Detecting Quantum Entanglement In Multiqubit Systems Using Covariance Matrix

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Contents

Why talk about characterization using covariance matrix?

1. **Requirement of CMC**:
   Devise techniques to detect entangled states present in the system and how they respond to certain operations.For eg.- To assess whether an operation preserves or destroys entanglement.
2. **How is CMC a better criteria ?**
   By measuring the quadratures in gaussian states[1] and their variances and covariances, we can build the experimental covariance matrix, encoding information about the means and fluctuations (variances) of the quadratures..
3. **Combining with State Tomography**:
   Quantum state tomography involves reconstructing the entire density matrix of the state, for better characterization.

[1] However, we confine our discussion to the discrete case.

Covariance Matrix

- Let ρ be a pure or mixed quantum state, described by a (positive) density operator in a d-dimensional Hilbert space $\mathcal{H}$ and let $\{M_k : k = 1, \ldots, N\}$ a suitable set of observables.

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Covariance Matrix

- Let ρ be a pure or mixed quantum state, described by a (positive) density operator in a d-dimensional Hilbert space $\mathcal{H}$ and let $\{M_k : k = 1, \ldots, N\}$ a suitable set of observables.
- These observables are orthonormal observables with respect to the Hilbert-Schmidt scalar product between observables, i.e., they can be defined using $Tr(M_i M_j)$.
- As an example for such a set of observables for the case of a single qubit, one can consider the (appropriately normalized) Pauli matrices, $\mathrm{M}_1 = \frac{\mathbf{I}}{\sqrt{2}}$, $M_2 = \frac{\sigma_x}{\sqrt{2}}$, $M_3 = \frac{\sigma_y}{\sqrt{2}}$, $M_4 = \frac{\sigma_z}{\sqrt{2}}$.

Bipartite case

- The $d^2 \times d^2$ covariance matrix $\gamma = \gamma(\rho, \{M_k\})$ and the $d^2 \times d^2$ symmetrized covariance matrix (CM) $\gamma^S = \gamma^S(\rho, \{M_k\})$ are defined by their matrix entries as -

$$\gamma_{i,j} = \langle M_i M_j \rangle - \langle M_i \rangle \langle M_j \rangle, \gamma^S_{i,j} = \frac{\langle M_i M_j \rangle + \langle M_j M_i \rangle}{2} - \langle M_i \rangle \langle M_j \rangle. \tag{1}$$

- γ is a complex Hermitian matrix. The matrix γ^S in turn is real and symmetric. Both γ and γ^S are positive semidefinite, i.e., $\gamma, \gamma^S \geq 0$

Block structure

Construction

Let ρ be a state of a bipartite system, and let $M_k = \{A_k \otimes \mathbb{I}, \mathbb{I} \otimes B_k\}$ be a set of observables and $\{M_k\}$ is a pairwise complete set. Then, the block covariance matrix $\gamma(\rho, \{M_k\})$ has the entries $\gamma_{i,j} = \langle M_i M_j \rangle - \langle M_i \rangle \langle M_j \rangle$ and consequently a block structure:

$$\gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix},$$

where

$$A = \gamma(\rho_A, \{A_k\}) \text{ and } B = \gamma(\rho_B, \{B_k\})$$

are CMs of the reduced states of systems A and B, and

$$C_{i,j}{}^2 = \langle A_i \otimes B_j \rangle - \langle A_i \rangle \langle B_j \rangle .$$

[2] Similarly, $\gamma^S\{M_k\}$.

Main Idea

- We want to exploit the structure of the covariance matrix and see whether if we notice a pattern.
- As we see later, a diagonal block structure for separable states is identified by using the constraints on each of these blocks dependent of the dimension of the quantum state.

The CM criterion:

Let $\kappa_a = \sum_k p_k|\phi_a\rangle\langle\phi_a|$ and $\kappa_b = \sum_k p_k|\psi_b\rangle\langle\psi_b|$ be spread over the Hilbert spaces H_A and H_B , then it is observed that-

$$\gamma(\rho, \{M_k\}) \geq \kappa_a \oplus \kappa_b, \tag{2}$$

$$\gamma(\rho, \{M_k\}) - t(\kappa_a \oplus \kappa_b) \geq 0, \tag{3}$$

for all covariance matrices over a seperable ρ. CMC works regardless of our choice of observables.

$$M_k \to \tilde{M}_k \equiv U^\dagger M_k U = \sum_l O_{k,l} M_l \tag{4}$$

Some Important Facts about the criterion

- The transformation of observables leave the *eigenvalues invariant*.

$$\gamma(\rho, \{\tilde{M}_k\}) = \mu\gamma(\rho, \{M_k\})\mu^T \tag{5}$$

- If ρ is a d-dimensional pure state and the $\{M_k\}$ are orthogonal, then $\gamma(\rho_A) = \frac{P}{2}$, where $P^2 = P$ is a projector onto a $2(d-1)$ dimensional subspace of a total d^2-dim space. It follows that $\mathrm{Tr}(\kappa_A) = d_A - 1$ and $\mathrm{Tr}(\kappa_B) = d_B - 1$.
- So for the 2- qubit case (i.e. with $dim(\mathcal{H}_A) = dim(\mathcal{H}_B) = 2$, we will have $\mathrm{Tr}(\kappa_A) = \mathrm{Tr}(\kappa_B) = 1$. This is a particularly useful result for two qubits when $\{\mathrm{A}_k\}$=$\{B_k\}$= $\{\mathrm{I},\sigma_x, \sigma_y, \sigma_z\}/\sqrt{2}$, an 8x8 matrix is reduced to a 6x6 matrix since the column and rows corresponding to identity are zero, leading to a $dim = 3$ value of $\kappa_{A/B}$.

Designing Covariance Matrix for 2 Qubits

$$A = \begin{bmatrix} (\frac{\mathbf{I}}{\sqrt{\mathbf{2}}} \otimes \mathbf{I})(\frac{\mathbf{I}}{\sqrt{\mathbf{2}}} \otimes \mathbf{I}) & (\frac{\mathbf{I}}{\sqrt{\mathbf{2}}} \otimes \mathbf{I})(\frac{\sigma_x}{\sqrt{2}} \otimes \mathbf{I}) & (\frac{\mathbf{I}}{\sqrt{\mathbf{2}}} \otimes \mathbf{I})(\frac{\sigma_y}{\sqrt{2}} \otimes \mathbf{I}) & (\frac{\mathbf{I}}{\sqrt{\mathbf{2}}} \otimes \mathbf{I})(\frac{\sigma_z}{\sqrt{2}} \otimes \mathbf{I}) \\ (\frac{\sigma_x}{\sqrt{2}} \otimes \mathbf{I})(\frac{\mathbf{I}}{\sqrt{\mathbf{2}}} \otimes \mathbf{I}) & (\frac{\sigma_x}{\sqrt{2}} \otimes \mathbf{I})(\frac{\sigma_x}{\sqrt{2}} \otimes \mathbf{I}) & (\frac{\sigma_x}{\sqrt{2}} \otimes \mathbf{I})(\frac{\sigma_y}{\sqrt{2}} \otimes \mathbf{I}) & (\frac{\sigma_x}{\sqrt{2}} \otimes \mathbf{I})(\frac{\sigma_z}{\sqrt{2}} \otimes \mathbf{I}) \\ (\frac{\sigma_y}{\sqrt{2}} \otimes \mathbf{I})(\frac{\mathbf{I}}{\sqrt{\mathbf{2}}} \otimes \mathbf{I}) & (\frac{\sigma_y}{\sqrt{2}} \otimes \mathbf{I})(\frac{\sigma_x}{\sqrt{2}} \otimes \mathbf{I}) & (\frac{\sigma_y}{\sqrt{2}} \otimes \mathbf{I})(\frac{\sigma_y}{\sqrt{2}} \otimes \mathbf{I}) & (\frac{\sigma_y}{\sqrt{2}} \otimes \mathbf{I})(\frac{\sigma_z}{\sqrt{2}} \otimes \mathbf{I}) \\ (\frac{\sigma_z}{\sqrt{2}} \otimes \mathbf{I})(\frac{\mathbf{I}}{\sqrt{\mathbf{2}}} \otimes \mathbf{I}) & (\frac{\sigma_z}{\sqrt{2}} \otimes \mathbf{I})(\frac{\sigma_x}{\sqrt{2}} \otimes \mathbf{I}) & (\frac{\sigma_z}{\sqrt{2}} \otimes \mathbf{I})(\frac{\sigma_y}{\sqrt{2}} \otimes \mathbf{I}) & (\frac{\sigma_z}{\sqrt{2}} \otimes \mathbf{I})(\frac{\sigma_z}{\sqrt{2}} \otimes \mathbf{I}) \end{bmatrix}$$

$$B = \begin{bmatrix} (\mathbf{I} \otimes \frac{\mathbf{I}}{\sqrt{\mathbf{2}}})(\mathbf{I} \otimes \frac{\mathbf{I}}{\sqrt{\mathbf{2}}}) & (\mathbf{I} \otimes \frac{\mathbf{I}}{\sqrt{\mathbf{2}}})(\mathbf{I} \otimes \frac{\sigma_x}{\sqrt{2}}) & (\mathbf{I} \otimes \frac{\mathbf{I}}{\sqrt{\mathbf{2}}})(\mathbf{I} \otimes \frac{\sigma_y}{\sqrt{2}}) & (\mathbf{I} \otimes \frac{\mathbf{I}}{\sqrt{\mathbf{2}}})(\mathbf{I} \otimes \frac{\sigma_z}{\sqrt{2}}) \\ (\mathbf{I} \otimes \frac{\sigma_x}{\sqrt{2}})(\mathbf{I} \otimes \frac{\mathbf{I}}{\sqrt{\mathbf{2}}}) & (\mathbf{I} \otimes \frac{\sigma_x}{\sqrt{2}})(\mathbf{I} \otimes \frac{\sigma_x}{\sqrt{2}}) & (\mathbf{I} \otimes \frac{\sigma_x}{\sqrt{2}})(\mathbf{I} \otimes \frac{\sigma_y}{\sqrt{2}}) & (\mathbf{I} \otimes \frac{\sigma_x}{\sqrt{2}})(\mathbf{I} \otimes \frac{\sigma_z}{\sqrt{2}}) \\ (\mathbf{I} \otimes \frac{\sigma_y}{\sqrt{2}})(\mathbf{I} \otimes \frac{\mathbf{I}}{\sqrt{\mathbf{2}}}) & (\mathbf{I} \otimes \frac{\sigma_y}{\sqrt{2}})(\mathbf{I} \otimes \frac{\sigma_x}{\sqrt{2}}) & (\mathbf{I} \otimes \frac{\sigma_y}{\sqrt{2}})(\mathbf{I} \otimes \frac{\sigma_y}{\sqrt{2}}) & (\mathbf{I} \otimes \frac{\sigma_y}{\sqrt{2}})(\mathbf{I} \otimes \frac{\sigma_z}{\sqrt{2}}) \\ (\mathbf{I} \otimes \frac{\sigma_z}{\sqrt{2}})(\mathbf{I} \otimes \frac{\mathbf{I}}{\sqrt{\mathbf{2}}}) & (\mathbf{I} \otimes \frac{\sigma_z}{\sqrt{2}})(\mathbf{I} \otimes \frac{\sigma_x}{\sqrt{2}}) & (\mathbf{I} \otimes \frac{\sigma_z}{\sqrt{2}})(\mathbf{I} \otimes \frac{\sigma_y}{\sqrt{2}}) & (\mathbf{I} \otimes \frac{\sigma_z}{\sqrt{2}})(\mathbf{I} \otimes \frac{\sigma_z}{\sqrt{2}}) \end{bmatrix}$$

$$C = \begin{bmatrix} (\frac{\mathbf{I}}{\sqrt{\mathbf{2}}} \otimes \mathbf{I})(\mathbf{I} \otimes \frac{\mathbf{I}}{\sqrt{\mathbf{2}}}) & (\frac{\mathbf{I}}{\sqrt{\mathbf{2}}} \otimes \mathbf{I})(\mathbf{I} \otimes \frac{\sigma_x}{\sqrt{2}}) & (\frac{\mathbf{I}}{\sqrt{\mathbf{2}}} \otimes \mathbf{I})(\mathbf{I} \otimes \frac{\sigma_y}{\sqrt{2}}) & (\frac{\mathbf{I}}{\sqrt{\mathbf{2}}} \otimes \mathbf{I})(\mathbf{I} \otimes \frac{\sigma_z}{\sqrt{2}}) \\ (\frac{\sigma_x}{\sqrt{2}} \otimes \mathbf{I})(\mathbf{I} \otimes \frac{\mathbf{I}}{\sqrt{\mathbf{2}}}) & (\frac{\sigma_x}{\sqrt{2}} \otimes \mathbf{I})(\mathbf{I} \otimes \frac{\sigma_x}{\sqrt{2}}) & (\frac{\sigma_x}{\sqrt{2}} \otimes \mathbf{I})(\mathbf{I} \otimes \frac{\sigma_y}{\sqrt{2}}) & (\frac{\sigma_x}{\sqrt{2}} \otimes \mathbf{I})(\mathbf{I} \otimes \frac{\sigma_z}{\sqrt{2}}) \\ (\frac{\sigma_y}{\sqrt{2}} \otimes \mathbf{I})(\mathbf{I} \otimes \frac{\mathbf{I}}{\sqrt{\mathbf{2}}}) & (\frac{\sigma_y}{\sqrt{2}} \otimes \mathbf{I})(\mathbf{I} \otimes \frac{\sigma_x}{\sqrt{2}}) & (\frac{\sigma_y}{\sqrt{2}} \otimes \mathbf{I})(\mathbf{I} \otimes \frac{\sigma_y}{\sqrt{2}}) & (\frac{\sigma_y}{\sqrt{2}} \otimes \mathbf{I})(\mathbf{I} \otimes \frac{\sigma_z}{\sqrt{2}}) \\ (\frac{\sigma_z}{\sqrt{2}} \otimes \mathbf{I})(\mathbf{I} \otimes \frac{\mathbf{I}}{\sqrt{\mathbf{2}}}) & (\frac{\sigma_z}{\sqrt{2}} \otimes \mathbf{I})(\mathbf{I} \otimes \frac{\sigma_x}{\sqrt{2}}) & (\frac{\sigma_z}{\sqrt{2}} \otimes \mathbf{I})(\mathbf{I} \otimes \frac{\sigma_y}{\sqrt{2}}) & (\frac{\sigma_z}{\sqrt{2}} \otimes \mathbf{I})(\mathbf{I} \otimes \frac{\sigma_z}{\sqrt{2}}) \end{bmatrix}$$

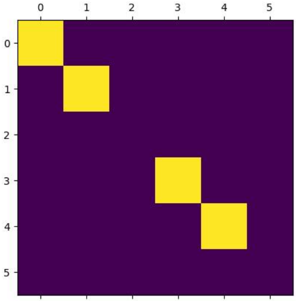
2 Qubit Separable State

- $|\psi\rangle = |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Co-variance Matrix :

$$\begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pictorial Representation of Blocks :



$t = 1$

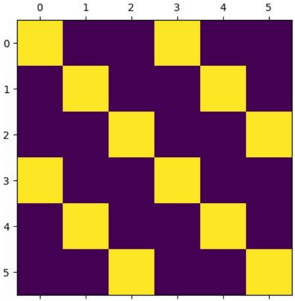
2 Qubit Bell (Werner) State

- $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \begin{bmatrix} 0.707 \\ 0 \\ 0 \\ 0.707 \end{bmatrix}$

Co-variance Matrix :

$$\begin{bmatrix} 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 \end{bmatrix}$$

Pictorial Representation of Blocks :



$t < 1$

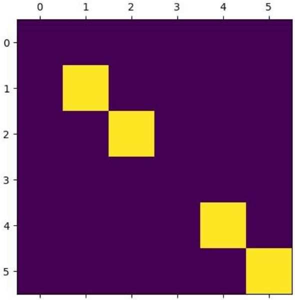
2 Qubit Separable Werner State

- $|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$

Co-variance Matrix :

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}$$

Pictorial Representation of Blocks :



$t = 1$

Three qubit extension

Block structure for 3 qubits

For three qubit, the block structure with $\{A_k\}$=$\{B_k\}$= $\{C_k\}$= $\{\mathrm{I}, \sigma_x, \sigma_y, \sigma_z\}/\sqrt{2}$

$$\gamma(\rho_{ABC}, \{M_k\}) = \begin{pmatrix} A & D & E \\ D^T & B & F \\ E^T & F^T & C \end{pmatrix} \tag{6}$$

The CMC is then given by-

$$\gamma - (\kappa_A \oplus \kappa_B \oplus \kappa_C) \geq 0 \tag{7}$$

- The above equation is modified to $\gamma - t(X_a \oplus X_b \oplus X_c) \geq 0$ and is treated as a feasibility problem.
  Hence, t<1 $\implies$ Entangled state.

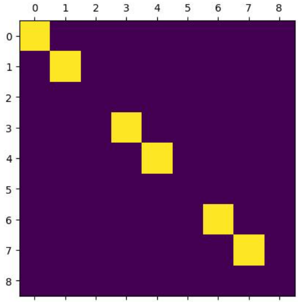
3 Qubit Separable State

- $|\psi\rangle = |011\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Co-variance Matrix :

$$\begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pictorial Representation of Blocks :



$t = 1$

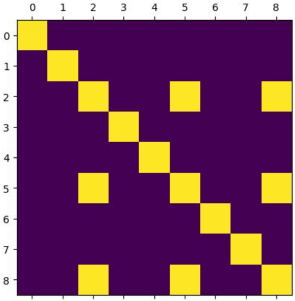
3 Qubit GHZ State

- $|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) = \begin{bmatrix} 0.707 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.707 \end{bmatrix}$

Co-variance Matrix :

$$\begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0.5 \end{bmatrix}$$

Pictorial Representation of Blocks :



t < 1

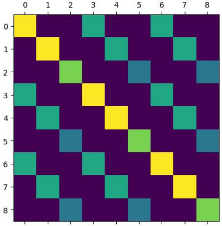
3 Qubit W State

- $|\psi\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle) = \begin{bmatrix} 0 \\ 0.577 \\ 0.577 \\ 0 \\ 0.577 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Co-variance Matrix :

$$\begin{bmatrix} 0.5 & 0 & 0 & 0.3 & 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.3 & 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0.4 & 0 & 0 & -0.2 & 0 & 0 & -0.2 \\ 0.3 & 0 & 0 & 0.5 & 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0.5 & 0 & 0 & 0.3 & 0 \\ 0 & 0 & -0.2 & 0 & 0 & 0.4 & 0 & 0 & -0.2 \\ 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & -0.2 & 0 & 0 & -0.2 & 0 & 0 & 0.4 \end{bmatrix}$$

Pictorial Representation of Blocks :



$t < 1$

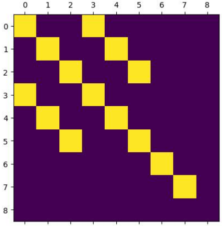
3 Qubit Bi separable State

- $|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |110\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes |0\rangle = \begin{bmatrix} 0.707 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.707 \\ 0 \end{bmatrix}$

Co-variance Matrix :

$$\begin{bmatrix} 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & -0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pictorial Representation of Blocks :



$t < 1$

CM for 4-qubits

Consider the Hilbert space $H_A \otimes H_B \otimes H_C \otimes H_D$ where the observable
$M_k = \{A_k \otimes I \otimes I \otimes I, I \otimes B_k \otimes I \otimes I, \ldots, I \otimes I \otimes I \otimes D_k\}$, where M_k is a pairwise distinct set and γ is defined as usual . The CM would now be seen as follows-

$$\gamma(\rho_{ABCD}, \{M_k\}) = \begin{pmatrix} A & E & F & G \\ E^T & B & H & I \\ F^T & H^T & C & J \\ G^T & I^T & J^T & D \end{pmatrix}$$

where A,B,C and D are the reduced covariance matrices, with $E_{kl} = \langle A_k B_l \rangle - \langle A_k \rangle \langle B_l \rangle$ and so on for the rest of the blocks.

CMC for 4-qubits

Formalism for 4-qubits

Let $\kappa_a = \sum_k p_k|\phi_a\rangle\langle\phi_a|$, $\kappa_b = \sum_k p_k|\psi_b\rangle\langle\psi_b|$,$\kappa_c = \sum_k p_k|\phi_c\rangle\langle\phi_c|$, $\kappa_d = \sum_k p_k|\psi_d\rangle\langle\psi_d|$ be spread over the Hilbert spaces $\mathcal{H}_A$, $\mathcal{H}_B$, $\mathcal{H}_C$ and $\mathcal{H}_d$, Then it is observed that-

$$\gamma(\rho_{ABCD}, \{M_k\}) \geq \kappa_a \oplus \kappa_b \oplus \kappa_c \oplus \kappa_d$$

- $\gamma(\rho, \{\tilde{M}_k\}) = \mu\gamma(\rho, \{M_k\})\mu^T$
- $M_k \rightarrow \tilde{M}_k \equiv U^\dagger M_k U = \sum_l O_{k,l} M_l$
- We know that the reduced covariance matrices form a Projector (P/2), where P is a projector onto the 2(d-1) dim subspace out of total d^2 dimensions.

More about the criterion

For a pure state with the state dimension two (qubit case) we readily see that-

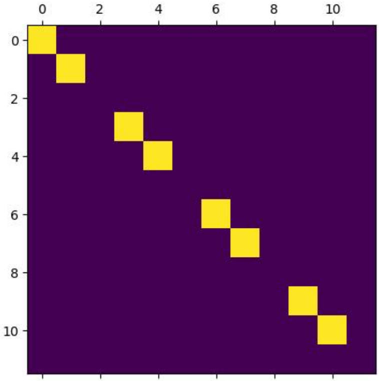
$$\gamma(|a\rangle\langle a|, \{A_k\}) = \frac{P}{2} = \frac{I_3 - |\phi_a\rangle\langle\phi_a|}{2} = \kappa_a$$

Thus allowing us to comment that $\mathrm{Tr}(\kappa_{a/b/c/d}) = 1$.
We restate the problem again as a feasibility one, i.e.,
$\gamma - t(X_a \oplus X_b \oplus X_c \oplus X_d) \geq 0$.
The objective of this would be to try and propose a measure of quantification for entanglement.

4 Qubit Separable State

- $$|\psi\rangle = |0101\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Pictorial Representation of Blocks :



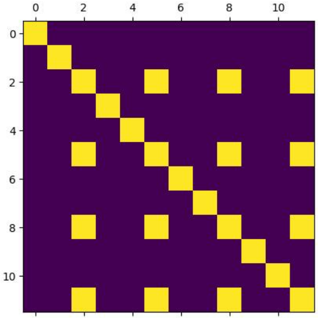
$t = 1$

4 Qubit GHZ like State

- $$|\psi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$$

$$= \begin{bmatrix} 0.707 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.707 \end{bmatrix}$$

Pictorial Representation of Blocks :



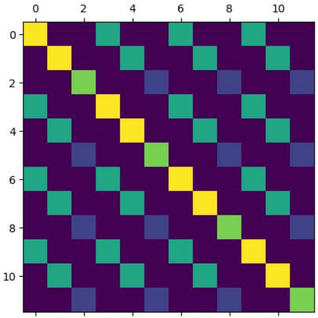
$t < 1$

4 Qubit W like State

- $|\psi\rangle = \frac{1}{2}(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle)$

$$= \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \\ 0 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Pictorial Representation of Blocks :



$t < 1$

4 Qubit W like State

Co-variance Matrix :

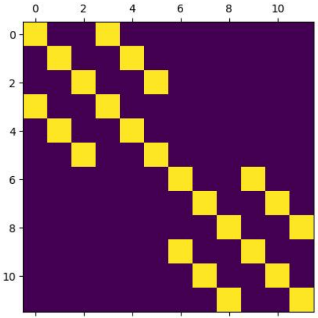
$$\begin{bmatrix}
0.5 & 0 & 0 & 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.3 & 0 & 0 \\
0 & 0.5 & 0 & 0 & 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.3 & 0 \\
0 & 0 & 0.4 & 0 & 0 & -0.1 & 0 & 0 & -0.1 & 0 & 0 & -0.1 \\
0.3 & 0 & 0 & 0.5 & 0 & 0 & 0.3 & 0 & 0 & 0.3 & 0 & 0 \\
0 & 0.3 & 0 & 0 & 0.5 & 0 & 0 & 0.3 & 0 & 0 & 0.3 & 0 \\
0 & 0 & -0.1 & 0 & 0 & 0.4 & 0 & 0 & -0.1 & 0 & 0 & -0.1 \\
0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.5 & 0 & 0 & 0.3 & 0 & 0 \\
0 & 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.5 & 0 & 0 & 0.3 & 0 \\
0 & 0 & -0.1 & 0 & 0 & -0.1 & 0 & 0 & 0.4 & 0 & 0 & -0.1 \\
0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.5 & 0 & 0 \\
0 & 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.5 & 0 \\
0 & 0 & -0.1 & 0 & 0 & -0.1 & 0 & 0 & -0.1 & 0 & 0 & 0.4
\end{bmatrix}$$

4 Qubit Bi Separable State

- $|\psi\rangle = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle) =$

$$\frac{(|00\rangle+|11\rangle)\otimes(|00\rangle+|11\rangle)}{2} = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ 0 \\ 0.5 \end{bmatrix}$$

Pictorial Representation of Blocks :



$t < 1$

4 Qubit Bi Separable like State

Co-variance Matrix :

$$\begin{bmatrix}
0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.5 & 0 & 0 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & -0.5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.5 & 0 & 0 & 0.5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5
\end{bmatrix}$$

Let's look at the feasibility formulation

Remember
$\gamma - t(\kappa_a \oplus \kappa_b \oplus \kappa_c \oplus \kappa_d) \geq 0$

Entanglement Parameter

Let ρ be a 4 party quantum state with CM $\gamma(\rho)$. We define a function $V(\rho)$ as
$\mathrm{V}(\rho) = \max_{t,\kappa_A,\kappa_B}\{t \leq 1 : \gamma(\rho) - t\kappa_A \oplus \kappa_B \oplus \kappa_C \oplus \kappa_D \geq 0\}$.
The entanglement parameter $E(\rho)$ is then defined as
$\mathrm{E}(\rho) = 1 - V(\rho)$.

The entanglement parameter $E(\rho)$ is invariant under local unitary transformations and is convex in the state, that is for $\rho = p\rho_1 + (1-p)\rho_2$ we have that $E(\rho) \leq pE(\rho_1) + (1-p)E(\rho_2)$.

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Thanks For Your Attention!