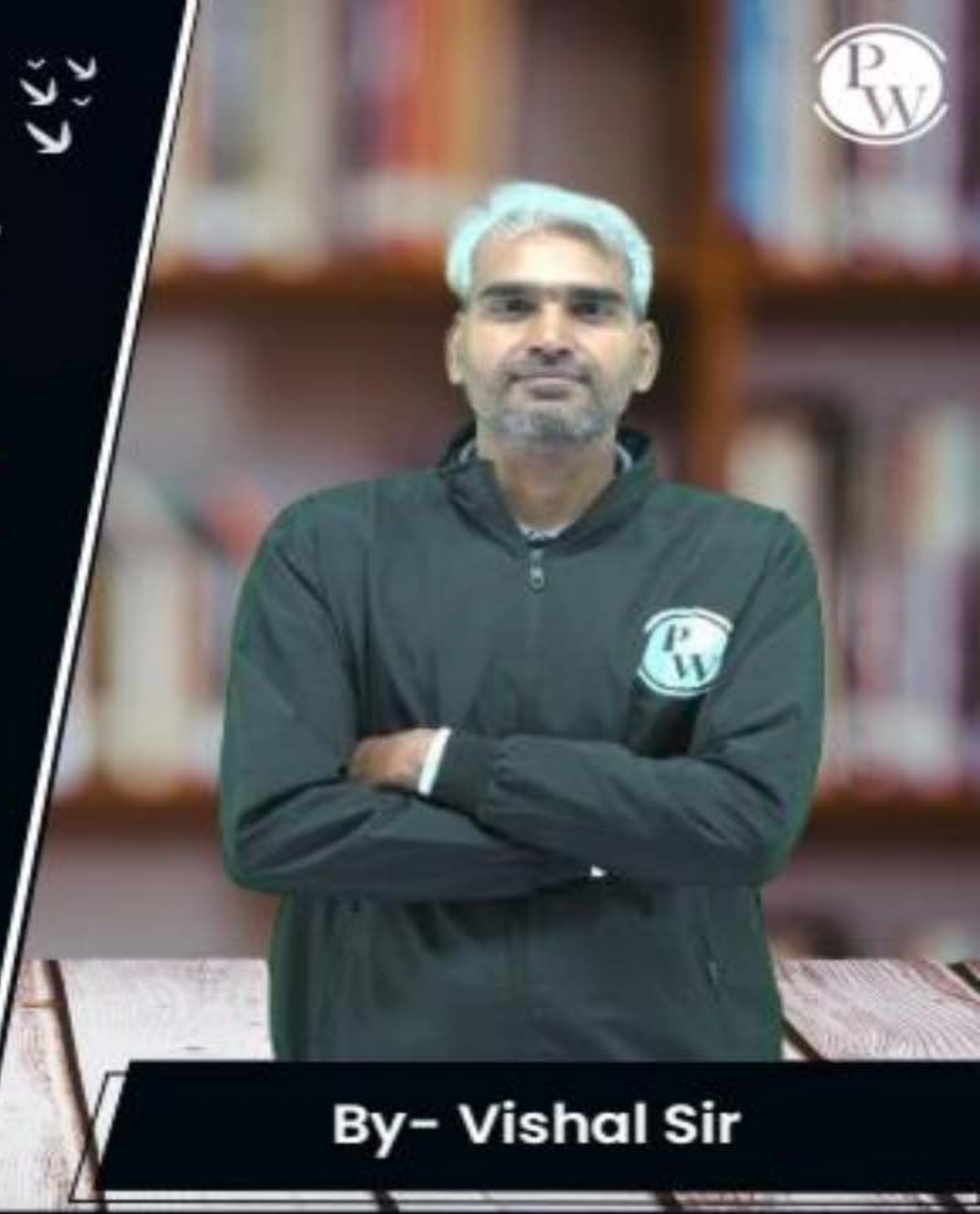
Computer Science & IT

Discrete Mathematics

Graph Theory

Lecture No. 07



By- Vishal Sir

Recap of Previous Lecture

Topic Planar graph

Slide

Topics to be Covered

Topic Planar graph

Slide

Topic : Simple planar graph

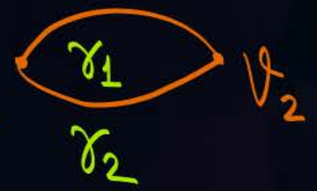
{ A planar graph with no loop & no parallel edges is called a simple planar graph. }

Loop:

"$\deg(r_1) = 1$"

{ degree of a region can be one only w.r.t. loop }

Parallel Edges:



"$\deg(r_1) = 2$"

{ degree of a region can be '2' only w.r.t. parallel edges }

i.e, In a simple planar graph degree of region can never be "1" or "2"

Topic : Simple planar graph

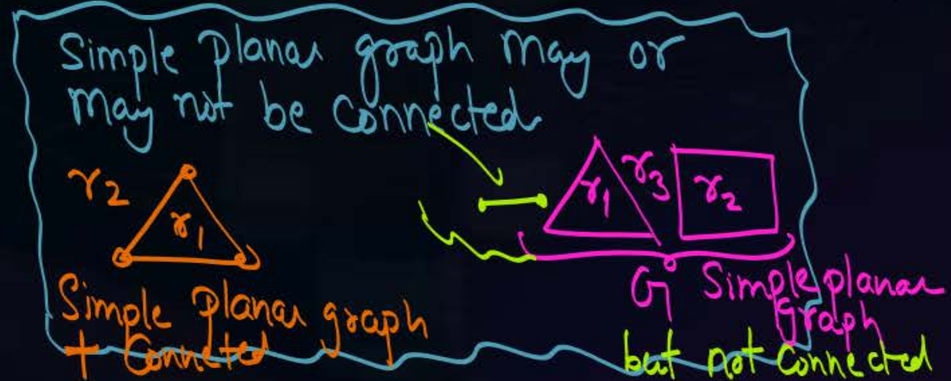
- In a simple planar graph with two or more regions degree of each region is at least "3"

i.e. In a simple planar graph

$3|R| \leq 2|E|$ will always hold true

{ if degree of each region is at least 'k' then, $k \cdot |R| \leq 2|E|$ }

$$3|R| \leq 2|E| \quad \text{— eq}^{n} \ (1)$$



(For Connected Planar graph)

Topic : Euler's equation

May or may not be simple

For any Connected Planar graph,

$$|V| + |R| = |E| + 2$$ will hold true

Where,

$|V|$ = No. of vertices

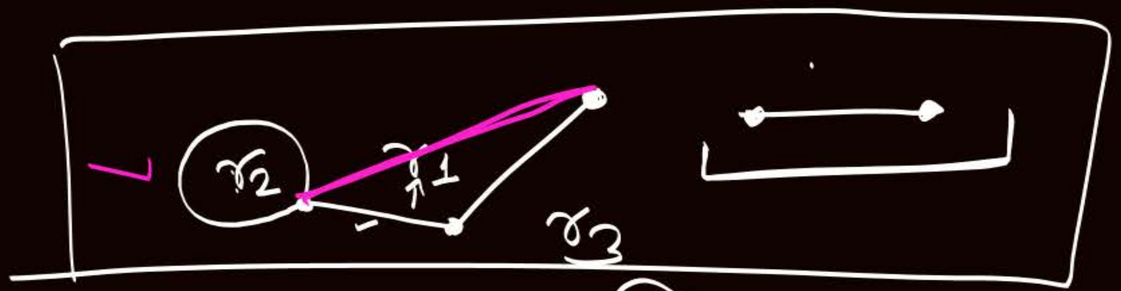
$|R|$ = No. of regions

$|E|$ = No. of Edges

let n = No. of vertices
e = No. of edges
f = No. of faces (regions)
then, $n + f = e + 2$
or $n - e + f = 2$

$$|V| + |R| = |E| + 2$$ — eqn (2)

(Will not hold in a disconnected planar graph)



$|V| + |R| = |E| + 2$

$1 + 1 = 0 + 2$ ✓

$1 + (1+1) = 1 + 2$ ✓

$(1+1) + (1+1) = (1+1) + 2$

$(1+1+1) + (1+1) = (1+1+1) + 2$

$(1+1+1) + (1+1+1) = (1+1+1+1) + 2$

Topic : Simple connected planar graph

Simple Connected Planar graph = (Simple planar graph + Connected Planar graph)

∴ $3|R| \leq 2|E|$ — eq^n ①
will hold true.

∴ $|V| + |R| = |E| + 2$ — eq^n ②
will also hold true

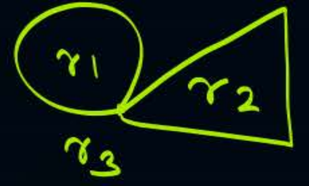
Note: ① In a simple planar graph eq^n ① will always hold true, but eq^n ② may or may not hold true

because need not be connected

② In a connected planar graph eq^n ② will always hold true, but eq^n ① may or may not hold true

because it need not be simple

Topic : Note



③ In a simple connected planar graph both eq^n ① as well as eq^n ② will always hold true

$3|R| \leq 2|E|$ — eq^n ① & $|V|+|R| = |E|+2$ — eq^n ②

By eq^n ① $|R| \leq \frac{2|E|}{3}$

Put this value of $|R|$ in eq^n ②

$$|V| + \frac{2|E|}{3} \geq |E|+2$$

$$|E| \leq 3|V|-6 \quad \text{— eq}^n \text{ ③}$$

$$e \leq 3n-6$$

By eq^n ① $|E| \geq \frac{3|R|}{2}$

Put this value of $|E|$ in eq^n ②

$$|V| + |R| \geq \frac{3|R|}{2} + 2$$

$$|V| \geq \frac{|R|}{2} + 2 \quad \text{or} \quad |R| \leq 2|V|-4 \quad \text{— eq}^n \text{ ④}$$

Topic : Note

④ In a simple connected planar graph all four equations eqⁿ ①, eqⁿ ②, eqⁿ ③ & eqⁿ ④ will always hold true.

Topic : Note

In a simple connected planar graph G, there exist at least one vertex $v \in G$, such that $\deg(v) \leq 5$

In a simple Connected Planar graph G
$\delta(G) \leq 5$

let G be a simple connected planar graph such that degree of each vertex is at least k

$\therefore k|V| \leq 2|E| \quad -(x)$

For a simple connected planar graph

$|E| \leq 3|V| - 6 \quad -(y)$ {from eqn (3)}

From eqn (x) & eqn (y)

$k|V| \leq 2*(3|V|-6) \Rightarrow k \leq \frac{6|V|}{|V|} - \frac{12}{|V|} \Rightarrow k \leq 6 - \frac{12}{|V|}$

$\frac{12}{|V|} > 0$

$\Rightarrow k < 6$

i.e. $k \leq 5$

Topic : Polyhedral graph

∴ eq^n ①, ②, ③ & ④ holds true

Polyhedral graph is a Simple Connected Planar graph in which degree of every vertex is at least '3'.

Because deg. of every vertex is at least '3'

$$\therefore \quad 3|V| \leq 2|E| \qquad \text{——— eq}^n \text{ ⑤}$$

In a Polyhedral graph eq^n ①, ②, ③, ④ & ⑤ will hold true

Topic : Note

Polyhedral graph :- degree of each vertex is at least '3' + Simple Connected planar graph

$3|V| \leq 2|E|$ — eq^n ⑤

$\therefore |V| + |R| = |E| + 2$ — eq^n ②

From eq^n ⑤ $|V| \leq \frac{2|E|}{3}$

Put this value of $|V|$ in eq^n ②

$|V| + |R| = |E| + 2$

$\frac{2|E|}{3} + |R| \geq |E| + 2$

$|R| \geq \frac{|E|}{3} + 2$

$\boxed{|E| \leq 3|R| - 6}$ — eq^n ⑥

From eq^n ⑤ $|E| \geq \frac{3|V|}{2}$

Put this value of $|E|$ in eq^n ②

$|V| + |R| = |E| + 2$

$|V| + |R| \geq \frac{3|V|}{2} + 2$

$|R| \geq \frac{|V|}{2} + 2$

$\boxed{|V| \leq 2|R| - 4}$ — eq^n ⑦

Topic : Note

* In a polyhedral graph all seven equations will hold true

Topic : Euler's equation for disconnected planar graph

Let G be a planar graph (May or may not be simple.) with 'k' Connected Components then,

$$|V| + |R| = |E| + K + 1$$

will hold true

Because overall graph is planar, ∴ Every Component will also be Planar.

|V| = Total no. of vertices
|E| = Total no. of edges
|R| = Total No. of regions = (Total no. of Interior regions + 1) ← Exterior region
K = No. of Components.

Let 'G' be a planar graph with 'K' Components

Graph 'G'

1st Component
Connected + Planar

Let $|V_1|$ = No. of vertices in first Component

$|E_1|$ = No. of Edges in 1st Component

$|R_1|$ = No. of regions w.r.t. first Component

Individually it is a Connected planar graph

$\therefore |V_1| + |R_1| = |E_1| + 2$ will hold

2nd Component
Connect + Planar

Let $|V_2|$ = No. of vertices in 2nd Component

$|E_2|$ = No. of Edges in 2nd Component

$|R_2|$ = No. of regions w.r.t. 2nd Component

$|V_2| + |R_2| = |E_2| + 2$

3rd Component
Connected + Planar

Let $|V_3|$ = No. of vertices in 3rd Component

$|E_3|$ = No. of Edges in 3rd Component

$|R_3|$ = No. of regions w.r.t. 3rd Component

$|V_3| + |R_3| = |E_3| + 2$

- - - - - -

Kth Component
Connected + Planar

Let $|V_K|$ = No. of vertices in Kth Component

$|E_K|$ = No. of Edges in Kth Component

$|R_K|$ = No. of regions w.r.t. Kth Component

$|V_K| + |R_K| = |E_K| + 2$

$|V_1| + |V_2| + |V_3| + \cdots + |V_K| = |V|$

$|V|$ ← Total no. of vertices in graph G_2

$|E_1| + |E_2| + |E_3| + \cdots + |E_K| = |E|$

$|E|$ ← Total no. of Edges in graph G_2

$|R_1| + |R_2| + |R_3| + \cdots + |R_K| = |R| + K - 1$

$|R|$ ← Total no. of regions in graph G_2

$[(|R_1| - 1) + (|R_2| - 1) + (|R_3| - 1) + \cdots + (|R_K| - 1)] + 1 = |R|$

No. of interior regions in 1st Component

No. of interior regions in 2nd Component

1 ← Exterior region

$|R|$ ← Total no. of regions in graph G_2

$= |R_1| + |R_2| + \cdots |R_K| - K + 1 = |R|$

$$\boxed{|R_1| + |R_2| + \cdots |R_K| = |R| + K - 1}$$

$$
\begin{aligned}
|V_1| + |R_1| &= |E_1| + 2 \\
|V_2| + |R_2| &= |E_2| + 2 \\
|V_3| + |R_3| &= |E_3| + 2 \\
&\vdots \\
|V_k| + |R_k| &= |E_k| + 2
\end{aligned}
$$

$$
(|V_1| + |V_2| + \cdots |V_k|) + (|R_1| + |R_2| + \cdots |R_k|) = (|E_1| + |E_2| + \cdots |E_k|) + 2k
$$

$$
|V| + |R| + k - 1 = |E| + 2k
$$

$$
\boxed{|V| + |R| = |E| + k + 1}
$$

Topic : Homeomorphic graphs

Two graphs G and G' are homeomorphic if there is a graph isomorphism from some subdivision of G to some subdivision of G'.

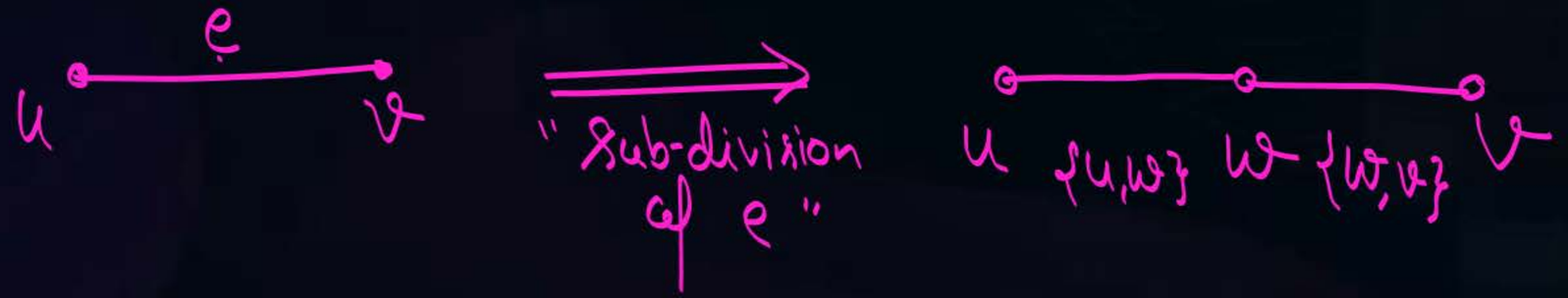
(Or)

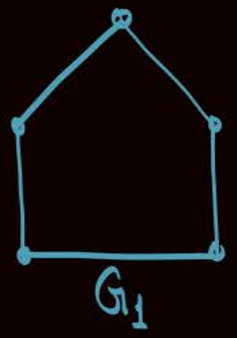
If some sub-division of graph G is isomorphic to some sub-division of graph G' then G & G' are said to be homeomorphic.

Topic : Homeomorphic graphs

A subdivision of a graph G is a graph resulting from the subdivision of some edges in G.

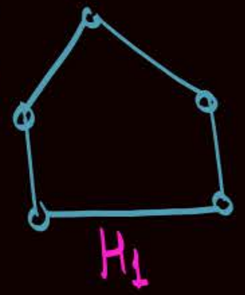
The subdivision of some edge e with endpoints {u,v} yields a graph containing one new vertex w, and with an edge set replacing e by two new edges, {u,w} and {w,v}



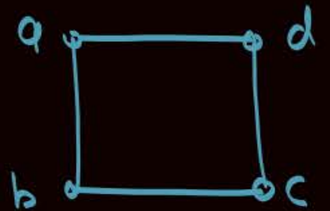


G_1

{Every graph is a sub-division of itself}

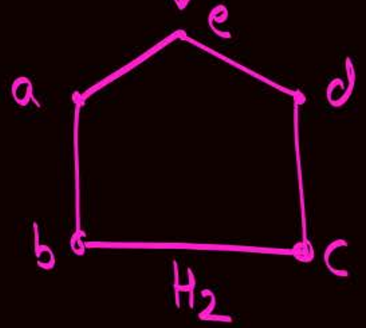


H_1



G_2

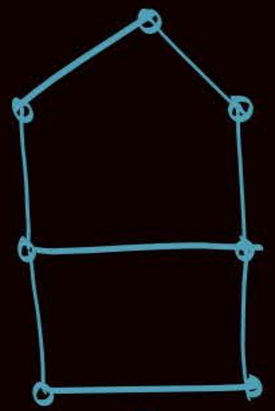
Sub-divide edge {a, d} with a new vertex e

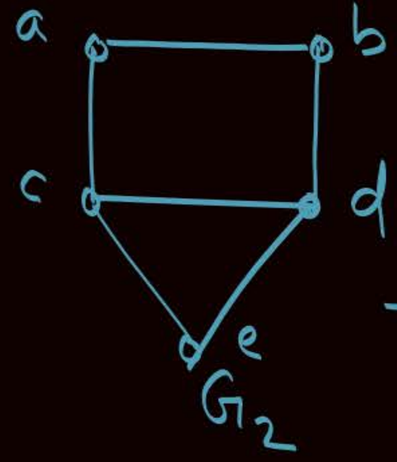


H_2 is sub-division of G_2

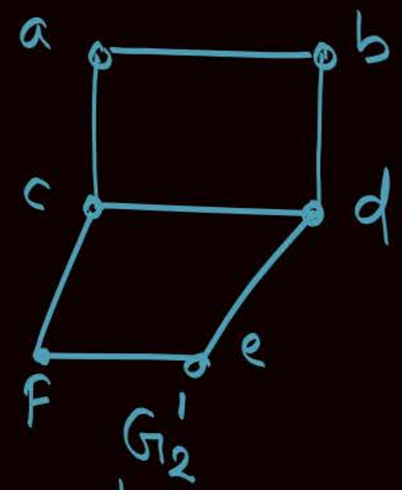
Some Subdivision of G_1 (i.e H_1) is isomorphic to some sub-division of G_2 (i.e. H_2)

↳ Hence G_1 & G_2 are Homeomorphic

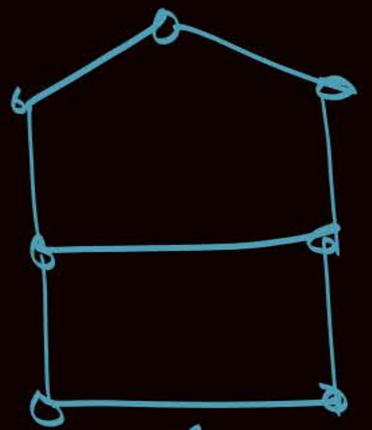




Sub-divide {c, e}



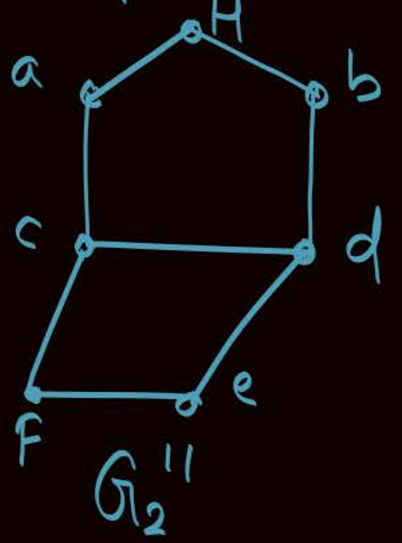
Every graph is sub-division of itself



$G_1 \cong G_2''$

i.e G_1 & G_2 are Homeomorphic

Sub-divide {a, b}

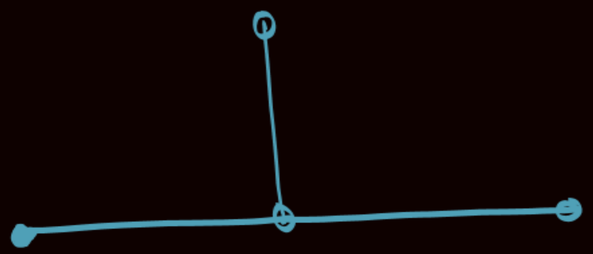


- If two graphs are isomorphic, then they are homeomorphic as well, but converse of the statement need not be true.

* we can perform multiple sub-division of edges in a sequence and all of them are sub-divisions of original graph

Topic : Homeomorphic graphs

* opposite of sub-division is smoothing



"G_1"

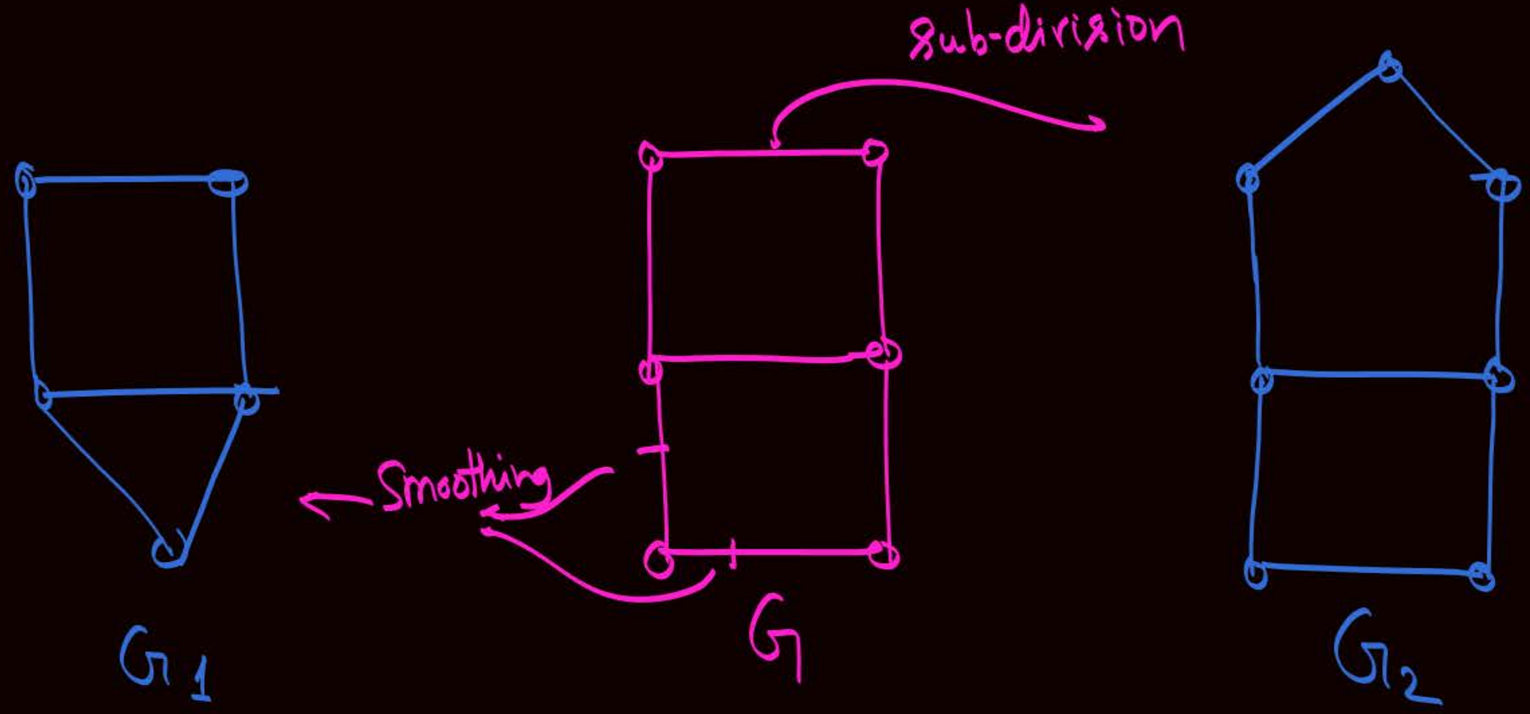
↳ Do we have any smoothing of graph G_2 { Ans = No }

Topic : Homeomorphic graphs

Two graphs G and G' are homeomorphic if there is a graph isomorphism from some subdivision of G to some subdivision of G'.

(Or)

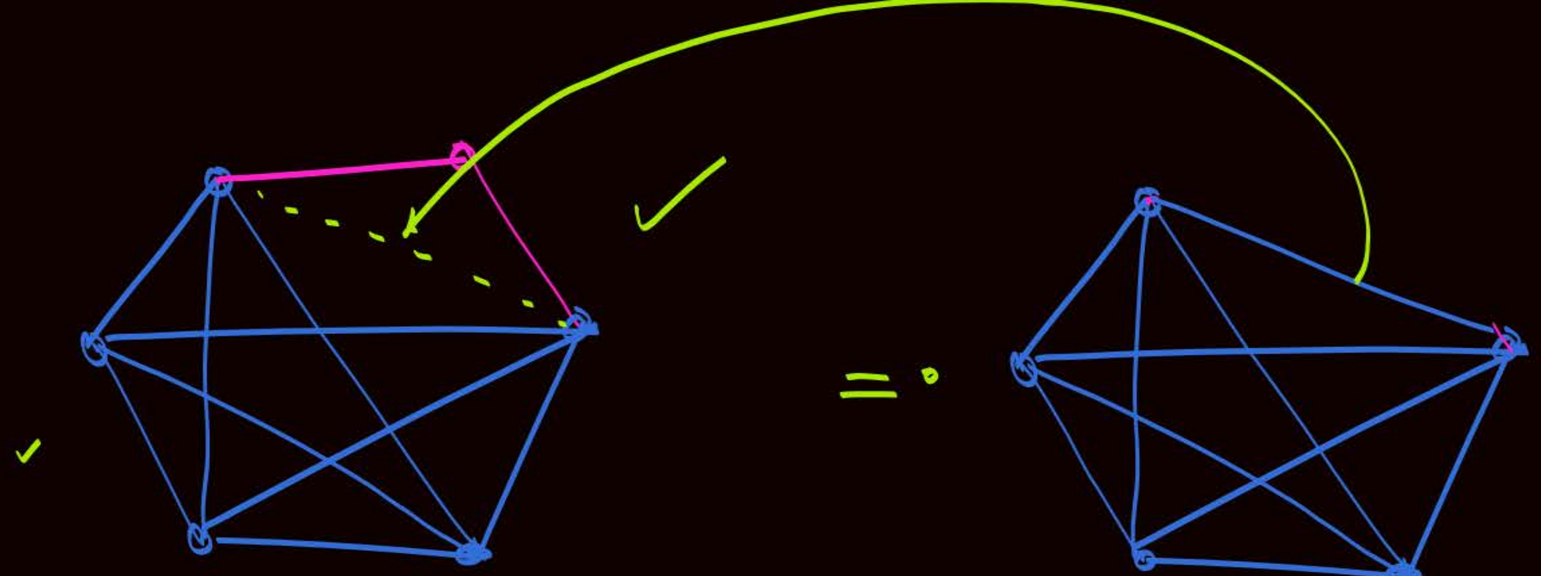
- If two graphs G_1 & G_2 can be obtained from the same graph G, by subdivision and/or smoothing of some edges of graph G, then G_1 & G_2 are said to be Homeomorphic



Both G_1 & G_2 can be obtained from G by subdivision and/or smoothing of edges of G

* Non-planar graph with Min no. of vertices is K_5

* Non-planar graph with Min no. of edges is $K_{3,3}$



$G \cong K_5$

G is a non-planar graph

Topic : Kuratowski's Theorem

A graph is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$

or

G is nonplanar if and only if G contains a subgraph that is homeomorphic to K_5 or $K_{3,3}$

2 mins Summary

Topic Planar graph

THANK - YOU