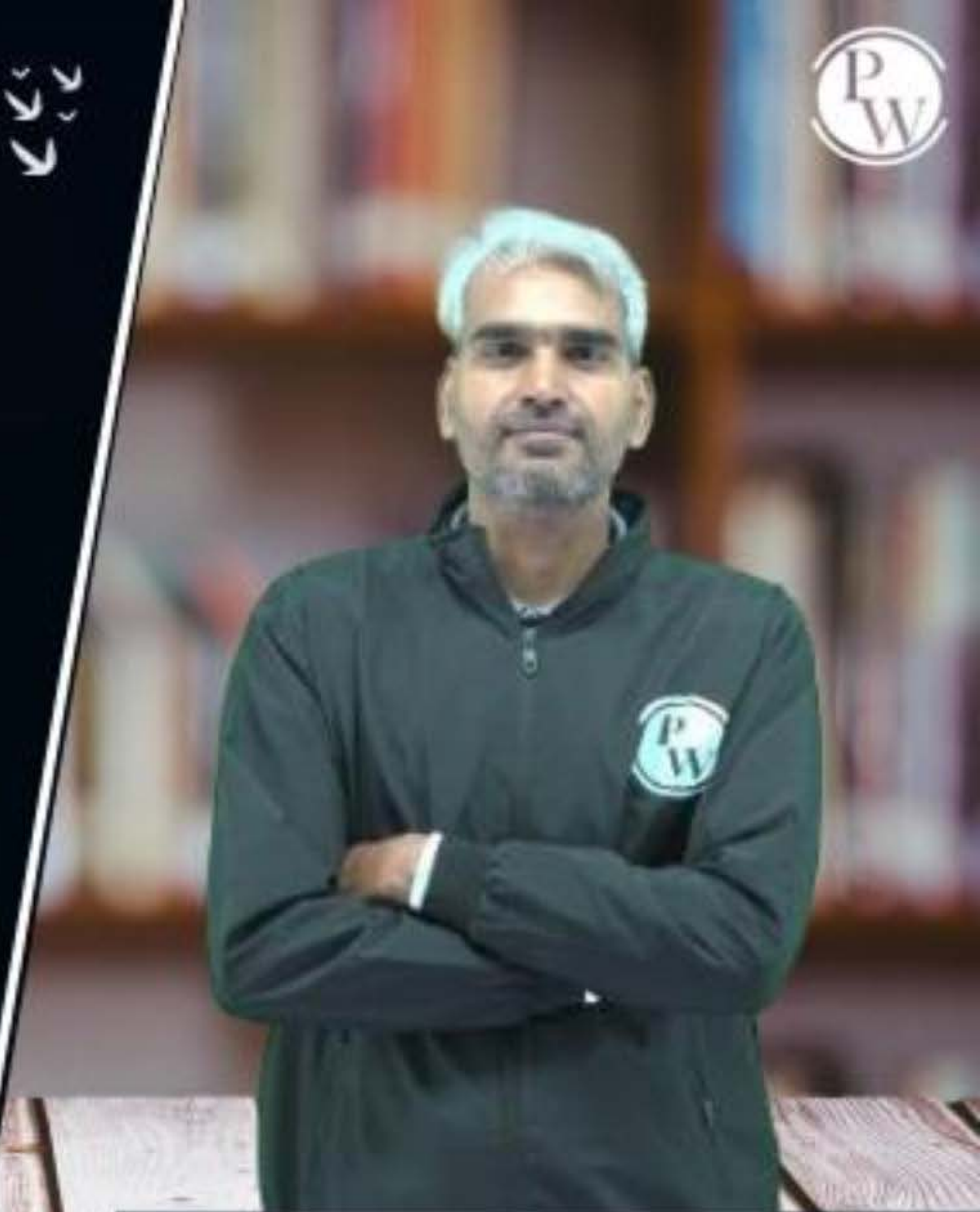
Computer Science & IT

Discrete Mathematics

Set Theory & Algebra

Lecture No. 16



By- Vishal Sir

Recap of Previous Lecture

- **Topic** Boolean lattice / Boolean algebra
- **Topic** Functions
- **Topic** Range of a function
- **Topic** Injective (one-one) function

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Topics to be Covered

- Topic — Surjective (onto) function
- Topic — Number of onto functions
- Topic — Bijective function
- Topic — Different types of functions
- Topic — Inverse of a function

Slide

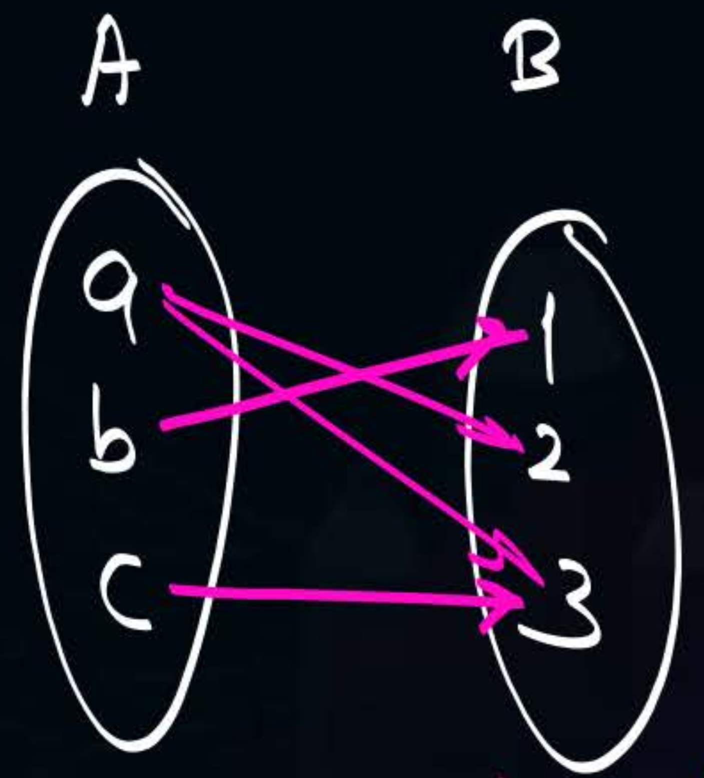
Topic : Function

A relation from set A to set B is called a function from set A to set B if and only if

Every element of set A relates with exactly one element of set B

{ Function from set A to set B is also a relation from set A to set B }

{ Every function is a relation, but every relation need not be a function }



It is a valid relation from A to B but is not a function from A to B

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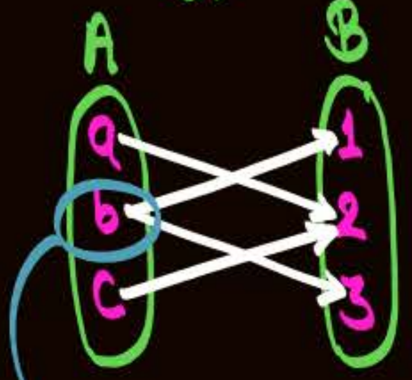
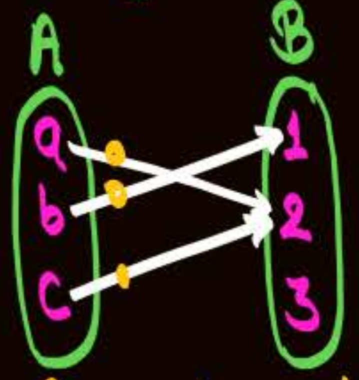
(i)



element 'b' of set A relates with two elements of set B
∴ Not a function from A to B

(ii)

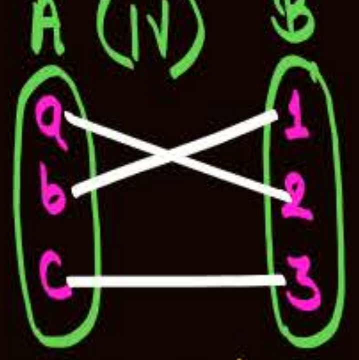


Every element of set A relates with exactly one element of set B
∴ It is a function from A to B

(iii)

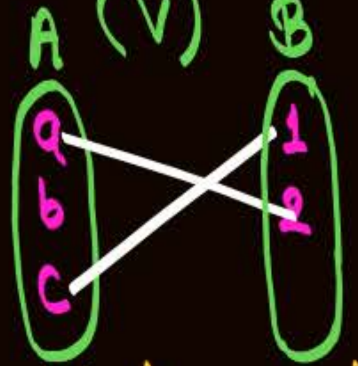
'c' relates with '0' elements of set B.
∴ Not a function from A to B

(iv)



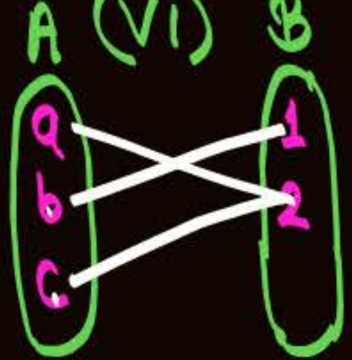
Every element of set A relates with exactly one element of set B
∴ It is a function from A to B

(v)



'b' does not relate with any element of set B
∴ Not a function from A to B

(vi)



Every element of set A relates with exactly one element of set B
∴ It is a function from A to B

(i) Neither a function from A to B nor a function from B to A

(ii) function from A to B but not a function from B to A

(iii) Neither a function from A to B nor a function from B to A

(iv) Function from A to B as well as function from B to A.

(v) Not a function from A to B, but a function from B to A

(vi) function from A to B, but not a function from B to A

Topic : Function

- A function 'f' from set A to B is denoted by function $f: A \longrightarrow B$

- Let $f: A \rightarrow B$ is a function, then

  Set A is called domain of the function

  & Set B is called Co-domain of the function

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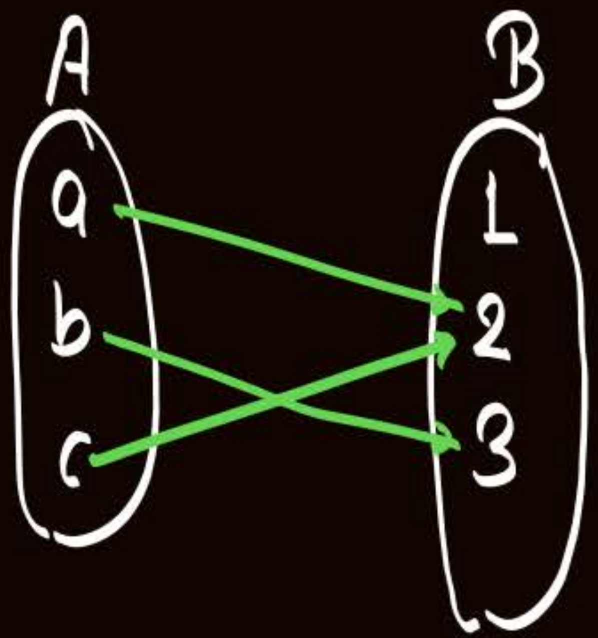
eg:



function $f: A \to B$

Domain = $\{a, b, c\}$

Co-domain = $\{1, 2, 3\}$

In a function

it is not necessary for every element of the Co-domain to be mapped by at least one element of the domain

eg: '1' is not mapped by any element of domain

Topic : Range of a function

It is a set of all the elements of the co-domain that are mapped by at least one element of domain

In general

$$\boxed{\text{Range of function} \subseteq \text{Co-domain of function}}$$

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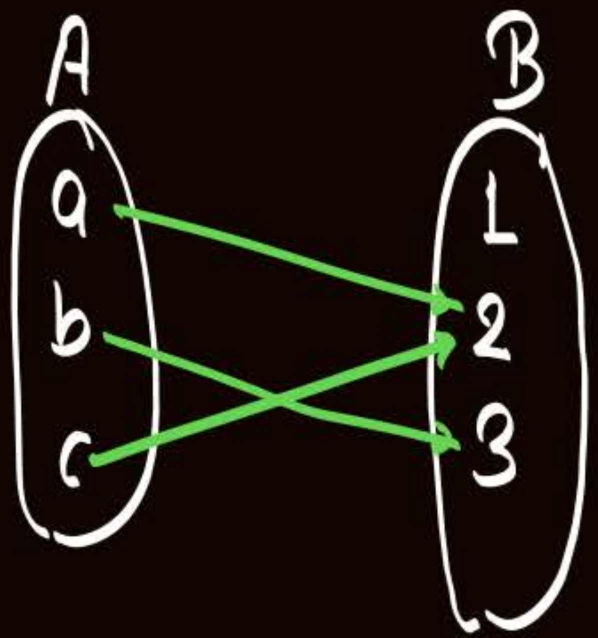
eg:



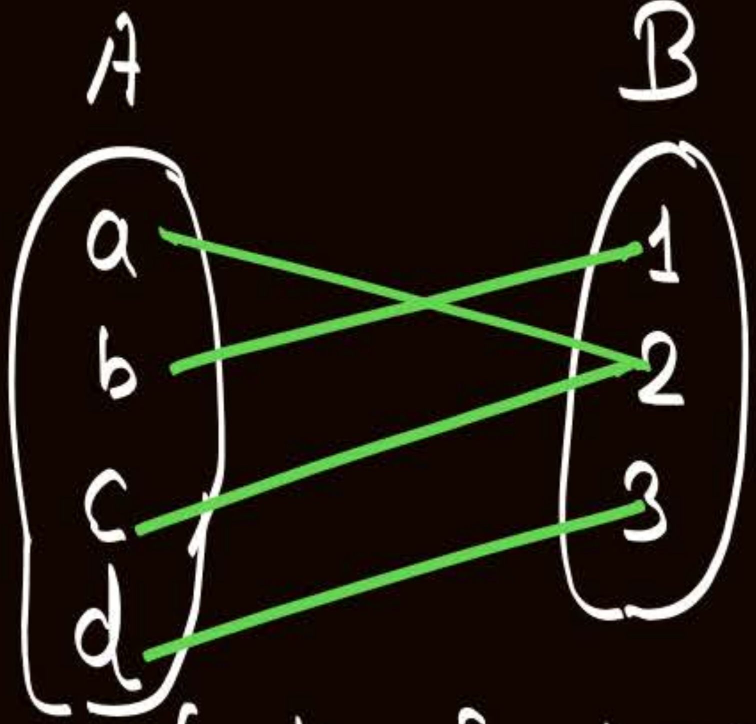
function $f_1 : A \to B$

Domain = $\{a, b, c\}$

Co-domain = $\{1, 2, 3\}$

Range = $\{2, 3\}$

Range $\subset$ Co-domain



function $f_2 : A \to B$

Domain = $\{a, b, c, d\}$

Co-domain = $\{1, 2, 3\}$

Range = $\{1, 2, 3\}$

Range = Co-domain

Note:-

1. Function must be defined for every element of the domain

2. The result of the function on the input from its domain can not acquire a value which is not present in Co-domain

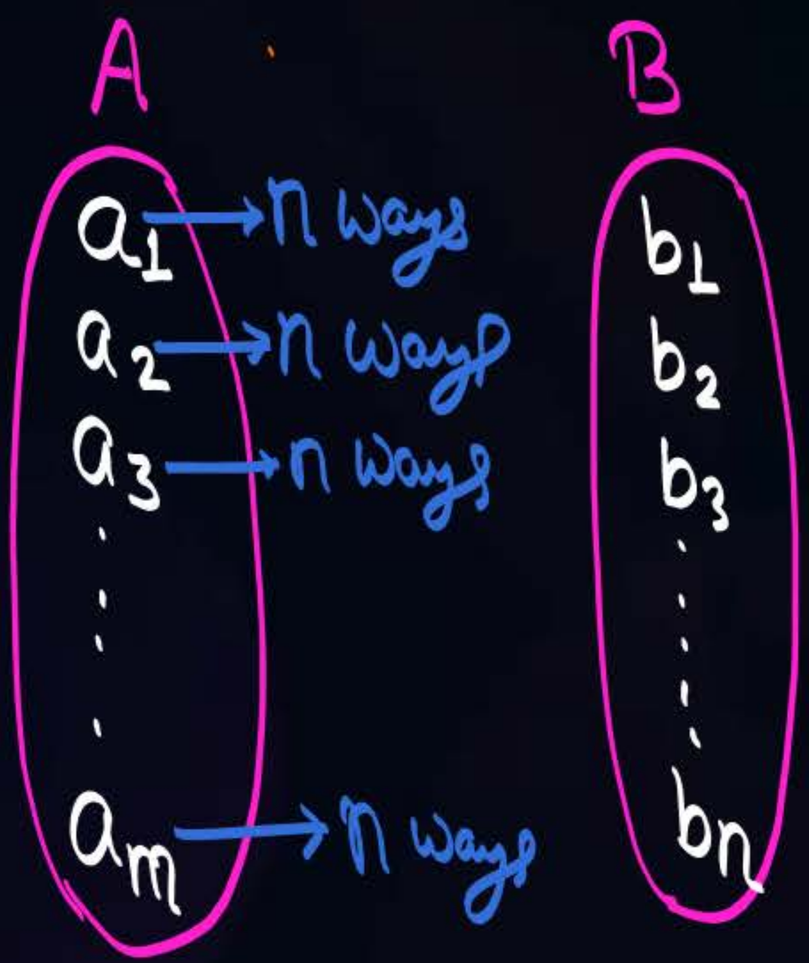
Topic : Total number of functions

let $|A| = m$ & $|B| = n$, then

total Number of functions Possible from set A to set B $= n^m = (\text{Size of Co-domain})^{\text{Size of domain}}$

$$= \underbrace{n * n * n * \cdots * n}_{m \text{ times}}$$

$$= n^m$$



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Note:- A function from set A to set A itself is called a function on set A.

Note: if $|A| = n$, then

Total number of functions possible on set A $= n^n$

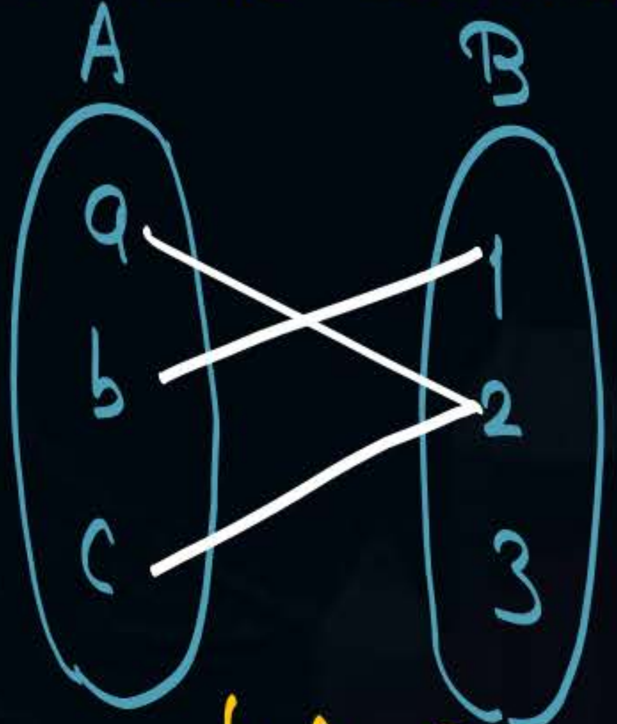
Topic : Injective (one-one) function

A function $f : A \to B$ is called an injective (one-one) function

Only if distinct elements of domain have distinct images in co-domain

i.e, if $\underbrace{a \neq b}_{\text{i.e., } a \text{ \& } b \text{ are distinct}}$, then $\underbrace{f(a) \neq f(b)}_{\text{images of 'a' \& 'b' must be distinct}}$

i.e, a & b are distinct → then → images of 'a' & 'b' must be distinct



f(a) is called image of element 'a' w.r.t function 'f'

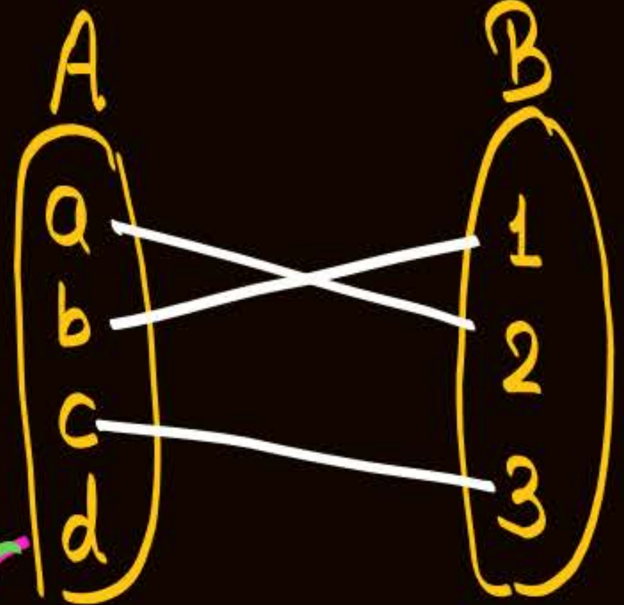
{'a' is called Pre-image of f(a)}

$f : A \to B$

it is a function but not a one-one function

as $a \neq c$ but $f(a) = f(c) = 2$

$\boxed{\text{Two pre-images of element '2'}}$



if there is no image of 'd' then it will not be a function

if there is an image of 'd' then it will be same as one of a, b or c

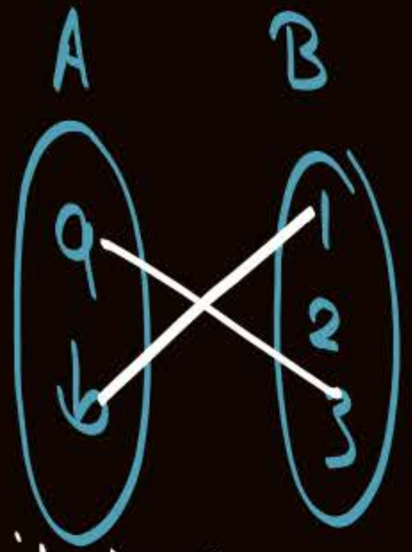
∴ it will not be one-one function

If $|A| > |B|$, then

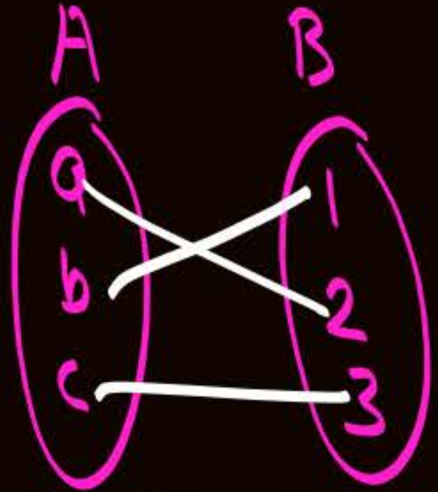
One-one function is not possible from set A to set B

A one-one function from set A to set B is possible only if,

$$|A| \leq |B|$$



it is one-one function



it is one-one function

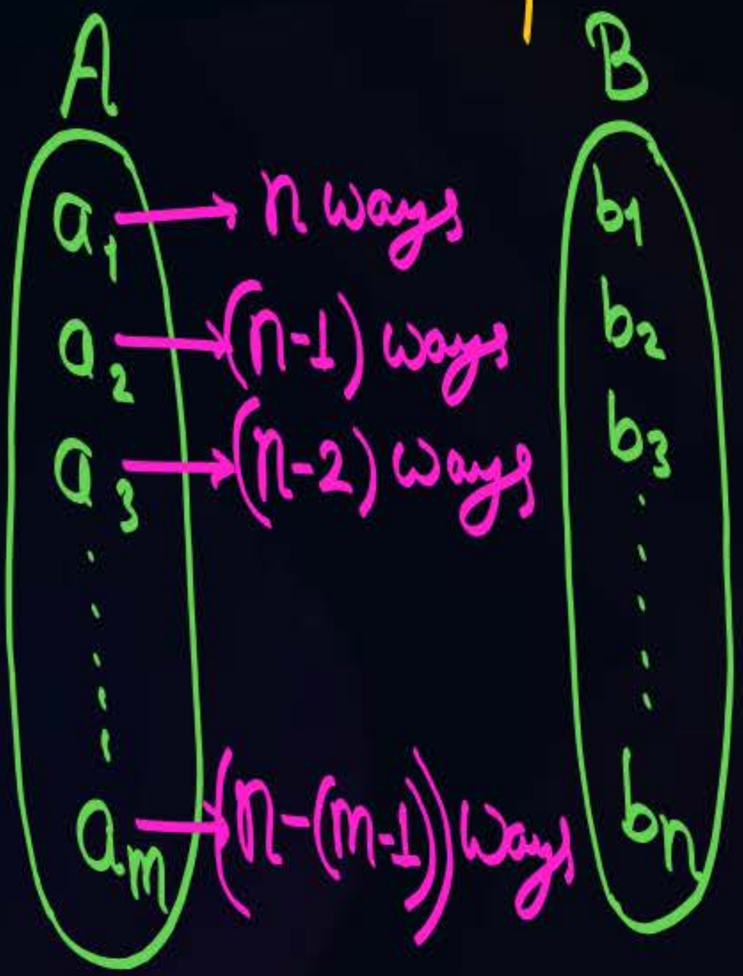
Note: In a one-one function, every element of co-domain will have at most one pre-image

Topic : Number of one-one function

let $|A| = m$ & $|B| = n$ such that $m \leq n$,

then,

total number of one-one functions possible from A to B $= {}^{n}P_{m} = \dfrac{n!}{(n-m)!}$



$$= n \times (n-1) \times (n-2) \times \cdots (n-m+1) \times \frac{(n-m) \times (n-m-1) \times \cdots 3 \times 2 \times 1}{(n-m) \times (n-m-1) \times \cdots 3 \times 2 \times 1}$$

$$= \frac{n!}{(n-m)!} = {}^{n}P_{m}$$

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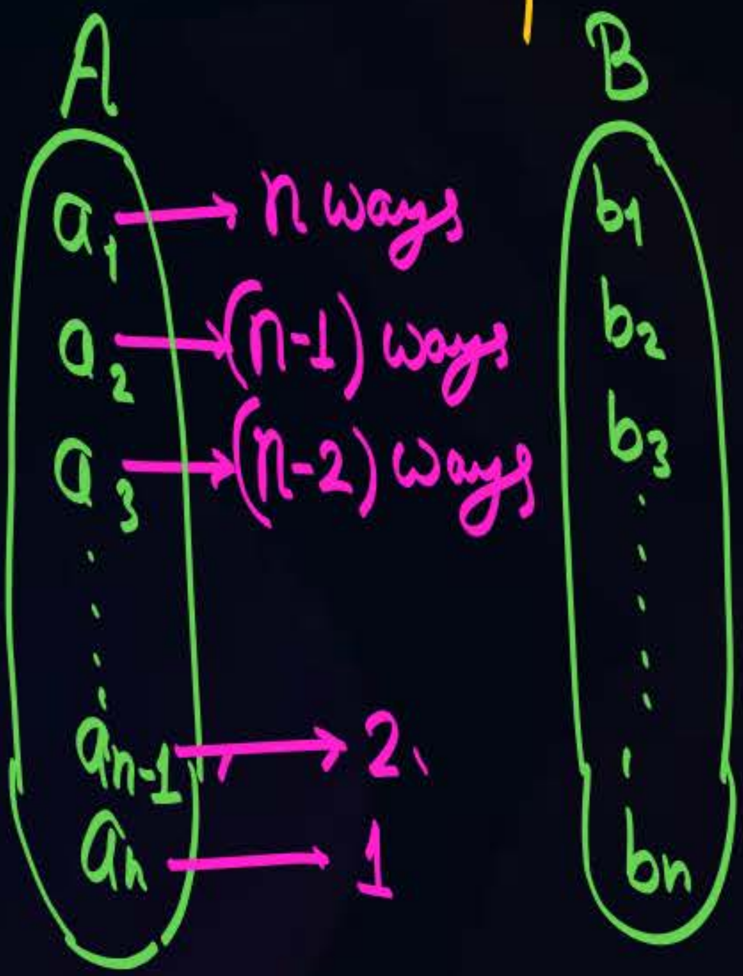
Topic : Number of one-one function

let $|A| = |B| = n$,

then,

total number of one-one functions possible from A to B $= n! \Rightarrow \boxed{{}^{n}P_{n} = \frac{n!}{(n-n)!} = n!}$



$= n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$

$= n!$

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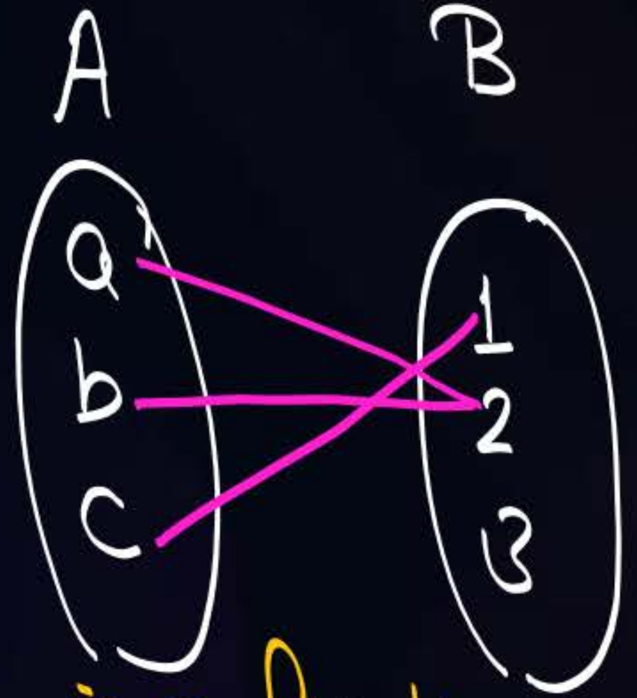
Topic : Surjective (onto) function

A function $f: A \to B$ is called surjective (onto function) if and only if every element of Co-domain is mapped by atleast one element of domain



element '3' is not mapped by any element of domain

it is a function, but not an 'Onto' function

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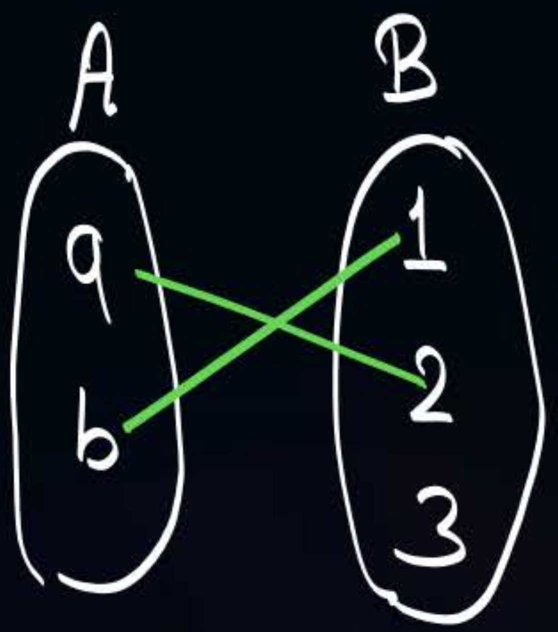
Topic : Surjective (onto) function



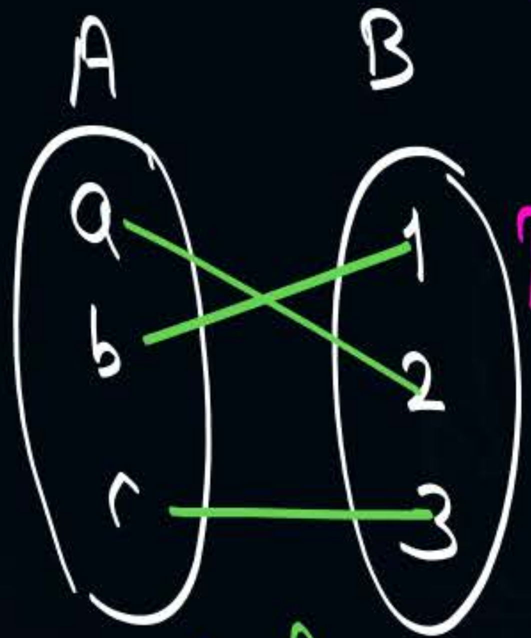
Range $\subset$ Codomain

$f: A \rightarrow B$

it is a function, but it can not be onto, if we try to map '3' with one of the elements of set A, then it will not remain a function

An onto function is possible from set A to set B, only if $|A| \geq |B|$

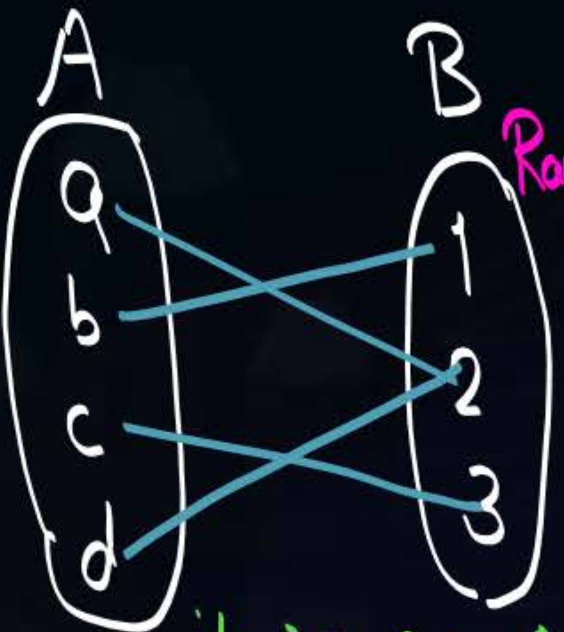
$|A| = |B|$



Range = Co-domain

onto function
{Also one-one function}

$|A| > |B|$



Range = Co-domain

it is an onto-function (but not one-one function)

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1. In an onto function.

$$\boxed{\text{Range of function} = \text{Co-domain of function}}$$

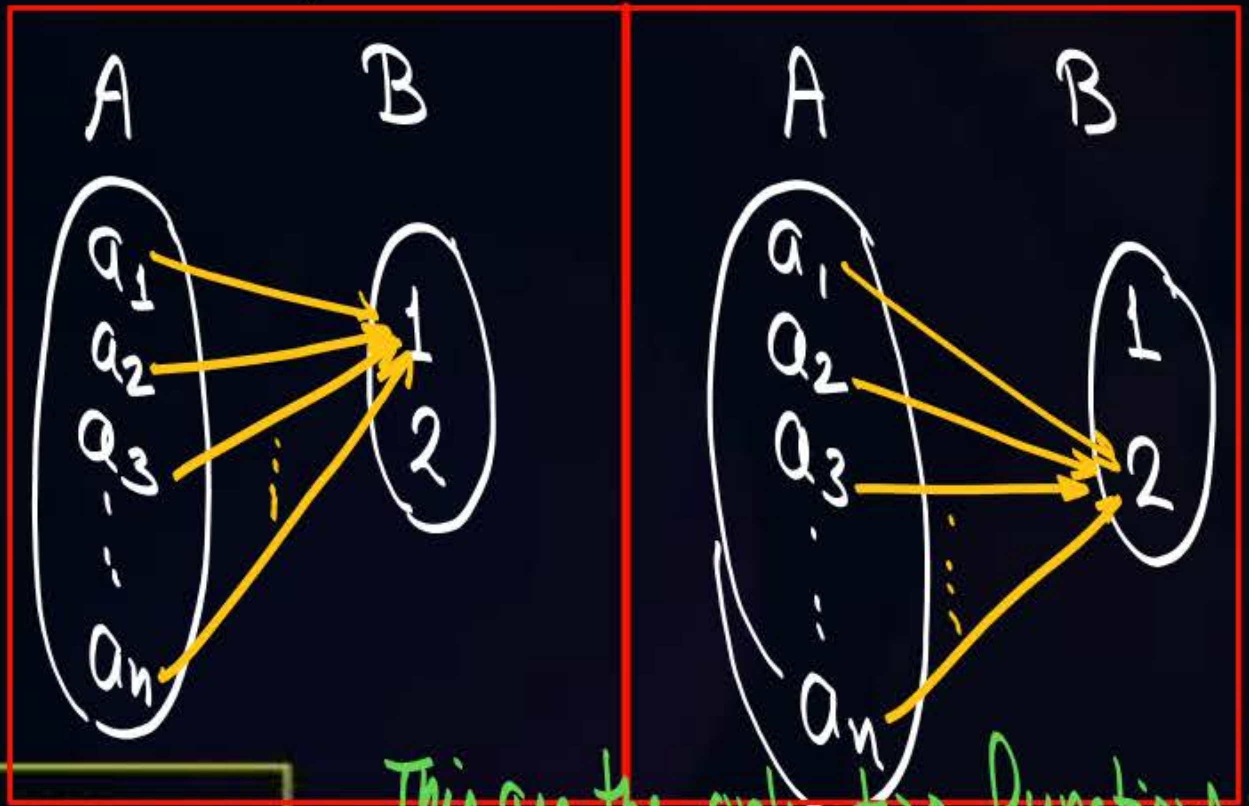
2. In an onto function every element of Co-domain will have at least one pre-image

Topic : Number of onto functions

Case 1 let $|A| = n$ $(n \geq 2)$ & $|B| = 2$.

Number of onto-functions Possible from A to B $=$ (Total no. of functions from A to B) $-$ (No. of functions which are not onto)

$$= 2^n - 2$$



This are the only two functions from Set A to Set B which are not "onto"

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Topic : Number of onto functions

Case ① let $|A| = n$ $(n \geq 2)$ & $|B| = 2$.

$$\boxed{\text{Number of onto-functions Possible from } A \text{ to } B = 2^n - 2}$$

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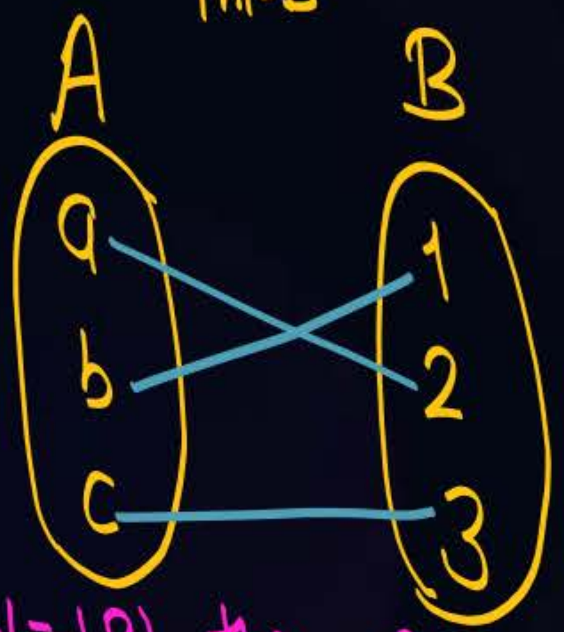
Topic : Number of onto functions

Case ②   If $|A| = |B| = n$.

Number of onto functions Possible from A to B = Number of one-one functions from A to B $= {}^{n}P_{n} = n!$

$|A| = B$



if $|A| = |B|$, then every one-one function from A to B will also be "Onto".
& Every Onto function from A to B will also be one-one

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Topic : Number of onto functions

Q: Let $\underbrace{|A| = 4}_{(n)}$ & $\underbrace{|B| = 3}_{(n-1)}$,
then how many onto functions are possible from set A to set B $= {}^4C_2 * 3!$

$= 6 * 6 = \underline{\underline{36}}$

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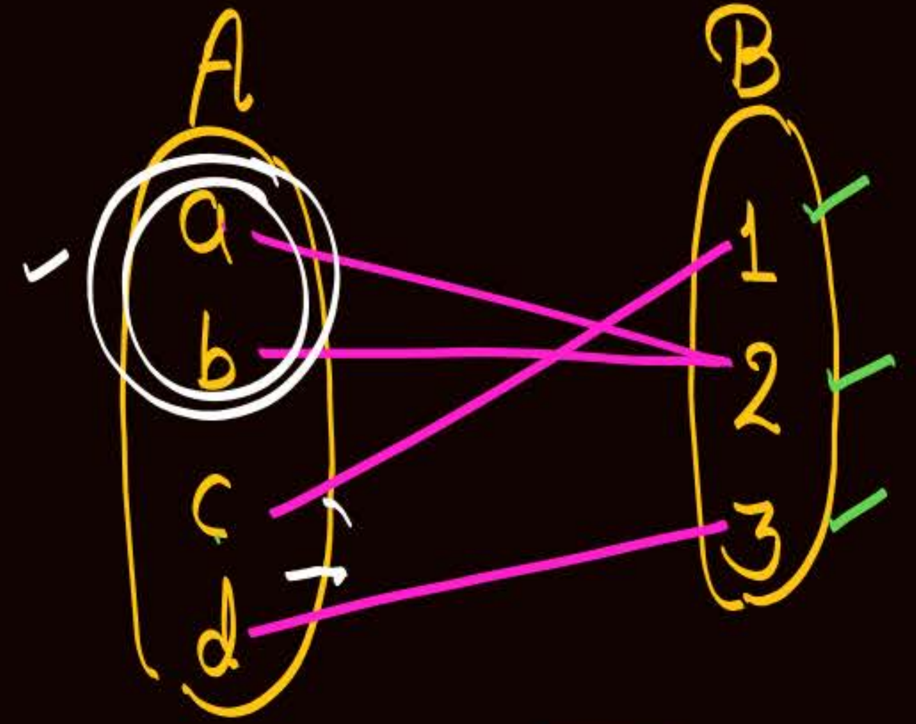

∴ No. of onto functions from A to B $= {}^4C_2 * 3 * 2!$

$= {}^4C_2 * 3!$

In any onto function from A to B there will be exactly two elements in set A which will map to the same element of set B.

Every other element of set A will map with distinct elements of set B

Topic : Number of onto functions

Case ③ $|A| = n$ & $|B| = (n-1)$

i.e. $\boxed{|A| = |B| + 1}$

Number of onto functions possible from set A to set B $= {}^{n}C_{2} \cdot (n-1)!$

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Topic : Number of onto functions

Case (4) let $|A| = m$ & $|B| = n$ s.t. $m \geq n$

No. of onto functions Possible from set A to set B = Total no. of functions from set A to set B − (No. of functions in which one element of Co-domain is not mapped by any element of domain + No. of functions in which two elements are not mapped + - - - - + (n-1) elements are not mapped)

No. of functions which are not onto

$$= n^m - {}^nC_1 * (n-1)^m + {}^nC_2 * (n-2)^m - {}^nC_3 * (n-3)^m + \cdots + (-1)^{n-1}\, {}^nC_{n-1} (n-(n-1))^m$$

$$= {}^nC_0 \cdot (n-0)^m - {}^nC_1 * (n-1)^m + {}^nC_2 (n-2)^m - {}^nC_3 \cdot (n-3)^m + - - - + (-1)^{n-1} \cdot {}^nC_{n-1} (n-(n-1))^m$$

$$\boxed{= \sum_{i=0}^{n-1} (-1)^i \cdot {}^nC_i * (n-i)^m}$$

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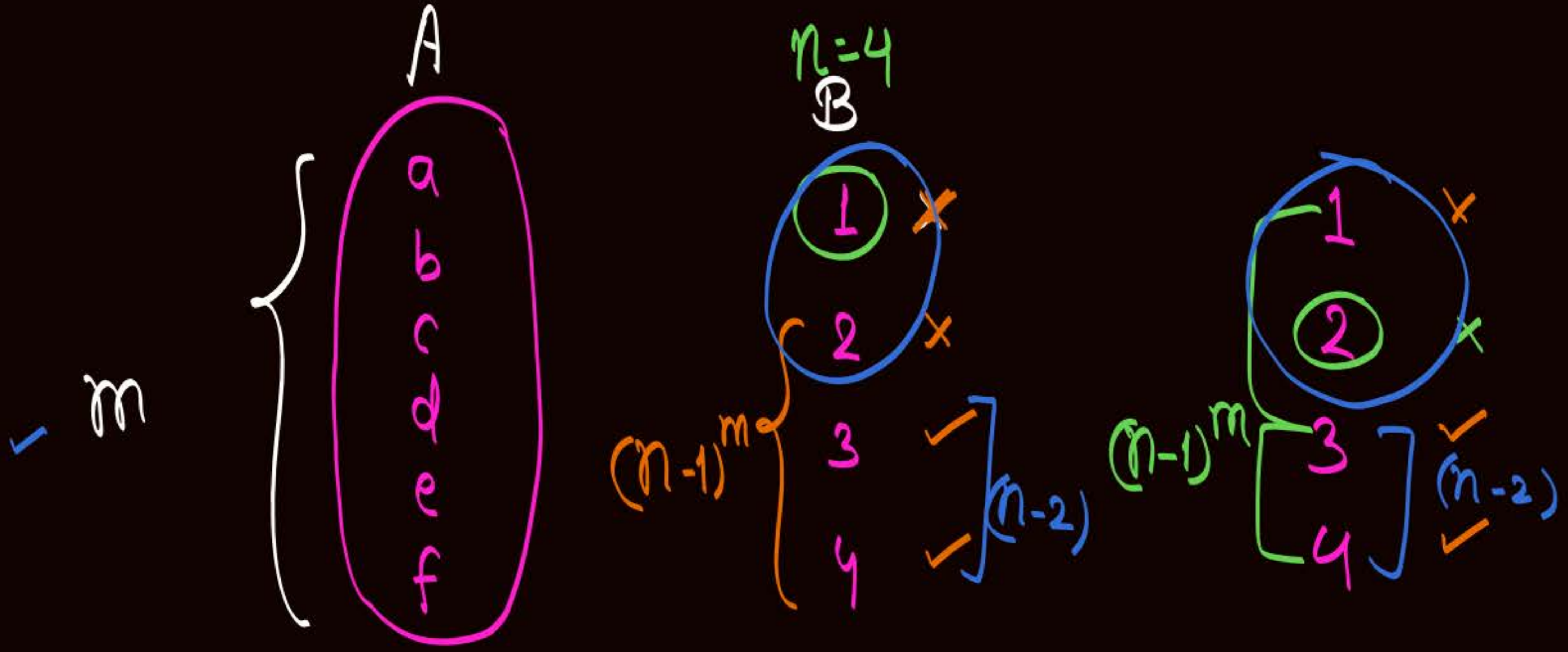
m
A
a
b
c
d
e
f
n=4
B
1
2
3
4
$(n-1)^m$
$(n-2)$
1
2
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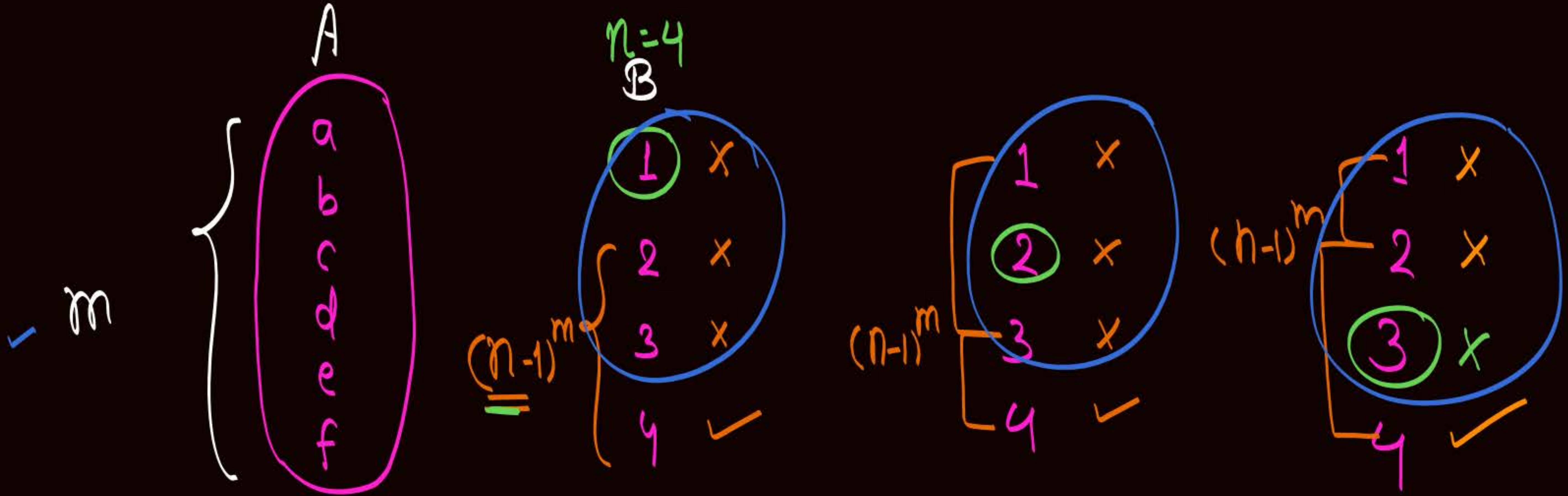
4
$(n-1)^m$
$(n-2)$


A
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B
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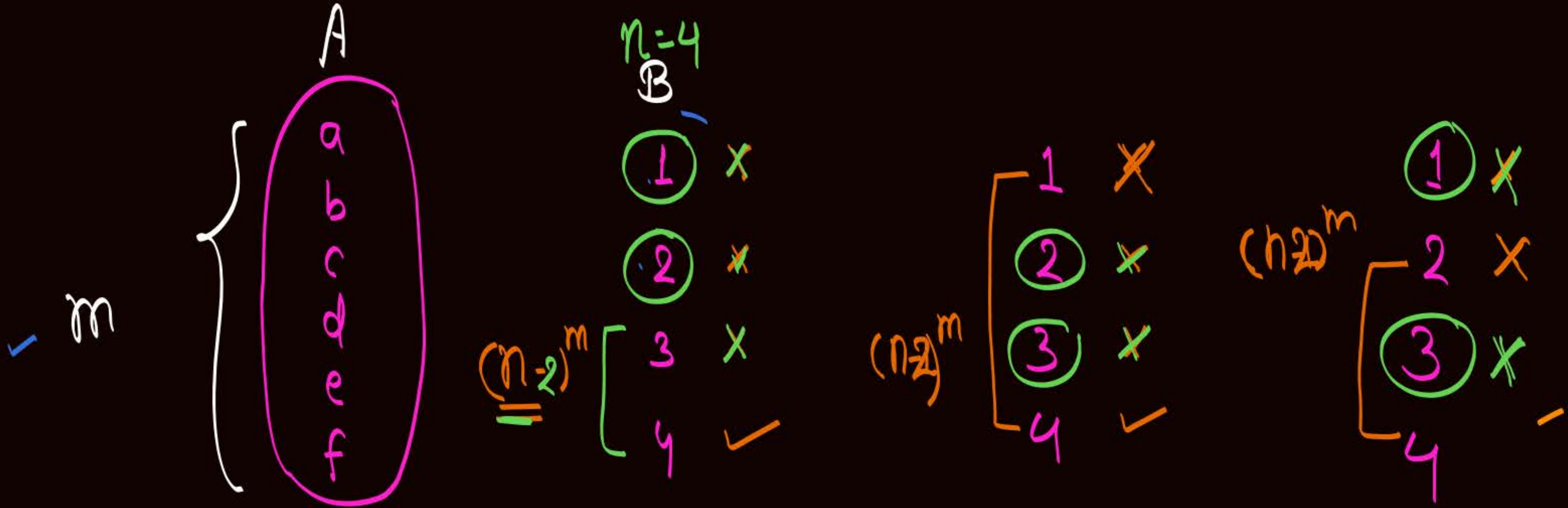
(n-1)^m



A
m
a
b
c
d
e
f
n=4
B
1
2
3
4
(n-2)^m
1
2
3
4
(n-2)^m
1
2
3
4
(n-2)^m

Case		No. of onto functions
1	$\lvert A\rvert = n$ & $\lvert B\rvert = 2$	$2^n - 2$
2	$\lvert A\rvert = \lvert B\rvert = n$	$n!$
3	$\lvert A\rvert = n$ & $\lvert B\rvert = n-1$	${}^nC_2 \times (n-1)!$
4	$\lvert A\rvert = m$ & $\lvert B\rvert = n$ $m \geq n$	$\sum_{i=0}^{n-1} (-1)^i \cdot {}^nC_i \cdot (n-i)^m$

Topic : Bijective Function

- A function $f: A \to B$ is called a bijective function if and only if, 1. $f: A \to B$ is one-one (Injective)
  & 2. $f: A \to B$ is onto (Surjective)

- A function is bijective iff it is one-one (i.e. $|A| \le |B|$) as well as onto (i.e., $|A| \ge |B|$)

- A bijective function from set A to set B is possible only if $|A| = |B|$

- If $|A| = |B| = n$, then
  Number of bijective functions possible from A to B $= n!$

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Note: If there exist a bijective function from set A to set B (or from set B to set A), then set A and set B are said to have one-one correspondance

$\therefore$ One-one Correspondance $\Longrightarrow$ One-one as well as onto

- if One-one Correspondance exists between set A & set B,
  then $|A| = |B|$

2 mins Summary

- **Topic** Surjective (onto) function
- **Topic** Number of onto functions
- **Topic** Bijective function
- **Topic** Different types of functions
- **Topic** Inverse of a function

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THANK - YOU

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