$$

$$|\cos| x = \begin{cases} -(x+3)(x-1) & 0 \le x \le 1 \\ (x+3)(x-1) & 1 \le x \le 2 \end{cases}$$

$$I = \int_0^1 (-x^2 - 2x + 3) dx + \int_1^2 (x^2 + 2x - 3) dx$$

$$= \left[\frac{-x^3}{3} - \frac{2x^2}{2} + 3x\right]_0^1 + \left[\frac{x^3}{3} + \frac{2x^2}{2} - 3x\right]_1^2$$

$$I = \left(\frac{-3}{3} - 1 + 3\right) + \left(\frac{8}{3} + 4 - 6\right) - \left(\frac{1}{3} + 1 - 3\right) = 4$$

Example: Find the value of $\int_0^n [x] dx$

Solution: [x] = step function

$$I = \int_0^n [x] dx = \int_0^1 0dx + \int_1^2 1dx + \int_2^3 2dx +$$

$$\int_3^4 3dx + \ldots\ldots\ldots + \int_{n-1}^n (n-1) dx$$

$$I = 1 + 2 + 3\ldots\ldots\ldots + (n-1) = \frac{n(n-1)}{2}$$

Example:

Determine the integral $I = \int_0^{\pi/2} \log(\tan x) dx$?

Solution: $I = \int_0^{\pi/2} \log(\cot x) dx$

$$\left[\because \int_0^{\pi/2} \log(\tan x) = \int_0^{\pi/2} \log\left(\tan\left(\frac{\pi}{2} - x\right)\right)\right]$$

$$I + I = \int_0^{\pi} [\log(\tan x) + \log(\cot x) dx] = \int_0^{\pi/2} \log(1) dx = 0$$

Example:

Determine the integral $\int_0^{\pi} \frac{\sec^2 x dx}{a^2 + b^2 \tan^2 x} dx$

Solution: In this case, $2a = \pi \text{ and } a = \frac{\pi}{2}$

Since, $\frac{\sec^2 x}{a^2 + b^2 \tan^2 x} = \frac{\sec^2(\pi - x)}{a^2 + b^2 \tan^2(\pi - x)}$

$$\int_0^{\pi} \frac{\sec^2 x dx}{a^2 + b^2 \tan^2 x} dx = 2\int_0^{\pi/2} \frac{\sec^2 x dx}{a^2 + b^2 \tan^2 x} dx$$

Assume t = tanx and dt = $\sec^2$ xdx

$$\int_0^{\pi} \frac{\sec^2 x dx}{a^2 + b^2 \tan^2 x} dx = 2\int_0^{\infty} \frac{1}{a^2 + b^2 t^2} dt$$

$$= 2 \times \frac{1}{a} \tan^{-1}\left[\frac{bt}{a}\right] \times \frac{1}{b}\Bigg|_0^{\infty} = \frac{2}{ab}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{ab}$$

Example: Find the value of $\int_0^{\pi/2} \sin^8 x dx$

Solution: Since, $\int_0^{\pi b} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx$

$$= \left[\frac{n-1}{n} \times \frac{n-3}{n-2} \times \ldots\ldots\ldots\ldots \frac{2}{3}\left(\text{ or } \frac{1}{2}\right)\right] K$$

$$\text{where } K = \begin{cases} 1 & \text{if } n \text{ is odd} \\ \frac{\pi}{2} & \text{if } n \text{ is even} \end{cases}$$

In this case, n = 8 (even)

$$\int_0^{\pi/2} \sin^8 x dx = \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{35\pi}{256}$$

Example: Determine the value of $\int_0^{\pi/2} \cos^7 x dx$

Solution: Since n = 7 (odd)

$$\int_0^{\pi/2} \sin^7 x dx = \int_0^{\pi/2} \sin^7\left(\frac{\pi}{2} - x\right) dx$$

$$= \int_0^{\pi/2} \cos^7 x dx = \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times 1 = \frac{16}{35}$$

Example: Find the value of integral

$$\int_0^{\pi/2} \sin^7 x \cos^9 x \, dx$$

Solution: $\int_0^{\pi/2} \sin^m x \cos^n x dx$

$$= \frac{[(m-1)(m-3) \ldots 2(\text{ or})1][(n-1)(n-3) \ldots 2(\text{ or}) x K]}{[(m+n)(m+n-2) \ldots 2(\text{or})1]}$$

Here, m = 7 and n = 9 which are odd

$$I = \frac{(6 \cdot 4 \cdot 2)(8 \cdot 6 \cdot 4 \cdot 2)}{16 \cdot 14 \cdot 12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \times 1 = \frac{1}{448}$$

Example: Calculate the value of

$$\int_0^{\pi/2} \sqrt{\sin x} \cos^3 x dx$$

Solution: Let sin x = t

cosx dt = dt

$$I = \int_0^1 \sqrt{t}\left(1 - t^2\right) dt = \int_0^1 \left(t^{1/2} - t^{5/2}\right) dt$$

$$= \left[\frac{t^{3/2}}{3/2} = \frac{t^{7/2}}{7/2}\right]_0^1$$

$$I = \frac{2}{3} - \frac{2}{7} = \frac{8}{21}$$

Improper Integrals:

Certain integrals cannot be computed directly using the rules of definite integrals. First kind:

$$\int \int_a^b f(x) dx \text{ if } a = -\infty \text{ (or) } b = \infty \text{ (or) both}$$

$$\text{i.e.} \int \int_{-\infty}^b f(x) dx \text{ (or) } \int_a^{\infty} f(x) dx \text{ (or) } \int_{-\infty}^{\infty} f(x) dx$$

Second kind: $\int_a^b f(x) dx$

When a and b are finite, but f(x) is infinite for some $x \in [a, b]$

Example:

1. $\int_0^1 \log(1-x) dx \rightarrow$ infinite for $x = 1$
2. $\int_{-1}^1 \sqrt{\frac{1-x}{1+x}} dx \rightarrow$ infinite for $x = -1$
3. $\int_0^2 \frac{1}{1-x} dx = \int_0^1 \frac{1}{1-x} dx + \int_1^2 \frac{1}{1-x} dx$

   $\rightarrow$ infinite for $x = 1$

Convergence:

1. $\int_a^b f(x)dx =$ finite, then it is a convergent improper integral

2. $\int_a^b f(x)dx =$ infinite, then it is a divergent improper integral

Solved Examples

Example: Find the convergence of following integrals.

1. $\int_1^\infty \frac{1}{x\sqrt{x^2-1}}dx$

2. $\int_0^\infty x\sin x dx$

Solution:

1. $\int_1^\infty \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1} x\Big]_1^\infty = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

Since, the integral is finite, it is convergent

2. $\int_0^\infty x\sin x dx = [x[-\cos x] + (1)[\sin x]]_0^\infty = \infty$

Since, the integral becomes infinite, it is divergent improper integral

Example:

The integral $\int_{-\infty}^0 \frac{1}{(1-3x)^2}dx$ converges to_____

Solution:

$$\int_{-\infty}^0 \frac{1}{(1-3x)^2}dx = \frac{-1}{1-3x} \times \frac{-1}{3}\Big]_{-\infty}^0 = \frac{1}{3} - 0 = \frac{1}{3}$$

Example: Check the convergence of the following integral $\int_1^\infty \log\left(\frac{1}{x}\right)dx$

Solution:

$$\int_1^\infty \log\left(\frac{1}{x}\right)dx = \int_1^\infty -\log x dx = -[x(\log x - 1)]_0^\infty = \infty$$

Thus, the following integral is divergent improper integral.

Example: Determine the integral

$\int_{-1}^1 \sqrt{\frac{1+x}{1-x}}dx =$ _______

Solution: Multiply both numerator and denominator by $\sqrt{1+x}$

$$\int_{-1}^1 \sqrt{\frac{1+x}{1-x}}dx = \int_{-1}^1 \frac{1+x}{\sqrt{1-x^2}}dx$$

$$= \int_{-1}^1 \frac{1}{\sqrt{1-x^2}}dx + \int_{-1}^1 \frac{x}{\sqrt{1-x^2}}dx$$

Since, the function in second integral is odd.

Thus, $\int_{-1}^1 \frac{x}{\sqrt{1-x^2}}dx = 0$

$$\int_{-1}^1 \sqrt{\frac{1+x}{1-x}}dx = 2\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = 2\left(\sin^{-1} x\right)_0^1 = 2 \times \left[\frac{\pi}{2} - 0\right] = \pi$$

Example: Calculate the value of integral $\int_{-1}^1 \frac{1}{x^2}dx$

Solution:

$$\int_{-1}^1 \frac{1}{x^2}dx = \int_{-1}^0 \frac{1}{x^2}dx + \int_0^1 \frac{1}{x^2}dx$$

$$\int_{-1}^1 \frac{1}{x^2}dx = \left[\frac{-1}{x}\right]_{-1}^0 + \left[\frac{-1}{x}\right]_0^1 = \infty$$

Thus, it is a divergent improper integral.

Example: Determine the integral

$$\int_0^3 \frac{1}{x^2-3x+2}dx$$

Solution:

$$\int_0^3 \frac{1}{x^2-3x+2}dx = \int_0^3 \frac{1}{(x-1)(x-2)}dx$$

$$\int_0^3 \frac{1}{x^2-3x+2}dx = \int_0^1 \frac{dx}{(x-1)(x-2)}$$

$$+\int_1^2 \frac{dx}{(x-1)(x-2)} + \int_2^3 \frac{dx}{(x-1)(x-2)}$$

$$I = \log\left(\frac{x-2}{x-1}\right)\Big]_0^1 + \log\left(\frac{x-2}{x-1}\right)\Big]_1^2 + \log\left(\frac{x-2}{x-1}\right)\Big]_2^3$$

$$= \infty$$

The first integral goes to infinity as x goes to 1. Hence, this integral if divergent improper integral.

Comparison Test

For first kind of improper integrals, we use this test,

1st Method:

Let $0 \le f(x) \le g(x)$ then

1. $\int_a^b f(x)dx$ converges if $\int_a^b g(x)dx$ is convergent.
2. $\int_a^b g(x)dx$ diverges if $\int_a^b f(x)dx$ is divergent.

Note: The following comparisons hold good for independent variable x

- x < finite ⇒ x is finite
- x > infinite ⇒ x is infinite
- x > finite ⇒ x may be finite (or) infinite
- x < infinite ⇒ x may be finite or infinite

2nd Method [Limit form]:

Let f(x) and g(x) be two positive function such that $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)} = l$ [non zero, finite]

Then $\int_a^b f(x)dx$ and $\int \int_a^b g(x)dx$ both diverge (or) converge together.

Solved Examples

Example: Find the convergence of following improper integrals

1. $\int_1^\infty e^{-x^2}dx$
2. $\int_2^\infty \frac{1}{\log x}dx$
3. $\int_1^\infty \frac{1}{x^2(e^{-x}+1)}dx$
4. $\int_1^\infty \frac{x\tan^{-1}x}{\sqrt{4+x^3}}dx$

Solution:

1. $e^{-x^2} \le e^{-x} \forall x \ge 1$

Thus, we can check for the convergence of

$$\int_1^\infty e^{-x}dx = \frac{e^{-x}}{-1}\Big]_1^\infty = \frac{1}{e}$$

Since, this integral is convergent, the given integral is convergent.

2. $\int_2^\infty \frac{1}{\log x}dx$

Since, $\log x < x \ \forall \ x \ge 2$

Thus, $\frac{1}{\log x} > \frac{1}{x}$

Checking the convergence of the integral,

$$\int_2^\infty \frac{1}{x}dx = \log x\Big]_2^\infty = \infty$$

Since, this integral is divergent. The integral $\int_2^\infty \frac{1}{\log x}dx$ is divergent.

3. $\int_1^\infty \frac{1}{x^2(e^{-x}+1)}dx$

The function $x^2\left(1+e^{-x}\right) > (0+1)x^2$

Thus, $\dfrac{1}{x^2\left(1+e^{-x}\right)} \le \dfrac{1}{x^2}$

$$\frac{f(x)}{g(x)} = \frac{1}{e^{-x}+1} \Rightarrow \lim_{x\to\infty}\frac{f(x)}{g(x)} = \frac{1}{0+1} = 1$$

$$g(x) = \frac{1}{x^2}$$

$$\int_1^\infty \frac{1}{x^2}dx = \left[\frac{-1}{x}\right]_1^\infty = 1$$

Both integrals will converge together. Thus, the integral $\displaystyle\int_1^\infty \frac{1}{x^2\left(e^{-x}+1\right)}dx$ is convergent.

4. $\displaystyle\int_1^\infty \frac{x\tan^{-1}x}{\sqrt{4+x^3}}dx =$ ______

$$f(x) = \frac{x\tan^{-1}x}{x\sqrt{x}\sqrt{\frac{4}{x^3}+1}}$$

Let $g(x) = \dfrac{1}{\sqrt{x}}$

$$\frac{f(x)}{g(x)} = \frac{\tan^{-1}x}{\sqrt{\frac{4}{x^3}+1}} \to \lim_{x\to\infty}\frac{f(x)}{g(x)} = \frac{\pi/2}{\sqrt{0+1}} = \frac{\pi}{2}$$

Thus, the integrals $\displaystyle\int_1^\infty \frac{x\tan^{-1}x}{\sqrt{4+x^3}}dx$ and $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}dx$ t converge and diverge together

$$\int_1^\infty \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\Big|_1^\infty = \infty$$

Since, one integral diverges. Both the integrals will diverge.

Comparison Test for Second Kind of Improper Integrals

1st method remains same for both kind of improper integrals. For the second method i.e. the limit form,

Let f(x) and g(x) be two +ve functions such that

1. 'a' is a point of discontinuity and $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)} = l_1$ [non zero, finite] or

2. 'b' is a point of disontunity and and $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)} = l_2$ [Non zero, finite then $\displaystyle\int_a^b f(x)dx$ and $\displaystyle\int_a^b g(x)dx$ both converge (or) diverge together.

Solved Examples

Example: Check the convergence of the following integral $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x\sqrt{x}}dx$

Solution: Since, $\dfrac{\sin x}{x} \le 1$

Thus, $\dfrac{\sin x}{x}\cdot\dfrac{1}{\sqrt{x}} \le \dfrac{1}{\sqrt{x}}$

Checking the convergence of the integral

$$\int_0^{\pi/2} \frac{1}{\sqrt{x}}dx$$

$$\int_0^{\pi/2} \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\Big]_0^{\pi/2} = 2\sqrt{\frac{\pi}{2}}$$

Thus, this integral is convergent.

$\therefore \displaystyle\int_0^{\pi/2} \frac{\sin x}{x\sqrt{x}}dx$ is convergent

Example: Check the convergence of the integral $\displaystyle\int_1^2 \frac{\sqrt{x}}{\log x}dx$

Solution: Since, logx < x

Thus, $\dfrac{1}{\log x} > \dfrac{1}{x} \Rightarrow \dfrac{\sqrt{x}}{\log x} > \dfrac{\sqrt{x}}{x} > \dfrac{1}{\sqrt{x}}$

Checking the convergence of the integral

$$\int_1^2 \frac{1}{\sqrt{x}}dx$$

$$\int_1^2 \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\Big]_1^2 = 2\sqrt{2} - 2$$

Since, this integral is finite so it converges.

Thus, $\int_1^2 \frac{\sqrt{x}}{\log x}dx > \int_1^2 \frac{1}{\sqrt{x}}dx$ may be convergent or divergent.

Applying Method II

Now let $g(x) = \frac{1}{x\log x} \Rightarrow \frac{f(x)}{g(x)} = x\sqrt{x}$

$$\lim_{x\to 1^+} \frac{f(x)}{g(x)} = 1$$

Checking the convergence of $\int_1^2 \frac{1}{x\log x}dx$

Let log x = t

$$\frac{1}{x}dx = dt$$

$$\therefore \int_0^1 \frac{1}{x\log x}dx = \int_0^{\log 2} \frac{1}{t}dt = \log t\Big]_0^{\log 2} = \infty$$

Thus, $\int_1^2 \frac{\sqrt{x}}{\log x}dx$ also diverges.

Example: Calculate the integral $\int_0^1 x\log x\,dx$

Solution: Applying Integration by parts

$$\int_0^1 x\log x dx = \log x\frac{x^2}{2} - \int \frac{1}{x}\frac{x^2}{2}dx = \left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right]_0^1$$

$$I = \frac{-1}{4} - \lim_{x\to 0}\frac{x^2}{2}\log x = \frac{-1}{4} - \lim_{x\to 0}\frac{\log x}{2/x^2}$$

Using L' Hospital's Rule $\lim_{x\to 0}\frac{\log x}{2/x^2} = \lim_{x\to 0}\frac{1/x}{-4/x^3}$

$$I = -\frac{1}{4} - \left[\lim_{x\to 0}\frac{-x^2}{4}\right] = -\frac{1}{4} - 0 = -\frac{1}{4}$$

Gamma Function

$$\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx$$

Note:

1. $\Gamma 1 = 1$
2. $\Gamma \frac{1}{2} = \sqrt{\pi}$
3. $\Gamma(n+1) = n\Gamma n \quad \forall n > 0$
4. $\Gamma(n+1) = n! \quad \forall n \in Z^+$
5. $\int_0^\infty e^{-ax}x^{n-1}dx = \frac{\Gamma n}{a^n}$

Solved Examples

Example: Calculate the integral $\int_0^\infty e^{-x^2}dx$?

Solution: Let $x^2 = t \Rightarrow 2xdx = dt$

$$dx = \frac{1}{2}t^{-1/2}dt$$

$$\int_0^\infty e^{-x^2}dx = \int_0^\infty e^{-t}\cdot\frac{1}{2}t^{-1/2}dt = \frac{1}{2}\int_0^\infty e^{-t}t^{\frac{1}{2}-1}dt = \frac{1}{2}\Gamma\frac{1}{2} = \frac{\sqrt{\pi}}{2}$$

Example: Calculate the integral $\int_0^\infty e^{-y^3}y^{1/2}dy$

Solution: Let $y^3 = t \Rightarrow 3y^2dy = dt$

$$dy = \frac{1}{3}t^{-2/3}dt$$

Thus, $I = \int_0^\infty e^{-t}t^{1/6}\frac{1}{3}t^{-2/3} = \frac{1}{3}\int_0^\infty e^{-t}t^{-1/2}dt$

$$I = \frac{1}{3}\Gamma\frac{1}{2} = \frac{\sqrt{\pi}}{3}$$

Example: Determine the value of integral

$$\int_0^1 (x\log x)^4 dx$$

Solution: Let $\log x = -t \Rightarrow x = e^{-t}$

$$dx = -e^{-t}dt$$

$$I = \int_\infty^0 \left(e^{-t}(-t)\right)^4\left(-e^{-t}\right)dt = \int_\infty^0 e^{-5t}t^{5-1} = \frac{\sqrt{5}}{5^5} = \frac{4!}{5^5}$$

Example: Determine the integral $\int_0^\infty 5^{-4x^2}dx$

Solution: Let $5^{-x2} = e^{-t}$

$$-4x^2\log 5 = -t$$

$$x = \frac{1}{2\sqrt{\log 5}}\sqrt{t}$$

$$dx = \frac{1}{2\sqrt{\log 5}} \times \frac{1}{2\sqrt{t}} dt$$

$$I = \int_0^\infty e^{-t} \frac{1}{4\sqrt{\log 5}} t^{-1/2} dt = \frac{1}{4\sqrt{\log 5}} \left\lceil \frac{1}{2} = \frac{\sqrt{\pi}}{4\sqrt{\log 5}}\right.$$

Beta Function:

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \quad (m>0, n>0)$$

Note:

1. $\beta(m,n) = \beta(n,m)$
2. $\beta(m,n) = \dfrac{\lceil m \lceil n}{\lceil m+n}$
3. $\beta(m,n) = \displaystyle\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$
4. $\beta(m,n) = 2\displaystyle\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$ is

   $\displaystyle\int_0^{\pi/2} \sin^p\theta \cos^q\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

Solved Examples

Example: Determine the integral

$$\int_0^2 x^7\left(16 - x^4\right)^{10} dx ?$$

Solution: Let $x^4 = 16t \Rightarrow 4x^3dt = 16dt$

$$I = \int_0^1 16t(16-16t)^{10} 4dt = 4 \times 16^{11}\int_0^1 t^{2-1}(1-t)^{11-1}dt$$

$$I = 4 \times 16^{11} \times \beta(2,11)$$

$$I = 4 \times 16^{11} \times \frac{\lceil 2 \times \lceil 11}{\lceil 13} = 4 \times 16^{11} \times \frac{1 \times 10!}{12!}$$

Example: Calculate the integral $\displaystyle\int_0^\infty \frac{x^3\left(1+x^3\right)}{(1+x)^{13}} dx$?

Solution: $I = \displaystyle\int_0^\infty \frac{x^3}{(1+x)^{13}}dx + \int_0^\infty \frac{x^8}{(1+x)^{13}}dx$

$$I = \int_0^\infty \frac{x^{4-1}}{(1+x)^{4+9}}dx + \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+4}}dx$$

$$I = \beta(4,9) + \beta(9,4)$$

$$I = 2\beta(4,9) = 2\left(\frac{\lceil 4 \cdot \lceil 9}{\lceil 13}\right)$$

$$I = 2\left[\frac{3! \times 8!}{12!}\right]$$

Example: Calculate the value of $\displaystyle\int_0^\infty \left[\frac{x}{1+x^2}\right]^3 dx$

Solution: Let $x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta$

$$I = \int_0^{\pi/2}\left[\frac{\tan\theta}{\sec^2\theta}\right]^3 \sec^2\theta d\theta = \int_0^{\pi/2} \frac{\tan^3\theta}{\sec^4\theta}d\theta$$

$$= \int_0^{\pi/2} \sin^3\theta \cos\theta d\theta$$

Let $\sin\theta = t \Rightarrow \cos\theta d\theta = dt$

$$I = \int_0^1 t^3 dt = \frac{1}{4}$$

Multiple Integrals

For multi-variable function, integral is defined in more than one dimensions and then it is called as Multiple Integral.

Double Integral

Consider the function f (x, y) of the independent variables x, y defined at each point in the finite region R of the xy-plane. Divide R into n elementary areas δA_1, δA_2 ,........., δA_n. Let (x_r, y_r) be any point in the r^{th} elementary area δA_r. Consider the sum

$$f(x_1,y_1)\delta A_1 + f(x_2,y_2)\delta A_2 + \ldots\ldots + f(x_n,y_n)\delta A_n$$

i.e., $\displaystyle\sum_{r=1}^{n} f(x_r,y_r)\delta A_r$

the limit of this sum, if it exists, as the number of sub-divisions increases indefinitely and area of each sub-division decreases to zero, is defined as the double integral of f(x, y) over the region R and is written as $\iint_R f(x,y)dA$

Thus, $\displaystyle\iint_R f(x,y)dA = \underset{\substack{n\to\infty \\ \delta A\to 0}}{\text{Lt}} \sum_{r=1}^{n} f(x_r,y_r)\delta A_r$

Thus, double integral can be evaluated as the limit of a sum but the process in such a case becomes tedious and hence we try to evaluate double integral in terms of single integrals.

Evaluating a Double Integral

Case 1: When the limits are $y = \phi_1$ (x) to $y = \phi_2$ (x) and $x = c_1$ to $x = c_2$

$$\therefore \iint_{P_1} f(x,y)dxdy = \int_{c_1}^{c_2}\left[\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy\right]dx$$

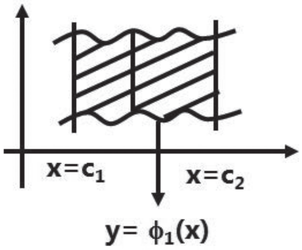


Fig. 2.7

In this case, we first keep x-constant and evaluate the integral with respect to y but the limits of y are a function of 'x' and hence we obtain a function of x which we again integrate with constant limits with respect to 'x'.

This can be understood graphically as we are determining area by summing the area of vertical strips.

Case 2 When the limits are $x = \phi_1$ (y) to $x = \phi_2$ (y) and $y = d_1$ to $y = d_2$

$$\therefore \iint_{P_1} f(x,y)dxdy = \int_{y=d_1}^{d_2}\left[\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dx\right]dy$$

In this case, we first keep y-constant and evaluate the integral with respect to x but the limits of x are a function of 'y' and hence we obtain a function of y which we again integrate with constant limits with respect to 'y'.

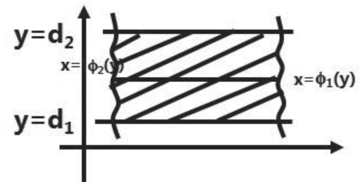


Fig. 2.8

This can be understood graphically as we are determining area by summing the area of horizontal strips.

Case 3: When the limits are $x = c_1$ to $x = c_2$ and $y = d_1$ to $y = d_2$

$$\iint_{P_1} f(x,y)dxdy$$

$$= \int_{x=c_1}^{c1}\left[\int_{y=d_1}^{d_1} f(x,y)dx\right]dy = \int_{y=d_1}^{d_1}\left[\int_{y=c_1}^{c_1} f(x,y)dx\right]dy$$

Here, since both limits are independent of each other, the order of integration does not matter.

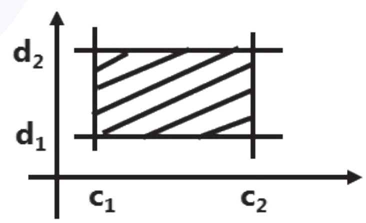


Fig. 2.9

The domain in this case is rectangular. So we can determine the area in terms of horizontal as well as vertical strips.

Solved Examples

Example: Evaluate the following integral

$$\int_1^2\int_3^4 \frac{1}{(x+y)^2}dxdy$$

Solution: Since both limits are independent of each other, the order of integration does not matter. Integrating first w.r.t. x

$$I = \int_1^2\left[\frac{-1}{x+y}\right]_3^4 dy = \int_1^2\left[\frac{1}{3+y} - \frac{1}{4+y}\right]dy$$

$$I = \left[\log\left(\frac{3+y}{4+y}\right)\right]_1^2 = \log\frac{5}{6} - \log\frac{4}{5} = \log\frac{25}{24}$$

Example: Evaluate the following integral

$$\int_0^3\int_0^x (6-x-y)dydx$$

Solution: Here, the limit of y depends on x. So, we first have to integrate w.r.t. y and then integrate w.r.t. x

$$\int_0^3\int_0^x (6-x-y)dydx = \int_0^3\left(6y - xy - \frac{y^2}{2}\right)_0^x dx$$

$$= \int_0^3\left(6x - x^2 - \frac{x^2}{2}\right)_0^x dx$$

$$I=\left[\frac{6x^2}{2}-\frac{3x^3}{6}\right]_0^3=27-\frac{27}{2}=\frac{27}{2}$$

Example: Evaluate the following integral $\int_0^4\int_0^{y^2} e^{x/y}dxdy$

Solution: Here, the limit of x is dependent on y. So, we first have to integrate w.r.t. x

$$\int_0^4\int_0^{y^2} e^{x/y}dxdy=\int_0^4\left[\frac{e^{x/y}}{1/y}\right]_0^{y^2}dy=\int_0^4 y\left[e^y-1\right]dy$$

$$I=e^y(y-1)-\frac{y^2}{2}\Big]_0^4=\left(3e^4-8\right)+1=3e^4-7$$

Example: The value of $\iint_R xy\ dxdy$ where R is the region of the 1st quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is______________.

Solution: In first quadrant both x and y are positive. Evaluating the limits of x in terms of y

x can go to a maximum value of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ i.e. } x=\frac{a}{b}\sqrt{b^2-y^2}$$

$$\int_0^b\int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy\,dxdyy=\int_0^b\left[y\frac{x^2}{2}\right]_0^{\frac{a}{b}\sqrt{b^2-y^2}}dy$$

$$I=\int_0^b\left[\frac{y}{2}\frac{a^2}{b^2}\left(b^2-y^2\right)\right]dy$$

$$I=\frac{a^2}{2b^2}\left[\frac{b^2y^2}{2}-\frac{y^4}{4}\right]_0^b=\frac{a^2}{2b^2}\left[\frac{b^4}{2}-\frac{b^4}{4}\right]=\frac{a^2b^2}{8}$$

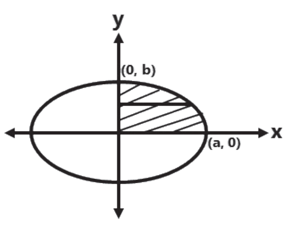


Fig. 2.10

Example: The value of $\iint_R y\ dxdy$ where R is the area bounded by x = 0, y = x^2 , x + y = 2 in the 1st quadrant is?

Solution: The region bounded by these curves is shown below, For a fixed x, y varies from x^2 to (2 − x)

$$\iint_R y\,dxdy=\int_0^1\int_{x^2}^{2-x} y\,dydx=\int_0^1\left[\frac{y^2}{2}\right]_{x^2}^{2-x}dx$$

$$I=\int_0^1\left[\frac{(2-x)^2}{2}-\frac{x^4}{2}\right]dx=\frac{1}{2}\int_0^1\left[4-x^2-4x-x^4\right]dx$$

$$I=\frac{1}{2}\left[4x+\frac{x^3}{3}-\frac{4x^2}{2}-\frac{x^5}{5}\right]_0^1=\frac{1}{2}\left[4+\frac{1}{3}-2-\frac{1}{5}\right]_0^1$$

$$=\frac{16}{15}$$

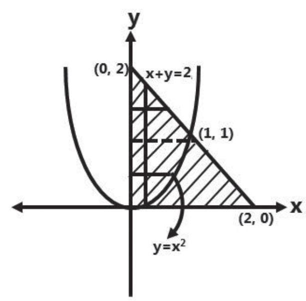


Fig. 2.11

Example: The value of $\iint_R r^2\ \sin\theta\ drd\theta$ for the curve r = 2a cosθ in the first quadrant.

Solution: In the first quadrant,

$r=0$ to $2a\cos\theta$

$\theta=0$ to $\frac{\pi}{2}$

$$\iint_R r^2\sin\theta\,drd\theta=\int_0^{\pi/2}\int_0^{2\cos\theta} r^2\sin\theta\,drd\theta$$

$$=\int_0^{\pi/2}\sin\theta\left[\frac{r^3}{3}\right]_0^{2a\cos\theta}d\theta$$

$$I=\frac{1}{3}\int_0^{\pi/2}8a^3\cos^3\theta\sin\theta\,d\theta$$

$$=\frac{8a^3}{3}\left[\frac{-\cos^4\theta}{4}\right]_0^{\pi/2}=\frac{2a^3}{3}$$

Note: The plots for the some of the regions in x-y plane are,

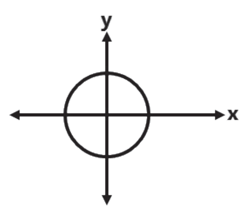


Fig. 2.12

r = a

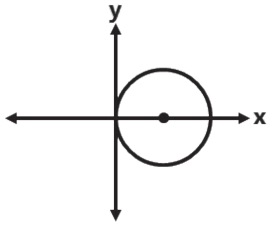


Fig. 2.13

r = acosθ

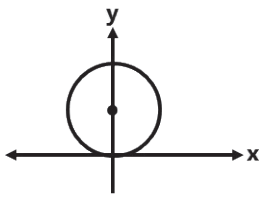


Fig. 2.14

r = asinθ

Area of the Region

The area of the region bounded by y = f(x) & y=g(x) between x = c_1 and x = c_2 is

$$A = \int_{x=c_1}^{c_2} [g(x) - f(x)]dx$$

It can also be computed in terms of Double Integral as,

$$A = \int_{c_1}^{c_2} \int_{f(x)}^{g(x)} 1\,dy\,dx$$

In polar form,

$$\text{Area} = \int_{\theta_1}^{\theta_2} \int_{f=f_1(\theta)}^{f_2(\theta)} r\,dr\,d\theta$$

Solved Examples

Example: The area bounded by the curve y = x^2 & y = x is ?

Solution: The point of intersection of these two curves have been marked in the figure below,

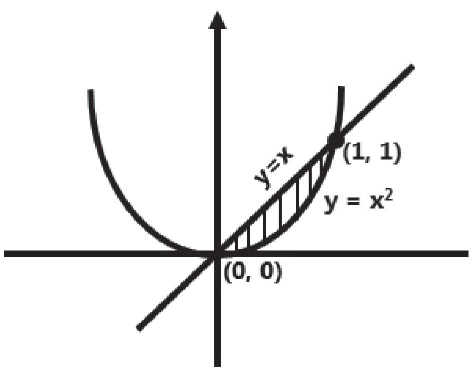


Fig. 2.15

The shaded area marks the area bounded by the two curves.

$$A = \int_0^1 \left[x - x^2\right]dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Example: The area bounded by 2y = x^2 and x = y – 4 is ?

Solution: To determine the points of intersection of these two curves,

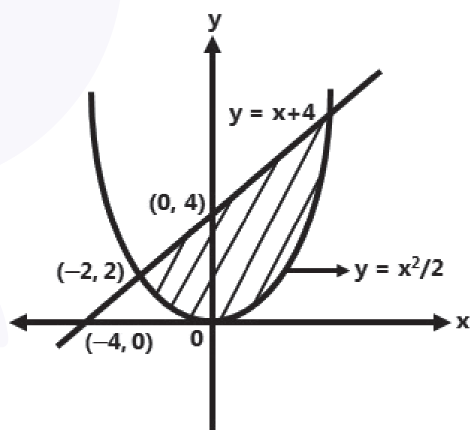


Fig. 2.16

$$x = \frac{x^2}{2} - 4 \Rightarrow x^2 - 2x - 8 = 0 \Rightarrow (x-4)(x+2) = 0$$

x = 4, –2 and y = 8, 2

$$A = \int_{-2}^{4} \left(x + 4 - \frac{x^2}{2}\right)dx = \left[\frac{x^2}{2} + 4x - \frac{x^3}{6}\right]_{-2}^{4}$$

$$A = \left(8 + 16 - \frac{64}{6}\right) - \left(2 - 8 + \frac{8}{6}\right) = 18$$

Example: Area in between the circle r = 2sinθ and r = 4sinθ is?

Solution: The given two curves are r = 2sinθ to r = 4sinθ

$\theta = 0$ to π

$$A = \int_0^{\pi}\int_{2\sin\theta}^{4\sin\theta} r\,dr\,d\theta = \int_0^{\pi}\left[\frac{r^2}{2}\right]_{2\sin\theta}^{4\sin\theta} d\theta = \frac{1}{2}\int_0^{\pi} 12\sin^2\theta d\theta$$

$$A = 3\left[\int_0^{\pi}(1-\cos 2\theta)d\theta\right] = 3\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} = 3\pi$$

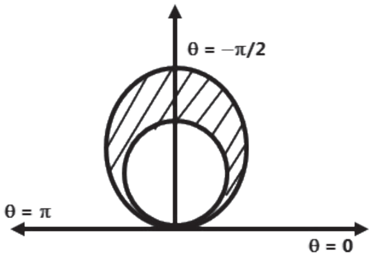


Fig. 2.17

or

Comparing these equations with equation of circle r = 2asinθ

where, a is the radius of the circle.

$r_1 = 1$, $r_2 = 2$

Since area of any circle is $A = \pi r^2$

$A_1 = \pi$, $A_2 = 4\pi$

$\therefore A_2 - A_1 = 3\pi$

Changing of Order of Integration

If we wish to reverse the order of integration i.e. if the integration is given first w.r.t. x and then w.r.t. y and we need to change the order i.e. first w.r.t. y and then w.r.t. x. This is called as change in order.

Essentially, we are converting the problem from finding the area in terms of Vertical Strips to a problem where we need to calculate the area in terms of Horizontal Strips.

This method is illustrated in the problems below,

Solved Examples

Example: Determine the value of integral

$$\int_0^{\infty}\int_x^{\infty}\frac{e^{-y}}{y}dydx$$

Solution: The domain of above integration is shown in the figure below,

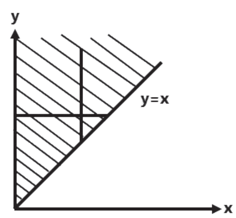


Fig. 2.18

y=x to y=∞

x=0 to x=∞

From the figure, same area can be expressed as,

x=0 to x=y

y=0 to y=∞

$$I = \int_0^{\infty}\int_0^{y}\frac{e^{-y}}{y}dxdy$$

$$I = \int_0^{\infty}\frac{e^{-y}}{y}[x]_0^y dy = \int_0^{\infty}\frac{e^{-y}}{y}ydy = \left.\frac{e^{-y}}{-1}\right]_0^{\infty} = 1$$

Here, by changing the order of integration the integration is simplified.

Example: By reversing the order of integration, the integral $I = \int_0^8\int_{x/4}^2 f(x,y)dydx$ reduces to $I = \int_r^s\int_p^q f(x,y)dydx$ then q =___.

Solution: Given limits are $y = \frac{x}{4}$ to $y = 2$ and x = 0 to x = 8

If we reverse the order, the limits can be written as,

x = 0 to x = 4y

y = 0 to y = 2

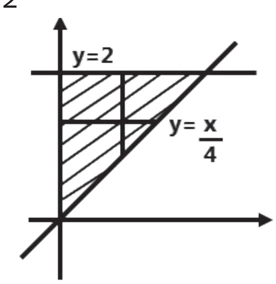


Fig. 2.19

$$I = \int_0^2 \int_0^{4y} f(x, y)dxdy$$

Thus, q = 4y

Triple Integral

Consider a function f(x, y, z) defined at every point the 3-dimentional finite region V. Divide V into n elementary volumes δV_1, δV_2,........., δV_n

Let (x_r, y_r, z_r) be any point within the r_{th} sub-division of the volume

Consider the sum

$$\sum_{r=1}^{\infty} f\left(x_r, y_r, z_r\right)\delta V_r$$

The limit of this sum, if it exists, as n → ∞ and $\delta V_r \to 0$ is called the triple integral of f(x, y, z) over the region V and is denoted by

$$\iiint_V f(x, y, z)dV$$

For purposes of evaluation, it can also be expressed as the repeated integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f\left(x_1 y_1 z\right)dxdydz$$

If x_1, x_2 are constant, y_1, y_2 are either constants or functions of x and z_1, z_2 are either constants or functions of x and y.

Thus, $\iiint_R f(x, y, z)dxdydz$

$$= \int_{x-c_1}^{c_2}\left[\int_{y=g_1(x)}^{g_1(x)}\left[\int_{g=h(x,y)}^{h(x,y)} f(x, y, z)dz\right]dy\right]dx$$

Solved Examples

Example: Determine the integral $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}$ dzdydx

Solution: Integrate first w.r.t. z keeping x and y as constant

$$I = \int_0^a\int_0^x \left(e^{x+y+z}\right)_0^{x+y} dydx$$

$$= \int_0^a\int_0^x \left[e^{2(x+y)} - e^{(x+y)}\right]dydx$$

Now, integrate w.r.t. y keeping x as constant

$$I = \int_0^a \left[\frac{e^{2(x+y)}}{2} - e^{x+y}\right]_0^x dx$$

$$= \int_0^a \left[\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^x\right]dx$$

$$I = \left[\frac{e^{4x}}{8} - 3\frac{e^{2x}}{4} + e^x\right]_0^a dx$$

$$= \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a - \left(\frac{1}{8} - \frac{3}{4} + 1\right)$$

$$= \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a - \frac{3}{8}$$

Example: Evaluate the integral over region $R\iiint_R ydxdydz$ where the region R is defined by, x = 0, y = 0, z = 0, x + y + z = 1 ?

Solution: Expressing the limits of z in terms of x and y

z goes from 0 to (1-x-y);

Similarly, maximum value of y occurs when z = 0 and then y = 1-x.

Thus, x goes from 0 to 1

$$I = \int_0^1\int_0^{1-x}\int_0^{1-x-y} ydzdydx = \int_0^1\int_0^{1-x} y[z]_0^{1-x-y} dydx$$

$$I = \int_0^1\int_0^{1-x} y(1-x-y)dydx = \int_0^1 \left[(1-x)\frac{y^2}{2} - \frac{y^3}{3}\right]_b^{1-x} dx$$

$$I = \int_0^1 \left[\frac{(1-x)^3}{2} - \frac{(1-x)^3}{3}\right]dx$$

$$= \frac{1}{6}\int_0^1 (1-x)^3 dx = \frac{1}{6}\left[\frac{(1-x)^4}{-4}\right]_0^1$$

$$I = \frac{1}{6}\left[0 + \frac{1}{4}\right] = \frac{1}{24}$$

Change of Variables

In double integral

Let x = f(u, v), y = g(u, v)

Suppose, we wish to change the integral in terms of u,v variables from x,y variables.

$$\iint_R \phi(x, y)dxdy = \iint_R \phi(u, v)\,|\,\mu\,|\,dudv$$

|J| = Jacobian of transformation

$$|J| = J\left[\frac{x,y}{u,v}\right] = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

Case 1: Cartesian Form → Polar Form

$(x, y) \to (r, \theta)$

$x = r\cos\theta,\ y = r\sin\theta$

$x^2 + y^2 = r^2$

$$|J| = J\left[\frac{x,y}{r,\theta}\right] = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$|J| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\therefore \int\int_R \phi(x,y)dxdy = \int\int_R \phi\left(r\cos\theta_{,r}r\sin\theta\right)rdrd\theta$$

If we wish to change the variables in triple integral,

Case 2: Cartesian → cylindrical

$(x, y, z) \to (r, \theta, z)$

$x = r\cos\theta,\ y = r\sin\theta,\ z = z$

$x^2 + y^2 = r^2,\ z = z$

$$|J| = J\left[\frac{x,y,z}{r,\theta,z}\right] = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$$\therefore \int\iint_V \phi(x,y,z)dxdydz$$

$$= \int\iint_V \phi(r\cos\theta, r\sin\theta, z)rdrd\theta d$$

Case 3: Cartesian → spherical polar form

$(x, y, z) \to (r, \theta, \phi)$

$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$

$x^2 + y^2 + z^2 = r^2$

$$|\mu| = J\left[\frac{x,y,z}{r,\theta,\phi}\right] = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta$$

Different variables have the following ranges for Cylindrical and Spherical Systems:

Cylinder	Sphere
$r : 0$ to r	$r : 0$ to r
$\theta : 0$ to 2π	$\Phi : 0$ to 2π
$z : z_1$ to z_2	$\theta : 0$ to π

Solved Examples

Example: Evaluate the integral $\int_0^\infty\int_0^\infty e^{-(x^2+h^2)}dxdy$

Solution: Changing to Polar Co-ordinates,

$x = r\cos\theta$ and $y = r\sin\theta$

$|J| = r$

Limits: $r : 0$ to ∞

$\theta : 0$ to $\frac{\pi}{2}$ (Since, x and y lie only in first quadrant)

$$I = \int_0^{\pi/2}\int_0^\infty e^{-r^2}rdrd\theta$$

$$\text{Let } r^2 = t \Rightarrow rdr = \frac{dt}{2}$$

$$I = \int_0^{\pi/2}\int_0^\infty e^{-t}\frac{dt}{2}d\theta = \frac{1}{2}\int_0^{\pi/2}\left[\frac{e^{-t}}{-1}\right]_0^\infty d\theta = \frac{1}{2}\times 1\times\frac{\pi}{2} = \frac{\pi}{4}$$

Example: Express the following integral by change of variables to Spherical Polar Coordinates,

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dzdydx}{\sqrt{1-x^2-y^2-z^2}} ?$$

Solution: Cartesian Coordinates in terms of Spherical Polar Coordinates are given as,

Let y = r sinθ sinϕ, x = r sinθ cosϕ, z = r cosθ

Thus, $x^2 + y^2 + z^2 = r^2$ and |J| = r^2 sinθ

Since, z varies from

$$z = 0 \text{ to } z = \sqrt{1-x^2-y^2} \Rightarrow x^2 + y^2 + z^2 = 1$$

Thus, the limits of Spherical Polar Co-ordinates are,

r : 0 to 1

$\phi : 0$ to $\frac{\pi}{2}$

$\theta : 0$ to $\frac{\pi}{2}$

$$I = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \frac{r^2 \sin\theta}{\sqrt{1-r^2}} dr d\theta d\phi$$

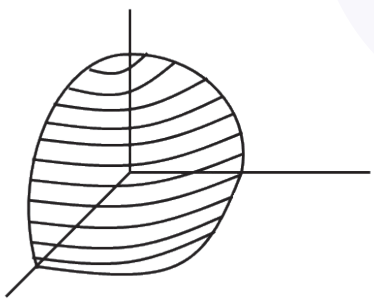


Fig. 2.20

Example: If the following relation exists between (x,y) and (u,v) variables

$$x(u,v) = uv, y(u,v) = \frac{v}{u}.$$

Then, In a double integral f(x, y) changes to

$$f\left(uv, \frac{v}{u}\right)\phi(u,v) \text{ then } \phi(u,v) = \underline{\qquad\qquad} ?$$

Solution:

$$\phi(u,v) = |J| = J\left[\frac{x,y}{u,v}\right] = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

$$= \begin{vmatrix} v & u \\ \frac{-v}{u^2} & \frac{1}{u} \end{vmatrix} = \frac{v}{u} + \frac{v}{u} = \frac{2v}{u}$$

Length of a Curve

Suppose we have to determine the length of curve y = f (x) between $x = x_1$ and $x = x_2$

$$L = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

In polar form, $V = \int_{x1}^{x_2} \pi y^2 dx$

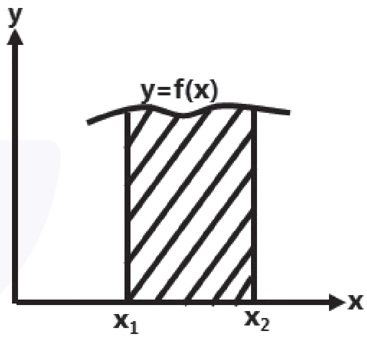


Fig. 2.21

Volume Of Solid Revolution

Volume generated by revolving the area enclosed by y = f(x) between $x = x_1$ and $x = x_2$ about x or u.

$$V = \int_{x1}^{x_2} \pi y^2 dx$$

Similarly volume generated by revolution about y or v

$$V = \int_{x_1}^{x_2} \pi x^2 dy$$

In polar form,

About initial line, $V = \int_{\theta_1}^{\theta_2} \frac{2\pi}{3} r^3 \sin\theta \, d\theta$

About the line $\theta = \frac{\pi}{2}, V = \int_{\theta_1}^{\theta_2} \frac{2\pi}{3} r^3 \cos\theta \, d\theta$

Solved Examples

Example: The length of the curve $y = \frac{2}{3}x^{\frac{3}{2}}$ between x = 0 and x = 1 is______.

Solution: $\frac{dy}{dx} = \frac{2}{3} \times \frac{3}{2} x^{\frac{1}{2}} = \sqrt{x}$

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \left.\frac{(1+x)^{3/2}}{3/2}\right]_0^1 = \frac{2}{3}(2\sqrt{2} - 1)$$

Example: Find the length of the curve defined by Cardioid r = a(1 + cosθ) where 0 < θ < π?

Solution:

$$L = \int_0^\pi \sqrt{r^2 + (-a\sin\theta)^2} d\theta$$

$$= \int_0^\pi \sqrt{a^2\left(1 + \cos^2\theta + 2\cos\theta\right) + a^2\sin^2\theta d\theta}$$

$$L = a\int_0^\pi \sqrt{2(1+\cos\theta)} d\theta$$

$$L = a\int_0^\pi 2\cos\frac{\theta}{2} d\theta = 2a\left[\frac{\sin\frac{\theta}{2}}{\frac{1}{2}}\right]_0^\pi = 4a[1-0] = 4a$$

Example: Determine the volume of revolution generated by revolving the curve $\frac{x^2}{4} + \frac{y^2}{4} = 1$ about x-axis.

Solution: $v = \int_{x_1}^{x_2} \pi y^2 dx = \int_{-2}^{2} \pi\left(4 - x^2\right) dx$

$$V = 2\pi \times \left[4x - \frac{x^3}{3}\right]_0^2$$

$$V = 2\pi\left[\frac{16}{3}\right] = \frac{32\pi}{3}$$

Example: The volume obtained by moving the area bounded by the parabola $y^2 = 8x$ and x = 2 and y axis is _____.

Solution: The volume generated by rotating the straight line x = 2 between y = –4 to +4 about y

$$V_1 = \int_{-4}^{4} \pi x^2 dy = \int_{-4}^{4} \pi(2)^2 dy = 4\pi[y]_{-4}^{4} = 32\pi$$

The volume generated by rotating the parabola $x = \frac{y^2}{8}$ between x = –4 to 4 about y axis is

$$V_2 = \int_{-4}^{4} \pi x^2 dy = \int_{-4}^{4} \pi\left(\frac{y^4}{64}\right) dy = \frac{\pi}{64}\left[\frac{y^5}{5}\right]_{-4}^{4} = \frac{32\pi}{5}$$

$\therefore$ required volume $= V_1 - V_2 = 32\pi - \frac{32\pi}{5} = \frac{128\pi}{5}$

Note: Some of the equations of different curves are given as follows.

Circle: Cartesian Form

1. $x^2 + y^2 = a^2$: Circle with centre (0, 0) and radius a.
2. $(x - h)^2 + (y - k)2 = a^2$: Circle with centre (h, k) and radius a.

Polar Form:

1. r = a : Circle with centre (0, 0) and radius a.
2. r = a sinθ : Circle with centre $\left(0, \frac{a}{2}\right)$ and radius $a/2$
3. r = a cosθ : Circle with centre $\left(\frac{a}{2}, 0\right)$ and radius $a/2$

Parabola:

1. $x^2 = 4ay$: Parabola with vertex at (0, 0) and focus at (0, a) and latus rectum = 4a
2. $x^2 = -4ay$: Parabola with vertex at (0, 0) and focus at (0, -a) and latus rectum = 4a
3. $y^2 = 4ax$: Parabola with vertex at (0, 0) and focus at (a, 0) and latus rectum = 4a
4. $y^2 = -4ax$: Parabola with vertex at (0, 0) and focus at (-a, 0) and latus rectum = 4a
5. $(x - h)^2 = 4a(y - k)^2$: Parabola with centre at (h, k) and focus at (0 + h, a + k) and latus rectum = 4a

Ellipse:

1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$: Ellipse with centre at (0, 0) and major axis = 2a and minor axis = 2b .
2. $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$: Ellipse with centre at (h, k) and major axis = 2a and minor axis = 2b .

Hyperbola:

1. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$: Hyperbola with vertex at (a, 0) and (-a, 0) and centre at (0, 0).

2. $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$: Hyperbola with vertex at (0, b) and (0, -b) and centre at (0, 0).

Vectors

There are two types of quantities: scalars and vectors. Scalars are those quantities which only have only magnitude like length, temperature and voltage. Vectors are those quantities which have magnitude as well as direction like Force is a vector which has a magnitude and direction of application and velocity which has a magnitude i.e. speed and the direction of motion.

A vector has a tail i.e. initial point and a tip or terminal point. The length (or magnitude) of a vector a (length of the arrow) is also called the norm (or Euclidean norm) of and is denoted by |a|.

A vector of length 1 is called a unit vector.

Components of a Vector

We choose an xyz Cartesian coordinate system in space, that is, a usual rectangular coordinate axes. Then if a given vector 'a' has initial point P : (x_1, y_1, z_1) and terminal point Q : (x_2, y_2, z_2) the three numbers.

$a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$, $a_3 = z_2 - z_1$; are called the components of the vector a with respect to that coordinate system, and we write simply $a = [a_1, a_2, a_3]$

By definition, the length |a| of a vector a is the distance between its initial point P and terminal point Q. Then, $|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

If the initial point of the vector is taken as origin, then it is called as Position Vector. The position vector of point A(x, y, z) is a vector with initial point (0, 0, 0) and terminal point (x, y, z) thus, it has the components $a_1 = x$, $a_2 = y$, $a_3 = z$

Vector Addition

The sum a + b of two vectors $a = [a_1, a_2, a_3]$ and $b = [b_1, b_2, b_3]$ is obtained by adding $a + b = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$

Geometrically, the two vectors can be added using Triangle Law or Parallelogram Law. By Parallelogram Law, the resultant vector has a magnitude of,

$$|a+b| = \sqrt{|a|^2 + |b|^2 + 2|a||b|\cos\phi}$$

where, Φ is the angle between the two vectors.

The angle of resultant vector with respect to A is given by,

$$\theta = \tan^{-1}\left(\frac{b\sin\phi}{a + b\cos\phi}\right)$$

Properties:

(a) $\dot{a} + \dot{b} = \dot{b} + \dot{a}$ (commutativity)

(b) $(\dot{U} + \dot{V}) + \dot{W} = \dot{U} + (\dot{V} + \dot{W})$ (associativity).

(c) $\dot{a} + 0 = 0 + \dot{a} = a$

(d) $\dot{a} + (-\dot{a}) = 0$

where −a denotes the vector having the length |a| and the direction opposite to that of a.

Vector Subtraction

To subtract one vector from the other we first reverse the direction of one vector and then add it to another vector.

To perform a − b , we first determine −b which has the same length as the b vector but has opposite direction as compared to b.

By component method, $a - b = [a_1 - b1, a_2 - b_2, a_3 - b_3]$

By parallelogram law, the resultant vector has a magnitude of

$$|a-b| = \sqrt{|a|^2 + |b|^2 - 2|a||b|\cos\phi}$$

The angle of resultant vector with respect to A is given by, $\theta = \tan^{-1}\left(\frac{-b\sin\phi}{a - b\cos\phi}\right)$

Note: Vector Subtraction is not commutative i.e. $a - b \neq b - a$

Both these vectors have the same magnitude but have opposite directions and for two vectors to be equal they must have same magnitude as well as direction.

Scalar Multiplication

The product of any vector 1 2 3 a = [a_1, a_2, a_3] and any scalar c (real number c) is the vector obtained by multiplying each component of a by c

ca = [ca_1, ca_2, ca_3]

Geometrically, if a ≠ 0, the ca with c > 0 has the direction a and with c < 0 the direction opposite to a. In any case, the length of ca is ca | ca | = | c | | a |, and ca = 0 if a = 0 or c = 0 (or both).

Unit Vectors

Any vector whose length is 1 is a unit vector i, j, k are example of special unit vectors, which are along x, y and z coordinate axes.

| i | = | j | = | k | = 1

In two dimensions, u = cosθi + sinθj

In three dimensions, u = sinθcosθi + sinθsinθ j + cosθk

The unit vector in the direction of any vector has the same direction as the original vector but has a magnitude of 1.

The unit vector in the direction of a, $a = \frac{a}{|a|}$.

Inner Product of two Vectors

The inner product or dot product a.b (read "a dot b") of two vectors a and b is the product of the lengths times the cosine of their angle.

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\phi \qquad \text{...(i)}$$

The angle ϕ, 0 ≤ ϕ ≤ π, between a and b is measured when the vectors have their initial points coinciding, as in fig. below.

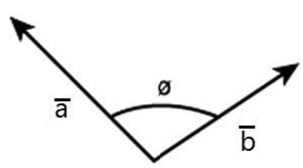


Fig. 2.22

In terms of components, a = [a_1, a_2, a_3], b = [b_1, b_2, b_3], and $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ Since the cosine in (i) may be positive, zero or negative, so may be the inner product.

If the angle between two vectors is 90° i.e. both vectors are orthogonal to each other, then the inner product or dot product of the vectors will be zero.

If the vector is taken as inner product with itself, then Φ = 0.

Then, $\vec{a}\cdot\vec{a} = |\vec{a}||\vec{a}|\cos 0 = |\vec{a}|^2$

Thus, length of a vector can be determined from the dot product,

$$|\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}}$$

Similarly, the angle between the two vectors can be calculated by means of inner product,

$$\cos\phi = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

Vector product (cross product)

The vector product (cross product) a × b of two vectors a = [a_1, a_2, a_3] and b = [b_1, b_2, b_3] is a vector.

v = a × b | a | | b | sinϕ ň where Φ is the angle between a and b.

Here, ň is a unit vector in a direction perpendicular to both a and b.

The direction of cross product can be determined by Right Hand Thumb Rule. We keep the fingers in the direction of 'a' and then curl them in the direction of 'b', then the direction of thumb represents the direction of cross product.

If a and b have the same or opposite direction or if one of these vectors is the zero vectors, then v = a × b = 0. In any other case, v = a × b has the length.

$$|\vec{v}| = |\vec{a}||\vec{b}|\sin\phi$$

In components, 1 2 3 v = [v_1, v_2, v_3] = a × b is

$$v_1 = a_2b_3 - a_3b_2$$
$$v_2 = a_3b_1 - a_1b_3$$
$$v_3 = a_1b_2 - a_2b_1$$

In terms of determinants,

$$v_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix};\ v_2 = \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix};\ v_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Hence v = [v_1, v_2, v_3] = v_1i, v_2i, v_3k is the expansion of the symbolical third-order determinant

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

A unit vector perpendicular to two given vectors a and b is given by

$$n = \frac{a \times b}{|a|b\#\sin\gamma} = \frac{a \times b}{|a \times b|}$$

Properties

Vector Product has the property that for every scalar l,

(la) × b = l(a × b) = a × l(b)

It is distributive with respect to vector addition, that is,

$$\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$$

$$(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c})$$

It is not commutative but anti-commutative, that is,

b × a = −(a × b)

It is not associative, that is,

a × (b × c) ≠ (a × b) × c

So that the parentheses cannot be omitted

Applications of Cross Product

- Area of triangle OAB = $\frac{1}{2}\left|\overline{OA} \times \overline{OB}\right| = \frac{1}{2}\left|\bar{a} \times \bar{b}\right|$

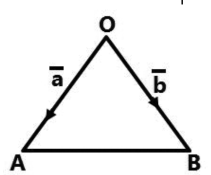


Fig. 2.23

- Area of triangle ABC =

$$\frac{1}{2}\left|\overline{AB} \times \overline{AC}\right| = \frac{1}{2}\left|(\bar{b} \cdot \bar{a}) \times (\bar{c} \cdot \bar{a})\right|$$

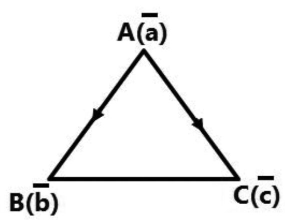


Fig. 2.24

- Area of parallelogram = $|\bar{a} \times \bar{b}|^2$

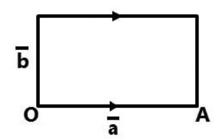


Fig. 2.25

Vector Products of the Standard Basis Vectors

Since i, j, k are orthogonal (mutually perpendicular) unit vectors. the definition of vector product gives some useful formulas for simplifying vector products in right – handed coordinates these are

i × j = k j × k = i k × i = j

j × i = −k k × j = −i i × k = −j

Scalar Triple Product

The scalar triple product or mixed triple product of three vectors.

a = [a_1, a_2, a_3]; b = [b_1, b_2, b_3]; c = [c_1, c_2, c_3]

Is denoted by (a, b, c) and is defined by (a b c) and is defined by (a b c) = a(b × c)

We can write this as a third – order determinant

For this we set b × c = v = [v_1, v_2, v_3]. Then from the dot product in components we obtain

$$a \cdot (b \times c) = a \cdot v = a_1v_1 + a_2v_2 + a_3v_3 =$$

$$a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\left(-\begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix}\right) + a_3\begin{vmatrix} b_1 & b_2 \\ c_2 & c_2 \end{vmatrix}$$

The expression on the right is the expansion of a third-order determinant by its first row.

Thus (a b c) = a · (b × c) = $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

The absolute value of the scalar triple product is the volume of the parallelepiped with a, b, c as edge vectors Also, a · (b × c) = (a × b) · c = c × (a × b) i.e. the value of triple product depends upon the cycle order of the vectors, but is independent of the position of dot and cross.

However if the order is non –cycle, then value changes. i.e., a · (b × c) ≠ b · (a × c)

Note: Three vectors form a linearly independent set if and only if their scalar triple product is not zero. The scalar triple product is the most important “repeated product”. Other repeated product exist, but are used only occasionally.

Vector Triple Product

If a, b and c are three vectors then the vector triple product is written as a × (b × c)

It can be proved that $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

Similarly, $(\vec{a}\times\vec{b})\times\vec{c}=(\vec{c}\cdot\vec{a})\,\vec{b}-(\vec{c}\cdot\vec{b})\,\vec{a}$

Gradient of a Scalar Field

Gradient is an operator by which we can derive vector fields from the scalar fields.

The gradient grad f of a given scalar function f(x, y, z) is the vector function defined by

$$\text{grad } f=\frac{\partial f}{\partial x}i+\frac{\partial f}{\partial y}j+\frac{\partial f}{\partial z}k$$

Here we must assume that f is differentiable. It has become popular, particularly with physicists and engineers, to introduce the differential operator.

$$\nabla=\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k$$

(read nabla or del) and to write

$$\text{grad } f=\nabla f=\frac{\partial f}{\partial x}i+\frac{\partial f}{\partial y}j+\frac{\partial f}{\partial z}k$$

In multiple dimensions, the derivative can be defined in multiple directions and thus the concept of directional derivative arises.

Directional Derivative

The rate of change of f at any point P in any fixed direction given by a vector b is defined as in calculus. We denote it by $D_b f$ or df/ds, call it the directional derivative of f at P in the direction of b. To determine the directional derivative we take the component of gradient in the direction where directional derivative is to be computed.

Thus, $D_b f=\frac{b}{|b|}\cdot\text{gradf}$

where, $\frac{\vec{b}}{|b|}$ represents the unit vector in the direction of b.

Note: The gradient of a surface at a point represents normal to the surface at that point.

Gradient also represents the direction of maximum rate of increase of the scalar field.

Solved Examples

Example: The unit vector normal to the surface $xy^3z^2=4$ at (–1, –1, 2) is ____________.

Solution: Let $\phi=xy^3z^2$

$$\nabla\phi=\hat{i}\left[y^3z^2\right]+\hat{j}\left[3xy^2z^2\right]+\hat{k}\left[2xy^3z\right]$$

At $(-1,-1,2)$

$$\nabla\phi=\hat{i}\left[-1^3\times4\right]+\hat{j}\left[3\times-1\times1\times4\right]+\hat{k}\left[2\times-1\times-1^3\times2\right]$$

$$=-4\hat{i}-12\hat{j}+4\hat{k}$$

$$\nabla\phi=4\hat{i}-12\hat{j}+4\hat{k}$$

$$\frac{\nabla\phi}{|\nabla\phi|}=\frac{\hat{i}-3\hat{j}+\hat{k}}{\sqrt{1+9+1}}=\frac{i-3j+k}{\sqrt{11}}$$

Example: The sphere of unit radius is centered at origin. Then a unit vector normal to the surface of the sphere at any point P(x, y, z) is the vector.

Solution: The given equation of the sphere is $x^2+y^2+z^2=1$

Let $\phi=x^2+y^2+z^2$

$$\nabla\phi=2xi+2yj+2zk$$

$$\frac{\nabla\phi}{|\nabla\phi|}=\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\nabla\phi}{|\nabla\phi|}=\frac{x\hat{i}+y\hat{j}+z\hat{k}}{1}$$

Note: Normal to the sphere with center at origin at any point P(x, y, z) is the position vector of the point itself.

Example: The directional derivative of $f=xy^2z$ at (1, –1, 1) in the direction of the vector $\bar{a}=\hat{i}+\hat{j}-2k$ is ____________.

Solution: $\nabla f=\hat{i}\left(y^2z\right)+\hat{j}\left(2xyz\right)+\hat{k}\left(xy^2\right)$

$$\nabla f\big|_{(1,-1,1)}=\hat{i}-2\hat{j}+k$$

$$\therefore D_{fa}=\nabla f\cdot\frac{\bar{a}}{|\bar{a}|}=\left(\hat{i}-2\bar{j}+\hat{k}\right)\cdot\frac{\left(\hat{i}+\hat{j}-2k\right)}{\sqrt{1+1+4}}$$

$$=\frac{1-2-2}{\sqrt{6}}=-\frac{3}{\sqrt{6}}$$

Example: The directional derivative of $\phi = xy^2 + yz^2 + zx^2$ at (1, 1, 1) along the direction of tangent to the curve $x = t$, $y = t^2$, $z = t^3$ is ______________.

Solution: The position vector of any point on the curve is,

$$r(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$$

$$\frac{dr}{dt} = \hat{i} + 2t\hat{j} + 3t^2\hat{k}$$

At (1, 1, 1) it $t = 1$

The tangent is given by,

$$\frac{dr}{dt} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\nabla\phi = \hat{i}\left(y^2 + 2xz\right) + \hat{j}\left(z^2 + 2xy\right) + \hat{k}\left(x^2 + 2yz\right)$$

$$\nabla\phi]_{(1,1,1)} = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

Thus, directional derivative in the direction of tangent is,

$$\therefore D = \nabla\phi \cdot \frac{\frac{dr}{dt}}{\left|\frac{dr}{dt}\right|} = \frac{3+6+9}{\sqrt{1+4+9}} = \frac{18}{\sqrt{14}}$$

Example: Let $f = x^{2/3} + y^{2/3}$ be a scalar point function. Then the derivative of f, along the line $y = x$ directed away from the origin at the point (8, 8) is ______________.

Solution: The given scalar field is,

$$f = x^{2/3} + y^{2/3}$$

$$\bar{r} = x\hat{i} + y\hat{j} = (r\cos\theta)\hat{i} + (r\sin\theta)\hat{j}$$

$$\frac{\bar{r}}{|r|} = \cos\theta\hat{i} + \sin\theta\hat{j}$$

Here $\theta = \frac{\pi}{4}$ since the given line is $y = x$

$$\hat{e} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$

Gradient of scalar field is,

$$\nabla f = \hat{i}\left(\frac{2}{3}x^{-1/3}\right) + \hat{j}\left(\frac{2}{3}y^{-1/3}\right)$$

$$\nabla f\big|_{(8,8)} = \frac{1}{3}\hat{i} + \frac{1}{3}\hat{j}$$

$$\therefore D = \nabla f.\hat{e} = \frac{1}{3\sqrt{2}} + \frac{1}{3\sqrt{2}} = \frac{2}{3\sqrt{2}} = \frac{\sqrt{2}}{3}$$

Example: The angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 - z = 3$ at (2, –1, 2).

Solution: Let $\phi_1 = x^2 + y^2 + z^2$

$$\nabla\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\nabla\phi_1\big]_{(2,-1,2)} = 4\hat{i} - 2\hat{j} + 4\hat{k}$$

Let $\phi_2 = x^2 + y^2 - z$

$$\nabla\phi_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$\nabla\phi_2\big]_{(2,-1,2)} = 4\hat{i} + 2\hat{j} - \hat{k}$$

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \frac{16+4-4}{\sqrt{16+4+16}\sqrt{16+4+1}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

Divergence

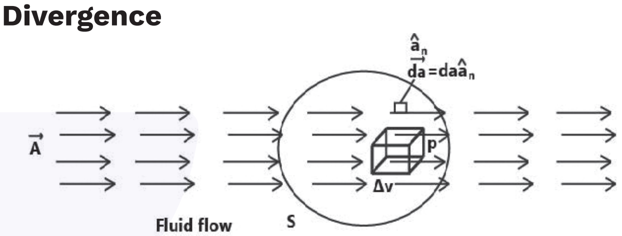


Fig. 2.26

Fluid flow=inlet outward flux per unit volume when volume shrinks to a point.

Closed surface always encloses a volume

$$\underset{\Delta Lt \to 0}{Lt} \frac{\oiint_s \vec{A} \cdot \vec{da}}{\Delta V} = \text{Div}\,\vec{A} = \nabla \cdot \vec{A}$$

The different cases of outflow and inflow have been shown below,

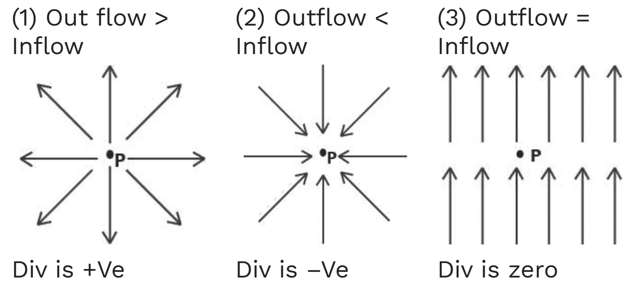


Fig. 2.27

$$\vec{\nabla} \cdot \vec{A} = \left(\frac{d}{dx}\hat{a}_x + \frac{d}{dy}\hat{a}_y + \frac{d}{dz}\hat{a}_z\right) \cdot \left(A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z\right)$$

$$\vec{\nabla} \cdot \vec{A} = \frac{dA_x}{dx} + \frac{dA_y}{dy} + \frac{dA_z}{dz}$$

Always a scalar quantity

If $\nabla.\vec{A} = 0 \Rightarrow \vec{A}$ is known as a "solenoidal fields"

Curl of a Vector Field

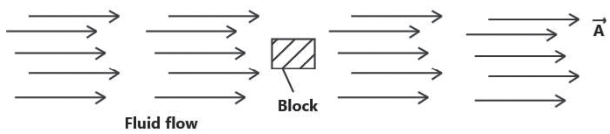


Fig. 2.28

If net rotational effect on block is present then field is solid to have curl.

$$\underset{\Delta S \to 0}{\mathrm{Lt}} \left(\frac{\oint_c \vec{A} \cdot \vec{dl}}{\Delta S} \right) \overset{\Delta}{=} \nabla \times \vec{A} \overset{\Delta}{=} \text{Curl of } \vec{A}$$

Net circulation per unit area when area shrinks to zero.

The curl in terms of graphical representation of different fields is shown below

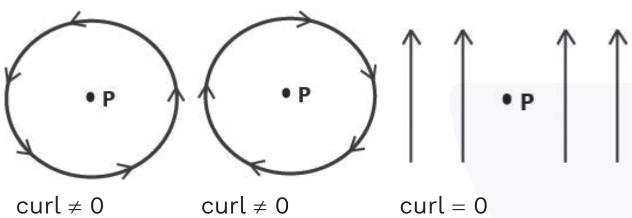


Fig. 2.29

The curl can be computed by the determinant shown below,

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ d/dx & d/dx & d/dx \\ A_x & A_y & A_z \end{vmatrix}$$

If $\nabla \times \vec{A} = 0 \Rightarrow \vec{A}$ is known as a "Irrotational field" or "Conservative field"

Note: If $\vec{v} \to$ Linear velocity

$\vec{\omega} \to$ Angular velocity

Then $\vec{V} = \vec{\omega} \times \vec{r}$

curl $\vec{v} = \nabla \times (\vec{\omega} \times \vec{r}) = 2\vec{\omega}$

Thus, $\vec{\omega} = \frac{1}{2}$ curl $\vec{v}$

Important Relations

1. div grad $f = \nabla^2 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
2. curl grad $f = \nabla \times \nabla f = 0$
3. div curl $f = \nabla \cdot (\nabla \times F) = 0$
4. curl curl f = grad div $F - \nabla^2 F = \nabla(\nabla.F) - \nabla^2 F$
5. grad div f = curl curl $F + \nabla^2 F = \nabla \times \nabla \times F + \nabla^2 \times F$

Solved Examples

Example: Suppose $\bar{F} = 4x^2 z\hat{i} - (7y^2 + 2xz)\hat{j} + 3yz^2\hat{k}$ represents a velocity vector

(i) Then Div. of $\bar{F}$ at (1, –1, 2) is ____________

(ii) Corresponding angular velocity at (2, 1, –2) is ______________.

Solution: The divergence of velocity is given by,

$$\text{Div } \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 8xz + (-14y) + 6yz$$

At (1, –1, 2)

Div $\vec{F} = 16 + 14 - 12 = 18$

Angular Velocity is given by, $\vec{\omega} = \frac{1}{2}$ Curl $\vec{F}$

$$\text{Curl } \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4x^2 z & -(7y^2 + 2xz) & 3yz^2 \end{vmatrix}$$

$$= i\left[3z^2 + 2x\right] - j\left[0 - 4x^2\right] + \bar{k}[-2z - 0]$$

At (2, 1, –2)

Curl $\bar{F} = 16\hat{i} + 16\hat{j} + 4\hat{k}$

$\therefore \omega = \frac{1}{2}$ Curl $\bar{F} = 8\hat{i} + 8\hat{j} + 2\hat{k}$

Example: The value of λ such that the vector function $\bar{F} = (\lambda x^2 y - yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz + y^2x^2)\hat{k}$ is solenoid is ____________.

Solution: For solenoidal fields the divergence is zero

Thus, div $\bar{F} = 0$

$(2\lambda xy - 0) + (2xy - 0) + (2xy + 0) = 0 \quad \Rightarrow \lambda = -2$

Example: If $\bar{F} = (x - 2y + az)\hat{i} + (bx - 3y + 4z)\hat{j} + (2x + cy - 5z)\hat{k}$ is irrotational then a, b, c is?

Solution: For irrotational fields, Curl $\bar{F} = 0$

$$\text{Curl } \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x - 2y + az & bx - 3y + 4z & 2x + cy - 5z \end{vmatrix} = 0$$

$i(c-4) - j(2-a) + k(b+2) = 0$

$c = 4,\ a = 2,\ b = -2$

Example: The directional derivative of $\text{div}\,\bar{F}$ in the direction of outer normal to the surface of the sphere with curve at origin and radius = 3, where $\bar{F} = x^4\hat{i} + y^4\hat{j} + z^4\hat{k}$ is

Solution: The outward normal to a surface is the gradient to the surface at that particular point.

$\text{div}\,\bar{F} = 4x^3 + 4y^3 + 4z^3 = f$

Now, we need to determine the gradient of f.

$\nabla f = 12x^{2\hat{i}} + 12y^2\hat{j} + 12z^2\hat{k}$

Let's take any point on the surface of the sphere P(1,2,2)

$[\nabla f]_{(1,2,2)} = 12\hat{i} + 48\hat{j} + 48\hat{k}$

$\nabla f = 12(\hat{i} + 4\hat{j} + 4\hat{k})$

Example: If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, and $r = |\bar{r}|$ then $\text{div}\,(r^n\bar{r}) =$ ____________

Solution: $r^n\bar{r} = \underset{F_1}{r^n x}\hat{i} + \underset{F_2}{r^n y}\hat{j} + \underset{F_3}{r^n z}\hat{k}$

$$\text{div}\left(r^n\bar{r}\right) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x}\left(r^n x\right) = r^n(1) + xnr^{n-1}\frac{\partial r}{\partial x}$$

Since, $r = \sqrt{x^2 + y^2 + z^2}$

Thus, $\dfrac{\partial r}{\partial x} = \dfrac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \dfrac{x}{r}$

Thus, $\dfrac{\partial F_1}{\partial x} = r^n + xn^{n-1} \times \dfrac{x}{r}$

$$\frac{\partial F_1}{\partial x} = r^n + xn^{n-1} \times \frac{x}{r}$$

Similarly $\dfrac{\partial F_x}{\partial y} = r^n + nr^{n-2}y^2$ and $\dfrac{\partial F_x}{\partial z} = r^n + nr^{n-2}z^2$

$$\therefore \text{div}\left(r^n r\right) = 3r^n + nr^{n-2}\underbrace{(x^2 + y^2 + z^2)}_{r^2}$$

$$\therefore \text{div}\left(r^n r\right) = 3r^n + nr^n = (n+3)r^n$$

If $n = -3\,\text{div}\left(r^n\bar{r}\right) = 0$

$\dfrac{\bar{r}}{r^3}$ is solenoidal

Line Integral of Vector functions

Let $\bar{F}(x, y, z) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a different vector function defined along the curve 'c' then its line integral is

$$\int_c \bar{F}\cdot d\bar{r}$$

In Cartesian form, $\int_c \bar{F}.d\bar{r} \int_c \left(F_1dx + F_2dy + F_3dz\right)$

Note: If c is a closed curve then the line integral of $\bar{F}$ along C is called circulation of $\bar{F}$ divided by $\int_c \bar{F}.\overline{dr}$

We represent the curve C by a parametric representation.

$r(t) = [x(t), y(t), z(t)] = x(t)i + y(t)j + z(t)k$

$(a \le t \le b)$

We call C the path of integrating A : r(a) its initial point, and B : r(b) its terminal point. The direction from A to B, in which t increases, is called the positive direction on C. The points A and B may coincide. Then C is called a closed path as initial and final points are identical.

We call C a smooth curve if C has a unique tangent at each of its points whose direction varies continuously as we move along C.

Work done by a force

The total work done by force $\bar{F}$ in moving some particle along the curve c is W.D. = $\int_c \bar{F}.\overline{dr}$

Note: The line integral of an irrotational vector function is independent of the path of the curve.

If $\bar{F}$ is irrotational then $\bar{F} = \nabla\phi$ where ϕ is a scalar potential function. Then $\int_A^B \bar{F}.dr = \phi_B - \phi_A$

For irrotational fields, always there exists a scalar potential function such that $\bar{F} = \nabla\phi$.

This can easily be proved as for irrotational field, $\nabla \times \vec{F} = 0$

By Vector identity, $\nabla \times \nabla\phi = 0$

Thus, $\bar{F} = \nabla\phi$

Solved Examples

Example: The value of $\int_c \bar{F}.d\bar{r}$ where $\bar{F} = 2x^2\hat{y} - y^2xj$ c is the curve $y = 2x^2$ joining the points (0,0) and (1,2) is ?

Solution: $\int_c \bar{F}\,d\bar{r} = \int_c 2x^2ydx - y^2xdy$

On curve C, $y = 2x^2 \Rightarrow dy = 4x\,dx$

$$\int_c \bar{F}.d\bar{r} = \int_0^1 2x^2\left(2x^2\right)dx - \left(4x^4\right)x.4x\,dx$$

$$= \int_0^1 \left[4x^4 - 16x^6\right]dx$$

$$\int_c \bar{F}.d\bar{r} = \frac{4}{5} - \frac{16}{7} = \frac{28-80}{85} = -\frac{52}{85}$$

Example: The value of $\int_c^{\bar{F}} dr$ where $\bar{F} = 3xy\hat{i} - y^2\hat{j}$ and c is the curve bounded by $y = x^2$ and $y = x$ is?

Solution: The curve C can be resolved into two parts C_1 and C_2

$$\int_c \bar{F}\cdot dr = \int_{c_1}(\;) + C\int_{c2}(\;)$$

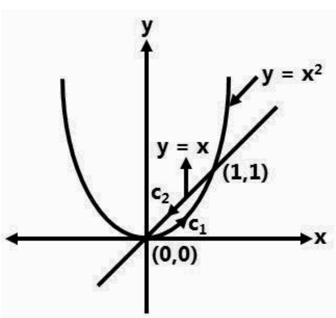


Fig. 2.30

Along c_1: $y = x^2$

$dy = 2x\,dx$

$$\int_{c_1} \bar{F}\,d\bar{r} = \int_c 3xydx - y^2dy = \int_0^1 3x\cdot x^2dx - x^4 \times 2xdx$$

$$\int_{c_1} \bar{F}\,d\bar{r} = \frac{3x^4}{4} - \frac{2x^5}{6}\Bigg]_0^1 = \frac{3}{4} - \frac{2}{6} = \frac{5}{12}$$

Along c_2: $y = x$ and x goes from 1 to 0 as we traverse C_2

$dy = dx$

$$\int_{c_1} \bar{F}\,d\bar{r} = \int_c 3xydx - y^2dy = \int_1^0 3x^2dx - x^2dx$$

$$\int_{c_2} \bar{F}\,d\bar{r} = \frac{3x^3}{3} - \frac{x^3}{3}\Bigg]_1^0 = -\frac{2}{3}$$

$$\therefore \int_c \bar{F}\cdot d\bar{r} = \frac{5}{12} - \frac{2}{3} = \frac{5-8}{12} = \frac{-1}{4}$$

Note: The line integral of an irrotational vector function over any closed curve is zero because line integral is independent of path and depends on end points. But if end points are identical, then the line integral is zero.

Example: The value of $\int_c (3x + 4y)dx + (2x - 3y)$ dy where c is a circle with center at the origin and radius as 2 in x - y plane is?

Solution: The equation of circle in x-y plane with center at origin and radius 2 is,

$x^2 + y^2 = 4$

Parameterizing the curve C,

Let $x = 2\cos t$ & $y = 2\sin t$

$dx = -2\sin t dt$ and $dy = 2\cos t dt$

$$\int_c (3x+4y)dx + (2x-3y)dy = \int_0^{2\pi} (6\cos t + 8\sin t)$$

$$(-2\sin t dt) + (4\cos t - 6\sin t)(2\cos t dt)$$

$$I = \int_0^{2\pi} (-24\sin t\cos t - 16\sin^2 t + 8\cos^2 t)\,dtI$$

$$= \int_0^{2\pi} (-12\sin 2t - 8(1-\cos 2t) + 4(1+\cos 2t))dt$$

$$= \int_0^{2\pi} [-12\sin 2t - 4 + 12\cos 2t]dt$$

$$I = \frac{12\cos 2t}{2} - 4t + \frac{12\sin 2t}{2}\Bigg]_0^{2\pi}$$

$$I = -4 \times 2\pi = -8\pi$$

Example: Determine the value of

$$\int_c \bar{F}\cdot d\bar{r}, \bar{F} = (2y+3)\hat{i} + xz\hat{j} - 3y^2z\hat{k}$$ and c is the line joining the points.

i) (0, 0, 1) to (0, 1, 1)

ii) (0, 1, 1) to (2, 1, 1)

Solution: (i) In this line, the values of x and z are constant x = 0,z = 1

dx = 0,dz = 0

But, y : 0 to 1

$$\int_c \bar{F}\cdot \overline{dr} = \int_c \bar{F}_1 dx + \bar{F}_2 dy + \bar{F}_3 dz = \int_0^1 xzdy = \int_0^1 0dy = 0$$

(ii) On this line, the values of y and z are constant y = 1, z = 1, dy = 0,dz = 0, x : 0 to 2

$$\therefore \int_c^{\bar{F}} \cdot d\bar{r} = \int_0^2 F_1 dx = \int_0^2 (2y+3)dx = \int_0^2 5dx = 10$$

Example: The total work done by the force $\bar{F} = \left(3x^2+6y\right)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$ in moving a particle along the straight line joining the set of points (0, 0, 0) & (1, 1, 2) is ________________.

Solution: The equation of the line joining two points in three dimensions is given by,

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t$$

Substituting the points we get,

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{2-0} = t$$

$$x = t, y = t, z = 2t$$

$$dx = dt, dy = dt, dz = 2dt$$

$$\text{W.D.} = \int_c \bar{F}d\bar{s} = \int_c \left(3x^2+6y\right)dx - 14yzdy + 20xz^2dz$$

$$= \int_0^1 \left(3t^2+6t\right)dt - 28t^2dt + 160t^3dt$$

$$\text{W.D.} = \left.\frac{3t^3}{3} + \frac{6t^2}{2} - \frac{28t^3}{3} + 160\frac{t^4}{4}\right]_0^1$$

$$= \frac{-25}{3} + \frac{6}{2} + \frac{160}{4} = 43 - \frac{25}{3}$$

$$\text{W.D.} = \frac{104}{3}$$

Example: Find

$\int \bar{v}\cdot dr$ where $\bar{v} = y\hat{i} + (xz+1)\hat{j} + xy\hat{k}$ from (0, 1, 0) to (2, 1, 4).

(a) 7

(b) 8

(c) 9

(d) Cannot be determined without specifying the path.

Solution: Here path is not specified check whether the function is irrotational. If it is irrotational proceed to line integral as line integral is independent of the path.

If not irrotational, the line integral cannot be determined without the specified path.

$$\text{Curl } \bar{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = i(x-x) - j(y-y) + k(z-z) = 0$$

Thus, $\bar{V}$ is irrotational. Hence, it can be expressed as gradient of scalar potential function, $\bar{v} = \nabla\phi$

If we traverse from (a, b, c) to (x, y, z)

$$\phi = \int_a^x F_1(x,y,z)dx + \int_b^y F_2(a,y,z)dy + \int_c^z F_3(a,b,z)dz$$

This integral represents that we first travel in x-direction keeping y and z constant and then in y-direction keeping x and z constant and lastly in z-direction keeping x and y as constant.

$$\phi = \int_a^x yzdx + \int_b^y (az+1)dy + \int_c^z abdz = yz[x]_a^x +$$

$$\left.(az+1)y\right]_b^y + \left.abz\right]_c^z$$

$$\phi = xyz - ayz + ayz + y - a\,|\,bz - b\,| + abz - abc$$

$$\phi(x,y,z) = xyz + y + k$$

$$\therefore \int \bar{v}\cdot dr = \phi(2,1,4) - \phi(0,1,0) = (8+1+k) - (0+1+k) = 8$$

Green's Theorem in a Plane

Let M(x, y) and N(x, y) be the continuous function having constant first order partial derivatives defined in a closed region 'R' bounded by closed curve 'c' then

$$\int_C Mdx + Ndy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]dxdy$$

Solved Examples

Example: Determine the value of $\int_c e^{-x} \sin y dx + e^{-x} \cos y dy$ where c is a rectangle with vertices $(0,0),(\pi,0),\left(\pi,\frac{\pi}{2}\right),\left(0,\frac{\pi}{2}\right)$.

Solution: The curve C and the region bounded by the curve are shown in the figure below,

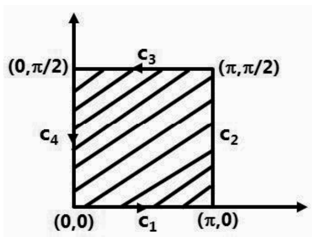


Fig. 2.31

$$M = e^{-x} \sin y \Rightarrow \frac{\partial M}{\partial y} = e^{-x} \cos y$$

$$N = e^{-x} \cos y \Rightarrow \frac{\partial N}{\partial x} = -e^{-x} \cos y$$

$$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -2e^{-x} \cos y$$

$$\therefore \int Mdx + Ndy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dxdy$$

$$I = \int_0^{\pi/2}\int_0^{\pi} -2e^{-x} \cos y dxdy = \int_0^{\pi/2} -2\cos y \left[\frac{e^{-x}}{-1}\right]_{00}^{\pi} dy$$

$$I = 2\left(e^{-\pi} - 1\right)[\sin y]_0^{\pi/2} = 2\left[e^{-\pi} - 1\right]$$

Example: Find $\int_c (y - \sin x)dx + \cos x dy$ where c is the curve bounded by $y = 0, x = \frac{\pi}{2}, y = \frac{2x}{\pi}$

Solution: Considering the vertical strip.

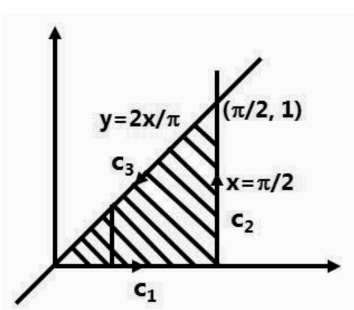


Fig. 2.32

$$y = 0 \text{ to } y = \frac{2x}{\pi}$$

$$x = 0 \text{ to } x = \frac{\pi}{2}$$

$$\frac{\partial M}{\partial y} = 1, \frac{\partial N}{\partial x} = -\sin x$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -\sin x - 1$$

$$I = \int_0^{\pi/2}\int_0^{2x/\pi} -(1 + \sin x)dydx = \int_0^{\pi/2} -(1 + \sin x)\frac{2x}{\pi} dx$$

Apply Integration by Parts,

$$I = \frac{-2}{\pi}\left[\frac{x^2}{2} + x(-\cos x) - (1)(-\sin x)\right]_0^{\pi/2} = \frac{-2}{\pi}\left[\frac{\pi^2}{8} + 1\right]$$

Example: Determine the value of $\int_c \left(3x^2 - 8y^2\right)dx + (4y - 6xy)dy$ where C is the curve bounded by $y = \sqrt{x}$ and $y = x^2$

Solution: The closed curve and region bounded by the two curves is shown in the figure below, Consider the vertical strip.

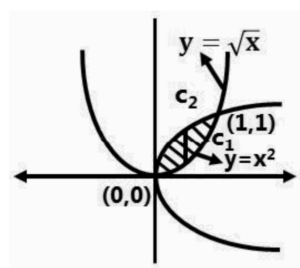


Fig. 2.33

$$y = x^2 \text{ to } y = \sqrt{x}$$

$$x = 0 \text{ to } x = 1$$

$$\frac{\partial M}{\partial y} = -16y, \frac{\partial N}{\partial x} = -6y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 10y$$

$$\int_0^1\int_{x^2}^{\sqrt{x}} 10ydydx = 10\int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx = 5\int_0^1 \left(x - x^2\right)dx =$$

$$5\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{5}{6}$$

Surface Integral

Let 1 2 3 $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a differentiable vector function defined over a surface s, then its surface integral is $\int_s \bar{F}\cdot\overline{ds}$ or $\int_s \bar{F}\cdot\bar{n}\, ds$.

Where $\bar{n} \rightarrow$ unit outward drawn normal to the surface S.

In Cartesian form

$$\int_s \bar{F}\cdot\bar{n}\, ds = \int\int_s \left[F_1 dydz + F_2 dxdz + F_3 dxdy\right]$$

To calculate the value of Surface Integral, we have to take the projection of region R onto xy plane or yz plane or xz plane.

If R_1 is the projection of 's' onto xy plane then

$$\int_s \bar{F}\,\bar{n}\, ds = \int\int_{R_1} \bar{F}\cdot\bar{n}\frac{dxdy}{|\bar{n}\cdot\bar{k}|}$$

If R_2 is the projection of 's' onto yz plane then

$$\int_s \bar{F}\,\bar{n}\, ds = \int\int_{R_2} \bar{F}.\bar{n}\frac{dydz}{|\bar{n}.\bar{i}|}$$

If R^3 is the projection of 's' onto xz plane then

$$\int_s \bar{F}.\bar{n}\, ds = \int\int_{R_2} \bar{F}.\bar{n}\frac{dxdz}{|\bar{n}.\bar{j}|}$$

Solved Examples

Example: The value of $\int_s \bar{F}.\bar{n}$ ds, where $\bar{F} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ and S = surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5 is ______________.

Solution: The cylinder in the first octant is shown in the figure below,

Let $\phi = x^2 + y^2$

$\nabla\phi = 2x\hat{i} + 2y\hat{j}$

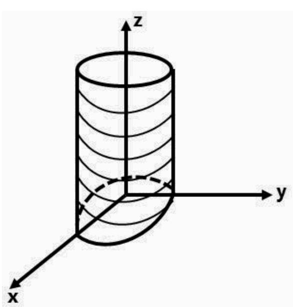


Fig. 2.34

The normal vector to the surface is unit vector in the direction of gradient

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\bar{n} = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}} = \frac{x\hat{i} + y\hat{j}}{4}$$

$$\bar{F}.\bar{n} = \frac{xz}{4} + \frac{xy}{4} = \frac{x}{4}(z + y)$$

Let R be the projection of S onto yz plane.

$$\int_s \bar{F}.\,\bar{n}ds = \iint_R \bar{F}\cdot\bar{n}\frac{dydz}{|\bar{n}\cdot\hat{i}|} = \iint_R \frac{x}{4}(y+z)\frac{dydz}{x/4}$$

$$I = \int_{z=0}^{5}\int_{y=0}^{4}(y+z)dydz = \int_{z=0}^{5}\left[\frac{y^2}{2} + zy\right]_0^4 dz$$

$$I = \int_0^5 [8 + 4z]dz = 8z + \left.\frac{4z^2}{2}\right]_0^4 = 40 + 50$$

$$I = 90$$

Example: The value of $\int_s \bar{F}.\,\bar{n}ds$ where $\bar{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and S is a surface of the cube bounded by x = 0, x = 1, y = 0, y = 1 and z = 0, z = 1 is ______________.

Solution: To integrate over the entire surface of cube, we can resolve the integral into 6 parts one over each surface of the cube.

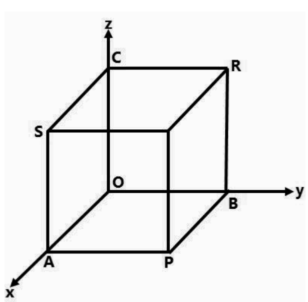


Fig. 2.35

$$\int_s \bar{F}.\,\bar{n}ds = \int_{s_1} + \int_{s_2} \ldots\ldots\ldots\ldots\ldots\ldots + \int_{s_1}$$

Over S_1: In xy plane (OAPB)

$z = 0, \bar{n} = -\bar{k}$

$\bar{F}.\bar{n} = -yz = 0$(Since, $z = 0$ for OAPB)

$$\int_{s_1} \bar{F}.\,\bar{N}\,ds = 0$$

Over S_2: parallel to xy plane (SQRC)

$z = 1, \bar{n} = \bar{K}, \bar{F}.\bar{n} = yz = y$

$$\int_s \bar{F}.\,\bar{n}\,ds = \int_s y\,ds = \iint_R y\frac{dxdy}{|\bar{n}\cdot\bar{K}|} = \int_0^1\int_0^1 y\,dydx = \frac{1}{2}$$

Over S_3: In yz plane (OBRC)

$x = 0, \bar{n} = -\bar{i}, \bar{F}.\bar{n} = -4xz = 0$

$$\int_s \bar{F}.\,\bar{n}\,ds = 0$$

Over S_4: Parallel to yz plane (APQS)

$x = 1, \bar{n} = \bar{i}, \bar{F}.\bar{n} = 4xz = 4z$

$$\int_{s_4} \bar{F}.\,\bar{n}\,ds = \int_0^1\int_0^1 4z\,dzdy = 2$$

Over S_5: In xz plane (OCSA)

$y = 0, \bar{n} = -\bar{j}, \bar{F}.\bar{n} = y^2 = 0$

$$\int_{s_5} \bar{F}.\,\bar{n}\,ds = 0$$

Over S_6: Parallel to xz plane (BRQP)

$y = 1, \bar{n} = \bar{j}\,\bar{F}.\bar{n} = -y^2 = -1$

$$\int_{s_6} \bar{F}.\,\bar{n}\,ds = \int_{s_6} -1\,ds = -1$$

$$\therefore \int_s \bar{F}.\,\bar{n}\,ds = 0 + \frac{1}{2} + 0 + 2 + 0 - 1 = \frac{3}{2}$$

Or if we use Gauss's Divergence Theorem (to be discussed later)

$$\int_S \bar{F}.\,\bar{n}\,ds = \int_V \operatorname{div}\bar{F}\,dv = \int_V (4z - 2y + y)\,dv =$$

$$\int_0^1\int_0^1\int_0^1 (4z - y)\,dzdydxz$$

$$\int_S \bar{F}.\,\bar{n}\,ds = \frac{3}{2}$$

Volume Integral

Let ϕ (x, y, z) be a differentiable scalar to and let $\bar{F}$ (x, y, z) be a differentiable vector function defined over a region whose volume bounded is V.

Then the volume integration are $\int_V \phi(x,y,z)dv$

Or, $\int_V \bar{F}(x,y,z)dv = \int_V F_1 dv + j\int_V F_2 dv + k\int_V F_3 dv$

Solved Examples

Example: Find $\int_V (2x + y)$ dv where V is the volume bounded by x = 0,x = 2, y = 0, y = 2, z = x ,z = 4

Solution:

$$\int_V (2x+y)\,dv = \int_{y=0}^{2}\int_{x=0}^{2}\int_{z=x^2}^{4} (2x+y)\,dzdxdy$$

$$I = \int_0^2\int_0^2 (2x+y)\left(4-x^2\right)dxdy = \int_0^2\int_0^2\left[8x - 2x^3 + 4y - yx^2\right]dxdy$$

$$I = \int_0^2\left[8.\frac{x^2}{2} - \frac{2x^4}{4} + 4yx - y.\frac{x^3}{3}\right]_0^2 dy = \int_0^2\left[16 - 8 + 8y - \frac{8y}{3}\right]dy$$

$$I = \int_0^2\left[8 + \frac{16y}{3}\right]dy = 8y + \frac{16y^2}{6}\Bigg]_0^2$$

$$I = 16 + \frac{32}{3} = \frac{80}{3}$$

Example: The volume of an object expressed in spherical coordinates is given by

$V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 r^2\sin\theta\,drd\phi d\theta$ then the value of V is _______________.

Solution: Integrating with respect to r

$$I = \int_0^{2\pi}\int_0^{\pi/3}\left[\sin\phi\,\frac{r^3}{3}\right]_0^1 d\phi d\theta = \int_0^{2\pi}\int_0^{\pi/3}\frac{1}{3}\sin\phi\,d\phi d\theta$$

$$I = \frac{1}{3}\int_0^{2\pi}\left[-\cos\phi\right]_0^{\pi/3} d\theta = \frac{1}{3}\int_0^{2\pi}\left(-\frac{1}{2}+1\right)d\theta$$

$$I = \frac{1}{3}\times\frac{1}{2}\times 2\pi = \frac{\pi}{3}$$

Gauss Divergence Theorem

Let S be a closed surface enclosing a volume 'V' and $\bar{F}(x,y,z)$ be a differentiable vector function defined over the surface S.

Then, $\int_s \bar{r}.\bar{n}\,ds = \int_s \text{div}.\bar{F}\,dv$

In Cartesian form.

$$\int_s F_1 dydz + F_2 dxdz + F_3 dxdy = \int_s \left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right] dv$$

Solved Examples

Example: The value of $\int_s \bar{r}.\bar{N}\,ds$ where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and S is the closed surface enclosing a volume V is ?

Solution: To apply Gauss's Divergence Theorem,

$$\int_s \bar{F}.\bar{n}\,ds = \int_s \text{div}\,\bar{F}\,dv$$

$$\text{div}\,\bar{r} = 1+1+1 = 3$$

$$\int_s \bar{r}\,\bar{N}ds = \int_v \text{div}\,\bar{r}dv = \int_v 3dv = 3V$$

Example: The value of $\int_s xdydz + ydxdz + zdxdy$ where S is the surface of

i. Cylinder $x^2 + y^2 = 9$, $y = 0$, $y = 4$

ii. Sphere $x^2 + y^2 + z^2 = 16$

Solution: The given vector function is,

$\vec{F} = xi + yj + zk$

Divergence of given vector function is,

$$\nabla \cdot \vec{F} = 1+1+1 = 3$$

$$\int_v (1+1+1)dv = 3V$$

ı) For the case of cylinder,

$$3V = 3\times \pi r^2 h = 3\times\pi\times 3^2\times 4 = 108$$

ıı) For the case of sphere,

$$\pi 3V = 3\times\frac{4}{3}\times\pi r^3 = 4\pi\times(4)^3 = 256\pi$$

Example: The value of $\int_s \left(x^2 + 2y + 3z^2\right)dS$ where S is the surface of a unit sphere with center at the origin.

Solution: The equation of sphere with center at origin and radius 1.

$$x^2 + y^2 + z^2 = 1$$

$$\bar{N} = \frac{\nabla\phi}{|\nabla\phi|} = \hat{i} + y\hat{j} + z\hat{k}$$

Let $\bar{F} = \bar{F}_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$\therefore \bar{F}\cdot\bar{n} = \bar{F}_1\hat{i} + \bar{F}_2\hat{j} + \bar{F}_3\hat{k} = x^2 + 2y^2 + 3z^2$

$\bar{F}_1 = x, \bar{F}_2 = 2y, \bar{F}_3 = 3z$

$\therefore \bar{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$

$\text{div}\,\bar{F} = 1+2+3 = 6$

$$\therefore \int_s \left(x^2 + 2y^2 + 3z^2\right)ds = \int_V \text{div}\,\bar{F}dV = \int_V 6dV$$

$$\therefore \int_s \left(x^2 + 2y^2 + 3z^2\right)ds = 6V = 6\times\frac{4}{3}\pi r^3 = 8\pi(1)^3 = 8\pi$$

Example: Value of $\int_s \bar{F}.\bar{n}ds$ where $\bar{F} = 8x^2z\hat{i} - y^2\hat{j} + 2xz^2\hat{k}$ and S is the surface bounded by $0\le x\le 1$, $0\le y\le 2$, $0\le z\le 3$ is ______________.

Solution: $\text{div}\bar{\bar{F}} = 16xz - 2y + 4xz = 20xz - 2y$

$$\int_s \bar{F}.\bar{n}ds = \int_V \text{div}\,\bar{F}\,dV = \int_V (20xz - 2y)dV$$

$$\int_s \bar{F}.\bar{n}ds = \int_{x=0}^{1}\int_{y=0}^{2}\int_{z=0}^{3}(20xz - 2y)dzdydx$$

$$= \int_0^1\int_0^2\left[20x.\frac{9}{2} - 2y\times 3\right]dydx$$

$$\int_s \bar{F}.\bar{n}ds = \int_0^1 [90x\times 2 - 6.2]dx = \int_0^1 (180x - 12)dx$$

$$\int_s \bar{F}.\bar{n}ds = 90 - 12 = 78$$

Example: The value of $\int_S \bar{F}\cdot\bar{n}ds$ where $\bar{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ where S is the surface bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$ is ______________.

Solution: The divergence of given vector field,

$$\text{div}\,\bar{F} = 4 - 4y + 2z$$

$$\int_S \bar{F}.\bar{n}\,ds = \int_V \text{div}\,\bar{F}\,dv = \int_V (4 - 4y + 2z)\,dV$$

$$\int_S \bar{F}.\bar{n}\,ds = \int_{z=0}^{3}\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(4 - 4y + 2z)\,dydxdz$$

Let $x = r\cos\theta,\ y = r\sin\theta,\ z = z$

$|J| = r, r = 0 \text{ to } 2$

$\theta = 0 \text{ to } 2\pi$

$z = 0 \text{ to } 3$

$$\int_S \bar{F}.\bar{n}ds = \int_0^3\int_0^{2\pi}\int_0^2 (4 - 4r\sin\theta + 2z)rdrd\theta dz$$

$$\int_S \bar{F}.\bar{n}ds = \int_0^3\int_0^{2\pi}\left[4\times\frac{r^2}{2} - 4\sin\theta\cdot\frac{r^3}{3} + 2z\cdot\frac{r^2}{2}\right]_0^2 d\theta dz$$

$$= \int_0^3\int_0^{2\pi} 8 - \frac{32}{3}\sin\theta + 4z \Bigg] d\theta dz$$

$$\int_S \bar{F}.\bar{n}ds = \int_0^3 (8+4z)\theta - \frac{32}{3}(-\cos\theta)\Big]_0^{2\pi} dz$$

$$\int_S \bar{F}.\bar{n}ds = \int_0^3 (8+4z)2\pi dz = 2\pi\left[8z + 2z^2\right]_0^3$$

$$= 2\pi[24 + 18] = 84\pi$$

Stokes Theorem

This theorem is used to convert Line Integral to Surface Integral and vice versa.

Let S be an open surface bounded by a closed curve 'c' and $\bar{F}(x, y, z)$ be a differentiable vector function defined along the curve 'c' then $\int_c \bar{F}\cdot d\bar{r} = \int_S \text{Curl}\,\bar{F}\cdot\bar{n}\,ds$

$$\int_c F_1dx + F_2dy + F_3dz = \int_S (\nabla\times\bar{F}).\bar{n}\,ds$$

In two dimensions, $\text{Curl}\,\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$

Thus, this theorem will reduce to Green's Theorem.

Solved Examples

Example: The value of $\int \bar{F}d\bar{r}$ where $\bar{F} = yz\hat{i} + xz\hat{j} + xy\hat{k}$ and C is the curve bounded by x + y = 2, x = 0, y = 0 in xy plane is ____________.

Solution: The curl of given vector field is,

$$\text{Curl}\ \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$\nabla\times\bar{F} = \hat{i}(x-x) - \hat{j}(y-y) + \hat{k}(z-z)$$

Curl $\bar{F} = 0 \Rightarrow \bar{F}$ is irrotational

$$\int \bar{F}d\bar{r} = \int_s Curl\bar{F}\cdot\bar{n}ds = \int_s \bar{0}\cdot\bar{n}ds = 0$$

Example: The value of $\int \bar{F}d\bar{r}$, , where $\bar{F} = -y^3\hat{i} + x^3\hat{j}$ and C is the boundary of circular disc $x^2 + y^2 \le 1$, z = 0

Solution: The curl of given vector field is,

$$\text{Curl}\ \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}\left(3x^2 + 3y^2\right)$$

$$\therefore\ \text{Curl}\ \bar{F} = 3\left(x^2 + y^2\right)\bar{k}$$

The normal vector to the given surface area is,

$\bar{n} = \bar{k}$

$$\text{Curl}\ \bar{F}\cdot\bar{n} = 3\left(x^2 + y^2\right)$$

$$\int_c \bar{F}\cdot d\bar{r} = \int_s Curl\bar{F}\cdot\bar{n}ds = \int_3 3\left(x^2 + y^2\right)ds$$

$$= \iint_R 3\left(x^2 + y^2\right)dxdy$$

Changing the co-ordinate system to polar co-ordinates $x = r\cos\theta$, $y = r\sin\theta$

$|J| = r,\ x^2 + y^2 = r^2$

$$\int_c \bar{F}.d\bar{r} = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} 3r^3drd\theta = 3\times\frac{1}{4}\times 2\pi = \frac{3\pi}{2}$$

3 Differential Equations

Objective

Upon completion of this chapter you will be able to:

- Solve differential equations of first order.
- Determine complimentary function for higher order differential equations.
- Determine particular integral for higher order differential equations.

Introduction

Differential equations form a very important area of engineering applications. Most of the physical problems involve modelling a physical system in terms of differential equations and then determining the solution of differential equations to find system parameters. Like in electrical engineering any circuit involving R, L and C is solved using differential equations. In Mechanical Engineering Heat Equation and even basic equations of motion are an example of differential equations. So, while dealing with a physical system, differential equations are inescapable.

Types of Differential Equations

Ordinary Differential Equations

These equations have all the differential co4 efficients with respect to a single independent variable.

Example:

$$\frac{dy}{dx}=\frac{x}{y};\frac{d^2y}{dx^2}+ky=0;$$

$$\left(\frac{d^2y}{dx^2}\right)^4+3\frac{dy}{dx}+5y=x^2$$

Partial Differential Equations

These equations have two or more independent variables and differential coefficients are with respect to any of them.

Example:

$$\frac{dy}{dx}=x\frac{dy}{dz};\frac{dy}{dt}+ky=4\left(\frac{dy}{dx}\right)^2+2\sin 4x$$

Order and Degree of a Differential Equation

Order: It is the order of the highest derivative appearing in a differential equation.

Degree: It is the degree of the highest derivative appearing in a differential equation after the equation has been expressed in a form free from radicals and fractions as far as derivatives are concerned.

Example: $\left(\frac{d^2y}{dx}\right)^4+3\frac{dy}{dx}+5y=x^2$,

this equation has order 2 and degree 4.

Solved Examples

Example: The order and degree of the differential equation

$$\frac{d^3y}{dx^3}+4\sqrt{\left(\frac{dy}{dx}\right)^3+y^2}=0\text{ are ?}$$

Solution: Removing radicals from the equation

$$\left(\frac{d^3y}{dx^3}\right)^2=16\left[\left(\frac{dy}{dx}\right)^3+y^2\right]$$

The order is 3 as the highest derivative is of 3rd order. The degree is 2 as the power of highest derivative is 2.

Solution of a Differential Equation

The solution of a differential equation refers to the relation between dependent and independent variable in the equation which can satisfy the given differential equation.

The solution involves arbitrary constants and thus the solution of a differential equation represents a family of curves where a different curve is defined for each value of the arbitrary constant.

Any particular solution or curve can be obtained by substituting the value of arbitrary constant.

Example: $y=\frac{x^2}{2}+c$ is a solution of the differential equation $\frac{dy}{dx}=x$.

Here, c represents an arbitrary constant.

A singular solution of a differential equation refers to a solution that cannot be obtained by substituting arbitrary constants in general solution.

Equations of first order and first degree

Variable Separation Method

The general form of a 1^{st} order 1^{st} degree differential equation is given by

$$M+N\frac{dy}{dx}=0 \qquad ...(1)$$

$$Mdx+Ndy=0 \qquad ...(2)$$

Where M, N are functions of variables x and y. Suppose it is possible to express (1) as

$$\frac{dy}{dx}=\frac{f(x)}{g(y)}$$

Then, f (x) dx = g (y) dy

$\int f(x)dx=\int g(y)dy$ gives the solution of differential equation.

Solved Examples

Example: A spherical naphthalene ball exposed to air, losses its volume proportional to its instantaneous surface area due to evaporation. If the initial diameter of the ball is 2 cm, reduces to 1 cm after 3 months , the ball completely evaporates in ___________?

Solution: Volume, $V_1=\frac{4}{3}\pi r^3$

Area, $A=4\pi r^2$

Assume at time t_1 volume is V_1 and at time t_2

Volume is V_2 Since, it is given

$$\frac{dv}{dt}\propto A \text{ i.e. } \frac{dv}{dt}=-KA$$

Negative sign is used as area decreases when naphthalene evaporates.

$$\text{i.e. } \frac{d}{dt}\left(\frac{4}{3\pi r^3}\right)=-K\left(4\pi r^2\right)$$

$$\frac{4\pi}{3}\times 3r^2\times\frac{dr^2}{dt}=-K4\pi r^2$$

$$\frac{dr}{dt}=-K \text{ .... (1)}$$

i) At t = 0 $\to$ r = 1 cm

ii) At t = 3 months $\to$ $r=\frac{1}{2}$cm

iii) At t = n $\to$ r = 0

From (1) $\frac{dr}{dt}=-K$

dr = –Kdt

Integrating both sides,

r = –Kt + c .... (2)

Substitute (i) in (2) $\Rightarrow$ 1 = c

Substitute (ii) in (2) 3K= $c-\frac{1}{2} \Rightarrow K=\frac{1}{6}$

Substitute (iii) in (2)

$$0=-\frac{1}{6}n+1$$

n = 6 months

Example: A body originally 60°C cools down to 40°C in 15 minutes while kept in air at a temperature of 25°C. Then what will be the temperature of body after 30 minutes?

Solution: To solve the above problem we can apply Newton's law of cooling. According to this law, the temperature of a body changes proportional to the difference between the body temperature and surrounding medium temperature. Assume that θ be the temperature of body and θ_0 the surrounding medium temperature then

$$\frac{d\theta}{dt}\alpha\ \theta-\theta_0$$

$$\frac{d\theta}{dt}=K\left(\theta-\theta_0\right) \qquad ...(1)$$

Where K $\rightarrow$ proportionality constant and '-ve' sign indicates body temperature decreases as time increases.

i) at $t = 0$, $\theta = 60°C$ ...(i)

$t = 15$, $\theta = 40°C$ ...(ii)

$t = 30$, $\theta = ?$ ...(iii)

From (1) $\frac{d\theta}{dt} = -K(\theta - \theta_0)$

i.e. $\frac{d\theta}{\theta - \theta_0} = -Kdt$

$$\int \frac{d\theta}{\theta - \theta_0} = -K\int dt$$

$\theta_0 = 25 \log(\theta - 25) = -Kt + c$ ...(2)

Substitute (i) in (2) $\log 35 = c$ ...(3)

$\log(\theta - 25) = -Kt + \log 35$

$Kt = \log 35 - \log(\theta - 25)$ ...(4)

Substitute (ii) in (4) $15K = \log 35 - \log 15$ ...(5)

$$(4)\ \frac{t}{15} = \frac{\log\left(\frac{35}{\theta - 25}\right)}{\log\left(\frac{35}{15}\right)}$$

put $t = 30$

$$\frac{30}{15}\log\frac{35}{15} = \log\frac{35}{\theta - 25}$$

$$\left(\frac{35}{15}\right)^2 = \frac{35}{\theta - 25}$$

i.e. $\theta - 25 = \frac{45}{7} = 6.428$

$\theta = 31.43°C$

Example: If a thermometer is taken outdoor where the temperature is 0°C from a room in which temperature is 21°C and the reading drops to 10°C in 1 minute. Then how long after its removal will the temperature reading be 5°C.

Solution: To solve the above problem we can apply Newton's law of cooling.

$$\frac{d\theta}{dt} = K(\theta - \theta_0) \quad ...(1)$$

The temperature at various instants of time is given by,

i) at $t = 0, \theta = 21°C$

ii) at $t = 1\,\text{min}$, $\theta = 10°C$

iii) at $t = ?$, $\theta = 5°C$

$$\int \frac{d\theta}{\theta - \theta_0} = -K\int dt$$

$\theta_0 = 0 \Rightarrow \log\theta = -Kt + c$ ...(1)

Substitute (ii) in (1)

$\log 21 = c$

Hence, $Kt = \log 21 - \log\theta = \log\frac{21}{\theta}$ ...(2)

Substitute (ii) in (2)

$K \times 1 = \log 21 - \log 10 = \log\frac{21}{10}$ ...(3)

When temperature is 5°C

$\log\frac{21}{5}$

$t = \log\frac{21}{10} = 1.96\text{ min}$

Example: Solve the following differential equation: $x^4\frac{dy}{dx} + x^3y + \text{cosec}\,xy = 0$

Solution: $x^3\left(x\frac{dy}{dx} + y\right) + \text{cosec}\,xy = 0$

Let xy=z

$$\frac{dz}{dx} = x\frac{dy}{dx} + y$$

$$x^3\frac{dz}{dx} + \text{cosec}\,z = 0$$

$$x^3\frac{dz}{dx} = -\text{cosec}\,z$$

$$\frac{dx}{x^3} = \frac{dz}{-\text{cosec}\,z} = \sin z\,dz$$

Integrating both sides,

$$\int \sin z\,dz = -\int\frac{dx}{x^3}$$

$$-\cos z = -\left(\frac{-1}{2}x^{-2}\right) + c$$

$$\cos z = \frac{-1}{2}x^2 + c$$

Example: Solve the following differential equation: $\frac{dz}{dx} + 1 - x\tan z = 1$

Solution: $\frac{dz}{dx} = x\tan z$

$$\frac{dz}{\tan z} = xdx$$

$$\int \cot z dx = \int xdx$$

$$\ln(\sin z) = \frac{x^2}{2} + c$$

Homogenous Equations

The differential equations of the form

$$\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$$

Where f (x, y) and ϕ(x, y) are homogenous functions of same degree in x and y.

A homogenous function of degree 'n' is represented as, $a x^n + a_1 x^{n-1}y + a_2 x^{n-2}y^2$ .... + $a_n y^n$. Here each term has the degree 'n'.

Example: $(x^2 - y^2)\tan\frac{y}{x}$ is a homogenous function of degree 2.

To solve such kind of differential equations, we substitute,

$$y = vx \text{ and } \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Solve resultant differential equation by variable separation.

Solved Examples

Example: Solve $(y^2 - x^2)\, dx - 2xydy = 0$

Solution: Given equation can be represented as,

$\frac{dy}{dx} = \frac{y^2 + x^2}{2xy}$ which is homogenous in x and y

Substitute, $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Then, the equation becomes

$$v + x\frac{dv}{dx} = \frac{1}{2}\left[\frac{v^2 - 1}{v}\right]$$

$$x\frac{dv}{dx} = -\left[\frac{v^2 + 1}{2v}\right]$$

Separating the variables, $\frac{2v}{v^2+1}dv = -\frac{dx}{x}$

Integrate both sides of the equation,

$$\int \frac{2v}{v^2+1}dv = -\frac{dx}{\int x}$$

$$\ln\left(v^2 + 1\right) = -\ln x + c = \ln\frac{1}{x} + c = \ln\frac{k}{x}$$

$$v^2 + 1 = \frac{k}{x}$$

Replace v by $\frac{y}{x}$, we get

$$\left(\frac{y}{x}\right)^2 + 1 = \frac{k}{x},$$

$$x^2 + y^2 = kx$$

This represents a family of circles with co-ordinates and radius dependent on k.

Linear Differential Equations

The general form of 1st order linear differential equation in variable y is given by,

$$\frac{dy}{dx} + Py = Q \qquad \text{...(1)}$$

where P and Q are functions of x

Multiply (1) with R where R is some other function of x

$$R\frac{dy}{dx} + RPy = RQ$$

$$\text{Let } RP = \frac{dR}{dx}$$

$$\therefore RP = \frac{dR}{dx}$$

$$\frac{dR}{R} = Pdx$$

$$\therefore R = \exp\left(\int Pdx\right)$$

$$\frac{d[Ry]}{dx} = RQ$$

$$Ry = \int RQdx$$

The solution of above differential equation is then,

$$y = e^{-\int Pdx} \times \int Qe^{\int Pdx}dx + c$$

The factor $R = \exp\left(\int P dx\right)$ is called as integrating factor.

Solved Examples

Example: Determine the solution of $t\frac{dx}{dt} + x = t$ satisfying the condition x(1) = 0.5

Solution: $t\frac{dx}{dt} + x = t$

Divide both sides by 't'

$$\frac{dx}{dt} + \frac{X}{t} = 1$$

$$IF = e^{\int \frac{1}{t} dt} = e^{\log t} = t$$

Solution of the differential equation is then,

$$x \times t = \int 1 \times t\, dt = \frac{t^t}{2} + c$$

Substituting initial conditions,

$$0.5 = \frac{1}{2} + c \Rightarrow c = 0$$

$$xt = \frac{t^2}{2} \Rightarrow x = \frac{t}{2}$$

Example: Determine the solution of $\frac{dy}{dx} + 2y$ tanx = sinx given y = 0 when $x = \frac{\pi}{3}$.

Solution:

$$IF = e^{\int 2\tan x} dx = e^{2\log \sec x} = e^{\log \sec^2 x} = \sec^2 x$$

$y \times \sec^2 x = \int \sin x \sec^2 x\, dx + c = \int \tan x \sec x\, dx + c = \sec x + c$

Substitute Initial Conditions,

0 = 2 + c

$\therefore c = -2$

$y = \cos x - 2\cos^2 x$

Example: Determine the solution of

$$x(x-1)\frac{dy}{dx} - y = x^2(x-1)^2$$

Solution: $\frac{dy}{dx} - \frac{y}{x(x-1)} = x(x-1)$

$$IF = e^{\int \frac{-1}{x(x-1)}} dx = e^{\int \frac{(x-1)-x}{x(x-1)}} dx = e^{\log x - \log(x-1)} = \frac{x}{x-1}$$

The solution of differential equation is given by,

$$y\left(\frac{x}{x}\right) = \int x(x-1) \times \frac{x}{x-1} dx$$

$$y\left(\frac{x}{x-1}\right) = \frac{x^3}{3} + c$$

Example: Determine the solution of following differential equation

$(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$

Solution: This equation can be rephrased as,

$$\frac{dx}{dy} = \frac{\tan^{-1} y - x}{1 + y^2}$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$$

$$IF = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

The solution of this differential equation is,

$$x \times e^{\tan-1\, y} = \int \frac{\tan^{-1} y}{1+y^2} \times e^{\tan-1y} dy$$

$t = \tan^{-1} y$

$$dt = \frac{1}{1+y^2} dy$$

$xe^t = \int t \times e^t dt = t \times e^t - \int e^t = e^t(t-1)$

Thus, the solution is $x = t - 1 = \tan^{-1} y - 1$

Example: Determine the solution of the equation: $2\frac{dy}{dx} - y\sec x = y^3 \tan x$

Solution: Divide both sides by y^3

$$\frac{2\, dy}{y^3 dx} - \frac{\sec x}{y^2} = \tan x$$

Multiply both sides by -1

$$\frac{-2\, dy}{y^3\, dx} + \frac{\sec x}{y^2} = -\tan x$$

$$\text{Assume } \frac{1}{y^2} = z$$

$$\frac{-2\, dy}{y^3\, dx} = \frac{dz}{dx}$$

Thus, this equation can be re-written as,

$$\therefore \frac{dz}{dx} + \sec x \times z = -\tan x$$

$$IF = e^{\int \sec x dx} = e^{\log(\sec x + \tan x)} = \sec x + \tan x$$

The solution of above differential equation is then,

z (secx + tanx) = ∫(secxtanx + $\tan^2$ x) dx

Since, $\tan^2 x = \sec^2 x - 1$

z (secx + tanx) = ∫(–secxtanx – $\sec^2$ x + 1) dx

z (secx + tanx) = –secx – tanx + x + c

$$\frac{1}{\tan x\, y^2} \sec x = \left(-\sec x - \tan x + x + c\right)$$

Inspection method

Consider the equation Mdx + Ndy = 0

Suppose it is possible to rearrange the terms on LHS of (1) by observing the following formulae we can solve the equation through inspection.

Some useful formulae

$$d\left[\log\left(\frac{x}{y}\right)\right] = \frac{xdy + ydx}{xy}$$

$$d\left[\log\left(\frac{y}{x}\right)\right] = \frac{xdy - ydx}{xy}$$

$$d\left[\tan^{-1}\left(\frac{x}{y}\right)\right] = \frac{ydx_2 - xdy_2}{x + y}$$

$$d\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = \frac{xdy - ydx}{x^2 + y^2}$$

$$d\left[\log\left(x^2 + y^2\right)\right] = 2\left(xdx + ydy\right)$$

$$d\left(\frac{e^x}{y}\right) = \left(\frac{ye^x dx - e^x dy}{y^2}\right)$$

$$d\left(\frac{e^y}{x}\right) = \left(\frac{xe^y dy - e^y dx}{x^2}\right)$$

$$d\left(\frac{x^2}{y}\right) = \left(\frac{2xydx - x^2 dy}{y^2}\right)$$

$$d\left(\frac{y^2}{x}\right) = \left(\frac{2xydy - y^2 dx}{x^2}\right)$$

$$d\left(\frac{x^2}{y^2}\right) = \left(\frac{2xy^2 dx - 2x^2 ydy}{y^4}\right)$$

$$d\left(\frac{y^2}{x^2}\right) = \left(\frac{2x^2 ydy - 2xy^2 dx}{x^4}\right)$$

Solved Examples

Example: Determine the solution of following differential equation

$$\left[y + \cos y + \frac{1}{2\sqrt{x}}\right]dx + \left(x - x\sin y - 1\right)dy = 0$$

Solution: (y dx + x dy) + (cosy dx – xsiny dy)

$$+\frac{1}{2\sqrt{x}}dx - dy = 0$$

This can be expressed as,

$$d\left(xy\right) + d\left(x\cos y\right) + \frac{1}{2\sqrt{x}}dx - dy = 0$$

Integrating the above equation we get,

$$xy + x\cos y + \sqrt{x} - y = c$$

Example: Determine the solution of

(y + 1 + x^2) dx + (x^2 siny – x) dy = 0

Solution: The above equation can be rewritten as,

(ydx – xdy) + dx + x^2dx + x^2 siny dy = 0

Divide both sides by x^2

$$\frac{y\,dx - x\,dy}{x^2} + \frac{dx}{x^2} + dx + \sin y\; dy = 0$$

$$-d\left(\frac{y}{x}\right) - d\left(\frac{1}{x}\right) + dx - d\left(\cos y\right) = 0$$

Integrating both sides,

$$-\left(\frac{y}{x}\right) - \frac{1}{x} + x - \cos y = c$$

Example: Determine the solution of differential equation (x^4e^x – 2mxy^2) dx + 2mx^2y dy = 0

Solution: This equation can be rearranged as,

x^4e^xdx – m(2xy^2) dx – 2x^2ydy = 0

Divide both sides by x^4

$$e^x dx + m\left(\frac{-2xy^2dx + 2x^2ydy}{x^4}\right) = 0$$

This equation can be expressed in form of differentials as,

$$e^x dx + m \times d\left(\frac{y^2}{x^2}\right) = 0$$

Integrating both sides as,

$$e^x + m\left(\frac{y^2}{x^2}\right) = c$$

Example: Solve the following differential equation,

$$\frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{a^2 - x^2 - y^2}{x^2 + y^2}}$$

Solution: The above equation can be expressed in form of differentials as,

$$d\left(\frac{x^2 + y^2}{(xdy^2 - ydx)}\right) = \sqrt{\frac{a^2 - (x^2 + y^2)}{x^2 + y^2}}$$

$$\frac{1}{2} - \frac{\sqrt{(x^2 + y^2)}}{\sqrt{a^2 - (x^2 + y^2)}} d(x_2 + y_2) = xdy - ydx$$

Multiply both sides by, $\frac{1}{x^2 + y^2}$

$$\frac{1}{2}\sqrt{x^2 + y^2}\,\frac{1}{\sqrt{a^2 - (x^2 + y^2)}} d(x^2 + y^2)$$

$$= \frac{xdy - ydx}{x^2 + y^2}$$

$$\text{Let}\sqrt{x^2 + y^2} = z \Rightarrow dz = \frac{1}{2\sqrt{x^2 + y^2}} \times d(x^2 + y^2)$$

$$\therefore \frac{dz}{\sqrt{a^2 - (x^2 + y^2)}} = d\tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{dz}{\sqrt{a^2 - z^2}} = d\tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{1}{a}\sin^{-1}\left(\frac{z}{a}\right) = \tan^{-1}\left(\frac{y}{x}\right) + c$$

$$\frac{1}{a}\sin^{-1}\left(\frac{\sqrt{x^2 + y^2}}{a}\right) = \tan^{-1}\left(\frac{y}{\cancel{x}}\right) + c$$

Bernoulli's Equation

The equation of the form $\frac{dy}{dx} + Py = Qy^n$

Where, P and Q are functions of x is called as Bernoulli's equation. To solve this type of equation we have to divide both sides by yn The equation now becomes, $y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$

Substitute, $z = y^{1-n}$

Thus, $\frac{dz}{dx} = 1(-n\,y)^{-n}\frac{dy}{dx}$

Thus, this equation reduces to

$$\frac{1\,dz}{1-n\,dx} + Pz = Q$$

$$\frac{dz}{dx} + P(1-n)z = Q(1-n)$$

This is a linear equation that can be solved easily.

Exact Differential Equation

Consider the equation Mdx + Ndy = 0 .... (1)

If it is possible to express the above equation as df (x, y) = 0 Then by Integration f (x, y) = c is the solution of (1)

Then we say that the above differential equation is an exact differential equation. The necessary and sufficient condition for an equation to be exact is,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

To get the required solution we can apply any one of the formulae.

∫Mdx + ∫terms of N not containing 'x' dy = c

Treat y as constant

∫Ndy + ∫terms of M not containing 'y' dx = c

Treat x as constant

Solved Examples

Example: Solve the following differential equation:

$$\frac{2x}{y^3}dx + \left(\frac{y^2 - 3x^2}{y^4}\right)dy = 0$$

Solution:

$$\frac{\partial M}{\partial y} = 2x \times \frac{-3}{y^4} = -\frac{6x}{y^4}$$

$$\frac{\partial N}{\partial x} = 0 - \frac{3}{y^4} \times 2x = -\frac{6x}{y^4}$$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ this equation is an exact differential equation.

Solution can be given as,

∫Mdx + ∫terms of N not containing 'x' dy = c

$$\frac{2}{y^3} \times \frac{x^2}{2} + \int y^{-2} dy = c$$

$$\frac{x^2}{y^3} \frac{1}{y} = c$$

Note: even if you apply another formula for the solution you will get the same expression.

Example: Solve the following differential equation (2xy + y – tany) dx + (x^2 – xtan^2 y + sec^2 y) dy = 0

Solution: $\frac{\partial M}{\partial y} = 2x + 1 - \sec^2 y = 2x - \tan^2 y$

$$\frac{\partial N}{\partial x} = 2x - \tan^2 y$$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ this equation is exact

The solution can be computed as,

∫Mdx + ∫terms of N not containing 'x' dy = c

$$y\frac{X^2}{2} + yx - x\tan y + \tan y = c$$

Example: Solve the following differential equation ($y^2e^{xy^2}$ + $4x^3$) dx + ($2xye^{xy^2}$ – $3y^2$) dy = 0

Solution:

$$\frac{\partial M}{\partial y} = y^2 e^{xy^2} \times 2xy + e^{xy^2} \times 2y$$

$$\frac{\partial N}{\partial x} = 2xy \times e^{xy^2}\left(y^2\right) + e^{xy^2} \times 2y = \frac{\partial M}{\partial y}$$

Thus, this equation is exact. Solution is given by,

∫Mdx + ∫terms of N not containing 'x' dy = c

$$\frac{y^2 e^{xy^2}}{y^2} + 4\frac{x^4}{4} + \left(\frac{-3y^3}{3}\right) = c$$

$$e^{xy^2} + x^4 - y^3 = c$$

Example: Solve the following differential equation (xycosxy + sinxy) dx + x^2 cos (xy) dy = 0

Solution: $\frac{\partial M}{\partial y} = -xy\sin xy \times x + x\cos xy + x\cos xy$

= –x^2ysinxy + 2xcosxy

$$\frac{\partial M}{\partial x} = -x^2 y \sin xy + 2x\cos xy = \frac{\partial M}{\partial y}$$

Solution is given by,

∫Ndy + ∫terms of M not containing 'y' dx = c

$$x^2 \times \left[\frac{\sin xy}{x}\right] = c$$

Thus, x sinxy = c is the solution.

Example: Solve $\left(1 + e^{x/y}\right)dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$

Solution:

$$\frac{\partial M}{\partial y} = e^{x/y} \times \frac{-x}{y^2}$$

$$\frac{\partial N}{\partial X} = e^{x/y} - \left(\frac{1}{y}\right) + \left(1 - \frac{x}{y}\right)e^{x/y} \times \frac{1}{y}$$

$$= -x\frac{e^{x/y}}{y} = \frac{\partial M}{\partial y}$$

Thus, this equation is exact.

Solution is given by,

∫Mdx + ∫terms of N not containing 'x' dy = c

$$x + \frac{e^{x/y}}{1} = c$$

$$x + \frac{y}{ye^{x/y}} = c$$

Equations reducible to exact equation form

Suppose, the differential equation given is ydx – x dy =0 ...(2)

$$\frac{ydx - xdy}{y^2} = 0$$

$$d\left(\frac{x}{y}\right) = 0$$

$$\frac{x}{y} = c$$

In the above equation (2) if the given equation is made an exact equation by multiplying by $\frac{1}{y^2}$, therefore it is called an integrating factor (IF).

We can also observe that $-\frac{1}{x^2}, \frac{1}{xy}, \frac{-1}{xy}, \frac{1}{(x^2+y^2)}, \frac{-1}{x^2+y^2}$ etc. are suitable IFs of (2). Therefore the IF of a differential equation is not unique.

If the given equation Mdx + Ndy = 0 is not exact equation.

i.e. $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

then it can be made exact equation by manipulating with a suitable IF. But to get the appropriate IF of (1) we can follow certain rules.

Rule I

If the given equation Mdx + Ndy = 0 isn't an exact equation but it is a homogenous equation and Mx + Ny ≠ 0, then $\frac{1}{Mx+Ny}$ is an integrating factor of the given equation.

Example: Determine the solution of $(x^2y - 2xy^2)\,dx + (3x^2y - x^3)\,dy = 0$

Solution: If we observe that every term of both M and N of the problem is with same degree, then we can recognize it is a homogenous equation.

$Mx + Ny = x^3y - 2x^2y^2 + 3x^2y^2 - x^3y = x^2y^2$

$$IF = \frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$$

Multiply IF into the given differential equation

$$\frac{x^2y-2xy^2}{x^2y^2}dx + \frac{3x^2y-x^3}{x^2y^2}dy = 0$$

$$\left(\frac{1}{y}-\frac{2}{x}\right)dx + \left(\frac{3}{y}-\frac{x}{y}\right)dy = 0$$

$$\left(-\frac{2}{x}\right)dx + \left(\frac{3}{y}\right)dy + d\left(\frac{x}{y}\right) = 0$$

Integrating both sides

$$\frac{x}{y} - 2\log x + 3\log y = c$$

Example: Solve the differential equation: $x^2ydx - (x^3 + y^3)\,dy = 0$

Solution: This is a homogenous equation of order 3 but it is not exact $Mx + Ny = x^3y - x^3y - y^4 = -y^4 \neq 0$

$$IF = \frac{1}{Ny} = \frac{1Mx}{y^4}$$

$$\frac{-x^2y}{y^4}dx + \frac{x^3+y^3}{y^4}dy = 0$$

$$\frac{-x^2}{y^3}dx + \left(\frac{x^3}{y^4}+\frac{1}{y}\right)dy = 0$$

$$\frac{dy}{y} - d\left(\frac{x^3}{3y^3}\right) = 0$$

Integrating both sides, we get $-\frac{x^3}{3y^3} + \log y = c$

Example: Solve the differential equation: $x^2ydx - (x^3 + y^3)dy = 0$

Solution: This is a homogenous equation of order 3 but it is not exact

$Mx + Ny - x^3y - x^3y - y^4 = -y^4 \neq 0$

$$IF = \frac{1}{Ny} = \frac{1}{y^4}Mx$$

$$\frac{-x^2y^4}{-x^2}dx + \frac{x^3+y^3}{X^3y^4\,\frac{1}{y}}dy = 0$$

$$\frac{-x^2}{y^3}dx + y^4 + dy = 0$$

$$\frac{dy}{y} - d\left(\frac{x^3}{3y^3}\right) = 0$$

Integrate both sides, we get $\frac{x^3}{3y^3} + \log y = c$

Rule II

In the given equation Mdx + Ndy = 0 is not an exact equation but it is in the form yf (xy) dx + xg (xy)dy = 0 and (Mx–Ny) ≠ 0

Then $IF = \frac{1}{Mx - Ny}$

Solved Examples

Example: Solve $(x^2y^2 + xy + 1)\,ydx + (x^2y^2 - xy + 1)\,x\,dy = 0$

Solution: The above differential equation is not exact but it is in the form, yf (xy)dx + xg(xy)dy = 0

Mx – Ny = $x^3y^3 + x^2y^2 + xy - (x^3y^3 - x^2y^2 + xy)$ = $2x^2y^2$

$$IF = \frac{1}{2x^2y^2}$$

$$\frac{x^2y^2 + xy + 1}{2x^2y^2} ydx + \frac{x^2y^2 - xy + 1}{2x^2y^2} xdy = 0$$

$$\left(y + \frac{1}{x} + \frac{1}{x^2y}\right)dx + \left(x - \frac{1}{y} + \frac{1}{xy^2}\right)dy = 0$$

$$\frac{dx}{x} - \frac{dy}{y} - d\left(\frac{1}{xy}\right) + d(xy) = 0$$

Integrate both sides

$$xy + \log x - \frac{1}{xy} - \log y = c$$

Example: Solve

$$[xy\cos(xy) + \sin(xy)]ydx$$
$$[xy\cos(xy) + \sin(xy)]ydx = 0$$

Solution: The above differential equation is not exact but it is in the form,

yf (xy) dx + xg(xy)dy = 0

Mx – Ny = 2xysinxy

$$IF = \frac{1}{2xy\sin xy}$$

$$\frac{xy\cos(xy) + \sin xy}{2xy\sin xy} ydx + \frac{xy\cos(xy)}{2xy\sin xy} xdy = 0$$

$$\frac{1}{2}[y\cot(xy)dx + x\cot(xy)dy]$$

$$+ \frac{1}{2x}dx - \frac{1}{2y}dy = 0$$

$$\frac{1}{2}d\log\text{sim}(xy) + \frac{1}{2x}dx - \frac{1}{2}\log\frac{x}{y} = \frac{1}{2}\log c$$

$$\frac{1}{2}\log[y\sin(xy)] = \frac{1}{2}\log c$$

$$x\sin xy = cv$$

Rule III

If the given equation Mdx + Ndy = 0 is not exact equation but $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ [only a function of x] then $e\int^{f(x)dx}$ is the integrating factor of the differential equation.

Rule IV

If the given equation Mdx + Ndy = 0 is not exact equation but $1\left(\frac{\partial N}{M} - \frac{\partial M}{\partial}\right) = g(y)$ (only a function of y) then $e\int^{g(y)dy}$ is the Integrating Factor of differential equation.

Note: While applying rule III or IV first we consider we must confirm that the equation is not exact.

Solved Examples

Example: Solve $(x^2 + y^2 + x)\,dx + xydy = 0$

Solution: Checking for the exactness of the differential equation

$$\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = 2y - y \neq 0$$

$$\text{Rule III}: \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{y}{xy} = \frac{1}{x} = f(x)$$

$$IF = e\int^{1/xdx} = e^{\log x} = x$$

Multiply IF into the differential equation,

$(x^3 + xy^2 + x^2)\,dx + x^2ydy = 0$

$$x^3dx + x^2dx + d\left(\frac{x^2y^2}{2}\right) = 0$$

Integrate both sides

$$\frac{X^4}{4} + \frac{x^3}{3} + \frac{x^2y^2}{2} = C$$

Example: Solve $(xy^3 + y)\,dx + 2\,(x^2y^2 + x + y^4)\,dy = 0$

Solution: Checking for exactness

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = x \times 3y^2 + 1 - (4xy^2 + 2) = -(xy^2 + 1)$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial X} - \frac{\partial M}{\partial y}\right) = \frac{xy^2 + 1}{xy^2 + 1} \times \frac{1}{y} = g(y)$$

$$\text{Rule IV}: \therefore\ IF = e^{\int \frac{1}{y}dy} = y$$

Multiply IF on both sides of the equation,

$(xy^4 + y^2)\,dx + 2\,(x^2y^3 + xy + y^5)\,dy = 0$

$(y^2dx + 2xydy) + 2y^5dy + (xy^4dx + 2x^2y^3dy) = 0$

$$d(xy^2) + 2d\left(\frac{y^2}{6}\right) + d\left(\frac{x^2y^4}{2}\right) = 0$$

Integrate both sides,

$$Xy^2 + \frac{y^6}{3} + \frac{x^2y^4}{2} = 0$$

Clairaut's Equation

The differential equations of the form y = px + f(p) where $p = \frac{dy}{dx}$ dy is known as Clairaut's equation.

Differentiate both sides with respect to x

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\left[x + f'(p)\right]\frac{dp}{dx} = 0$$

$$\text{Thus, either } \frac{dp}{dx} = 0 \text{ or } \left[x + f'(p)\right] = 0$$

$$\text{IF } \frac{dp}{dx} = 0 => p = c$$

Thus, solution of this differential equation is y = cx + f(c) where c is an arbitrary constant. Solution of a higher order linear differential equation with constant coefficient form.

Linear Differential Equation of n^{th} order

The general form is given by

$$\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + \dots\dots\dots k_{n-2}\frac{d^2y}{dx^2}$$

$$+ k_{n-1}\frac{dy}{dx} + k_ny = x$$

Where K_1, K_2.................K_n are constants and X = function of x.

Such equations are most important in the study of electromechanical vibrations and other engineering problems.

If y_1, y_2, y_3 ...., yn are only two solutions of the equations

$$\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + \dots\dots\dots k_{n-2}\frac{d^2y}{dx^2}$$

$$+ k_{n-1}\frac{dy}{dx} + k_ny = 0$$

Then, u = c_1y_1 + c_2y_2 + c_3y_3 .... + c_ny_n is also its solution. This means,

$$\frac{d^nu}{dx^n} + k_1\frac{d^{n-1}u}{dx^{n-1}} + k_2\frac{d^{n-2}u}{dx^{n-2}} + \dots\dots\dots k_{n-2}\frac{d^2u}{dx^2}$$

$$+ k_{n-1}\frac{du}{dx} + k_nu = 0$$

If y = v be any particular solution of

$$\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^n} + \dots\dots\dots\dots k_ny = x$$

$$\text{Then, } \frac{d^nv}{dx^n} + k_1\frac{d^{n-1}v}{dx^{n-1}} + \dots\dots\dots\dots k_nv = x$$

$$\text{Then, } \frac{d^n(u+v)}{dx^n} + k_1\frac{d^{n-1}(u+v)}{dx^{n-1}}$$

$$+ \dots\dots\dots K(u+v) = x$$

Then, y = u + v is complete solution of the differential equation. The part 'u' is called as Complimentary Function (CF) and v is called as Particular Integral (PI)

$$\text{Let } D = \frac{d}{dx} : \text{Then, } D^n = \frac{d^n}{dx^n}$$

Thus, the differential equation can be represented as,

(D^n + K_1 D^{n-1} + K_2 D^{n-2} + + ... ($K^{n-2}D^n$ + Kn + D) y = X

i.e. f (D) y = X

Suppose X = 0

We get f(D) y = 0

Which is called homogeneous linear differential equation. Otherwise, if (X ≠ 0), it is called non-homogeneous linear differential equation.

Rules for finding the Complimentary Function

To solve the equation $\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}}{dx^{n-1}} + \dots\dots\dots\dots$

$+ k_ny = 0$ where k's are constants.

This equation can be represented as,

(Dn + K_1 D^{n-1} + ... + K_n) y = 0

This equation can be represented in symbolic co-efficient form as

D^n + K_1D^{n-1} + ... + Kn = 0

This is called as Auxiliary equation (A.E.). Let m1, m2, m3, ..., mn be its roots.

Case 1: If all roots are real and different. Then, this equation is represented as,

(D–m_1) (D–m_2) (D–m_3) .... (D–m_n) y = 0

If (D–m_n) y = 0, then

$$\frac{dy}{dx} - m_ny = 0$$

This is a linear equation, and the integrating factor is e^{-mnx}

The solution is $y = ce^{mnx}$

complimentary function is $y = c_1e_{m1x} + c_2e^{m2x} + \dots\dots\dots + c_ne^{mnx}$

Case 2: If two roots are equal i.e. $m_1 = m_2 = m$ and others are distinct Then, the complimentary function is given by

$y = (c_1 + c_2\ x)\ e^{mx} + c_3\ e^{m3x} \dots\dots\dots\dots\dots\dots + c_n\ e^{mnx}$

However, if there are 3 equal roots then the complimentary function is given by.

$y = (c_1 + c_2x + c_3\ x^2)e^{mx} + c_4e^{m4x} \dots\dots\dots\dots\dots\dots + c_ne^{mnx}$

Case 3: If one pair of roots are complex

$m_1 = \alpha + i\beta;\ m_2 = \alpha - i\beta$

Then, the solution is given by

$y = e^{\alpha x}\ (c_1\ \cos\beta x + c_2\ \sin\beta x) + c_3e^{m3x} \dots\dots\dots\dots\dots\dots + c_ne^{mnx}$

Case 4: If two pair of imaginary roots be equal

$m_1 = m_2 = \alpha + i\beta$

$m_3 = m_4 = \alpha - i\beta$

Then, the complimentary function is given by,

$y = e^{ax}\left[(c_1x+c_2)\cos\beta x+(c_3x+c_4)\sin\beta x\right]+c_5e^{m5x} \dots\dots\dots\dots\dots\dots+c_ne^{mnx}$

Solution of f(D) y = 0 form:

Given Equations	General forms
1. $(D - a)\ y = 0$	1. $y = ce^{ax}$
2. $(D + a)\ b = 0$	2. $y = ce^{-bx}$
3. $Dy = 0$	3. $y = c$
4. $(D - a)(D - b)\ y = 0$	4. $y = c_1e^{ax} + c_2e^{bx}$
5. $(D - a)(D + b)(D - c)\ y = 0$	5. $y = c_1e^{ax} + c_2e^{-bx} + c_3e^{cx}$
6. $(D - a)(D - b)y = 0$ Where $a = \alpha + j\beta; b = \alpha - j\beta$	6. $y = e^{\alpha x}\left[k_1 \cos\beta x + k_2 \sin\beta x\right]$
7. $(D - a)(D - b)y = 0$ Where $a = \alpha + \sqrt{\beta}; b = \alpha - \sqrt{\beta}$	7. $y = e^{\alpha x}\left[k_1 \cos\sqrt{\beta}x + k_2 \sin\sqrt{\beta}x\right]$
8. $(D - a)^2\ y = 0$	8. $y = (c_1 + c_2x)\ e^{ax}$
9. $(D - a)^3\ y = 0$	9. $y = (c_1 + c_2x + c_3x^2)\ e^{ax}$
10. $(D - a)^4\ y = 0$	10. $y = (c_1 + c_2x + c_3x^3 + c_4x^4)\ e^{ax}$
11. $(D - a)^2\ (D + b)^3\ y = 0$	11. $y = (c_1 + c_2x)e^{ax} + (c_3x + c_4x + c_5x^2)\ e^{-bx}$
12. $(D-a)^2(D+b)^2 y = 0$, $a = \alpha + j\beta$, $b = \alpha - j\beta$	12. $y = e^{\alpha x}\left[(k_1 + k_2x)\cos\beta x + (k_3 + k_4x)\sin\beta x\right]$
13. $(D-a)^2(D-b^2)y = 0$, $a = \alpha + j\sqrt{\beta}$, $b = \alpha - j\sqrt{\beta}$	13. $y = e^{\alpha x}\dfrac{\left[(k_1 + k_2x)\cosh\sqrt{\beta}x\right]}{\left[+(k_3 + k_4x)\sinh\sqrt{\beta}x\right]}$

14. $(D-a)^2(D-b)(D-c)(D+d)y=0$ $b=\alpha+j\beta$ and $c=\alpha-j\beta$	14. $y=(c_1+c_2x)e^{ax}+e^{\alpha x}(c_3\cos\beta+c_4\sin\beta x)$ $+c_5e^{-dx}$

Note: In writing the general solution for complete solution of the given differential equation it must be observed that the number of arbitrary constants in the solution is equal to the order of the differential equation given.

Solved Examples

Example: Solve the differential equation

$$\frac{d^3y}{dx^3}-\frac{3dy}{dx}+2y=0$$

Solution: Auxiliary equation is $(D^3 - 3D + 2)\ y = 0$

$(D^3 - D - 2D + 2)\ y = 0$

$$\left[D\left(D^2-1\right)-2\left(D-1\right)\right]y=0$$

$$\left(D-1\right)\left[D\left(D+1\right)-2\right]y=0$$

$$\left(D-1\right)\left[\left(D^2+D-2\right)\right]y=0$$

$$\left(D-1\right)\left[\left(D+2\right)\left(D-1\right)\right]y=0$$

$$\left(D-1\right)^2\left(D+2\right)y=0$$

$$D=1,1,-2$$

$$\therefore y=\left(c_1+c_2x\right)e^x+c_2e^{-2x}$$

Example: Solve the differential equation

$$\frac{d^4y}{dx^4}+8\frac{d^2y}{dx^2}+16y=0$$

Solution: The auxiliary equation is $(D^4 + 8D^2 + 16)y = 0$

$(D^4 + 4D^2 + 4D^2 + 16)\ y = 0$

$$\left[D^2\left(D^2+4\right)+4\left(D^2+4\right)\right]y=0$$

$$\left(D^2+4\right)^2=0$$

$$D^2=-4,-4$$

$$D=\pm 2i,\pm 2i$$

Thus, complementary function is given by

$$y_1=\left(c_1+c_2x\right)\cos 2x+\left(c_3+c_4x\right)\sin 2x$$

Example: Solve the differential equation

$$\frac{d^4y}{dx^4}-2\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=0$$

Solution: The auxiliary equation is $(D^4 - 2D^3 - 3D^2 + 4D + 4)\ y = 0$

$(D + 1)(D^3 - 3D^2 + 4)\ y = 0$

$(D + 1)(D + 1)(D^2 - 4D + 4) = 0$

$(D + 1)^2\ (D - 2)^2 = 0$

$D = -1, -1, 2, 2$

$\therefore\ y = (c_1 + c_2\ x)e^{-x} + (c_3 + c_4\ x)e^{2x}$

Example: The solution of $y'' - 2y' + 10y = 0$ satisfying $y(0) = 4$ and $y'(0) = 1$

Solution: The auxiliary equation is $(D^2 - 2D + 10)\ y = 0$

$D^2 - 2D + 10 = 0$

$$D=\frac{2\pm\sqrt{4-40}}{2}=\frac{2\pm 6i}{2}=1+3i$$

$$y=e^x\left(c_1\cos 3x+c_2\sin 3x\right)$$

$$y\left(0\right)=4$$

$$\therefore c_1=4$$

$$\frac{dy}{dx}=e^x\left(-4\times 3\sin 3x+3c_2\cos 3x\right)$$

$$+\left(4\cos 3x+c_2\sin 3x\right)e^x$$

$$y'\left(0\right)=1$$

$$4+3c_2=1 => c_2=-1$$

$$\therefore y=e^x\left(4\cos 3x-\sin 3x\right)$$

Example: The solution of $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+17y=0$ satisfying the condition $y\left(0\right)=1, y'\left(\frac{\pi}{4}\right)=0$

Solution: The auxiliary equation is $(D^2 + 2D + 17)y = 0$

$D^2 + 2D + 17 = 0$

$$\text{i.e. } D=\frac{-2+\sqrt{4-68}}{2}=\frac{-2\pm 8i}{2}=-1\pm 4i$$

$$y=e^{-x}\left(c_1\cos 4x+c_2\sin 4x\right)$$

$$y\left(0\right)=1$$

$$\frac{dy}{dx}=e^{-x}\left(4c_1\sin 4x+4c_2\cos 4x\right)+$$

$-e^{-x}(c_1\cos 4x + c_2 \sin 4x)$

$$\frac{dy}{dx}\left(\frac{\pi}{4}\right)=0$$

i.e. $c_2 = \frac{1}{4}$

Complimentary Function is $y = e^{-x}\begin{pmatrix}\cos 4x + \\ \frac{1}{4}\sin 4x\end{pmatrix}$

Rules for finding the Particular Integral

The differential equation $\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}}{dx^{n-1}} + \ldots\ldots + k_n y = x$ where k's are constants can be represented as, $(D^n + K_1D^{n-1} + \ldots\ldots\ldots + K_n)\ y = X$ or $f(D)y = X$

From this we can observe that $y = \frac{x}{f(D)}$ is also a solution to it which is called particular integral or particular solution and is denoted by y_P.

An example of this is $\frac{X}{D-a} = e^{ax}\int Xe^{-ax}dx$

Assume $\frac{X}{D-a} = y$

Thus, $(D-a)y = x$

This can be expressed in differential form as,

$$\frac{dy}{dx} - ay = x$$

Integrating factor is e^{-ax}, its solution

is $ye^{-ax} = \int xe^{-ax}dx$

Thus, $y = \frac{x}{D-a} = e^{ax}\int Xe^{-ax}dx$

Case 1: Consider the equation $f(D)\ y = e^{ax+b}$ then

$$y_p = \frac{e^{ax+b}}{f(D)} = \frac{e^{ax+b}}{f(a)}$$

since, $D(e^{ax+b}) = a(e^{ax+b})$

$D^2(e^{ax+b}) = a^2(e^{ax+b})$

$D^n(e^{ax+b}) = a^{(n)}(e^{ax+b})$

Thus, we can express in general form as

$f(D)e^{ax+b} = f(a)e^{ax+b}$

i.e. $\frac{e^{ax+b}}{f(a)} = \frac{e^{ax+b}}{f(D)}$

If $f(a) = 0$ then above rule fails,

$$\frac{e^{ax+b}}{f(D)} = x\frac{e^{ax+b}}{f'(a)}$$

Again if $f'(a) = 0$ then $\frac{e^{ax+b}}{f(D)} = x^2\frac{e^{ax+b}}{f''(a)}$

Some examples of this case are,

i) $\frac{e^{2-3x}}{(D+1)(D+2)} = \frac{e^{2-3x}}{(-3+1)(-3+2)} = \frac{e^{2-3x}}{2}$

ii) $\frac{e^{2-3x}}{(D-a)} = e^{ax}\int e^{ax+b}\times e^{-ax}dx = xe^{ax+b}$

iii) $\frac{e^{ax+b}}{(D-a)^2} = \frac{1}{(D-a)}xe^{ax+b} = e^{ax}\int xe^{ax+b}\times e^{-ax}dx$

$= \frac{x^2e^{ax+b}}{2}$

iv) $\frac{e^{2-3x}}{(D-a)^3} = \frac{1}{(D-a)}\times\frac{x^2}{2}e^{ax+b} = e^{ax}\int\frac{x^2e^{ax+b}}{2}$

$\times e^{-ax}dx = \frac{x^3e^{ax+b}}{3!}$

v) $\frac{e^{2-3x}}{(D-a)^k} = \frac{x^ke^{ax+b}}{k!}$

vi) $\frac{e^{ax+b}}{(D-a)^m(D-n)(D+P)^r} = \frac{x^m}{m!}\frac{e^{ax+b}}{(a-n)(a+p)^r}$

Case 2:

$f(D)y = \sin(ax+b)$ or $\cos(ax+b)$

$$y_P = \frac{\sin(ax+b)}{s(D)}$$

$D(\sin(ax+b)) = a\cos(ax+b)$

$D^2(\sin(ax+b)) = -a^2\sin(ax+b)$

$D^2 = -a^2$

$$y_P = \frac{\sin(ax+b)}{f(D^2)} = \frac{\sin(ax+b)}{f(-a^2)} \quad \text{Given } f(-a^2)\neq 0$$

Similarly

$$y_P = \frac{\cos(ax+b)}{f(D^2)} = \frac{\cos(ax+b)}{f(-a^2)} \quad [f(-a^2)\neq 0]$$

Some examples of this case are,

i) $\frac{\cos(3x+4)}{(D^2+4)} = \frac{\cos(3x+4)}{(-9+4)} = \frac{\cos(3x+4)}{-5}$

ii) $\frac{\sin 2-x}{D^3+1}=\frac{\sin(2-x)}{D\times D^2+1}=\frac{\sin(2-x)}{D\times -1+1}$

$=\frac{\sin(2-x)}{1-D}$

$=(1+D)\frac{\sin(2-x)}{1}\frac{\sin(2-x)}{D^3+1}$

$=(1+D)\frac{\sin(2-x)}{1+1}$

$=\frac{1}{2}[\sin(2-x)-\cos(2-x)]$

iii) $\frac{\cos(ax+b)}{D^2+a^2}=\frac{x}{2a}\sin(ax+b)$

Since $f(-a^2)=-a^2+a^2=0$

$\cos(ax+b)=\frac{x}{f'(D)}\cos(ax+b)$

$=\left(\frac{x}{2D}\right)\cos(ax+b)=\frac{x}{2a}\sin(ax+b)$

iv) $\frac{\sin(ax+b)}{D^2+a^2}=\frac{-x}{2a}\cos(ax+b)$

Since, $f(-a^2)=-a^2+a^2=0\ \frac{\sin(ax+b)}{D^2+a^2}$

$=\frac{x}{f'(D)}\sin(ax+b)$

v) $\frac{\sin(3x+4)}{D^2+9}=\frac{-x}{2\times 3}\cos(3x+4)$

$=\frac{-x}{6}\cos(3x+4)$

vi) $\frac{\cos(2x-3)}{D^2+4}=\frac{x}{2}\sin\frac{(2x-3)}{2}$

$=\frac{x}{4}\sin(2x-3)$

Example: Solve the differential equation: $(D^3 – 5D^2 + 8D – 4)\ y = e^{2x}$

Solution: The homogenous equation for the following differential equation can be written as,

$(D^3 – 5D^2 + 8D – 4) = 0$

$(D – 1)\ (D2 – 4D + 4) = 0$

$(D – 1)\ (D – 2)^2 = 0$

The roots of this equation are,

$D = 1, 2, 2$

$y_c = c_1 e^x + (c_2 + c_3 x)e^{2x}$

$y_P=\frac{x}{f(D)}=\frac{e^{2x}}{(D-1)(D-2)^2}=x^2\frac{e^{2x}}{2!}$

Since, $f(2)=0$ and $f'(2)=0$

$y_P=\frac{x}{f(D)}=x^2\frac{e^{2x}}{f''(a)}$

$f''(D)=6D-10$

Thus, $f''(2)=6\times 2-10=2$

Example: Determine the solution of $y''–8y'+16y = 3e^{4x}$ satisfying $y = 0$ at $x = 0$ and $x = 2$

Solution: The given differential equation can be represented as,

$(D^2 – 8D + 16)\ y = 0$

$(D – 4)^2 = 0$

$D = 4, 4$

Complimentary Function is given by, $y_c = (c_1 + c_2 x)e^{4x}$

Particular integral can be calculated as,

$y_P=\frac{3e^{4x}}{(D-4)^2}=\frac{x^2}{2!}3e^{4x}$

Since, $f(4)=0$ and $f'(4)=0$

$y_P=\frac{3e^{4x}}{(D-4)^2}=\frac{x^2}{f''(4)}3e^{4x}=\frac{x^2}{2}3e^{4x}$

Complete solution is, $y_c=y_c+y_P$

$=(c_1+c_2x)e^{4x}+\frac{3x^2}{2}e4x$

Satisfying the initial conditions,

$y = 0$ at $x = 0$

$0 = c_1 + 0$

$c_1 = 0$

$y = 0$ at $x = 2$

$0 = 2c_2e^8 + 6e^8$

$c_2 = –3$

$\therefore y=-3xe^{4x}+\frac{3x^2}{2}e^{4x}$

Example: Solve the differential equation

$\frac{d^4y}{dx^4}-y=15\cos 2x$

Solution: The homogenous part of the differential equation can be represented as,

$(D^4 – 1)y = 0$

$(D^2)^2 - 1^2 = 0$

$(D^2 + 1)(D^2 - 1)^2 = 0$

$D = \pm 1, \pm i$

$y_c = c_1e^x + c2e^{-x} + e^0 (c_3\cos x + c_4\sin x)$

$y_c = c_1e^x + c2e^{-x} + c_3\cos x + c_4\sin x$

$$y_P = \frac{15\cos 2x}{(D^2+1)(D^2-1)} = \frac{15\cos 2x}{(-2^2+1)(-2^2-1)}$$

$$= \frac{15\cos 2x}{-3\times -5} = \cos 2x$$

$$y = y_c + y_P = c_1e^x + c_2e^{-x} + c_3 + \cos x + c_4 \sin x + \cos 2x$$

Example: Solve the following differential equation $\frac{d^2y}{dx^2} - 4y = \cosh(2x-1+3)^x$

Solution: Differential form of the given equation is $(D^2 - 4)\, y = \cosh\,(2x - 1) + 3^x$

The homogenous part of the equation is, $(D^2 - 4) = 0$

$(D + 2)(D - 2) = 0$

$D = \pm 2$

$y_c = \cosh(2x-1) + 3^x$

Particular Integral can be computed as,

$$y_P = \frac{\cosh(2x-1)+3^x}{(D+2)(D-2)}$$

$$y_P = \frac{1}{2}\frac{\left[e^{2x-1}+e^{-2x+1}\right]}{(D+2)(D-2)} - \frac{e^{x\log 3}}{(D+2)(D-2)}$$

$$y_P = \frac{x}{2}\frac{\left[e^{2x-1}+e^{-2x+1}\right]}{f'(D)} + \frac{3^x}{(2+\log 3)(-2+\log 3)}$$

$$= \frac{x}{4}\frac{\left[e^{2x-1}+e^{-2x+1}\right]}{D} + \frac{3^x}{(2+\log 3)(-2+\log 3)}$$

$$y_P = \frac{x}{8}\left[e^{2x-1}+e^{-2x+1}\right] + \frac{3^x}{(2+\log 3)(-2+\log 3)}$$

$$= \frac{x}{4}\sinh(2x-1) + \frac{3^x}{(\log 3)^2 - 4}$$

Example: Solve $y'' + y = \sin x \sin 2x$

Solution: The homogenous part of the differential equation is $D^2 + 1 = 0$

$D = \pm i$

Thus, complementary function is $y_c = c_1 \cos x + c_2 \sin x$

$$X = \sin\sin 2x = \frac{1}{2}(2\sin x\sin 2x)$$

$$= \frac{1}{2}\left[\cos x - \cos 3x\right]$$

Particular Integral can be computed as,

$$y_P = \frac{\sin_2 \sin 2x}{D^2+1} = \frac{1}{2}\left[\frac{\cos x}{D^2+1} - \frac{\cos 3x}{D^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{x}{f'(D)}\cos x - \frac{\cos 3x}{-3^2+1}\right]$$

$$y_P = \frac{1}{2}\left[\frac{x}{2D}\cos x - \frac{\cos 31}{-3^2+1}\right]$$

$$y_P = \frac{1}{2}\left[\frac{x}{2}\sin x + \frac{\cos 3x}{8}\right]$$

Case 3:

Consider $f(D)y = x^m \left(m \in Z^+\right)$

$$\text{Then } y_P = \frac{x^m}{f(D)} = f(D)^{-1}x^m$$

$$\text{Then } y_P = \frac{x^m}{f(D)} = f(D)^{-1}x^m$$

By using binomial expansion and then apply on x^m. Some of the common binomial expansions are given as,

$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots\ldots\ldots\ldots$

$(1 + x)^{-1} = 1 - x + x^2 + x^3 + \ldots\ldots\ldots\ldots\ldots$

$(1 \mp x)^{-2} = 1 \pm 2x + 3x^2 \pm 4x^3 + \ldots\ldots\ldots\ldots\ldots$

$(1 \mp x)^{-3} = 1 \pm 3x + 6x^2 \pm 10x^2 + \ldots\ldots\ldots\ldots\ldots$

Solved Examples

Example: Solve the differential equation $y'' - 4y' + 4y = x^3$

Solution: The given differential equation can be represented in differential form as,

$(D^2 - 4D + 4)y = x^3$

To determine the complementary function

$(D - 2)^2 = 0$

$D = 2, 2$

Thus, the complimentary function is given by,

$y_c = (c_1 + c_2 x)e^{2x}$

To determine particular integral,

$$y_P = \frac{x^3}{(D-2)^2} = \frac{x^3}{\left(3D^2 - \frac{D^2}{2}\right)} = \frac{1}{4}\left(1 - \frac{D}{2}\right)^{-2}$$

$$y_P = \frac{1}{4}\left[1 + 2\times\frac{D}{2} + \frac{1}{4} + \frac{4D^3}{8}\right]x^3 = \frac{1}{4}\left[\begin{matrix} x^2 + 3x^2 + \frac{3}{4} \\ \times 6x + \frac{1}{2}\times 6 \end{matrix}\right]$$

$$= \frac{1}{4}\left[x^3 + 3x^2 + \frac{9}{2}x + 3\right]$$

Example: Solve the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$$

Solution: The homogenous part of differential equation is $(D^2 + D) = 0$

$D(D + 1) = 0$

$D = 0, -1$

$y_c = c_1 + c_2e^{-x}$

To determine the particular integral

$$y_P = \frac{x + 2x + 4}{D(D+1)} = \frac{1}{D}(D+1)^{-1}\left(x^2 + 2x + 4\right)$$

$$= \frac{1}{D}\left(1 - D + D^2 - D^3\right)\left(x^2 + 2x + 4\right)$$

$$y_P = \left(\frac{1}{D} - 1 + D - D^2\right)\left(x^2 + 2x + 4\right)$$

$$= \frac{x^3}{3} + \frac{2x^2}{2} + 4x - x^2 - 2x - 4 + 2x + 2 - 2$$

We are not considering higher powers as higher order derivatives will go to zero,

$$y_P = \frac{x^3}{3} + 4x - 4$$

Example: Solve the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2 + x$$

Solution: The homogenous part of differential equation is $(D^2 - 3D + 2) = 0$

$(D - 1)(D - 2) = 0$

$D = 1, 2$

$$y_c = c_1e^x + c_2^{2x} = x^2 + x$$

$$y_P = \frac{x^2}{\left(D^2 - 3D + 2\right)} = \left(1 - \frac{3}{2}D + \frac{D^2}{2}\right)$$

$$= \frac{1}{2}\left[1 - \left(\frac{3}{2} - \frac{D^2}{2}\right)\right]^{-1}\left(x^2 + x\right)$$

$$y_P = \frac{1}{2}\left[1 + \left(\frac{3}{2}D - \frac{D^2}{2} + \frac{9}{4}D^2\right)\right]\left(x^2 + x\right)$$

$$= \frac{1}{2}\left[x^2 + x + \frac{3}{2}(2x+1) - \frac{1}{2}\times 2 + \frac{9}{4}\times 2\right]$$

$$y_P = \frac{1}{2}\left[x^2 + 4x + 5\right]$$

Example: Determine the solution of

$$\frac{d_2y}{dx^2} = 3x - 2 \text{ satisfying } y(0) = 2 \text{ and } y'(1) = -3$$

Solution:

$$\frac{dy}{dx} = \frac{3x^2}{2} - 2x + c_1$$

$$y = \frac{3x^3}{2\times 3} - \frac{2x^2}{2} + c_1x + c$$

$$\text{Since, } y'(1) = -3$$

$$-3 = \frac{3}{2} - 2 + c_1$$

$$c_1 = \frac{5}{2}$$

$$\text{Since, } y(0) = 2$$

$$\text{i.e. } 2 = \frac{0}{2} + 0 + c$$

$$c = 2$$

$$\text{Thus, the solution is, } y = \frac{x^3}{2} - x^2 - \frac{5}{2}x + 2$$

Case 4:

Consider the equations $f(D)\,y = e^{ax}.V$ where V is also a function of x. It may be $\sin(bx + c)$ or $\cos(ax + b)$ or x^m etc.

$f(D)y = e^{ax}.V$

$$y^P = \frac{e^{ax}.V}{f(d)} = e^{ax}\left[\frac{V}{f(D+a)}\right]$$

$$D\left[e^{ax}.V\right] = e^{ax}.D(V) + aVe^{ax} = e^{ax}\left[D + a\right]V$$

$$D^2\left[e^{ax}.V\right] = e^{ax}\left[D^2V + aDV\right] + (D + a)Ve^{ax}$$

$$= e^{ax}\left[D^2 + 2aD + a^2\right]V = e^{ax}(D+a)^2\,V$$

Solved Examples

Example: Solve the following differential equation $(D2 - 5D + 6)y = e^{2x}.x^3$

Solution: The homogenous part of the differential equation is $(D - 2)(D - 3) = 0$

$D = 2, 3$

Thus, the complementary function is

$y_c = c_1e^{2x} + c_2\,e^{3x}$

To determine the particular integral,

$$y_P = \left(D - e^2\right)\left(xD_3 - 3\right) = e^{2x}\left|\begin{matrix}\left(-D+2-2\right)x^3 \\ \left(D+2-3\right)\end{matrix}\right|$$

$$-e^{2x}\left[\frac{D^1\left(1-D\right)}{x^3}\right]$$

$$y_P = -e^{2x}\left[\frac{1}{D}\left(1-D\right)^{-1}x^3\right] = -e^{2x}\left[\begin{matrix}\frac{1}{D}1+D+D^2 \\ +D^3+D^4\end{matrix}\right]x^3$$

$$y_P = e^{-2x}\left[\left(\frac{1}{2}+1+D+D^2+D^3\right)x^3\right]$$

$$= e^{-2x}\left[\frac{x^4}{4}+x^3+3x^2+6x+6\right]$$

Example: Solve the differential equation y" + 4y = $2e^x \sin^2 x$

Solution: The differential equation can be expressed in differential form as, $(D^2 + 4)y = e^x (1 - \cos 2x) = ex - ex \cos 2x$

The homogenous part of differential equation is,

$(D^2 + 4)y = 0$

$D = \pm 2i$

Thus, complimentary function is $y_c = c_1 \cos 2x + c^2 \sin 2x$

Particular Integral can be calculated as,

$$y_{P1} = \frac{e^x}{D^2+4} = \frac{e^x}{5}$$

$$y_{P2} = \frac{e^x \cos 2x}{D^2+4} = e^x \times \frac{\cos 2x}{\left(D+1\right)^2+4}$$

$$= e^x \frac{\cos 2x}{D^2+2D+5} = e^x \frac{\cos 2x}{-4+2D+5}$$

Here, we have replaced D^2 by $-a^2$ as per **Case 5:**

$$y_{P2} = e^x \frac{\left(2D-1\right)\cos 2x}{4D^2-1} = e^x \frac{\left(2D-1\right)\cos 2x}{-17}$$

$$= \frac{-e^x}{17}\left[-4\sin 2x - \cos 2x\right]$$

$$y_{P2} = \frac{e^x}{17}\left[\cos 2x + 4\sin 2x\right]$$

$$y_P = y_{P1} - y_{P2}\left(X = e^x - e^x \cos 2x\right)$$

$$\text{Thus, solution is } y = y_c + y_P = c_1 \cos 2x$$

$$+c_2 \sin 2x + \frac{e^x}{5} - \frac{e^x}{17}\left[\cos 2x + 4\sin 2x\right]$$

Example: Determine the solution of (D2 – 2D + 4) y = ex cosx

Solution: The homogenous part of the differential equation is, $(D^2 - 2D + 4)\, y = 0$

$$D = \frac{2+\sqrt{4-4\times 4}}{2} = 1 \pm i\sqrt{3}$$

Thus, complementary function is,

$$y_c = e^x\left|c_1 \cos\sqrt{3}x + c_2 \cos\sqrt{3}x\right|$$

The Particular Integral can be computed as,

$$y_P = \frac{e^x \cos x}{D^2-2D+4} = e^x \frac{\cos x}{\left(D+1\right)^2-2D+4-2}$$

$$= e^x \frac{\cos x}{D^2+2D+2-2D+4-2}$$

$$\left[y_P = \frac{e^{ax}\cos bx}{f\left(D\right)} = e^{ax}\frac{\cos bx}{f\left(D+a\right)}\right]$$

$$y_P = e^x \frac{\cos x}{D^2+4} = e^x \frac{\cos x}{-1+4} = \frac{e^x \cos x}{3}$$

Thus, the complete solution is,

$$y = e^x\left[c_1 \cos\sqrt{3}x + c_2 \cos\sqrt{3}x\right] + e^x \frac{\cos x}{3}$$

Example: Determine the solution of y" – 7y' + 6y = e^{2x} (1 + x)

Solution: The homogenous part of differential equation is $(D^2 - 7D + 6)y = 0$

$(D - 6)(D - 1) = 0$

$D = 6, 1$

Thus, complementary function is $y_c = c_1 e_{6x} + c_2 e^x$

The particular integral is,

$$y_P = \frac{e^{2x}\left(1+x\right)}{D^2-7D+6} = \frac{e^{2x}\left(1+x\right)}{\left(D-6\right)\left(D-1\right)}$$

$$= \frac{e^{2x}1-x}{\left(D-4\right)\left(D+1\right)}\left[y_P = \frac{e^{2x}V\left(x\right)}{\left(D\right)} = e^{ax}\frac{V\left(x\right)}{f\left(D+a\right)}\right]$$

$$y_P = e^{2x}\left[\frac{\left(1+x\right)}{D^2-3D-4}\right] = 4e^{2x}\left[\left(D+1\right)\times\left(1-\frac{D}{4}\right)^{-1}\right]$$

$$\left(1+x\right)$$

$$y_P = -4e^{2x}\left[\left(1-D\right)\left(1+\frac{D}{4}\right)\right]\left(1+x\right)$$

$$y_P = -4e^{2x}\left[1-D+\frac{D}{4}-\frac{D^2}{4}\right]\left(1+x\right)$$

$$y_P = -4e^{2x}\left[1+x-\frac{3}{4}\right] = 4e^{2x}\left[x+\frac{1}{4}\right]$$

Method of variation of parameters

Consider the equation $\frac{d^2y}{dx^2}+k_1\frac{dy}{dx}+k_2y=x$

Where K_1, k_2 are constants and x is a function of x. Now we can have the complimentary function $y_c = c_1y_1 + c_2y_2$

Where c_1 and c_2 are arbitrary constants or parameters and y_1, y_2 are functions of x.

By the method of variation of parameters it is possible to write the Particular Integral as in the form of complimentary function given by $y_P = Ay_1 + By_2$ where

$$A=\int\frac{xy_2}{W}dx$$

$$B=\int\frac{xy_1}{W}dx$$

$$W=\begin{vmatrix} y_1, & y_2 \\ y_1' & Y_2' \end{vmatrix}[\text{called Wronskian of } y_1, y_2]$$

Therefore the complete solution of differential equation is given by

$y = y_c + y_P = (c_1 + A)\, y_1 + (c_2 + B)y_2$

Solved Examples

Example: Determine the solution of $(D^2 + a^2)\, y = \tan ax$

Solution: The homogenous part of differential equation is, $(D^2 + a^2)\, y = 0$

$D = \pm ai$

Thus, complimentary function is,

$$y_c = c_1\underbrace{\cos ax}_{y_1} + c_2\underbrace{\sin ax}_{y_2}$$

$$W=\begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}=a\times 1=a$$

$$A=-\int\frac{Xy_2}{W}dx=-\int\frac{\tan ax\times\sin ax}{a}dx$$

$$=\frac{-1}{a}\int\frac{\sin^2 ax}{\cos ax}=\frac{-1}{a}\int\frac{1-\cos^2 ax}{\cos ax}dx$$

$$A=\frac{-1}{a}\left[\left[\frac{\log(\sec ax+\tan ax)}{a}\right]-\frac{\sin ax}{a}\right]$$

$$=\frac{-1}{a^2}\left[\log(\sec ax+\tan ax)-\sin ax\right]$$

$$B=\frac{1}{a}\int\tan ax\times\cos ax\,dx=\frac{1}{a}\int\sin ax\,dx$$

$$=\frac{1}{a}\times\frac{-\cos ax}{a}=\frac{-1}{a^2}\cos ax$$

$$y_P = Ay_1+By_2=\frac{-1}{a^2}\left[\log(\sec ax+\tan ax)-\sin ax\right]$$

$$\cos ax-\frac{1}{a^2}\cos ax\sin ax$$

Example: Solve the differential equation

$$\left(D^2-6D+9\right)y=\frac{e^{3x}}{x^2}$$

Solution: The differential equation is represented as

$$(D-3)^2\,y=\frac{e^{3x}}{x^2}$$

$D = 3, 3$

Complimentary function is $y_c=(C_1+c_2x)e^{3x}$

$$=c_1\underbrace{e^{3x}}_{y_1}+c_2\underbrace{xe^{3x}}_{y_2}$$

Wronskian is given by,

$$W=\begin{vmatrix} e^{3x} & e^{3x} \\ 3e^{3x} & 3e^{3x}+e^{3x} \end{vmatrix}=e^{6x}$$

$$A=-\int\frac{e^{3x}}{x^2}\times\frac{xe^{3x}}{e^{6x}}dx=-\log x$$

$$B=\int\frac{e^{3x}}{x^2}\times\frac{xe^{3x}}{e^{6x}}dx=-\frac{1}{x}$$

$$y_P=Ay_1+By_2=-e^{3x}\log x-\frac{e^{3x}}{x}$$

Example: Determine the solution of $(D^2 - 1)\, y = e^{2x}\sin(e^{-x})$

Solution: The homogenous part of differential equation is $D^2 - 1 = 0$

$D = \pm 1$

Complimentary Function is given by,

$$y_c=c_1\underbrace{e^x}_{y^1}+c_2\underbrace{e^{-x}}_{y^2}$$

$$W=\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}=-1-1=-2$$

$$A=\int\frac{e^{-2x}\sin(e^{-x})e^{-x}}{2}dx$$

$e^{-x}=z$

$dz=-e^{-x}dx$

$$A=-\frac{1}{2}\int z^2\sin z\,dz=\frac{-1}{2}\left[-z^2\cos z+\int 2z\cos z\,dx\right]$$

$$A=-\frac{1}{2}\left[-z^2\cos z+2\times(z\sin z+\cos z)\right]$$

$$=\frac{1}{2}\left[e^{-2x}\cos e^{-x}-2e^{-x}\sin e^{-x}+2\cos e^{-x}\right]$$

$$B = \frac{-1}{2}\int e^{-2x}\sin e^{-x} \times e^{x}dx$$

$e^{-x} = z;\quad dz = -e^{-x}dx$

$$B = \frac{1}{2}\int \sin z dz = \frac{-1}{2}\cos\left(e^{-x}\right)$$

Particular integral is, $y_P = Ay_1 + By_2$

Euler-Cauchy Equation

The differential equation of the form,

$$x^m \frac{d^m y}{dx^m} + k_1 x^{m-1}\frac{d^{m-1}y}{dx^{m-1}} + \ldots\ldots\ldots + k_{m-1}x\frac{dy}{dx}$$

$+k_m y = x$ is called Euler-Cauchy equation.

Substitute $x = e^z \Rightarrow z = \log x D = \frac{d}{dz}$

Then, $x\frac{dy}{dx} = x\frac{dy}{dz}\frac{dz}{dx} = x \times \frac{dy}{dz} \times \frac{1}{x} = Dy$

Similarly, $x^2\frac{d^2y}{dx^2} = D\left(D-1\right)y$

Also, $X^3\frac{d^3y}{dx^3} = D\left(D-1\right)\left(D-2\right)y$

Then, the equation becomes,

$D\left(D-1\right)\ldots.\left(D-m+1\right)y + k_1 D\left(D-1\right)\ldots.\left(D-m+2\right)$

$y + \ldots\ldots\ldots + k_m y = X$

This equation can be solved in terms of 'z' by complimentary function and particular integral.

Solved Examples

Example: Solve the equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0$

Solution: If we substitute $x = e^z$, the differential equation becomes,

D (D – 1) y + Dy – 4y = 0

$(D^2 - D + D - 4)\, y = 0$

(D + 2) (D – 2) y = 0

$D = \pm 2$

$$y = c_1e^{-2z} + c_2e^{2z} = \frac{c_1}{x^2} + c_2 \times x^2$$

Applying initial conditions, $y\left(0\right) = 0$

$$0 = \frac{c_1}{x^2} + c_2 \times 0 \Rightarrow c_1 = 0$$

Since, $y\left(1\right) = 1$

$1 = c_2 \times 1$

$\therefore c_2 = 1$

$\therefore y = x^2$

Example: Solve the differential equation $x^2\frac{d^2y}{dx^2} - 2y = x^2 + \frac{1}{x}$

Solution: If we substitute $x = e^z$, the differential equation becomes,

$D\left(D-1\right)y - 2y = e^{2z} + e^{-z}$

$\left(D^2 - D - 2\right)y = e^{2z} + e^{-z}$

The homogenous part of differential equation is (D – 2)(D + 1) y = 0

D = 2, –1

Thus, complimentary function is $y_c = c_1e^{2z} + c_2e^{-z}$

The particular integral is $y_P = \frac{e^{2z} + e^{-z}}{\left(D-2\right)\left(D+1\right)}$

$$= \frac{z}{3}e^{2z} - \frac{ze^{-z}}{3}$$

Complete solution is given by, $y = y_c + y_P$

$$\left(c_1x^2 + \frac{c_2}{x}\right) + \frac{\log x}{3}\left(x^2 - \frac{1}{x}\right)$$

4 Complex Variables

Objective

Upon completion of this chapter you will be able to:

- Analyze the function of complex variable.
- Understand the basics of analytic functions
- Determine the line integrals of complex functions
- Determine the line integrals by Residue Method
- Determine the singularities of complex functions

Introduction

Complex analysis, traditionally known as the theory of complex variables, is the branch of mathematical analysis that investigates the functions of complex numbers. Complex numbers are ordered pairs of real numbers (x, y). Two complex numbers are said to be equal if they are exactly same i.e. (x, y) = (u, v) which implies x = u and y = v. A complex function is one in which the dependent as well as independent variables are complex numbers or we can say that the domain and range of complex functions is the subset of the complex plane.

Complex Number

If x, y are two real numbers and 'i' is an imaginary unit such that $i^2 = -1$ or $i = \sqrt{-1}$ then the number of the form z = x + iy is called complex number.

Therefore, $z = x + iy$ where $x = Re(z)$ & $y = Im(z)$
If $z = x + iy$ then $z = x - iy$.

If $z = x + iy$ is a complex number then

$|z| = |x + iy| = \sqrt{x^2 + y^2}.\sqrt{\cdot}$ This is called as the magnitude of a complex number.

Complex exponential can be represented as, $e^{i\theta} = \cos\theta + i\sin\theta$. The magnitude of this exponential is always 1. To represent any general complex number, amplitude and phase terms can be combined together as,

$z = x + iy = re^{i\theta} = r\{\cos\theta\}$, where $r|z$

$= \sqrt{x^2 + y^2}, \theta = \tan^{-1}\left(\frac{y}{x}\right)$

θ = argument of $z = x + iy$.

If $z = x + iy$ and $z_0 = x_0 + iy_0$ are two complex numbers then the distance between z and z_0 is given by $|z - z_0|$ or $|z_0 - z|$

$$\therefore |z - z_0| = |x + iy - (x_0 + iy_0)| = |$$

$$(x - x_0) + i(y - y_0)| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

The equation of a circle in rectangular co-ordinates is,

$x^2 + y^2 = r^2$ or $r = \sqrt{x^2 + y^2}$

Here, 'r' represents distance of (x + iy) from origin.

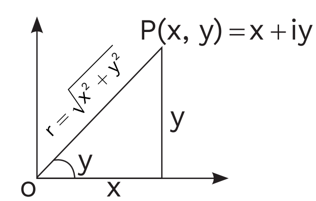
This equation in complex form can be expressed as, |z| = r.

This equation represents locus of all the points which are at a constant distance from the origin. Thus it is an equation of circle with center at origin and radius r.

The equation of a circle with center at (x0 , y0) and radius 'r' is,

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

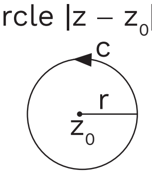
$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = r$$



$|z - z_0| = r$ is the equation of a circle with center at z_0 and radius r.

$|z - z_0| < r$ represents a set of all points lying within the circle $|z - z_0| = r$.

$|z - z_0| > r$ represents a set of all points lying outside the circle $|z - z_0| = r$.



Complex Function

If A and B are two sets of complex numbers and every element of the form z=x+iy in a set A is associated with the unique element of the form w=u+iv in a set B, then w=u+iv is called complex function of a complex variable z=x+iy and it is denoted by w=f(z) where, z=x+iy and w=u+iv.

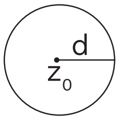
Therefore $w = f(z) = f(x + iy) = u(x, y) + iv(x, y)$

$w = f(z) = f(rei) = u(r,\theta) + iv(r,\theta)$

Neighborhood of a z_0

The set of all points within the circle having a center at z_0 but not on the circle is called neighborhood of a point z_0 and it is also called open circular disc (region).

Therefore $N_d(Z_0) = N(d, Z_0) = \{Z : |z - z|_0 < d\}$



Analytic Function

If a complex function f(z) is differentiable at a point z_0 and also differentiable at every point in some neighborhood of a point z_0 then the function f(z) is called Analytic function at a point z_0 and the point z_0 is called Analytic point of f(z).

Singular Point

If a function f(z) is not defined or not differentiable or not analytic, at a point z_0 then z_0 is called singular point of f(z).

Suppose a complex function is given as,

$$f(z) = \frac{z+4}{z-2}$$

$\therefore z = 2$ is a singular point of f(z).

Let us take another function, $f(z) = \sqrt{z-4}$

This function is defined for all values of 'z'. Consider derivative of the function

$$f'(z) = \frac{1}{2}\sqrt{z-4}a_n = (3+4i)^n$$

$$r = \frac{1}{\underset{n\to\infty}{\mathrm{lt}} |a_n|^{1/n}} = \underset{n\to\infty}{\mathrm{Lt}} \frac{1}{|(3+4i)|^{1/n}}$$

1 1

z = 4 is a singular point of f(z) as f(z) is not analytic at z = 4.

Entire Function

If a complex function f(z) is differentiable or analytic at every point throughout a complex plane, then the function f(z) is called an entire function and it is also called integrable function.

Euler's Theorem

The trigonometric functions of real variables can be defined as,

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}; \cos y = \frac{e^{iy} + e^{-iy}}{2}$$

Similarly, trigonometric functions of complex variables can be defined,

$$\sinh z = \frac{e^{z} - e^{-z}}{2}; \cosh y = \frac{e^{z} + e^{-z}}{2}$$

Euler's Theorem states that,

$\cos z + i\sin z = e^{iz}$

Exponential Function

Exponential Function of real variable is very well defined and similarly we can define the exponential function of a complex variable.

$e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y)$

De Moivre's Theorem

De Moivre's Theorem states that,

$(\cos z + i\sin z)^n = (e^{iz})^n = e^{inz} = \cos nz + i\sin nz$

Hyperbolic Functions

Different hyperbolic functions are defined as,

$$\sinh z = \frac{e^{z} - e^{-z}}{2}; \cosh y = \frac{e^{z} + e^{-z}}{2}$$

We can also define,

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}$$

$$\coth z = \frac{\cosh z}{\sinh z} = \frac{e^{z} + e^{-z}}{e^{z} - e^{-z}}$$

$$\mathrm{sech} z = \frac{1}{\cosh z} = \frac{2}{e^{z} + e^{-z}}$$

$$\mathrm{cosech} z = \frac{1}{\sinh z} = \frac{2}{e^{z} - e^{-z}}$$

Logarithmic Function of a Complex Variable

If z= (x + iy) and w=(u + iv) be so related that $e^w = z$, then w is said to be a logarithm of z to the base e and is written as $w = \log_e z$.

Also, $e^{w+2in\pi} = e^w e^{2in\pi} = z$

Therefore, log z = w + 2inπ

Thus, the logarithm of a complex number has an infinite number of values and is therefore a multivalued function.

The general value of logarithm of z is represented as log z to distinguish it from principal value which is written as log z.

In Cartesian coordinates log (x + iy) = 2inπ + log (x + iy) = 2inπ + $\log\left[\left[r(\cos\theta + i\sin\theta)\right]\right]$

$$\text{Log}(x+iy) = 2in\pi + \log\left[\left[re^{i\theta}\right]\right]$$

$$= 2in\pi + \log r + i\theta$$

$$= \log\sqrt{x^2+y^2} + i\left[\left[2n\pi + \tan^{-1}(y/x)\right]\right]$$

The logarithm of a negative quantity is complex and can be evaluated as,

$$\text{Log}_e(-x) = \log_e x + \log_e(-1) + \log x + i\pi$$

Analyticity of a Complex Function

If f (z) = u(x, y) = iv(x, y) is analytic function at a point z_0, then u_x ,u_y , v_x , v_y exists and satisfies the Cauchy Riemann equations.

$u_x = v_y$ & $v_x = -u_y$

At every point in some neighborhood of z_0.

$$u_x = \frac{\partial u}{\partial x}; u_y = \frac{\partial u}{\partial y}; v_x =; v_y = \frac{\partial v}{\partial y}$$

Sufficient condition for a function f(z) to be analytic. If

1. f (z) = u(x, y) + iv (x, y) is defined at every point in some neighborhood of z_0.
2. u and v satisfy the C-R equations at every point in some neighborhood z_0.
3. u, v, u_x, u_y, v_x, v_y are continuous at every point in some neighborhood z_0.

Then the function f(z) = u +iv is analytic at z_0 and $f'(z) = u_x + iv_x$

Note: ex, sinx, cosx, sinhx, coshx and every polynomial of the form $a_0 + a_1 x + a_2 x^2 + \ldots.. a_n x^n$ ($a \neq 0$ & $n \in N$) are every where defined, continuous, differentiable and also integrable.

If f, g are two continuous functions, then

a) $f \pm g$ is also continuous

b) f.g is also continuous

c) f/g (g≠0) is also continuous

Solved Examples

Example: Test the analyticity of the function: f (z) = x + e^x cosy + iy + ie^x siny

Solution: u + iv = f (z) = (x + e^x cosy) + i(y + e^x siny)

u = x + e^x cosy

v = y + e^x siny

$$u_x = \frac{\partial u}{dx} = 1 + e^x \cos y; \quad v_x = \frac{\partial v}{\partial y} = e^x \sin y$$

$$u_y = \frac{du}{dy} = -\left(e^x \sin y\right); \quad v_y = \frac{\partial v}{\partial y} = 1 + e^x \cos y$$

Here $u_x = v_y$ and $v_x = -u_y$ at entry point and u, v, u_x, u_y, v_x, v_y are continuous at every point.

∴ f(z) is not an analytic function.

Example: Determine whether the following function is analytic: f (z) = z^-

Solution: u + iv = f (z) = x – iy

u = x, v = –y

$u_x = 1, u_y = 0$ & $v_x = 0$, $v_y = -1$

$u_x \neq v_y$ and $v_x = -u_y$

∴ f(z) is not analytic function.

Note:

1. f(z) = $|z|^2$ is differentiable only at the origin but not analytic at any point.
2. f(x) = z^- is not differentiable and not analytic at any point.

Example: If $x = \sqrt{-1}$ then $x^x = ?$

Solution: $x = \sqrt{-1} = i$

$$x = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{\frac{-p}{2}}$$

$$x^x = \left(e^{\frac{-p}{2}}\right) = e^{\frac{-p}{2}}$$

Note: C-R equations in polar form are given by $u_r = \frac{1}{r}v_\theta$ and $v_r = \frac{-1}{r}u_\theta$

The derivative formula in polar form is given by $f'(z) = (u_r + |iv_r|)e^{-i\theta}$

Determining an analytic Function

1. Given v (or u). Find v_x, v_y (or u_x, u_y)
2. Consider $f'(z) = u_x + iv_x = u_y + iv_y$
3. Replace x by z and y by '0' i.e. $f'(z) = g(z)$
4. $f(x) = \int g(x)dx + c$ where $c = c_1 + ic_2$

Solved Examples

Example: If $v(r,\theta) = 3r^2 \sin|2\theta| + 2r\sin\theta + 7$ then find analytic function f (z) = u + iv where v is an imaginary part of analytic function f(z).

Solution: $v_r = 6r\sin 2\theta + 2\sin\theta$ & $v_\theta = 6r^2\cos 2\theta + 2r\cos\theta$

Consider $f'(z) = (u_r + iv_r)e^{i\theta} = \left(\frac{1}{r}v_\theta + iv\right)e^{-i\theta}$

$$f'(z) = \begin{Bmatrix} (6r\cos 2\theta + 2\cos\theta) \\ +i[6r\sin 2\theta + 2\sin\theta] \end{Bmatrix} e^{-i\theta}$$

Replace 'r' by 'z' and 'θ' by '0'.

$f'(z) = 6z + 2$

$\therefore f(z) = 3z^2 + 2z + c$ where $c = c_1 + ic_2$

$f(z) = 3z^2 + 2z + c_1 + ic_2 = 3(re^{i\theta})^2 + 2(re^{i\theta}) + (c_1 + ic_2)$

$$f(z) = 3r^2[\cos 2\theta + \sin 2\theta] + 2r[\cos\theta + \sin\theta] + c_1 + ic_2$$

$$f(z) = (3r^2\cos 2\theta + 2r\cos\theta + c_1) + i(3r^2\sin 2\theta + 2r\sin\theta + c_2)$$

Harmonic Conjugate Function

If u_x, u_y, u_{xx} & u_{yy} are continuous functions and $u_{xx} + u = 0$ or $\nabla^2 u = 0$ then u(x, y) is called harmonic function.

$u_{xx} + u_{yy} = 0$ or $\nabla^2 u = 0$ is called Laplace equation

Note:

1. If f(x) = u + iv is analytic function, then u and v satisfy Laplace equations.
2. If u and v are harmonic functions then u+iv may or may not be an analytic function.

If u and v are harmonic function as u+iv is also analytic function, then v is called harmonic conjugate function of 'u'. Similarly, '-u' is the harmonic conjugate function of 'v'.

Method

Step 1: If v(x, y) is given to find u(x, y) the consider

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Step 2: $du = u_x dx + u_y dy = (v_y)dx + (-v_x)\,dy$

$(\because u_x = v_y \ \&\ v_x = -uy)$

Step 3:

$$u = \int (v_y)dx + \int [\text{terms not containing x in } (-v_x)dy] + k$$

Treating y as constant, $K \to$ real integral constant.

Solved Examples

Example: If $v(r,\theta) = 3r^4\sin(4\theta) + 4$, then find its harmonic conjugate function?

Solution: Consider

$$du = (u_r)dr + (u\theta)d_\theta = \left(\frac{1}{r}v_\theta\right)dr + (-rv_r)d\theta$$

$$\left(\because u_r = \frac{1}{r}v_\theta \ \&\ v_r = \frac{-1}{r}u_e\right)$$

$$du = [12r^3\cos(4\theta)]dr + [-12_r^4\sin(4\theta)]d\theta$$

$$u = \int [12r^3\cos(4\theta)]dr + \int [-12_r^4\sin(4\theta)]d\theta + k$$

$$\therefore u_{(r,\theta)} = 3r^4\cos(4\theta) + k$$

Note:

f (z) = u(x,y) + iv (x,y) is analytic function.

$\downarrow$ complex potential function $\downarrow$ velocity potential function $\downarrow$ stream function

$\uparrow$ f(z) = $\uparrow$ $\Phi(x, y)$ + $\uparrow$ $i\psi(x, y)$

Example:

If $f(z) = x^3 - 3xy^2 + i\varphi(x,y)$ where $i = \sqrt{-1}$ and $f(x+iy)$ is analytic function, then find the stream function ψ.

Solution: Given $u = x^3 = 3xy^2$, $v = \varphi$

$u_x = 3x^2 - 3y^2$, $u_y = 6xy$

Consider $dv = v_x\, dx + v_y\, dy$

$dv = -u_y\, dx + u_x\, dy$

$dv = 6xydx + (3x^2 - 3y^2)\, dy$

$v = \int(v_x)\, dx + \int$[terms not containing x in (v_y) dy] + K

$v = \int(6y\, xdx + \int(-3y^2)\, dy + K$

$v = 3x^2 y - y^3 + K$

Complex Integration

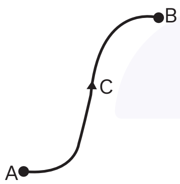
If a complex function f(z) is defined at every point on the curve C from a point A to B, then the evaluation of integral of complex function f(z), is called line integral of a complex function f(z) and is denoted by $\int_c f(z)\, dz$ where C is called path of integratio.

The relation between real line integral and complex line integral.

If f(z) is given by $f(z) = u + iv$ and $dz = dx + idy$ where $z = x + iy$ then

$$\int_C f(z)\, dz = \int_C (u+iv)(dx+idy)$$

$$= \int_C (udx - vdy) + i\int_C (vdx + udy)$$



Parameterization of the Curve

The curve for line integral can be represented by a parametric representation as $z(t) = x(t) + iy(t)$.

The sense of increasing 't' is called as positive sense on C, and we assume that C is a smooth curve i.e. C has acontinuous and non-zero derivative $\dot{z} = \frac{dz(t)}{dt}$ at each point.

Solved Examples

Example: Evaluate $\int_0^{1+i} zdz$ along a curve C where C is the curve y = x

Solution: Assume x =t and y = t

$$\int_0^{1+i} zdz = \int_{(x,y)=(0,0)}^{(1,1)} (x+iy)(dx+idy)$$

$$= \int_0^1 (t+it)(dt+idt)$$

Example: Evaluate $\int_0^{1+i} zdz\,(x^2 - iy)dz$ along the curve C where C is (i) y = x (ii) y = x²?

Solution:

1. Assume x = t and y= t

$$\int_{0,0}^{(1,1)} (x^2 - iy)(dx+idy) = \int_0^1 (t^2 - it)(dt+idt)$$

$$\int_0^{1,i} (x^2 - iy)dz = \int_0^1 t^2dt + it^2dt - itdt + tdt$$

$$= \left[\frac{t^2}{3} + i\frac{t^3}{3} - i\frac{t^2}{2} + \frac{t^2}{2}\right]_0^1 = \frac{1}{3} + \frac{i}{3} - \frac{i}{2} + \frac{1}{2} = \frac{5-i}{6}$$

2. $$\int_{(0,0)}^{(1,1)} (x^2 - iy)(dx+idy) = \int_0^1 (t^2 - it^2)(dt + i2tdt)$$

$x = t;\ y = t^2$

$dx = dt$

$dy = 2tdt$

$$\int_{(0,0)}^{(1,1)} (x^2 - iy)(dx+idy) = \int_0^1 (1-i)t^2 \times dt(1 + i \times 2t)$$

$$\int_{(0,0)}^{(1,1)} (x^2 - iy)(dx+idy) = (1-i)\int_0^1 t^2dt + i2t^3dt$$

$$= (1-i)\left[\frac{t^3}{3} + i\frac{2t^4}{4}\right]_0^1$$

$$\int_{(0,0)}^{(1,1)} (x^2 - iy)(dx+idy) = (1-i)\left[\frac{1}{3} + \frac{i}{2}\right] = \frac{5+i}{6}$$

Note: If the integrant function is analytic, then the value of the integral depends only on the end points of the paths not on the path.

i.e. $\int_C f(z)\, dz = \int_A^B f(z)\, dz$

Example: $I = \int_{z=0}^{1+i} zdz = \left.\frac{z^2}{2}\right]_0^{1+i} = \frac{(1+i)^2}{2} = \frac{2i}{2} = i$

Parameterisation of Circle

$$x^2 + y^2 = r^2 \Rightarrow |z| = r\sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$$x = r\cos\theta \quad x = x_0 + r\cos\theta$$

$$y = r\sin \quad y = y_0 + r\cos\theta$$

$$r = |z - z_0| \quad z = z_0 + re^{i\theta}$$

Note: The parameter equation of a circle $|z - z_0| = r$ is $z = z_0 + re^{i\theta}$ where, $\to$ 0 to 2π for total circular path.

Solved Examples

Example: Evaluate $\int_{?} \frac{2z+3}{z} dz$ along a curve C, where C is $|z| = 3$

Solution: $C = |z| = 3$

$z = 3e^{i\theta}$

$dz = 3e^{i\theta}d\theta$

Here $\theta = 0$ to 2π

$$I = \int_C \frac{2z+3}{z} dz = \int_{\theta=0}^{2\pi} \frac{2 \times 3e^{i\theta} + 3}{3e^{i\theta}} \times i3e^{i\theta} d\theta$$

$$I = \int_0^{2\pi} \left(6ie^{i\theta} + 3i\right) d\theta = \left[\frac{6ie^{i\theta}}{i} + 3i\theta\right]_0^{2\pi}$$

$$I = \left(6e^{i2\pi} + 6i\pi\right) - \left(6e^0 + 0\right) = 6i\pi$$

Simple Connected Domain

Let f(z) be analytic in a simple connected domain D. A domain D is called simple connected if every closed curve without self-intersections encloses points only in D. Then there exists an indefinite integral of f(z) in the domain D, that is, an analytic function F(z) such that F'(z) = f(z) in D and for all the paths in D joining two points z_0 and z_1 in D we have,

$$\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0)$$

Example: $\int_{z_0}^{1+i} z^2 dz = \frac{1}{3}z^3\Big|_0^{1+i} = \frac{1}{3}(1+i)^3 = -\frac{2}{3} + \frac{2}{3}i$

Cauchy's Integral Theorem

If a function f(z) is analytic at every point within and on a simple closed curve C, then integral over C.

$$\oint_C f(z)dz = 0$$

Where $\oint_C f(z)dz$ represents integral of f(z) over a closed curve C.

This theorem is actually an extension of the fact that integral of an analytic function depends only on the endpoints. In case of closed curve the initial and end points are same so integral is zero.

If a function f(z) is analytic everywhere within and on a triply connected region R bounded by 3 simply closed curve C_1, C_2, C_3 but not analytic within C_1, C_2 and analytic only in C_3, then

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz$$

Since, f(z) is analytic in C_3,

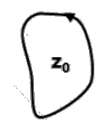
$$\oint_{C_3} f(z)dz = 0$$

Thus, $\oint_C (z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz$

Cauchy's Integral Formula

If f(z) is analytic at every point within and on a simple closed curve C and z_0 is any point within C then

1. $\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$
2. $\oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$



Thus, if a function is analytic on the simple closed curve C then the values of function and all its derivatives can be found at any point of C.

Method to use Cauchy's integral formula,

- Let $\oint f(z)dz = I$, first find the singular points of f(z) i.e. points where f(z) goes to infinity.
- Check which of these points lie inside the closed curve given.
- Apply CIF only at those points and if none of the points lie inside the given curve, $\int f(z)dz = 0$

Solved Examples

Example: Evaluate

$$\int_c \frac{2z+\sin z+e^x}{(z-4)^{10}(z-6)^{100}}dz \text{ where C is } |z| = \frac{3}{2}?$$

Solution: Let $f(z) = \dfrac{2z+\sin z+e^z}{(z-4)^{10}(z-6)^{100}}$

Singular points: z = 4, 6

The given curve is, $|z| = \dfrac{3}{2}$

Since the curve is a circle centered at origin and radius 3/2. Both the points lie outside the circle.

$$\therefore \int_c f(z)dz = 0$$

Example: Evaluate $\oint \dfrac{2z+3}{z}dz$ along C where C : |z| = 3 ?

Solution: $g(z) = \dfrac{2z+3}{2}$

Singular Point of this function is, z = 0

The given curve is |z| = 3 . Since z = 0 lies inside the given curve.

$$\therefore \text{By CIF } \oint \frac{2z+3}{z-0}dz = \oint \frac{f(z)}{z}dz = \frac{2\pi_i}{1}\times f(0)$$

$$= 2\pi i\times(2\times 0+3) = 6\pi i$$

Example: Evaluate $\oint_c \dfrac{z}{(z-1)(z-2)}dz$ along curve C where $C:|Z-2| = \dfrac{1}{2}$?

Solution: $g(z) = \dfrac{z}{(z-1)(z-2)^2}$

Singular point: z= 1, 2

$$C:|Z-2| = \frac{1}{2}?$$

If z = 1

$|-1| = 1 > \dfrac{1}{2}$, so it lies outside the given circle.

z = 2

$$|0| = 1 < \frac{1}{2},$$

$$\frac{\frac{z}{(z-1)}}{(z-2)^2} = \frac{f(z)}{(z-z_0)^{n+1}} \text{ where } f(z) = \frac{z}{z-1}$$

Example: Evaluate $\int_c \dfrac{e^z+\cos z}{(z-3)(z-2)}dz$, $C:|z| = 5$

Solution: $f(z) = \dfrac{e^z+\cos z}{(z-3)(z-2)}$

$$\frac{1}{(z-a)(z-v)} = \frac{1}{(a-b)(z-a)} - \frac{1}{(a-b)(z-b)}$$

Singular point: z = 2,3

$$f(z) = \frac{e^z+\cos z}{z-3} - \frac{e^z+\cos z}{z-2}$$

C : |z| < 5 so both singularities lie inside the given curve.

$$\therefore \oint_c \Phi(z)dz = \int_c \frac{e^z+\cos z}{z-3}dz - \int_c \frac{e^z+\cos z}{z-2}$$

$$= 2\pi i[f(3) - f(2)]$$

Example: Evaluate $\int_c \dfrac{\bar{z}}{z}dz$ along a unit circle?

Solution: Let $\Phi(z) = \dfrac{\bar{z}}{z}$

C : |z| = 1

The singular point of the function is z=0

$$g(z) = \frac{\bar{z}}{z-0} = \frac{f(z)}{z-z_0}$$

Since, $\bar{z}$ is not analytic anywhere so we have to calculate this integral as given below,

Method 1:

$$z.\,\bar{z} = |z|^2$$

$$\bar{z} = \frac{|z|^2}{z} \Rightarrow \frac{\bar{z}}{z} = \frac{|z|^2}{z^2} = \frac{1}{z^2} \quad \therefore |z| = 1$$

$$\bar{z} = \frac{1}{(z-0)^2} = \frac{f(z)}{(z-2_0)^{n+1}}$$

$\therefore$ By Cauchy integral formula we have

$$\int_c \frac{\bar{z}}{z}dz = \frac{2\pi i}{1!}f'(o) = 2\pi i\left[\frac{d}{dz}f(z)\right] = 0$$

Example: Evaluate $\int_c \dfrac{z}{z-2}dz$

along a circle |z| = 2

Solution: Let $\Phi(z) = \dfrac{z}{z-2} = \dfrac{f(z)}{2-z_0}$

Singular point of this function is, z = 2

The given curve is, C : |z| = 2

$z = 2e^{i\theta}$

$dz = 2e^{i\theta} d\theta$

$\theta = 0 \text{ to } 2\pi$

$$I = \int_c \frac{z}{z-2} dz = \int_0^{2\pi} \frac{2e^{i\theta}}{2e^{i\theta} - 2} 2ie^{i\theta} d\theta = 2i \int_0^{2\pi} \frac{e^{2i\theta}}{e^{2i\theta} - 1} d\theta$$

The Singular point lies on the curve C. So the function cannot be evaluated.

Complex Power Series

An infinite series of the form

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots\dots\dots$$
$$+ a_0(z - z_0)^n + \dots\dots\dots\dots$$
$$(\text{or}) f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

In powers of $(z - z_0)$ z or about a point z_0. In the above power series a_n is a real or complex constant which is called coefficient of power series, z is a complex variable and z_0 is a fixed complex constant which is called center of the power series.

$$\text{for } a_n = 1, f(z) = \sum_{n=0}^{\infty} (z - z_0)^n$$

Region of convergence (ROC)

The set of all values of z for which the power series converges is called region of convergence.

$$\text{Eg. } 1 + z + z^2 + \dots\dots\dots = (1 - z)^{-1}; |z| < 1$$

Here $|z| < 1$ is an ROC, $|z| = 1$ is a circle of convergence (COC) and radius r = 1 is a radius of convergence of the power series.

$$\text{Similarly, } 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\dots = e^z, \forall z \in C$$

Here an entire complex plane is an ROC of power series.

$$\text{If } f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{then}|$$

The radius of convergence of the above power series is given by

$$r = \frac{1}{\lim\limits_{n\to\infty} |a_n|^{1/n}} (\text{or}) r = \lim_{n\to\infty} \left| \frac{a_n}{a^{n+1}} \right| x$$

The circle of convergence of above series is given by $|z - z_0| = r$

The region of convergence ROC of above power series is given by $|z - z_0| < r$

Solved Examples

Example: Find the radius of convergence, COC and ROC of the given power series.

$$\sum_{n=0}^{\infty} n! z^n$$

Solution: Compare the given series with

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Here $a_n = n!$ & $z_0 = 0$

Radius of convergence

$$\therefore r = \underset{n\to\infty}{\text{Lt}} \left| \frac{n!}{(n+1)!} \right| = \underset{n\to\infty}{\text{Lt}} \left| \frac{1}{n+1} \right| = 0$$

$$\text{ROC} |z - 0| < 0$$

Circle of convergence $|z - 0| = 0$

Example: Find the radius of convergence, COC and ROC of the given power series.

$$\sum_{n=0}^{\infty} (3 + 4i)(z + 2i)^n$$

Solution: $\sin y = \dfrac{e^{iy} - e^{-iy}}{2i}; \cos y = \dfrac{e^{iy} + e^{-iy}}{2}$

Example: Find the radius of convergence, COC and ROC of the given power series.

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} (z - 3)^n$$

Solution: Compare given power series with

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Here $a_n = \frac{2^n}{n!}$ & $z_0 = 3$

$$r = \underset{n\to\infty}{Lt}\left|\frac{a_n}{a_{n+1}}\right| = \underset{n\to\infty}{Lt}\left|\frac{2^n}{n!}\cdot\frac{(n+1!)}{2^{n+1}}\right|$$

$$= \underset{n\to\infty}{Lt}\left|\frac{2^n}{n!}\cdot\frac{(n+1)n!}{2^n\times 2}\right| = \underset{n\to\infty}{Lt}\left|\frac{n+1}{2}\right| = \infty$$

Circle of convergence: $|z-3| = \infty$

Region of convergence: $|z-3| < \infty$

Note:

(1) If f(z) is an analytic function at z0 (not a Singular Point) $\Rightarrow$ Taylor series.

(2) If f(z) is not an analytic function at z0 (Singular Point) $\Rightarrow$ Laurent series.

Taylor's Theorem

If a function f(z) is analytic at every point within a circle C, then for every point z within the circle C, the function f(z) can be expressed as a power series in +ve powers of $(z - z_0)$ or about $z = z_0$.

$$\text{i.e. } f(z) = f(z_0) + (z-z_0)f'(z) + \frac{(z-z_0)^2}{2!}$$

$$f''(z_0) + \dots\dots\dots + \frac{(z-z_0)^n}{n!}f^n(z_0) + \dots\dots\dots$$

$$f(z) = \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{n!}f^n(z_0) = \sum_{n=0}^{\infty}a_n(z-z_0)^n$$

$$\text{Where } a_n = \frac{f^n(z_0)}{n!}$$

The RHS of above is called Taylors series about 0 $z = z_0$. The ROC of Taylor series is given by $|z-z_0| = r$. Where the radius of convergence r is a distance from a center of the power series z0 to its nearest singular point of the same function f(z).

Example: Find the Taylor series expansion of $f(z) = \frac{1}{z-2}$ about a point z=1. Hence find radius of convergence, COC and ROC

Solution: Given $f(z) = \frac{1}{z-2}, z = 1$

Here $z_0 = 1$ and singular point, $z = 2$

$r = |S.pt. - z_0| = |2-1| = 1$

COC : $|z - z_0| = r$

i.e. $|z-1| = 1$

ROC : $|z-z_0| < r \Rightarrow |z-1| < 1$

Expansion :

$f(z) = \frac{1}{z-2}, z = 1$

Let $z - 1 = t$ then $z = 1 + t$

$f(z) = \frac{1}{t-1} = -(1-t)^{-1}\ |t| < 1$

$f(z) = [1 + t + t^2 + t^3 \dots\dots\dots + t^n + \dots\dots\dots]$

$f(z) = (-1)\left[1 + (z-1) + (z-1)^2 + (z-1)^3 + \dots\dots\dots\right]$

Laurent's Theorem

If a function f(z) is analytic at every point within a ring shaped region R bounded by two concentric circles C_1, C_2 having center at z_0, with radii r_1, r_2 such that $r_2 < r_1$, then for every point z within R, the function f(z) can be represented by a power series in both +ve and –ve powered of 0 $z - z_0$ or about $z = z_0$.

$$\text{i.e. } f(z) = \sum_{n=0}^{\infty}a_n(z-z_0)^n + \sum_{n=1}^{\infty}b_n(z-z_0)^{-n}$$

Here second summation is known as principle part of Laurent's series

$$\text{Where } a_n = \frac{1}{2\pi i}\oint_{c_1}\frac{f(z)}{(z-z_0)^{n+1}dz}$$

$$\text{and } b_n = \frac{1}{2\pi i}\oint_{c_2}\frac{f(z)}{(z-z_0)^{-n+1}dz}$$

The RHS of the above is called Laurent's series about $z = z_0$ and the ROC of a Laurent's series is given by $r_2 < |z - z_0| < r_1$.

Solved Examples

Example: Expand $f(z) = \frac{e^{2z}}{(z-1)^2}$ as an infinite series about z = 1 and also find ROC.

Solution: Let $z - 1 = t$ then $z = 1 + t$

$$f(z)=\frac{e^{2(1+t)}}{t^2}=e^2\times\frac{e^{2t}}{t^2}\left[1+2t+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}+\ldots\ldots\right]$$

$$f(z)=e^2\left[\frac{1}{t^2}+\frac{2}{t}+2+\frac{4t}{3}+\ldots\ldots\right]$$

$$f(z)=e^2\left[\frac{1}{(z-1)^2}+\frac{2}{(z-1)}+\frac{2^2}{2!}+\frac{2^3}{3!}(z-1)+\ldots\ldots\right]$$

Therefore the above series is a Laurent's series about $z=1$ and entire complex plane is an ROC except $z=1$.

Example: Expand $f(z)=(z-3)\sin\left(\frac{1}{z+2}\right)$ as an infinite series about z = –2 and also find ROC.

Solution: Let z – (–2) = t

Then z = t – 2

$$f(z)=(t-5)\sin\frac{1}{t}$$

$$\sin x=x-\frac{x^3}{3!}+\frac{X^5}{5!}+\ldots\ldots$$

$$f(x)=(t-5)\left[\frac{1}{t}-\frac{1}{t^3}\times\frac{1}{3!}+\left(\frac{1}{t}\right)^5\frac{1}{5!}+\ldots\ldots\right]$$

$$=1-\frac{5}{t}-\frac{1}{3!t^2}+\frac{5}{3!t^3}+\ldots\ldots$$

$$f(x)=1-\frac{5}{z+2}-\frac{1}{3!(z+2)^2}+\frac{5}{3!(z+2)^3}+\ldots\ldots$$

Therefore the above series is Laurent's serie. Entire complex plane is an ROC except $z=-2$.

Example: Expand $f(z)=\frac{z}{(z+1)(z+2)}$ as an infinite series about z = –2

Solution: Let z – (–2) = t ⇒ z = t – 2

$$f(z)=\frac{t-2}{(t-1)t}=\frac{-1}{t-1}+\frac{2}{t}$$

$$\frac{2}{t}+\frac{1}{1-t}=\frac{2}{t}+(1-t)^{-1}=\frac{2}{t}+1+t+t^2+t^3+\ldots\ldots\left[|t|<1\right]$$

$$\text{i.e. } f(z)=\frac{2}{z+2}+\left[1+(z+2)+(z+2)^2+(z+2)^3+\ldots\ldots\right]$$

Therefore, the ROC of above power series is $0<|z+2|<1$

Zeros and Singularities of Complex Function

If f(z) is analytic at a point z_0 and f (z_0) = 0, then the point z_0 is called zero of the function f(z).

Suppose, $f(z)=(z-3)^4$. Here, the function is analytic at z = 3 and f(3) = 0, z = 3 is a zero of f(z).

If f(z) is an analytic function at z_0 and $f(z_0)=0, f'(z_0)=0, f''(z_0)=0$ and so on………… $f^{(m-1)}(z_0)=0$ but $f^m(z_0)\neq 0$ then the point z_0 is called zero of order m.

Solved Examples

Example: Find the order of zero z = 2 of the function f(z) = $(z-2)^3$

Solution:

$$f'(z)=3(z-2)^2;\ f'(2)=0$$

$$f''(z)=6(z-2);\ f''(2)=0$$

$$f'''(2)=6 \qquad f^{iii}(2)\neq 0$$

Therefore z = 2 is a zero of order 3.

Example: Determine the order z = nπ, n∈I for the function f (z) = sinz

Solution: $\sin z=\frac{e^{iz}-e^{-iz}}{2i}$

$$\cos z=\frac{e^{iz}-e^{-iz}}{2i}$$

$$f(z)=\sin z,\ z=n\pi, n\in I$$

$\therefore z=n\pi, n\in I$ are first order zeros.

Example: Determine the order of zero $z=(2n+1)\frac{\pi}{2}, n\in I$ for the function f (z) = cosz

Solution: $f'(z)=\sin z\neq 0$ at $z=(2n+1)\frac{\pi}{2}, n\in I$

$z=(2n+1)\frac{\pi}{2}, n\in I$ are simple zeros of f(z)

Similarly, f(z) = sinhz, z = nπi, n∈I (simple zeros)

$s(z)=\cosh z, z=i(2n+1)\frac{\pi}{2}, n\in I$(simple zeros)

Types of Singularities

Isolated Singular Point

If z_0 is a singular point of f(z) and f(z) is analytic at every point except z_0 or in at least 1 neighborhood of z_0, then the point z_0 is called isolated singular point of f(z).

Example : $f(z)=\frac{(z+4)^3}{z-2}$

$z=2 \rightarrow$ Isolated singular point

Similarly, $f(z)=\frac{(z-2)^3(z-4)}{(z-5)^2(z-6)^3}$

singular point is z = 5, 6

At least one region exists, so 5, 6 are isolated singular points.

$f(z)=\frac{1}{\sin z}$

$\text{Singular Point}: z=n\pi, n\in I \Rightarrow \underbrace{z=0,\pm\pi,\pm2\pi}_{\text{Isolated point}}......$

We can't find any other singular point in this region. So isolated singular point.

Removable Singular Point

If the principal part of Laurent's series expansion of f(z) about z – z0 does not exist then the singular point z0 is

called removable Singular Point of f(z).

Example : $f(z)=\frac{\sin z}{z}$

Pole (of order m)

If the principal part of Laurent's expansion of f(z) about $z-z_0$ contains finite number of –ve powers of $(z-z_0)$ then

the singular point 'z_0' is called pole of order m i.e. say m terms.

Example : $f(z)=\frac{e^z}{z}$

z = 0 is the singular point.

Now expand about this point.

$$f(z)=\frac{e^2}{z}=\frac{1}{2}\left\{1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+..................\right\}$$

$$=\frac{1}{z-0}+1+\frac{z-0}{2!}+\frac{(z-0)^2}{3!}+.............$$

$$f(z)=0\times\frac{1}{(z-0)^2}+0\times\frac{1}{(z-0)}+\frac{1}{(z-0)}..........$$

$\therefore$ z = 0 is a pole of order 1 (simple pole)

Essential Singular Points

If the principal part of Lourent's series expansion of f(z) about $z-z_0$ contains infinite number of negative powersof $(z-z_0)$ then the regular point z_0 is called essential singular point of f(z).

Example : $f(z)=(z-4)\sin\left(\frac{1}{z-4}\right)$

$$=(z-4)\left\{\frac{1}{z-4}-\frac{1}{(z-4)^3 3!}+\frac{1}{(z-4)^5 5!}+.........\right\}$$

$$f(z)=1-\frac{1}{(z-4)^2 3!}+\frac{1}{(z-4)^4 5!}+...........$$

Classification of Singular Points

1. Suppose $f(z)=\frac{N^r}{D^r}$
2. Find the singular points of f(z) [i.e. zeros of Dr function].
3. N^r (at singular points)
$$\begin{cases} \neq 0 \Rightarrow \text{Poles} \\ = 0 \Rightarrow \text{pole or removable singuar point} \end{cases}$$
4. If m > n $\Rightarrow$ pole order = m – n
   Else $m \le n \Rightarrow$ removable singular point
   where n $\rightarrow$ order of zero of Nr and m $\rightarrow$ order of zero of Dr

Solved Examples

Example: Determine the type of singular points for the function

$$f(z)=\frac{(z-4)^3(z-6)^2}{(z-5)^{10}(z-7)^5}$$

Solution: Singular points: z = 5 (order = 10) and z = 7 (order = 5)

Since, numerator is non-zero at both these singular points, these are poles.

Example: Determine the type of singular points for the function $f(z)=\frac{\sin x}{z-\frac{\pi}{2}}$

Solution: Singular point $z = \frac{\pi}{2}[\text{order } 1]$

Since, numerator is non-zero at $z = \frac{\pi}{2}$ pole of order 1 (simple pole)

Example: Determine the type of singular points for the function f(z) = tanz

Solution: $f(z) = \frac{\sin z}{\cos z}$

Singular point: $z = \frac{\pi}{2}(2n+1)\ \ n \in I$

Thus, there are infinite number of singular points.

Since, Numerator is non-zero at all these points $\therefore\ z - (2n+1)\frac{\pi}{2}$ is a pole of order 1.

Example: Determine the type of singular points for the function $f(z) = \frac{\cos z}{z - \frac{\pi}{2}}$

Solution: $f(z) = \frac{\cos z}{z - \frac{\pi}{2}} = \frac{\phi(z)}{(z - z_0)^m}\ m = 1$

Singular Point = $\frac{\pi}{2}$

Since, numerator is zero at $z = \frac{\pi}{2}$ this is a removable singularity or a pole. To determine the order of zero of N^r

f(z) cosz 0 at $z = \frac{\pi}{2}$

f'(z) cosz 0 at $z = \frac{\pi}{2}$

Therefore $z = \frac{\pi}{2}$ is a pole of order 1.

$\therefore$ n = 1

$\therefore\ m = n = 1 \Rightarrow z\ z = \frac{\pi}{2}$ is a removable Singular Point

Example: Determine the type of singular points for the function $f(z) = \frac{1 - \cos z}{z}$

Solution: singular point: z = 0, Since Nr = 0 at z = 0

$$f(z) = \frac{1 - \cos z}{z} = \frac{g(z)}{(z - z_0)^m}\ m = 1$$

$$g(z) = 1 - \cos z = 0 \text{ at } z = 0$$

$$g'(z) = \sin z = 0 \text{ at } z = 0$$

$$g''(z) = \cos z \neq 0 \text{ at } z = 0$$

Thus, n = 2 i.e. z = 0 is a zero of numerator of order 2

Since n > m $\therefore$ z = 0 is a removable singular point

Example: Determine the type of singular points for the function $f(z) = (z-4)\sin\left(\frac{2}{z-4}\right)$

Solution: $f(z) = (z-4)\left[\frac{2}{z-4} - \frac{2^3}{3!(z-4)^3} + \frac{2^5}{5!(z-4)^5} \ldots\ldots\right]$

$$f(z) = 2 - \frac{2^3}{3!(z-4)^2} + \frac{2^5}{(z-4)^4\, 4!} \ldots\ldots\ldots$$

Singular Point z = 4 $\Rightarrow$ Essential singular point.

Residue of a Complex Function

If z0 is an isolated singular point of f(z), then the coefficient of $\frac{1}{z - z_0}$ in Laurent's series of f(z) about z = z_0 is called residue of f(z) and it is denoted by Res [f(z) : z = z_0]

Therefore, Res[f(z) : z = z0] = The coefficient of $\frac{1}{z - z_0}$ in Laurent's series.

The coefficient is given by, $b_1 = \frac{1}{2\pi i}\oint f(z)dz$

Cauchy's Residue Theorem

If f(z) is analytic at every point within and on a simple closed curve X except at a finite number of isolated singular

points z, z_2, ... z_n within C, then

$$\oint_c f(z)dz = 2\pi i\left(\sum_{j-1}^{n} R_j\right)$$

where $R_j = \text{Res}[f(z) : z = z]$

This theorem is useful in computing line integrals of complex functions over a closed curve.

Methods to find Residues

Removable Singular Point

If z_0 is a removable Singular Point of f(z) then Res(f(z) : z = z_0) = b1 = 0

Essential Singular Point

If the point z_0 is an essential Singular Point of f(z) then expand f(z) as a Laurent's series, about $z = z_0$ and collectthe coefficient of $\frac{1}{z - z_0}$ in the Laurent's series which gives the residue of f(z).

Pole

(A) If $f(z) = \frac{P(z)}{Q(z)}$ has simple pole at z_0, then.

$$\left[\operatorname{Res}\left[f(z) : z = z_0\right] = \underset{z \to z_0}{\text{Lt}}\left[(z - z_0) f(z)\right]\right]$$

(B) If $f(z) = \frac{\phi(z)}{\psi(z)}$ has simple pole at z_0

then $\left[\operatorname{Res}\left(f(z) : z = z_0\right) = \frac{\phi(z_0)}{\psi(z_0)}\right]$

where $\phi(z_0) \neq 0$ and $\psi'(z_0) \neq 0$

(C) $f(z) = \frac{g(z)}{z - z_0}$ has simple pole at z_0, then

$$\left[\operatorname{Res}\left[f(z) : z = z_0\right] = g(z_0)\right] \text{where } g(z_0) \neq 0$$

(D) If f(z) has pole at z0 of order m, then

$$\operatorname{Res}\left[f(z) : z = z_0\right] = \frac{1}{(m-1)!}$$

$$\underset{z \to z_0}{\text{Lt}}\left[\frac{d^{m-1}}{dz^{m-1}}\left\{(z - z_0)^m f(z)\right\}\right]$$

Solved Examples

Example: Find the residue of following function. $f(z) = \frac{z}{z^2 + 4}$

Solution: To determine the singular point

$z^2 + 4 = 0$

$z = +2i \rightarrow$ singular points, Simple poles.

$$R_1 : \operatorname{Res}\left\{f(z) : z = 2i\right\} = \underset{z \to 2i}{\text{Lt}}\left[(z - 2i) f(z)\right]$$

$$= \underset{z = 2i}{\text{Lt}} \frac{(z - 2i) \times z}{(z - 2i)(z + 2i)} = \frac{2i}{4i} = \frac{1}{2}$$

$$R_2 : \operatorname{Res}\left\{f(z) : z = -2i\right\} = \underset{z \to 2i}{\text{Lt}}\left[(z - 2i) f(z)\right]$$

$$= \underset{z = 2i}{\text{Lt}} \frac{(z + 2i) \times z}{(z + 2i)(z - 2i)} = \frac{2i}{-2i - 2i} = \frac{1}{2}$$

Example: Find the residue of following function.

Solution: Singular Point $z = \pi \rightarrow$ simple pole.

$\therefore$ Residue = $g(\pi)$ = -1

Example: Find the residue of following function. f (z) = cot z

Solution: $f(z) = \cot z = \frac{\cos z}{\sin z} = \frac{\phi(z)}{\psi(z)}$

Singular Point $z = n\pi$ $n \in I$

$$\operatorname{Res}\left\{f(z) : z = z_0\right\} = \frac{\phi(n\pi)}{\psi'(n\pi)} = \frac{\cos n\pi}{\cos n\pi} = 1$$

Example: Find the residue of following function. $f(z) = \frac{\cos z}{z - \frac{\pi}{2}}$

Solution:

$$f(z) = \frac{\cos z}{z = \frac{\pi}{2}} = \frac{g(z)}{(z - z_0)^m} m = 1$$

$z - \frac{\pi}{2} \rightarrow$ pole of order 1.

$N^r = 0$ at $z = \frac{\pi}{2}$

$g(z) = \cos z = 0$

$g'(z) = -\sin z \neq 0 \therefore n = 1$

$m = n = 1 \Rightarrow z = \frac{\pi}{2} \rightarrow$ Removable Singular Point

$$\therefore R_1 = \operatorname{Res}\left[f(z) : z = \frac{\pi}{2}\right] = 0$$

Example: Find the residue of following function. f(z) = (z – 2) $^{3/z-2}$

Solution: $f(z) = (z - 2)\left[1 + \frac{3}{z-2} + \frac{3^2}{2!(z-2)} + \frac{3^3}{3!(z-2)} + \dots\dots\dots\right]$

z = 2 essential singularity point. (∞ number of negative powers of z)

$$\therefore R_1 = \operatorname{Res}\left\{f(z) : z = 2\right\} = b_1 = \frac{3^2}{2!} = \frac{9}{2}$$

Method to calculate Complex Integrals by Residue Method

1. Let f(z) = Integrand function
2. Find singular point of f(z)
3. Consider the region R enclosed by curve C
4. Check whether the singular point is within the region or not.

No singular point within the region $\Rightarrow \text{CIT} \oint f(z)dz = 0$

One or more singular point within the region,

1. Classify the singular points
2. Find the residues
3. Substitute residue in Cauchy integral theorem $\oint f(z)dz = 2\pi i(R_1 + R_2 + \dots\dots\dots + R_n)$

Solved Examples

Example: Evaluate $\int_c \frac{z}{(z-1)(z-2)^2} dz, c : |z-2| = \frac{1}{2}$

Solution: Let $f(z) = \frac{z}{(z-1)(z-2)^2}$

Singular point : z = 1, 2

$c : |z-2| = \frac{1}{2}$

$z = 1$ outside $c > \frac{1}{2}$

$z = 1$ inside $c > \frac{1}{2}$

$z = 2$ pole of order 2

$N^r \neq 0$ at $z = 2$

$R_1 = \text{Res}\{f(z) : z = 2\} = \frac{1}{(2-1)!}$

$\underset{z\to 2}{\text{Lt}} \left[\frac{d}{dz}(z-2)^2 \times \frac{z}{(z-1)(z-2)^2}\right]$

Example: Evaluate $\oint z^2 e^{\frac{1}{z}} dz$ along a unit circle $|z| = 1$

Solution:

$f(z) = z^2 e^{\frac{1}{z}}$

$f(z) = z^2\left[1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots\dots\dots\dots\right]$

$f(z) = z^2 + z + \frac{1}{2} + \frac{1}{3!z^2} + \frac{1}{4!z^3} + \dots\dots\dots\dots$

Singular point = z = 0 → Essential singular point

$C : |z| = 1$

z = 0 lies inside C

Coefficient of $\frac{1}{z - z_0} = \frac{1}{3!}$

$\therefore R_1 = \text{Res}\{f(z) : z = 0\} = \frac{1}{3!} = \frac{1}{6}$

$\therefore$ by c.r.t we have $\oint_c f(z)dz = 2\pi i \times R_1 = \frac{\pi i}{3}$

5 Probability and Statistics

Objective

Upon completion of this chapter you will be able to:

- Understand the key terms associated with Probability i.e. Sample Space, Event etc.
- Determine the probability of occurrence of an event.
- Understand different discrete and continuous probability distributions.
- Understand the basics of statistics i.e. mean, standard deviation and variance.

Introduction

The foundations of probability theory were built by a mathematical study of games of chance. Today a huge gambling industry rests on the foundations of probability theory. Casinos make sure that people win just enough to keep them hooked but the odds are slightly balanced in favor of casinos so that they always come ahead. Similarly, in the Stock Market investors are always engaged in predicting the random fluctuations in the market. also, insurance business relies heavily on the probability theory. similarly in communication engineering digital communication is based on probability theory.

Definitions

Sample Space and Event:

If the output of any experiment is not known with certainty like tossing a coin or rolling a dice, then such an experiment is called as random experiment. But we can assume that the set of all possible outcomes is known like the outcome of a coin toss can be either heads or tails and the outcome of dice rolling can be from 1-6.

The set of all possible outcomes of an experiment is called as the sample space of an experiment and is represented by S. Thus, the set {H,T} for toss outcome will be the sample space for coin toss. Also, the set {1,2,3,4,5,6} is the sample space for dice rolling.

Any subset of sample space is called as an Event. As for example, the occurrence of head on a coin toss is an event. Similarly, the possibility of an outcome of an even number i.e. {2,4,6} is an event for rolling a dice.

Union and intersection:

For any two events A and B we define a new event $A \cup B$ called as "A union B" which consists of outcomes that are either in A or B or in both A and B. This also means $A \cup B$ will occur if A or B or both have occurred.

Similarly, we define an event $A \cap B$ called as "A intersection B" which consists of outcomes that are in both A and B. This means $A \cap B$ will occur if both A and B occur.

Complementary event:

The event E^C is called a complementary event for the event E. It consists of all outcomes not in E, but in S. As for an example, the occurrence of head is a complementary event to occurrence of tail. Similarly, the occurrence of even number on a dice roll is complementary to the occurrence of an odd number on a dice roll.

Equally likely events:

Two events are said to be equally likely if their probability of occurrence is equal. As for an example the probability of occurrence of tail on a coin toss is the same as occurrence of head and both are equal to ½ so both events are equally likely.

Mutually Exclusive Events:

Two events are mutually exclusive if they have no common outcome. As an example occurrence of a head on a coin toss and the occurrence of tail are both mutually exclusive events. Similarly, the occurrence of odd number and an even number on a coin toss are both mutually exclusive events. But the occurrence of a number divisible by 2 on a dice roll i.e. {2,4,6} and the occurrence of a number divisible by 3 i.e. {3,6} have the common outcome {6}. Thus, they are not mutually exclusive events.

In mathematical terms, we can say A and B are two mutually exclusive events if, $P(A \cap B) = 0$

Collectively Exhaustive Events

Two events E and F are collective exhaustive if they combined form the sample space of an event. For example occurrence of head on a coin toss and occurrence of tail are collective exhaustive events as they form the complete sample space i.e. {H,T} of a coin toss event.

Independent events:

Two events A and B are said to be independent if the probability of occurrence of one event does not depend on the probability of occurrence of the second. This, is expressed in terms of conditional probability, $P(A \mid B)$ which represents the probability of occurrence of A given that B has occurred.

If A and B are independent, then $P(A \mid B) = P(A)$ and in a similar manner $P(B \mid A) = P(B)$.

This is also expressed as, $P(A \cap B) = P(A) \times P(B)$

De Morgan's:

This law is used to find the probability of complement of a group of events as shown below,

$$\left(\bigcup_{i=1}^{n} E_1\right)^C = \bigcap_{i=1}^{n} E_i^C$$

This indicates that the complement of the event that at least one of the events occur is the same as the probability that none of the events occur.

$$\left(\bigcap_{i=1}^{n} E_i\right)^C = \bigcup_{i=1}^{n} E_1^C$$

Axioms of probability:

Consider an event A for an experiment where the sample space is represented as S. Then, the following axioms must be satisfied,

Axiom-1: $0 \le P(A) \le 1$

Here, $P(A) = 0$ indicates an impossible event.

$P(A) = 1$ indicates a certain or sure event.

Axiom-2: $P(S) = 1$, which indicates that the probability of sample space is 1 which means that certainly one of the events from the sample space will occur.

Axiom-3: If there are 'n' mutually exclusive events A_1 , A_2 , ... , A_n which means $P(A_i \cap A_j) = 0$. Then, P

$$\text{Then, } P\left(\bigcap_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

There are certain rules of the probability of calculation of compound event involving events A and B,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

If A and B are mutually exclusive events, $P(A \cap B) = 0$

Then, $P(A \cup B) = P(A) + P(B)$

Also, the probability of intersection can be expressed in terms of conditional probability,

$P(A \cap B) = P(A) \times P(B \mid A) = P(B) \times P(A \mid B)$

Here P(A) and P(B) are called as marginal probabilities of A and B and $P(A \cap B)$ is called as joint probability of A and B.

If A and B are independent events, then joint probability is the same as product of marginal probabilities.

$P(A \cap B) = P(A) \times P(B)$

Some rules on complementary probability are,

$$P(A^c) = 1 - (A)$$

$$P(A^c \cap B) = P(B) - P(A \cap B)$$

$$P(A^c \cap B^c) = P(\overline{A \cup B}) = 1 - P(A \cup B)$$

$$P(A \cap B)^c = P(A \cap B^c) + P(A^c \cap B)$$

$$P(A^c | B = \frac{P(A^C \cap B)}{P(B)} = \frac{P(B) - (A \cap B)}{P(B)}$$

$$= \frac{1 - P(A \cap B)}{P(B)} = 1 - P(A / B)$$

$$P(A | B^c = \frac{P(A \cap B^c)}{P(B^c)} = \frac{P(A) - P(A \cap B)}{1 - P(B)} \because (P(B) \neq 1$$

$$P(A^c | B^c) = \frac{P(A^C \cap B^c)}{P(B^C)} = \frac{1 - P(A \cup B)}{1 - P(B)} \because P(B) \neq 1$$

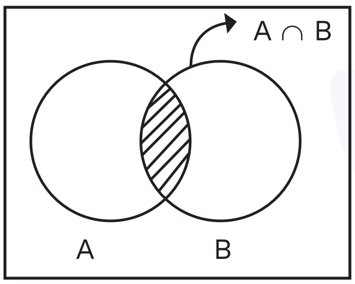


Fig. 5.1 A intersection B

If A and B are independent events, the probability of ($A \cap B^c$), P($A^C \cap B$) and P($A^C \cap B^C$) are also zero and these events are also independent.

Probability of an Event

The probability of an event can be computed by the classical approach in terms of favorable cases and sample space i.e. total outcomes.

Assume, m represents the number of favorable cases for an event E happening and n represents the total number of possible cases.

Then, the probability of occurrence of E is,

$$P(E) = \frac{m}{n}$$

As for an example, if we wish to compute the probability of occurrence of an odd number on a dice roll then the set {1,3,5} represents favorable cases and the set {1,2,3,4,5,6} represents sample space.

Thus, P (Odd Number) $= \frac{3}{6} = \frac{1}{2}$

Solved Examples

Example: If 3 coins are tossed at a time find P(getting at most one head)?

Solution: The 3 coin tosses are independent events. So we can multiply their probabilities.

Probability of occurrence of at most one Head is equal to the probability that no Head occurs and probability that exactly one Head occurs.

If a fair coin is tossed, P (Head) = P (Tail)= $\frac{1}{2}$

P (No-Head) = P (All Tails) $= \frac{1}{2} \times \frac{1}{1} \times \frac{1}{2} = \frac{1}{8}$

If exactly one Head occurs then there can be 3 cases that Head occurs on first toss, on second toss or on third toss.

$$P(\text{One} - \text{Head}) = 3 \times P(\text{OneHead} \,\&\, 2\,\text{Tails})$$

$$= 3 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{8}$$

$$P(\text{At Most One} - \text{Head}) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$$

Example: Find the probability of occurrence of at least one Tail if 3 coins are tossed simultaneously.

Solution: Here also all tosses are independent so the probability of intersection can be calculated by multiplication of individual probabilities.

The complementary event of at least one tail would be no tail and thus, we can write,

P(At Least One - Tail) = 1 - P(No Tail) = 1 - P (All Heads)

P (At Least - One Tail) $= 1 - \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{7}{8}$

Example: In the same problem find the probability of getting at least one head and at least one tail?

Solution: The sample space of 3 coin tosses is {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT} The favorable cases are {HHT, HTH, HTT, THH, THT, TTH}

P (At Least One Head and One Tail) $= \frac{6}{8} = \frac{3}{4}$

This can also be computed as

1 -P(No Head) -P (No Tail) $= 1 - \frac{1}{8} - \frac{1}{8} = \frac{3}{4}$

Example: A player tosses 6 coins, Find the probability that number of heads is more than number of tails.

Solution: The favorable cases are when there are 4 Heads and 2 Tails or when there are 5 Heads and 1 Tails or when there are all 6 heads.

P (4 Heads & 2 Tails) $= {}^6C_2 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^2 = \frac{15}{64}$

Here, we have considered the combination term as the two tails can occur on any 2 of the 6 tosses.

P (5 Heads & 1 Tails) ${}^6C_1 \left(\frac{1}{2}\right)\left(\frac{1}{1}\right) = \frac{6}{64}$

$$P(6\ \text{Heads}) = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$

$$\text{Thus}, P(\text{Heads} > \text{Tails})$$

$$= \frac{15}{64} + \frac{6}{64} + \frac{1}{64} = \frac{22}{64} = \frac{11}{32}$$

This problem can also be dealt with in terms of binomial random variable which will be covered later.

Example: Two dice are thrown 2 times. Find the probability of getting a sum of 7.

(A) At least once

(B) Only once

(C) Twice

Solution: Consider A as the event when a sum of 7 occurs on the first throw and B as the event when a sum of 7 occurs on the second throw. Here A and B are both independent events.

There are 36 possible outputs when two dice are thrown as 6 outputs are possible on each dice.

The following combinations will yield a sum of 7: {1,6}, {2,5}, {3,4}, {4,3}, {5,2}, {6,1} i.e. a total of 6 favorable cases.

$$P(A) = \frac{6}{36} = \frac{1}{6} \ \& \ P\left(A^c\right) = 1 - P\left(A\right) = \frac{5}{6}$$

$$P\left(B\right) = \frac{6}{36} = \frac{1}{6} \ \& \ P\left(B^c\right) = 1 - P\left(B\right) = \frac{5}{6}$$

(A) P (At least One)

$= P(A \cup B) = 1 - P(A^c \cap B^c) = 1 [P(A^c).P(B^c)]$

P(At lease Once) $= 1 - \frac{5}{6} \times \frac{5}{6} = \frac{11}{36}$

(B) $P\left(\text{only once}\right) P\left(A \cap B^c\right) + P\left(A^c \cap B\right)$

$$= P\left(A\right)P\left(B^c\right) + P\left(A^c\right)P\left(B\right) = \frac{1}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{1}{6} = \frac{10}{36}$$

(C) $P\left(\text{twice}\right) = P\left(A \cap B\right) = P\left(A\right)P\left(B\right) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

Example: Two dice are rolled simultaneously. Find the probability of getting a prime number on the first or a sum of 8 on both.

Solution: There are 3 possible prime number on a dice roll viz. {2,3,5}

$$P\left(A\right) = \left(\text{prime number}\right) = \frac{3}{6} = \frac{1}{2}$$

There are a total of 36 combinations on the two dice roll and the following combinations will yield a sum of 8: {2,6}, {3,5}, {4,4}, {5,3}, {6,2} i.e. a total of 5 combinations

$$P\left(B\right) = P\left(\text{sum of } 8\right) = \frac{5}{36}$$

The probability that first dice shows prime number and sum is 8 has three cases {3,5}, {5,3}, {2,6}

In our case, we need to compute

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$=\frac{1}{2}+\frac{5}{36}-\frac{1}{12}=\frac{5}{9}$$

Example: A determinant is chosen from a set of all determinants of order 2 with the elements 0 (&) or 1. Find the P (the chosen determinant is non zero)

Solution: Each element in 2×2 matrix can be zero or one. Thus there are 16 such possible matrices.

$$n(S)=16$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \ \Delta = ad-bc\neq 0$$

$$\text{Case (i)}: \Delta = +1[a=d]$$

$$=1 \text{ and at least one of } b=c=0$$

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix},\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix},\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}=3$$

$$\text{Case (ii)}: \Delta = -1[b=c]$$

$$=1 \text{ at least one of } a=d=0$$

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix},\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix},\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}=3$$

$$P(\text{non zero } \Delta)=\frac{6}{16}=\frac{3}{8}$$

Example: Four cards are drawn at random from the pack of 52 cards.

(A) Find P(all 4 cards are drawn from the same suit)

(B) P(no two cards are drawn from the same suit)

Solution: If all four cards are from same suit then there are four possibilities that all four are from spades, diamonds, hearts, clubs.

$$P(\text{All 4 from suit})=\frac{{}^{13}C_4+{}^{13}C_4+{}^{13}c_4}{{}^{52}C_4}=\frac{44}{4165}$$

If all 4 cards are from different suits then each card is drawn from a different suit

P(no two cards from the same suit)

$$=\frac{{}^{13}c_1\times{}^{13}c_1\times{}^{13}c_1\times{}^{13}c_1}{{}^{52}c_4}=\frac{2197}{20825}$$

Example: A card is drawn from the pack of 52 cards. Find the probability that it is neither a diamond nor an ace card.

Solution: There are 13 possible cards in diamond, so the probability that the card drawn is a diamond,

$P(\text{Diamond})=\frac{13}{52}$

There are 4 aces, one in each suit, so the probability that the card drawn is ace

$P(\text{ace})=\frac{4}{52}=\frac{1}{13}$

The probability that the drawn card is an ace of diamonds

$P(D\cap \text{ace})=\frac{1}{52}$

$$P(D^c\cap \text{ace}^c)=1-(D\cup \text{ace})=1-\left[\frac{13}{52}+\frac{4}{52}-\frac{1}{52}\right]$$

$$=\frac{36}{52}=\frac{9}{13}$$

Example: A and B are the two players rolling a dice on the condition that one who gets the 2 first win the game. If A starts the game what are the winning chances of player A, B.

Solution: Probability of getting a 2 on a dice roll is,

$P(2)=\frac{1}{6}=p$

Probability of getting any number other than 2 on dice roll is,

$P(2^c)=\frac{5}{6}=q$

If A starts the game then he can win in the following cases,

- A gets 2 on the first roll.
- A gets any number other than 2 on first roll, then B gets any number other than 2 on second roll then A gets 2 on the third roll.

- Similarly, A can get 2 on odd number of rolls and on rolls previous to winning one there should be any number other than 2.

$$P\left(A_{win}\right) = p + q^2p + q^4p + q^6p + ...$$

$$P\left(A_{win}\right) = \frac{p}{1-a^2} = \frac{1/6}{1-25/36} = \frac{6}{11}$$

If we wish to calculate the probability of B winning,

$$P\left(B_{win}\right) = 1 - P\left(A_{win}\right) = 1 - \frac{6}{11} = \frac{5}{11}$$

Example: A, B & C are tossing a coin on the condition that one who gets the head first wins the game. If A starts the game, what are the winning chances of C in the 3rd trial?

Solution: If a fair coin is tossed then, probability of head and teal occurrence is,

$$P\left(H\right) = \frac{1}{2} = p \qquad p\left(T\right) = \frac{1}{2} = q$$

Now, if C wins on the third trial, then all must get tails on first and second trial and then on third trial A and B must get Tails but C must get Heads.

$$P\left(winC\right) = q^3q^3\,qqp =$$

$$\left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^3 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{512}$$

Conditional Probability

Conditional Probability represents the probability of an event occurrence given that some other event has already occurred.

P(A|B) represents the probability of occurrence of A given that B has already occurred.

Total probability theorem:

Suppose there are n events which that are mutually exclusive and collectively exhaustive then probability of occurrence of an event E is given by,

$P(E) = P(A_1 \cap E) + P(A_2 \cap E) + ... + P(An \cap E)$

in terms of conditional probability,

$P(E) = P(A_1)P(E \mid A_1) + P(A_2)P(E \mid A_2) + ... P(A_n)P(E \mid A_n)$

Bayes' Theorem

This theorem is used for finding reverse probability,

$$P\left(A_i \middle| E\right) = \frac{\left(A_1 \cap E\right)}{P(E)}$$

If we represent P(E) in terms of the total probability theorem

$$P\left(A_i \middle| E\right) =$$

$$\frac{p\left(A_i \cap E\right)}{P\left(A_1\right)P\left(E \middle| A_1\right) + P\left(A_2\right)P\left(E \middle| A_2\right) + ...P\left(A_n\right)P\left(E \middle| A_n\right)}$$

Solved Examples

Example: A number is drawn from the 100 numbers those are 0, 1, 2, 3, ... 99. Let x denote the sum of digitson a number and y denotes the product of the digits on the number. Find P(x=9), given y=0.

Solution: The following cases can be formed where the product of digits of a number drawn is 0.

00	10
01	20
02	30
03	40
04	50
05	60
06	70
07	80
08	90
09	

In the following cases the sum of digits is 9; 09 and 90 i.e. 2 cases are there where the sum of digits are 9 and product of digits is 0

$$P\left(x = 9 \middle| y = 0\right) = \frac{P\left(x = 9 \cap y = 0\right)}{P\left(y = 0\right)} = \frac{2/100}{19/100} = \frac{2}{19}$$

Example: 60% of the employees of the company are college graduates of these 10% are in the sales department. Of the employees who didn't graduate from the college 80% are in the sales department. A person is selected at random. Find the probability that

(A) The person is in the sales department.

(B) Neither in the sales department nor a college graduate.

Solution: Assuming that the total number of employees is 100. Then, college graduates are 60.

Number of college graduates in sales department = 10% of 60 = 6

Number of people who are not college graduates = 40

Number of non college graduates in sales department = 80% of 40 = 32

Total number of employees in sales department = 32+ 6 =38

(A) Probability of a randomly selected person to be from the Sales Department,

P(sales) = 38/100

(B) Number of people not from college and not in sales = 40 -32 = 8

Probability that the randomly selected person is non college graduate and not in sales, P(N-S & N-G) = 8/100

Example: In answering a question on multiple choice test, the students either know the answer or guess the answer. Let P be the probability that student know the answer and 1-P that of guessing the answer. Assume that the student guesses the answer to a question will be correct with a probability of 1/5. What is the conditional probability that the student knew the answer to a question given that he answered correctly?

Solution: Probability of knowing the answer

P(K) = P

Probability of guessing the answer

P(G) = 1-P

Let us define the event E as the answer is correct.

Probability that the answer is correct given that student knows the answer P(E|K) = 1

Probability that the answer is correct given that student is guessing the answer

P(E|G)= $\frac{1}{5}$

By Total Probability Theorem,

$$P(E) = (K \cap E) + P(G \cap E) = P(K)P(E|K) + P(G)P(E|G)$$

$$P(E) = P \times 1 + (1-P)\frac{1}{5} = \frac{4P+1}{5}$$

By Baye's theorem $P(K|E) =$

$$\frac{P(K \cap E)}{P(E)} = \frac{P(K)P(E|K)}{P(E)} = \frac{P \times 1}{\frac{4P+1}{5}} = \frac{5P}{4P+1}$$

Example: There are 3 coins out of which 2 are unbiased and one is biased coin with two heads. A coin is drawn at random and tossed 2 times. If head appears on at both the times, find the probability that the selected coin is a biased coin?

Solution: Number of Unbiased Coins = 2 Number of biased coins = 1

Probability of selecting an unbiased or a biased coin

$$P(UB) = \frac{2}{3}, P(B) = \frac{1}{3}$$

Let us define an event E as getting two heads on two tosses Probability that we get two heads using an unbiased coin,

$$P(E) = P(UB \cap E) + P(B \cap E) =$$

$$P(UB)P(E|UB) + P(B)P(E|B)$$

$$P(E) = \frac{2}{3} \times \frac{1}{4} \times \frac{1}{3} \times 1 = \frac{2}{12} + \frac{1}{3} = \frac{6}{12} = \frac{1}{2}$$

$$\frac{P(B \cap E)}{P(E)} = \frac{\frac{1}{3} \times 1}{\frac{1}{2}} = \frac{2}{3}$$

Example: Player A speaking truth $\frac{4}{7}$ times and the card is drawn from the pack of 52 cards, he reports that there is a diamond. What is the probability that actually there was a diamond?

Solution: Probability of a person speaking the truth and lie are,

$$P(T) = \frac{4}{7}; P(L) = \frac{3}{7}$$

Let us define an event E as the Player A reports that there is a diamond. Probability that there is a diamond given that Player A is speaking the truth,

$$P(E|T) = \frac{13}{52} = \frac{1}{4}$$

Probability that there is a diamond given that he is lying,

$$P(E|T) = \frac{39}{52} = \frac{3}{4}$$

By total probability theorem,

$$P(E) = P(T \cap E) + P(L \cap E) = P(T)P(E|T)$$

$$+P(L)P(E|L) = \frac{4}{7} \times \frac{1}{4} \times \frac{3}{7} \times \frac{3}{4} = \frac{13}{27}$$

By Baye's theorem,

$$P(T|E) = \frac{P(T \cap E)}{P(E)} = \frac{\frac{4}{7} \times \frac{1}{4}}{\frac{13}{28}} = \frac{4}{13}$$

Example: A letter is known to have come from Tatanagar (or) Calcutta on the envelope, the just two consecutive letters T and A are visible. Find the probability that the letter has come from Calcutta?

Solution: Probability that the word selected is Tatanagar or Calcutta is,

$$P(T) = \frac{1}{2} \quad P(C) = \frac{1}{2}$$

There are 8 pairs of consecutive letters in Tatanagar i.e. "TA"; "AT"; "TA"; "AN"; "NA"; "AG"; "GA"; "AR". There are two cases where "TA" are consecutive

Similarly, in Calcutta there are 7 pairs of consecutive letters i.e. "CA"; "AL"; "LC"; "CU"; "UT"; "TT"; "TA" There is only one case where "TA" are consecutive.

Define an event E: Getting a 'TA'

$$P(E|T) = \frac{2}{8}$$

$$P(E|C) = \frac{1}{7}$$

By Total Probability Theorem,

$$P(E) = \frac{1}{2} \times \frac{2}{8} + \frac{1}{2} \times \frac{1}{7} = \frac{11}{56}$$

$$P(C|E) = \frac{P(C \cap E)}{P(E)} = \frac{\frac{1}{2} \times \frac{1}{7}}{\frac{11}{56}} = \frac{4}{11}$$

Example: There are 3 bags which contain blue, red and green color balls in the form of

	B	R	G
(A)	1	2	3
(B)	2	3	1
(C)	3	1	2

A bag is drawn at random and two balls are drawn from it. They are found to be 1 blue and 1 red. Find the probability that the selected balls are from bag C?

Solution: Since all bags are equally likely to be selected

$$P(A) = P(B) = P(C) = \frac{1}{3}$$

Let us define on event E : Getting one blue and one red ball

$$P(E|A) = \frac{1C1 \times {}^2C_1}{C_2} = \frac{2}{15}$$

$$P(E|B) = \frac{6C2 \times {}^3C_1}{{}^6C_2} = \frac{6}{15}$$

$$P(E|C) = \frac{{}^3C_1 \times {}^1C_1}{{}^6C_2} = \frac{3}{15}$$

By Total Probability Theorem,

$$P(E) = P(A \cap E) + P(B \cap E) + P(C \cap E) =$$

$$P(A)P(E|A) + P(B)P(E|B) + P(C)P(E|C)$$

$$P(E) = \frac{1}{3}\left[\frac{2}{15} + \frac{6}{15} + \frac{3}{15}\right] = \frac{11}{45}$$

By Bayes' theorem

$$P(C|E) = \frac{P(C \cap E)}{P(E)} = \frac{\frac{1}{3} \times \frac{3}{15}}{\frac{11}{45}} = \frac{3}{11}$$

Statistics

Statistics is a branch of mathematics that deals with summarizing large quantities of data into some meaningful parameters like mean, standard deviation etc.

Quantities like Mean, Median and Mode quantify the central value around which the data points are centered and quantities like standard deviation, variance and coefficient of variation quantify how scattered the data points are around central point.

Types of data

- Grouped and ungrouped
- Open and closed

Grouped data: If the data is in the form of class intervals and frequency together, then the data is known as grouped data or distributing the frequencies to their corresponding class intervals is known as frequency distribution.

Closed data: If the class intervals are in continuous form without any discontinuity, the data is known as closed data. Otherwise open data.

Ungrouped data: If the data contains only observations without any class intervals, then the data is known as ungauged data or raw data.

Mean (Average)

x_i represents the individual observation. Then, the mean value for grouped and ungrouped data is given by,

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \text{ for ungrouped data}$$

For grouped data we associate a frequency to each interval represented by 'f'. In any interval we also define the lower limit and upper limit of an interval.

$$\bar{x} = \frac{\sum_{i=1}^{n} fx_i}{N} \text{ for grouped data}$$

Where

$$X_i = \frac{\text{Upper limit} + \text{lower limit}}{2}$$

n= no. of observations

N= sum of frequencies

Median

Mean represents a central value in the sense that positive and negative deviations from the arithmetic mean balance each other.

On the other hand, median is the central value of the distribution such that a number of values less than median is equal to number of value greater than the median.

If n is odd → The middle observation itself is the median

If n is even → Average between the middle observations, provided

1. Data is rearranged either in ascending or in a descending order.
2. The number of observations above the middle is equal to the number of observations below the middle.
3. For the grouped data, the median is given by

$$\text{Median} = L + \left[\frac{\frac{N}{2} - F}{f_m}\right] h$$

Where,

L → lower limit for the median class

f_m → frequency for median class

F → cumulative frequency for above the median class.

h → size of the median class

Solved Examples

Example: Find the median for the following grouped data?

Class Interval	Frequency	Cumulative frequency
0 – 10	3	3
10 – 20	5	8
20 – 30	7	15
30 – 40	2	17
40 – 50	1	18
	N = 18	

Table 5.1

Solution: The middle value i.e. 9^{th} and 10^{th} values both lie in the class 20-30 and hence 20-30 is the median class. Lower limit of median class, L=20

Frequency of median class = 7

Cumulative frequency of class above median class = 8 Width of median class = 30-20 = 10

$$\text{Median} = 20 + \frac{9-8}{7} \times 10 = 20 + 1.4 = 21.4$$

Note: If the first class itself is median class, the cumulative frequency and frequency (f) are equal (m = f).

Mode

The most frequently repeated observation is known as the mode If there is only one value with the highest frequency then it is unimodal.

If there are 2 such values having the highest but equal frequencies of occurrence then it is bimodal signal. Eg: 1, 2, 3, 4, 5, 2, 3, 6, 7, 2, 3, 11, 14, 2, 21, 23, 3, 36

In this dataset 2 and 3 both occur 4 times so it is bimodal having two mode values i.e. 2 and 3. For grouped data, mode can be computed as,

$$\text{Mode} = L + \left[\frac{\Delta_1 + h}{\Delta_1 \Delta_2}\right]$$

Modal class is defined as the class having highest frequency.

L = lower limit of modal class

h = size of modal class

f = frequency of modal class

$\Delta_1 = f - f_{-1}$

$\Delta_2 = f = f_{+1}$

f-1 = frequency of class preceding the modal class

f+1 = frequency of class succeeding modal class

For grouped data, the relation between mean, median and mode is given by, Mode = 3 Median – 2 Mean

Solved Examples

Example: Find the mode for the following frequency and data

Class Interval	Frequency
0-2	11
2-4	14
4-6	17
6-8	08
8-10	03

Table 5.2

Solution: The class 4-6 has the highest frequency of 17 and thus it is the modal class.

Lower Limit = 4

$f_{-1} = 14$

$f_{+1} = 8$

$\Delta_1 = f - f_{-1} = 17 - 14 = 3$

$\Delta_2 = f - f_{+1} = 17 - 8 = 9$

$$M_0 = 4 + \left[\frac{3}{3+9}\right]2 = 4 + \frac{6}{12} = 4.5$$

Note: If all the frequencies are equal mode is undefined $\left(\frac{0}{0}\text{ form}\right)$

Measures of Dispersion / Variability

To check the consistency, uniformity etc of the data measures of dispersion is used. They are

- Range
- Quartile Deviation (QD)
- Mean Deviation (MD)
- Standard Deviation (SD)
- Coefficient of Deviation (c, u)

Range: It is defined as the difference between the maximum and minimum values.

Standard Deviation

Instead of taking absolute deviation from the mean of the data, we square each deviation and obtain arithmetic mean of squared deviations. The mean of squared deviations is called as Variance of the data and square root of variance is called as standard deviation.

Standard Deviation SD= $\sqrt{\text{Vanance}}$

Variance = $(SD)^2$

$$\text{Standard Deviation}, \sigma_x = \sqrt{\frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}{n}}$$

Lesser value of standard deviation or variance indicates that data is more uniform. Thus, the variance of constant is 0.

Note:

- Variance can never be negative.
- Sum of the deviations from the mean is always zero. $\sum\left(x_i - x\right) = 0$
- Sum of the squares of the deviation from mean is minimum.
- If the variance are equal for the different groups greater mean is more consistent
- For grouped data, variance is given by

$$\sigma_x^2 = \frac{1}{N}\sum f_i x_i^2 - \left(\frac{1}{N}\sum f_i x_i^2\right)^2$$

- Relation between quartile deviation, mean deviation and standard deviation

6QD=5MD=4SD

$QD = \frac{4}{6}$ SD $= \frac{2}{3}\sigma$

5MD=4SD

$MD = \frac{4}{5}$ SD

- Comparison of variability of two sets of data is done in terms of coefficient of variation.

Coefficient of variation,

$$C.V = \frac{\text{Standard Deviation}}{\text{mean}} \times 100 = \frac{\sigma}{\mu} \times 100$$

- Coefficient of variation is the best measure to determine data consistency.

Solved Examples

Example: Find the mean and variance of 1st 'n' natural numbers?

$$\bar{x} = \frac{1+2+3+n}{n}$$

$$= 1/n\left[\frac{n(n+1)}{2}\right] = \frac{n+1}{2}$$

$$\sigma_x{}^2 = \frac{1}{n}\sum_{i=1}^{n}(xi)^2 - \left(\frac{1}{n}\sum_{i=1}^{n}2x\right)^2$$

$$\frac{1}{n}\sum xi^2 = \frac{1}{n}\left[12 + 22 + \dots\dots\dots\dots\dots + n^2\right]$$

$$= \frac{1}{n}\frac{n(n+1)(2n+1)}{6}$$

$$\sigma^{x_2}\frac{(n+1)(2n+1)}{6} = \left(\frac{n+1}{2}\right)^2$$

$$\sigma^{x_2} = \frac{n+1}{2}\left[\frac{2n+1}{3} - \frac{n+1}{2}\right]$$

$$= \frac{n+1}{2}\left[\frac{4n+2-3n-3}{6}\right]$$

$$\sigma^{x_2} = \frac{n+1}{2}\left[\frac{n-6-n^2-2^1}{6}\right]$$

Skewness

It is the measurement of lack of symmetry. Pearson's Coefficient of skewness.

$$S_{kp} = \frac{M - M_0}{\sigma}\left(\frac{3\,\text{Mean} - \text{Median}}{\sigma}\right)$$

$$-3 \le S_{kp} \le +3$$

If $-3 \le S_{kp} \le 0$, the data is said to be negatively skewed.

S_{kp} =0, the data is said to be symmetric

$0 \le S_{kp}$ 3, the data is said to be positively skewed.

Random Variables

It is frequently the case when an experiment is performed that we are mainly interested in some function of the outcome as opposed to the actual outcome itself.

For instances, in tossing dice we are often interested in the sum of two dice and are not really concerned about the separate value of each die. That is, we may be interested in knowing that the sum is 7 and not be concerned over whether the actual outcome was (1, 6) or (2, 5) or (3, 4) or (4, 3) or (5, 2) or (6, 1). These quantities of interest, or more formally, these real valued functions defined on the sample space, are known as random variables.

Because the value of the random variable is determined by the outcome of the experiment, we may assign probabilities to the possible values of the random variable.

Connecting the outcomes of an expectation with the real values is known as random variable (R-V) (1-D, R-V). The corresponding data is known as univariate data.

2-D R-V i.e. Two dimensional random variable is defined as connecting the two outcomes to a real value provided those two outcomes are drawn from the same sample space. The corresponding data is known as bivariate data.

Types of Random Variable

Random variable may be discrete or continuous.

Discrete random variable: A variable that can take one value from a discrete set of values.

Let x denotes sum of 2 dice. Number x is a discrete random variable as it take one value from the set {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}, since the sum of 2 dice can only be one of these values.

Continuous random variable: A variable that can take one value from a continuous range of values. Let X denotes the volume of Pepsi in a 500ml cup. Now x may be a number from 0 to 500, any of which value, x may take.

Note: Similar to Random Variables we can define Discrete and Continuous Distributions.

Properties of Discrete Distribution

- $\sum P(x) = 1$
- $E(x) = \sum xP(x)$
- $V(x) = E(x^2) - (E(x))^2 = \sum x^2P(x) - \left[\sum xP(x)\right]^2$

E(x) denotes the expected value or average value of the random variable x, while V(x) denotes the variance of the random variable x.

Probability

Distribution and Cumulative Distribution Function

The probability distribution function or the probability density function represents the distribution of probability

i.e. how the probability varies as the random variable takes on different values. It is represented as f(x). Cumulative Distribution Function represents the probability that the random variable is less than a certain value.

$\frac{d}{dx}F(x) = f(x)$ Probability Density Function (PDF)

$F(x) = \int_{-\infty}^{x} f(x)\,dx =$ Cumulative Distribution Function (CDF)

Note: Both these terms are defined for continuous distributions.

Properties of Continuous Distribution

- $\int_{-\infty}^{\infty} f(x)\,dx = 1 \text{ or } F(\infty) = 1$
- $E(x) = \int_{-\infty}^{\infty} xf(x)\,dx$
- $E(g(x)) = \int_{-\infty}^{\infty} g(x)f(x)f(x)\,dx$
- $V(x) = E(x^2) - [E(x)]^2$

  $= \int_{-\infty}^{\infty} x^2f(x)\,dx - \left[\int_{-\infty}^{\infty} xf(x)\,dx\right]^2$
- $P(a < x < b) = P(a \le x < b)$

  $= P(a < x \le b) = \int_{a}^{b} f(x)\,dx$

Properties of Expectation

- If ‘X’ is a random variable and ‘a’ constant E(ax) = aE(x)
- if X and Y are two random variables

  E(X + Y) = E(X) + E(Y)

  E(X - Y) = E(X) – E(Y)

  E(X . Y) = E(X) E(Y | X)= E(Y) E(X | Y)

Here, E(Y | X) and E(X | Y) are called as Conditional Expectations.

- If X and Y are independent Random Variables E(X. Y) = E(X) E(Y)
- If Y = aX + b, a, b → constants

  E(Y) = E(aX + b) = E(aX) +E(b)= a E(X) + b

  E(constant) = constant
- E(E(E(x))) = E(x)

Properties of Variance

- If X is a Random Variable and 'a' is constant $V(aX) = a^2 v(X)$

  V(-Y) = (-1)2 V(Y) = V(Y)
- If X and Y are independent Random Variables V(X + Y) = V(X) + V(Y)

  V(X – Y) = V(X) + V(-Y) = V(X) + V(Y)

  V(X.Y) = V(X) .V(Y)
- If a and b are constants, X and Y are independent Random Variables

  $$V\left(\frac{x}{a}-\frac{Y}{b}\right)=\frac{1}{a^2}V(X)+\frac{1}{b^2}V(y)$$
- If Y = aX + b, where a, b are constants

  V(Y) = V(aX+b) = V(aX) + V(b) = a2V(x) because V(constant)=0
- If X and Y are two Random Variables V(X . Y) = V(X) +V(Y) + 2cov(X, Y)

  cov(X, Y) = E(X.Y) – E(X) E(Y)

  cov(a, b) = E(ab) – E(a) E(b) = ab – ab = 0 where a and b → constant
- If x and y are independent Random Variables, covariance of x and y cov(x, y)=0. But the converse of statement is not true.

Thus, E(X . Y) = E(X).E(Y)

Note:

- Variance and covariance are independent of change of origin and dependent on change of scale.
- Expectation (mean) dependent on change of origin as well as dependent on change of scale.

Skewness

Skewness is defined as the lack of symmetry.

$$\beta_1=\frac{\mu^2{}_3}{\mu^3{}_2}$$

μ_3 : 3^{rd} central moment $\mu_3=E\left((x-\mu)^3\right)$

$\mu2$: variance

- If μ_3 value is negative, then the curve is known as negatively skewed.
- If μ_3 value is positive, then the curve is known as positively skewed.
- If $\mu_3=0$, then the curve is known as symmetric $\beta_1 = 0$

Solved Examples

Example: Find the expectation of the number on a die when it is thrown?

x R.V	1	2	3	4	5	6
P(x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Solution: E(X) = mean

$$=\sum_{x=1}^{6}xP(x)=$$

$$1\times P(1)+2P(2)+3P+4P(4)+5P(5)+6P(6)$$

$$E(x)=\frac{1}{6}[1+2+3+4+5+6]=\frac{21}{6}=\frac{7}{2}$$

Example: Find the variance of outcome of a single dice throw.

$$\text{Solution}: V(x)=E(x^2)-[E(x)]^2$$

$$E(x^2)=\sum_{x=1}^{6}x^2P(x)=\frac{1}{6}$$

$$[1^2+2^2+3^2+4^2+5^2+6^2]=\frac{91}{6}$$

$$V(x)=\frac{91}{6}-\left(\frac{7}{2}\right)^2=\frac{91}{6}-\frac{49}{4}=\frac{182-147}{12}=\frac{35}{12}$$

Note: The mean and variance for the sum of the numbers on the dice is

$$E(x)=\frac{7n}{2}$$

$$V(x)=\frac{35n}{12}$$

where, n represents the number of dice thrown.

To derive this, we need to construct a probability distribution table, where we list down the probability of occurrence of each value of sum. As for an example, for two dice, the sum can vary from 2 to 12.

Example: 3 unbiased dice are thrown. Find the mean and variance of the sum of the numbers on them?

Solution:

$$E(x)=\frac{7}{2}\times=\frac{21}{2}$$

$$V(x)=\frac{35}{12}\times 3=\frac{35}{4}$$

Example: When two dice are rolled, find the expectation for sum 7?

Solution: E(x) = x.P(x)

The sum 7 can be obtained for the following outcome (1,6); (2,5); (3,4); (4,3); (5,2); (6,1). Thus, 6 cases out of 36 yield the sum of 7.

$$E(7)=7.P(7)=7.\frac{6}{36}=\frac{7}{6}$$

Example: A player tosses 3 coins. He wins Rs.500 if 3 heads occur. Rs.300 if 2 heads occur, Rs.100 if only 1 head occur. On the other hand, if 3 tails occur he losses Rs.1500. Find expected value of the game?

Solution: Constructing the Probability distribution table,

x	+500	+300	+100	-1500
No. of head	3	2	1	${}^3C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3$
P(x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
	${}^3C_3\left(\frac{1}{2}\right)^3$	${}^3C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1$	${}^3C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2$	

Value of the game = Gain – loss

$$\text{Value}=500\times\frac{1}{8}+300\times\frac{3}{8}+100\times\frac{3}{8}-1500\times\frac{1}{8}$$

$$=\frac{1700}{8}-\frac{1500}{8}=\text{Rs. }25$$

Note: If the game is said to be balanced or fair, then the value of game = 0 (No loss and no gain)

Example: A man was given n keys of which 1 fits the lock. He tries them successively without replacement to open the lock. What is the probability that the lock will be open after the nth trial. Also determine mean and variance?

Solution: There can be two cases for opening the lock, With replacement ⇒ Independent trials

Without replacement ⇒ dependent trials

P(opening lock), 1^{st} trial = $\frac{1}{n}$, Since there are n keys available

P(opening lock), 2^{st} trial = $\frac{1}{n-1}$, Since there are (n-1) keys available

P(opening lock), 3^{rd} trial = $\frac{1}{n-2}$, Since there are (n-2) keys available

P(opening lock, 1^{st} failure, 2^{nd} trial)

$$\left(1\frac{1-1}{n-1}\right)=\frac{n-1}{n}\times\frac{1}{n-1}=1$$

P(opening lock, 1^{st} failure, 2^{nd} failure 3^{rd} trial)

$$\left(\frac{1-1}{n}\right)\left(\frac{1-1}{n-1}\right)\frac{1}{n-2}=\frac{1}{n}$$

Similarly, P(opening lock 1st success, 9th trial) = $\frac{1}{n}$

Thus, the probability of success on each trial is the same and we can call it uniform distribution.

$$E(x)=\sum_{i=1}^{n} i\times P(x=i)=\sum_{i=1}^{n} i\times\frac{1}{n}=\left(\frac{n-n+n}{2n}\right)$$

$$=\frac{n+1}{2}$$

Similiarly, $V(x)=\frac{n^2-1}{12}$

Example: A man was given 900 keys. Find the probability that the lock will be open at 450th trial by with and without replacement.

Solution: From the previous example,

450th trial lock open without replacement= $\frac{1}{n}=\frac{1}{900}$

Lock open 450th trial with replacement, then there must be failure on previous 449 trials and since the trials are with replacement, the number of keys in each trial is 900.

P (success) $= \frac{1}{900}$

P (failure) = 1-1/900 $\frac{1}{900} = \frac{899}{900}$

P(450^{th} trial with replacement)

$$= \left(1 - \frac{1}{900}\right)^{499} \times \frac{1}{900} = 6.74 \times 10^{-4}$$

Example: If X is continuous Random Variable and $f(x) = ke^{-x^2}/2 \quad -\infty < X < +\infty$

Find

(A) K

(B) E(x), V(x)

Solution: Probability of sample space is 1 or the total probability is always one.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} Ke^{-x^2/2}dx = 1$$

$$K\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = 1$$

Since, the integrand is an even function,

$$2K\int_{0}^{\infty} e^{-x^2/2}\,dx = 1$$

Put $x^2 = 2u$

$2xdx = 2du$

$$dx = \frac{2du}{2x} = \frac{du}{\sqrt{2u}}$$

$$2K\int_{0}^{\infty} e^{-u}u^{-1/2}\,\frac{du}{\sqrt{2}} = 1$$

$$\sqrt{2K}\int_{0}^{\infty} e^{-u}\,u^{-1/2}\,du = 1$$

$$\sqrt{2K}\int_{0}^{\infty} e^{-u}\,u^{1/2-1}\,du = 1$$

By definition, gamma function is defined as,

This is due to the fact that integrand is an odd function and hence the integration in positive and negative limits cancels out.

$$\overline{|n} = \int_{0}^{\infty} e^{-u}u^{n-1}\,du$$

Therefore, $\sqrt{2K}\int_{0}^{\infty} e^{-u1} - u\,2^{-1}du = \sqrt{2K}\overline{\left|\frac{1}{2}\right.} = 1$

$$\sqrt{2K}\sqrt{\pi} = 1$$

$$K = \frac{1}{\sqrt{2\pi}}$$

Calculating expectation and variance of the random variable,

$$E(x)\int_{\infty}^{0} xf(x)\,dx = \frac{-1}{\sqrt{2\square}}\int_{-\infty}^{\infty} x.^{-x^2}x^{-2}\,dx = 0$$

This is due to the fact that integrand is an odd function and hence the integration in positive and negative limits cancels out.

$$E(x) = \int_{-\infty}^{\infty} x.^2\,f(x)\,dx = \frac{1}{\sqrt{1Å}}\int_{-\infty}^{\infty} x^{2-x^2}e^{-2}dx$$

Again substitute $x^2 = 2^1$ ud

$$\left(Ex^2\right) = \frac{\sqrt{2ue^{-u}}}{\sqrt{\pi}}\int_{0}^{\infty} \frac{2}{\sqrt{2}} =$$

$$\frac{2}{\sqrt{\pi}}\int_{0}^{\infty} u^{-2}e\,du\,\frac{2}{\sqrt{\square}}\left|\frac{3}{2}\right. \times \frac{2}{\sqrt{\pi}} \times \frac{1}{2} \times \left|\frac{1}{2}\right. = 1$$

$$V(x) = \left(x^2\right) - \left[E(x)\right]^2 = 1 - 0 = 1$$

Example: If X is a continuous Random Variable and f(x) = |x|, =1 < x < 1 find V(X)?

Solution: $E(x) = \int_{-1}^{1} xf(x)\,dx = \int_{-1}^{1} x|x|\,dx = 0$

Because, here the integrand is an odd function of x. $\left[\frac{x^4}{4}\right]_0^1 = \frac{2}{4} = \frac{1}{2}$

$$E\left(x^2\right) = \int_{-1}^{1} x^2 f(x)\,dx = \int_{-1}^{1} x^2\,|x|\,dx = 2\int_{0}^{1} x^2.x\,dx = 2$$

$$V(x) = E\left(x^2\right) - \left(E(x)\right)^2 = \frac{1}{2} - 0\frac{1}{2}$$

Example: If x and y are the R.V, mean(x) = 10, variance (x) = 25, find +ve values of a, b such that Y=aX-b has E(Y)

= 0, V(Y) = 1

Solution: Variance of the two random variables are related as,

$V(Y) = a^2 V(X)$

$1 = 25a^2$

$a^2 = \frac{1}{25}$

$a = \frac{1}{5}$

$\text{Since}, Y = aX - b$

$E(Y) = aE(X) - b$

$0 = 10a - b$

$b = 10a = 10 \times \frac{1}{5} = 2$

Bivariate Data

Case 1: Continuous Random Variable

If x and y are 2-D continuous Random Variables and its probability function is known as joint probability density function and is denoted by f(x, y).

The marginal density functions are

$$f(x) = \int_y f(x, y)\, dy$$

$$f(y) = \int_x f(x, y)\, dy$$

The marginal density functions represent the 1-D density function for 2-D random variable.

If X and Y are 2-D, continuous, independent Random Variables if f(x,y) = f(x).f (y) is

JPDF = mdf (x) mdf (y)

Where JPDF = Joint Probability Density Function MDF = Marginal Density Function

Relation between JDF and JPDF

$$\frac{d^2}{d \times dy} F(x, y) = f(x, y)$$

$$\text{or} F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x, y)\, dxdy$$

Conditional PDF is $f\left(\frac{x}{y}\right) = \frac{f(x, y)}{f(y)} \left(d(y) \neq 0\right)$

Conditional expectation

$$E\left(\frac{x}{y}\right) = \frac{E(xy)}{E(y)} \left(E(y) \neq 0\right)$$

Case-2: Discrete Random Variable

If x and y are 2-D discrete Random Variable and its probability function is known as joint probability mass function denoted by P(x y).

The marginal mass function is

$$P(x) = \sum_y P(x, y)$$

$$P(y) = \sum_x P(x, y)$$

Covariance

Covariance is a measure of the joint variability of two random variables For Discrete random variables

$$\mathrm{Cov}(X, Y) = \frac{\sum_{i=1}^{n} (x_i - \cancel{x})(y1 - \cancel{y})}{n}$$

For Continues real valued random variables

$$\mathrm{Cov}(X, Y) = E(XY) - E(X)E\,E(Y)$$

Solved Examples

Example: If x and y are 2-D continuous Random variables and their corresponding joint probability function is

f (x, y) = xy . x → 0 to 1, y → 0 to 1

i) Find the mean and variance of y

ii) E(x, y), Cov(x, y)

iii) f(x/y) = E(x/y)

iv) Check whether x and y are independent variable or not

Solution: The marginal PDF can be obtained from Joint PDF as,

$$f(x) = \int_y f(x, y)\, dy = \int_0^1 xy\, dy = x \int_0^1 y\, dy = x \left[\frac{y^2}{2}\right]_c^1$$

$$= \frac{x}{2}$$

$$\text{Similarly}, f(y) = \frac{y}{2}$$

$$E(X) = \int_0^1 x f(x) dx = \int_0^1 x.\frac{x}{2} dx = \frac{1}{6}$$

$$E(Y) = \frac{1}{6}$$

Variance can be calculated as $V(Y)$

$$V(Y) = E(Y^2) - [E(Y)]^2$$

$$E(y^2) = \int_0^1 y^2 f(y) dy = \int^1 yy^2 \frac{y}{2} dy = \frac{1}{8}$$

$$V(Y) = \frac{1}{8} - \left(\frac{1}{6}\right)^2 = \frac{1}{8} - \frac{1}{36} = \frac{9-2}{72} = \frac{7}{72}$$

$$E(XY) = \int_0^1 \int_0^1 xyf(x,y) dxdy = \int_0^1 \int_0^1 x^2 y^2 dxdy$$

$$= \int_0^1 x^2 \left[\frac{y^3}{3}\right]_0^1 = \frac{1}{9}$$

Conditional PDF and Conditional Expectation are,

$$f\left(\frac{x}{y}\right) = \frac{f(x,y)}{f(y)} = \frac{xy}{y/2} = 2x$$

$$E\left(\frac{x}{y}\right) = \frac{E(XY)}{E(Y)} = \frac{1/9}{1/6} = \frac{2}{3}$$

Since,

$$f(x,y) \neq f(x) f(y)$$

$$xy \neq \frac{x}{y} . \frac{y}{2}$$

$\therefore$ x and y are dependent Random Variable

Example: If x and y are 2-D discrete R.V, the corresponding probability function is

xy	- 1	0	+ 1
-1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Find $P\left(\frac{x+y=2}{x}\right)$

Solution:

$$P\left(\frac{x+y=2}{x-y=0}\right) = \frac{P(x+y=2 \cap x-y=0)}{P(x-y=0)}$$

$$P\left(\frac{x+y=0}{x-y=0}\right) =$$

$$\frac{P(x=1, y=1)}{P(x=-1, y=-1) + P(x=0, y=0) + P(x=1, y=1)}$$

$$P\left(\frac{x+y=2}{x-y=0}\right) = \frac{1}{4} - \frac{1-1-1}{4+2+4} = \frac{1}{4}$$

Binomial distribution

Suppose that a trial or an experiment, whose outcome can be classified as either a success or a failure is performed. As an example, when we toss a coin we can call occurrence of Head as success and occurrence of tail as failure.

Suppose now that n independent trials, each of which results in a success with probability p and in a failure with probability 1-p are to be performed. If X represents the number of successes that occur in the n trials, then X is said to be binomial random variable with parameters (n, p).

The Binomial distribution occurs when the experiment performed satisfies the three assumptions of bernoulli trials, which are:

1. Only 2 outcomes are possible, success and failure
2. Probability of success (p) and failure (1-p) remains the same from trial to trial
3. The trials are statistically independent. i.e. The outcome of one trial does not influence subsequent trials. i.e., No memory.

The Probability Mass Function for x-success in n-trials is,

$P(x) = {}^nC_x p^x (1-p)^{n-x}$

Properties

$$E(x) = \text{mean} = np$$

$$V(x) = \mu_2 = npq$$

$$\mu_s = npq(q-p) = npq(1-2p)$$

$$\text{Skewness, } \beta_1 = \frac{\mu^{3^2}}{\mu_2} = \frac{n^2p^2q^2(1-2p)^2}{n_3p_3q_3} = \frac{(1-2p)^2}{npq}$$

Moment generating function

$m_x(t) = E(e^{tx}) = (q+pe^t)^n$

Characteristic Function

$\phi_x = E(e^{itx}) = (q+pe^{it})^n$

Note:

For $p = \frac{1}{2}$, then the binomial distribution is symmetric ($\mu_3 = 0$).

For $p < \frac{1}{2}$, $\Rightarrow \mu_3$ is positive, then the curve is positively skewed.

For $p < \frac{2}{2}$, $\Rightarrow \mu_3$ is negative, then the curve is negatively skewed.

Sum of the independent binomial Random Variables is also a Binomial Random Variable.

The moment generating function is used to find addition and difference between Random Variable with their probability function.

The characteristic function is used for finding the convolution between the Random Variable and ratio between the Random Variable.

Solved Examples

Example: Find the probability of getting a 9 exactly twice in 3 times with a pair of dice?

Solution: Number of Trials, n = 3

Number of Success, x = 2

The sum can be 9 for the following combinations (3, 6); (4, 5); (5,4); (6, 3)

$$P(\text{sum}=9) = \frac{4}{36} = \frac{1}{9}$$

$$q = \left(1 - \frac{1}{9}\right) = \frac{8}{9}$$

$$P(x=2) = {}^3C_2 \left(\frac{1}{9}\right)^2 \left(\frac{8}{9}\right) = \frac{8}{243}$$

Example: The probability of a man hitting the target is $\frac{1}{3}$.

(i) If he fires 5 times, what is the probability of his hitting the target at least twice.

(ii) How many times must he fire so that the probability of his hitting the target at least once is more than 90%.

Solution: The probability of success is, $p = \frac{1}{3}$

Thus, the probability of failure is, $q = (1-p) = \frac{2}{3}$

(i) Number of Trials, n = 5

$$P(x \geq 2) = 1 - P(x <_5 2) =$$

$$1 - \left[\left[P_1(x =_4 0) + P(x=1)\right]\right]$$

$$P(x \geq 2) = 1 - \left[\left(\frac{2}{3}\right) + {}^5C_1\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\right] = \frac{131}{243}$$

(iI) $P(x \geq 1) > 90\%$

$$1 - P(x=0) > 0.9$$

$$P(x=0) < 0.1$$

$${}^nC_0\, P_n^{\,0} q^n < 0.1$$

$$\left(\frac{2}{3}\right) < 0.1$$

$$n > 5.67 = 6$$

Example: Two dice are rolled 120 times. Find the average number of times in which the number on the first dice exceeds the number on 2nd dice.

Solution: Number of Trials, n = 120

The cases in which number of first dice exceeds the number on 2nd dice

(2, 1); (3, 1); (3, 2); (4, 1); (4, 2); (4, 3); (5, 1); (5, 2); (5, 3); (5, 4); (6, 1); (6, 2); (6, 3); (6, 4); (6, 5);

Thus, there are 15 cases out of 36 in which we have success

Probability of Success,

$$P = \frac{15}{36} = \frac{5}{12}$$

$$E(x) = np = 120 \times \frac{15}{36} = 50$$

Example: If x and y are the binomial Random Variables x is B(2, P), y is B(4, P) if $P(x \geq 1) = \frac{5}{9}$ find $P(y \geq 1)$?

Solution: The number of trials for both experiments,

$$n_x = 2 \text{ and } n_y = 4$$

$$\text{Given}, P(x \geq 1) = \frac{5}{9}$$

$$P(x \geq 0) = 1 - P(x = 0) = \frac{5}{9}$$

$$\text{Thus}, P(x = 0) = \frac{4}{9}$$

In terms of Binomial Random Variable,

$$P(x = 0) = {}^nC_0 p^0 q^n$$

$$\text{Therefore}, q^n = \frac{4}{9}$$

$$\text{For 2 trials}, q^2 = \frac{4}{9}$$

$$q = \frac{2}{3} \Rightarrow P = \frac{1}{3}$$

$$P(y \geq 1) = 1 - P(y = 0) = 1 - q^n = 1 - \left(\frac{2}{3}\right)^4$$

$$= 1 - \frac{16}{81} = \frac{65}{81}$$

Example: If x is a binomial Random Variable, then find the value of $\sum_{x=0}^{n} \frac{x}{n} {}^nC_x p^x q^{n-x}$

Solution: $\sum_{x=0}^{n} \frac{x}{n} {}^nC_x p^x q^{n-x} = \frac{1}{n} \sum_{x=0}^{n} x \, {}^nC_x p^x q^{n-x}$

In terms, of Binomial Random Variable,

$$\frac{1}{n} \sum_{x=0}^{n} x \, {}^nC_x p^x q^{n-x} = \frac{1}{n}\left[\sum_{x=0}^{n} x.P(x)\right] = \frac{1}{n} E(x) = \frac{1}{n} np = p$$

Example: If x is a binomial Random Variable and E(x) = 4, V(x) = $\frac{4}{3}$. Find P (x ≤ 2)

Solution:

$$E(x) = np = 4$$

$$V(x) = npq = 4q = \frac{4}{3}$$

$$\therefore q = \frac{1}{3}$$

Thus, Probability of Success, p = $\frac{2}{3}$

np = 4

Thus, the number of trials, $n = \frac{3}{2} \times 4 = 6$

$$P(x \leq 2) = P(x = 0) + P(x = 1) + P(x = 2) = \left(\frac{1}{3}\right)^6$$

$$+ {}^6C_1 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right) + {}^6C_2 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)2$$

$$P(x \leq 2) = \frac{1}{729}[1 + 12 + 60] = \frac{73}{729}$$

Example: There are 10 markers on a table, of which 6 are defective and 4 are not defective. If 3 are randomly taken from the above lot, what is the probability that exactly 1 of markers is defective?

Solution: Random Variable, X represents the number of defected markers from the selected lot. D is defective and ND is non defective.

$$P(x = 1) = \frac{6C_1 \times 4C_2}{10C_3} = 0.3$$

X can now take the values 0, 1, 2 or 3.

$$P(X = x) = \frac{6C_x \times 4C_{3-x}}{10C_3}$$

From the above formula, we can calculate the following:

$$P(X = 1) = \frac{6C_1 \times 4C_2}{10C_3}$$

$$P(x \geq 1) = P(x = 0) + P(x = 1) = \frac{6C_0 \times 4C_3}{10C_3}$$

$$+ \frac{6C_1 \times 4C_2}{10C_3}$$

$$P(x \geq 1) = 1 - P(x = 0) = 1 - \left[\frac{6C_0 \times 4C_3}{10C_3}\right]$$

Poisson Distribution

If a Random Variable can take the values from the set of Natural Numbers, then it is modeled as a Poisson Random Variable. It can be used to model the,

- Arrival Rate
- Rare occurrence
- Defect probability
- Evolutionary process

If x is a Poisson Random Variable defined in the interval 0 →∞ with a parameter λ (>0) and its probability mass

$$\text{function is } P(x:\lambda>0)=P(x)=\begin{cases}\dfrac{e^{-\lambda x}}{x!} \\ 0\end{cases}$$

$\lambda > 0$

$0 \le x \le 0$ other wise

Conditions for Poisson Random Variable

- Observation are infinitely large $(n \to \infty)$
- Probability of success is very small $(p \to 0)$
- $np = \lambda \Rightarrow p = \dfrac{\lambda}{n}$
- $p(x:np) = \dfrac{e^{-np}(np)^x}{x!}$

Poisson Random Variable is an approximation of Binomial Random Variable as the number of observations or trials tend to infinite.

Properties

- E(x) = mean = λ
- V(x) $\mu_2=\lambda$
- $\mu_3=\lambda$
- Skewness, $\beta_1 = \dfrac{\mu_3^{\ 2}}{\mu_2^{\ 3}} = \dfrac{\lambda^2}{\lambda^3} = \dfrac{1}{\lambda}$
- Moment Generating Function, $m_x(t) = e^{\lambda(e^t-1)}$
- Characteristic Function, $\phi_x(t) = e^{\lambda(eit-1)}$

Note:

- In Poisson distribution mean = variance = parameter = λ
- It is always positively skewed.
- Sum of the independent Poisson Random Variable is also Poisson Random Variable
- The difference between the independent Poisson Random Variable is not a Poisson Random Variable.

Solved Examples

Example: A telephone switchboard receives 20 calls on an average during an hour. Find the probability that for a period of 5 minutes

i) no call is received

ii) exactly 3 calls are received

iii) Atleast 2 calls are received

Solution: Average number of calls in a period of 5 minutes,

$\lambda = \dfrac{20\times5}{60} = 1.67$

i) $P(x=0 = \dfrac{e^{-1.67}(1.67)^0}{0!} = e^{-1.67})$

ii) $P(x=3) = \dfrac{e^{-1.67}(1.67)^3}{3!}$

iii) $P(x\ge2) = 1-P(x<2) = 1-$

$[P(x=0)+P(x=1)]$

$P(x\ge2) = 1-\left[e^{-1.65}+(1.65)e^{-1.65}\right] = 0.491$

Example: If x and y are two independent Poisson Random Variable such that P(x=1)=P (x=2)=P & P (y=2)=P (y=3), Find V (3x-4y)?

Solution: It is given that, P(X = 1) = P (X = 2)

Thus, $\dfrac{e^{-\lambda 1}}{1!} = \dfrac{e^{-\lambda 2}}{2!}$

$1 = \dfrac{\lambda}{2} => \lambda = 2$

Also, $P(Y=2) = P(y=3)$

Thus, $\dfrac{e^{-\theta}\theta^2}{2!} = \dfrac{e^{-\theta}\theta^3}{3!}$

$1 = \dfrac{\theta}{3} => \theta = 3$

Thus, $E(X) = V(X) = 2$ and $E(Y) = 3$

$V(3x-4y) = 3^2V(x)+(-4)^2V(y)$

$= 9\times2+16\times3 = 66$

Example: If x_1 and x_2 are two independent Random Variables with variances 1, 2. Find $P(x_1 + x_2 = 4)$

Solution:

$$P(x_1+x_2=k) = \frac{e^{-\lambda_1\lambda_2}\left(\lambda_1+\lambda^{2k}\right)}{k!}$$

Since sum of Poisson Random Variable is a Poisson Random Variable

$\lambda_1 = 1, \lambda_2 = 2$

$$P(x_1 + x_2 = 4) = \frac{e^{-(1+2)}(1+2)^4}{4!} = \frac{e^{-3}3^4}{4!}$$

Example: If x is a Poisson Random Variable then find the value of $\sum_{x=0}^{\infty} \frac{x}{\lambda} \frac{e^{-\lambda}\lambda^x}{x!}$

Solution:

$$\sum_{x=0}^{\infty} \frac{x}{\lambda} \frac{e^{-\lambda}\lambda^x}{x!} = \frac{1}{\lambda}\sum_{x=0}^{\infty} \frac{xe^{-\lambda}\lambda^x}{x!} = \frac{1}{\lambda}\sum_{x=0}^{\infty} xP(x)$$

$$\sum_{x=0}^{\infty} \frac{x}{\lambda} \frac{e^{-\lambda}\lambda^x}{x!} = \frac{1}{\lambda}E(x) = \frac{1}{\lambda} \times \lambda = 1$$

Example: If x is a Poisson Random Variable and $E(x^2) = 6$ find V(x).

Solution: Variance can be expressed as,

$$V(x) = E(x^2) - [E(x)^2]$$

$$\lambda = 6 - \lambda^2 \qquad \because v(x) = E(x) = \lambda$$

$$\lambda^2 + \lambda - 6 = 0$$

$$(\lambda + 3)(\lambda - 2) = 0$$

$$\lambda = -3, 2$$

$$\lambda = 2 \Rightarrow v(x) = 2$$

(Because variance can never be a negative value)

Uniform Distribution

In general we say that X is a uniform random variable on the interval (a, b) if its probability density function is given by:

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{if } \alpha < x < \beta \text{ otherwise} \\ 0 \end{cases}$$

Since f(x) is a constant, all values of x between α and β are equally likely (uniform).

$$\text{Mean} = E[X] = \frac{(\beta + \alpha)}{2}$$

$$\text{Variance} = v(X) = \frac{(\beta + \alpha)^2}{12}$$

Solved Examples

Example: If X is uniformly distributed over (0, 10), calculate the probability that:

a) X < 3

b) X > 6

c) 3 < X < 8

Solution: The value of Probability Density Function is given by,

(i) $P\{x < 3\} = \int_0^3 \frac{1}{10}dx = \frac{3}{10}$

(ii) $P\{x < 6\} = \int_6^{10} \frac{1}{10}dx = \frac{4}{10}$

(iii) $P\{3 < X < 8\} = \int_3^8 \frac{1}{10}dx = \frac{1}{2}$

Exponential Distribution

A continuous random variable whose probability density function is given for some $\lambda > 0$ by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Is said to be exponential random variable with parameter λ. The cumulative function F(a) of an exponential random variable is given by:

$$F(a) = P(x \leq a) = \int_0^a \lambda e^{-\lambda x}dx$$

$$= (e^{-\lambda x})_0^a = 1 - e^{-\lambda a} \quad a \geq 0$$

$$\text{Mean} = E[X] = \frac{1}{\lambda}$$

$$\text{Variance} = v(x) = \frac{1}{\lambda^2}$$

Solved Examples

Example: Suppose that the length of a phone call in minutes is an exponential random variable with parameter $\lambda = \frac{1}{10}$. If someone arrives immediately ahead of you at a public telephone booth, find the probability that you will have to wait.

(a) More than 10 times

(b) Between 10 and 20 minutes

Solution: Letting X denote the length of the call made by the person in the booth, we have the desired probabilities:

(a) $P\{x > 10\} = 1 - P(x < 10) = 1 - F(10)$

$$= 1 - \left(1 - e^{-\lambda \times 10}\right)$$

$$P\{X > 10\} = e^{-10\lambda} = e^{-1} = 0.368$$

(b) $P\{10 < X < 20\} = F(20) - F(10)$

$$= \left(1 - e^{-20\lambda}\right) - \left(1 - e^{-10\lambda}\right)$$

$$P\{10 < X < 20\} = e^{-1} - e^{-2} = 0.233$$

Normal Gaussian Distribution

It is called parent distribution because it covers the entire set of natural numbers.

If x is said to be a normal Random Variable defined in (-∞,∞) with mean = μ and variance = σ^2, then its density function is

$$N\left(x\mu : \sigma^2,\right) = f(X) = \begin{cases} \dfrac{1}{\sigma\sqrt{2\pi}\, e} - \dfrac{1}{2}\left(\dfrac{x-\mu}{\sigma}\right)^2 \\ 0 \end{cases}$$

$$-\infty < X < +\infty$$

$$-\infty < \mu < +\infty$$

$$0 < \sigma < \infty$$

otherwise

Standard Normal Random Variable

If x is a normal Random Variable with mean = 0 and variance = 1, then the Random Variable is known as standard normal Random Variable and its density function is

$$N(0,1) = f(X) = \frac{1}{\sqrt{2\pi^e}} - \frac{1}{2}x^2$$

Mathematically the standard normal variable is denoted by 'z' and is given by.

$$z = \frac{x - E(x)}{\sqrt{V(x)}}; -3 \le z \le +3$$

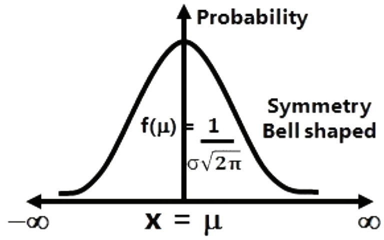


Fig. 5.2

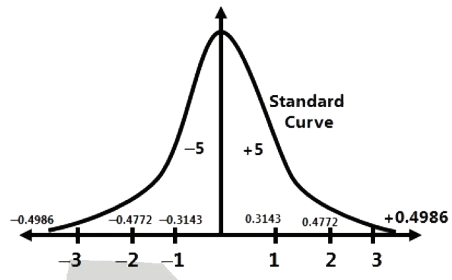


Fig. 5.3

Areas under Normal curve

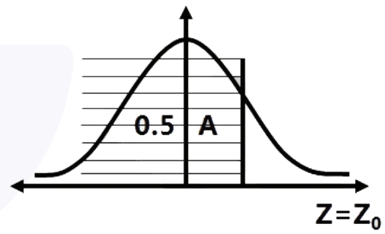


Fig. 5.4

$$P\left(Z \le Z_0\right) = 0.5 + A\left(0 < Z < Z_0\right)$$

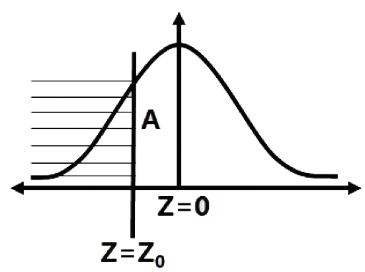


Fig. 5.5

$$P\left(Z \le Z_0\right) = 0.5 - A\left(Z_0 < Z < Z0\right)$$

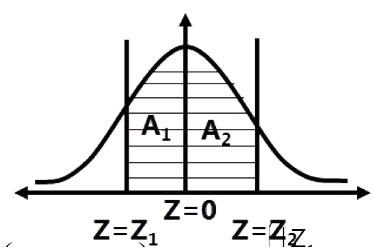


Fig. 5.6 Normal Probability distribution

$$P\left(Z_1 \le Z \le Z_0\right)^1 = A1 + A2 \begin{cases} Z_1 - ve \\ Z_2 + ve \end{cases}$$

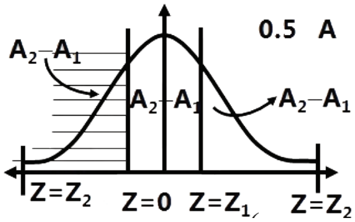


Fig. 5.7

$$P\left(Z_1 \le Z \le Z_0\right) = A_1 - A_2 \begin{cases} Z_1 \text{ \& } Z_2 \\ \left(+\text{ve or} - \text{v}^2\text{e}\right) \end{cases}$$

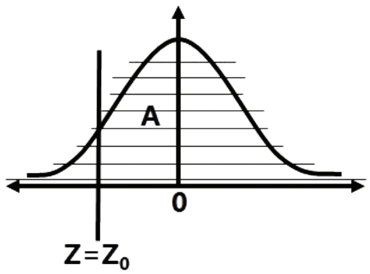


Fig. 5.8

$$P\left(Z \le Z_0\right) = 0.5 + A\left(Z_0 < Z < 0\right)$$

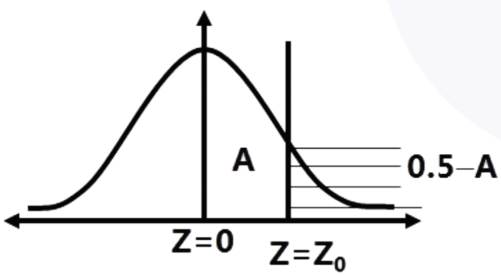


Fig. 5.9

$$P\left(Z \le Z_0\right) = 0.5 - A\left(0 < Z < Z_0\right)$$

Properties

- $E(x)$ = mean $= \mu$
- $V(x) = \mu 2 = \sigma^2$
- Third Central Moment, $\mu_3 = 0$
- $\beta_1 = 0$, hence symmetric

Moment Generating Function: $m_x(t) = \dfrac{e^{tu + t^2\sigma^2}}{2}$

Characteristic function: $\phi_x(t) = \dfrac{e^{itu - t^2\sigma^2}}{2}$

If X is a standard Normal Variable,

$$m_x(t) = e\frac{t^2}{2} \text{ and } \phi x(t) = e - \frac{t^2}{2}$$

Sum of the independent Random Variable is also a normal Random Variable

The difference between normal Random Variable is also a normal Random Variable.

Solved Examples

Example: If 6 is distributed with S.D = 3.33, μ = 20, find the probability that 21.11 and 26.66. The area under the curve Z=0 to Z=0.33 is 0.1293 and The area under the curve Z=0 to Z=2 is 0.4772?

Solution: We need to determine P(21.11 ≤ x ≤ 26.66)

$z = 0 \text{ to } z = 0.33 \text{ is } 0.1293$

$z = 0 \text{ to } z = 2 \text{ is } 0.4772$

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{21.11 - 20}{3.33} = \frac{1.11}{3.33} = 0.33$$

$$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{26.66 - 20}{3.33} = \frac{6.66}{3.33} = 2$$

$$P\left(21.11 \le x \le 26.66\right) = P\left(0.33 \le Z \le 2\right) =$$

$$P\left(0 \le x \le 2\right) - \left(0 \le Z \le 0.33\right)$$

$$P\left(21.11 \le x \le 26.66\right) = 0.4772 - 0.1293 = 0.3479$$

Example: A die is rolled 180 times, using the normal distribution find the probability that the face 4 will turn up at least 35 times. The area under the normal curve z = 0 to z = 1 is 0.3413

Solution: Number of Trials, n = 180

The probability of success, p = $\dfrac{1}{6}$

The probability of failure, q= $\dfrac{5}{6}$

Mean of the distribution = $np = \dfrac{1}{6} \times 180 = 30$

Variance of the distribution =

$$npq = \frac{1}{6} \times 180 \times \frac{5}{6} = 25$$

The variable z is,

$$= \frac{X - \mu}{\sigma} = \frac{35 - 30}{\sqrt{25}} = \frac{5}{5} = 1$$

$$P\left(X \ge 35\right) = P\left(Z \ge 1\right) = 0.5 - P\left(0 \ge Z \ge 1\right)$$

$$P\left(X \ge 35\right) = 0.5 - 0.3413 = 0.1587$$

Example: If x is normally distributed with mean = 30 and standard deviation = 5. Find P $(|x - 30| > 5)$ given that

P(0 ≤ z ≤ +1) = 0.3413?

$$P(|x-30|>5)=1-P(|x-30|\le 5)$$
$$P(|x-30|<5)=P(-5\le x-30\le 5)$$
$$=P(25\le x\le 35)$$
$$Z_1=\frac{x_1-\mu}{\sigma}=\frac{25-30}{5}=-1,$$
$$Z_2=\frac{35-30}{5}=1$$
$$P(25\le x\le 35)=P(-1\le z\le +1)$$
$$=0.3413+0.3413=0.6826$$
$$P(|x-30|>5)=1-P(|x-30|\le 5)$$
$$=1-0.6826=0.3174$$

Correlation and Regression

Correlation: The relation between the 2-D Random Variable in bivariable data is known as correlation. That means the changes in one variable is affecting the changes in other variable in parallel, then those variables are known as correlated variable.

Types of correlation

1. +ve correlation
2. -ve correlation
   a) If the changes in both variable are in same direction i.e. either both increasing or both decreasing then these variables are known as positively correlated variables.
   b) If the changes in the one variable are affecting the changes of the other variable parallel in the reverse direction i.e. increase in one variable causes decrease in the other then those variables are known as negatively correlated variables.

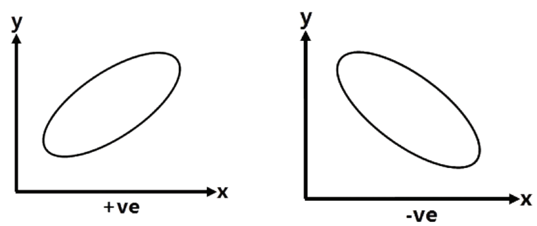


Fig. 5.10

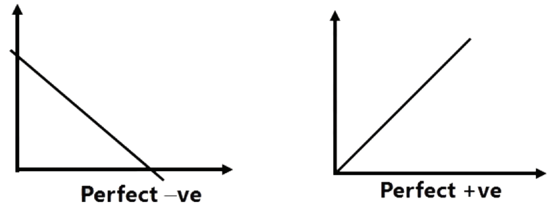


Fig. 5.11

Karl Pearson's Correlations

$$r(X,Y)=\frac{\operatorname{cov}(X,Y)}{\sigma_X\sigma_Y}\ \text{Such that } -1\le r\le 1$$

Where $\operatorname{cov}(X,Y)=E(XY)-E(X)E(Y)$

Scatter Diagram

It is a graphical representation of correlation. If the points are very closer and very thick on the x-y plane, then their points are known as correlated points.

If the points are widely separated, then they are said to be uncorrelated.

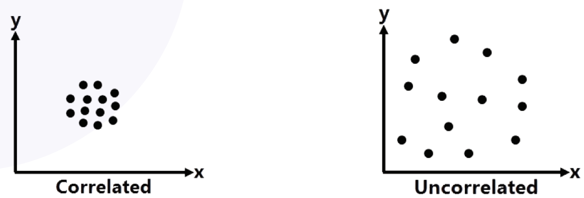


Fig. 5.12

If the two Random Variables are independent $\Rightarrow \operatorname{cov}(x, y) = 0$ $r(x, y) = 0$

They are highly uncorrelated.

Regression

The linear relationship between the 2-D Random Variables in bivariable data is known as regression. Lines of Regression

- Y on X

$$(y-\bar{y})=r\times\frac{\sigma_y}{\sigma_x}(x-\bar{x})$$

- X on Y

$$(x-\bar{x})=\gamma\times\frac{\sigma_x}{\sigma_y}(y-\bar{y})=b_{xy}(x-\bar{x})$$

Properties

Correlation coefficient is the geometric mean between

$$b_{yx}.b_{xy}=r\times\frac{\sigma_y}{\sigma_x}\times\gamma\frac{\sigma_x}{\sigma_y}=r^2 \qquad \therefore r=\pm\sqrt{b_{xy}\times b_{yx}}$$

Note:

- Both the regression coefficient must have the same sign.

  i.e if both are positive $\Rightarrow$ r is positive Both are negative $\Rightarrow$ r is negative
- If $b_{yx} > 1 \Rightarrow b_{xy} < 1$ & vice versa
- If the regression coefficient are equal $\Rightarrow$ variance also equal

$$b_{xy} = b_{yx} \Rightarrow r \times \frac{\sigma_y}{\sigma_x} = r\frac{\sigma_x}{\sigma_y}$$

$$\sigma_y^{\,2} = \sigma_x^{\,2}$$

- Regression lines pass through the points

$$\bar{x}, \bar{y}\; \theta = \tan^{-1}\left[\frac{1-r^2 \;\; \sigma_x \sigma y}{r\,\sigma_x^{\,2} + \sigma_y^{\,2}}\right]$$

r = 0 $\Rightarrow \theta\frac{\pi}{2}$. Thus, both lines are perpendicular.

r = 1 $\Rightarrow \theta = \pi$. Thus, both lines are parallel.

Solved Examples

Example: The regression equations are

$$x + 2y = 0 \;\&\; 2 + y = 1. \quad \text{Find i) } r \quad \text{ii) } \bar{x}\,\bar{y}$$

Solution: The equation for regression line of Y on X is, x + 2y = 0

$$2y = -x => y = -\frac{x}{2}$$

$$\text{Thus, } b_{yx} = -\frac{1}{2}$$

The equation for regression line of X on Y is, 2x + y = 1

$$2x = 1 - y => x = \frac{1}{2} - \frac{y}{2}$$

$$\text{Thus, } b_{xy} = -\frac{1}{2}$$

$$r = -\sqrt{\frac{1}{2} \times \frac{1}{2}} = -\frac{1}{2}$$

To determine the mean values, we will find the intersection points of both lines. Solving both equations we get,

$$\left(\bar{x}, \bar{y}\right) = \left(\frac{2}{3}, -\frac{1}{3}\right)$$

Example: The equation for the two regression lines are,

$$x - 3y = 4 \;(y \text{ on } x)$$

$$2x - y = 1 \;(x \text{ on } y)$$

Find i) r ii) $\bar{x}, \bar{y}$

Solution: The regression line for Y on X is, x - 3y = 4

$$-3y = 4 - x$$

$$y = \frac{-4}{3} + \frac{x}{3}$$

$$\text{Thus, } b_{yx} = \frac{1}{3}$$

The regression line for X on Y is, 2x - y = 1

$$2x = 1 + y$$

$$x = \frac{1}{2} + \frac{y}{2}$$

$$\text{Thus, } b_{xy} = \frac{1}{2}$$

$$r = \sqrt{\frac{1}{3} \times \frac{1}{2}} = \sqrt{\frac{1}{6}}$$

To determin the mean, we again find the intersection point of both lines,

$$\left(\bar{x}, \bar{y}\right) = \left(\frac{1}{5} - \frac{7}{5}\right)$$

Chebyshev's Inequality

In probability theory, Chebyshev's inequality says that the fraction of data in any distribution that lies within k standard deviations of the mean is at least $1 - \frac{1}{k^2}$

Let x be a random variable with finite expected values μ and finite non – zero variance σ^2 . Then for any real number k > 0

$$P\left(\mu - k\sigma \le x \le \mu + k\sigma\right) \ge 1 - \frac{1}{k^2}$$

$$P\left(\left|x - \mu\right| \le k\sigma\right) \ge 1 - \frac{1}{k^2} \qquad \text{for } k > 0$$

$$\text{This is also leads to, } P\left(\left|x - \mu\right| > k\sigma\right) \le \frac{1}{k^2}$$

Chapter Summary

Types of events

- **Complementary events**

  {E^C} = {S} = {E}

  The complement of an event E is set of all outcomes not in E.

- **Mutually Exclusive Events**

  Two events E & F are mutually exclusive iff $P\ (E \cap F)$ =0

- **Collectively exhaustive events**

  Two events E & F are mutually exhaustive $(E \cup F)$ = S Where S is sample space.

- **Independent events**

  If E & F are two independent events

  $P\ (E \cap F) = P(E)*P(F)$

De Morgan's Law

- $\left(\bigcup_{i-1}^{n} E_i\right)^c = \bigcap_{i-1}^{n} E^c{}_i$
- $\left(\bigcap_{i-1}^{n} E_i\right)^c = \bigcup_{i-1}^{n} E^c{}_i$

Axioms of Probability

$E_1 = E_2 = \ldots\ldots\ldots\ldots\ldots\ E_n$ are possible events & S is the sample space

a) $0 \le P(E) \le 1$

b) $P(S)=1$

c) $P \quad \left(\bigcup_{i-1}^{n} E_1\right) = \sum_{i-1}^{n} P(E_i)$

for mutually exclusive events

Some important rules of probability

$P(A \cup B)=P(A)+P(B)-P(A \cap B)$

$P(A \cap B)=P(A)*P(B \mid A)=P(B)*P(A \mid B)$

$P(A \mid B)$ is conditional probability of A given B

If A & B are independent events

$P(A \cap B) = P(A)*P(B)$

$P(A \mid B)=P(A)$

$P(B \mid A)=P(B)$

Total Probability Theorem

$$P(A \cap B)=P(A \cap E)+P(B \cap E)$$
$$=P(A)*P(E \mid A)+P(B)*P(E \mid B)$$

Baye's Theorem

$$P(A \mid E)=P(A \cap E)+P(B \cap E)$$
$$= P(A)*P(E \mid A)+P(B)*P(E \mid B)$$

Statistics

- Arithmetic Mean of Raw data

  $$\bar{x} = \frac{\sum x}{n}$$

  $\bar{x}$ = arithmetic mean; x = value of observation, n = number of observation

- Median of Raw data

  Arrange all the observation in ascending order

  $x_1 < x_2 < \ldots\ldots\ldots\ldots\ldots\ldots < xn$

  If n is odd, median

  If n is odd, median = $\frac{(n+1)}{2}$ th value

  If n is even, Median =

  $$\frac{\left(n/2\right)^{th} \text{value} + \left(n/2 + 1\right) \text{th value}}{2}$$

Mode of Raw data.

Most frequently occurring observation in the data.

Properties of discrete distributions

a) LP(x)=1

b) $E(X)$ = LxP(x)

c) V(x) =E(x2) - E(x)}

Properties of continuous distributions

- $^{ID} f(x)dx = 1$
- $F'(x) = J f\{x\}dx \quad = \text{cumulative distributiobn}$
- $E(x) = :if(x)dx \quad = \text{expected value of } x$
- $V(x) = E(x^2) - [E(x)]^2 \quad = \text{variance of } x$

- **Properties Expectation & Variance**

  $E(ax+b) = aE(x)+b$

  $V(ax+b) = a^2V(x)$

  $E(ax_1+bx_2) = aE(x_1)+bE(X_2)$

  $V(ax_1+bx_2) = a^2V(x_1)+b^2V(x_2)$

  $Cov(x,y)=E(x\,y)-E(x)E(y)$

- **Binomial Distribution**

  No of trials =n

  Probability of failure = (1-P)

  $P(X=x)={}^nC_xP^x(1-P)^{n-x}$

  Mean = $E(X)=nP$

  Variance = $V[x]=nP(1-P)$

Poisson Distribution

A random variable x, having possible values 0, 1, 2, 3, ..., is Poisson Variable is

$$P(X=x)=\frac{e^{-\lambda}x}{x!}$$

$$\text{Mean}=E(x)=\lambda$$

$$\text{Variance}=V(x)=\lambda$$

Continuous Distributions

Uniform Distribution

$$f(x)=\begin{cases}\frac{1}{b-a} \text{ if } a\le x\le b\\ 0 \quad \text{otherwise}\end{cases}$$

$$\text{Mean}=E(x)=\frac{b=a}{2}$$

$$\text{Variance}=V(x)=\frac{(b-a^2)}{12}$$

Exponential Distribution

$$f(x)=\begin{cases}e^{-\lambda x} \text{ if } x\ge 0\\ 0 \text{ if } x<0\end{cases}$$

$$\text{Mean}=E(x)=\frac{1}{\lambda}$$

$$\text{Variance}=V(x)=\frac{1}{\lambda^2}$$

Normal Distribution

$$f(x)=\frac{1}{\sqrt{2}}\exp\left(\frac{-(x-\mu)}{2\sigma^2}\right), -\infty<x<\infty$$

$$\text{Mean}=E(x)=\mu$$

$$\text{Variance}=V(x)=\sigma^2$$

Coefficient of Correlation

$$P=\frac{\text{cov}(x,y)}{\sqrt{\text{var}(x)\text{var}(y)}}$$

x & y are linearly related, if $p=\pm1$

x & y are un–correlated if $p=0$

Regression lines

- $(x-\bar{x})=b_{xy}(x-\bar{y})$
- $(x-\bar{y})=b_{xy}(x-\bar{x})$

Where $\bar{x}$ & $\bar{y}$ are mean values of x & y respectively

$$b_{xy}=\frac{\text{cov}(x,y)}{\text{var}(y)};\ b_{yx}=\frac{\text{cov}(x,y)}{\text{var}(x)}$$

$$P=\sqrt{b_{xy}b_{yx}}$$

Objective Questions of ESE (Prelims) EE

1. A fair dice is rolled twice. The probability that an odd number will follow an even number is

(A) $\frac{1}{2}$

(B) $\frac{1}{4}$

(C) $\frac{1}{3}$

(D) $\frac{1}{6}$

2. A urn contain 5 red ball and 5 black balls. In the first draw, one ball is picked at random and discarded with noticing its colour. The probability to get a red ball in the second draw is.

(A) $\frac{1}{2}$

(B) $\frac{6}{9}$

(C) $\frac{5}{9}$

(D) $\frac{4}{9}$

3. The probability that two friends share the same birth-month is

(A) $\frac{1}{6}$

(B) $\frac{1}{12}$

(C) $\frac{1}{144}$

(D) $\frac{1}{24}$

4. A box contains 5 black and 5 red balls. Two balls are randomly picked one after another form the box, without replacement. The probability for both balls being red is

(A) $\frac{1}{90}$

(B) $\frac{19}{90}$

(C) $\frac{1}{2}$

(D) $\frac{2}{9}$

5. An unbiased coin is tossed three times. The probability that the head turns up in exactly two cases is

(A) $\frac{1}{9}$

(B) $\frac{1}{8}$

(C) $\frac{3}{8}$

(D) $\frac{2}{3}$

6. From a pack of regular playing cards, two cards are drawn at random. What is the probability that both cards will be Kings, if first card is NOT replaced?

(A) $\frac{1}{26}$

(B) $\frac{1}{52}$

(C) $\frac{1}{169}$

(D) $\frac{1}{221}$

7. A single die is thrown twice. what is the probability that the sum is neither 8 nor 9?

(A) $\frac{1}{9}$

(B) $\frac{1}{4}$

(C) $\frac{5}{36}$

(D) $\frac{3}{4}$

8. A box contain two washers, 3 nuts and 4 bolts. Items are drawn from the box at random one at a time without replacement. The probability of drawing 2 washers first followed by 3 nuts and subsequently the 4 bolts is

(A) $\frac{2}{315}$

(B) $\frac{1}{630}$

(C) $\frac{1}{1260}$

(D) $\frac{1}{2520}$

9. 9. An unbiased coin is tossed five times. The outcome of each toss is either a head or a tail. The probability of getting at least one head is:

(A) $\frac{1}{32}$

(B) $\frac{13}{32}$

(C) $\frac{16}{32}$

(D) $\frac{31}{32}$

10. Out of all 2-digit integers between 1 and 100, a 2-digit number has to be selected at random. What is the probability that the selected number is not divisible by 7?

(A) $\frac{13}{90}$

(B) $\frac{12}{90}$

(C) $\frac{78}{90}$

(D) $\frac{77}{90}$

Answer Key

1 – b	**2 – a**	**3 – b**	**4 – d**	**5 – c**
6 – d	**7 – d**	**8 – c**	**9 – d**	**10 – d**

6 Numerical Methods

Objective

Upon completion of this chapter you will be able to:

- Solve Algebraic and Transcendental Equations by Numerical Methods.
- Perform Definite Integral using Numerical Techniques.
- Determine Solution of a Differential Equation by Numerical Methods

Introduction

There are two methods by which a mathematical problem like a differential equation, transcendental or a linear equation can be solved.

Analytical Methods:

The method by which solution of an equation can be directly obtained in terms of coefficients present in the equation.

Like Quadratic formula can be used to find the solution of a quadratic equation. An example of Integration by analytical method is shown below,

$$\int x\,dx = \frac{X^2}{2} = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$$

Numerical Methods:

Here, instead of directly writing the solution in terms of some formulae we perform a stepwise calculation using an algorithm to arrive at the result. These methods are more popular as they can be implemented on computers to solve a wider class of problems than what can be solved by analytical methods. Like, we do not have analytical methods for solution of a polynomial of degree 4 or more. But numerical methods work successfully on such type of problems. The only disadvantage with numerical methods is that exact solutions cannot be obtained and there is some degree of error in the solution.

Causes of Errors

The error in any numerical calculation can be quantified as, Absolute Error = | Exact Value – Approximate Value |

$$\text{Relative Error} = \left[\frac{Exact - Appromiximate}{Exact}\right]$$

Percentage Error =

$$\left[\frac{Exact - Appromiximate}{Exact}\right] \times 100\%$$

There are two main causes of Errors in There are two main causes of errors in numerical methods, which is round-off error and truncation error. Round-off error is mainly due to the limited storage capacity of the device, as we can only save the result a to few significant digits. only a few terms in an infinite series. These errors can be reduced by reducing the tolerance limit and performing more number of iterations.

Thus, there is always a trade-off between the speed of calculation and the accuracy of computation.

Solution of Algebraic and Transcendental Equations

An equation that involves trigonometric functions is called transcendental equation.

Example: $f(x) = x - \cos x = 0$

Roots of Algebraic Equation

- An algebraic equation of nth order can be represented in the form shown below:

  $$f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots\ldots + a_{n-1} x + a_0 = 0$$

  This equation will have n roots. Certain properties of the roots of this algebraic equation are,
- Complex roots always occur in pairs. Therefore, if (a+ib) is a root of the equation then (a-ib) is also a root of the equation.
- Surd roots also occur in pairs. Therefore, if $a + \sqrt{b}$ then $a - \sqrt{b}$ is also the root of the equation.

Descarte's Rule of Signs

An equation f(x)=0 cannot have more positive roots then there are changes of sign in f(x) and cannot have more negative roots then there are changes of sign in f(-x).

Intermediate Mean Value Theorem (I.M.T)

Let f(x) is continuous function defined in the interval [a, b].

f(a) and f(b) having opposite signs (say f(a)<0, f(b)>0), then there exists at least one root f(x)=0 in the interval [a, b].

Example : $f(x) = x^3 - 4x - 9 = 0 \quad [2,3]$

$f(2) = 2^3 - 4 \times 2 - 9 = -9 < 0$

$f(3) = 3^3 - 4(3) - 9 = 6 > 0$

Since f (2) < 0 and f (3) > 0 . So at least one root of $x^3 - 4x - 9 = 0$ is in $[2,3]$

Bisection Method

Let f(x) is continuous function defined in [a, b]

1. Let f(a) < 0 and f(b) > 0

   Using IMT there exists at-least one root f(x) = 0 in [a, b]

2. Let x_1 is first approximation root of f(x) = 0 and $x_1 = \frac{1+b}{2}$

Case 1: If f(x) = 0 then x_1 is root, stop the process

Case 2: If $f(x_1) > 0$, then a root of f(x) = 0 lies in $[x_1, b]$ then compute x_2 using $x_2 = \frac{x_1 + b}{2}$

Case 3: If $f(x_1)$> 0, then a root of f(x)= 0 lies in $[a_1, x]$ then compute x_2 using $x_2 = \frac{a + x_1}{2}$. The length of new interval $[a_1, x]$ or $[x_1, b]$ is exactly half of the previous interval i.e. $\frac{b-1}{2}$

Continue the above process until the desired accuracy of root is found. After, n iterations the length of the interval will be $\frac{b-1}{2^n}$

If $\frac{b-a}{2^n} \leq \varepsilon$ where ε is a small positive number indicating the desired accuracy of root then we will stop the process.

So, the number of steps required to achieve a desired accuracy will be, $n \geq \frac{\log_e \frac{b-a}{\varepsilon}}{\log_e 2}$

Solved Examples

Example: Find x_2 and x_3 using bisection, where $f(x) = x^3 - 4x - 9 = 0$ in $[2, 3]$?

Solution: f (2) = −9 < 0

f (3) = 6 > 0

Let $x_1 = \frac{a+b}{2} = \frac{2+3}{2} = 2.5$

Now $x_2 = \frac{x_1 + b}{2} = \frac{2.5+3}{2} = 2.75$

Now $x_2 = \frac{x_1 + b}{2} = \frac{2.5+3}{2} = 2.75$

$f(x_2) = (2.75)^3 - 4(2.75) - 9 = 0.7069 > 0$

$\therefore x_3 = \frac{x_1 + x_2}{2} = \frac{2.5 + 2.75}{2} = 2.625$

$x_3 = 2.62; \quad x_2 = 2.75 \ \& \ x_3 = 2.62$

Regula-Falsi Method

Let f(x) is continuous function. x_0, x_1 are initial guess values such that $f(x_0)$ and $f(x_1)$ having opposite signs i.e. $f(x_0)$

$f(x_1) < 0$ (say $f(x_0) < 0$ and $f(x_1) > 0$).

Regula-Falsi iteration formula for finding roots of f(x)=0 is

$$x_{n+1} = \frac{f_n x_{n-1} - f_{n-1} x}{f_n - f_{n-1}}$$

In particular for $n = 1$

$$x_2 = \frac{f_1 x_0 - f_0 x_1}{f_1 - f_0}$$

Case 1: If (x_2) = 0, then $x_2 \rightarrow$ root, stop the process

Case 2: If $f(x_2)$< 0 then compute x_3 by replacing x_0 by (x_2) and x_0 by f_2

$$X3 = \frac{f_1x_2 - f_2x_1}{f_1 - f_2}$$

Case 3: If $f(x_1) > 0$ compute x_3 by replacing x_1 by x_2 and f_1 by f_2

$$x_3 = \frac{f_2x_0 - f_0x_2}{f_2 - f_0}$$

Continue the above process the until desired accuracy of root is foun the d. Like bisection method this is also 100% reliablei.e. root will always be found. Both bisection and Regula-falsi method have linear convergence.

Solved Examples

Example: Find x_2 and x_3 using Regula-Falsi Method. $f(x) = x^3 + x - 1 = 0$ $x_0 = 0.5, x_1 = 1$

Solution: $f_0 = f(0.5) = -0.375 < 0$

$f_1 = f(1) = 1 > 0$

Then, x_2

$$= f_1x_0 - f_0x_1 = \frac{1\times(0.5)-(-0.375)(1)}{1-(-0.375)} = 0.636$$

$$f_2 = f(x_2) = f(0.64) = -0.0979 < 0$$

Then, the root will lie in (0.64,1). Next value x3 can be computed as,

$$X_3 = \frac{f_1x_2 - f_2x_1}{f_1 - f_2} = \frac{1\times 0.64-(-0.0979)(1)}{1-(-0.0979)}$$

$$X_3 = 0.672$$

Secant Method

Secant method is similar to Regula-Falsi except that in secant method initial values x_0, x_1 need not satisfy the condition $f(x_0)\ f(x_1) < 0$ ie. These two need not have opposite signs. Secant method does not provide a guarantee for existence of root in the interval $[x_0, x_1]$. So it is unreliable.

Secant method iteration formula for finding roots of the equation of f(x)=0, is x

$$xn = \frac{f_nx_{n-1} - f_{n-1}x_n}{f_n - f_{n-1}}$$

Similar to Regula-Falsi method the formula for calculating x_2 is,

$$X2 = \frac{f_1x_0 - f_0x_1}{f_1 - f_0}$$

In the secant method to compute x_3 replace x_0 by x_1 and x_1 by x_2. Thus, x_3 can be calculated as,

$$X_3 = \frac{f_2x_1 - f_1x_2}{f_2 - f_1}$$

Solved Examples

Example: Find x_2 and x_3 using the secant $f(x) = x^3 - 2x - 5 = 0, x_0 = 2, x_1 = 3$?

Solution: $f_0 = f(2) = -1$

$$f_1 = f(x_1) = f(3) = 16$$

$$X_2 = \frac{f_1x_0 - f_0x_1}{f_1 - f_0} = \frac{16\times 2 + 1\times 3}{16+1} = 2.058$$

$$f_2 = f(x_1) = f(2.058) = 0.3907$$

$$X_3 = \frac{f_2x_1 - f_1x_2}{f_2 - f_1} = \frac{-0.3907\times 3 - 16\times 2.058}{-0.3907 - 16}$$

$$= 2.0804$$

Newton Raphson Method

This method gives the better result as compared to previous two methods. Let x_0 be the initial guess for the root of f(x)=0 and let $x_1 = x_0 + h$ be the correct root such that $f(x_1)=0$.

By Taylor's Series we obtain,

$$f(x) = f(x_0) + hf(x_0) + f(x_0) + \ldots = 0$$

Neglecting second order and higher order derivatives.

$$\text{Then, } x_1 = x_0 - \frac{f'(x_0)}{f(x_0)}$$

Thus, Newton-Raphson formula for finding root of the equation f (x) 0 is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Newton-Raphson method has a second-order or quadratic convergence. Geometrically, in Newton-Raphson method a tangent to the curve y = f(x) is drawn at first root and the intersection of the tangent with x-axis is taken as the second root and same procedure continues till there is a small difference between successive roots.

Different methods of solving algebraic equations have the following order of convergence,

Method	Order of Convergence
Bisection method	1
Regula-falsi method	1
Secant method	1.62
Newton-raphson method	2

Solved Examples

Table 6.1

Example: Find N-R iteration formula for square root of C, where C > 0?

Solution: Let $x = \sqrt{C}$

Squaring both sides

$x^2 = c$

$f(x) = x^2 - c = 0$

$f'(x) = 2x$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x^n)} = x_n - \frac{x_n^2 - C}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + c}{2x_n}$$

Example: Solve by N-R iteration formula till two iterations for f (x)= $x^2 - 117 = 0$ given $x_0 = 10$?

Solution: The function for the Newton-Raphson application can be formed as,

$f(x) = x^2 - 117 = 0$ $f'(x) = 2x$

Newton-Raphson formula is,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n \frac{x_n^2 - 117}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + 117}{2x_n}$$

Since initial guess is, $x_0 = 10$

$$x_1 = \frac{x_0^2 + 117}{2x_0} = \frac{100 + 117}{20} = 10.85$$

$$x_2 = \frac{x_1^2 + 117}{2x_1} = \frac{10.85^2 + 117}{2 \times 10.85} = 10.8167$$

Example: Find the N-R iteration formula for $\sqrt[3]{C}$ where C > 0.

Solution: Let $x = \sqrt[3]{C}$

$f(x) = x^3 - c = 0$

$f(x) = 3x^2$

$$x_{n=1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - C}{3x_n^2} \frac{2x_n^3 + C}{3x_n^2}$$

The iteration formula for $\sqrt{C}$ is,

$$x_{n+1} = \frac{x_n^2 + C}{2x_n}$$

The iteration formula for $\sqrt[3]{C}$ is,

$$x_{n+1} = \frac{2x_n^3 + C}{3x_n^2}$$

Thus, the iteration formula for

$\sqrt[k]{c}$ will be $x_{n+1} + \dfrac{(k-1)x_n^k + C}{kx_n^{k-1}}$

Example: f (x) = $x^2 - 2 = 0$, $x_0 = -1$ then N-R iteration formula will converge to?

Solution: The Newton-Raphson iteration formula for this equation will be,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x^n)} = x_n - \frac{x_n^2 - 2}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + 2}{2x_n}, x_o = -1$$

$$x_1 = \frac{x_0^2 + 2}{2x_n} = \frac{(-1) + 2}{2 \times (-1)} + -1.5$$

$$x_2 = \frac{x_1^2 + 2}{2x_1} = \frac{(-1.5) + 2}{2 \times (-1.5)} = -1.416$$

$$x_3 = \frac{x_2^2 + 2}{2x_2} = -1.414 = -\sqrt{2}$$

If $x_0 = 1$ then it converges to $\sqrt{2}$

If x_0 is not provided then it is converging to $\pm\sqrt{2}$ which are two roots of the given equation. So, the root of which

the method can converge also depends on the initial guess.

Example: If $x_{n+1} = \frac{x^n}{2} + \frac{9}{8x_n}$ and $x_0 = 0.5$ then N-R iteration formula will converge to

Solution: When the method converges, successive iterations will yield identical results i.e.

$$x_{n+1} = x_n = x$$

$$x_{n+1} = \frac{x_n}{2} + \frac{9}{8x_n}$$

$$x = \frac{x}{2} + \frac{9}{8x} => x^2 = \frac{9}{4} => x = \pm\frac{3}{2}$$

$$\therefore x = +1.5, x = -1.5$$

Since x_0 =0.5, $x = 1.5$ as the method converges to the root nearest to the initial guess.

Example: Find x_1 & x_2 using the N-R formula for reciprocal of a, where a =7, x_0 = 0.2?

Solution: Let $x = \frac{1}{a}$. This can also be expressed as $\frac{1}{x}$ = a because for $x = \frac{1}{a} = 0, f''(x)$ doesn't exist.

$$f(x) = \frac{1}{x} - a = 0$$

$$f'(x) = -\frac{1}{x_2}$$

$$x_{n+1} + x_n - \frac{\left(\frac{1}{x} - a\right)}{\left(\frac{-1}{x_n^{\ 2}}\right)} = x_n + x_n^2\left(\frac{1-a}{x_n}\right) = 2x_n - ax_n^2$$

$$x_{n+1} = 2x_n - 1x_n^2$$

$$n = 0, x_1 = 2x_0 - ax_0^2 = 2 \times 0.12 - 7(0.2)^2 = 0.12$$

$$n = 1, x_2 = 2x_1 - ax_1^2 = 2 \times 0.12 - 7(0.12)^2$$
$$= 0.1392$$

Note: NR applicable for a function f (x) only when f''(x) ≠ 0.

Rate of Convergence

The fastness of convergence to the root is called rate of convergence.

Assume x = 2 is the exact root of f (x) = 0. The various results of iterations by different hypothetical methods have been given in the table below,

Iteration	Method 2	Method 3	Method 1
1	3	6	6
2	5	4	5
3	7	2.01	4
4	10		3
5	12		2.01

Table 6.2

In the above table rate of convergence of method 2 is higher than method 3 as it converges to root in lesser number of iterations. In method 1 when iterations are increasing it is moving away from the root. So method 1 is said to be not converging to the root.

Order of Convergence

A method is said to be convergence of order 'p' if $\in_{n+1} = K \in_n^P$ where K is constant.

Suppose if f(x) = 0 and x = 2 is exact root of f(x) = 0.

Method 1:

$$\left\{\begin{matrix} x_n = 2.004\ ;\ \in_n = -0.004 \\ x_{n+} = 2.002\ ;\ \in_{n+1} = -0.002 \end{matrix}\right\} \in_{n+1} = \frac{1}{2} \in \text{.........} (1)$$

Method 2:

$$\left\{\begin{matrix} x_n = 2.004\ ;\ \in_n = -0.004 \\ x_{n+} = 2.000016\ ;\ \in = -0.000016 \end{matrix}\right\}$$

$$\in_{n+1} = k \in_n^2 \text{.........} (2)$$

Note:

- N-R method is better than the remaining methods but it is applicable only for the curves which are having large slope values that is where the graph a crossing x the required root otherwise apply any of the remaining 3 methods.
- Secant method is better than Regula False and bisection. But in the secant method, there is a possibility that iteration formula may be invalid.

Suppose f (x) $= x^2$

$$X_0 = -1 \rightarrow f_0(-1) = (-1)^2 = 1$$

$$X_0 = 1 \rightarrow f_1 = f(1) = 1^2 = 1$$

$$X_2 = \frac{f_1 x_0 - f_0 x_1}{f_1 - f_0}$$

$\therefore f_1 - f_2 = 0$ so the above formula is invalid, so secant method is inapplicable.

Numerical Methods of Integration

The area bounded by f(x) and x-axis between the limits a and b is denoted by $\int_a^b f(x)dx$

Divide the interval (a, b)

into 'n' equal subintervals where the length of each interval is h (step size).

i.e. $[a,b] = \{a = x_0, x_1, x_2, \ldots\ldots\ldots\ldots, x_n = b\}$

where $a = x_0$

$x_1 = x_0 + h$

$x_2 = x_1 + x_0 + 2h$

:

:

$x_n = x_0 + nh$

i.e. $b = a + nh$

$n = \dfrac{b-a}{h}$ or $h = \dfrac{b-a}{n}$

To calculate these integrals, the function f(x) is approximated by a polynomial using the interpolation technique. If f(x) is approximated using Newton's forward difference formula the integral becomes,

$$I = \int_a^b (x)dx = \int_{x0}^{xn} (x)dx$$

$$I = \int_{x0}^{xn} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \ldots\ldots\ldots \right] dx$$

$\Delta y_i = y_{i+1} - y_i$

$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i = y_{i+1} - 2y_i + y_{i-1}$

Similarly, $\Delta^n y_i = \Delta^{n-1} y_{i+1} - \Delta^{n-1} y_i$

Each of the terms above is called as forward difference.

$y_i = f(x_i)$

where $i \in [1, n]$

if $x = x_0 + ph$ t hen $dx = hdp$ then above integral comes out to be,

$$I = nh\left[\begin{array}{l} y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 \\ + \frac{n(n-2)^2}{24}\Delta^3 y_0 \ldots\ldots\ldots \end{array} \right]$$

Trapezoidal Rule

By this rule, the area under the curve is approximated by a trapezoid or multiple trapezoids. Each trapezoid has a width 'h' and the two parallel sides have length y_{i-1} & y_i.

This can also be understood in terms that the curve between two successive points will be approximated by a straight line or linear approximation.

Thus, if we put n=1, the forward differences higher than first will go to zero.

$$I = \int_{x0}^{x1} ydx = h\left[y_0 + \frac{1}{2}\Delta y_0 \right] = h\left[y_0 + \frac{1}{2}y(_1 - y_0) \right]$$

$$= \frac{h}{2}[y_0 + y_1]$$

$$\text{Similarly, } I = \int_{x}^{x_{i+1}} ydx = h\left[y_i + \frac{1}{2}\Delta y_i \right] = \frac{h}{2}[y_i + y_{i+1}]$$

Adding all such integrals we get,

$$\int_a^b (x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \ldots\ldots\ldots + y_{n-1})]$$

Simpson's 1/3 Rule or Simpson's Rule (N divisible by 2)

In this rule, the curve is approximated by n/2 arcs of second order polynomials. So, we consider three points at a time to construct a second order polynomial.

This can be obtained by substituting n=2 in the general formula,

$$I = \int_{x0}^{x2} ydx = 2h\left[y_0 + \Delta y_0 \frac{1}{6}\Delta^2 y_0 \right]$$

$$= \frac{h}{3}[6y_0 + 6(y_1 - y_0) + (y_2 - 2y_1 + y_0)]$$

$$I = \frac{h}{3}[y_0 + 4y_1 + y_2]$$

All forward difference higher than the second are zero.

Generally,

$$I = \int_{x_{i-1}}^{x_{i+1}} ydx = \frac{h}{3}[6y_{i-1} + 6(y_i - y_{i-1}) + (y_{i+1} - 2y_{i-1})]$$

$$= \frac{h}{3}[y_{i-1} + 4y_{i+1}]$$

Summing a|h| such terms we get,

$$\int_a^b f(x)dx = 3\left[\begin{array}{l}(y_0+y_n)+2(y_2+y_4+\ldots\ldots+y_{n-2}) \\ +4(y_1+y_3+y_5+\ldots\ldots+y_{n-1})\end{array}\right]$$

The only limitation with this rule is that we need to divide the interval into even number of subintervals.

Simpson's 3/8 Rule (N divisible by 3)

This rule is obtained by putting n=3 in the general formula. Though this rule is not so accurate as Simpson's 1/3 rule.

$$\int_a^b f(x)dx = \frac{3h}{8}\left[\begin{array}{l}(y_0+y_n)+2 \\ (y_3+y_6+\ldots..+y_{n-3}) \\ +3(y_1+y_2+y_4+y_5\ldots..+y_{n-1})\end{array}\right]$$

Solved Examples

Example: The table below gives values of a function F(x) obtained for values of x at intervals of 0.25. Evaluate

$\int_0^1 f(x)dx$ using Simpson's Rule.

x	0	0.25	0.5	0.75	1
f(x)	1	0.9412	0.8	0.64	0.5

Table 6.3

The value of integral between the limits 0 and 1 using Simpson's Rule is

Solution: By default Simpson's $\frac{1^{rd}}{3}$ rule is called Simpson's Rule.

$$\text{Simpson}\frac{1}{3}\text{Rule}$$

$$= \frac{h}{3}[(y_0+y_4)+2y_2+4(y_1+y_3)]$$

$$= \frac{0.25}{3}[(1+0.5)+2(0.8)+4(0.9412+0.64)]$$

$$= 0.7854$$

Example: The integral $\int_1^3 \frac{1}{x}dx$ when evaluated using Simpson's Rule on two equal intervals with length of each interval is 1, equals

Solution: Evaluating all the subintervals and values of function at the sub-interval boundaries,

x	$\frac{\mathbf{1}}{\mathbf{x}}$
$x_0 = 1$	1 y_0
$x_1 = 2$	1/2 y_1
$x_2 = 3$	1/3 y_2

Table 6.4

$$\text{simpson}\frac{1}{3}\text{Rule} = \frac{1}{3}[(y_0+y_2)+4y_1]$$

$$= \frac{1}{3}\left[1+\frac{1}{3}+\right] = \frac{10}{9}$$

Example: The integral $\int_0^{2\pi}$ sinx dx0 is evaluated using Trapezoidal Rule on 8 equal intervals with '5' significant digits.

Solution: $h = \frac{b-1}{n} = \frac{2\pi-0}{8} = \frac{\pi}{4}$

$$T-\text{Rule} = \frac{h}{2}$$

$$[(y_0+y_8)+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7)]$$

$$= \frac{\pi}{8}[0+0+2(0.7071+1+0.7071-1-0.7071)] = 0$$

5 significant digits are required = 0.00000

x	**sin x**
x_0	y_0
$x_1 = \frac{\pi}{4}$	$y_1 = 0.70710$
$x_2 = \frac{\pi}{2}$	$y_2 = 1$
$x_3 = \frac{3\pi}{4}$	$y_3 = 0.70710$
$x_4 = \pi$	$y_4 = 0$
$x_5 = \frac{5\pi}{4}$	$y_5 = -0.70710$
$x_6 = \frac{6\pi}{4}$	$y_6 = -1$
$x_7 = \frac{7\pi}{4}$	$y_7 = -0.70710$
$x_8 = \frac{8\pi}{4}$	$y_8 = 0$

Table 6.5

Truncation Error

Let f(x) is a function defined in the interval $[x_0, x]$ where $(x \ x_0) = h$.

Expand f(x) about x_0 using Taylor series expansion.

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + (x - x_0)^2$$

$$f''(x_0) + \ldots\ldots + \frac{(x - x_0)^n}{n!} f^n(x_n) + R$$

$$\text{where } R = \frac{(x - x_0)^{n+1}}{(n+1)!} f^{n+1}$$

If we consider only terms upto 'n' then we neglect R. Thus, R will be called as Truncation Error.

Truncation error bound denoted by |R| and

$$|R| \le \left| \max\left(\frac{(x - x_0)^{n+1}}{(n+1)!} f^{n+1}(\phi) \right)_{(x_{0,x})} \right|$$

Here, $\phi \in [x_0, x]$

Solved Examples

Example: If we approximate $e^x = 1 + x + \frac{x^2}{2!}$ then find (i) T.E. (ii) |T. E| in [2, 3]?

Solution: The Taylor's Series for e^x

$$e^x = 1 + x = \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\ldots$$

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + (x - x_0)^2$$

$$f''(x_0) + \ldots\ldots.$$

Comparing $f(x) = e^x$ and $x_0 = 0$

$$f(x_0) = e^0 = 1$$

$$f'(x_0) = e^0 = 1$$

The truncation error is given by,

$$R = \frac{(x - x_0)^{n+1}}{(n+1)!} f^{n+1}(\Phi),\ x \frac{\le}{0} \Phi \le x$$

$$R = \frac{(x - 0)^3}{3!} f^4(\Phi)$$

$$|TE| = R = \frac{x^3}{3!} 2^{\Phi}$$

$$x_0 \le \Phi \le x$$

$$\text{ii) } |TE| \le \left| \max\left(\frac{x^3}{3!} e^{\Phi} \right) \right|_{[2,3]} \le \frac{3^5}{e^3}$$

$$|TE| \le \frac{27}{6} 3^3$$

Truncation Error in Numerical Integration

Trapezoidal Rule

$$T_{E(max)} = -\frac{h^3}{12} N f''(x) = \frac{(b-a)}{12} h^{2f''}(x)$$

Where, $x \in [x_0, x_n]$ and $h = \frac{b-1}{N}$ is the step size

The maximum value of Truncation Error can be,

$$|T_E|_{(max)} = \frac{h^2}{12}(b-a) \times \max|f''(x)|$$

Simpson's 1/3 Rule

The Truncation Error for simple Simpson's Rule with 3 points is given by,

$$T_E = \frac{n}{90} f^{(iv)}(x)$$

If there are N intervals then,

$$T_{E(max)} = \frac{h^5}{90} N f^{(iv)}(x)$$

$$\text{Where, } N = f = \frac{N^i}{2}$$

The maximum value of Truncation Error can be,

$$T_E = \frac{h^4}{180}(b-a) \times \max\left|f^{(iv)}(x)\right|$$

Simpson's 3/8 Rule

The Truncation Error for simple Simpson's 3/8 Rule is given by, $T_E = \frac{h^4}{80}(b-a) f^{(iv)}(x)$

Where, $x \in [x_0, x_0]$ and $h = \frac{b-1}{N}$ is the step size.

The maximum value of Truncation Error can be,

$$\left|T_E\right|_{(max)} = \frac{h^4}{80}(b-1)\times \max\left|f^{(iv)}(x)\right|$$

Important Points

- Trapezoidal Rule evaluates the polynomial with exact results and they are having upto degree 1 (0 or 1).
- In Simpsons rule we truncated 4th order derivative. It evaluates the polynomials with exact results if they are having degree upto 3 (0, 1, 2 or 3).
- T Rules is applicable on any number of intervals.
- Simpson's 1/3 rule is applicable only if number of intervals are multiples of 2. (n=2 (or) 4 (or) 6 ....).
- Simpson's 3/8 rule is applicable if the number of intervals are multiples of 3. (n = 3 (or) 6 (or) 9 ....)
- Error order in Trapezoidal Rule is order of h3
- Error order in Simpson 1/3 Rule is order of h5
- Error order in Simpson 3/8 Rule is order of h5

Solved Examples

Example: Minimum number of equivalent subintervals needed to approximate $\int_1^2 xe^x dx$ to an accuracy at least $\frac{1}{3}\times 10^{-6}$

Solution: $f(x) = e^x$

$a = 1, b = 2$

$f'(x) = e^x + xe^x$

$f''(x) = 2e^x + xe^x$ which is an increa sing function.

$\text{Max}\left|f''(x)\right|_{[1,2]} = 4e^2\ (x = 2)$

Since, Accuracy $\geq \frac{1}{3}\times 10^{-6}$

Thus, $\left|T_E\right| \leq \frac{1}{3}\times 10^{-6}$

For Trapezoidal Rule,

$$\left|\frac{b-1}{12}\times h^2 \times \max f''(x\right| \leq \left|\frac{1}{3}\times 10^{-6}\right.$$

$$\frac{2-1}{12}\times h^2 \times 4e^2 \leq \frac{1}{3}\times 10^{-6}$$

$$h^2 \times e^2 \leq 10^{-6}$$

$$h = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

$$\frac{1}{n^2}\times e^2 \leq 10^{-6}$$

$$e^2 \times 10^6 \leq n^2$$

$$n \geq 1000e$$

Example: The minimum number of subintervals needed to approximate $\frac{8}{45}\times 10^{-8}$ using Simpson's Rule is?

Solution: $f(x) = e^{2x}$

$a = 0, b = 2$

Successive derivatives of the function are,

$f'(x) = 2e^{2x}$

$f''(x) = 4e^{2x}$

$f'''(x) = 8e^{2x}$

$f^{(IV)}(x) = 16e^{2x}$

Max $f^{(iv)}(x)_{[[0,2]]} = 16e^4$ as it is an increasing function so maxima occur at the end point of interval.

Since, Accuracy $\geq \frac{8}{45}\times 10^{-8}$

$$\left|T_E\right| \geq \frac{8}{45}\times 10^{-8}$$

For Simpson's 1/3 rule

$$\left|\frac{(b-1)}{180}h^4 \times f^{iv}(x\right| \leq \frac{8}{45}\times 10^{8}$$

$$\frac{2}{180}\times h^4 \times 16e^4 \leq \frac{8}{45}\times 10^{8}$$

$$h^4 e^4 \leq 1^{-8}$$

$$n = \frac{b-a}{h} = \frac{2}{h}$$

$$\left(\frac{2}{n}\right)^4 e^4 \leq 1^{-8}$$

$$\frac{2^4}{n^4}e^4 \leq 10^{-8}$$

$$n^4 \geq 2^4 \times e^4 \times 10^8$$

$$n \geq 2\times 2\times 10^2$$

$$n \geq 200e$$

Numerical Solution of Differential Equations

Analytically, only a limited set of differential equations can be solved. The most common type of differential equations occurring practically cannot be solved by analytical methods. They have to be solved numerically. The differential equation can be represented as,

$\frac{dy}{dx} = f(x,y)$ with $(x_0) = y_0$

The condition $y(x_0) = y_0$ is called as Initial Conditions and this problem is called as Initial Value Problem. The problems where values are specified at more than one point are called as Boundary Value Problems.

Euler's Method

Consider $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$ ....(*)

$y_{i+1} = y_i + hf(x_i, y_i)$

In particular for $i = 0, y_1 = y_0 + hf(x_0, y_0)$

Here, h represents the step size for the numerical integration.

In Euler's Method, we approximate the curve of solution by the tangent in each interval and thus for the method to be accurate h must be small. Else, the tangent may deviate considerably from the curve.

Solved Examples

Example: $\frac{dy}{dx} = x + y, y(0) = 1, h = 0.1$ find y(0.2) using Euler's method?

Solution: $\frac{dy}{dx} = x + y, y(0) = y', x_0 = 0, y_0 = 1$

x	y	Comment
$x_0 = 0$	$y_0 = 1$	Initial condition
$x_1 = 0.1$	$y_0 = 1.1$	$y_1 = y_0 + hf(x_0, y_0)$ $= 1 + 0.1(x_0, +y_0)$ $= 1 + 0.1(0 + 1) = 1.1$
$x_2 = 0.2$	$y_2 = 1.22$	$y_1 = y_1 + hf(x_1, y_1)$ $= 1.1 + 0.1(x_1, +y_1)$ $= 1 + 0.1(0.1 + 1.1) = 1.22$

Example: $\frac{dy}{dx} - y = x, y(0) = 0, h = 0,1.$ Find y (0.3)using Euler's method?

Solution $\frac{dy}{dx} = f(x,y) = x + y$

$y_0 = 0, x_0 = 0, h = 0,1$

$y_1 = y_0 + hf(x_0, y_0) = 0 + 0.1(x_0 + y_0) = 0$

$y_2 = y_1 + hf(x_1, y_1) = 0 + 0.1(0.1 + 0) = 0.01$

$y_3 = y_2 + hf(x_2, y_2) = 0.01 + 0.1(0.2 + 0.01)$

$= 0.031$

Backward Euler's Method (Implicit Euler's Method)

$y_{i+1} = y_i + hf(x_{i+1}, y_{i+1})$

y_{i+1} appears on LHS as well as RHS

Any Numerical method where y_{i+1} appears on LHS and RHS is called an implicit method. In Euler's Method y_{i+1} only appears on LHS so it is called as explicit method.

Solved Examples

Example: $\frac{dy}{dx} = x + y, y(x_0) = y_0, y(0) = 1, h = 0.1,$ find y (0.2) using Implicit Euler's Method.

Solution: $\frac{dy}{dx} = f(x,y) = x + y, x_0 = 0, y_0$

$y_{i+1} = y_i + hf(x_{i+1}, y_{i1})$

$y_{i+1} = y_i + h(x_{i+1}, +y_{i1})$

$(1-h)y_{i+1} = y_i + hx_{i+1}$

$y_{i+1} = \frac{y_i + hx_{i+1}}{(1-h)}$

x	y	Comment
$x_0 = 0$	$y_0 = 1$	Initial condition
$x_1 = 0.1$	$y_0 = 1.1$	$y_1 = \frac{y_0 + hx_1}{(1-h)}$ $= \frac{1 + 0.1[\lfloor 0.1 \rfloor]}{1 - 0.1} = 1.12$

$x_2 = 0.2$	$y_2 = 1.22$	$y_1 = \dfrac{y_1 + hx_2}{(1-h)} = \dfrac{1.12 + 0.1[0.2]}{1-0.1} = 1.26$

Table 6.6

Example: $\frac{dy}{dx} = 0.25y^2, y(0) = 1, h = 1$. Find $y(1)$ using implicit Euler's Method?

Solution: $y_{i+1} = y_i + hf(x_{i+1}, y_{i+1})$

$y_0 = 1, y_1 = y_0 + h(0.25)y_1^2$

$0.25y_1^2 - y_1 + 1 = 0$

$y_1 = 2$

Runge-Kutta Method

Runge-Kutta Method is most commonly used to find numerical solution of differential equations and it has different orders. The accuracy of approximation increases as the order of Runge-Kutta Method.

Second Order Method or Modified Euler method

$$y = y + \frac{1}{2}(k_1 + k_2)$$

Where, $k_1 = hf(x_i, y_i)$

$k_2 = hf(x_i + h, y_i + k_1)$

Third Order Method

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

Where, $k_1 = hf(x_i, y_i)$

$$k_2 = hf\left(x_i + \frac{f}{2}, y_i + \frac{k_1}{2}\right)$$

$k_3 = hf(x_i + h, y_i + k_2)$

Fourth Order Method

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + 2k_4)$$

Where $k_1 = hf(x_i, y_i)$

$$k_2 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{k_2}\right)$$

$$k_3 = hf\left(x_i + \frac{h}{2}, y_i + \frac{2}{2}\right)$$

$h_4 = hf(x_i + h, y_i + k_3)$

Solved Examples

Example: $\frac{dy}{dx} = x - y, y(0) = 0, h = 0.2\ dx$

Find y(0.2) using second order Runge Kutta?

Solution:

$$\frac{dy}{dx} = f(x,y) = x - y, x_0 = 0, y_0 = 0, h = 0.2$$

Second order Formula is $y_0 = y_1 + \frac{1}{2}(k_1 + k_2)$

Where $k_1 = hf(x_0, y_0) = 0.2(x_0 - y_0) = 0$

$K_2 = hf(x_0 + h, y_0 + k_1) = 0.2((x_0 + h) - (y_0 + k_1))$

$= 0.2((0 + 0.2) - (0 + 0)) = 0.04$

$$y_1 = 0 + \frac{1}{2}(0 + 0.04) = 0.02$$

Example: $\frac{dy}{dx} = x + y, y(0) = 1, h = 0.2$

find y(0.2) using

(a) Fourth Order RK

(b) Third order RK

Solution: Third Order Runge-Kutta

$k_1 = hf(x_0, x_0) = 0.2 \times (x_0 + y_0) = 0.2 \times (0 + 1)$

$= 0.2$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.2 \times \left(0 + \frac{0.2}{2} + 1 + \frac{0.2}{2}\right) = 0.24$$

$k_3 hf(x_0 + h, y_0 + k_2) = 0.2 \times (0 + 0.2 + 1 + 0.24)$

$= 0.288$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

$$= 1 + \frac{1}{6}(0.2 + 4 \times 0.24 + 0.88)$$

$y_1 = 1.1151$

Fourth Order Runge-Kutta

$k_1 = hf(x_0, y_0) = 0.2(x_0 + y_0) = 0.2 \times (0 + 1)$

$= 0.2$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y + \frac{k_1}{2}\right)$$

$$= 0.2 \times \left(0 + \frac{0.2}{2} + 1 + \frac{0.2}{2}\right) = 0.24$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y + \frac{k_2}{2}\right)$$

$$= 0.2 \times \left(0 + \frac{0.2}{2} + 1 + \frac{0.24}{2}\right) = 0.244$$

$$k_4 = hf\left(x_0 + h, +k_3\right)$$

$$= 0.2 \times \left(0 + 0.2 + 1 + 0.244\right) = 0.2888$$

$$y_1 = 1 + \frac{1}{6}\left(0.2 + 2 \times 0.24 + 2 \times 0.244 + 0.288\right)$$

$$y_1 = 1.24267$$

Stability Analysis in Euler's Method

Consider Euler's equation $y_{i+1} = y_i + hf\left(x_i, y_i\right)$ Convert the equation into the form $y_{i+1} = Ey_i$ $+hk$ Where K is constant or it contains $'x_i'$ terms. The above equation is said to be stable if $E \le 1$

Solved Examples

Example: $\frac{dx}{dt} = \frac{1-x}{T}$ with step size ΔT > 0 is evaluated using Euler's method. What is the maximum permissible value of ΔT to ensure the stability in the solution?

Solution: $\frac{dx}{dt} = \frac{1-x}{T}$ The given differential equation is,

This can be reframed as,

$$\frac{dy}{dx} = f\left(x, y\right) = \frac{1-y}{T}$$

By Euler's Method we have,

$$y_{i+1} = y_i + hf\left(x_i, y_i\right)$$

$$y_{i+1} = \left(\frac{1-y_i}{T}\right)$$

$$y_{i+1} = \left(1 - \frac{h}{T}\right)y_i + \frac{h}{T}$$

Where $E = 1 - \frac{h}{T}, k = \frac{1}{T}$

System is stable if $\left|E\right| \le 1$

$$\left|1 - \frac{h}{T}\right| \le 1$$

$$\left|1 - \frac{\Delta T}{T}\right| \le 1$$

$$-1 \le 1 - \frac{\Delta T}{T} \le 1$$

$$-2 \le -\frac{\Delta T}{T} \le 1$$

$$0 \le \Delta T \le 2$$

Hence, the maximum permissible value of ΔT is 2T.

Note: The order of error by different methods in Numerical Integration is given below,

Method	Error Order
Euler's (First Order RK)	$0(h^2)$
2^{nd} Order R.K	$0(h^3)$
3^{rd} Order R.K	$0(h^4)$
4^{th} Order R.K	$0(h^5)$

Table 6.7

Chapter Summary

Numerical solution of algebraic equations

- **Descartes Rule of sign**

  An equation f(x) = 0 cannot have more positiv roots then the number of sign changes in (x) & cannot have more negative roots then the number of sign changes in f(–x)

- **Bisection Method**

  If a function f(x) is cuntinuous between a & f(a) & f(b) are of opposite sign, then thereat leastone of f(x) between 1 & b

Since root lies between a & n, we assume root $x_0 = \frac{\left(1+b\right)}{2}$

If $f\left(x_0\right) = 0; x_0$ is the root

Else, if f(x0) has same sign as f(a), then roots lies between x0 & b and we assume $x_1 = \frac{x_0 + b}{2}$ and follow same procedure if $f(x_0)$

has same sign as f(b), then root lies between a & x_0 & we assume $x_1 = \frac{a + x_0}{2}$ & follow same procedure.

We keep on doing is, till is close zero.

No. of step required to achieve and accuracy $\in$

$$n \geq \frac{\log_e\left(\frac{|b-a|}{\in}\right)}{\log_e 2}$$

- **Regula-falsi Method**

  This method is similar to bisection method, as we assume two value x_0 & x_1 Such that $f(x_0)f(x_1) < 0$

  $$x_2 = \frac{f(x_1) - x_0 - f(x_0) - x_1}{f(x_1) - f(x_0)}$$

  If $f(x_2) = 0$ then x_2 is the root, stop the process

  If $f(x_2) > 0$ then

  $$x_3 = \frac{f(x_2)x_0 - f(x_0)x_2}{f(x_2) - f(x_0)}$$

  If $f(x_2) < 0$ then

  $$x_3 = \frac{f(x_1)x_2 - f(x_2)x_1}{f(x_1) - f(x_2)}$$

- **Regula-falsi Method**

  IID. secant method, we remove the condition that $f(x_0)f(4) < 0$ and it doesn't provide the guarantee for existence of the root **m** the given interval, So it is called in reliable method.

  and to compute X_3 replace eve:ry variable by its variable **m** -12

  $$x_3 = \frac{f(x_2)..x_1 - f(x_1)..x_2}{f(x_2) - f(x_1)}$$

  Continue the above process till required mot not found

- **Newton-Raphson Method**

  $$x_{n+1} = xt - !\frac{f(x_n)}{f|(x\,t\})}$$

Note: Since R iteration method is qu this formufa must exist f(x)

Order of convergence

Bisection	Linear
Regula Falsi	Linear
Secant	Superliners
Newton. Raphson.=	Quadratic

- **Numerical Intergration Trapezaidal Rule**

  $\int_a^b f(x)\,dx$,can be calculated as

  Divide interval(a,n) into n sub-intervals such that width od each interval

  $$h = \frac{(b-1)}{n}$$

  we have (n+1) point at edged of each intervals

  $(x_0, x_1, x_2, \ldots\ldots\ldots x_n)$

  $y_0 = f(x_0) = y_1 = f(x_1), \ldots\ldots\ldots y_n = f(x_n)$

  $$\int_a^b f(x)\,dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + \ldots\ldots\ldots + y_{n-1})y_n\right]$$

- **Simpson's 1/3rd Rule**

  Here the number of intervals should be even

  $$h = \left(\frac{b-1}{n}\right)$$

  $$\int_a^b f(x)dx = \frac{h}{3}\left[\begin{array}{l} y_0 + 4(y_i + y_3 + y_3 + \ldots\ldots y_{n-1}) + 2 \\ (y_2 + y_4 + \ldots\ldots + y_{n-1}) + y_n \end{array}\right]$$

  Simpson's $\frac{3}{8}$ th Rule

  Here the number of intervals should be even

  $$\int_a^b f(x)\,dx = \frac{3h}{8}\left[\begin{array}{l} y_0 + 3(y_1 + y_2 + y_4 + y_5 \ldots\ldots u_{n-1}) \\ +2(y_3 + y_6 + y_9 \ldots\ldots + y_{n-3}) + y_n \end{array}\right]$$

- **Simpson's 1/3rd Rule**

  Trapezoidal Rule : $|T_e|_{bound} = \frac{(b-a)}{12}h^2$ max

  $|f''(\in)|$ and order of error = 2

  Simpson's $\frac{1}{3}$ Rule : $|T_e|_{bound} = \frac{(b-a)}{180}h^4$ max

  $|f^{(iv)}(\in)|$ and order of error = 4

  Simpson's $\frac{3}{8}$ Rule : $|T_e|_{bound} = \frac{3(b-a)}{n180}h^4$ max

  $|f^{(iv)}(\in)|$ and order of error = 5

  Where $x_0 \leq \varepsilon \leq x\ln$

Note: If truncation error occurs at nth order derivation then given exact result while integrating the polynomial up to degree (n-)

Numberical solution of Differential equation

Euler's Method

$$\frac{dy}{dx} = f(x, y)$$

To solve differential equation by numberical method, we define a step size h

We can calculate of y at (x_0+h,x_0+2h,....x_0+nh) & not any intermediate points

$$y_{i+1} = y_i + hf(x_i, y_i)$$

$$y_i = y(x_i); y_{i+1} + y(x_{i+1}); X_{i+1} = X_i + h$$

Modified Euler's Method (Heun's method)

$$y_i = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0 + h, y_0 + h)\right]$$

Runge-Kutta Method

$$y_i = y_0 + k$$

$$k = \frac{1}{6}(k_1 + 2k_2 + k_4)$$

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Similar method for other iterations

7 Transform Theory

Objective

Upon completion of this chapter you will be able to:

- Analyze a function in terms of Laplace Transform.
- Determine the Z-Transform of a discrete function.
- Determine Fourier Transform of a continuous function.

Introduction

The transforms are used to transform the complicated differential or difference equation into a simple algebraic equation. This is called as operational calculus. In the case of Laplace and Z-Transforms especially we have the advantage that we can consider initial conditions directly without too much of an effort. Fourier Transform is a special case of Laplace Transform and it converts a function from time domain to frequency domain. It is helpful in analyzing the frequency spectrum of a function i.e., the strength of different frequency components in a function.

Laplace Transform

Response of a LTI system with impulse response h(t) for an input x(t) = e^{st} is y(t) = H(s) e^{st}

where, H(s) is the Laplace Transform of Impulse response h(t).

Analysis equation: $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$

Where, s = $(\sigma + j\,\omega)$

$$X\,(\sigma + j\omega) = \int_{-\infty} x(t)e^{(-\sigma + j\omega t)}dt$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt$$

If the real part of s, $\sigma = 0$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \text{Fourier Transform}$$

So, for imaginary values of 's' Laplace Transform converges to Fourier Transform.

Region of convergence (ROC)

ROC is a range of values of s for which Laplace Transform converges is known as region of convergence. Region of Convergence makes the Laplace Transform of a signal unique.

Types of Laplace Transform

1. Unilateral Laplace Transform
2. Bilateral Laplace Transform

Unilateral Laplace Transform (ULT)

The unilateral Laplace transform is defined by the analysis equation

$$H(s) = \int_{0}^{\infty} h(t)e^{-st}dt$$

- The unilateral Laplace transform is restricted to causal time functions and take initial condition into account in the solution of differential equation and in the analysis of systems.
- ULT is obtained for only right sided signal as ULT of left sided signal is always '0'
- ULT is obtained for signal where BLT cannot be obtained

Example: x(t) = $e^{-2t}u(t)$

$$X(s) = \int_{0}^{\infty} e^{-2t}e^{-st}dt = \frac{1}{(s+2)}$$

Bilateral Laplace Transform (BLT)

The bilateral Laplace transform is defined by the analysis equation

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

- The bilateral Laplace transform can represent both causal and non-causal time functions. Initial conditions are accounted by including additional inputs. It is also used to describe frequency response and stability.
- It is obtained for right sided, left sided, 2 sided signals.
- BLT & ULT are the same for right sided signal.

Inverse Laplace Transform

To recover a signal from its Laplace Transform we have to use the following synthesis equation which is also called as inverse Laplace Transform.

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

Solved Examples

Example: Find the Laplace transform of given signals & indicate the region of convergence for the following signals.

(A) $x(t) = e^{-at}u(t)$

Solution:

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-(s+a)t}dt$$

If $s + a > 0$; $e^{-\infty} = 0$

If $s + a < 0$; $e^{-(\infty)} = \infty$

Hence, s + a should be greater than

Region of convergence $R_e\{s + a\} > 0 => R_e(s) > -a$

$$X(s)=\frac{1}{-(s+a)}e^{-(s+a)t}\Big|_0^{\infty}$$

$$X(s)=\frac{1}{(s+a)}$$

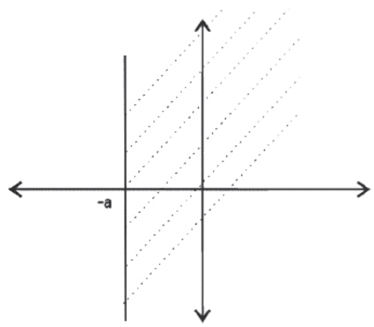


Fig. 7.1

(B) $x(t) = -e^{-at}u(-t)$

Solution:

$$X(s) = -\int_{-\infty}^{\infty} e^{-at}u(-t)e^{-st}dt = -\int_{-\infty}^{0} e^{-(s+a)t}dt$$

Region of convergence

$R_e(s + a) < 0 => R_e(s) < -a$

$$X(s)=\frac{1}{(s+a)}e^{-(s+a)}\Big|_{-\infty} = \frac{1}{(s+a)}$$

Note: For the above two signals the expression for the Laplace Transform is same but their ROCs are different. So ROC makes the Laplace Transform of a signal unique.

Example: Find the ROC of the continuous time signal $x(t) = 3e^{-2t}u(t) + 4e^{t}u(-t)$.

Solution:

$$X(s)=\int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty}\left[3e^{-2t}u(t)+4e^{t}u(-t)\right]e^{-st}dt$$

$$X(s) = 3\int_{0}^{\infty} e^{-2t}e^{-st}dt+4\int_{-\infty}^{\infty} e^{t}e^{-st}dt$$

$$= 3\int_{0}^{\infty} e^{-(s+2)t}dt + 4\int_{-\infty}^{0} e^{-(s-1)}dt$$

$$= 3\underbrace{\left[\frac{e^{-(s+2)t}}{-(s+2)}\right]_0^{\infty}}_{\mathrm{I}} + 4\underbrace{\left[\frac{e^{-(s-1)}}{-(s-1)}\right]_{-\infty}^{0}}_{\mathrm{II}}$$

$$X(s) = \frac{3}{(s+2)} - \frac{4}{(s-1)}$$

For I integral to be converge Re(s + 2)>0 or Re(s)>-2

For II integral to be converge Re (s – 1)<0 or Re(s)<1

Therefore, ROC: -2<Re(s)<1

Poles & Zeroes

If the Laplace Transform of a signal can be expressed in rational form as shown below

$$G(s) = \frac{K(s-s_1)(s-s_2).....(s-s_n)}{(s-s_a)(s-s_b)...(s-s_m)}$$. Where k is constant.

- If in the Laplace Transform we put $s = s_a, s_b, ..., s_m$ the value of Laplace Transform becomes infinity & thus these are called as poles of Laplace Transform.
- If in Laplace Transform we put $s = s_1, s_2, ..., s_n$, the value of Laplace Transform is zero & these are called as zeroes of Laplace Transform.

Properties of ROC

1. ROC of Laplace Transform consists of strips parallel to the imaginary axis
2. ROC should not contains any poles & ROC doesn't depend on zeroes.
3. If x(t) is of finite duration then the ROC is the entire s-plane except possibly $s = \infty$ or $s = 0$

Example: $x(t) = \delta(t+1)$

$$X(s) = \int_{-\infty}^{\infty} \delta(t+1) e^{-st} dt = e^{-s(-1)}$$

$$= e^{s} = 1 + s + \frac{s^2}{2!} + ...$$

∴ x(t) is defined by only when 's' is finite. If $s = \infty$ then $x(s) = \infty$

∴ ROC is the entire s–plane except, if x(t) is finite duration.

Example:

$$x(t) = \begin{Bmatrix} \left| e^{-at} \right. ; 0 < t < T \\ 0 \quad ; \text{elsewhere} \end{Bmatrix}$$

$$x(s) = \lim_{0-2} e^{-at} e^{-st} dt = \left\{ \frac{e^{-(s+a)t}}{-(s+a)} \right\}_0^T = \frac{1 - e^{-(s+a)T}}{s+a}$$

$$\lim_{s-a} X(s) = \lim_{s-a} \frac{Te^{-(s+a)T}}{1} = T$$

ROC is the entire s-plane.

4. If x(t) is a right sided signal then all values of 's' for which $Re\{s\} = \sigma_0$ is include the ROC. Then the right side signal is $Re\{s\} > \sigma_0$ and if the signal is left side ROC is $Re\{s\} < \sigma_0$

Example: $e^{-at}u(t) \xleftrightarrow{\text{L.T.}} \frac{1}{s+a}; Re(s) > -a$

$$\text{or } -e^{-at}u(-t) \xleftrightarrow{\text{L.T.}} \frac{1}{s+a}; Re(s) < -a$$

5. If x(t) is two sided signal then we will consider common ROC or If signal x(t) is infinite 2 sides signal then ROC is a strip between 2 poles.
6. If x(s) is rational and if the signal is right sided then the ROC is right of the right most pole and if the signal is left sided ROC is left of the left most pole.

Example:

$$x(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

Poles are $s = -1$, $s = -2$

If x(t) is right sided then

$Re\{s\} > -1$

$x(t) = e^{-t}u(t) - e^{-2t}u(t)$

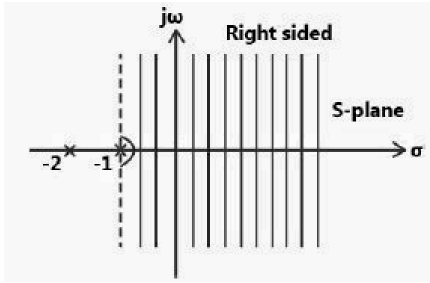


Fig. 7.2

If x(t) is left side then

$Re\{s\} < -2$

$x(t) = -e^{-t} u(-t) - e^{-2t} u(-t)$

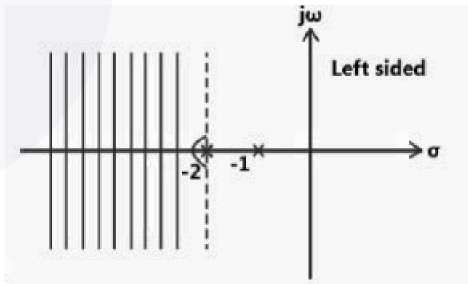


Fig. 7.3

If x(t) two sided then

$Re\{s\} > -2$ but $Re\{s\} < -1$

$-2 < Re\{s\} < -1$

$\therefore x(t) = -e^{-t} u(-t) - e^{-2t} u(t)$

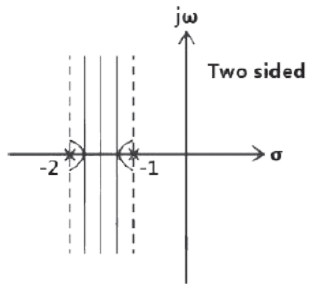


Fig. 7.4

7. If x(t) is band limited signal then ROC is the entire s-plane except 0 & ∞.

Note: If there is no common ROC between the poles for an infinite 2 sided signal then Laplace transform does not exist.

Gamma Function

$$L\{t^n\} = \begin{cases} \dfrac{n!}{s^{n+1}} & ;n \in Z^+ \\ \dfrac{n+1}{s^{n+1}} & ;n \notin Z^+ \end{cases} \quad \text{For } s > 0$$

Gamma Function $\lceil\overline{n+1}$, use only when n> –1

$\lceil\overline{n+1}\, n \lceil\overline{n}$ = n (n–1)(n–2) ... for +ve values of n.

$\lceil\overline{n+1} = n!,\ n \in Z^+$

$\lceil\overline{1} = 1 \;\&\; \lceil\overline{\tfrac{1}{2}} = \sqrt{\pi}$

Some common Laplace Transform with ROC

Signal	Laplace Transform	ROC
$\delta(t)$	1	Entire s-plane
$u(t)$	$\dfrac{1}{s}$	$R_e(s) > 0$
$tu(t)$	$\dfrac{1}{s^2}$	$R_e(s) > 0$
$t^n u(t)$	$\dfrac{n!}{s^{n+1}}$	$R_e(s) > 0$
$-u(-t)$	$\dfrac{1}{s}$	$R_e(s) < 0$
$-tu(-t)$	$\dfrac{1}{s^2}$	$R_e(s) < 0$
$-t^n u(-t)$	$\dfrac{n!}{s^{n+1}}$	$R_e(s) < 0$
$e^{-at}u(t)$	$\dfrac{1}{s+a}$	$R_e(s) > -a$
$te^{-at}u(t)$	$\dfrac{n!}{(s+a)^{n+1}}$	$R_e(s) > -a$
$t^n e^{-at}u(t)$	$\dfrac{1}{s+a}$	$R_e(s) > -a$
$-e^{-at}u(t)$	$\dfrac{n!}{s^2+a^2}$	$R_e(s) < -a$
$-t^n e^{-at}u(-t)$	$\dfrac{n!}{(s+a)^{n+1}}$	$R_e(s) > 0$
$\cos(at)\,u(t)$	$\dfrac{s}{s^2+a^2}$	$R_e(s) > 0$
$\sin(at)\,u(t)$	$\dfrac{a}{s^2-a^2}$	$R_e(s) > a$
$\cosh(at)\,u(t)$	$\dfrac{s}{s^2-a^2}$	$R_e(s) > a$
$\sinh(at)\,u(t)$	$\dfrac{a}{s^2-a^2}$	$R_e(s) > a$

$e^{-at}\cos(bt)\,u(t)$	$\dfrac{s+a}{(s+a)^2+b^2}$	$R_e\,(s) > -a$
$e^{-at}\sin(bt)\,u(t)$	$\dfrac{b}{(s+a)^2+b^2}$	$R_e\,(s) > -a$

Table 7.1

Properties of Laplace Transform

1. **Linearity:**

If $x_1\,(t) \xleftrightarrow{L.T.} e_0^{-st}\; X_1(s)$ with ROC $= R_1$

If $x_2\,(t) \xleftrightarrow{L.T.} e_0^{-st}\; X_2(s)$ with ROC $= R_2$

Then $ax_1\,(t) + bx_2(t) \xleftrightarrow{L.T.} e_0^{-st}\; aX_1(s) + bX_2\,(s)$

ROC $= R_1 \cap R_2$

2. **Time scaling:**

If $x(t) \xleftrightarrow{L.T.} e_0^{-st}\; X(s)$ with ROC $= R$

Then $x(at) \xleftrightarrow{L.T.} e_0^{-st}\; \dfrac{1}{|a|} \times \left(\dfrac{s}{a}\right)$ with ROC $= \dfrac{R}{a}$

3. **Time shifting:**

If $x(t) \xleftrightarrow{L.T.} e_0^{-st}\; X(s)$ with ROC $= R$

Then $x(t - t) \xleftrightarrow{L.T.} e_0^{-st}\, X(s)$ with ROC $= R$

4. **Shifting in s-domain:**

If $x(t) \xleftrightarrow{L.T.} e_0^{-st}\; X(s)$ with ROC $= R$

Then $e^{s_0 t}x(t) \xleftrightarrow{L.T.} e_0^{-st}\; X(s-s_0)$ with ROC $=$ $R + \text{Re}\,\{s_0\}$

5. **Time reversal:**

If $x(t) \xleftrightarrow{L.T.} e_0^{-st}\; X(s)$ with ROC $= R$

Then $x(-t) \xleftrightarrow{L.T.} e_0^{-st}\; X(s)$ with ROC $= -R$

6. **Convolution:**

If $x_1(t) \xleftrightarrow{L.T.} e_0^{-st}\; X_1(s)$ with ROC $= R_1$

If $x_2(t) \xleftrightarrow{L.T.} e_0^{-st}\; X_2(s)$ with ROC $= R_2$

Then In time: $x_1\,(t) * x_2\,(t) \xleftrightarrow{L.T.} e_0^{-st}\; X_1\,(s) \times X_2\,(s)$ with ROC $= R_1 \cap R_2$

And In s-domain:

$x_1\,(t)x_2(t) \xleftrightarrow{L.T.} e_0^{-st}\; \dfrac{1}{2\pi}\left\lceil X_1(s) * X_2(s) \right\rfloor$ with

ROC $= R_1 \cap R_2$

7. **Differentiation in s-domain:**

If $x(t) \xleftrightarrow{L.T.} e_0^{-st}\; X(s)$ with ROC $= R$

If $tx(t) \xleftrightarrow{L.T.} e_0^{-st}\; \dfrac{d}{ds}X(s)$ with ROC $= R$

In general $(-t)^n\; x(t) \xleftrightarrow{L.T.} e_0^{-st}\; \dfrac{d^n}{ds^n}X(s)$

8. **Differentiation in time domain:**

Valid only for ULT

If $x(t) \xleftrightarrow{L.T.} e_0^{-st}\; X(s)$

$\dfrac{d}{dt}x(t) \xleftrightarrow{L.T.} e_0^{-st}\; sX(s)-x(0^-)$

In general $\dfrac{d^n}{dt}x(t) \xleftrightarrow{L.T.} e_0^{-st}\; s^nX(s) + s^{n-1}X(0^-)$ $-\; s^{n-2}x(0^-)\ldots -X^{n-1}(0^-)$

9. **Integration in time:**

If $x(t) \xleftrightarrow{L.T.} e_0^{-st}\; X(s)$

$\underset{0-1}{\text{limit}}\; x\,(\tau)\,d\tau \xleftrightarrow{L.T.} X(s)/s\ldots$ (i)

$\underset{\infty-t}{\text{limit}}\; x\,(\tau)\,d\tau \xleftrightarrow{L.T.} \left(\dfrac{X^s s}{s}\right) + \dfrac{\int_{-\infty}^{0^-} x(\tau)\,d\tau}{s}$ ....(ii)

10. **Integration in s-domain:**

If $x(t) \xleftrightarrow{L.T.} e_0^{-st}\; X(s)$

If $\dfrac{x(t)}{t} \xleftrightarrow{L.T.} \underset{s-\infty}{\text{limit}}\; X(\lambda)\,d\lambda$

11. **Initial value theorem:**

Valid only for ULT

$x(0^+) = \lim\limits_{s\to\infty} sX(s)$

12. **Final value theorem:**

Valid only for ULT

$x(\infty) = \lim\limits_{s\to 0} sX(s)$

Basic Types of Functions

1. Unit Impulse or Kronecker delta or Dirac delta function $\left[\delta(t)\;\&\;\delta\left[\lceil n \rfloor\right]\right]$

An ideal impulse function is a function that is zero everywhere but at the origin, it is infinitely high. However, the area of the impulse is finite. $\left[\lceil \delta n \rfloor\right]$ is also known as Kronecker delta or sample function.

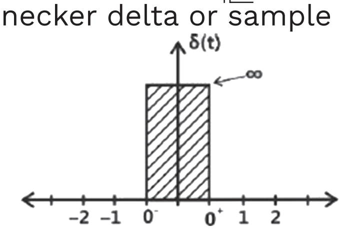


Fig. 7.5

$$\int_{0+}^{0-} \delta(t)\,dt = 1;\; \delta(t) = 0 \;\forall\; t \neq 0$$

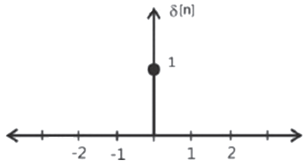


Fig. 7.6

$$\delta[n] = \begin{Bmatrix} 1, n = 0 \\ \hline 0, n \neq 0 \end{Bmatrix}$$

Properties of δ (t)

a) x(t) δ (t) = x(0) δ (t)

b) $\int_{-\infty}^{\infty} x(t)\delta(t)\,dt = x(0)$

c) $x(t)\delta(t)dt = x(t_0)\delta(t - t_0)$

d) $\int_{-\infty}^{\infty} x(t)\delta(t - t)dt = x(t_0)$

e) $\int_{-\infty}^{\infty} x(t)\frac{d^k}{dt^k}(\delta(t - t_0))dt^0$

$$= (-1)^k \frac{d^k}{dt^k}(x(t))\Big|_{t=t_0}$$

Time – Scaling

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

Similarly $\delta(at + b) = \frac{1}{|a|}\delta\left(t + \frac{b}{a}\right)$

Time – Scaling

a) δ(an) = δ (n) (Time scaling is unaffected)

$$\delta[n] = \begin{cases} 1; & n = 0 \\ 0; & n \neq 0 \end{cases}$$

$$\text{But } \delta(Kn) = \begin{cases} 1; & Kn = 0 \\ 0; & Kn \neq 0 \end{cases}$$

$$\delta(Kn) = \delta(n)$$

2. Unit or Heaviside step function

u (t) & u (n)

The unit step function is just a piecewise function with a jump discontinuity at t = a

$$Mu(t - a) = \begin{cases} M, & t > a \\ 0, & t < a \end{cases}$$

Where M in the function represents the height of the jump and a is the number of units shifted to the right.

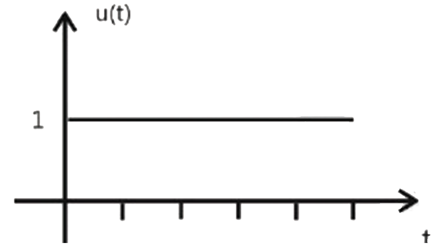


Fig. 7.7 Uniform probability distribution

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

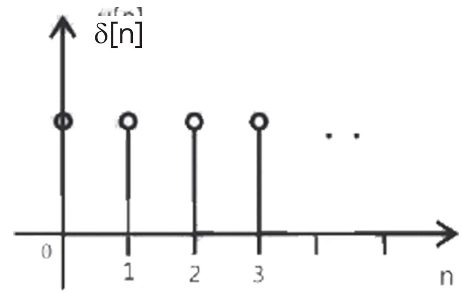


Fig. 7.8

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Heaviside Unit Step function u(t)

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \\ \frac{1}{2}, & t = 0 \end{cases}$$

Relation between unit step and impulse function:

a) $\frac{d}{dt}u(t) = \delta(t)$

b) δ(n) = (n) – u (n – 1)

c) $u(t) = \int_{-\infty}^{t} \delta(t)dt$

d) $u(n) = \sum_{K=0}^{\infty} \delta\, n - K$

3. Ramp Function r (t) or r (n)

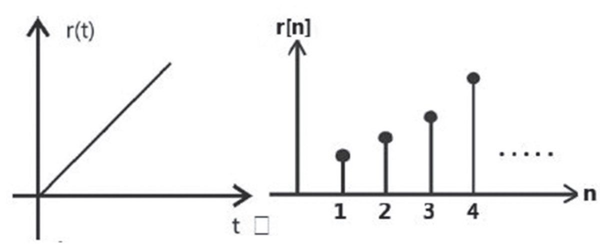


Fig. 7.9

$$r(t)\begin{cases} t, & t>0 \\ 0, & t<0 \end{cases} \qquad r[n]=\begin{cases} n, & n\geq 0 \\ 0, & n\leq 0 \end{cases}$$

a) $\frac{d}{dt}r(t)=u(t)$

b) $\frac{d^2}{dt}r(t)=\delta(t)$

c) $\int_{-\infty}^{t} u(t)\,dt=r(t)$

d) $u(n)=r(n+1)-r(n)$

e) $\sum_{K=\infty}^{n-1}|u(K)\sum_{K=0}^{\infty}u(n-K-1)=r(n)$

4. Parabola p [t]

$$p(t)\begin{cases} t^2, & t>0 \\ 0, & t>0 \end{cases}$$

a) $\frac{d}{dt}p(t)=r(t)$

b) $\int_{-\infty}^{t} r(t)\,dt=p(t)$

c) $\frac{d^2}{dt}p(t)=u(t)$

d) $\frac{d^3}{dt}p(t)=\delta(t)$

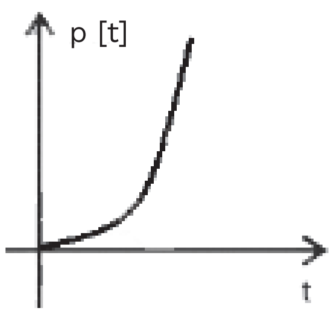


Fig. 7.10

5. Gate Function or Rectangular function

$$A\prod\left(\frac{t}{T}\right) \text{or Arect}\left(\frac{t}{T}\right)$$

$$\text{Arect}\left(\frac{t}{T}\right)=\begin{cases} A & ;\frac{T}{2}<t<\frac{T}{2}, \\ 0 & ;\text{elsewhere} \end{cases}$$

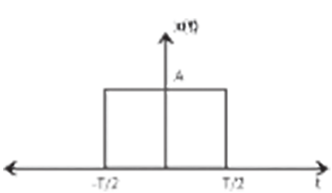


Fig. 7.11

6. Triangular Function

$$A\Delta\left(\frac{t}{T}\right)=\begin{cases} A\left(1+\frac{t}{T}\right) & ;-T<t<0 \\ A\left(1-\frac{t}{T}\right) & ;\ 0<t<T \\ 0 & ;\ \text{elsewhere} \end{cases}$$

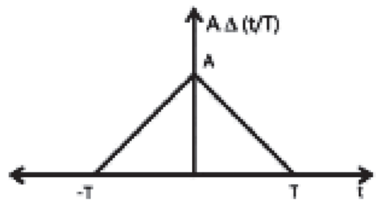


Fig. 7.12

7. Periodic Function

Cycle: A complete set of values of an alternating quantity is called as cycle T=indicate the fundamental period

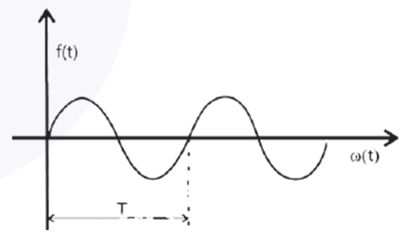


Fig. 7.13

The time taken to complete one cycle of an alternating quantity is known as fundamental period.

f (t) = Asin (ωt + θ) Where $\omega=2\pi f=\frac{2\pi}{T}$

For discrete time signals,

f [n] = Asin (ωn + θ); $\frac{\omega}{2\pi}=\frac{m}{N}$

where N is fundamental period and ‘m’ and ‘N’ have no common factors

8. Continuous exponential

x(t) = Ae^{-at}

Case 1: a > 0 exponential decay

a = 1 x(t) = Ae^{-t}

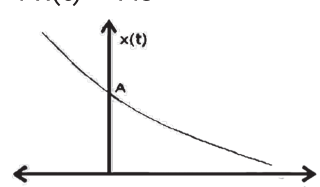


Fig. 7.14

Case 2: a < 0, exponential rising

a=-1 $x(t) = Ae^{t}$

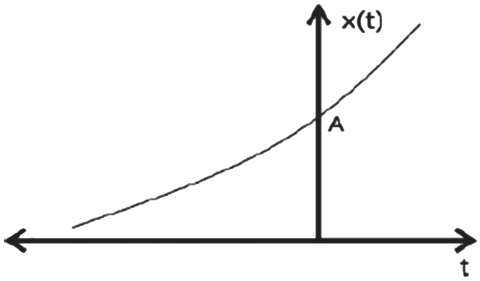


Fig. 7.15

9. Discrete exponential

$X(n) = a^{n}$

Case 1: 0 < a < 1

a = 0.2 $\quad (n) = (0.2)^{n}$

For $|a| < 1$, the signal is exponentially decaying.

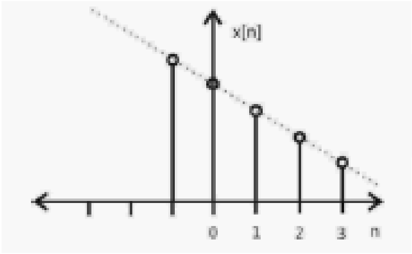


Fig. 7.16

Case 2: 0 < a < 1

a = 2 $\qquad [n] = 2^{n}$

For $|a| > 1$, the signal is exponentially increasing.

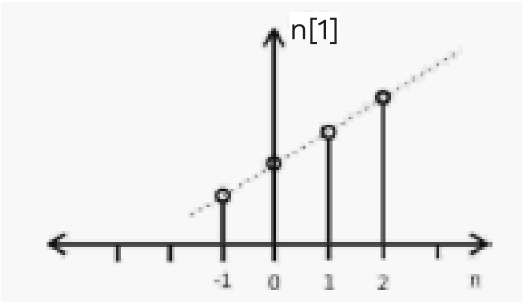


Fig. 7.17

10. Signum Function

$$\text{sgn}(t) = \begin{cases} 1 & ,t > 0 \quad \text{or} \quad 2u(t) - 1 \\ -1 & ,t < 0 \end{cases}$$

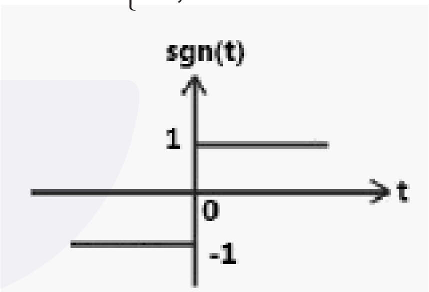


Fig. 7.18

Solved Examples

Example: Determine the Laplace Transform of a^{t}; a > 0

Solution: Laplace Transform of e^{-at} will be $\frac{1}{s+a}$

$$F(s) = L\left\{e^{\log a^{t}}\right\} = \left\{e^{t \log a}\right\} = \frac{1}{s - \log a}$$

Example: Determine Laplace Transform of $\left\{t^{1/2} + t^{1/2} + t^{3}\right\}$

$$F(s) = L\left\{t^{7/2} + t^{-1/2} + t^{3}\right\} = \frac{\Gamma\left(\frac{7}{2}+1\right)}{\frac{-1}{s^{2}}+1} + \frac{\Gamma\left(\frac{-1}{2}+1\right)}{\frac{-1}{s^{2}}+1} + \frac{3!}{s^{4}}$$

$$F(s) = \frac{7/2.\,5/2.\,1/2.\,\Gamma(1/2)}{s^{9/2}} + \frac{\Gamma(1/2)}{s^{1/2}} + \frac{6}{s^{4}}$$

$$= \frac{105\sqrt{\pi}}{16s^{9/2}} + \frac{\sqrt{\pi}}{s^{1/2}} + \frac{6}{s^{4}}$$

Example: Determine Laplace Transform of sin $\sqrt{t}$.

Solution:

$$F(s) = L\left\{\sin\sqrt{t}\right\} = \left\{t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \ldots\right\}$$

$$F(s) = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{3/2}} - \frac{\Gamma\left(\frac{3}{2}+1\right)}{3!\,s^{5/2}} + \frac{\Gamma\left(\frac{5}{2}+1\right)}{5!\,s^{7/2}} \ldots\ldots$$

$$= \frac{1/2\sqrt{\pi}}{s^{3/2}} - \frac{3/2.\,1/2\sqrt{\pi}}{3!\,s^{3/2}} + \frac{5/2.\,1/2\sqrt{\pi}}{5!\,s^{7/2}} + \ldots$$

$$F(s) = \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{2!(4s)^{2/3}} - \frac{1}{3!(4s)}\right] + \ldots$$

$$= \frac{\sqrt{\pi}}{2s^{e-1/4s}}(s > 0)$$

Example: Determine Laplace Transform of et^2.

Solution:

$$F(s)=L\left\{e^{t^2}\right\}=\left\{1+t^2+\frac{t^4}{2!}+\frac{t^6}{3!}+......\right\}$$

Does not exist or integral is not absolutely converging

$$F(s)=\frac{1}{s}+\frac{2!}{s^3}+\frac{4!}{2!s^5}+\frac{6!}{3!s^7}+.......$$

$$\sum_{n=0}^{\infty}\frac{(2n)!}{n!s^{2n+1}}\rightarrow \text{Divergent series}$$

Since all are increasing terms. These transformations do not exist.

Example: Determine the Laplace Transform of $\sin 3t \times \cos 4t$

Solution:

$$F(s)=L\{\sin 3t\cos 4t\}=\frac{1}{2}L\{\sin 7t+\sin(-t)\}$$

$$F(s)=\frac{1}{2}\left[\frac{7}{s^2+49}-\frac{1}{s^2+1}\right]=\frac{1}{2}$$

$$\left\{\frac{6s^2-42}{(s^2+49)(s^2+1)}\right\}=\frac{3s^2-21}{(s^2+49)(s^2+1)}$$

Example: Determine the Laplace Transform of {cost cos2t cos8t}

Solution:

$$F(s)=L\{\cos t\ \ \cos 2t\ \ \cos 8t\}$$

$$=\frac{1}{2}L\{(\cos 3t+\cos(-t))\cos 3t\}$$

$$F(s)=\frac{1}{2}L\{\cos^2 3t+\cos t\ \ \cos 3t\}$$

$$=\frac{1}{4}L\{1+\cos 6t+\cos 4t+\cos(-2t)\}$$

$$F(s)=\frac{1}{4}\left[\frac{1}{s}+\frac{s}{s^2}+\frac{s}{2}+\frac{s}{2}\right]$$

Example: If

$$L\{f(t)\}=\frac{e^{-1/s}}{s}\text{ then }L\left\{e^{3t}\int_0 f(3t)\,dt\right\}=\_\_\_\_\_\_?$$

Solution:

$$L\{f(3t)\}=\frac{1}{3}\left(\frac{s}{3}\right)=\frac{3e^{-3/5}}{3s}=\frac{e^{-3/5}}{s}$$

$$L\left\{\int_0^t f(3t)\,dt\right\}=\frac{1e^{-3/5}}{s}=\frac{e^{-3/5}}{s^2}$$

$$L\left\{e\int_0^{-3t} f(3t)\,dt\right\}=\frac{e^{-3/(s+3)}}{(s+3)^2}$$

Example: Determine Laplace Transform of

$$f\ (t)\ =\begin{cases}(t-2)^2 & ;t\geq 2\\ 0 & ;t<2\end{cases}\ ?$$

Solution: $f\ (t) = (t-2)^2\ u(t-2)$

$$L\{t^2\}=\frac{2}{s^3}$$

$$L\left\{(t-2)^2 u(t-2)\right\}=e^{-2s}\times\frac{2}{s^3}$$

Example: Determine Laplace Transform of given wave.

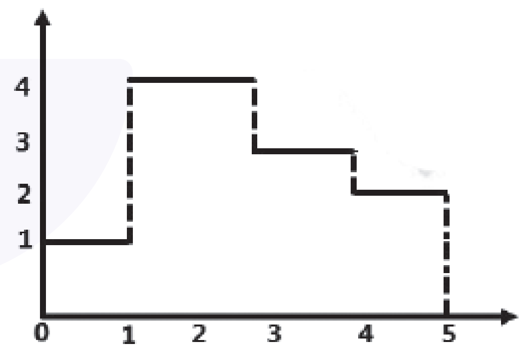


Fig. 7.19

Solution:

$$f(t)\left[\left[u(t)-u(t-1)\right]\right]+4\left[\left[u(t-1)-u(t-3)\right]\right]$$
$$+3\left[\left[u(t-3)-u(t-4)\right]\right]+2\left[\left[u(t-4)-u(t-5)\right]\right]$$

$$f(t)=u(t)+3(u(t-1))-u(t-3)-2u(t-5)$$

$$F(S)=\frac{1}{s}\left[\left[1+3e^{-s}-e^{-3s}-e^{-4s}-2e^{-5s}\right]\right]$$

Staircase/Step/Integral Function

$[[x]] = n\ ;\ n \leq x < n+1 \qquad n \in Z$

$f(t) = u(t-1) + u(t-2) + u(t-3) + ...$

$$L\{f(t)\}=\frac{e^{-s}}{s}+\frac{e^{-2s}}{s}+\frac{e^{-3s}}{s}+...........$$

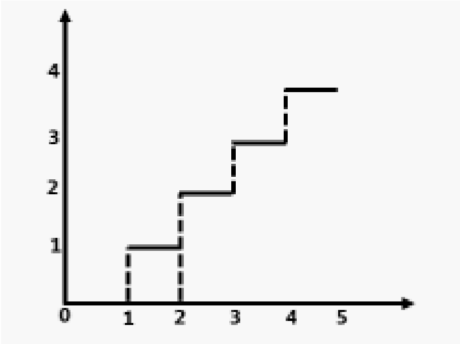


Fig. 7.20

Laplace Transform of periodic signals

- Bilateral LT doesn't exist for periodic signals rather unilateral LT exits as there is no common ROC between poles.
- $F(s) = \frac{X(s)}{1-e^{-st}}$ Where X(s) is Laplace transform of one period of the signal. Where T represents the time period of the signal

Solved Examples

Example: Determine the Laplace Transform of the following periodic function

$$f(t)\begin{cases} \sin\omega t & ;0 \le t < \frac{\pi}{\omega} \\ 0 & ;\frac{\pi}{\omega} \le t < \frac{2\pi}{\omega} \end{cases}$$

Solution: Since this function is periodic with a period of $\frac{2\pi}{\omega}$

$$f\left(t+\frac{2\pi}{\omega}\right) = f(t)\,v\,t|$$

$$f(t) = \sin\omega t\left(u(t) - ut - \frac{\pi}{\omega}\right)$$

$$L\{f(t)\} = \left(\frac{1}{1-e\frac{s2\pi}{\omega}}\right)L\left\{\sin\omega t\, u(t) - u\left(t-\frac{\pi}{\omega}\right)\right]$$

$$L\{f(t)\} = \left(\frac{1}{1-e^{\frac{s2\pi}{\omega}}}\right)\begin{Bmatrix} \sin\omega t \times u(t) + \sin\omega\left[t-\frac{\pi}{\omega}\right] \\ \times u\left[t-\frac{\pi}{\omega}\right] \end{Bmatrix}$$

$$L\{s(t)\} = \left(\frac{1}{1-e^{\frac{-s2\pi}{\omega}}}\right)\left\{\frac{\omega}{s_2+\omega_2}\left(1+e^{-\frac{s\pi}{\omega}}\right)\right\}$$

$$= \frac{1}{\left(1+e^{\frac{-s\pi}{\omega}}\right)\left(1-e^{\frac{-s\pi}{\omega}}\right)}\left\{\frac{\omega}{s_2+\omega_2}\left(1+e^{-\frac{s\pi}{\omega}}\right)\right\}$$

$$L\{f(t)\} = \left\{\frac{1}{1-e^{-\frac{s\pi}{\omega}}}\frac{\omega}{s^2+\omega^2}\right\}$$

Example: If

$$L\left\{\sin\sqrt{t}\right\} = \frac{\sqrt{\neq e^{-1/4s}}}{2s^{3/2}} \text{ find } L\left\{\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right]\right\}$$

Solution: $f(t) = \sin\sqrt{t} \Rightarrow f(0) = 0$

$$f'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$$

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

$$L\left[\frac{\cos\sqrt{t}}{2\sqrt{t}}\right] = \frac{s\sqrt{\pi}e^{-1/4s}}{2s^{3/2}} = \frac{1}{2}s\sqrt{\frac{\pi}{s}}e^{-1/4s}$$

$$L\left\{\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right]\right\} = \sqrt{\frac{\pi}{s}}e^{-1/4s}$$

Example: Determine inverse Laplace Transform of $\frac{1}{\sqrt{4s-3}}$

Solution:

$$L^{-1}\left\{\frac{1}{\sqrt{4s-3}}\right\} = \frac{1}{\sqrt{4}}L^{-1}\left\{\left[\left(\frac{1_{1/2}}{s\frac{3}{4}}\right)\right]\right\} = \frac{1}{\sqrt{4}}e^{4^{t^3}}$$

$$L^{-1}\left\{\left[\left(\frac{1_1}{s^2}\right)\right]\right\} = \frac{1}{2}e^{\frac{3}{4}t}\frac{t^{2^{1-1}}}{\lceil 1/2} = \frac{e^{\frac{3}{4}t}}{2\sqrt{\neq t}}$$

Therefore, $L^{-1}\left\{\frac{1}{(as+b)}\right\} = \frac{1}{a^n} - \frac{b}{e^{at}}\frac{t^{n-1}}{\sqrt{n}}$

Example: Determine the inverse Laplace Transform of $\frac{1}{s}\cot^{-1}(s)$

Solution: Let F(s) = $\cot^{-1}(s)$

$$\frac{d}{ds}F(s) = \frac{-1}{s^2+1}$$

$$L^{-1}\left\{\frac{d}{ds}\right\}F(s) = -t\,L^{-1}\{F(s)\}$$

$$L^{-1}\left\{\frac{1}{s^2}\right\} = tL^{-1}\{\cot^{-1}(s)\} \text{ i.e., } \frac{\sin t}{t} = L^{-1}\{\cot^{-1}(s)\}$$

$$\therefore\ L_1\left[\frac{1}{s}\cot^{-1}s\right] = \int_0^t \frac{\sin t}{t}dt$$

Example: $L\{f(t)\} = \dfrac{2s+3}{s^2-2s+2}$.

Then determine $\underset{t\to\infty}{\text{Lt}}\ f(t) = ?$

Solution: $\dfrac{2s+3}{s^2-2s+2} = \dfrac{2s+3}{(s-(1+i)\{s-(1-i)\})}$

Final value theorem is not applicable as poles are on right side. So the system is unstable. Hence, final value is undefined.

Example: Determine the solution of following differential equation using Laplace Transform.

$$\frac{dy}{dt^2} - 3\frac{dy}{dt} + 2y = 8t;\ \ y(0) = 0,\ \ \frac{dy}{dt}$$
When $t = 0 = 1$.

Solution: Take Laplace Transform on both sides

$$s^2y(s) - sy(0) - y'(0) - 3[[sy(s) - y(0)]]$$
$$+2y(s) = 1$$
$$(s^2 - 3s + 2)y(s2) = 1 + y'(0) = 1 + 1 = 2$$

Fourier Transform

For continuous time signals, Fourier Transform is called as Continuous-Time Fourier Transform (CTFT) and for discrete signals it is called as Discrete-Time Fourier Transform (DTFT).

Note:

- The spectrum of Fourier Transform is always continuous i.e., non-periodic signals are converted into continuous frequency.

- In spite of having a number of application of Fourier Transform, Fourier Transform could analyse only the bounded signal & stable systems. So Laplace and Z-transform are used to analyse both bounded & unbounded signals; stable & unstable systems.
- Laplace transform is used for continuous time signals & Z-transform for discrete time signals.

Continuous time fourier transform:

The Fourier transform is a reversible, linear transform with many important properties. For any function f(x) (which is usually real-valued, but f(x) may be complex), the Fourier transform can be denoted F(s), where the product of x and s is dimensionless. Often x is a measure of time t (i.e., the time-domain signal) and so s corresponds to inverse time or frequency f (i.e., the frequency-domain signal).

These equations are used to find the Fourier Transform of a signal and analyse the frequency components of a signal and hence these are termed as Analysis equation.

$$\underset{\text{Continuous}}{x(t)} \xleftrightarrow{\ \text{LT}\ } \left.\begin{matrix} X(j\omega) \\ X(\omega) \\ X(f) \end{matrix}\right\}\ \text{Continous}$$

$$X(\omega) = \int_{-\infty}^{\infty} X(t)e^{-j\omega t}dt$$

$$X(f) = \int_{-\infty}^{\infty} X(t)e^{-j2\pi ft}dt$$

Inverse Fourier Transform

These equations are used to derive signal in time domain from the frequency domain and hence these are termed as Synthesis Equation.

$$x(t) = \frac{1}{2}{}_{-\infty}^{\infty} \times(\omega)e^{j\omega t}d\omega$$

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

Dirichlet's Conditions

Like in Fourier Series these conditions must be satisfied for Fourier Transform to converge.

1. x(t) must be absolutely integrable
$\int_{-\infty}^{\infty} X|t|dt < \infty$
2. x(t) must have a finite number of discontinuities.
3. X(t) must have a finite number of maxima and minima.
So, here expect the periodicity condition on x(t), all other conditions must be satisfied for Fourier Transform to exist.

Note:
1. The spectrum of Fourier Transform is always continuous frequency response
2. Its magnitude response is symmetrical and phase response is anti-symmetrical

Fourier Sine and Cosine Transformations

For odd functions only the sine transform is defined and its cosine transform is zero.

$$F_s\{f(t)\} = F_s\{\omega\} = 2\int_0^{\infty} f(t)\sin\omega t\, dt$$

(Fourier Sine Transformation)

$$f(t) = F_s^{-1}\{F_3(\omega)\} = \frac{1}{\pi}\int_0^{\infty} F_s(\omega)\sin\omega t\, d\omega$$

(Inverse Fourier Sine Transformation)

For even functions only cosine transform is defined and its sine transform is zero.

$$F_c\{f(t)\} = F_c(\omega) = 2\int_0^{\infty} f(t)\cos\omega t\, dt$$

(Fourier Cosine Transformation)

$$f(t) = (F_c(\omega)) = \frac{1}{\pi}\int_0^{\infty} F_c(\omega)\cos\omega t d\omega$$

(Inverse Cosine Transformation)

Note:
- $A\text{rect}\left(\frac{t}{T}\right) \xleftrightarrow{\text{F.T.}} AT \times Sa\left(\frac{\omega T}{2}\right)$
  or $At \times \text{Since}(fT)$
- $\text{Sinc}(X) = \frac{\sin \pi.x}{\pi.x}$
- $Sa(x) = \frac{\sin x}{x}$
- $\Delta\left(\frac{t}{\tau}\right) \xleftrightarrow{\text{LT}} \tau \sin c^2\left(\frac{\omega\tau}{2}\right)$

Solved Examples

Example: Find Fourier Transform of f (t)

$$= \begin{cases} 1 - t^2 & ; |t| \le 1 \\ 0 & ; |t| > 1 \end{cases}$$

Solution:

$$F\{f(t)\} = \int_{-1}^{1}(1-t^2)e^{-j\omega t}dt = \left[\frac{e^{-j\omega t}}{-j\omega}(1-t)^2 - (-2t)\right.$$

$$\left.\left[\left|\frac{e^{-j\omega t}}{(-j\omega)^2}\right| + (-2)\frac{e^{-j\omega t}}{(-j\omega)^3}\right]\right]_{-1}^{1}$$

$$F\{f(t)\} = \left|\begin{matrix} \frac{2e^{-j\omega}}{2} - \frac{2e^{-j\omega}}{-j\omega^3} + \frac{2e^{j\omega}}{-j\omega^3} = \frac{-2e}{-j\omega^3} \\ = \frac{-2e^{-j\omega}}{\omega^2} \frac{2e^{j\omega}}{-j\omega^3} + \frac{2e - 2e^{j\omega}}{+j\omega^3} \end{matrix}\right|$$

$$F\{f(t)\} = (e^{\omega} + e^{-\omega}) + \frac{2}{j\omega^3}(e^{\omega} - e^{-\omega})$$

$$F\{f(t)\} = \left[\frac{\omega}{\omega^2}(2\cos\omega) + \frac{2}{j\omega^3}(2j\sin\omega)\right]$$

$$= \frac{4}{\omega^3}[\sin\omega - \cos\omega]$$

Example: Find Fourier Transform of e^{-at}; (a>0)

Solution:

$$F(\omega) = 1\int_{-\infty}^{\infty} e^{at}e^{j\omega t}dt + \int_0^{\infty} e^{-at}e^{j\omega t}dt$$

$$= \left|\frac{e^{(a+j\omega t)} + e^{-(a+j\omega t)}}{(a+j\omega)_{-\infty} - a(a-j\omega)_{-0}}\right|$$

$$F(\omega) = \frac{1}{a+j\omega} + \frac{1}{a-j\omega} = \frac{2a}{a^2+\omega^2}$$

Example: Find Fourier Sine Transform of e^{-ax}?

Solution:

$$F(\omega) = 2\int_0^{\infty} e^{-ax}\sin\omega x dx = 2\int_0^{\infty} e^{-ax}$$

$$\left(\frac{e^{j\omega x} - e^{-j\omega x}}{2j}\right)dx$$

$$F_s(\omega) = 2\int_0^\infty \frac{e^{-(a-j\omega x)} - e^{-(a+j\omega x)}}{2j}$$

$$dx = -j\left[\frac{e^{-(a-j\omega x)}}{a-j\omega} + \frac{e^{-(a+j\omega x)}}{a+j\omega}\right]_{-0}^{\infty}$$

$$= -j\left[\frac{1}{a-j\omega} - \frac{1}{a+j\omega}\right] = \left[\frac{2\omega}{a^2+\omega^2}\right]$$

Example: Find Fourier Sine Transform of xe^{-ax}

Solution: $F(\omega) = 2\ xe^{-ax}\ \sin x dx$

Integrate w.r.t

$$(\omega)d\omega = 2\int_0^\infty xe\left(\frac{-\cos\omega x}{x}\right)dx$$

$$F(\omega)d\omega = -2\int_0^\infty e^{-ax}\cos\omega x\ dx$$

$$\int_s^F(\omega)d\omega = -2\int_0^\infty e\Big|^{-ax}\frac{e^{j\omega x}+e^{-j\omega x}}{2}dx$$

$$\int_s^F(\omega)d\omega = -2\int_0^\infty e\Big|\frac{e^{-(-aj\omega)x}+e^{-(a+j\omega)x}}{2}dx$$

$$\int F_s(\omega)d\omega = \left[-\frac{e^{-(-aj\omega)x}}{a-j\omega} - \frac{e^{-(a+j\omega)x}}{a+j\omega}\right]_{-0}^{\infty}$$

$$= -\left[\frac{1}{a-j\omega} + \frac{1}{a+j\omega}\right] = -\left[\frac{2a}{a^2+\omega^2}\right]$$

Diff. w.r.t. ω

$$F_s(\omega)\left|\frac{4a\omega}{(\omega^2+a^2)}\right|$$

Self-Reciprocal Functions

If the transformation of a function is the function itself then it is called self-reciprocal.

$$F_s\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{\sqrt{\omega}}$$

Gauss Function is also self-reciprocal.

$$F\left(e\frac{x^2}{-2}\right) = e - \frac{\omega^2}{2}$$

Example: If

$$\int_0^\infty (x)\sin tx dx = \begin{cases} 1\ ;\ 0 \le t \le 1 \\ 2\ ;\ 1 \le t \le 2 \text{ Then find } f(x)? \\ 0\ ;\ t \le 2 \end{cases}$$

$$f(x) = F_s^{-1}(F_s(t)) = \frac{1}{\pi}\int_0^\infty F_s(t)\frac{1}{\pi}$$

$$\left[\int_0^1 1\sin tx dt + \int_1^2 2\sin tx dx + \int_2^\infty 0 dt\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-\cos tx}{x}\right)^1 + \left(\frac{2\cos tx}{x}\right)^2\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-\cos x}{x} + \frac{1}{x}^{t-0} - \frac{2\cos 2x}{x} + \frac{2\cos 2x}{x}\right)\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos x}{x} + \frac{1}{x} - \frac{2\cos 2x}{x}\right)\right]$$

Properties of Fourier Transform

1. **Linearity**

   if $x_1(t) \xleftrightarrow{F.T.} x_1(\omega)$

   Ifif $x_2(t) \xleftrightarrow{F.T.} x_2(\omega)$

   Then $ax_1(t) + bx_2(t)$ if $x_1(t) \xleftrightarrow{F.T.} ax_1(\omega)$

   $+bx_1(\omega)$ or $ax_1(f) + bx_1(f)$

2. **Time scaling**

   If $x(t) \xleftrightarrow{F.T.} X(\omega)$

   Then $x(at) \xleftrightarrow{F.T.} \frac{1}{|a|} \times \left(\frac{\omega}{a}\right)$

Example: Find the Fourier transform of

(i) $x_1(t) = A\text{rect}\left(\frac{t}{2T}\right)$ (ii) $x_2(t) = A\text{rect}\left(\frac{2t}{T}\right)$

Solution:

(i) $A\text{rect}(t/T) \xleftrightarrow{F.T.} AT\text{sinc}\left(\frac{\omega T}{2}\right)$ or $AT\text{sinc}(fT)$

$$x_1(t) = x(t/2) \Rightarrow a = \frac{1}{2}$$

$$\therefore x_1(\omega) = \frac{1}{\left|\frac{1}{2}\right|}x(2\omega) = 2AT \times Sa(\omega T)$$

(ii) $x_2(t) = x(2t) \Rightarrow a = 2$

$$\therefore x_2(\omega) = \frac{1}{|2|}X\left(\omega/2\right) = \frac{1}{|2|}AT \times Sa\left(\frac{\omega/2 \cdot T}{2}\right)$$

$$x_2(\omega) = \frac{1}{2}AT \times Sa\left(\frac{\omega T}{4}\right)$$

3. **Time shifting**

   If $x(t) \xleftrightarrow{F.T.} X(\omega)$

   Then $X(t - t_0) \xleftrightarrow{F.T.} e^{-j\omega t_0}X(\omega)$

4. **Symmetry or duality**

If $x(t) \xleftrightarrow{F.T.} X(\omega)$

Then $X(t) \xleftrightarrow{F.T.} 2\pi x(-\omega)$ or $X(t) \xleftrightarrow{F.T.} x(-f)$

5. **Shifting in frequency**

If $x(t) \xleftrightarrow{F.T.} X(\omega)$

Then $e^{j\omega_0 t}X(t) \xleftrightarrow{F.T.} X(\omega-\omega_0)$

6. **Differentiation in time**

If $x(t) \xleftrightarrow{F.T.} X(\omega)$

Then $\frac{d}{dt}X(t) \xleftrightarrow{F.T.} j\omega X(\omega)$ or $j2\pi f x(f)$

$\frac{d}{dt}X(t) \xleftrightarrow{F.T.} (j\omega)^n X(\omega)$

or $(j2\pi f)^n X(f)$

7. **Differentiation in frequency**

If $x(t) \xleftrightarrow{F.T.} X(\omega)$

Then $-jtx(t) \xleftrightarrow{F.T.} \frac{d}{d\omega}X(\omega)$

or $tx(t) \xleftrightarrow{F.T.} \frac{d}{d\omega}X(\omega)$

$(-jt)X(t) \xleftrightarrow{F.T.} \frac{d^n X(\omega)}{d\omega^n}$

8. **Integration in time**

If $x(t) \xleftrightarrow{F.T.} X(\omega)$

$$\text{Then } \int_{-\infty}^{t} X(\tau)d\tau \xleftrightarrow{F.T.} \begin{cases} \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega) & \text{if } X(0) \\ \frac{X(f)}{j2\pi f} + \frac{X(0)\delta(f)}{2} & \text{if } X(0) \neq 0 \end{cases}$$

9. **Convolution**

In Time:

If $x_1(t) \xleftrightarrow{F.T.} X_1(\omega)$

and $x_2(t) \xleftrightarrow{F.T.} X_2(\omega)$ then

$x_1(t) * x_2(t) \xleftrightarrow{F.T.} X_1(\omega)X_2(\omega)$

$x_1(t) * x_2(t) \xleftrightarrow{F.T.} X_1(f)X_2(f)$

10. **Conjugation**

If $x(t) \xleftrightarrow{F.T.} X_1(j\omega)$

Then $x^*(t) \xleftrightarrow{F.T.} X^*(-j\omega)$ or $X^*(-f)$

then $x^*(t) \xleftrightarrow{F.T.} X^*(-j\omega)X^*(-f)$

Where '*' denotes complex conjugate

Note: If x(t) is a real valued function, then its Fourier transform will be even conjugate or conjugate symmetric.

$X^*(j\omega) = X(-j\omega)$ or $X(j\omega) = X^*(-j\omega)$

Here, the magnitude of Fourier transform has even symmetry while the phase has odd symmetry. If x(t) is real valued, then Fourier transform $X(j\omega)$ is generally complex.

Solved Examples

Example: The Fourier transform of a conjugate symmetric function is always_____________.

Solution: According to duality property, if the Fourier transform of real signal is conjugate symmetric, then Fourier transform of conjugate symmetric will also be real.

11. **Time Reversal**

If $x(t) \xleftrightarrow{F.T.} X(j\omega)$

Then $x(-t) \xleftrightarrow{F.T.} X(-j\omega)$

12. **Parseval's Power Theorem**

If $x(t) \xleftrightarrow{F.T.} X(\omega)$ Then

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} \left|\frac{1}{2\pi}X(\omega)\right|^2 d\omega$$

or $\int_{-\infty}^{\infty} |X(f)|^2 df$

Some common Fourier Transform Pairs

Signal	Fourier Transform
$\sum_{K=-\infty}^{\infty} a_k e^{jk\omega 0t}$	$2\pi \sum_{K=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$
$e^{jk\omega 0t}$	$2\pi\delta(\omega-\omega_0)$

$\cos \omega_0 t$	$\pi\left[\delta\left(\omega+\omega_0\right)+\delta\left(\omega-\omega_0\right)\right]$
$\sin \omega_0 t$	$\frac{\pi}{j}\left[\delta\left(\omega-\omega_0\right)+\delta\left(\omega+\omega_0\right)\right]$
1	$2\pi\delta(\omega)$
$\sum_{K=-\infty}^{\infty} \delta\left(t-nT\right)$	$\frac{2\pi}{T}\sum_{K=-\infty}^{\infty} \delta\left(\omega-\frac{2\pi k}{T}\right)$
$x\left(t\right)=\left\{\begin{vmatrix}1\\0\end{vmatrix},\begin{vmatrix}t<T_1\\t>T_1\end{vmatrix}\right.$	$\frac{2\sin\omega T_1}{\omega}$
$\frac{\operatorname{Sin}\omega_0 t}{\pi t}$	$x\left(\omega\right)=\left\{\begin{vmatrix}1\\0\end{vmatrix},\begin{vmatrix}\omega<\omega_0\\\omega>\omega_0\end{vmatrix}\right.$
$\delta(t)$	1
$u(t)$	$\frac{1}{j\omega}+\pi\delta\left(\omega\right)$
$\delta(t-t_0)$	$e^{-j\omega t0}$
$e^{-at}u(t)$, Re(a) > 0	$\frac{1}{a+j\omega}$

Table 7.2

Fourier Transform of periodic signal

$$f\left(t\right)=\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$F\left(\omega\right)=\sum_{n=-\infty}^{\infty} 2\pi C_n \delta\left(\omega-n\omega_0\right)$$

$$F\left(\omega\right)=2\pi\sum_{n=-\infty}^{\infty} C_n \delta\left(\omega-n\omega_0\right)$$

Z Transform

Z transform is defined for unstable systems for which we cannot define DTFT. The Z-transform is simply a power series representation of a discrete-time sequence.

The response of a LTI Discrete Time System with an impulse response h[n] to a complex exponential input zn is given as

$y[n] = H(z)z^n$

Where H (z) is z-Transform of impulse response h[n].

$$H\left(z\right)=\sum_{n=-\infty}^{\infty} h\left[n\right]z^{-n}$$

Let $Z = re^{j\omega}$

$$X\left(re^{j\omega}\right)=\sum_{n=-\infty}^{\infty} x[n]\left(re^{-j\omega}\right)^n$$

$$=\sum_{n=-\infty}^{\infty} x[n]\left(re^{-j\omega}\right)^n$$

$$X\left(e^{j\omega}\right)^n \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\omega n}$$

If r = 1, which means |z| = 1

$$X\left(e^{j\omega}\right)^n=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}=\text{Fourier Transform}$$

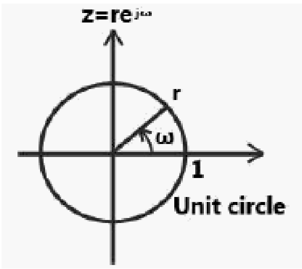


Fig. 7.21

- So, for all 'z' lying on the unit circle z-Transform converges to Fourier Transform.

Region of convergence (ROC)

ROC is a range of values of ‘z’ for which z-Transform converges is known as region of convergence. Region of Convergence makes the z-Transform of a signal unique.

Types of Z Transform

1. Unilateral Z Transform
2. Bilateral Z Transform

Unilateral Z Transform (UZT)

The unilateral Z transform is defined by the analysis equation

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n|}$$

The unilateral Z transform is restricted to causal functions and take the initial condition into account in the solution of difference equation and in the analysis of systems.

Bilateral Z Transform (BZT)

The bilateral Z transform is defined by the analysis equation

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

The bilateral Z transform can represent both causal and non-causal time functions. Initial conditions are accounted by including additional inputs. It is also used to describe frequency response and stability.

Inverse z-Transform

To recover a discrete time signal from its Z-Transform the following Synthesis Equation must be used which is also called as Inverse z-Transform.

Inverse ZT $\{X(z)\} = x[n]$

$$x[n] = \frac{1}{2\pi}\int x(z) z^{n-1} dz$$

Properties of ROC

1. ROC of Z transform consists of a ring in the Z-plane centred about origin
2. ROC does not contain any poles.
3. If x[n] is of finite duration then the ROC is the entire Z-plane, except z = 0 or $z = \infty$

   Example: $\delta[n+1] \xleftrightarrow{\text{Z.T.}} Z$

   ROC : entire Z-plane except at $z = \infty$
4. If x[n] is a right side sequence and if $|z| = r_0$ is in the ROC then, all values of ‘z’ for which $|z| > r0$ will also be in the ROC and if x[n] is left sided then ROC is $|z| < r0$
5. If X(z) is rational and if the signal is right sided, to define ROC consider the largest pole in magnitude and if it is left sided ROC is defined with the smallest pole in magnitude and if two-sided consider the common ROC.

Solved Examples

Example: Find Z-transform & ROC of

$x[n] = a^n u[n]$

Solution:

$$X(z) = \sum_{-\infty}^{\infty} a^n z^{-n} u[n] = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$X(z) = \frac{1}{1-\frac{a}{z}};\ |az^{-1}| < 1 => |z| > |a|$$

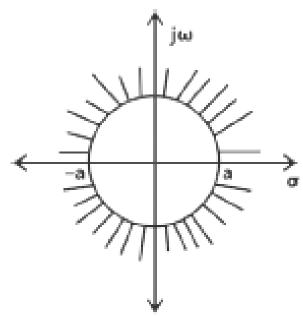


Fig. 7.22

Example: Find Z-transform & ROC of

$x[n] = a^n u[-n-1]$

$$X(z) = \sum_{-\infty}^{\infty} a^n z^{-n} u[-n-1] z^n$$

$$= \sum_{n=\infty}^{-1} -a^n z^n = -\sum_{n=\infty}^{-1} (az^{-1})^n$$

$$X(z) = \sum_{n=1}^{\infty} (az^{-1})^{-n}$$

$$X(z) = \sum_{n=1}^{\infty} (a^{-1}z)^n; |a^{-1}z| < 1$$

$$=> |z| < |a|$$

$$X(z) = \frac{z/a}{1 - z/a} = \left(\frac{z}{z-a}\right)$$

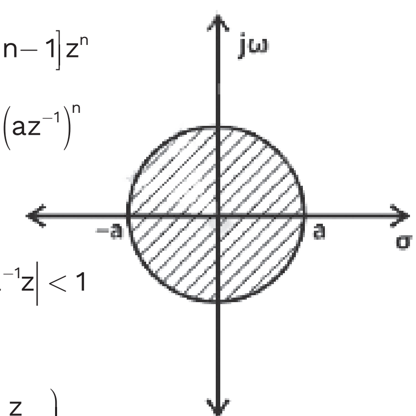


Fig. 7.23

Note: For the signals mentioned in the above examples the z-Transform turns out to be same but their ROCs are different. So, ROC makes a z-Transform unique.

	Domain	Tranform	ROC
$z\left[a^n u(n)\right]$	$n \geq 0$	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$z\left[a^{n-1} u(n-1)\right]$	$n > 0$	$\frac{1}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$z\left[a^n u(-n)\right]$	$n \leq 0$	$\frac{a}{a-z}$	$\lvert z\rvert < \lvert a\rvert$
$z\left[a^n u(-n-1)\right]$	$n < 0$	$\frac{z}{a-z}$	$\lvert z\rvert < \lvert a\rvert$

Table 7.3

Example: Determine the Z-Transform of the following signal $\frac{1}{n!}$

$$Z\left(\frac{1}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{z^{-n}} = 1 + \frac{1}{z} + \frac{1}{2!} + \frac{1}{3!} \ldots . n!$$

Similarly,

$$Z\left\{\frac{n!}{(n+1)!}\right\} = \sum_{n=0}^{\infty} \frac{1}{(n+1)} z^{-n} \lvert z\rvert > 0 \text{ or } \lvert z\rvert \neq 0$$

$$= 1 + \frac{1}{2!z^1} + \frac{1}{3!z^2} + e^z$$

$$z\left\{\frac{1}{(n+1)!}\right\} = z\left\{\frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \ldots\right\}$$

$$\text{Similarly } z\left\{\frac{1}{(n-1)!}\right\} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{-n} = \frac{1}{z} e^{1/z}$$

Z-Transform of Trigonometric Functions

Combining Trigonometric Function into a complex exponential

$$Z\{\cos n\theta + i \sin n\theta\} = Z\left\{e^{in\theta}\right\}$$

$$Z\left\{\left(e^{i\theta}\right)^n\right\} = \frac{z}{\left(z - e^{i\theta}\right)} = \frac{z\left(z - e^{-i\theta}\right)}{z\left(z - e^{-i\theta}\right) z\left(z - e^{-i\theta}\right)}$$

$$\text{for } \lvert z\rvert > \left\lvert e^{-i\theta}\right\rvert = 1$$

$$Z\left\{e^{-in\theta}\right\} = \frac{z^2 - z(\cos\theta - i\sin\theta)}{z^2 - z\left(e^{\theta} + e^{\theta}\right) + e^{i\theta} e^{-i\theta}}$$

$$= \frac{z^2 (z\cos\theta + iz\sin\theta)}{z^2 - 2z\cos\theta + 1}$$

Comparing Real and Imaginary parts.

$$Z\{\cos n\theta\} = \frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}$$

$$Z\{\sin n\theta\} = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$\text{similarly} z\{\sinh n\theta\} = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$Z\{\cosh n\theta\} = \frac{z^2 - z\cosh\theta}{z^2 - 2z\cos\theta + 1}$$

Solved Examples

Example: Determine Z-Transform of $\sin n\theta \cos n\theta$.

Solution:

$$Z\{\sin\theta\cos n\theta\} = \frac{1}{2} z(\sin(2\theta)) = \frac{1}{2}\left\{\frac{z\sin 2\theta}{z^2}\right\}$$

Example: Determine ROC for Z-Transform of the following function $f(n) = \left(\frac{1}{3}\right)^n + \left(\frac{1}{2}\right)^n u(n)$

Solution: $f(n) = \begin{cases} \left(\frac{1}{3}\right)^n + \left(\frac{1}{2}\right)^n & n \geq 0 \\ \left(\frac{1}{3}\right)^{-n} & n < 0 \end{cases}$

$$F(z) = \frac{z}{z - \frac{1}{3}} + \frac{z}{z - \frac{1}{2}} + \frac{z}{3 - z}$$

The ROC for each term in the above expression is,

$$\lvert z\rvert > \frac{1}{3} \ \& \ \lvert z\rvert > \frac{1}{2} \text{ and } \lvert z\rvert < 3$$

The intersection of these regions is,

$$\frac{1}{2} < \lvert z\rvert < 3$$

Some Common Z-Transform Pairs

Signal	z-Transform	ROC
Unit Impulse, $\delta[n]$	1	Entire z-plane
Unit step, $u[n]$	$\frac{z}{z-1}$	$\|z\| > 1$
$-u[-n-1]$	$\frac{z}{z-1}$	$\|z\| < 1$
$nu[n]$	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
$-nu[-n-1]$	$\frac{z}{(z-1)^2}$	$\|z\| < 1$
$a^n u[n]$	$\frac{z}{z-a}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{z}{z-a}$	$\|z\| < \|a\|$
$na^n u[n]$	$\frac{az}{(z-a)^2}$	$\|z\| > \|a\|$
$-na^n u[-n-1]$	$\frac{az}{(z-a)^2}$	$\|z\| < \|a\|$
$a^n, 0 \le n \le N-1$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$
$\cos\omega_0 n u[n]$	$\frac{z(z-\cos\omega_0)}{Z^2-2z\cos\omega_0+1}$	$\|z\| > 1$
$\sin\omega_0 n u[n]$	$\frac{z\sin\omega_0}{Z^2-2z\cos\omega_0+1}$	$\|z\| > 1$

Table 7.4

Properties of Z-transform

1. Linearity:

$x_1[n] \xleftrightarrow{\text{Z.T.}} X_1[z]$; ROC : R_1

$x_2[n] \xleftrightarrow{\text{Z.T.}} X_2[z]$; ROC : R_2

Then $ax_1 + x_1[n] + bx_2[n] \xleftrightarrow{\text{Z.T.}}$

$aX_1[z] + bx_2$; ROC, $R_1 \cap R_2$

2. Time shifting:

If $x[n] \xleftrightarrow{\text{Z.T.}} x(z)$; ROC : R

Then $x[n] \xleftrightarrow{\text{Z.T.}} z^{-n=0} x(z)$; ROC : R except deletion or addition of origin or inf infinity

3. Multiplication with an exponential:

If $x[n] \xleftrightarrow{\text{Z.T.}} x(z)$ with ROC $=$ R

Then $a^n x[n] \xleftrightarrow{\text{Z.T.}} x[a^{-1}z] = x\left[\frac{z}{a}\right]$

with ROC $= |aR|$

4. Time reversal:

If $x[n] \xleftrightarrow{\text{Z.T.}} x(z)$ with ROC $= R$

Then $x[-n] \xleftrightarrow{\text{Z.T.}} X(z^{-1})$

with ROC $= \frac{1}{R}$

5. Differentiation in z-domain:

If $x[n] \xleftrightarrow{\text{Z.T.}} x(z)$ with ROC $= R$

Then $nx[n] \xleftrightarrow{\text{Z.T.}} z\frac{d}{dz}X(z)$

with ROC $= R$

In general $(n)^k x[n] \xleftrightarrow{\text{Z.T.}} (-z)^k \frac{d^k}{dz^k}X(z)$

with ROC $= R$

6. Convolution in time domain:

$x_1[n] \xleftrightarrow{\text{Z.T.}} x_1(z)$ with ROC $= R_1$

$x_2[n] \xleftrightarrow{\text{Z.T.}} x_2(z)$ with ROC $= R_2$

Then $x_1[n] * x_2[n] \xleftrightarrow{\text{Z.T.}} X_1(z)X_2(z)$

with ROC $= R_1 \cap R_2$

7. Conjugate property:

If $x[n] \xleftrightarrow{\text{Z.T.}} x(z)$ with ROC $= R$

Then $x^*[n] \xleftrightarrow{\text{Z.T.}} x^*(z^*)$ with ROC $= R$

8. Time Accumulation:

Only valid for UZT

If $x[n] \xleftrightarrow{\text{Z.T.}} x(z)$ with ROC $= R$

Then $\sum_{k=-\infty}^{n} x[k] \xleftrightarrow{\text{Z.T.}} \frac{x(z)}{1-z^{-1}}$

with ROC $= R \cap |z| > 1$

9. Right Shift:

Only valid for UZT

If $x[n] \xleftrightarrow{\text{Z.T.}} x(z)$

Then $x[n-1] \xleftrightarrow{\text{Z.T.}} z^{-n}X(z) + x(-1)$

10. Left Shift:

Only valid for UZT

If $x[n] \xleftrightarrow{\text{Z.T.}} x(z)$

Then $x[n+1] \xleftrightarrow{\text{Z.T.}} zx(z) - zx(0)$

11. Initial Value Theorem:

Only valid for UZT

$x(0) = \lim_{z\to\infty} x(z)$

12. Final Value Theorem:

Only valid for UZT

$x(\infty) = \lim_{z\to 1}(1-z^{-1})x(z)$

or

$\lim_{z\to 1}(z-1)x(z)$

Note:

1. The z-transform X(z) must be a proper order i.e. order of numerator less than or equal to the order of denominator.
2. For system response H(z) to be stable all the poles must lie inside the unit circle & a simple pole is acceptable on unit circle, then final value theorem can be applied.
3. x (0) $\to$ initial value for causal

   ROC: $|z| > 0$

   x (∞) $\to$ final value for causal

   ROC: $|z| > 0$

   x ($-\infty$) $\to$ initial value for anti-causal

   ROC: $|z| < 0$

   x (0) $\to$ final value for anti-causal

   ROC: $|z| < 0$

Example: Let X(z) be the z-transform of a DT signal x[n] given as X(z) = $\frac{0.5z^2}{(z-1)(z-0.5)}$. The initial value of x[n] is ______________.

Solution: From initial value theorem

$$x[0] = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{0.5z}{(z-1)(z-0.5)} = \lim_{z\to\infty} \frac{0.5}{\left(1-\frac{1}{z}\right)\left(1-\frac{0.5}{z}\right)} = 0.5$$

Solved Examples

Example: Find z-Transform of the function

$$f(n)=\begin{cases}4^n & ; n<0\\ 5^n & ; n<0\end{cases}$$

Solution: The ROC for the first part of the signal i.e. defined for |z| < 4 n < 0 is The ROC for second part of the signal i.e. defined for |n| > 0 is z > 5

The intersection of these two sets is a null set and thus Z-Transform does not converge for any value of z.

Example: Determine the Z-Transform of ${}^nC_k$ where k is a scalar.

Solution: Z-Transform of a given function can be determined from the equation, $\sum_{n=k}^{\infty} n\, C_k Z^{-n}$

Since, ${}^nC_r$ = ${}^nC_{n-r}$

$$F(z)=\sum_{n=k}^{\infty} {}^nC_{n-k}Z^{-n}C_0Z^{-k}+^{(k+1)}+^{(k+2)}C_2z^{-(k+2)}+....$$

$$f(z)=z^{-k}\left\{{}^kC_0+^{(k+1)}C_1z^{-1}+^{(k+2)}C_2z^{-2}+..........\right\}$$

$$=z^{-k}\left\{\left(1-\frac{1}{z}\right)^{-(k+1)}\right\}\text{ for } |z|>1$$

Note:

$$(1-x)^n=1+nx+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+.....$$

$$(1-x)^{-(k+1)}=1+k+x+\frac{(k+1)}{2!}x^2+\frac{(k+1)(k+2)(k+3)}{3!}x^3+.....$$

Example: Find Z transform of $a^n nu(n)$

Solution: $z\{u(n)\}=\frac{z}{(z-1)}$

Then by differentiation in z-domain property

$$Z\{nu(n)\}=-z\frac{d}{dz}\left(\frac{z}{(z-1)}\right)=\frac{z}{(z-1)^2}$$

By Scaling Property,

$$z\left[a^n nu(n)\right]=\frac{\frac{z}{a}}{\left(\frac{z}{a}-1\right)^2}=\frac{az}{(z-a)^2}$$

Example:

If $z\{x(n)\}=\frac{z}{z-1}+\frac{z}{z^2+1}$ then $Z(x(n-2))=?$

Solution:

$$Z\{x(n-2)\}=z^{-2}X(z)=z^{-2}\left\{\frac{z}{z-1}+\frac{z}{z^2+1}\right\}$$

Example: If

$$Z(x(n))=\frac{z^2-3z+4}{(z-3)};\text{for}|z|>3\text{ then } x(3)=?$$

Solution:

$$X(z)=\frac{z^2-3z+4}{(z-3)^2}\left|\frac{3}{z}\right|<1$$

$$X(z)=\frac{Z^2-3z+4}{z^3\left(1-\frac{3}{2}\right)^3}=\frac{z^2-3z+4}{z^3}\left[1-\frac{3}{z}\right]^{-3}$$

$$X(z)=\frac{z^2-3z+4}{z^3}\begin{Bmatrix}1+3\times\frac{3}{z}+6\times\left(\frac{3}{z}\right)^2\\ +10\left(\frac{3}{2}\right)^3+...........\end{Bmatrix}$$

$$X(z)=x(0)+\frac{x(1)}{z}+\frac{x(2)}{z^2}+\frac{x(3)}{z^3}+..........$$

$$x(3)=\text{coefficient of }\frac{1}{z^3}=54-27+4=31$$

Solved Examples

Example: Let X(z) be the z-transform of a DT signal x[n] given as

$$X(z)=\frac{0.5z^2}{(z-1)(z-0.5)}$$

The initial value of x[n] is ______________

Solution: From the ainitial value theorem

$$x[0]=\lim_{z\to\infty}X(z)=\lim_{z\to\infty}\frac{0.5z^2}{(z-1)(z-0.5)}$$

$$\lim_{z\to\infty}\frac{0.5}{\left(1-\frac{1}{z}\right)\left(1-\frac{0.5}{z}\right)}=0.5$$

Inverse Z-transform by Partial Function Method

To find inverse z-Transform by Partial Fraction Method we first break the term in terms of Partial Fraction and then find Inverse z-Transform of each term. This can be seen in the example below:

Example:
$$x[n]=\frac{3-\frac{5}{6}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$
$$=\frac{2}{\left(1-\frac{1}{3}z^{-1}\right)}+\frac{1}{\left(1-\frac{1}{4}z^{-1}\right)}$$

Solution:
$$x[n]=2\left(\frac{1}{3}\right)^n u(n)+\left(\frac{1}{4}\right)^n u(n);|z|>\frac{1}{3}$$
$$\text{or } x[n]=2\left(\frac{1}{3}\right)^n u\,[-n-1]+\left(\frac{1}{4}\right)^n u(n);\frac{1}{4}<|z|<\frac{1}{3}$$
$$\text{or } x[n]=2\left(\frac{1}{3}\right)^n u[-n-1]+\left(\frac{1}{4}\right)^n u\,[-n-1];$$
$$|z|<\frac{1}{4}$$

Inverse Z-Transform by Cauchy Residue Method

$$X[n]=\frac{1}{2\pi j}\int_c^0 x(z)z^{n-1}dz$$

Where C is a closed contour in the counter clockwise direction enclosing all the singularities of function $X(z)z^{n-1}$.

$$x[n]=\sum(\text{residue})$$

For a pole at $z=\beta$ of order 'm'

Residue of

$$x(z)=\frac{1}{(m-1)!}\lim_{z\to\beta}\frac{d^{m-1}}{dz^{m-1}}\left[x(z)z^{n-1}(z-\beta)^m\right]$$

Solved Examples

Example: Find inverse transform of

$$X(z)=\frac{z}{(z-2)^3}$$

Solution: Pole at z = 2, order = 3

$$x[n]=\frac{1}{(3-1)!}\lim_{z\to 2}\frac{d^2}{dz^2}\left[z^n\right]=\frac{1}{2!}\lim_{z\to 2}(n-1)z^{n-2}$$
$$=n(n-1)z^{(n-3)}u[n];\ |z|>2$$
$$\text{If } z<2;[n]=-n(n-1)2^{(n-3)}u(-n-1)$$

Example: Find Inverse Z-Transform of $e^{3/z}$

Solution:
$$Z^{-1}\left(e^{3/z}\right)=z^{-1}\left\{1+\frac{3}{z}+\frac{1}{2!}\left(\frac{3}{z}\right)^2+\frac{1}{3!}\left(\frac{3}{z}\right)^3\ .....\right\}$$
$$x(n)=z^{-1}\left\{\sum_{n-0}^{\infty}\frac{3^n}{n!}\right\}=3^n\frac{1}{n!}$$

Example: Determine Inverse Z-Transform of
$$\frac{z}{(z-5)(z+4)}$$

Solution: $\frac{X(z)}{z}=\frac{1}{(z-5)(z+4)}=\frac{1/9}{z-5}-\frac{1/9}{z+4}$
$$=\frac{(5)^n}{9}u(n)-\frac{(-4)^n}{9}u(n)$$

This can also be solved by Residue Theorem We have to calculate the residue of the function
$$\left[\left(z-5^z\right)-z\left({}^n z^{-1}+4\right)\right]=(z-5)z\left({}^n z+4\right)$$

Residue at z = 5
$$\lim_{z\to 5}(z-5)\frac{z^n}{(z-5)(z+4)}=\frac{5^n}{9}$$

Residue at z = –4
$$\lim_{z\to 4}(z+4)\frac{z^n}{(z-5)(z+4)}=\frac{(-4)^n}{9}$$

Thus, the signal is,
$$x(n)=\left[\frac{5^n}{9}+\frac{(-4)^n}{9}\right]u(n)$$

Example: Determine the Inverse Z-Transform of $\frac{z}{(z-2)^2(z+2)}$

Solution: By Residue Method,

The function whose residue is to be calculated is,

$$\frac{z\times z^{n-1}}{(z-2)^2(z+2)}=\frac{z^n}{(z-2)^2(z+2)}$$

Residue at $z=-2$

$$\lim_{z\to 2}(z+2)\frac{z^n}{(z-2)^2(z+2)}=\frac{(-2)^n}{16}$$

Residue at $z=2$

$$\lim_{z\to 2}\frac{d}{dz}(z-2)^2\frac{z^n}{(z-2)^2-(z+2)}$$

$$=\lim_{z\to 2}\left[\frac{n(z+2)z^{n-1}-z^n(1)}{(z+2)}\right]=\frac{n2^{n+1}-2^n}{16}$$

$$x(n)=\frac{n2^{n+1}-2^n}{16}+\frac{(+2)^n}{16}$$

Example: If $\{f(n)\}=\left(\frac{1}{z-2}\right)\frac{z}{(z+8)(z-2)}$ in $|z|=1$ then $f(2)=$ __________?

Solution: By Residue Method,

The function whose residue is to be calculated is,

$$\frac{z\times z^{n-1}}{\left(z-\frac{1}{2}\right)(z+3)(z-2)}=\frac{z^n}{\left(z-\frac{1}{2}\right)(z+3)(z-2)}$$

Since, ROC: $|z|=1$

Only z = ½ lies inside ROC so residue will only be computed at this pole

$$x(n)=\lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{(z)^n}{\left(z-\frac{1}{2}\right)(3+z)(z-2)}=\frac{-4}{21}\left(\frac{1}{2}\right)^n$$

$$f(2)=\frac{-4}{21}\left(\frac{1}{2}\right)^2=-\frac{1}{21}$$

Practice Questions

1. The Laplace Transform of the function f(t) = e^{at} when t > 0 and where a is a constant is

(A) $\frac{1}{(s-a)}$ (B) $\frac{1}{(s+a)}$

(C) $\frac{1}{(s-a)^{-1}}$ (D) $\frac{1}{(s+a)^{-1}}$

2. The Laplace transform of f(t) is F(s) .

Given F(s) = $\frac{\omega}{s^2+\omega^2}$, the final value of f(t) is

(A) infinite (B) zero

(C) one (D) none

3. The inverse Laplace transform of $\frac{s+9}{s^2+6s+13}$ is

(A) cos 2 t + 9 sin 2t

(B) e^{-3t} cos 2t - $3e^{-3t}$ sin 2t

(C) e^{-3t} sin 2t + $3e^{-3t}$ cos 2t

(D) e^{-3t} cos 2t + $3e^{-3t}$ sin 2t

4. If L{f(c)} = $\frac{2(s+1)}{s^2+2s+1}$ then f(0+) and f(∝) given by ......

(A) 0, 2 respectively

(B) 2, 0 respectively

(C) 0, 1 respectively

(D) 2/5, 0 respectively

5. The inverse Laplace transform of the function $\frac{s+5}{(s+1)(s+3)}$ is ....

(A) $2e^{-t}-e^{-3t}$

(B) $2e^{-t}+e^{-3t}$

(C) $e^{-t}-2e^{-3t}$

(D) $e^{-t}+2e^{-3t}$

6. Laplace transform L(f’), where f’ is the derivative of function f, is given by

(A) L(f) – f(0) (B) sL(f) – f(0)

(C) s^2L(f) – f(0) (D) L(f)/s – f(0)

7. The Laplace transform of $e^{\alpha t} \cos \alpha t$ is equal to

(A) $\frac{s-\alpha}{(s-\alpha)^2+\alpha^2}$

(B) $\frac{s+\alpha}{(s+\alpha)^2+\alpha^2}$

(C) $\frac{1}{(s-\alpha)^2}$

(D) none

8. The Laplace transform of $(t^2 - 2t)$ u (t–1) is

(A) $\frac{2}{s^3}e^{-s} - \frac{2}{s^2}e^{-s}$

(B) $\frac{2}{s^3}e^{-2s} - \frac{2}{s^2}e^{-s}$

(C) $\frac{2}{s^3}e^{-s} - \frac{2}{s}e^{-s}$

(D) none

9. The Laplace transform of a unit step function $Ua(t)$, defined as

$U_a(t) = \{\frac{0 \text{ for } t < a}{1 \text{ for } t < a}$ is

(A) $\frac{e^{-as}}{s}$ (B) Se^{-as}

(C) s–u(0) (D) se^{-as}–1

10. $(S+1)^{-2}$ is the Laplace transform of

(A) t^2 (B) t^3

(C) e^{-2t} (D) te^{-t}

Answer Key

1 – A	2 – D	3 – D	4 – B	5 – A
6 – B	7 – A	8 – D	9 – A	10 – D